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Weak solutions for Navier–Stokes equations with initial data in weighted L^2 spaces.

Pedro Gabriel Fernández-Dalgo*†, Pierre Gilles Lemarié–Rieusset‡§

Abstract

We show the existence of global weak solutions of the 3D Navier-Stokes equations with initial velocity in the weighted spaces $L^2_{w_{\gamma}}$, where $w_{\gamma}(x) = (1+|x|)^{-\gamma}$ and $0 < \gamma \le 2$, using new energy controls. As application we give a new proof of the existence of global weak discretely self-similar solutions of the 3D Navier–Stokes equations for discretely self-similar initial velocities which are locally square integrable.

Keywords: Navier-Stokes equations, weighted spaces, discretely self-

similar solutions, energy controls

AMS classification: 35Q30, 76D05.

1 Introduction.

Infinite-energy weak Leray solutions to the Navier–Stokes equations were introduced by Lemarié-Rieusset in 1999 [8] (they are presented more completely in [9] and [10]). This has allowed to show the existence of local weak solutions for a uniformly locally square integrable initial data.

Other constructions of infinite-energy solutions for locally uniformly square integrable initial data were given in 2006 by Basson [1] and in 2007 by Kikuchi and Seregin [7]. These solutions allowed Jia and Sverak [6] to construct in 2014 the self-similar solutions for large (homogeneous of degree -1) smooth data. Their result has been extended in 2016 by Lemarié-Rieusset [10] to

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solutions for rough locally square integrable data. We remark that an homogeneous (of degree -1) and locally square integrable data is automatically uniformly locally L^2 .

Recently, Bradshaw and Tsai [2] and Chae and Wolf [3] considered the case of solutions which are self-similar according to a discrete subgroup of dilations. Those solutions are related to an initial data which is self-similar only for a discrete group of dilations; in contrast to the case of self-similar solutions for all dilations, such an initial data, when locally L^2 , is not necessarily uniformly locally L^2 , therefore their results are no consequence of constructions described by Lemarié-Rieusset in [10].

In this paper, we construct an alternative theory to obtain infinite-energy global weak solutions for large initial data, which include the discretely self-similar locally square integrable data. More specifically, we consider the weights

$$w_{\gamma}(x) = \frac{1}{(1+|x|)^{\gamma}}$$

with $0 < \gamma$, and the spaces

$$L_{w_{\gamma}}^2 = L^2(w_{\gamma} \, dx).$$

Our main theorem is the following one:

Theorem 1 Let $0 < \gamma \le 2$. If \mathbf{u}_0 is a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_{\gamma}}(\mathbb{R}^3)$ and if \mathbb{F} is a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,+\infty),L^2_{w_{\gamma}})$, then the Navier-Stokes equations with initial value \mathbf{u}_0

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

has a global weak solution \mathbf{u} such that :

- for every $0 < T < +\infty$, **u** belongs to $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and ∇ **u** belongs to $L^2((0,T), L^2_{w_{\gamma}})$
- the pressure p is related to \mathbf{u} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j - F_{i,j})$$

where, for every $0 < T < +\infty$, $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(u_i u_j)$ belongs to $L^4((0,T), L_{w_{\widehat{\gamma}}}^{6/5})$ and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ belongs to $L^2((0,T), L_{w_{\gamma}}^2)$

• the map $t \in [0, +\infty) \mapsto \mathbf{u}(t, .)$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_{\gamma}}$, and is strongly continuous at t = 0:

$$\lim_{t\to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

• the solution **u** is suitable: there exists a non-negative locally finite measure μ on $(0, +\infty) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left((\frac{|\mathbf{u}|^2}{2} + p)\mathbf{u} \right) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

In particular, we have the energy controls

$$\|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + 2 \int_{0}^{t} \|\nabla \mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} - \int_{0}^{t} \int \nabla |\mathbf{u}|^{2} \cdot \nabla w_{\gamma} dx ds + \int_{0}^{t} \int (|\mathbf{u}|^{2} + 2p)\mathbf{u} \cdot \nabla (w_{\gamma}) dx ds$$

$$-2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j}(\partial_{i}u_{j})w_{\gamma} + F_{i,j}u_{i}\partial_{j}(w_{\gamma}) dx ds$$

and

$$\|\mathbf{u}(t,.)\|_{L^2_{w\gamma}}^2 \leq \|\mathbf{u}_0\|_{L^2_{w\gamma}}^2 + C_{\gamma} \int_0^t \|\mathbb{F}(s,.)\|_{L^2_{w\gamma}}^2 ds + C_{\gamma} \int_0^t \|\mathbf{u}(s,.)\|_{L^2_{w\gamma}}^2 + \|\mathbf{u}(s,.)\|_{L^2_{w\gamma}}^6 ds$$

A key tool for proving Theorem 1 and for applying it to the study of discretely self-similar solutions is given by the following a priori estimates for an advection-diffusion problem :

Theorem 2 Let $0 < \gamma \le 2$. Let $0 < T < +\infty$. Let \mathbf{u}_0 be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let \mathbf{b} be a time-dependent divergence free vector-field $(\nabla \cdot \mathbf{b} = 0)$ such that $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.

Let ${f u}$ be a solution of the following advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

be such that:

• **u** belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and ∇ **u** belongs to $L^2((0,T),L^2_{w_{\gamma}})$

• the pressure p is related to **u**, **b** and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j})$$

where $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(b_i u_j)$ belongs to $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$ and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ belongs to $L^2((0,T), L_{w_{\gamma}}^2)$

• the map $t \in [0,T) \mapsto \mathbf{u}(t,.)$ is weakly continuous from [0,T) to $L^2_{w_{\gamma}}$, and is strongly continuous at t=0:

$$\lim_{t \to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

• there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu. \tag{1}$$

Then, we have the energy controls

$$\|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + 2 \int_{0}^{t} \|\nabla \mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} - \int_{0}^{t} \int \nabla |\mathbf{u}|^{2} \cdot \nabla w_{\gamma} dx ds + \int_{0}^{t} \int |\mathbf{u}|^{2} \mathbf{b} \cdot \nabla (w_{\gamma}) dx ds$$

$$+ 2 \int_{0}^{t} \int p\mathbf{u} \cdot \nabla (w_{\gamma}) dx ds - 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j}(\partial_{i}u_{j})w_{\gamma} + F_{i,j}u_{i}\partial_{j}(w_{\gamma}) dx ds$$

and

$$\begin{aligned} \|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \int_{0}^{t} \|\nabla \mathbf{u}\|_{L^{2}_{w_{\gamma}}}^{2} ds \\ \leq & \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{t} \|\mathbb{F}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds + C_{\gamma} \int_{0}^{t} (1 + \|\mathbf{b}(s,.)\|_{L^{3}_{w_{3\gamma/2}}}^{2}) \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds \end{aligned}$$

where C_{γ} depends only on γ (and not on T, and not on \mathbf{b} , \mathbf{u} , \mathbf{u}_0 nor \mathbb{F}).

In particular, we shall prove the following stability result:

Theorem 3 Let $0 < \gamma \le 2$. Let $0 < T < +\infty$. Let $\mathbf{u}_{0,n}$ be divergence-free vector fields such that $\mathbf{u}_{0,n} \in L^2_{w_{\gamma}}(\mathbb{R}^3)$ and \mathbb{F}_n be tensors such that $\mathbb{F}_n \in L^2((0,T),L^2_{w_{\gamma}})$. Let \mathbf{b}_n be time-dependent divergence free vector-fields such that $\mathbf{b}_n \in L^3((0,T),L^3_{w_{3\gamma/2}})$.

Let \mathbf{u}_n be solutions of the following advection-diffusion problems

$$(AD_n) \begin{cases} \partial_t \mathbf{u}_n = \Delta \mathbf{u}_n - (\mathbf{b}_n \cdot \nabla) \mathbf{u}_n - \nabla p_n + \nabla \cdot \mathbb{F}_n \\ \nabla \cdot \mathbf{u}_n = 0, & \mathbf{u}_n(0,.) = \mathbf{u}_{0,n} \end{cases}$$

such that:

- \mathbf{u}_n belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_n$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$
- the pressure p_n is related to \mathbf{u}_n , \mathbf{b}_n and \mathbb{F}_n by the formula

$$p_n = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_{n,i} u_{n,j} - F_{n,i,j})$$

• the map $t \in [0,T) \mapsto \mathbf{u}_n(t,.)$ is weakly continuous from [0,T) to $L^2_{w_\gamma}$, and is strongly continuous at t=0:

$$\lim_{t\to 0} \|\mathbf{u}_n(t,.) - \mathbf{u}_{0,n}\|_{L^2_{w_{\gamma}}} = 0.$$

• there exists a non-negative locally finite measure μ_n on $(0,T) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}_n|^2}{2}) = \Delta(\frac{|\mathbf{u}_n|^2}{2}) - |\nabla \mathbf{u}_n|^2 - \nabla \cdot \left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right) - \nabla \cdot (p_n\mathbf{u}_n) + \mathbf{u}_n \cdot (\nabla \cdot \mathbb{F}_n) - \mu_n.$$

If $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_{0,\infty}$ in $L^2_{w_{\gamma}}$, if the sequence \mathbb{F}_n is strongly convergent to \mathbb{F}_{∞} in $L^2((0,T),L^2_{w_{\gamma}})$, and if the sequence \mathbf{b}_n is bounded in $L^3((0,T),L^3_{w_{3\gamma/2}})$, then there exists p_{∞} , \mathbf{u}_{∞} , \mathbf{b}_{∞} and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in \mathbb{N} such that

- \mathbf{u}_{n_k} converges *-weakly to \mathbf{u}_{∞} in $L^{\infty}((0,T), L^2_{w_{\gamma}})$, $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}_{\infty}$ in $L^2((0,T), L^2_{w_{\gamma}})$
- \mathbf{b}_{n_k} converges weakly to \mathbf{b}_{∞} in $L^3((0,T), L^3_{w_{3\gamma/2}})$, p_{n_k} converges weakly to p_{∞} in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T), L^2_{w_{\gamma}})$

• \mathbf{u}_{n_k} converges strongly to \mathbf{u}_{∞} in $L^2_{loc}([0,T)\times\mathbb{R}^3)$: for every $T_0\in(0,T)$ and every R>0, we have

$$\lim_{k \to +\infty} \int_0^{T_0} \int_{|y| < R} |\mathbf{u}_{n_k}(s, y) - \mathbf{u}_{\infty}(s, y)|^2 \, ds \, dy = 0.$$

Moreover, \mathbf{u}_{∞} is a solution of the advection-diffusion problem

$$(AD_{\infty}) \begin{cases} \partial_t \mathbf{u}_{\infty} = \Delta \mathbf{u}_{\infty} - (\mathbf{b}_{\infty} \cdot \nabla) \mathbf{u}_{\infty} - \nabla p_{\infty} + \nabla \cdot \mathbb{F}_{\infty} \\ \nabla \cdot \mathbf{u}_{\infty} = 0, \qquad \mathbf{u}_{\infty}(0, .) = \mathbf{u}_{0, \infty} \end{cases}$$

and is such that:

• the map $t \in [0,T) \mapsto \mathbf{u}_{\infty}(t,.)$ is weakly continuous from [0,T) to $L^2_{w_{\gamma}}$, and is strongly continuous at t=0:

$$\lim_{t\to 0} \|\mathbf{u}_{\infty}(t,.) - \mathbf{u}_{0,\infty}\|_{L^2_{w_{\gamma}}} = 0.$$

• there exists a non-negative locally finite measure μ_{∞} on $(0,T) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}_{\infty}|^2}{2}) = \Delta(\frac{|\mathbf{u}_{\infty}|^2}{2}) - |\nabla\mathbf{u}_{\infty}|^2 - \nabla\cdot\left(\frac{|\mathbf{u}_{\infty}|^2}{2}\mathbf{b}_{\infty}\right) - \nabla\cdot(p_{\infty}\mathbf{u}_{\infty}) + \mathbf{u}_{\infty}\cdot(\nabla\cdot\mathbb{F}_{\infty}) - \mu_{\infty}.$$

Notations.

All along the text, C_{γ} is a positive constant whose value may change from line to line but which depends only on γ .

2 The weights w_{δ} .

We consider the weights $w_{\delta} = \frac{1}{(1+|x|)^{\delta}}$ where $0 < \delta$ and $x \in \mathbb{R}^3$. A very important feature of those weights is the control of their gradients:

$$|\nabla w_{\delta}(x)| = \delta \frac{w_{\delta}(x)}{1 + |x|} \tag{2}$$

Lemma 1 (Muckenhoupt weights) If $0 < \delta < 3$ and $1 , then <math>w_{\delta}$ belongs to the Muckenhoupt class \mathcal{A}_{p} .

Proof: We recall that a weight w belongs to $\mathcal{A}_p(\mathbb{R}^3)$ for 1 if and only if it satisfies the reverse Hölder inequality

$$\sup_{x \in \mathbb{R}^3, R > 0} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} w(y) \, dy \right)^{\frac{1}{p}} \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} \frac{dy}{w(y)^{\frac{1}{p-1}}} \right)^{1-\frac{1}{p}} < +\infty.$$
 (3)

For all $0 < R \le 1$ the inequality |x - y| < R implies $\frac{1}{2}(1 + |x|) \le 1 + |y| \le 2(1 + |x|)$, thus we can control the left side in (3) for w_{δ} by $4^{\frac{\delta}{p}}$.

For all R > 1 and |x| > 10R, we have that the inequality |x - y| < R implies $\frac{9}{10}(1 + |x|) \le 1 + |y| \le \frac{11}{10}(1 + |x|)$, thus we can control the left side in (3) for w_{δ} by $(\frac{11}{9})^{\frac{\delta}{p}}$.

Finally, for R > 1 and $|x| \le 10R$, we write

$$\left(\frac{1}{|B(x,R)|} \int_{B(x,R)} w(y) \, dy\right)^{\frac{1}{p}} \left(\frac{1}{|B(x,R)|} \int_{B(0,R)} \frac{dy}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} \\
\leq \left(\frac{1}{|B(0,R)|} \int_{B(x,11R)} w(y) \, dy\right)^{\frac{1}{p}} \left(\frac{1}{|B(0,R)|} \int_{B(0,11R)} \frac{dy}{w(y)^{\frac{1}{p-1}}}\right)^{1-\frac{1}{p}} \\
= \left(\frac{1}{R^3} \int_0^{11R} r^2 \frac{dr}{(1+r)^{\delta}}\right)^{\frac{1}{p}} \left(\frac{1}{R^3} \int_0^{11R} r^2 (1+r)^{\frac{\delta}{p-1}} \, dr\right)^{1-\frac{1}{p}} \\
\leq c_{\delta,p} \left(\frac{1}{R^3} \int_0^{11R} r^2 \frac{dr}{r^{\delta}}\right)^{\frac{1}{p}} \left(\left(\frac{1}{R^3} \int_0^{11R} r^2 dr\right)^{1-\frac{1}{p}} + \left(\frac{1}{R^3} \int_0^{11R} r^{2+\frac{\delta}{p-1}} \, dr\right)^{1-\frac{1}{p}}\right) \\
= c_{\delta,p} \frac{11^3}{(3-\delta)^{\frac{1}{p}}} \left(\frac{(11R)^{-\frac{\delta}{p}}}{3^{1-\frac{1}{p}}} + \frac{1}{(3+\frac{\delta}{p-1})^{1-\frac{1}{p}}}\right).$$

The lemma is proved.

Lemma 2 If $0 < \delta < 3$ and $1 , then the Riesz transforms <math>R_i$ and the Hardy–Littlewood maximal function operator are bounded on $L^p_{w_\delta} = L^p(w_\delta(x) dx)$:

 \Diamond

$$||R_j f||_{L^p_{ws}} \le C_{p,\delta} ||f||_{L^p_{ws}} \text{ and } ||\mathcal{M}_f||_{L^p_{ws}} \le C_{p,\delta} ||f||_{L^p_{ws}}.$$

Proof: The boundedness of the Riesz transforms or of the Hardy–Littlewwod maximal function on $L^p(w_{\gamma} dx)$ are basic properties of the Muckenhoupt class \mathcal{A}_p [5].

We will use strategically the next corollary, which is specially useful to obtain discretely self-similar solutions.

Corollary 1 (Non-increasing kernels) Let $\theta \in L^1(\mathbb{R}^3)$ be a non-negative radial function which is radially non-increasing. Then, if $0 < \delta < 3$ and $1 , we have, for <math>f \in L^p_{w_\delta}$, the inequality

$$\|\theta * f\|_{L^p_{w_{\delta}}} \le C_{p,\delta} \|f\|_{L^p_{w_{\delta}}} \|\theta\|_1.$$

Proof: We have the well-known inequality for radial non-increasing kernels [4]

$$|\theta * f(x)| \le ||\theta||_1 \mathcal{M}_f(x)$$

so that we may conclude with Lemma 2.

We illustrate the utility of Lemma 2 with the following corollaries:

Corollary 2 Let $0 < \gamma < \frac{5}{2}$ and $0 < T < +\infty$. Let \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,T),L^2_{w_{\gamma}})$. Let \mathbf{b} be a time-dependent divergence free vector-field $(\nabla \cdot \mathbf{b} = 0)$ such that $\mathbf{b} \in L^3((0,T),L^3_{w_{3\gamma/2}})$.

Let \mathbf{u} be a solution of the following advection-diffusion problem

$$\begin{cases}
\partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla q + \nabla \cdot \mathbb{F} \\
\nabla \cdot \mathbf{u} = 0,
\end{cases}$$
(4)

 \Diamond

be such that: \mathbf{u} belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$, and the pressure q belongs to $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

Then, the gradient of the pressure ∇q is necessarily related to \mathbf{u} , \mathbf{b} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$\nabla q = \nabla \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j}) \right)$$

and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(b_i u_j)$ belongs to $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$ and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ belongs to $L^2((0,T), L_{w_{\gamma}}^2)$.

Proof: We define

$$p = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j})\right).$$

As $0 < \gamma < \frac{5}{2}$ we can use Lemma 2 and (2) to obtain $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j)$ belongs to $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$ and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ belongs to $L^2((0,T), L_{w_{\gamma}}^2)$.

Taking the divergence in (4), we obtain $\Delta(q-p)=0$. We take a test function $\alpha \in \mathcal{D}(\mathbb{R})$ such that $\alpha(t)=0$ for all $|t|\geq \varepsilon$, and a test function $\beta \in \mathcal{D}(\mathbb{R}^3)$; then the distribution $\nabla q*(\alpha \otimes \beta)$ is well defined on $(\varepsilon, T-\varepsilon) \times \mathbb{R}^3$. We fix $t \in (\varepsilon, T-\varepsilon)$ and define

$$A_{\alpha,\beta,t} = (\nabla q * (\alpha \otimes \beta) - \nabla p * (\alpha \otimes \beta))(t,.).$$

We have

$$A_{\alpha,\beta,t} = (\mathbf{u} * (-\partial_t \alpha \otimes \beta + \alpha \otimes \Delta \beta) + (-\mathbf{u} \otimes \mathbf{b} + \mathbb{F}) \cdot (\alpha \otimes \nabla \beta))(t,.)$$

$$- (p * (\alpha \otimes \nabla \beta))(t,.).$$
(5)

Convolution with a function in $\mathcal{D}(\mathbb{R}^3)$ is a bounded operator on $L^2_{w_\gamma}$ and on $L^{6/5}_{w_{6\gamma/5}}$ (as, for $\varphi \in \mathcal{D}(\mathbb{R}^3)$ we have $|f * \varphi| \leq C_\varphi \mathcal{M}_f$). Thus, we may conclude from (5) that $A_{\alpha,\beta,t} \in L^2_{w_\gamma} + L^{6/5}_{w_{6\gamma/5}}$. If $\max\{\gamma, \frac{\gamma+2}{2}\} < \delta < 5/2$, we have $A_{\alpha,\beta,t} \in L^{6/5}_{w_{6\delta/5}}$.

In particular, $A_{\alpha,\beta,t}$ is a tempered distribution. As we have

$$\Delta A_{\alpha,\beta,t} = (\alpha \otimes \beta) * (\Delta(q-p))(t,.) = 0,$$

we find that $A_{\alpha,\beta,t}$ is a polynomial. We remark that for all $1 < r < +\infty$ and $0 < \delta < 3$, $L^r_{w_\delta}$ does not contain non-trivial polynomials. Thus, $A_{\alpha,\beta,t} = 0$. We then use an approximation of identity $\frac{1}{\epsilon^4}\alpha(\frac{t}{\epsilon})\beta(\frac{x}{\epsilon})$ and conclude that $\nabla(q-p)=0$.

Actually, we can answer a question posed by Bradshaw and Tsai in [2] about the nature of the pressure for self-similar solutions of the Navier–Stokes equations. In effect, we have the next corollary:

Corollary 3 Let $1 < \gamma < \frac{5}{2}$ and $0 < T < +\infty$. Let \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,T),L^2_{w_\gamma})$.

Let **u** be a solution of the following problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, \end{cases}$$

be such that : \mathbf{u} belongs to $L^{\infty}([0,+\infty), L^2)_{loc}$ and $\nabla \mathbf{u}$ belongs to $L^2([0,+\infty), L^2)_{loc}$, and the pressure q is in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

We suppose that there exists $\lambda > 1$ such that $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t, x)$ and $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$. Then, the gradient of the pressure ∇q is necessarily related to \mathbf{u} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$\nabla q = \nabla \left(\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j - F_{i,j}) \right)$$

and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(u_i u_j)$ belongs to $L^4((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$ and $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j F_{i,j}$ belongs to $L^2((0,T), L_{w_{\gamma}}^2)$.

Proof: We shall use Corollary 2, and thus we need to show that **u** belongs to $L^{\infty}((0,T), L^2_{w_{\gamma}} \cap L^3((0,T), L^3_{3\gamma/2}))$ and $\nabla \mathbf{u}$ belongs to $L^2((0,T), L^2_{w_{\gamma}})$. In fact,

$$||u||_{L^{\infty}((0,T),L^{2}_{w_{\gamma}})} \leq \sup_{0 \leq t \leq T} \int_{|x|<1} |\mathbf{u}(t,x)|^{2} dx + c \sup_{0 \leq t \leq T} \sum_{k \in \mathbb{N}} \int_{\lambda^{k-1} < |x| < \lambda^{k}} \frac{|\mathbf{u}(t,x)|^{2}}{\lambda^{\gamma k}} dx$$

and

$$\sup_{0 \le t \le T} \sum_{k \ge 1} \int_{\lambda^{k-1} < |x| < \lambda^k} \frac{|\mathbf{u}(t,x)|^2}{\lambda^{\gamma k}} \, dx \le \sup_{0 \le t \le T} \sum_{k \in \mathbb{N}} \lambda^{(1-\gamma)k} \int_{\lambda^{-1} < |x| < 1} |\mathbf{u}(\frac{t}{\lambda^{2k}},x)|^2 \, dx \\
\le c \sup_{0 \le t \le T} \int_{\lambda^{-1} < |x| < 1} |\mathbf{u}(t,x)|^2 \, dx < +\infty.$$

For $\nabla \mathbf{u}$, we compute for $k \in \mathbb{N}$,

$$\int_{0}^{T} \int_{\lambda^{k-1} < |x| < \lambda^{k}} |\nabla \mathbf{u}(t, x)|^{2} dt dx = \lambda^{k} \int_{0}^{\frac{T}{\lambda^{2k}}} \int_{\frac{1}{\lambda} < |x| < 1} |\nabla \mathbf{u}(t, x)|^{2} dx dt.$$

We may conclude that $\nabla \mathbf{u}$ belongs to $L^2((0,T),L^2_{w_\gamma})$, since for $\gamma > 1$ we have $\sum_{k\in\mathbb{N}} \lambda^{(1-\gamma)k} < +\infty$.

Now, we use the Sobolev embeddings described in next Lemma (Lemma 3) to get that **u** belongs to $L^2((0,T),L^6_{w_{3\gamma}})$, and thus (by interpolation with $L^{\infty}((0,T),L^2_{w_{\gamma}}))$ to $L^4((0,T),L^3_{w_{3\gamma/2}})$.

In particular, $\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(u_i u_j)$ belongs to $L^4((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$, since we have

$$\|(\mathbf{u} \otimes \mathbf{u})w_{\gamma}\|_{L^{6/5}} \leq \|\sqrt{w_{\gamma}}\mathbf{u}\|_{L^{2}}\|\sqrt{w_{\gamma}}\mathbf{u}\|_{L^{3}} \leq \|\sqrt{w_{\gamma}}\mathbf{u}\|_{L^{2}}^{\frac{3}{2}}\|\sqrt{w_{\gamma}}\mathbf{u}\|_{L^{6}}^{\frac{1}{2}}$$

 \Diamond

Lemma 3 (Sobolev embeddings) Let $\delta > 0$. If $f \in L^2_{w_\delta}$ and $\nabla f \in L^2_{w_\delta}$ then $f \in L^6_{w_{3\delta}}$ and

$$||f||_{L^6_{w_{3\delta}}} \le C_\delta(||f||_{L^2_{w_\delta}} + ||\nabla f||_{L^2_{w_\delta}}).$$

Proof: Since both f and $w_{\delta/2}$ are locally in H^1 , we write

$$\partial_i(fw_{\delta/2}) = w_{\delta/2}\partial_i f + f\partial_i(w_{\delta/2}) = w_{\delta/2}\partial_i f - \frac{\delta}{2} \frac{x_i}{|x|} w_{\delta/2} f$$

and thus

$$||w_{\delta/2}f||_2^2 + ||\nabla(w_{\delta/2}f)||_2^2 \le (1 + \frac{\delta^2}{2})||w_{\delta/2}f||_2^2 + 2||w_{\delta/2}\nabla f||_2^2.$$

Thus, $w_{\delta/2}f$ belongs to L^6 (since $H^1 \subset L^6$), or equivalently $f \in L^6_{w_{3\delta}}$.

3 A priori estimates for the advection-diffusion problem.

3.1 Proof of Theorem 2.

Let $0 < t_0 < t_1 < T$. We take a function $\alpha \in \mathcal{C}^{\infty}(\mathbb{R})$ which is non-decreasing, with $\alpha(t)$ equal to 0 for t < 1/2 and equal to 1 for t > 1. For $0 < \eta < \min(\frac{t_0}{2}, T - t_1)$, we define

$$\alpha_{\eta,t_0,t_1}(t) = \alpha(\frac{t-t_0}{\eta}) - \alpha(\frac{t-t_1}{\eta}).$$

We take as well a non-negative function $\phi \in \mathcal{D}(\mathbb{R}^3)$ which is equal to 1 for $|x| \leq 1$ and to 0 for $|x| \geq 2$. For R > 0, we define $\phi_R(x) = \phi(\frac{x}{R})$. Finally, we define, for $\epsilon > 0$, $w_{\gamma,\epsilon} = \frac{1}{(1+\sqrt{\epsilon^2+|x|^2})^{\delta}}$. We have $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \in \mathcal{D}((0,T)\times\mathbb{R}^3)$ and $\alpha_{\eta,t_0,t_1}(t)\phi_R(x)w_{\gamma,\epsilon}(x) \geq 0$. Thus, using the local energy

balance (1) and the fact that $\mu \geq 0$, we find

$$-\iint \frac{|\mathbf{u}|^{2}}{2} \partial_{t} \alpha_{\eta,t_{0},t_{1}} \phi_{R} w_{\gamma,\epsilon} \, dx \, ds$$

$$\leq -\sum_{i=1}^{3} \iint \partial_{i} \mathbf{u} \cdot \mathbf{u} \, \alpha_{\eta,t_{0},t_{1}} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) \, dx \, ds$$

$$-\iint |\nabla \mathbf{u}|^{2} \, \alpha_{\eta,t_{0},t_{1}} \phi_{R} w_{\gamma,\epsilon} dx \, ds$$

$$+\sum_{i=1}^{3} \iint \frac{|\mathbf{u}|^{2}}{2} b_{i} \alpha_{\eta,t_{0},t_{1}} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) \, dx \, ds$$

$$+\sum_{i=1}^{3} \iint \alpha_{\eta,t_{0},t_{1}} p u_{i} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) \, dx \, ds$$

$$-\sum_{i=1}^{3} \sum_{j=1}^{3} \iint F_{i,j} u_{j} \alpha_{\eta,t_{0},t_{1}} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) \, dx \, ds$$

$$-\sum_{i=1}^{3} \sum_{j=1}^{3} \iint F_{i,j} \partial_{i} u_{j} \, \alpha_{\eta,t_{0},t_{1}} \phi_{R} w_{\gamma,\epsilon} \, dx \, ds.$$

We remark that, independently from R > 1 and $\epsilon > 0$, we have (for $0 < \gamma \le 2$)

$$|w_{\gamma,\epsilon}\partial_i\phi_R| + |\phi_R\partial_i w_{\gamma,\epsilon}| \le C_\gamma \frac{w_\gamma(x)}{1+|x|} \le C_\gamma w_{3\gamma/2}(x).$$

Moreover, we know that \mathbf{u} belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})\cap L^2((0,T),L^6_{w_{3\gamma}})$ hence to $L^4((0,T),L^3_{w_{3\gamma/2}})$. Since $T<+\infty$, we have as well $\mathbf{u}\in L^3((0,T),L^3_{w_{3\gamma/2}})$. (This is the same type of integrability as required for \mathbf{b}). Moreover, we have $pu_i\in L^1_{w_{3\gamma/2}}$ since $w_{\gamma}p\in L^2((0,T),L^{6/5}+L^2)$ and $w_{\gamma/2}\mathbf{u}\in L^2((0,T),L^2\cap L^6)$. All those remarks will allow us to use dominated convergence.

We first let η go to 0. We find that

$$-\lim_{\eta \to 0} \iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta, t_0, t_1} \phi_R w_{\gamma, \epsilon} \, dx \, ds$$

$$\leq -\sum_{i=1}^3 \int_{t_0}^{t_1} \int \partial_i \mathbf{u} \cdot \mathbf{u} \left(w_{\gamma, \epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma, \epsilon} \right) \, dx \, ds$$

$$-\int_{t_0}^{t_1} \int |\nabla \mathbf{u}|^2 \, \phi_R w_{\gamma, \epsilon} \, dx \, ds$$

$$+\sum_{i=1}^3 \int_{t_0}^{t_1} \int \frac{|\mathbf{u}|^2}{2} b_i (w_{\gamma, \epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma, \epsilon}) \, dx \, ds$$

$$+\sum_{i=1}^3 \int_{t_0}^{t_1} \int p u_i (w_{\gamma, \epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma, \epsilon}) \, dx \, ds$$

$$-\sum_{i=1}^3 \sum_{j=1}^3 \int_{t_0}^{t_1} \int F_{i,j} u_j (w_{\gamma, \epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma, \epsilon}) \, dx \, ds$$

$$-\sum_{i=1}^3 \sum_{j=1}^3 \int_{t_0}^{t_1} \int F_{i,j} \partial_i u_j \, \phi_R w_{\gamma, \epsilon} \, dx \, ds.$$

Let us define

$$A_{R,\epsilon}(t) = \int |\mathbf{u}(t,x)|^2 \phi_R(x) w_{\gamma,\epsilon}(x) dx.$$

As we have

$$-\iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta, t_0, t_1} \phi_R w_{\gamma, \epsilon} \, dx \, ds = -\frac{1}{2} \int \partial_t \alpha_{\eta, t_0, t_1} A_{R, \epsilon}(s) \, ds$$

we find that, when t_0 and t_1 are Lebesgue points of the measurable function $A_{R,\epsilon}$

$$\lim_{\eta \to 0} - \iint \frac{|\mathbf{u}|^2}{2} \partial_t \alpha_{\eta, t_0, t_1} \phi_R w_{\gamma, \epsilon} \, dx \, ds = \frac{1}{2} (A_{R, \epsilon}(t_1) - A_{R, \epsilon}(t_0)).$$

Then, by continuity, we can let t_0 go to 0 and thus replace t_0 by 0 in the inequality. Moreover, if we let t_1 go to t, then by weak continuity, we find that $A_{R,\epsilon}(t) \leq \lim_{t_1 \to t} A_{R,\epsilon}(t_1)$, so that we may as well replace t_1 by $t \in (0,T)$. Thus we find that for every $t \in (0,T)$, we have

$$\int \frac{|\mathbf{u}(t,x)|^{2}}{2} \phi_{R} w_{\gamma,\epsilon} dx$$

$$\leq \int \frac{|\mathbf{u}_{0}(x)|^{2}}{2} \phi_{R} w_{\gamma,\epsilon} dx$$

$$- \sum_{i=1}^{3} \int_{0}^{t} \int \partial_{i} \mathbf{u} \cdot \mathbf{u} \left(w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon} \right) dx ds$$

$$- \int_{0}^{t} \int |\nabla \mathbf{u}|^{2} \phi_{R} w_{\gamma,\epsilon} dx ds$$

$$+ \sum_{i=1}^{3} \int_{0}^{t} \int \frac{|\mathbf{u}|^{2}}{2} b_{i} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds$$

$$+ \sum_{i=1}^{3} \int_{0}^{t} \int p u_{i} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds$$

$$- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j} u_{j} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds$$

$$- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j} \partial_{i} u_{j} \phi_{R} w_{\gamma,\epsilon} dx ds.$$
(6)

Thus, letting R go to $+\infty$ and then ϵ go to 0, we find by dominated convergence that, for every $t \in (0,T)$, we have

$$\|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + 2 \int_{0}^{t} \|\nabla \mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} - \int_{0}^{t} \int \nabla |\mathbf{u}|^{2} \cdot \nabla w_{\gamma} dx ds + \int_{0}^{t} \int (|\mathbf{u}|^{2} \mathbf{b} + 2p\mathbf{u}) \cdot \nabla (w_{\gamma}) dx ds$$

$$- 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j}(\partial_{i}u_{j})w_{\gamma} + F_{i,j}u_{i}\partial_{j}(w_{\gamma}) dx ds.$$

Now we write

$$\left| \int_0^t \int \nabla |\mathbf{u}|^2 \cdot \nabla w_{\gamma} \, ds \, ds \right| \leq 2\gamma \int_0^t \int |\mathbf{u}| |\nabla \mathbf{u}| \, w_{\gamma} \, dx \, ds$$
$$\leq \frac{1}{4} \int_0^t \|\nabla \mathbf{u}\|_{L^2_{w_{\gamma}}}^2 \, ds + 4\gamma^2 \int_0^t \|\mathbf{u}\|_{L^2_{w_{\gamma}}}^2 \, ds.$$

Writing

$$p_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(b_i u_j)$$
 and $p_2 = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(F_{i,j})$

and using the fact that $w_{6\gamma/5} \in \mathcal{A}_{6/5}$ and $w_{\gamma} \in \mathcal{A}_2$, we get

$$\begin{split} \left| \int_{0}^{t} \int (|\mathbf{u}|^{2}\mathbf{b} + 2p_{1}\mathbf{u}) \cdot \nabla(w_{\gamma}) \, dx \, ds \right| &\leq \gamma \int_{0}^{t} \int (|\mathbf{u}|^{2}|\mathbf{b}| + 2|p_{1}| \, |\mathbf{u}|) \, w_{\gamma}^{3/2} \, dx \, ds \\ &\leq \gamma \int_{0}^{t} \|w_{\gamma}^{1/2}\mathbf{u}\|_{6} (\|w_{\gamma}|\mathbf{b}| \|\mathbf{u}\|\|_{6/5} + \|w_{\gamma}p_{1}\|_{6/5}) ds \\ &\leq C_{\gamma} \int_{0}^{t} \|w_{\gamma}^{1/2}\mathbf{u}\|_{6} \|w_{\gamma}|\mathbf{b}| \|\mathbf{u}\|\|_{6/5} \, ds \\ &\leq C_{\gamma} \int_{0}^{t} \|w_{\gamma}^{1/2}\mathbf{u}\|_{6} \|w_{\gamma}^{1/2}\mathbf{b}\|_{3} \|w_{\gamma}^{1/2}\mathbf{u}\|_{2} \, ds \\ &\leq C_{\gamma} \int_{0}^{t} (\|\nabla\mathbf{u}\|_{L_{w\gamma}^{2}} + \|\mathbf{u}\|_{L_{w\gamma}^{2}}) \, \|\mathbf{b}\|_{L_{w_{3\gamma/2}}^{3}} \|\mathbf{u}\|_{L_{w\gamma}^{2}} \, ds \\ &\leq \frac{1}{4} \int_{0}^{t} \|\nabla\mathbf{u}\|_{L_{w\gamma}^{2}}^{2} \, ds + C_{\gamma}'' \int_{0}^{t} \|\mathbf{u}\|_{L_{w\gamma}^{2}}^{2} (\|\mathbf{b}\|_{L_{w_{3\gamma/2}}^{3}} + \|\mathbf{b}\|_{L_{w_{3\gamma/2}}^{3}}^{2}) \, ds \end{split}$$

and

$$\left| \int_{0}^{t} \int 2p_{2}\mathbf{u} \cdot \nabla(w_{\gamma}) \, dx \, ds \right| \leq 2\gamma \int_{0}^{t} \int |p_{2}| \, |\mathbf{u}| \, w_{\gamma} \, dx \, ds$$

$$\leq \gamma \int_{0}^{t} \|\mathbf{u}\|_{L_{w_{\gamma}}^{2}}^{2} + \|p_{2}\|_{L_{w_{\gamma}}^{2}}^{2} \, ds$$

$$\leq C_{\gamma} \int_{0}^{t} \|\mathbf{u}\|_{L_{w_{\gamma}}^{2}}^{2} + \|\mathbb{F}\|_{L_{w_{\gamma}}^{2}}^{2} \, ds.$$

Finally, we have

$$\left| 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{i,j}(\partial_{i}u_{j}) w_{\gamma} + F_{i,j}u_{i}\partial_{j}(w_{\gamma}) dx ds \right| \leq 2 \int_{0}^{t} \int |F| \left(|\nabla \mathbf{u}| + \gamma |\mathbf{u}| \right) w_{\gamma} dx ds$$

$$\leq \frac{1}{4} \int_{0}^{t} \|\nabla \mathbf{u}\|_{L_{w_{\gamma}}^{2}}^{2} ds + C_{\gamma} \int_{0}^{t} \|\mathbf{u}\|_{L_{w_{\gamma}}^{2}}^{2} + \|\mathbb{F}\|_{L_{w_{\gamma}}^{2}}^{2} ds.$$

We have obtained

$$\|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \int_{0}^{t} \|\nabla \mathbf{u}\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{t} \|\mathbb{F}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds + C_{\gamma} \int_{0}^{t} (1 + \|\mathbf{b}(s,.)\|_{L^{3}_{w_{3\gamma/2}}}^{2}) \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$(7)$$

3.2 Passive transportation.

From inequality (7), we have the following direct consequence:

Corollary 4 Under the assumptions of Theorem 2, we have

$$\sup_{0 < t < T} \|\mathbf{u}\|_{L^2_{w_{\gamma}}} \leq \left(\|\mathbf{u}_0\|_{L^2_{w_{\gamma}}} + C_{\gamma} \|\mathbb{F}\|_{L^2((0,T),L^2_{w_{\gamma}})} \right) e^{C_{\gamma}(T + T^{1/3} \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}^2)}$$

and

$$\|\nabla \mathbf{u}\|_{L^2((0,T),L^2_{w\gamma})} \le (\|\mathbf{u}_0\|_{L^2_{w\gamma}} + C_{\gamma} \|\mathbb{F}\|_{L^2((0,T),L^2_{w\gamma})}) e^{C_{\gamma}(T + T^{1/3} \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}^2)}$$

where the constant C_{γ} depends only on γ .

Another direct consequence is the following uniqueness result for the advection-diffusion problem with a (locally in time), bounded \mathbf{b} :

Corollary 5. Let $0 < \gamma \le 2$. Let $0 < T < +\infty$. Let \mathbf{u}_0 be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let \mathbf{b} be a time-dependent divergence free vector-field $(\nabla \cdot \mathbf{b} = 0)$ such that $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$. Assume moreover that \mathbf{b} belongs to $L^2_t L^\infty_x(K)$ for every compact subset K of $(0,T) \times \mathbb{R}^3$.

Let (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) be two solutions of the following advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

be such that, for k = 1 and k = 2,:

- \mathbf{u}_k belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_k$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$
- the pressure p_k is related to \mathbf{u}_k , \mathbf{b} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$p_k = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_{k,j} - F_{i,j})$$

• the map $t \in [0,T) \mapsto \mathbf{u}_k(t,.)$ is weakly continuous from [0,T) to $L^2_{w_\gamma}$, and is strongly continuous at t=0:

$$\lim_{t\to 0} \|\mathbf{u}_k(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

Then $\mathbf{u}_1 = \mathbf{u}_2$.

Proof: Let $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$ and $q = p_1 - p_2$. Then we have

$$\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - (\mathbf{b} \cdot \nabla) \mathbf{v} - \nabla q \\ \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v}(0, .) = 0 \end{cases}$$

Moreover on every compact subset K of $(0,T) \times \mathbb{R}^3$, $\mathbf{b} \otimes \mathbf{v}$ is in $L^2_t L^2_x$, while it belongs globally to $L^3_t L^{6/5}_{w_{6\gamma/5}}$. Writing, for $\varphi, \psi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ such that $\psi = 1$ on the neigborhood of the support of φ ,

$$\varphi q = q_1 + q_2 = \varphi \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (\psi b_i v_j) + \varphi \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j ((1 - \psi) b_i v_j)$$

we find that $||q_1||_{L^2L^2} \leq C_{\varphi,\psi} ||\psi \mathbf{b} \otimes \mathbf{v}||_{L^2L^2}$ and

$$||q_2||_{L^3L^\infty} \le C_{\varphi,\psi} ||\mathbf{b} \otimes \mathbf{v}||_{L^3L_{w_{6/5}}^{6/5}}$$

with

$$C_{\varphi,\psi} \le C \|\varphi\|_{\infty} \|1 - \psi\|_{\infty} \sup_{x \in \operatorname{Supp} \varphi} \left(\int_{y \in \operatorname{Supp} (1 - \psi)} \left(\frac{(1 + |y|)^{\gamma}}{|x - y|^3} \right)^6 \right)^{1/6} < +\infty.$$

Thus, we may take the scalar product of $\partial_t \mathbf{v}$ with \mathbf{v} and find that

$$\partial_t(\frac{|\mathbf{v}|^2}{2}) = \Delta(\frac{|\mathbf{v}|^2}{2}) - |\nabla \mathbf{v}|^2 - \nabla \cdot \left(\frac{|\mathbf{v}|^2}{2}\mathbf{b}\right) - \nabla \cdot (q\mathbf{v}).$$

Thus we are under the assumptions of Theorem 2 and we may use Corollary 4 to find that $\mathbf{v} = 0$.

3.3 Active transportation.

We begin with the following lemma:

Lemma 4 Let α be a non-negative bounded measurable function on [0,T) such that, for two constants $A, B \geq 0$, we have

$$\alpha(t) \le A + B \int_0^t \alpha(s) + \alpha(s)^3 ds.$$

If $T_0 > 0$ and $T_1 = \min(T, T_0, \frac{1}{4B(A+BT_0)^2})$, we have, for every $t \in [0, T_1]$, $\alpha(t) \leq \sqrt{2}(A+BT_0)$.

Proof: We write $\alpha \leq 1 + \alpha^3$. We define

$$\Phi(t) = A + BT_0 + B \int_0^t \alpha^3 ds \text{ and } \Psi(t) = A + BT_0 + B \int_0^t \Phi^3(s) ds.$$

We have, for $t \in [0, T_1]$, $\alpha \leq \Phi \leq \Psi$. Since Ψ is \mathcal{C}^1 , we may write

$$\Psi'(t) = B\Phi(t)^3 \le B\Psi(t)^3$$

and thus

$$\frac{1}{\Psi(0)^2} - \frac{1}{\Psi(t)^2} \le 2Bt.$$

We thus find

$$\Psi(t)^2 \le \frac{\Psi(0)^2}{1 - 2B\Psi(0)^2 t} \le 2\Psi(0)^2.$$

 \Diamond

The lemma is proven.

Corollary 6 Assume that \mathbf{u}_0 , \mathbf{u} , p, \mathbb{F} and \mathbf{b} satisfy assumptions of Theorem 2, Assume moreover that \mathbf{b} is controlled by \mathbf{u} : for every $t \in (0,T)$,

$$\|\mathbf{b}(t,.)\|_{L^3_{w_{3\gamma/2}}} \le C_0 \|\mathbf{u}(t,.)\|_{L^3_{w_{3\gamma/2}}}.$$

Then there exists a constant $C_{\gamma} \geq 1$ such that if $T_0 < T$ is such that

$$C_{\gamma}(1+C_0^4)\left(1+C_0^4+\|\mathbf{u}_0\|_{L^2_{w\gamma}}^2+\int_0^{T_0}\|\mathbb{F}\|_{L^2_{w\gamma}}^2\,ds\right)^2T_0\leq 1$$

then

$$\sup_{0 \le t \le T_0} \| \mathbf{u}(t,.) \|_{L^2_{w_\gamma}}^2 \le C_\gamma (1 + C_0^4 + \| \mathbf{u}_0 \|_{L^2_{w_\gamma}}^2 + \int_0^{T_0} \| \mathbb{F} \|_{L^2_{w_\gamma}}^2 \, ds)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{u}\|_{L^2_{w\gamma}}^2 ds \le C_{\gamma} (1 + C_0^4 + \|\mathbf{u}_0\|_{L^2_{w\gamma}}^2 + \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w\gamma}}^2 ds).$$

Proof: We start from inequality (7):

$$\begin{aligned} \|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \int_{0}^{t} \|\nabla \mathbf{u}\|_{L^{2}_{w_{\gamma}}}^{2} ds \\ \leq &\|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{t} \|\mathbb{F}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds + C_{\gamma} \int_{0}^{t} (1 + \|\mathbf{b}(s,.)\|_{L^{3}_{w_{3\gamma/2}}}^{2}) \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds \end{aligned}$$

We write

$$\|\mathbf{b}(s,.)\|_{L^3_{w_{3\gamma/2}}}^2 \le C_0^2 \|\mathbf{u}(s,.)\|_{L^3_{w_{3\gamma/2}}}^2 \le C_0^2 C_\gamma \|u\|_{L^2_{w\gamma}} (\|u\|_{L^2_{w\gamma}} + \|\nabla \mathbf{u}\|_{L^2_{w\gamma}}).$$

This gives

$$\begin{split} \|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \frac{1}{2} \int \|\nabla \mathbf{u}\|_{L^{2}_{w_{\gamma}}}^{2} ds \\ \leq & \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{t} \|\mathbb{F}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds \\ & + C_{\gamma} \int_{0}^{t} \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} + C_{0}^{2} \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{4} + C_{0}^{4} \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{6} ds \\ \leq & \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{t} \|\mathbb{F}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds + 2C_{\gamma} \int_{0}^{t} \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} + C_{0}^{4} \|\mathbf{u}(s,.)\|_{L^{2}_{w_{\gamma}}}^{6} ds. \end{split}$$

For $t \leq T_0$, we get

$$\|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \frac{1}{2} \int \|\nabla \mathbf{u}\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2} + C_{\gamma} \int_{0}^{T_{0}} \|\mathbb{F}\|_{L^{2}_{w_{\gamma}}}^{2} ds + C_{\gamma} (1 + C_{0}^{4}) \int_{0}^{t} \|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + \|\mathbf{u}(t,.)\|_{L^{2}_{w_{\gamma}}}^{6} ds$$

and we may conclude with Lemma 4.

4 Stability of solutions for the advection-diffusion problem.

4.1 The Rellich lemma.

We recall the Rellich lemma:

Lemma 5 (Rellich) If s > 0 and (f_n) is a sequence of functions on \mathbb{R}^d such that

- the family (f_n) is bounded in $H^s(\mathbb{R}^d)$
- there is a compact subset of \mathbb{R}^d such that the support of each f_n is included in K

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2(\mathbb{R}^d)$.

We shall use a variant of this lemma (see [9]):

Lemma 6 (space-time Rellich) If s > 0, $\sigma \in \mathbb{R}$ and (f_n) is a sequence of functions on $(0,T) \times \mathbb{R}^d$ such that, for all $T_0 \in (0,T)$ and all $\varphi \in \mathcal{D}(\mathbb{R}^3)$

- φf_n is bounded in $L^2((0,T_0),H^s)$
- $\varphi \partial_t f_n$ is bounded in $L^2((0,T_0),H^{\sigma})$

then there exists a subsequence (f_{n_k}) such that f_{n_k} is strongly convergent in $L^2_{loc}([0,T)\times\mathbb{R}^3)$: if f_{∞} is the limit, we have for all $T_0\in(0,T)$ and all $R_0>0$

$$\lim_{n_k \to +\infty} \int_0^{T_0} \int_{|x| \le R} |f_{n_k} - f_{\infty}|^2 dx dt = 0.$$

Proof: With no loss of generality, we may assume that $\sigma < \min(1, s)$. Define g by $g_n(t,x) = \alpha(t)\varphi(x)f_n(t,x)$ if t > 0 and $g_n(t,x) = \alpha(t)\varphi(x)f_n(-t,x)$ if t < 0, where $\alpha \in \mathcal{C}^{\infty}$ on (0,T), is equal to 1 on $[0,T_0]$ and equal to 0 for $t > \frac{T+T_0}{2}$, and $\varphi(x) = 1$ on $B(0,R_0)$. Then the support of g_n is contained in $[-\frac{T+T_0}{2},\frac{T+T_0}{2}] \times \operatorname{Supp} \varphi$. Moreover, g_n is bounded in $L_t^2H^s$ and $\partial_t g_n$ is bounded in $L_t^2H^s$ so that g_n is bounded in $H^{\rho}(\mathbb{R} \times \mathbb{R}^3)$ with $\rho = \frac{s}{s+1-\sigma}$ (just write $(1+\tau^2+\xi^2)^{\frac{s}{s+1-\sigma}} \leq ((1+\tau^2)(1+\xi^2)^{\sigma})^{\frac{s}{s+1-\sigma}} ((1+\xi^2)^s)^{\frac{1-\sigma}{s+1-\sigma}})$.. By the Rellich lemma, we know that there is a subsequence g_{n_k} which is strongly convergent in $L^2(\mathbb{R} \times \mathbb{R}^3)$, thus a subsequence f_{n_k} which is strongly convergent in $L^2((0,T_0)\times B(0,R_0))$.

We then iterate this argument for an increasing sequence of times $T_0 < T_1 < \cdots < T_N \to T$ and an increasing sequence of radii $R_0 < R_1 < \cdots < R_N \to +\infty$ and finish the proof. by the classical diagonal process of Cantor.

4.2 Proof of Theorem 3.

Assume that $\mathbf{u}_{0,n}$ is strongly convergent to $\mathbf{u}_{0,\infty}$ in $L^2_{w_{\gamma}}$ and that the sequence \mathbb{F}_n is strongly convergent to \mathbb{F}_{∞} in $L^2((0,T),L^2_{w_{\gamma}})$, and assume that the sequence \mathbf{b}_n is bounded in $L^3((0,T),L^3_{w_{3\gamma/2}})$. Then, by Theorem 2 and Corollary 4, we know that \mathbf{u}_n is bounded in $L^\infty((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T),L^2_{w_{\gamma}})$. In particular, writing $p_n = p_{n,1} + p_{n,2}$ with

$$p_{n,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(b_{n,i} u_{n,j})$$
 and $p_{n,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(F_{n,i,j})$

we get that $p_{n,1}$ is bounded in $L^3((0,T), L^{6/5}_{w_{\frac{6\gamma}{5}}})$ and $p_{n,2}$ is bounded in $L^2((0,T), L^2_{w_{\gamma}})$. If $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we find that $\varphi \mathbf{u}_n$ is bounded in $L^2((0,T), H^1)$ and, writing

$$\partial_t \mathbf{u}_n = \Delta \mathbf{u}_n - \left(\sum_{i=1}^3 \partial_i (b_{n,i} \mathbf{u}_n) + \nabla p_{n,1}\right) + (\nabla \cdot \mathbb{F}_n - \nabla p_{n,2}),$$

 $\varphi \partial_t \mathbf{u}_n$ is bounded in $L^2 L^2 + L^2 W^{-1,6/5} + L^2 H^{-1} \subset L^2((0,T),H^{-2})$. Thus, by Lemma 6, there exists \mathbf{u}_{∞} and an increasing sequence $(n_k)_{k \in \mathbb{N}}$ with values in \mathbb{N} such that \mathbf{u}_{n_k} converges strongly to \mathbf{u}_{∞} in $L^2_{\text{loc}}([0,T) \times \mathbb{R}^3)$: for every $T_0 \in (0,T)$ and every R > 0, we have

$$\lim_{k \to +\infty} \int_0^{T_0} \int_{|y| < R} |\mathbf{u}_{n_k}(s, y) - \mathbf{u}_{\infty}(s, y)|^2 \, dy \, ds = 0.$$

As \mathbf{u}_n is bounded in $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_{\gamma}})$, the convergence of \mathbf{u}_{n_k} to \mathbf{u}_{∞} in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ implies that \mathbf{u}_{n_k} converges *-weakly to \mathbf{u}_{∞} in $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}_{\infty}$ in $L^2((0,T), L^2_{w_{\gamma}})$.

By Banach–Alaoglu's theorem, we may assume that there exists \mathbf{b}_{∞} such that \mathbf{b}_{n_k} converges weakly to \mathbf{b}_{∞} in $L^3((0,T),L^3_{w_{3\gamma/2}})$. In particular $b_{n_k,i}u_{n_k,j}$ is weakly convergent in $(L^{6/5}L^{6/5})_{\text{loc}}$ and thus in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$; as it is bounded in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$, it is weakly convergent in $L^3((0,T),L^{6/5}_{w_{\frac{6\gamma}{5}}})$ to $b_{\infty,i}u_{\infty,j}$. Let

$$p_{\infty,1} = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_{\infty,i} u_{\infty,j}) \text{ and } p_{\infty,2} = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (F_{\infty,i,j}).$$

As the Riesz transforms are bounded on $L_{w_{\frac{6\gamma}{5}}}^{6/5}$ and on $L_{w_{\gamma}}^{2}$, we find that $p_{n_{k},1}$ is weakly convergent in $L^{3}((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5})$ to $p_{\infty,1}$ and that $p_{n_{k},2}$ is strongly convergent in $L^{2}((0,T), L_{w_{\gamma}}^{2})$ to $p_{\infty,2}$.

In particular, we find that in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$

$$\partial_t \mathbf{u}_{\infty} = \Delta \mathbf{u}_{\infty} - \sum_{i=1}^3 \partial_i (b_{\infty,i} \mathbf{u}_{\infty}) - \nabla (p_{\infty,1} + p_{\infty,2}) + \nabla \cdot \mathbb{F}_{\infty}.$$

In particular, $\partial_t \mathbf{u}_{\infty}$ is locally in L^2H^{-2} , and thus \mathbf{u}_{∞} has representative such that $t \mapsto \mathbf{u}_{\infty}(t,.)$ is continuous from [0,T) to $\mathcal{D}'(\mathbb{R}^3)$ and coincides with $\mathbf{u}_{\infty}(0,.) + \int_0^t \partial_t \mathbf{u}_{\infty} ds$. In $\mathcal{D}'((0,T) \times \mathbb{R}^3)$, we have that

$$\mathbf{u}_{\infty}(0,.) + \int_{0}^{t} \partial_{t} \mathbf{u}_{\infty} ds = \mathbf{u}_{\infty} = \lim_{n_{k} \to +\infty} \mathbf{u}_{n_{k}} = \lim_{n_{k} \to +\infty} \mathbf{u}_{0,n_{k}} + \int_{0}^{t} \partial_{t} \mathbf{u}_{n_{k}} ds = \mathbf{u}_{0,\infty} + \int_{0}^{t} \partial_{t} \mathbf{u}_{\infty} ds$$

Thus, $\mathbf{u}_{\infty}(0,.) = \mathbf{u}_{0,\infty}$, and \mathbf{u}_{∞} is a solution of (AD_{∞}) . Next, we define

$$A_n = -\partial_t \left(\frac{|\mathbf{u}_n|^2}{2}\right) + \Delta \left(\frac{|\mathbf{u}_n|^2}{2}\right) - \nabla \cdot \left(\frac{|\mathbf{u}_n|^2}{2}\mathbf{b}_n\right) - \nabla \cdot (p_n\mathbf{u}_n) + \mathbf{u}_n \cdot (\nabla \cdot \mathbb{F}_n) = |\nabla \mathbf{u}_n|^2 + \mu_n.$$

As \mathbf{u}_n is bounded in $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_n$ is bounded in $L^2((0,T), L^2_{w_{\gamma}})$, it is bounded in $L^2((0,T), L^6_{w_{3\gamma/2}})$ and by interpolation with $L^{\infty}((0,T), L^2_{w_{\gamma}})$ it is bounded in $L^{10/3}((0,T), L^{10/3}_{w_{5\gamma/3}})$. Thus, u_{n_k} is locally bounded in $L^{10/3}L^{10/3}$ and locally strongly convergent in L^2L^2 ; it is then strongly convergent in L^3L^3 . Thus, A_{n_k} is convergent in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$ to

$$A_{\infty} = -\partial_t(\frac{|\mathbf{u}_{\infty}|^2}{2}) + \Delta(\frac{|\mathbf{u}_{\infty}|^2}{2}) - \nabla \cdot \left(\frac{|\mathbf{u}_{\infty}|^2}{2}\mathbf{b}_{\infty}\right) - \nabla \cdot (p_{\infty}\mathbf{u}_{\infty}) + \mathbf{u}_{\infty} \cdot (\nabla \cdot \mathbb{F}_{\infty}).$$

In particular, $A_{\infty} = \lim_{n_k \to +\infty} |\nabla \mathbf{u}_{n_k}|^2 + \mu_{n_k}$. If $\Phi \in \mathcal{D}((0,T) \times \mathbb{R}^3)$ is non-negative, we have

$$\iint A_{\infty} \Phi \, dx \, ds = \lim_{n_k \to +\infty} \iint A_{n_k} \Phi \, dx \, ds \ge \limsup_{n_k \to +\infty} \iint |\nabla \mathbf{u}_{n_k}|^2 \Phi \, dx \, ds \ge \iint |\nabla \mathbf{u}_{\infty}|^2 \Phi \, dx \, ds$$

(since $\sqrt{\Phi}\nabla\mathbf{u}_{n_k}$ is weakly convergent to $\sqrt{\Phi}\nabla\mathbf{u}_{\infty}$ in L^2L^2). Thus, there exists a non-negative locally finite measure μ_{∞} on $(0,T)\times\mathbb{R}^3$ such that $A_{\infty} = |\nabla\mathbf{u}_{\infty}|^2 + \mu_{\infty}$, i.e. such that

$$\partial_t(\frac{|\mathbf{u}_{\infty}|^2}{2}) = \Delta(\frac{|\mathbf{u}_{\infty}|^2}{2}) - |\nabla \mathbf{u}_{\infty}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}_{\infty}|^2}{2}\mathbf{b}_{\infty}\right) - \nabla \cdot (p_{\infty}\mathbf{u}_{\infty}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}_{\infty}) - \mu_{\infty}.$$

Finally, we start from inequality (6):

$$\int \frac{|\mathbf{u}_{n}(t,x)|^{2}}{2} \phi_{R} w_{\gamma,\epsilon} dx \leq \int \frac{|\mathbf{u}_{0,n}(x)|^{2}}{2} \phi_{R} w_{\gamma,\epsilon} dx
- \sum_{i=1}^{3} \int_{0}^{t} \int \partial_{i} \mathbf{u}_{n} \cdot \mathbf{u}_{n} \left(w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon} \right) dx ds
- \int_{0}^{t} \int |\nabla \mathbf{u}_{n}|^{2} \phi_{R} w_{\gamma,\epsilon} dx ds
+ \sum_{i=1}^{3} \int_{0}^{t} \int \frac{|\mathbf{u}_{n}|^{2}}{2} b_{n,i} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds
+ \sum_{i=1}^{3} \int_{0}^{t} \int p_{n} u_{n,i} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds
- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{n,i,j} u_{n,j} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds
- \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{n,i,j} \partial_{i} u_{n,j} (w_{\gamma,\epsilon} \partial_{i} \phi_{R} + \phi_{R} \partial_{i} w_{\gamma,\epsilon}) dx ds$$

This gives

$$\begin{split} \limsup_{n_k \to +\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2} \phi_R w_{\gamma,\epsilon} \, dx + \int_0^t \int |\nabla \mathbf{u}_{n_k}|^2 \, \phi_R w_{\gamma,\epsilon} dx \, ds \\ & \leq \int \frac{|\mathbf{u}_{0,\infty}(x)|^2}{2} \phi_R w_{\gamma,\epsilon} \, dx \\ & - \sum_{i=1}^3 \int_0^t \int \partial_i \mathbf{u}_\infty \cdot \mathbf{u}_\infty \left(w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon} \right) dx \, ds \\ & + \sum_{i=1}^3 \int_0^t \int \frac{|\mathbf{u}_\infty|^2}{2} b_{\infty,i} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\ & + \sum_{i=1}^3 \int_0^t \int p_\infty u_{\infty,i} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\ & - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j} u_{\infty,j} (w_{\gamma,\epsilon} \partial_i \phi_R + \phi_R \partial_i w_{\gamma,\epsilon}) \, dx \, ds \\ & - \sum_{i=1}^3 \sum_{j=1}^3 \int_0^t \int F_{\infty,i,j} \partial_i u_{\infty,j} \, \phi_R w_{\gamma,\epsilon} \, dx \, ds. \end{split}$$

As we have

$$\mathbf{u}_{n_k} = \mathbf{u}_{0,n_k} + \int_0^t \partial_t \mathbf{u}_{n_k} \, ds$$

we see that $\mathbf{u}_{n_k}(t,.)$ is convergent to $\mathbf{u}_{\infty}(t,.)$ in $\mathcal{D}'(\mathbb{R}^3)$, hence is weakly convergent in L^2_{loc} (as it is bounded in $L^2_{w_{\gamma}}$), so that :

$$\int \frac{|\mathbf{u}_{\infty}(t,x)|^2}{2} \phi_R w_{\gamma,\epsilon} dx \le \limsup_{n_k \to +\infty} \int \frac{|\mathbf{u}_{n_k}(t,x)|^2}{2} \phi_R w_{\gamma,\epsilon} dx.$$

Similarly, as $\nabla \mathbf{u}_{n_k}$ is weakly convergent in $L^2 L_{w_{\gamma}}^2$, we have

$$\int_0^t \int \frac{|\nabla \mathbf{u}_{\infty}(s,x)|^2}{2} \phi_R w_{\gamma,\epsilon} \, dx \, ds \leq \limsup_{n_k \to +\infty} \int_0^t \int \frac{|\nabla \mathbf{u}_{n_k}(s,x)|^2}{2} \phi_R w_{\gamma,\epsilon} \, dx \, ds.$$

Thus, letting R go to $+\infty$ and then ϵ go to 0, we find by dominated convergence that, for every $t \in (0,T)$, we have

$$\|\mathbf{u}_{\infty}(t,.)\|_{L^{2}_{w_{\gamma}}}^{2} + 2\int_{0}^{t} \|\nabla \mathbf{u}_{\infty}(s,.)\|_{L^{2}_{w_{\gamma}}}^{2} ds$$

$$\leq \|\mathbf{u}_{0,\infty}\|_{L^{2}_{w_{\gamma}}}^{2} - \int_{0}^{t} \int \nabla |\mathbf{u}_{\infty}|^{2} \cdot \nabla w_{\gamma} dx ds + \int_{0}^{t} \int (|\mathbf{u}_{\infty}|^{2} \mathbf{b}_{\infty} + 2p_{\infty} \mathbf{u}_{\infty}) \cdot \nabla (w_{\gamma}) dx ds$$

$$-2\sum_{i=1}^{3} \sum_{j=1}^{3} \int_{0}^{t} \int F_{\infty,i,j}(\partial_{i} u_{\infty,j}) w_{\gamma} + F_{\infty,i,j} u_{\infty,i} \partial_{j}(w_{\gamma}) dx ds.$$

Letting t go to 0, we find

$$\limsup_{t \to 0} \|\mathbf{u}_{\infty}(t,.)\|_{L^{2}_{w\gamma}}^{2} \le \|\mathbf{u}_{0,\infty}\|_{L^{2}_{w\gamma}}^{2}.$$

On the other hand, we know that \mathbf{u}_{∞} is weakly continuous in $L^2_{w_{\gamma}}$ and thus we have

$$\|\mathbf{u}_{0,\infty}\|_{L^2_{w_{\gamma}}}^2 \le \liminf_{t \to 0} \|\mathbf{u}_{\infty}(t,.)\|_{L^2_{w_{\gamma}}}^2$$

This gives $\|\mathbf{u}_{0,\infty}\|_{L^2_{w_{\gamma}}}^2 = \lim_{t\to 0} \|\mathbf{u}_{\infty}(t,.)\|_{L^2_{w_{\gamma}}}^2$, which allows to turn the weak convergence into a strong convergence. Theorem 3 is proven.

5 Solutions of the Navier–Stokes problem with initial data in $L^2_{w_{\gamma}}$.

We now prove Theorem 1. The idea is to approximate the problem by a Navier–Stokes problem in L^2 , then use the a priori estimates (Theorem 2) and the stability theorem (Theorem 3) to find a solution to the Navier–Stokes problem with data in $L^2_{w_{\gamma}}$).

5.1 Approximation by square integrable data.

Lemma 7 (Leray's projection operator) Let $0 < \delta < 3$ and $1 < r < +\infty$. If \mathbf{v} is a vector field on \mathbb{R}^3 such that $\mathbf{v} \in L^r_{w_\delta}$, then there exists a unique decomposition

$$\mathbf{v} = \mathbf{v}_{\sigma} + \mathbf{v}_{\nabla}$$

such that

- $\mathbf{v}_{\sigma} \in L^{r}_{ws}$ and $\nabla \cdot \mathbf{v}_{\sigma} = 0$.
- $\mathbf{v}_{\nabla} \in L^r_{w_{\delta}}$ and $\nabla \wedge \mathbf{v}_{\nabla} = 0$.

We shall write $\mathbf{v}_{\sigma} = \mathbb{P}\mathbf{v}$, where \mathbb{P} is Leray's projection operator.

Similarly, if \mathbf{v} is a distribution vector field of the type $\mathbf{v} = \nabla \cdot \mathbb{G}$ with $\mathbb{G} \in L^r_{ws}$ then there exists a unique decomposition

$$\mathbf{v} = \mathbf{v}_{\sigma} + \mathbf{v}_{\nabla}$$

such that

- there exists $\mathbb{H} \in L^r_{w_{\delta}}$ such that $\mathbf{v}_{\sigma} = \nabla \cdot \mathbb{H}$ and $\nabla \cdot \mathbf{v}_{\sigma} = 0$.
- there exists $q \in L^r_{w_\delta}$ such that $\mathbf{v}_\nabla = \nabla q$ (and thus $\nabla \wedge \mathbf{v}_\nabla = 0$).

We shall still write $\mathbf{v}_{\sigma} = \mathbb{P}\mathbf{v}$. Moreover, the function q is given by

$$q = -\sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j(G_{i,j}).$$

Proof: As $w_{\delta} \in \mathcal{A}_r$ the Riesz transforms are bounded on $L^r_{w_{\delta}}$. Using the identity

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \wedge (\nabla \wedge \mathbf{v})$$

we find (if the decomposition exists) that

$$\Delta \mathbf{v}_{\sigma} = -\nabla \wedge (\nabla \wedge \mathbf{v}_{\sigma}) = -\nabla \wedge (\nabla \wedge \mathbf{v}) \text{ and } \Delta \mathbf{v}_{\nabla} = \nabla (\nabla \cdot \mathbf{v}_{\nabla}) = \nabla (\nabla \cdot \mathbf{v}).$$

This proves the uniqueness. By linearity, we just have to prove that $\mathbf{v} = 0 \implies \mathbf{v}_{\nabla} = 0$. We have $\Delta \mathbf{v}_{\nabla} = 0$, and thus \mathbf{v}_{∇} is harmonic; as it belongs to \mathcal{S}' , we find that it is a polynomial. But a polynomial which belongs to $L^r_{w_{\delta}}$ must be equal to 0. Similarly, if $\mathbf{v}_{\nabla} = \nabla q$, then $\Delta q = \nabla \cdot \mathbf{v}_{\nabla} = \nabla \cdot \mathbf{v} = 0$; thus q is harmonic and belongs to $L^r_{w_{\delta}}$, hence q = 0.

For the existence, it is enough to check that $v_{\nabla,i} = -\sum_{j=1}^{3} R_i R_j v_j$ in the first case and $\mathbf{v}_{\nabla} = \nabla q$ with $q = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (G_{i,j})$ in the second case fulfill the conclusions of the lemma.

Lemma 8 Let $0 < \gamma \le 2$. Let \mathbf{u}_0 be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2((0,+\infty), L^2_{w_\gamma})$. Let $\phi \in \mathcal{D}(\mathbb{R}^3)$ be a non-negative function which is equal to 1 for $|x| \le 1$ and to 0 for $|x| \ge 2$. For R > 0, we define $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R\mathbf{u}_0)$ and $\mathbb{F}_R = \phi_R\mathbb{F}$. Then $\mathbf{u}_{0,R}$ is a divergence-free square integrable vector field and $\lim_{R\to+\infty} \|\mathbf{u}_{0,R} - \mathbf{u}_0\|_{L^2_{w_\gamma}} = 0$. Similarly, \mathbb{F}_R belongs to L^2L^2 and $\lim_{R\to+\infty} \|\mathbb{F}_R - \mathbb{F}\|_{L^2((0,+\infty),L^2_{w_\gamma})} = 0$.

Proof: By dominated convergence, we have $\lim_{R\to+\infty} \|\phi_R \mathbf{u}_0 - \mathbf{u}_0\|_{L^2_{w\gamma}} = 0$. We conclude by writing $\mathbf{u}_{0,R} - \mathbf{u}_0 = \mathbb{P}(\phi_R \mathbf{u}_0 - \mathbf{u}_0)$.

5.2 Leray's mollification.

We want to solve the Navier–Stokes equations with initial value \mathbf{u}_0 :

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

We begin with Leray's method [11] for solving the problem in L^2 :

$$(NS_R) \begin{cases} \partial_t \mathbf{u}_R = \Delta \mathbf{u}_R - (\mathbf{u}_R \cdot \nabla) \mathbf{u}_R - \nabla p_R + \nabla \cdot \mathbb{F}_{\mathbb{R}} \\ \nabla \cdot \mathbf{u}_R = 0, & \mathbf{u}_R(0,.) = \mathbf{u}_{0,R} \end{cases}$$

The idea of Leray is to mollify the non-linearity by replacing $\mathbf{u}_R \cdot \nabla$ by $(\mathbf{u}_R * \theta_{\epsilon}) \cdot \nabla$, where $\theta(x) = \frac{1}{\epsilon^3} \theta(\frac{x}{\epsilon}), \ \theta \in \mathcal{D}(\mathbb{R}^3), \ \theta$ is non-negative and radially decreasing and $\int \theta \, dx = 1$. We thus solve the problem

$$(NS_{R,\epsilon}) \begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{u}_{R,\epsilon} * \theta_{\epsilon}) \cdot \nabla) \mathbf{u}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_R \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0, & \mathbf{u}_{R,\epsilon}(0,.) = \mathbf{u}_{0,R} \end{cases}$$

The classical result of Leray states that the problem $(NS_{R,\epsilon})$ is well-posed:

Lemma 9 Let $\mathbf{v}_0 \in L^2$ be a divergence-free vector field. Let $\mathbb{G} \in L^2((0, +\infty), L^2)$. Then the problem

$$(NS_{\epsilon}) \begin{cases} \partial_t \mathbf{v}_{\epsilon} = \Delta \mathbf{v}_{\epsilon} - ((\mathbf{v}_{\epsilon} * \theta_{\epsilon}) \cdot \nabla) \mathbf{v}_{\epsilon} - \nabla q_{\epsilon} + \nabla \cdot \mathbb{G} \\ \nabla \cdot \mathbf{v}_{\epsilon} = 0, \qquad \mathbf{v}_{\epsilon}(0, .) = \mathbf{v}_0 \end{cases}$$

has a unique solution \mathbf{v}_{ϵ} in $L^{\infty}((0,+\infty),L^2) \cap L^2((0,+\infty),\dot{H}^1)$. Moreover, this solution belongs to $\mathcal{C}([0,+\infty),L^2)$.

5.3 Proof of Theorem 1 (local existence)

We use Lemma 9 and find a solution $\mathbf{u}_{R,\epsilon}$ to the problem $(NS_{R,\epsilon})$. Then we check that $\mathbf{u}_{R,\epsilon}$ fulfills the assumptions of Theorem 2 and of Corollary 6:

- $\mathbf{u}_{R,\epsilon}$ belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_{R,\epsilon}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$
- the map $t \in [0, +\infty) \mapsto \mathbf{u}_{R,\epsilon}(t, .)$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_2}$, and is strongly continuous at t = 0:

$$\lim_{t\to 0} \|\mathbf{u}_{R,\epsilon}(t,.) - \mathbf{u}_{0,R}\|_{L^2_{w_{\gamma}}} = 0.$$

• on $(0,T) \times \mathbb{R}^3$, $\mathbf{u}_{R,\epsilon}$ fulfills the energy equality:

$$\partial_t (\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) = \Delta (\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) - |\nabla \mathbf{u}_{R,\epsilon}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2} \mathbf{b}_{R,\epsilon}\right) - \nabla \cdot (p_{R,\epsilon} \mathbf{u}_{R,\epsilon}) + \mathbf{u}_{R,\epsilon} \cdot (\nabla \cdot \mathbb{F}_R).$$
with $\mathbf{b}_{R,\epsilon} = \mathbf{u}_{R,\epsilon} * \theta_{\epsilon}$.

• $\mathbf{b}_{R,\epsilon}$ is controlled by $\mathbf{u}_{R,\epsilon}$: for every $t \in (0,T)$,

$$\|\mathbf{b}_{R,\epsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}} \le \|\mathcal{M}_{\mathbf{u}_{R,\epsilon}(t,.)}\|_{L^3_{w_{3\gamma/2}}} \le C_0 \|\mathbf{u}_{R,\epsilon}(t,.)\|_{L^3_{w_{3\gamma/2}}}.$$

Thus, we know that, for every time T_0 such that

$$C_{\gamma}(1+C_0^4)\left(1+C_0^4+\|\mathbf{u}_{0,R}\|_{L_{w_{\gamma}}^2}^2+\int_0^{T_0}\|\mathbb{F}_R\|_{L_{w_{\gamma}}^2}^2\,ds\right)^2\,T_0\leq 1$$

we have

$$\sup_{0 \le t \le T_0} \| \mathbf{u}_{R,\epsilon}(t,.) \|_{L^2_{w_{\gamma}}}^2 \le C_{\gamma} (1 + C_0^4 + \| \mathbf{u}_{0,R} \|_{L^2_{w_{\gamma}}}^2 + \int_0^{T_0} \| \mathbb{F}_R \|_{L^2_{w_{\gamma}}}^2 ds)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{u}_{R,\epsilon}\|_{L^2_{w\gamma}}^2 ds \le C_{\gamma} (1 + C_0^4 + \|\mathbf{u}_{0,R}\|_{L^2_{w\gamma}}^2 + \int_0^{T_0} \|\mathbb{F}_R\|_{L^2_{w\gamma}}^2 ds).$$

Moreover, we have that

$$\|\mathbf{u}_{0,R}\|_{L^2_{w\gamma}} \le C_{\gamma} \|\mathbf{u}_0\|_{L^2_{w\gamma}} \text{ and } \|\mathbb{F}_R\|_{L^2_{w\gamma}} \le \|\mathbb{F}\|_{L^2_{w\gamma}}$$

so that

$$\begin{aligned} \|\mathbf{b}_{R,\epsilon}\|_{L^{3}((0,T_{0}),L^{3}_{w_{3\gamma/2}}} &\leq C_{\gamma}\|\mathbf{u}_{R,\epsilon}\|_{L^{3}((0,T_{0}),L^{3}_{w_{3\gamma/2}}} \\ &\leq C_{\gamma}'T_{0}^{\frac{1}{12}}\left((1+\sqrt{T_{0}})\|\mathbf{u}_{R,\epsilon}\|_{L^{\infty}((0,T_{0}),L^{2}_{w_{\gamma}})} + \|\nabla\mathbf{u}_{R,\epsilon}\|_{L^{2}((0,T_{0}),L^{2}_{w_{\gamma}})}\right) \\ &\leq C_{\gamma}''\sqrt{1+C_{0}^{4}+\|\mathbf{u}_{0}\|_{L^{2}_{w_{\gamma}}}^{2}+\int_{0}^{T_{0}}\|\mathbb{F}\|_{L^{2}_{w_{\gamma}}}^{2}ds}. \end{aligned}$$

Let $R_n \to +\infty$ and $\epsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,R_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\epsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\epsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to \mathbf{u}_0 in $L^2_{w_{\gamma}}$, \mathbb{F}_n is strongly convergent to \mathbb{F} in $L^2((0,T_0),L^2_{w_{\gamma}})$, and the sequence \mathbf{b}_n is bounded in $L^3((0,T_0),L^3_{w_{3\gamma/2}})$. Thus there exists p, \mathbf{u} , \mathbf{b} and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in \mathbb{N} such that

- \mathbf{u}_{n_k} converges *-weakly to \mathbf{u} in $L^{\infty}((0,T_0),L^2_{w_{\gamma}})$, $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}$ in $L^2((0,T_0),L^2_{w_{\gamma}})$
- \mathbf{b}_{n_k} converges weakly to \mathbf{b} in $L^3((0,T_0),L^3_{w_{3\gamma/2}}), p_{n_k}$ converges weakly to p in $L^3((0,T_0),L^{6/5}_{w_{\frac{6\gamma}{5}}}) + L^2((0,T_0),L^2_{w_{\gamma}})$
- \mathbf{u}_{n_k} converges strongly to \mathbf{u} in $L^2_{loc}([0,T_0)\times\mathbb{R}^3)$.

Moreover, \mathbf{u} is a solution of the advection-diffusion problem

$$\begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

and is such that:

• the map $t \in [0, T_0) \mapsto \mathbf{u}(t, .)$ is weakly continuous from $[0, T_0)$ to $L^2_{w_{\gamma}}$, and is strongly continuous at t = 0:

$$\lim_{t \to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w\gamma}} = 0.$$

• there exists a non-negative locally finite measure μ on $(0, T_0) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

Finally, as $\mathbf{b}_n = \theta_{\epsilon_n} * (\mathbf{u}_n - \mathbf{u}) + \theta_{\epsilon_n} * \mathbf{u}$, we see that \mathbf{b}_{n_k} is strongly convergent to \mathbf{u} in $L^3_{loc}([0, T_0) \times \mathbb{R}^3)$, so that $\mathbf{b} = \mathbf{u}$: thus, \mathbf{u} is a solution of the Navier–Stokes problem on $(0, T_0)$. (It is easy to check that

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (u_i u_j - F_{i,j})$$

as $u_{i,n_k}u_{j,n_k}$ is weakly convergent to u_iu_j in $L^4((0,T_0),L_{w_{\frac{6\gamma}{5}}}^{6/5})$ and $w_{\frac{6\gamma}{5}}\in\mathcal{A}_{6/5})$.

5.4 Proof of Theorem 1 (global existence)

In order to finish the proof, we shall use the scaling properties of the Navier–Stokes equations: if $\lambda > 0$, then \mathbf{u} is a solution of the Cauchy initial value problem for the Navier–Stokes equations on (0,T) with initial value \mathbf{u}_0 and forcing tensor \mathbb{F} if and only if $\mathbf{u}_{\lambda}(t,x) = \lambda \mathbf{u}(\lambda^2 t, \lambda x)$ is a solution of the Navier–Stokes equations on $(0,T/\lambda^2)$ with initial value $\mathbf{u}_{0,\lambda}(x) = \lambda \mathbf{u}_0(\lambda x)$ and forcing tensor $\mathbb{F}_{\lambda}(t,x) = \lambda^2 \mathbb{F}(\lambda^2 t, \lambda x)$.

We take $\lambda > 1$ and for $n \in \mathbb{N}$ we consider the Navier–Stokes problem with initial value $\mathbf{v}_{0,n} = \lambda^n \mathbf{u}_0(\lambda^n \cdot)$ and forcing tensor $\mathbb{F}_n = \lambda^{2n} \mathbb{F}(\lambda^{2n} \cdot, \lambda^n \cdot)$. Then we have seen that we can find a solution \mathbf{v}_n on $(0, T_n)$, with

$$C_{\gamma} \left(1 + \|\mathbf{v}_{0,n}\|_{L^{2}_{w_{\gamma}}}^{2} + \int_{0}^{+\infty} \|\mathbb{F}_{n}\|_{L^{2}_{w_{\gamma}}}^{2} ds \right)^{2} T_{n} = 1.$$

Of course, we have $\mathbf{v}_n(t,x) = \lambda^n \mathbf{u}_n(\lambda^{2n}t,\lambda^n x)$ where \mathbf{u}_n is a solution of the Navier–Stokes equations on $(0,\lambda^{2n}T_n)$ with initial value \mathbf{u}_0 and forcing tensor \mathbb{F}

Lemma 10

$$\lim_{n \to +\infty} \frac{\lambda^n}{1 + \|\mathbf{v}_{0,n}\|_{L^2_{w_{\infty}}}^2 + \int_0^{+\infty} \|\mathbb{F}_n\|_{L^2_{w_{\infty}}}^2 ds} = +\infty.$$

Proof: We have

$$\|\mathbf{v}_{0,n}\|_{L^{2}_{w\gamma}}^{2} = \int |\mathbf{u}_{0}(x)|^{2} \lambda^{n(\gamma-1)} \frac{(1+|x|)^{\gamma}}{(\lambda^{n}+|x|)^{\gamma}} w_{\gamma}(x) dx.$$

We have

$$\lambda^{n(\gamma-1)} < \lambda^n$$

as $\gamma \leq 2$ and we have, by dominated convergence,

$$\lim_{n \to +\infty} \int |\mathbf{u}_0(x)|^2 \frac{(1+|x|)^{\gamma}}{(\lambda^n + |x|)^{\gamma}} w_{\gamma}(x) dx = 0.$$

Similarly, we have

$$\int_0^{+\infty} \|\mathbb{F}_n\|_{L^2_{w\gamma}}^2 ds = \int_0^{+\infty} \int |\mathbb{F}(s,x)|^2 \lambda^{n(\gamma-1)} \frac{(1+|x|)^{\gamma}}{(\lambda^n+|x|)^{\gamma}} w_{\gamma}(x) dx ds = o(\lambda^n).$$

Thus, $\lim_{n\to+\infty} \lambda^{2n} T_n = +\infty$.

Now, for a given T > 0, if $\lambda^{2n}T_n > T$ for $n \ge n_T$, then \mathbf{u}_n is a solution of the Navier-Stokes problem on (0,T). Let $\mathbf{w}_n(t,x) = \lambda^{n_T} \mathbf{u}_n(\lambda^{2n_T}t,\lambda^{n_T}x)$.

For $n \geq n_T$, \mathbf{w}_n is a solution of the Navier-Stokes problem on $(0, \lambda^{-2n_T}T)$ with initial value \mathbf{v}_{0,n_T} and forcing tensor \mathbb{F}_{n_T} . As $\lambda^{-2n_T}T \leq T_{n_T}$, we have

$$C_{\gamma} \left(1 + \|\mathbf{v}_{0,n_T}\|_{L^2_{w_{\gamma}}}^2 + \int_0^{+\infty} \|\mathbb{F}_{n_T}\|_{L^2_{w_{\gamma}}}^2 ds \right)^2 \lambda^{-2n_T} T \le 1.$$

By corollary 6, we have

$$\sup_{0 \le t \le \lambda^{-2n_T T}} \| \mathbf{w}_n(t,.) \|_{L^2_{w_\gamma}}^2 \le C_\gamma (1 + \| \mathbf{v}_{0,n_T} \|_{L^2_{w_\gamma}}^2 + \int_0^{\lambda^{-2n_T T}} \| \mathbb{F}_{n_T} \|_{L^2_{w_\gamma}}^2 \, ds)$$

and

$$\int_0^{\lambda^{-2n_T T}} \|\nabla \mathbf{w}_n\|_{L^2_{w_\gamma}}^2 ds \le C_\gamma (1 + \|\mathbf{v}_{0,n_T}\|_{L^2_{w_\gamma}}^2 + \int_0^{\lambda^{-2n_T T}} \|\mathbb{F}_{n_T}\|_{L^2_{w_\gamma}}^2 ds).$$

We have

$$\|\mathbf{w}_n\|_{L^2_{w_{\gamma}}}^2 = \int |\mathbf{u}_n(\lambda^{2n_T}t, x)|^2 \lambda^{n_T(\gamma - 1)} \frac{(1 + |x|)^{\gamma}}{(\lambda^{n_T} + |x|)^{\gamma}} w_{\gamma}(x) dx \ge \lambda^{n_T(\gamma - 1)} \|\mathbf{u}_n(\lambda^{2n_T}t, .)\|_{L^2_{w_{\gamma}}}^2.$$

and

$$\int_{0}^{\lambda^{-2n_{T}T}} \|\nabla \mathbf{w}_{n}\|_{L_{w_{\gamma}}^{2}}^{2} ds = \int_{0}^{T} \int |\nabla \mathbf{u}_{n}(s,x)|^{2} \lambda^{n_{T}(\gamma-1)} \frac{(1+|x|)^{\gamma}}{(\lambda^{n_{T}}+|x|)^{\gamma}} w_{\gamma}(x) dx ds$$

$$\geq \lambda^{n_{T}(\gamma-1)} \int_{0}^{T} \|\nabla \mathbf{u}_{n}\|_{L_{w_{\gamma}}^{2}}^{2} ds.$$

Thus, we have a uniform control of \mathbf{u}_n and of $\nabla \mathbf{u}_n$ on (0,T) for $n \geq n_T$. We may then apply the Rellich lemma (Lemma 6) and Theorem 3 to find a subsequence \mathbf{u}_{n_k} that converges to a global solution of the Navier–Stokes equations. Theorem 1 is proven.

6 Solutions of the advection-diffusion problem with initial data in $L^2_{w_{\gamma}}$.

The proof of Theorem 1 on the Navier–Stokes problem can be easily adapted to the case of the advection-diffusion problem :

Theorem 4 Let $0 < \gamma \le 2$. Let $0 < T < +\infty$. Let \mathbf{u}_0 be a divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_{\gamma}}(\mathbb{R}^3)$ and \mathbb{F} be a tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$

such that $\mathbb{F} \in L^2((0,T), L^2_{w_{\gamma}})$. Let **b** be a time-dependent divergence free vector-field $(\nabla \cdot \mathbf{b} = 0)$ such that $\mathbf{b} \in L^3((0,T), L^3_{w_{2\gamma/2}})$.

Then the advection-diffusion problem

$$(AD) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{b} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

has a solution **u** such that:

- **u** belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$
- the pressure p is related to \mathbf{u} , \mathbf{b} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j (b_i u_j - F_{i,j})$$

• the map $t \in [0,T) \mapsto \mathbf{u}(t,.)$ is weakly continuous from [0,T) to $L^2_{w_{\gamma}}$, and is strongly continuous at t=0:

$$\lim_{t\to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

• there exists a non-negative locally finite measure μ on $(0,T) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}\right) - \nabla \cdot (p\mathbf{u}) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

Proof: Again, we define $\phi_R(x) = \phi(\frac{x}{R})$, $\mathbf{u}_{0,R} = \mathbb{P}(\phi_R \mathbf{u}_0)$ and $\mathbb{F}_R = \phi_R \mathbb{F}$. Moreover, we define $\mathbf{b}_R = \mathbb{P}(\phi_R \mathbf{b})$. We then solve the mollified problem

$$(AD_{R,\epsilon}) \begin{cases} \partial_t \mathbf{u}_{R,\epsilon} = \Delta \mathbf{u}_{R,\epsilon} - ((\mathbf{b}_R * \theta_{\epsilon}) \cdot \nabla) \mathbf{u}_{R,\epsilon} - \nabla p_{R,\epsilon} + \nabla \cdot \mathbb{F}_{R,\epsilon} \\ \nabla \cdot \mathbf{u}_{R,\epsilon} = 0, \qquad \mathbf{u}_{R,\epsilon}(0,.) = \mathbf{u}_{0,R} \end{cases}$$

for which we easily find a unique solution $\mathbf{u}_{R,\epsilon}$ in $L^{\infty}((0,T),L^2)\cap L^2((0,T),\dot{H}^1)$. Moreover, this solution belongs to $\mathcal{C}([0,T),L^2)$.

Again, $\mathbf{u}_{R,\epsilon}$ fulfills the assumptions of Theorem 2:

• $\mathbf{u}_{R,\epsilon}$ belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_{R,\epsilon}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$

• the map $t \in [0,T) \mapsto \mathbf{u}_{R,\epsilon}(t,.)$ is weakly continuous from [0,T) to $L^2_{w_{\gamma}}$, and is strongly continuous at t=0:

$$\lim_{t \to 0} \|\mathbf{u}_{R,\epsilon}(t,.) - \mathbf{u}_{0,R}\|_{L^2_{w_{\gamma}}} = 0.$$

• on $(0,T) \times \mathbb{R}^3$, $\mathbf{u}_{R,\epsilon}$ fulfills the energy equality :

$$\partial_t(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) = \Delta(\frac{|\mathbf{u}_{R,\epsilon}|^2}{2}) - |\nabla \mathbf{u}_{R,\epsilon}|^2 - \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2}\mathbf{b}_{R,\epsilon}\right) - \nabla \cdot (p_{R,\epsilon}\mathbf{u}_{R,\epsilon}) + \mathbf{u}_{R,\epsilon} \cdot (\nabla \cdot \mathbb{F}_R).$$

with $\mathbf{b}_{R,\epsilon} = \mathbf{b}_R * \theta_{\epsilon}$.

Thus, by Corollary 4 we know that,

$$\sup_{0 < t < T} \| \mathbf{u}_{R,\epsilon} \|_{L^2_{w_{\gamma}}} \leq \left(\| \mathbf{u}_{0,R} \|_{L^2_{w_{\gamma}}} + C_{\gamma} \| \mathbb{F}_{R} \|_{L^2((0,T),L^2_{w_{\gamma}})} \right) e^{C_{\gamma} (T + T^{1/3} \| \mathbf{b}_{R,\epsilon} \|_{L^3((0,T),L^3_{w_{3\gamma/2}})})}$$

and

$$\|\nabla \mathbf{u}_{R,\epsilon}\|_{L^2((0,T),L^2_{w\gamma})} \leq \left(\|\mathbf{u}_{0,R}\|_{L^2_{w\gamma}} + C_{\gamma}\|\mathbb{F}_R\|_{L^2((0,T),L^2_{w\gamma})}\right) e^{C_{\gamma}(T+T^{1/3}\|\mathbf{b}_{R,\epsilon}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}^2)}$$

where the constant C_{γ} depends only on γ .

Moreover, we have that

$$\|\mathbf{u}_{0,R}\|_{L^2_{w\gamma}} \leq C_{\gamma} \|\mathbf{u}_0\|_{L^2_{w\gamma}}, \|\mathbb{F}_R\|_{L^2_{w\gamma}} \leq \|\mathbb{F}\|_{L^2_{w\gamma}}$$

and

$$\|\mathbf{b}_{R,\epsilon}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \le \|\mathcal{M}_{\mathbf{b}_R}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \le C_{\gamma}' \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$$

Let $R_n \to +\infty$ and $\epsilon_n \to 0$. Let $\mathbf{u}_{0,n} = \mathbf{u}_{0,T_n}$, $\mathbb{F}_n = \mathbb{F}_{R_n}$, $\mathbf{b}_n = \mathbf{b}_{R_n,\epsilon_n}$ and $\mathbf{u}_n = \mathbf{u}_{R_n,\epsilon_n}$. We may then apply Theorem 3, since $\mathbf{u}_{0,n}$ is strongly convergent to \mathbf{u}_0 in $L^2_{w_{\gamma}}$, \mathbb{F}_n is strongly convergent to \mathbb{F} in $L^2((0,T), L^2_{w_{\gamma}})$, and the sequence \mathbf{b}_n is strongly convergent to \mathbf{b} in $L^3((0,T), L^3_{w_{3\gamma/2}})$. Thus there exists p, \mathbf{u} and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in \mathbb{N} such that

- \mathbf{u}_{n_k} converges *-weakly to \mathbf{u} in $L^{\infty}((0,T),L^2_{w_{\gamma}})$, $\nabla \mathbf{u}_{n_k}$ converges weakly to $\nabla \mathbf{u}$ in $L^2((0,T),L^2_{w_{\gamma}})$
- p_{n_k} converges weakly to p in $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5}) + L^2((0,T), L_{w_{\gamma}}^2)$
- \mathbf{u}_{n_k} converges strongly to \mathbf{u} in $L^2_{\text{loc}}([0,T)\times\mathbb{R}^3)$.

We then easily finish the proof.

7 Application to the study of λ -discretely self-similar solutions

We may now apply our results to the study of λ -discretely self-similar solutions for the Navier–Stokes equations.

Definition 1 Let $\mathbf{u}_0 \in L^2_{loc}(\mathbb{R}^3)$. We say that \mathbf{u}_0 is a λ -discretely self-similar function (λ -DSS) if there exists $\lambda > 1$ such that $\lambda \mathbf{u}_0(\lambda x) = \mathbf{u}_0$.

A vector field $\mathbf{u} \in L^2_{loc}([0, +\infty) \times \mathbb{R}^3)$ is λ -DSS if there exists $\lambda > 1$ such that $\lambda \mathbf{u}(\lambda^2 t, \lambda x) = \mathbf{u}(t, x)$.

A forcing tensor $\mathbb{F} \in L^2_{loc}([0,+\infty) \times \mathbb{R}^3)$ is λ -DSS if there exists $\lambda > 1$ such that $\lambda^2 \mathbb{F}(\lambda^2 t, \lambda x) = \mathbb{F}(t,x)$.

We shall speak of self-similarity if \mathbf{u}_0 , \mathbf{u} or \mathbb{F} are λ -DSS for every $\lambda > 1$.

Examples:

• Let $\gamma > 1$ and $\lambda > 1$. Then, for two positive constants $A_{\gamma,\lambda}$ and $B_{\gamma,\lambda}$, we have : if $\mathbf{u}_0 \in L^2_{loc}(\mathbb{R}^3)$ is λ -DSS, then $\mathbf{u}_0 \in L^2_{w_{\gamma}}$ and

$$A_{\gamma,\lambda} \int_{1 < |x| \le \lambda} |\mathbf{u}_0(x)|^2 dx \le \int |\mathbf{u}_0(x)|^2 w_{\gamma}(x) dx \le B_{\gamma,\lambda} \int_{1 < |x| \le \lambda} |\mathbf{u}_0(x)|^2 dx$$

- $\mathbf{u}_0 \in L^2_{\text{loc}}$ is self-similar if and only if it is of the form $\mathbf{u}_0 = \frac{\mathbf{w}_0(\frac{x}{|x|})}{|x|}$ with $\mathbf{w}_0 \in L^2(S^2)$.
- \mathbb{F} belongs to $L^2((0,+\infty),L^2_{w_\gamma})$ with $\gamma > 1$ and is self-similar if and only if it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$ with $\int |\mathbb{F}_0(x)|^2 \frac{1}{|x|} dx < +\infty$.

Proof:

• If \mathbf{u}_0 is λ -DSS and if $k \in \mathbb{Z}$ we have

$$\int_{\lambda^k < |x| < \lambda^{k+1}} |\mathbf{u}_0(x)|^2 w_\gamma(x) \, dx \le \frac{\lambda^k}{(1+\lambda^k)^\gamma} \int_{1 < |x| < \lambda} |\mathbf{u}_0(x)|^2 \, dx$$

with $\sum_{k\in\mathbb{Z}} \frac{\lambda^k}{(1+\lambda^k)^{\gamma}} < +\infty$ for $\gamma > 1$.

• If \mathbf{u}_0 is self-similar, we have $\mathbf{u}_0(x) = \frac{1}{|x|}\mathbf{u}_0(\frac{x}{|x|})$. From this equality, we find that, for $\lambda > 1$

$$\int_{1<|x|<\lambda} |\mathbf{u}_0(x)|^2 \, dx = (\lambda - 1) \int_{S^2} |\mathbf{u}_0(\sigma)|^2 \, d\sigma$$

• If \mathbb{F} is self-similar, then it is of the form $\mathbb{F}(t,x) = \frac{1}{t}\mathbb{F}_0(\frac{x}{\sqrt{t}})$. Moreover, we have

$$\int_0^{+\infty} \int |\mathbb{F}(t,x)|^2 w_{\gamma}(x) dx ds = \int_0^{+\infty} \int |\mathbb{F}_0(x)|^2 w_{\gamma}(\sqrt{t} x) dx \frac{dt}{\sqrt{t}} = C_{\gamma} \int |\mathbb{F}_0(x)|^2 \frac{dx}{|x|}$$
with $C_{\gamma} = \int_0^{+\infty} \frac{1}{(1+\sqrt{\theta})^{\gamma}} \frac{d\theta}{\sqrt{\theta}} < +\infty.$

In this section, we are going to give a new proof of the results of Chae and Wolf [3] and Bradshaw and Tsai [2] on the existence of λ -DSS solutions of the Navier–Stokes problem (and of Jia and Šverák [6] for self-similar solutions):

Theorem 5 Let $4/3 < \gamma \le 2$ and $\lambda > 1$. If \mathbf{u}_0 is a λ -DSS divergence-free vector field (such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$) and if \mathbb{F} is a λ -DSS tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that $\mathbb{F} \in L^2_{\mathrm{loc}}([0,+\infty) \times \mathbb{R}^3)$, then the Navier-Stokes equations with initial value \mathbf{u}_0

$$(NS) \begin{cases} \partial_t \mathbf{u} = \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0, .) = \mathbf{u}_0 \end{cases}$$

has a global weak solution \mathbf{u} such that :

- **u** is a λ -DSS vector field
- for every $0 < T < +\infty$, **u** belongs to $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and ∇ **u** belongs to $L^2((0,T), L^2_{w_{\gamma}})$
- the map $t \in [0, +\infty) \mapsto \mathbf{u}(t, .)$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_{\gamma}}$, and is strongly continuous at t = 0:

$$\lim_{t \to 0} \|\mathbf{u}(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

• the solution **u** is suitable: there exists a non-negative locally finite measure μ on $(0, +\infty) \times \mathbb{R}^3$ such that

$$\partial_t(\frac{|\mathbf{u}|^2}{2}) = \Delta(\frac{|\mathbf{u}|^2}{2}) - |\nabla \mathbf{u}|^2 - \nabla \cdot \left((\frac{|\mathbf{u}|^2}{2} + p)\mathbf{u} \right) + \mathbf{u} \cdot (\nabla \cdot \mathbb{F}) - \mu.$$

7.1 The linear problem.

Following Chae and Wolf, we consider an approximation of the problem that is consistent with the scaling properties of the equations: let θ be a non-negative and radially decreasing function in $\mathcal{D}(\mathbb{R}^3)$ with $\int \theta \, dx = 1$; We define $\theta_{\epsilon,t}(x) = \frac{1}{(\epsilon\sqrt{t})^3} \theta(\frac{x}{\epsilon\sqrt{t}})$. We then will study the "mollified" problem

$$(NS_{\epsilon}) \begin{cases} \partial_t \mathbf{u}_{\epsilon} = \Delta \mathbf{u}_{\epsilon} - ((\mathbf{u}_{\epsilon} * \theta_{\epsilon,t}) \cdot \nabla) \mathbf{u}_{\epsilon} - \nabla p_{\epsilon} + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{u}(0,.) = \mathbf{u}_0 \end{cases}$$

and begin with the linearized problem

$$(LNS_{\epsilon}) \begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla) \mathbf{v} - \nabla q + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{v}(0,.) = \mathbf{u}_0 \end{cases}$$

Lemma 11 Let $1 < \gamma \le 2$. Let $\lambda > 1$ Let \mathbf{u}_0 be a λ -DSS divergence-free vector field such that $\mathbf{u}_0 \in L^2_{w_\gamma}(\mathbb{R}^3)$ and \mathbb{F} be a λ -DSS tensor $\mathbb{F}(t,x) = (F_{i,j}(t,x))_{1 \le i,j \le 3}$ such that, for every T > 0, $\mathbb{F} \in L^2((0,T), L^2_{w_\gamma})$. Let \mathbf{b} be a λ -DSS time-dependent divergence free vector-field $(\nabla \cdot \mathbf{b} = 0)$ such that, for every T > 0, $\mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}})$.

Then the advection-diffusion problem

$$(LNS_{\epsilon}) \begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{b} * \theta_{\epsilon,t}) \cdot \nabla) \mathbf{v} - \nabla q + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{v}(0,.) = \mathbf{u}_0 \end{cases}$$

has a unique solution \mathbf{v} such that :

- for every positive T, \mathbf{v} belongs to $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{v}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$
- the pressure p is related to \mathbf{v} , \mathbf{b} and \mathbb{F} through the Riesz transforms $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$ by the formula

$$p = \sum_{i=1}^{3} \sum_{j=1}^{3} R_i R_j ((b_i * \theta_{\epsilon,t}) v_j - F_{i,j})$$

• the map $t \in [0, +\infty) \mapsto \mathbf{v}(t, .)$ is weakly continuous from $[0, +\infty)$ to $L^2_{w_{\gamma}}$, and is strongly continuous at t = 0:

$$\lim_{t\to 0} \|\mathbf{v}(t,.) - \mathbf{u}_0\|_{L^2_{w_{\gamma}}} = 0.$$

This solution \mathbf{v} is a λ -DSS vector field.

Proof: As we have $|\mathbf{b}(t,.)*\theta_{\epsilon,t}| \leq \mathcal{M}_{\mathbf{b}(t,.)}$ and thus

$$\|\mathbf{b}(t,.) * \theta_{\epsilon,t}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})} \le C_{\gamma} \|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$$

we see that we can use Theorem 4 to get a solution \mathbf{v} on (0,T).

As clearly $\mathbf{b} * \theta_{\epsilon,t}$ belongs to $L_t^2 L_x^{\infty}(K)$ for every compact subset K of $(0,T) \times \mathbb{R}^3$, we can use Corollary 5 to see that \mathbf{v} is unique.

Let $\mathbf{w}(t,x) = \frac{1}{\lambda} \mathbf{v}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$. As $b * \theta_{\epsilon,t}$ is still λ -DSS, we see that \mathbf{w} is solution of (LNS_{ϵ}) on (0,T), so that $\mathbf{w} = \mathbf{v}$. This means that \mathbf{v} is λ -DSS. \diamond

7.2 The mollified Navier–Stokes equations.

The solution \mathbf{v} provided by Lemma 11 belongs to $L^3((0,T),L^3_{w_{3\gamma/2}})$ (as \mathbf{v} belongs to $L^\infty((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{v}$ belongs to $L^2((0,T),L^2_{w_{\gamma}})$). Thus we have a mapping $L_{\epsilon}: \mathbf{b} \mapsto \mathbf{v}$ which is defined from

$$X_{T,\gamma} = \{ \mathbf{b} \in L^3((0,T), L^3_{w_{3\gamma/2}}) / \mathbf{b} \text{ is } \lambda - DSS \}$$

to $X_{T,\gamma}$ by $L_{\epsilon}(\mathbf{b}) = \mathbf{v}$.

Lemma 12 For $4/3 < \gamma$, $X_{T,\gamma}$ is a Banach space for the equivalent norms $\|\mathbf{b}\|_{L^3((0,T),L^3_{w_{3\gamma/2}})}$ and $\|\mathbf{b}\|_{L^3((0,T/\lambda^2),\times B(0,\frac{1}{\lambda}))}$.

Proof: We have

$$\int_0^T \int_{B(0,1)} |\mathbf{b}(t,x)|^3 dx dt = \lambda^2 \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{b}(t,x)|^3 dx dt$$

and, for $k \in \mathbb{N}$,

$$\int_0^T \int_{\lambda^{k-1} < |x| < \lambda^k} |\mathbf{b}(t, x)|^3 \, dx \, dt = \lambda^{2k} \int_0^{\frac{1}{\lambda^{2k}}} \int_{\frac{1}{\lambda} < |x| < 1} |\mathbf{b}(t, x)|^3 \, dx \, dt.$$

We may conclude, since for $\gamma > 4/3$ we have $\sum_{k \in \mathbb{N}} \lambda^{k(2-\frac{3\gamma}{2})} < +\infty$.

Lemma 13 For $4/3 < \gamma \le 2$, the mapping L_{ϵ} is continuous and compact on $X_{T,\gamma}$.

Proof: Let \mathbf{b}_n be a bounded sequence in $X_{T,\gamma}$ and let $\mathbf{v}_n = L_{\epsilon}(\mathbf{b}_n)$. We remark that the sequence $\mathbf{b}_n(t,.) * \theta_{\epsilon,t}$ is bounded in $X_{T,\gamma}$. Thus, by Theorem 2 and Corollary 4, the sequence \mathbf{v}_n is bounded in $L^{\infty}((0,T), L^2_{w_{\gamma}})$ and $\nabla \mathbf{v}_n$ is bounded in $L^2((0,T), L^2_{w_{\gamma}})$.

We now use Theorem 3 and get that then there exists q_{∞} , \mathbf{v}_{∞} , \mathbf{B}_{∞} and an increasing sequence $(n_k)_{k\in\mathbb{N}}$ with values in \mathbb{N} such that

- \mathbf{v}_{n_k} converges *-weakly to \mathbf{v}_{∞} in $L^{\infty}((0,T), L^2_{w_{\gamma}})$, $\nabla \mathbf{v}_{n_k}$ converges weakly to $\nabla \mathbf{v}_{\infty}$ in $L^2((0,T), L^2_{w_{\gamma}})$
- $\mathbf{b}_{n_k} * \theta_{\epsilon,t}$ converges weakly to \mathbf{B}_{∞} in $L^3((0,T), L^3_{w_{3\gamma/2}})$,
- the associated pressures q_{n_k} converge weakly to q_{∞} in $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5}) + L^2((0,T), L_{w_{\alpha}}^2)$
- \mathbf{v}_{n_k} converges strongly to \mathbf{v}_{∞} in $L^2_{loc}([0,T)\times\mathbb{R}^3)$: for every $T_0\in(0,T)$ and every R>0, we have

$$\lim_{k \to +\infty} \int_0^{T_0} \int_{|y| < R} |\mathbf{v}_{n_k}(s, y) - \mathbf{v}_{\infty}(s, y)|^2 \, ds \, dy = 0.$$

As $\sqrt{w_{\gamma}}\mathbf{v}_n$ is bounded in $L^{\infty}((0,T),L^2)$ and in $L^2((0,T),L^6)$, it is bounded in $L^{10/3}((0,T)\times\mathbb{R}^3)$. The strong convergence of \mathbf{v}_{n_k} in $L^2_{\mathrm{loc}}([0,T)\times\mathbb{R}^3)$ then implies the strong convergence of \mathbf{v}_{n_k} in $L^3_{\mathrm{loc}}((0,T)\times\mathbb{R}^3)$.

Moreover, \mathbf{v}_{∞} is still λ -DSS (a property that is stable under weak limits). We find that $\mathbf{v}_{\infty} \in X_{T,\gamma}$ and that

$$\lim_{n_k \to +\infty} \int_0^{\frac{T}{\lambda^2}} \int_{B(0,\frac{1}{\lambda})} |\mathbf{v}_{n_k}(s,y) - \mathbf{v}_{\infty}(s,y)|^3 ds dy = 0.$$

This proves that L_{ϵ} is compact.

If we assume moreover that \mathbf{b}_n is convergent to \mathbf{b}_{∞} in $X_{T,\gamma}$, then necessarily we have $\mathbf{B}_{\infty} = \mathbf{b}_{\infty} * \theta_{\epsilon,t}$, and $\mathbf{v}_{\infty} = L_{\epsilon}(\mathbf{b}_{\infty})$. Thus, the relatively compact sequence \mathbf{v}_n can have only one limit point; thus it must be convergent. This proves that L_{ϵ} is continuous.

Lemma 14 Let $4/3 < \gamma \le 2$. If, for some $\mu \in [0,1]$, \mathbf{v} is a solution of $\mathbf{v} = \mu L_{\epsilon}(\mathbf{v})$ then

$$\|\mathbf{v}\|_{X_{T,\gamma}} \le C_{\mathbf{u}_0, \mathbb{F}, \gamma, T}$$

where the constant $C_{\mathbf{u}_0,\mathbb{F},\gamma,T}$ depends only on \mathbf{u}_0 , \mathbb{F} , γ and T (but not on μ nor on ϵ).

Proof: We have $\mathbf{v} = \mu \mathbf{w}$; with

$$\begin{cases} \partial_t \mathbf{w} = \Delta \mathbf{w} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla) \mathbf{w} - \nabla q + \nabla \cdot \mathbb{F} \\ \nabla \cdot \mathbf{w} = 0, & \mathbf{w}(0,.) = \mathbf{u}_0 \end{cases}$$

Multiplying by μ , we find that

$$\begin{cases} \partial_t \mathbf{v} = \Delta \mathbf{v} - ((\mathbf{v} * \theta_{\epsilon,t}) \cdot \nabla) \mathbf{v} - \nabla(\mu q) + \nabla \cdot \mu \mathbb{F} \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{v}(0,.) = \mu \mathbf{u}_0 \end{cases}$$

We then use Corollary 6. We choose $T_0 \in (0,T)$ such that

$$C_{\gamma} \left(1 + \|\mathbf{u}_0\|_{L^2_{w_{\gamma}}}^2 + \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_{\gamma}}}^2 ds \right)^2 T_0 \le 1.$$

Then, as

$$C_{\gamma} \left(1 + \|\mu \mathbf{u}_0\|_{L^2_{w_{\gamma}}}^2 + \int_0^{T_0} \|\mu \mathbb{F}\|_{L^2_{w_{\gamma}}}^2 ds \right)^2 T_0 \le 1$$

we know that

$$\sup_{0 \le t \le T_0} \| \mathbf{v}(t,.) \|_{L^2_{w_\gamma}}^2 \le C_\gamma (1 + \mu^2 \| \mathbf{u}_0 \|_{L^2_{w_\gamma}}^2 + \mu^2 \int_0^{T_0} \| \mathbb{F} \|_{L^2_{w_\gamma}}^2 \, ds)$$

and

$$\int_0^{T_0} \|\nabla \mathbf{v}\|_{L^2_{w_\gamma}}^2 ds \le C_\gamma (1 + \mu^2 \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + \mu^2 \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 ds).$$

In particular, we have

$$\int_0^{T_0} \|\mathbf{v}\|_{L^3_{w_3\gamma/2}}^3 ds \le C_\gamma T_0^{1/4} (1 + \|\mathbf{u}_0\|_{L^2_{w_\gamma}}^2 + \int_0^{T_0} \|\mathbb{F}\|_{L^2_{w_\gamma}}^2 ds)^{\frac{3}{2}}.$$

 \Diamond

As **v** is λ -DSS, we can go back from T_0 to T.

Lemma 15 Let $4/3 < \gamma \leq 2$. There is at least one solution \mathbf{u}_{ϵ} of the equation $\mathbf{u}_{\epsilon} = L_{\epsilon}(\mathbf{u}_{\epsilon})$.

Proof: Obvious due to the Leray–Schauder principle (and the Schaefer theorem), since L_{ϵ} is continuous and compact and since we have uniform a priori estimates for the fixed points of μL_{ϵ} for $0 \le \mu \le 1$.

7.3 Proof of Theorem 5.

We may now finish the proof of Theorem 5. We consider the solutions \mathbf{u}_{ϵ} of $\mathbf{u}_{\epsilon} = L_{\epsilon}(\mathbf{u}_{\epsilon})$.

By Lemma 14, \mathbf{u}_{ϵ} is bounded in $L^3((0,T),L^3_{w_{3\gamma/2}})$, and so is $\mathbf{u}_{\epsilon}*\theta_{\epsilon,t}$. We then know, by Theorem 2 and Corollary 4, that the family \mathbf{u}_{ϵ} is bounded in $L^{\infty}((0,T),L^2_{w_{\gamma}})$ and $\nabla \mathbf{u}_{\epsilon}$ is bounded in $L^2((0,T),L^2_{w_{\gamma}})$.

We now use Theorem 3 and get that then there exists p, \mathbf{u} , \mathbf{B} and a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ (converging to 0) with values in $(0,+\infty)$ such that

- \mathbf{u}_{ϵ_k} converges *-weakly to \mathbf{u} in $L^{\infty}((0,T), L^2_{w_{\gamma}})$, $\nabla \mathbf{u}_{\epsilon_k}$ converges weakly to $\nabla \mathbf{u}$ in $L^2((0,T), L^2_{w_{\gamma}})$
- $\mathbf{u}_{\epsilon_k} * \theta_{\epsilon_k,t}$ converges weakly to **B** in $L^3((0,T),L^3_{w_{3\gamma/2}})$
- the associated pressures p_{ϵ_k} converge weakly to p in $L^3((0,T), L_{w_{\frac{6\gamma}{5}}}^{6/5}) + L^2((0,T), L_{w_{\gamma}}^2)$
- \mathbf{u}_{ϵ_k} converges strongly to \mathbf{u} in $L^2_{loc}([0,T)\times\mathbb{R}^3)$.

Moreover we easily see that $\mathbf{B} = \mathbf{u}$. Indeed, we have that $\mathbf{u} * \theta_{\epsilon,t}$ converges strongly in $L^2_{loc}((0,T) \times \mathbb{R}^3)$ as ϵ goes to 0 (since it is bounded by $\mathcal{M}_{\mathbf{u}}$ and converges, for each fixed t, strongly in $L^2_{loc}(\mathbb{R}^3)$); moreover, we have $|(\mathbf{u} - \mathbf{u}_{\epsilon}) * \theta_{\epsilon,t}| \leq \mathcal{M}_{\mathbf{u} - \mathbf{u}_{\epsilon}}$, so that the strong convergence of \mathbf{u}_{ϵ_k} to \mathbf{u} is kept by convolution with $\theta_{\epsilon,t}$ as far as we work on compact subsets of $(0,T) \times \mathbb{R}^3$ (and thus don't allow t to go to 0).

Thus, Theorem 5 is proven.

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