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ON THE CONVERGENCE RATE OF SOME NONLOCAL ENERGIES

ANTONIN CHAMBOLLE, MATTEO NOVAGA, AND VALERIO PAGLIARI

ABSTRACT. We study the rate of convergence of some nonlocal functionals recently considered by Bourgain, Brezis, and Mironescu, and, after a suitable rescaling, we establish the Γ -convergence of the corresponding rate functionals to a limit functional of second order.

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1. INTRODUCTION

We are interested in the rate of converge, as $h \searrow 0$, of the nonlocal functionals

$$\mathcal{F}_h(u) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) dy dx,$$

to the limit functional

$$\mathcal{F}_0(u) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f(|\nabla u(x) \cdot \hat{z}|) dz dx.$$

Here and in the sequel, we set $K_h(z) := h^{-d}K(z/h)$ with $K: \mathbb{R}^d \rightarrow [0, +\infty)$ an even function in $L^1(\mathbb{R}^d)$ that has finite second moment, see (27) below. We let $f: [0, +\infty) \rightarrow [0, +\infty)$ be a convex function of class C^2 satisfying $f(0) = f'(0) = 0$, and, for $z \in \mathbb{R}^d \setminus \{0\}$, we put $\hat{z} = z/|z|$.

Functionals similar to \mathcal{F}_h and \mathcal{F}_0 were considered by Bourgain, Brezis, and Mironescu in [5]. For K radial and $f(t) = |t|^p$ with $p \geq 1$, they established convergence as $h \searrow 0$ to a multiple of $\|u\|_{W^{1,p}(\Omega)}^p$ whenever $u \in W^{1,p}(\Omega)$ with Ω a smooth, bounded domain in \mathbb{R}^d . Their result has been extended in several directions ([3, 9, 12, 13], see also [2, 4, 11]), and, among others, we would like to spend some words on the contributions by Ponce [16]. The author studied the case in which $\{K_h\}$ is a suitable family of functions in $L^1(\mathbb{R}^d)$ that approaches the Dirac delta in 0 and $f: [0, +\infty) \rightarrow [0, +\infty)$ is a generic convex function. When $u \in L^p(\Omega)$

for some $p \geq 1$ and the boundary of Ω is compact and Lipschitz, he showed pointwise convergence of some functionals that generalize the ones in [5]. The limit is a first order functional, which is given by a variant of \mathcal{F}_0 if $K_h(z) = h^{-d}K(z/h)$ for some $K \in L^1(\mathbb{R}^d)$ and $u \in W^{1,p}(\Omega)$. Further, when Ω is also bounded, in [16] Γ -convergence to the pointwise limit with respect to the L^1 -topology is proved too. For the definition and the properties of Γ -convergence, we refer to the monographs [6, 8].

Let now

(1)

$$\begin{aligned} \mathcal{E}_h(u) &:= \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2} \\ &= \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[K(z) f(|\nabla u(x) \cdot \hat{z}|) - K_h(z) f\left(\frac{|u(x+z) - u(x)|}{|z|}\right) \right] dz dx \end{aligned}$$

be the functional which measures the rate of convergence of \mathcal{F}_h to \mathcal{F}_0 . In this paper, under the assumptions that the function f is strongly convex (see condition (3) below) and that we restrict to functions that vanish outside a bounded, Lipschitz set Ω , we prove that the family $\{\mathcal{E}_h\}$ Γ -converges, with respect to the $H^1(\mathbb{R}^d)$ -topology, to the *second order* limit functional

$$\mathcal{E}_0(u) := \begin{cases} \frac{1}{24} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) |z|^2 f''(|\nabla u(x) \cdot \hat{z}|) |\nabla^2 u(x) \hat{z} \cdot \hat{z}|^2 dz dx & \text{if } u \in H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

The uniform convexity assumption on f , which is needed for the Γ -inferior limit inequality, excludes some interesting cases such as $f(x) = |x|^p$ with $p \geq 1$, $p \neq 2$. In particular, when $f(x) = |x|$ and K is radially symmetric, the analysis is related to a geometric problem considered in [14] in the context of a physical model for liquid drops with dipolar repulsion. We also observe that our study differs from a higher order Γ -limit of \mathcal{F}_h (see [7]), which would rather correspond to deal with the Γ -limit of the functionals

$$\frac{\mathcal{F}_h - \min \mathcal{F}_0}{h^\alpha} \quad \text{for some } \alpha > 0.$$

As a consequence of our result (see Remark 5) we also get that, if the rate of convergence of $\mathcal{F}_h(u)$ to $\mathcal{F}_0(u)$ is fast enough, more precisely if $|\mathcal{E}_h(u)| \leq M$ for all h 's sufficiently small, then $u \in H^2(\mathbb{R}^d)$.

We notice that our result is reminiscent to the one obtained by Peletier, Planqué, and Röger in [15]. There, motivated by a model for bilayer membranes, the authors considered the convolution functionals

$$\mathcal{G}_h(u) := \int_{\mathbb{R}^d} f(K_h * u) dx,$$

which converge to the functional $\mathcal{G}_0(u) = c \int_{\mathbb{R}^d} f(u) dx$ as $h \searrow 0$, with $c = c(K, d)$ a positive constant, and they showed that the corresponding rate functionals

$$(2) \quad \frac{\mathcal{G}_0(u) - \mathcal{G}_h(u)}{h^2} = \frac{1}{h^2} \int_{\mathbb{R}^d} (c f(u) - f(K_h * u)) dx$$

converge pointwise to the limit functional

$$\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) |z|^2 f''(u(x)) |\nabla u(x) \cdot \hat{z}|^2 dz dx \quad \text{for } u \in H^1(\mathbb{R}^d).$$

In particular, the rate functionals are uniformly bounded if and only if $u \in H^1(\mathbb{R}^d)$.

In the proof of our convergence result, we follow a strategy similar to the one in [10, 11]: we first consider a related 1-dimensional problem, and then reduce the general case to it by a *slicing* procedure. More precisely, in Section 2 we study the functionals

$$E_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \left[f(u(x)) - f \left(\int_x^{x+h} u(y) dy \right) \right] dx,$$

which are a particular case of (2), and we show their convergence (see Theorem 1) to the limit energy

$$E_0(u) := \frac{1}{24} \int_{\mathbb{R}} f''(u(x)) |u'(x)|^2 dx \quad \text{for } u \in H^1(\mathbb{R}).$$

Then, in Section 3 we consider the general functionals in (1) and we establish the Γ -convergence to \mathcal{E}_0 (see Theorem 2), which is the main result of the present paper. We first show the convergence for $d = 1$, using the result of Section 2, and then we reduce to the 1-dimensional case by means of a delicate slicing technique.

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2. FINITE DIFFERENCE FUNCTIONALS IN THE 1-DIMENSIONAL CASE

Let $f: \mathbb{R} \rightarrow [0, +\infty)$ be a strongly convex function of class C^2 such that $f(0) = f'(0) = 0$. By saying that f is strongly convex, we mean that

$$(3) \quad \text{there exists } \gamma > 0 \text{ such that } f''(t) \geq \gamma \text{ for all } t \in \mathbb{R}.$$

Let us fix an open interval $I := (a, b) \subset \mathbb{R}$. We introduce the closed subspace $Y \subset L^2(\mathbb{R})$ defined as

$$(4) \quad Y := \{ u \in L^2(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus I \},$$

and, for $u \in Y$ and $h > 0$, we define the energy

$$(5) \quad E_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} [f(u(x)) - f(D_h U(x))] dx,$$

where

$$(6) \quad U(x) := \int_0^x u(y) dy \quad \text{and} \quad D_h U(x) := \frac{U(x+h) - U(x)}{h} = \int_x^{x+h} u(y) dy.$$

By using some simple changes of variable and the positivity of f , one can find

$$\begin{aligned} \int_{\mathbb{R}} f(u(x)) dx &= \int_a^b f(u(x)) dx = \int_{a-h}^b \int_x^{x+h} f(u(y)) dy dx \\ &= \int_{\mathbb{R}} \int_x^{x+h} f(u(y)) dy dx, \end{aligned}$$

the integrals possibly diverging to $+\infty$. Then, by combining the previous identity with Jensen's inequality

$$\int_x^{x+h} f(u(y))dy \geq f(D_h U(x)),$$

we see that $E_h(u)$ ranges in $[0, +\infty]$ when $u \in Y$.

In the present section we compute the Γ -limit of $\{E_h\}$ regarded as a family of functionals on Y endowed with the L^2 -topology. Let us set

$$(7) \quad E_0(u) := \begin{cases} \frac{1}{24} \int_{\mathbb{R}} f''(u(x)) |u'(x)|^2 dx & \text{if } u \in Y \cap H^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

We prove the following:

Theorem 1. *Let us assume that $f: \mathbb{R} \rightarrow [0, +\infty)$ is a function of class C^2 such that $f(0) = f'(0) = 0$ and (3) holds. Then, the restriction to Y of the family $\{E_h\}$ Γ -converges, as $h \searrow 0$, to E_0 w.r.t. the $L^2(\mathbb{R})$ -topology, that is, for every $u \in Y$ the following properties hold:*

(1) *For any family $\{u_h\} \subset Y$ that converges to u in $L^2(\mathbb{R})$ we have*

$$E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h).$$

(2) *There exists a sequence $\{u_h\} \subset Y$ converging to u in $L^2(\mathbb{R})$ such that*

$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

The Γ -upper limit is established in Proposition 1, while Proposition 2 takes care of the lower limit. In turn, the latter is achieved by exploiting a suitable lower bound on the energy (see Lemma 1) and a compactness result (see Lemma 2), which are a consequence of the strong convexity of f .

2.1. Pointwise limit and upper bound. We now compute the limit of $E_h(u)$, as $h \searrow 0$, for a function $u \in Y \cap C^2(\mathbb{R})$. We observe that strong convexity of f is not needed for the next proposition to hold.

Proposition 1. *Let $f: \mathbb{R} \rightarrow [0, +\infty)$ be a C^2 function such that $f(0) = f'(0) = 0$, and let $u \in Y \cap C^2(\mathbb{R})$. Then, there exists a continuous, bounded, and increasing function $m: [0, +\infty) \rightarrow [0, +\infty)$ such that $m(0) = 0$ and*

$$(8) \quad |E_h(u) - E_0(u)| \leq c m(h),$$

where $c := c(b-a, \|u\|_{C^2(\mathbb{R})}, \|f\|_{C^2([- \|u\|_{C^2(\mathbb{R})}, \|u\|_{C^2(\mathbb{R})])}) > 0$ is a constant. In particular, $\lim_{h \searrow 0} E_h(u) = E_0(u)$ and for every $u \in Y$ there exists a sequence $\{u_h\} \subset Y$ that converges to u in $L^2(\mathbb{R})$ and satisfies

$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

Proof. Since $u \in Y \cap C^2(\mathbb{R})$ and $f \in C^2(\mathbb{R})$, it is easy to see that $h^2 E_h(u)$ and $E_0(u)$ are uniformly bounded in h . Thus, there exists a constant $c_\infty > 0$ such that

$$(9) \quad |E_h(u) - E_0(u)| \leq c_\infty \quad \text{for } h > 1.$$

Next, we focus on the case $h \in (0, 1]$. If $x \notin (a - h, b)$, then $D_h U(x) = 0$, and hence

$$h^2 E_h(u) = \int_a^b [f(u(x)) - f(D_h U(x))] dx - \int_{a-h}^a f(D_h U(x)) dx.$$

Being u regular, for any $x \in (a - h, b)$ we have the Taylor's expansion

$$D_h U(x) = u(x) + \frac{h}{2} u'(x) + \frac{h^2}{6} u''(x_h), \quad \text{with } x_h \in (x, x + h),$$

which we rewrite as

$$(10) \quad D_h U(x) = u(x) + h v_h(x), \quad \text{with } v_h(x) := \frac{u'(x)}{2} + \frac{h}{6} u''(x_h);$$

note that v_h converges uniformly to $u'/2$ as $h \searrow 0$.

Plugging (10) into the definition of E_h , we get

$$\begin{aligned} h^2 E_h(u) &= - \int_a^b [f(u(x) + h v_h(x)) - f(u(x))] dx - \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \\ &= - h \int_a^b f'(u(x)) v_h(x) dx - \frac{h^2}{2} \int_a^b f''(w_h(x)) v_h(x)^2 dx \\ &\quad - \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx, \end{aligned}$$

where w_h fulfils $w_h(x) \in (u(x), u(x) + h v_h(x_h))$ for all $x \in (a, b)$.

In view of the regularity of f and u , we can utilize the Mean Value Theorem to obtain

$$\left| \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \right| \leq c_1 h^5$$

for a constant $c_1 > 0$ that depends only on $N := \|u\|_{C^2(\mathbb{R})}$ and on $\|f''\|_{L^\infty([-N, N])}$. Moreover, recalling the definition of v_h , we have

$$\int_a^b f'(u(x)) v_h(x) dx = \frac{h}{6} \int_a^b f'(u(x)) u''(x_h) dx,$$

and therefore

$$(11) \quad \begin{aligned} |E_h(u) - E_0(u)| &\leq \frac{1}{6} \left| - \int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| \\ &\quad + \frac{1}{2} \left| \frac{1}{4} \int_a^b f''(u(x)) u'(x)^2 dx - \int_a^b f''(w_h(x)) v_h(x)^2 dx \right| + c_1 h^5. \end{aligned}$$

Since $u \in Y \cap C^2(\mathbb{R})$, u'' admits a uniform modulus of continuity $m_{u''} : [0, +\infty) \rightarrow [0, \infty)$. An integration by parts gives that

$$\begin{aligned} \left| - \int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| &\leq \int_a^b |f'(u(x))| |u''(x) - u''(x_h)| dx \\ &\leq c_2 m_{u''}(h), \end{aligned}$$

where $c_2 := (b - a) \|f'\|_{L^\infty([-N, N])}$.

In a similar manner, denoting by $m_{f''}$ the modulus of continuity of the restriction of f'' to the interval $[-N, N]$, we also find

$$\begin{aligned} & \left| \frac{1}{4} \int_a^b f''(u(x)) u'(x)^2 dx - \int_a^b f''(w_h(x)) v_h(x)^2 dx \right| \\ & \leq \int_a^b |f''(u(x))| \left| \frac{1}{4} u'(x)^2 - v_h(x)^2 \right| dx + \int_a^b |f''(u(x)) - f''(w_h(x))| v_h(x)^2 dx \\ & \leq c_3(h + m_{f''}(h)), \end{aligned}$$

with c_3 depending on $b - a$, N , and $\|f''\|_{L^\infty([-N, N])}$.

By combining (11) with the inequalities above, we obtain

$$(12) \quad |E_h(u) - E_0(u)| \leq c_0(m_{u''}(h) + m_{f''}(h) + h + h^5) \quad \text{for } h \in (0, 1],$$

for a suitable constant $c_0 > 0$. At this stage, (8) follows by combining (9) and (12).

As for the existence of a family that fulfils the upper limit inequality, we apply a standard density argument that we sketch in the following lines. Let $u \in Y$. If $u \notin H^1(\mathbb{R})$, the inequality holds trivially; otherwise, by rescaling the domain and mollifying, we construct a sequence of smooth functions $\{u_\ell\} \subset Y$ that converges to u both uniformly and in $H^1(\mathbb{R})$. Then, since f'' is a continuous function, we get $\lim_{\ell \nearrow +\infty} E_0(u_\ell) = E_0(u)$. Besides, we know that $\lim_{h \searrow 0} E_h(u_\ell) = E_0(u_\ell)$ for any $\ell \in \mathbb{N}$, because u_ℓ is smooth. We conclude that there exists a subsequence $\{h_\ell\}$ such that

$$\lim_{\ell \nearrow +\infty} E_{h_\ell}(u_\ell) = E_0(u).$$

□

Remark 1. Notice that, as a consequence of Proposition 1, the Γ -limit of the rate functionals

$$hE_h(u) = \frac{1}{h} \int_{\mathbb{R}} [f(u(x)) - f(D_h U(x))] dx$$

is equal to zero.

2.2. Lower bound in the strongly convex case. In view of Proposition 1, to accomplish the proof of Theorem 1, it only remains to establish statement (1), that is, for any $u \in Y$ and for any family $\{u_h\} \subset Y$ converging to u in $L^2(\mathbb{R})$ it holds

$$E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h).$$

In the current subsection we utilize the strong convexity of the function f . We exploit this hypothesis to provide a lower bound on the energy E_h , by means of which we prove that sequences with equibounded energy are relatively compact w.r.t. the L^2 -topology.

Lemma 1 (Lower bound on the energy). *Let us assume that $f: \mathbb{R} \rightarrow [0, +\infty)$ is a function of class C^2 such that $f(0) = f'(0) = 0$ and (3) is fulfilled. Then, for any $u \in Y$, it holds*

$$(13) \quad E_h(u) \geq \sup_{\varphi \in C_c^\infty(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}} \int_x^{x+h} \left(\frac{u(y) - D_h U(x)}{h} \varphi(x, y) - \frac{\varphi(x, y)^2}{4\lambda_h(x, y)} \right) dy dx \right\},$$

with

$$(14) \quad \lambda_h(x, y) := \int_0^1 (1 - \vartheta) f''((1 - \vartheta)D_h U(x) + \vartheta u(y)) d\vartheta.$$

Moreover,

$$(15) \quad E_h(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}} \int_{-h}^h J_h(r) \left(\frac{u(y+r) - u(y)}{h} \right)^2 dr dy,$$

where

$$J(r) := (1 - |r|)^+ \quad \text{and} \quad J_h(r) := \frac{1}{h} J\left(\frac{r}{h}\right).$$

Proof. For a given $h > 0$, let us consider $u \in Y$ such that $E_h(u)$ is finite. We write

$$E_h(u) = \frac{1}{h^2} \int_{\mathbb{R}} e_h(x) dx, \quad \text{where} \quad e_h(x) := \int_x^{x+h} [f(u(y)) - f(D_h U(x))] dy.$$

Thanks to the identity

$$f(s) - f(t) = f'(t)(s - t) + (s - t)^2 \int_0^1 (1 - \vartheta) f''((1 - \vartheta)t + \vartheta s) d\vartheta,$$

we find

$$(16) \quad e_h(x) = \int_x^{x+h} \lambda_h(x, y) (u(y) - D_h U(x))^2 dy,$$

where $\lambda_h(x, y)$ is as in (14). Observe that for any $\varphi \in C_c^\infty(\mathbb{R}^2)$ we have the pointwise inequality

$$\lambda_h(x, y) (u(y) - D_h U(x))^2 \geq \frac{u(y) - D_h U(x)}{h} \varphi(x, y) - \frac{\varphi(x, y)^2}{4\lambda_h(x, y)},$$

from which we infer (13).

The strong convexity of f grants that $\lambda_h(x, y) \geq \gamma/2$ for all $(x, y) \in \mathbb{R}^2$ and $h > 0$, thus we also deduce that

$$e_h(x) \geq \frac{\gamma}{2} \int_x^{x+h} (u(y) - D_h U(x))^2 dy.$$

Hence, we get

$$\begin{aligned} E_h(u) &\geq \frac{\gamma}{2} \int_{\mathbb{R}} \int_x^{x+h} \left(\frac{u(y) - D_h U(x)}{h} \right)^2 dy dx \\ &\geq \frac{\gamma}{4} \int_{\mathbb{R}} \int_x^{x+h} \int_x^{x+h} \left(\frac{u(z) - u(y)}{h} \right)^2 dz dy dx, \end{aligned}$$

where the last inequality follows from the identity

$$\int |\varphi(y)|^2 d\mu(y) = \left| \int \varphi(y) d\mu(y) \right|^2 + \frac{1}{2} \int \int |\varphi(z) - \varphi(y)|^2 d\mu(z) d\mu(y),$$

which holds whenever μ is a probability measure and $\varphi \in L^2(\mu)$. By Fubini's Theorem and neglecting contributions near the boundary, we find the lower bound

on the energy:

$$\begin{aligned} E_h(u) &\geq \frac{\gamma}{4} \int_{\mathbb{R}} \int_{y-h}^y \int_x^{x+h} \left(\frac{u(z) - u(y)}{h} \right)^2 dz dx dy \\ &= \frac{\gamma}{4h} \int_{\mathbb{R}} \int_{y-h}^{y+h} \left(1 - \frac{|z-y|}{h} \right) \left(\frac{u(z) - u(y)}{h} \right)^2 dz dy. \end{aligned}$$

The conclusion (15) is now achieved by the change of variables $r = z - y$. \square

Remark 2. Let $u \in Y \cap H^1(\mathbb{R})$. If along with the previous assumptions we also require that f'' is bounded above by a constant $c > 0$, then the family $\{E_h(u)\}$ is uniformly bounded. Indeed, it follows from (16) and the definition of λ_h that

$$E_h(u) \leq \frac{c}{2h^2} \int_{\mathbb{R}} \int_x^{x+h} (u(y) - D_h U(x))^2 dy dx.$$

Then, since $u \in H^1(\mathbb{R})$, when $y \in (x, x+h)$ it holds

$$|u(y) - D_h U(x)|^2 \leq h \int_x^{x+h} |u'(z)|^2 dz = h^2 \int_0^1 |u'(x+hz)|^2 dz,$$

and we derive the estimate

$$(17) \quad E_h(u) \leq \frac{c}{2} \int_{\mathbb{R}} |u'(x)|^2 dx.$$

Lemma 2 (Compactness). Let the function f be as in Lemma 1 and let $\{u_h\} \subset Y$ be a sequence of functions such that $E_h(u_h) \leq M$ for some $M \geq 0$. Then, there exist a subsequence $\{u_{h_\ell}\}$ and a function $u \in Y \cap H^1(\mathbb{R})$ such that $u_{h_\ell} \rightarrow u$ in $L^2(\mathbb{R})$.

Proof. We adapt the strategy of [1, Theorem 3.1].

By Lemma 1, we infer that

$$(18) \quad \frac{\gamma}{4} \int_{\mathbb{R}} \int_{-h}^h J_h(r) \left(\frac{u_h(y+r) - u_h(y)}{h} \right)^2 dr dy \leq M.$$

Observe that $J_h(r)dr$ is a probability measure on $[-h, h]$.

We now introduce the mollified functions $v_h := \rho_h * u_h$, where $\{\rho_h\}$ is the family

$$\rho_h(r) := \frac{1}{ch} \rho\left(\frac{r}{h}\right), \quad \text{with } c := \int_{\mathbb{R}} \rho(r) dr.$$

Here, $\rho \in C_c^\infty(\mathbb{R})$ is an even kernel, and it is chosen in such a way that its support is contained in $[-1, 1]$,

$$0 \leq \rho \leq J, \quad \text{and} \quad |\rho'| \leq J.$$

Note that, for all $h > 0$, $v_h: \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function whose support is a subset of $(a-h, b+h)$. Moreover, the family of derivatives $\{v_h'\}_{h \in (0,1)}$ is uniformly

bounded in $L^2(\mathbb{R})$; indeed, since $\int_{\mathbb{R}} \rho'(r) dr = 0$, it holds

$$\begin{aligned} \int_{\mathbb{R}} |v'_h(y)|^2 dy &= \int_{\mathbb{R}} \left| \int_{-h}^h \rho'_h(r) [u_h(y+r) - u_h(y)] dr \right|^2 dy \\ &\leq \int_{\mathbb{R}} \left(\int_{-h}^h |\rho'_h(r)| |u_h(y+r) - u_h(y)| dr \right)^2 dy \\ &\leq \frac{1}{c^2} \int_{\mathbb{R}} \left(\int_{-h}^h J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right| dr \right)^2 dy \\ &\leq \frac{1}{c^2} \int_{\mathbb{R}} \int_{-h}^h J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right|^2 dr dy, \end{aligned}$$

and thus

$$(19) \quad \int_{\mathbb{R}} |v'_h(y)|^2 dy \leq \frac{4M}{c^2 \gamma}.$$

For all $h \in (0, 1)$, let \tilde{v}_h be the restriction of v_h to the interval $(a-1, b+1)$. By Poincaré inequality, (19) entails boundedness in $H_0^1((a-1, b+1))$ of the family $\{\tilde{v}_h\}_{h \in (0, 1)}$, and, in view of Sobolev's Embedding Theorem, this grants in turn that there exists a subsequence $\{\tilde{v}_{h_\ell}\}$ uniformly converging to some $\tilde{u} \in H_0^1([a-1, b+1])$. Since each \tilde{v}_{h_ℓ} is supported in $(a-h_\ell, b+h_\ell)$, we see that $\tilde{u} \in H_0^1(\bar{I})$; therefore, if we set

$$u(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{I}, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that $\{v_{h_\ell}\}$ converges uniformly to $u \in Y \cap H^1(\mathbb{R})$.

Lastly, to achieve the conclusion, we provide a bound on the L^2 -distance between u_h and v_h . Similarly to the previous computations, we have

$$\begin{aligned} \int_{\mathbb{R}} |v_h(y) - u_h(y)|^2 dy &= \int_{\mathbb{R}} \left| \int_{-h}^h \rho_h(r) [u_h(y+r) - u_h(y)] dr \right|^2 dy \\ &\leq \int_{\mathbb{R}} \int_{-h}^h \rho_h(r) |u_h(y+r) - u_h(y)|^2 dr dy \\ &\leq \frac{1}{c} \int_{\mathbb{R}} \int_{-h}^h J_h(r) |u_h(y+r) - u_h(y)|^2 dr dy, \end{aligned}$$

and, by (18), we get

$$(20) \quad \int_{\mathbb{R}} |v_h(y) - u_h(y)|^2 dy \leq \frac{4M}{c\gamma} h^2.$$

Since there exists a subsequence $\{v_{h_\ell}\}$ uniformly converging to a function $u \in Y \cap H^1(\mathbb{R})$, (20) gives the conclusion. \square

Now we can prove statement (1) of Theorem 1.

Proposition 2. *Let the function f be as in Lemma 1. Then, for any $u \in Y$ and for any family $\{u_h\} \subset Y$ that converges to u in $L^2(\mathbb{R})$, it holds*

$$(21) \quad E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h).$$

Proof. Fix $u, u_h \in Y$ in such a way that $u_h \rightarrow u$ in $L^2(\mathbb{R})$. We can suppose that the inferior limit in (21) is finite, otherwise the conclusion holds trivially. Consequently, up to extracting a subsequence, which we do not relabel, there exists $\lim_{h \searrow 0} E_h(u_h)$ and it is finite. In particular, there exists $M \geq 0$ such that $E_h(u_h) \leq M$ for all $h > 0$, and, by Lemma 2, this yields that $u \in Y \cap H^1(\mathbb{R})$.

We use formula (13) for each u_h , choosing, for $(x, y) \in \mathbb{R}^2$,

$$\varphi(x, y) = \psi\left(x, \frac{y-x}{h}\right), \quad \text{with } \psi \in C_c^\infty(\mathbb{R}^2).$$

We get

$$(22) \quad \begin{aligned} E_h(u_h) &\geq \int_{\mathbb{R}} \int_x^{x+h} \frac{u_h(y) - \int_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dy dx \\ &\quad - \frac{1}{4} \int_{\mathbb{R}} \int_x^{x+h} \frac{\psi\left(x, \frac{y-x}{h}\right)^2}{\lambda_h(x, y)} dy dx, \end{aligned}$$

where, coherently with (14),

$$\lambda_h(x, y) := \int_0^1 (1-\vartheta) f'' \left((1-\vartheta) \int_x^{x+h} u_h(z) dz + \vartheta u_h(y) \right) d\vartheta \geq \frac{\gamma}{2}.$$

Let us focus on the first quantity on the right-hand side of (22). We have

$$\begin{aligned} &\frac{1}{h} \int_{\mathbb{R}} \int_x^{x+h} \left(\int_x^{x+h} u_h(z) dz \right) \psi\left(x, \frac{y-x}{h}\right) dy dx \\ &= \frac{1}{h^3} \int_{\mathbb{R}} \int_x^{x+h} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dy dz dx \\ &= \frac{1}{h^3} \int_{\mathbb{R}} \int_{z-h}^z \int_x^{x+h} u_h(z) \psi\left(x, \frac{y-x}{h}\right) dy dx dz, \end{aligned}$$

and, by similar computations, we obtain

$$(23) \quad \begin{aligned} &\int_{\mathbb{R}} \int_x^{x+h} \frac{u_h(y) - \int_x^{x+h} u_h}{h} \psi\left(x, \frac{y-x}{h}\right) dy dx \\ &= \frac{1}{h} \int_{\mathbb{R}} \int_{y-h}^y \int_x^{x+h} u_h(y) \left[\psi\left(x, \frac{y-x}{h}\right) - \psi\left(x, \frac{z-x}{h}\right) \right] dz dx dy. \end{aligned}$$

By a simple change of variable, we get

$$\begin{aligned} \int_{y-h}^y \int_x^{x+h} \psi\left(x, \frac{y-x}{h}\right) dz dx &= \int_0^1 \int_0^1 \psi(y-hr, r) dq dr, \\ \int_{y-h}^y \int_x^{x+h} \psi\left(x, \frac{z-x}{h}\right) dz dx &= \int_{y-h}^y \int_0^1 \psi(x, r) dr dx \\ &= \int_0^1 \int_0^1 \psi(y-hq, r) dq dr, \end{aligned}$$

hence

$$\begin{aligned}
& \frac{1}{h} \int_{y-h}^y \int_x^{x+h} \left[\psi \left(x, \frac{y-x}{h} \right) - \psi \left(x, \frac{z-x}{h} \right) \right] dz dx \\
&= \int_0^1 \int_0^1 \frac{\psi(y-hr, r) - \psi(y-hq, r)}{h} dq dr \\
&= - \int_0^1 \int_0^1 \int_q^r \partial_1 \psi(y-hs, r) ds dq dr \\
&= - \int_0^1 \int_0^1 (r-q) \int_q^r \partial_1 \psi(y-hs, r) ds dq dr.
\end{aligned}$$

Being ψ smooth, we have that $\partial_1 \psi(y-hs, r) = \partial_1 \psi(y, r) + O(h)$ as $h \searrow 0$, uniformly for $s \in [0, 1]$. Consequently,

$$\frac{1}{h} \int_{y-h}^y \int_x^{x+h} \left[\psi \left(x, \frac{y-x}{h} \right) - \psi \left(x, \frac{z-x}{h} \right) \right] dz dx = - \int_0^1 \left(r - \frac{1}{2} \right) \partial_1 \psi(y, r) dr + O(h).$$

Plugging this equality in (23) yields

$$\int_{\mathbb{R}} \int_x^{x+h} \frac{u_h(y) - \int_x^{x+h} u_h}{h} \psi \left(x, \frac{y-x}{h} \right) dy dx = - \int_{\mathbb{R}} u_h(y) \int_0^1 \left(r - \frac{1}{2} \right) \partial_1 \psi(y, r) dr + O(h).$$

It is possible to take the limit $h \searrow 0$ in the previous formula, since $u_h \rightarrow u$ in $L^2(\mathbb{R})$. We then get

$$(24) \quad \lim_{h \searrow 0} \int_{\mathbb{R}} \int_x^{x+h} \frac{u_h(y) - \int_x^{x+h} u_h}{h} \psi \left(x, \frac{y-x}{h} \right) dy dx = - \int_{\mathbb{R}} \int_0^1 u(y) \left(r - \frac{1}{2} \right) \partial_1 \psi(y, r) dr dy.$$

Now, we turn to the second addendum on the right-hand side of (22). By Fubini's Theorem and a change of variables, we have

$$\int_{\mathbb{R}} \int_x^{x+h} \frac{\psi \left(x, \frac{y-x}{h} \right)^2}{\lambda_h(x, y)} dy dx = \int_{\mathbb{R}} \int_0^1 \frac{\psi(y-hr, r)^2}{\lambda_h(y-hr, y)} dr dy.$$

The function ψ has compact support and $\lambda_h \geq \gamma/2$ for all $h > 0$, therefore we can apply Lebesgue's Convergence Theorem to let $h \searrow 0$ in the previous expression, and we get

$$\begin{aligned}
& \lim_{h \searrow 0} \int_{\mathbb{R}} \int_0^1 \frac{\psi(y-hr, r)^2}{\int_0^1 (1-\vartheta) f'' \left((1-\vartheta) \int_{y-hr}^{y+(1-r)h} u_h(z) dz + \vartheta u_h(y) \right) d\vartheta} dr dy \\
&= \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{\int_0^1 (1-\vartheta) f''(u(y)) d\vartheta} dr dy,
\end{aligned}$$

thus

$$(25) \quad \lim_{h \searrow 0} \int_{\mathbb{R}} \int_x^{x+h} \frac{\psi \left(x, \frac{y-x}{h} \right)^2}{\lambda_h(x, y)} dy dx = 2 \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{f''(u(y))} dr dy.$$

Summing up, by (24) and (25), we deduce

$$(26) \quad \liminf_{h \searrow 0} E_h(u_h) \geq - \int_{\mathbb{R}} \int_0^1 \left[u(y) \left(r - \frac{1}{2} \right) \partial_1 \psi(y, r) dr dy + \frac{1}{2} \int_{\mathbb{R}} \int_0^1 \frac{\psi(y, r)^2}{f''(u(y))} dr dy \right] dr dy,$$

for all $\psi \in C_c^\infty(\mathbb{R}^2)$.

We can reach the conclusion from the last inequality by a suitable choice of the test function ψ . To see this, we let $\eta \in C_c^\infty(\mathbb{R})$ and we choose a standard sequence of mollifiers $\{\rho_k\}$. We then set

$$\psi(x, y) = \psi_k(x, y) := \eta(x) \left(\zeta_k(y) - \frac{1}{2} \right), \quad \text{with } \zeta_k(y) := \int_{\mathbb{R}} \rho_k(z - y) z dz,$$

so that (26) reads

$$\begin{aligned} \liminf_{h \searrow 0} E_h(u_h) &\geq - \int_0^1 \left(r - \frac{1}{2} \right) \left(\zeta_k(r) - \frac{1}{2} \right) dr \int_{\mathbb{R}} u(y) \eta'(y) dy \\ &\quad - \frac{1}{2} \int_0^1 \left(\zeta_k(r) - \frac{1}{2} \right)^2 dr \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy. \end{aligned}$$

Because of the identity $\int_0^1 (r - 1/2)^2 dr = 1/12$, letting $k \rightarrow +\infty$ yields

$$\begin{aligned} \liminf_{h \searrow 0} E_h(u_h) &\geq - \frac{1}{12} \left[\int_{\mathbb{R}} u(y) \eta'(y) dy + \frac{1}{2} \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy \right] \\ &= \frac{1}{12} \left[\int_{\mathbb{R}} u'(y) \eta(y) dy - \frac{1}{2} \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy \right], \end{aligned}$$

where $u' \in L^2(\mathbb{R})$ is the distributional derivative of u , which exists since $u \in H^1(\mathbb{R})$. Recall that, in the previous formula, the test function η is arbitrary, thus, to recover (21), it suffices to take the supremum w.r.t. $\eta \in C_c^\infty(\mathbb{R})$. \square

3. Γ -LIMIT IN ARBITRARY DIMENSION

Let us fix the assumptions and the notation that we use in the current section. We consider an open, bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a measurable function $K: \mathbb{R}^d \rightarrow [0, +\infty)$ such that

$$(27) \quad \int_{\mathbb{R}^d} K(z) (1 + |z|^2) dz < +\infty.$$

We require that $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^d$ and that the support of K contains a sufficiently large annulus centered at the origin. More precisely, let us set

$$(28) \quad \sigma_d := \begin{cases} 1 & \text{when } d = 2, \\ \frac{d-2}{d-1} & \text{when } d > 2; \end{cases}$$

we suppose that there exist $r_0 \geq 0$ and $r_1 > 0$ such that $r_0 < \sigma_d r_1$ and

$$(29) \quad \text{ess inf} \{ K(z) : z \in B(0, r_1) \setminus B(0, r_0) \} > 0.$$

The simplest case for which (29) holds is when there exists $k > 0$ such that $K(z) \geq k$ for all $z \in B(0, r_1)$. Finally, we let $f: [0, +\infty) \rightarrow [0, +\infty)$ be a C^2 function such that $f(0) = f'(0) = 0$ and the strong convexity condition (3) is satisfied.

For $u \in H^1(\mathbb{R}^d)$, we define the functionals

$$\begin{aligned} \mathcal{F}_h(u) &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_h(y - x) f \left(\frac{|u(y) - u(x)|}{|y - x|} \right) dy dx, \\ \mathcal{F}_0(u) &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f(|\nabla u(x) \cdot \hat{z}|) dz dx, \end{aligned}$$

where $\hat{z} := z/|z|$ for $z \neq 0$ and $K_h(z) := h^{-d} K(z/h)$.

Remark 3. By appealing to the results in [16], one can show that $\mathcal{F}_h(u)$ tends to $\mathcal{F}_0(u)$ as $h \searrow 0$ when u is smooth enough and vanishes in $\mathbb{R}^d \setminus \Omega$, and also that \mathcal{F}_0 is the Γ -limit of the family $\{\mathcal{F}_h\}$. Indeed, if $u = 0$ a.e. in $\mathbb{R}^d \setminus \Omega$, we have

$$\begin{aligned} \mathcal{F}_h(u) &:= \int_{\Omega} \int_{\Omega} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) dydx \\ &\quad + 2 \int_{\Omega} \int_{\mathbb{R}^d \setminus \Omega} K_h(y-x) f\left(\frac{|u(x)|}{|y-x|}\right) dydx. \end{aligned}$$

By [16], we know that the first addendum on the right-hand side converges and Γ -converges to \mathcal{F}_0 . It is clear that this is also the Γ -inferior limit of $\{\mathcal{F}_h\}$, because the term

$$\tilde{\mathcal{F}}_h(u) := 2 \int_{\Omega} \int_{\mathbb{R}^d \setminus \Omega} K_h(y-x) f\left(\frac{|u(x)|}{|y-x|}\right) dydx$$

is positive and may be dropped. As for the pointwise limit, we pick a function $u \in C^1(\mathbb{R}^d)$ that equals 0 in $\mathbb{R}^d \setminus \Omega$, and we observe that the quotient $|u(x)|/|y-x|$ is bounded above by $\|u\|_{C^1(\Omega)}$. It follows that

$$\tilde{\mathcal{F}}_h(u) \leq c \int_{\Omega} \int_{\mathbb{R}^d \setminus \Omega} K_h(y-x) dydx,$$

with $c := 2 \|f\|_{L^\infty([0, \|u\|_{C^1(\mathbb{R}^d)}])}$, whence $\lim_{h \searrow 0} \tilde{\mathcal{F}}_h(u) = 0$ (recall that $K \in L^1(\mathbb{R}^d)$).

Analogously to the 1-dimensional case, we define

$$\mathcal{E}_h(u) := \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2}$$

and we study the asymptotics of this family as $h \searrow 0$. Notice that the functionals \mathcal{E}_h are positive (see Lemma 3 below). Let us set

$$(30) \quad X := \{u \in H^1(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\}$$

and

$$(31) \quad \mathcal{E}_0(u) := \begin{cases} \frac{1}{24} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) |z|^2 f''(|\nabla u(x) \cdot \hat{z}|) |\nabla^2 u(x) \hat{z} \cdot \hat{z}|^2 dzdx & \text{if } u \in X \cap H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

We observe that if $u \in X \cap H^2(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ or if f'' has quadratic growth at infinity and $u \in X \cap H^2(\mathbb{R}^d)$, then $\mathcal{E}_0(u)$ is finite.

Remark 4 (Radial case). When K is radial, that is $K(z) = \bar{K}(|z|)$ for some $\bar{K}: [0, +\infty) \rightarrow [0, +\infty)$, we have

$$\begin{aligned} \mathcal{F}_0(u) &= \|K\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(|\nabla u(x) \cdot e|) d\mathcal{H}^{d-1}(e) dx, \\ \mathcal{E}_0(u) &= \frac{1}{24} \left(\int_{\mathbb{R}^d} K(z) |z|^2 dz \right) \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f''(|\nabla u(x) \cdot e|) |\nabla^2 u(x) e \cdot e|^2 d\mathcal{H}^{d-1}(e) dx. \end{aligned}$$

This Section is devoted to the proof of the following:

Theorem 2. Let Ω , K , and f satisfy the assumptions stated at the beginning of the current section. Then, there hold:

- (1) For any family $\{u_h\} \subset X$ such that $\mathcal{E}_h(u_h) \leq M$ for some $M > 0$, there exists a subsequence $\{u_{h_\ell}\}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $\nabla u_{h_\ell} \rightarrow \nabla u$ in $L^2(\mathbb{R}^d)$.
- (2) For any family $\{u_h\} \subset X$ that converges to $u \in X$ in $H^1(\mathbb{R}^d)$

$$\mathcal{E}_0(u) \leq \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$

- (3a) For any $u \in X \cap W^{1,\infty}(\mathbb{R}^d)$ there exists a family $\{u_h\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$ with the property that

$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \leq \mathcal{E}_0(u).$$

- (3b) If f'' is bounded, for any $u \in X$ there exists a family $\{u_h\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$ with the property that

$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \leq \mathcal{E}_0(u).$$

Statements 2, (3a), and (3b) amounts to saying that \mathcal{E}_0 is the Γ -limit of $\{\mathcal{E}_h\}$ with respect to the $H^1(\mathbb{R}^d)$ -convergence if either we restrict to functions in $X \cap W^{1,\infty}(\mathbb{R}^d)$ or f'' is bounded.

3.1. Slicing. When the dimension is 1, by virtue of the analysis in Section 2, it is not difficult to derive the Γ -convergence of the functionals \mathcal{E}_h .

Corollary 1. *Let $K: \mathbb{R} \rightarrow [0, +\infty)$ be an even function such that (27) holds. For $h > 0$ and $u \in H^1(\mathbb{R})$, we define the family*

$$\mathcal{E}_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_h(z) \left[f(|u'(x)|) - f\left(\left|\frac{u(x+z) - u(x)}{z}\right|\right) \right] dz dx.$$

We also let $\Omega = (a, b)$ be an open interval, $f: [0, +\infty) \rightarrow [0, +\infty)$ be a C^2 function satisfying $f(0) = f'(0) = 0$ and $f''(t) \geq \gamma$ with $\gamma > 0$ for all $t \in \mathbb{R}$, and $X \subset H^1(\mathbb{R})$ be as in (30). Then, the restrictions of the functionals \mathcal{E}_h to X Γ -converge w.r.t. the $H^1(\mathbb{R})$ -topology to

$$\mathcal{E}_0(u) := \begin{cases} \frac{1}{24} \left(\int_{\mathbb{R}} K(z) z^2 dz \right) \int_{\mathbb{R}} f''(u'(x)) |u''(x)|^2 dx & \text{if } u \in X \cap H^2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. A change of variables gives

$$\mathcal{E}_h(u) = \int_{\mathbb{R}} K(z) z^2 \left[\frac{1}{(hz)^2} \int_{\mathbb{R}} f(|u'(x)|) - f\left(\left|\frac{u(x+hz) - u(x)}{hz}\right|\right) dx \right] dz.$$

Recalling (5), we notice that the quantity between square brackets is equal to $E_{hz}(u')$, therefore the conclusion follows by a straightforward adaptation of the proof of Theorem 1 (see also the proof of Proposition 3). \square

Corollary 1 concludes the analysis when $d = 1$, so we may henceforth assume that $d \geq 2$. Our aim is proving that the restrictions to X of the functionals \mathcal{E}_h Γ -converge w.r.t. the $H^1(\mathbb{R}^d)$ -topology to \mathcal{E}_0 . The gist of our proof is a slicing procedure, which amounts to express the d -dimensional energies \mathcal{E}_h as superpositions of the 1-dimensional energies E_h , regarded as functionals on each line of \mathbb{R}^d .

Hereafter we tacitly assume that Ω , K , and f satisfy the hypotheses made at the beginning of the section. When $z \in \mathbb{R}^d \setminus \{0\}$, we set

$$\hat{z}^\perp := \{ \xi \in \mathbb{R}^d : \xi \cdot \hat{z} = 0 \}.$$

Lemma 3 (Slicing). *For $u \in X$, $z \in \mathbb{R}^d \setminus \{0\}$, and $\xi \in \hat{z}^\perp$, we define $w_{\hat{z},\xi}: \mathbb{R} \rightarrow \mathbb{R}$ as $w_{\hat{z},\xi}(t) := u(\xi + t\hat{z})$. Then, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z}$ and*

$$(32) \quad \mathcal{E}_h(u) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 E_{h|z|}(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz,$$

where $E_{h|z|}$ is as in (5) (note that the function f in (5) must be replaced here by $f(|t|)$).

Proof. Formula (32) is an easy consequence of Fubini's Theorem. Indeed, once the direction $\hat{z} \in \mathbb{S}^{d-1}$ is fixed, we can write $x \in \mathbb{R}^d$ as $x = \xi + t\hat{z}$ for some $\xi \in \mathbb{R}^d$ such that $\xi \cdot \hat{z} = 0$ and $t \in \mathbb{R}$. Using this decomposition, we have

$$\begin{aligned} \mathcal{F}_h(u) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f\left(\frac{|u(x+hz) - u(x)|}{h|z|}\right) dz dx \\ &= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) f\left(\frac{|w_{\hat{z},\xi}(t+h|z|) - w_{\hat{z},\xi}(t)|}{h|z|}\right) dt d\mathcal{H}^{d-1}(\xi) dz, \end{aligned}$$

whence

$$\mathcal{E}_h(u) = \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) \left[f(|w'_{\hat{z},\xi}(t)|) - f\left(\frac{|w_{\hat{z},\xi}(t+h|z|) - w_{\hat{z},\xi}(t)|}{h|z|}\right) \right] dt d\mathcal{H}^{d-1}(\xi) dz.$$

To obtain (32), it now suffices to multiply and divide the integrands by $|z|^2$. \square

The connection with the 1-dimensional case provided by Lemma 3 suggests that the Γ -convergence of the functionals E_h might be exploited to prove Theorem 2. Though, to be able to apply the results of Section 2, we need the functions $w_{\hat{z},\xi}$ in (32) to admit a second order weak derivative for a.e. z and ξ . This poses no real problem for the proof of the upper limit inequality, because we may reason on regular functions; as for the lower limit one, we shall tackle the difficulty in the next subsection by means of a compactness criterion, see Lemma 6 below. For the moment being, we are able to establish the following:

Proposition 3. *Let $u \in X \cap H^2(\mathbb{R}^d)$. Then:*

(1) *For any family $\{u_h\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$, there holds*

$$\mathcal{E}_0(u) \leq \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$

(2) *If $u \in X \cap C^3(\mathbb{R}^d)$, then*

$$\mathcal{E}_0(u) = \lim_{h \searrow 0} \mathcal{E}_h(u).$$

Proof. We prove both the assertions by using the slicing formula (32).

(1) For all $h > 0$, $z \in \mathbb{R}^d \setminus \{0\}$, and $\xi \in \hat{z}^\perp$, we let $w_{h;\hat{z},\xi}: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $w_{h;\hat{z},\xi}(t) := u_h(\xi + t\hat{z})$. Then,

$$\mathcal{E}_h(u_h) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 E_{h|z|}(w'_{h;\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz,$$

and, by Fatou's Lemma,

$$(33) \quad \liminf_{h \searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 \left[\liminf_{h \searrow 0} E_{h|z|}(w'_{h;\hat{z},\xi}) \right] d\mathcal{H}^{d-1}(\xi) dz.$$

Let $w_{\hat{z},\xi}$ be as in Lemma 3. Note that for any kernel $\rho: \mathbb{R}^d \rightarrow [0, +\infty)$ such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$ we may write

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u_h - \nabla u|^2 &\geq \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} |(\nabla u_h(\xi + t\hat{z}) - \nabla u(\xi + t\hat{z})) \cdot \hat{z}|^2 dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} |w'_{h;\hat{z},\xi}(t) - w'_{\hat{z},\xi}(t)|^2 dt d\mathcal{H}^{d-1}(\xi) dz. \end{aligned}$$

Since the left-hand side vanishes as $h \searrow 0$, it follows that there exists a subsequence of $\{w'_{h;\hat{z},\xi}\}$, which we do not relabel, that converges in $L^2(\mathbb{R})$ to $w'_{\hat{z},\xi}$ for \mathcal{L}^d -a.e. $z \in \mathbb{R}^d$ and \mathcal{H}^{d-1} -a.e. $\xi \in \hat{z}^\perp$. In particular, by assumption, $w'_{\hat{z},\xi} \in H^1(\mathbb{R})$ for a.e. (z, ξ) and it equals 0 on the complement of some open interval $I_{\hat{z},\xi}$.

From the previous considerations, we see that Proposition 2 can be applied on the right-hand side of (33), yielding

$$\liminf_{h \searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 E_0(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz = \mathcal{E}_0(u).$$

- (2) For any fixed $z \in \mathbb{R}^d \setminus \{0\}$ and $\xi \in \hat{z}^\perp$, we define the function $w_{\hat{z},\xi} \in C^3(\mathbb{R})$ as above. Since Ω is bounded, there exists $r > 0$ such that, for any choice of z , $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z} = 0$ whenever $\xi \in z^\perp$ satisfies $|\xi| \geq r$, while $w'_{\hat{z},\xi}(t)$ is supported in an open interval $I_{\hat{z},\xi}$ if $|\xi| < r$.

By virtue of the slicing formula (32), we obtain

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 |E_{h|z|}(w'_{\hat{z},\xi}) - E_0(w'_{\hat{z},\xi})| d\mathcal{H}^{d-1}(\xi) dz$$

Proposition 1 gives the existence of a constant $c > 0$ and of a continuous, bounded, and increasing function $m: [0, +\infty) \rightarrow [0, +\infty)$ such that $m(0) = 0$ and

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq c \int_{\mathbb{R}^d} \int_{z^\perp} K(z) |z|^2 m(h|z|) d\mathcal{H}^{d-1}(\xi) dz.$$

We remark that here m can be chosen depending only on ∇u , and not on \hat{z} and ξ .

Recalling (27), to achieve the conclusion it now suffices to appeal to Lebesgue's Convergence Theorem. □

3.2. Lower bound, compactness, and proof of the main result. Similarly to the 1-dimensional case, we shall prove the compactness of functions with equi-bounded energy by establishing at first a lower bound on the functionals \mathcal{E}_h . More precisely, Lemma 4 below shows that, when f is strongly convex, $\mathcal{E}_h(u)$ is greater than a double integral which takes into account, for each $z \in \mathbb{R}^d \setminus \{0\}$, the squared projection of the difference quotients of ∇u in the direction of z . Thanks to the slicing formula, the inequality follows with no effort by applying Lemma 1 on each line of \mathbb{R}^d .

We point out that our approach results in the appearance of an effective kernel \tilde{K} in front of the difference quotients. This function stands as a multidimensional counterpart of the kernel J in Lemma 1; actually, \tilde{K} depends both on K and on J (see (34) for the precise definition). In Lemma 5, we shall collect some properties of the effective kernel that will be useful in the proof of Lemma 6.

Lemma 4 (Lower bound on the energy). *Let us set*

$$(34) \quad \tilde{K}(z) := \int_{-1}^1 J(r)K_{|r|}(z)dr \quad \text{for a.e. } z \in \mathbb{R}^d,$$

with J as in Lemma 1. Then, it holds

$$(35) \quad \mathcal{E}_h(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{(\nabla u(x+hz) - \nabla u(x)) \cdot \hat{z}}{h} \right]^2 dx dz.$$

Proof. Thanks to Lemma 3, we can reduce to the 1-dimensional case, and we take advantage of the lower bound provided by Lemma 1. Keeping the notation of Lemma 3, we find

$$\begin{aligned} \mathcal{E}_h(u) &\geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r)K(z) |z|^2 \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h|z|} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r)K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz. \end{aligned}$$

To cast this bound in the form of (35), we change variables and use Fubini's Theorem:

$$\begin{aligned} I &:= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r)K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-1}^1 J(r)K(z) \left(\frac{w'_{\hat{z},\xi}(t+h|z|r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r)K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t-h|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\ &\quad + \int_0^1 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r)K_r(z) \left(\frac{w'_{\hat{z},\xi}(t+h|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \end{aligned}$$

Note that

$$\begin{aligned} &\int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r)K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t+h|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\ &= \int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r)K_{-r}(z) \left(\frac{w'_{-\hat{z},\xi}(-t-h|z|) - w'_{-\hat{z},\xi}(-t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\ &= \int_{-1}^0 \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r)K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t+h|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr, \end{aligned}$$

because $w'_{-\hat{z},\xi}(-s) = -w'_{\hat{z},\xi}(s)$ for all $s \in \mathbb{R}$. Thus, we conclude that

$$\begin{aligned} I &= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \left(\int_{-1}^1 J(r)K_{|r|}(z)dr \right) \left(\frac{w'_{\hat{z},\xi}(t+h|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{(\nabla u(x+hz) - \nabla u(x)) \cdot \hat{z}}{h} \right]^2 dx dz, \end{aligned}$$

which concludes the proof. \square

Let us remind that, by assumption, the kernel K is bounded away from 0 in a suitable annulus. The next lemma shows that the effective kernel appearing \tilde{K} in (35) inherits a similar property.

Lemma 5. *Let $\tilde{K}: \mathbb{R}^d \rightarrow [0, +\infty)$ be as in (34). Then,*

$$(36) \quad \int_{\mathbb{R}^d} \tilde{K}(z) (1 + |z|^2) dz < +\infty.$$

Moreover, if σ_d and r_1 are the constants in (28) and (29), then,

$$(37) \quad \text{ess inf} \left\{ \tilde{K}(z) : z \in B(0, \sigma_d r_1) \right\} > 0.$$

Proof. The convergence of the integral in (36) follows easily from (27). Indeed, by the definition of \tilde{K} , we see that

$$\int_{\mathbb{R}^d} \tilde{K}(z) dz = \int_{-1}^1 \int_{\mathbb{R}^d} J(r) K_{|r|}(z) dz dr = \int_{\mathbb{R}^d} K(z) dz;$$

analogously, one finds that

$$\int_{\mathbb{R}^d} \tilde{K}(z) |z|^2 dz = c \int_{\mathbb{R}^d} K(z) |z|^2 dz,$$

for some $c > 0$.

For what concerns (37), let us set $k := \text{ess inf} \{ K(z) : z \in B(0, r_1) \setminus B(0, r_0) \}$. In view of (29), $k > 0$.

We distinguish between the case $z \in B(0, r_0)$ and the case $z \in B(0, r_1) \setminus B(0, r_0)$. In the first situation, for a.e. $z \in \mathbb{R}^d$,

$$\begin{aligned} \tilde{K}(z) &\geq 2 \int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} J(r) K_r(z) dr \geq 2k \int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} \frac{1}{r^d} J(r) dr \\ &= \frac{2k}{|z|^{d-1}} \int_{r_0}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s} \right) ds. \end{aligned}$$

When $z \in B(0, r_1) \setminus B(0, r_0)$, instead, similar computations get

$$\tilde{K}(z) \geq 2 \int_{\frac{|z|}{r_1}}^1 J(r) K_r(z) dr = \frac{2k}{|z|^{d-1}} \int_{|z|}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s} \right) ds \quad \text{for a.e. } z \in \mathbb{R}^d,$$

so that we obtain

$$(38) \quad \tilde{K}(z) \geq \frac{2k}{|z|^{d-1}} \int_{\max(r_0, |z|)}^{r_1} s^{d-2} \left(1 - \frac{|z|}{s} \right) ds \quad \text{for a.e. } z \in \mathbb{R}^d.$$

When $d = 2$, the estimate above becomes

$$\tilde{K}(z) \geq 2k \left[\frac{r_1 - \max(r_0, |z|)}{|z|} - \log \left(\frac{r_1}{\max(r_0, |z|)} \right) \right] \quad \text{for a.e. } z \in \mathbb{R}^d.$$

Exploiting the concavity of the logarithm, we see that the lower bound that we have obtained is strictly positive if $|z| < r_1 = \sigma_2 r_1$.

On the other hand, putting $M := \max(r_0, |z|)$ for shortness, if $d \geq 3$, the right-hand side in (38) equals

$$\frac{2k}{(d-1)(d-2)|z|^{d-1}} \left[(d-2)(r_1^{d-1} - M^{d-1}) - (d-1)|z|(r_1^{d-2} - M^{d-2}) \right],$$

and therefore

$$\tilde{K}(z) \geq \frac{2kM^{d-2}}{(d-1)(d-2)|z|^{d-1}} \cdot \left\{ \left(\frac{r_1}{M} \right)^{d-2} [(d-2)r_1 - (d-1)|z|] - [(d-2)M - (d-1)|z|] \right\}$$

for a.e. $z \in \mathbb{R}^d$. When $|z| < \frac{d-2}{d-1}r_1 = \sigma_d r_1$, the quantity between braces is strictly positive if

$$\frac{(M - |z|)d - (2M - |z|)}{(r_1 - |z|)d - (2r_1 - |z|)} < \left(\frac{r_1}{M} \right)^{d-2}.$$

Observe that both the left-hand side and the right-hand one are strictly increasing in d ; also, the left-hand side is bounded above by $(M - |z|)/(r_1 - |z|)$, so the last inequality holds if

$$\frac{M - |z|}{r_1 - |z|} < \frac{r_1}{M},$$

which, in turn, is true for all $z \in B(0, r_1)$. \square

We are now in the position to prove that families with equibounded energy are compact in $H^1(\mathbb{R}^d)$, and that their accumulation points admit second order weak derivatives.

Lemma 6 (Compactness). *If $\{u_h\} \subset X$ satisfies $\mathcal{E}_h(u_h) \leq M$ for some $M \geq 0$, there exist a subsequence $\{u_{h_\ell}\}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $u_{h_\ell} \rightarrow u$ in $H^1(\mathbb{R}^d)$.*

Proof. Let $\tilde{k} := \text{ess inf } \{ \tilde{K}(z) : z \in B(0, \sigma_d r_1) \}$; Lemma 5 ensures that $\tilde{k} > 0$. We consider a function $\rho \in C_c^\infty([0, +\infty))$ such that

$$\rho(r) = 0 \quad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty \right),$$

and we further require that

$$0 \leq \rho(r) \leq \tilde{k} \quad \text{and} \quad |\rho'(r)| \leq \tilde{k}.$$

For $h > 0$ and $y \in \mathbb{R}^d$, we set

$$\rho_h(y) := \frac{1}{ch^d} \rho\left(\frac{|y|}{h}\right), \quad \text{with } c := \int_{\mathbb{R}^d} \rho(|y|) dy,$$

and we introduce the functions $v_h := \rho_h * u_h$, as before.

Each function v_h is a smooth function and, for all $\tilde{h} \in (0, 1)$, its support is contained in

$$\Omega_{\tilde{h}} := \{x : \text{dist}(x, \Omega) \leq 2^{-1/2} \tilde{h} \sigma_d r_1\}$$

if $h \in (0, \tilde{h})$. In particular, we can choose \tilde{h} so small that $\partial\Omega_{\tilde{h}}$ is still Lipschitz. For such an \tilde{h} , we assert that the family $\{v_h\}_{h \in (0, \tilde{h})}$ is relatively compact in $H_0^1(\Omega_{\tilde{h}})$. In order to prove this, we first remark that

$$(39) \quad \int_{\Omega_{\tilde{h}}} |\nabla^2 v_h|^2 = \int_{\Omega_{\tilde{h}}} |\Delta v_h|^2,$$

and next we show that the right-hand side is uniformly bounded.

We observe that $\int_{\mathbb{R}^d} \nabla \rho_h(y) dy = 0$ for all $h > 0$, because ρ is compactly supported. Hence,

$$\begin{aligned} \|\Delta v_h\|_{L^2(\Omega_{\tilde{h}})}^2 &= \int_{\mathbb{R}^d} |\Delta v_h|^2 \\ &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla \rho_h(y) \cdot (\nabla u_h(x+y) - \nabla u_h(x)) dy \right|^2 dx \\ &\leq \int_{\mathbb{R}^d} \left[\frac{1}{ch^{d+1}} \int_{\mathbb{R}^d} \left| \rho' \left(\frac{|y|}{h} \right) \right| |(\nabla u_h(x+y) - \nabla u_h(x)) \cdot \hat{y}| dy \right]^2 dx. \end{aligned}$$

By our choice of ρ and (37), we find

$$\begin{aligned} \|\Delta v_h\|_{L^2(\Omega_{\tilde{h}})}^2 &\leq \int_{\mathbb{R}^d} \left[\frac{1}{ch} \int_{\mathbb{R}^d} \tilde{K}_h(y) |(\nabla u_h(x+y) - \nabla u_h(x)) \cdot \hat{y}| dy \right]^2 dx \\ &\leq \int_{\mathbb{R}^d} \left[\frac{1}{ch} \int_{\mathbb{R}^d} \tilde{K}(z) |(\nabla u_h(x+hz) - \nabla u_h(x)) \cdot \hat{z}| dz \right]^2 dx \end{aligned}$$

Further, since $\tilde{K} \in L^1(\mathbb{R}^d)$, Jensen's inequality and Fubini's Theorem yield

$$\|\Delta v_h\|_{L^2(\Omega_{\tilde{h}})}^2 \leq \frac{\|\tilde{K}\|_{L^1(\mathbb{R}^d)}}{c^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{(\nabla u_h(x+hz) - \nabla u_h(x)) \cdot \hat{z}}{h} \right]^2 dx dz.$$

The lower bound (35) entails

$$\|\Delta v_h\|_{L^2(\Omega_{\tilde{h}})}^2 \leq \frac{4}{c^{2\gamma}} \|\tilde{K}\|_{L^1(\mathbb{R}^d)} \mathcal{E}_h(u_h),$$

so that, in view of the assumption $\mathcal{E}_h(u_h) \leq M$ and of (39), we get

$$(40) \quad \|\nabla^2 v_h\|_{L^2(\Omega_{\tilde{h}})}^2 \leq \frac{4M}{c^{2\gamma}} \|\tilde{K}\|_{L^1(\mathbb{R}^d)}.$$

We argue as in the proof of Lemma 2. We recall that, for $h \in (0, \tilde{h})$, each v_h vanishes on the complement of $\Omega_{\tilde{h}}$, and thus, by Poincaré inequality, (40) implies a uniform bound on the norms $\|v_h\|_{H_0^2(\Omega_{\tilde{h}})}$. As a consequence, by Rellich-Kondrachov Theorem, the family $\{\tilde{v}_h\}_{h \in (0, \tilde{h})}$ of the restrictions of the functions v_h to $\Omega_{\tilde{h}}$ admits a subsequence $\{\tilde{v}_{h_\ell}\}$ that converges in $H_0^1(\Omega_{\tilde{h}})$ to a function $\tilde{u} \in H_0^2(\Omega_{\tilde{h}})$. Actually, the support of \tilde{u} is contained in $\bar{\Omega}$, and, if we put,

$$u(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{\Omega}, \\ 0 & \text{otherwise,} \end{cases}$$

we infer that $\{v_{h_\ell}\}$ converges in $H^1(\mathbb{R}^d)$ to $u \in X \cap H^2(\mathbb{R}^d)$.

To accomplish the proof, it suffices to show that the L^2 distance between ∇u_h and ∇v_h vanishes when $h \searrow 0$. Since ρ_h has unit $L^1(\mathbb{R}^d)$ -norm and is radial, we have

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_h(y) (\nabla u_h(x+y) - \nabla u_h(x)) dy \right|^2 dx \\ &= \frac{1}{4} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \rho_h(y) (\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)) dy \right|^2 dx \\ &\leq \frac{1}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_h(y) |\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)|^2 dy dx. \end{aligned}$$

We remark that for any fixed $y \in \mathbb{R}^d \setminus \{0\}$ and for all $p \in \mathbb{R}^d$, the identity $|p|^2 = |p \cdot y|^2 + |(\text{Id} - y \otimes y)p|^2$ can be reformulated as

$$(41) \quad \begin{aligned} |p|^2 &= |p \cdot y|^2 + \int_{\hat{y}^\perp} \pi(|\eta|) |p \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta) \\ &= |p \cdot y|^2 + \frac{1}{h^2} \int_{\hat{y}^\perp} \pi_h(\eta) |p \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta), \end{aligned}$$

where $\pi: [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that

$$\int_{e^{\frac{1}{d}}} \pi(|\eta|) |\eta|^2 d\mathcal{H}^{d-1}(\eta) = 1,$$

and $\pi_h(\eta) := h^{-d+1} \pi(|\eta|/h)$. We further prescribe that

$$\pi(r) = 0 \quad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty \right)$$

and that $\lim_{r \searrow 0} \pi(r)/r \in \mathbb{R}$.

We apply the formula (41) to $p_h(x, y) := \nabla u_h(x + y) + \nabla u_h(x - y) - 2\nabla u_h(x)$ and we find that

$$(42) \quad \int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 dx \leq \frac{1}{4} (I_1 + I_2),$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_h(y) |y|^2 |p_h(x, y) \cdot \hat{y}|^2 dy dx, \\ I_2 &:= \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho_h(y) \pi_h(\eta) |p_h(x, y) \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta) dy dx. \end{aligned}$$

We first consider I_1 . Keeping in mind that ρ is compactly supported and $\rho(|y|) \leq \tilde{k} \leq \tilde{K}(y)$ for a.e. $y \in B(0, 2^{-1/2} \sigma_d r_1)$, we get

$$\begin{aligned} I_1 &\leq \frac{(\sigma_d r_1)^2}{c} \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}_h(y) |(\nabla u_h(x + y) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx \right. \\ &\quad \left. + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}_h(y) |(\nabla u_h(x - y) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx \right], \end{aligned}$$

and, by (35),

$$(43) \quad I_1 \leq \frac{8(\sigma_d r_1)^2 M}{c\gamma} h^2.$$

As for I_2 , we assert that there exist a constant $L > 0$, depending on $d, \sigma_d, r_1, \tilde{k}$, and c , such that

$$(44) \quad I_2 \leq \frac{LM}{\gamma} h^2.$$

To prove the claim, we write the integrand appearing in I_2 as follows:

$$\begin{aligned}
p_h(x, y) \cdot \eta &= (\nabla u_h(x + y) + \nabla u_h(x - y) - 2\nabla u_h(x)) \cdot \eta \\
&= (\nabla u_h(x + y) + \nabla u_h(x - y) - 2\nabla u_h(x - \eta)) \cdot \eta \\
&\quad + 2(\nabla u_h(x - \eta) - \nabla u_h(x)) \cdot \eta \\
&= (\nabla u_h(x + y) - \nabla u_h(x - \eta)) \cdot (\eta + y) \\
&\quad + (\nabla u_h(x - y) - \nabla u_h(x - \eta)) \cdot (\eta - y) \\
&\quad - (\nabla u_h(x + y) - \nabla u_h(x - y)) \cdot y + 2(\nabla u_h(x - \eta) - \nabla u_h(x)) \cdot \eta.
\end{aligned}$$

We plug this identity in the definition of I_2 and we find that

$$\begin{aligned}
I_2 &\leq \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x + hy) - \nabla u_h(x - h\eta)) \cdot (\eta + y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
&\quad + \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x - hy) - \nabla u_h(x - h\eta)) \cdot (\eta - y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
&\quad + \frac{8}{c} \|\pi\|_{L^1(e_d^\perp)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(|y|) |y|^2 |(\nabla u_h(x + hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx \\
&\quad + \frac{16}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x + h\eta) - \nabla u_h(x)) \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta) dy dx.
\end{aligned}$$

We estimate separately each of the contributions on the right-hand side.

Let us set $\mathbb{S}_+^{d-1} := \{e \in \mathbb{S}^{d-1} : e \cdot e_d > 0\}$ and $\mathbb{S}_-^{d-1} := \{e \in \mathbb{S}^{d-1} : e \cdot e_d < 0\}$. Hereafter, we denote by L any strictly positive constant depending only on d , σ_d , r_1 , and on the norms of ρ and π .

Taking advantage of the Coarea Formula, we rewrite the first addendum as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x + hy) - \nabla u_h(x - h\eta)) \cdot (\eta + y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x + h(\eta + y)) - \nabla u_h(x)) \cdot (\eta + y)|^2 d\mathcal{H}^{d-1}(\eta) dx dy \\
&= \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \int_{e^\perp} r^{d-1} \rho(r) \pi(|\eta|) |(\nabla u_h(x + h(\eta + re)) - \nabla u_h(x)) \cdot (\eta + re)|^2 d\mathcal{H}^{d-1}(\eta) dr dx d\mathcal{H}^{d-1}(e) \\
&= \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 |y \cdot e|^{d-1} \rho(|y \cdot e|) \pi(|(\text{Id} - e \otimes e)y|) |(\nabla u_h(x + hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx d\mathcal{H}^{d-1}(e).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x - hy) - \nabla u_h(x - h\eta)) \cdot (\eta - y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
&= \int_{\mathbb{S}_-^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 |y \cdot e|^{d-1} \rho(|y \cdot e|) \pi(|(\text{Id} - e \otimes e)y|) |(\nabla u_h(x + hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx d\mathcal{H}^{d-1}(e),
\end{aligned}$$

and thus

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) |(\nabla u_h(x+hy) - \nabla u_h(x-h\eta)) \cdot (\eta+y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
& \quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) |(\nabla u_h(x-hy) - \nabla u_h(x-h\eta)) \cdot (\eta-y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
& = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y \cdot e|^{d-1} |y|^2 \rho(|y \cdot e|)\pi(|(\text{Id} - e \otimes e)y|) |(\nabla u_h(x+hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx d\mathcal{H}^{d-1}(e).
\end{aligned}$$

Let us recall that $\rho(r) = \eta(r) = 0$ if $r \notin [0, 2^{-1/2}\sigma_d r_1]$, whence, for any $e \in \mathbb{S}^{d-1}$, the product $\rho(|y \cdot e|)\pi(|(\text{Id} - e \otimes e)y|)$ vanishes outside the cylinder

$$C_e := \{y \in \mathbb{R}^d : |y \cdot e|, |(\text{Id} - e \otimes e)y| \in [0, 2^{-1/2}\sigma_d r_1]\} \subset B(0, \sigma_d r_1).$$

We therefore see that the last multiple integral equals

$$\begin{aligned}
& \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{C_e} |y \cdot e|^{d-1} |y|^2 \rho(|y \cdot e|)\pi(|(\text{Id} - e \otimes e)y|) |(\nabla u_h(x+hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx d\mathcal{H}^{d-1}(e) \\
& \leq L \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^d} \int_{C_e} \tilde{K}(y) |(\nabla u_h(x+hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx d\mathcal{H}^{d-1}(e) \\
& \leq \frac{LM}{\gamma} h^2.
\end{aligned}$$

We then obtain

$$\begin{aligned}
(45) \quad & \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) |(\nabla u_h(x+hy) - \nabla u_h(x-h\eta)) \cdot (\eta+y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
& + \frac{4}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) |(\nabla u_h(x-hy) - \nabla u_h(x-h\eta)) \cdot (\eta-y)|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
& \leq \frac{LM}{\gamma} h^2.
\end{aligned}$$

Next, we have

$$(46) \quad \frac{8}{c} \|\pi\|_{L^1(e_d^\perp)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(|y|) |y|^2 |(\nabla u_h(x+hy) - \nabla u_h(x)) \cdot \hat{y}|^2 dy dx \leq \frac{LM}{\gamma} h^2,$$

$$(47) \quad \frac{16}{c} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|) |(\nabla u_h(x+h\eta) - \nabla u_h(x)) \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta) dy dx \leq \frac{LM}{\gamma} h^2.$$

The bound in (46) may be deduced as the one in (43), so, to establish (44), we are only left to prove (47). To this aim, let $\psi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be a test function. By a

standard argument and Fubini's Theorem we have that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) \psi(y, \eta) d\mathcal{H}^{d-1}(\eta) dy \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|y|}{2\varepsilon} \chi_{\{t < \varepsilon\}}(|\eta \cdot y|) \rho(|y|) \pi(|\eta|) \psi(y, \eta) d\eta dy \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \frac{\pi(|\eta|)}{|\eta|} \left(\int_{\mathbb{R}^d} \frac{|y|}{2\varepsilon} \chi_{\{t < \varepsilon\}}(|\eta \cdot y|) \rho(|y|) |y| \psi(y, \eta) dy \right) d\eta \\
&= \int_{\mathbb{R}^d} \int_{\hat{\eta}^\perp} \frac{\pi(|\eta|)}{|\eta|} \rho(|y|) |y| \psi(y, \eta) d\mathcal{H}^{d-1}(y) d\eta
\end{aligned}$$

(recall that we assume $\lim_{r \searrow 0} \pi(r)/r$ to be finite). It follows that

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{y}^\perp} \rho(|y|) \pi(|\eta|) |(\nabla u_h(x + h\eta) - \nabla u_h(x)) \cdot \eta|^2 d\mathcal{H}^{d-1}(\eta) dy dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\hat{\eta}^\perp} \frac{\pi(|\eta|)}{|\eta|} \rho(|y|) |y| |(\nabla u_h(x + h\eta) - \nabla u_h(x)) \cdot \eta|^2 d\mathcal{H}^{d-1}(y) d\eta dx \\
&\leq L \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(\eta) |(\nabla u_h(x + h\eta) - \nabla u_h(x)) \cdot \eta|^2 d\eta dx.
\end{aligned}$$

In view of the bound on the energy, we retrieve (47).

The proof is now concluded, because from (42), (43), and (44) we obtain

$$\int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 dx \leq \frac{LM}{\gamma} h^2,$$

as desired. \square

Remark 5. *The choice $u_h = u$ in Lemma 6 provides a criterion for a function in $H^1(\mathbb{R}^d)$ to belong to $H^2(\mathbb{R}^d)$. Namely, when Ω , K , and f fulfil the assumptions of the current section and f'' is bounded, a function $u \in X$ is in $H^2(\mathbb{R}^d)$ if and only if $\mathcal{E}_h(u) \leq M$ for some $M > 0$ and for all h 's small enough. One implication is a byproduct of Lemma 6, while the other follows by exploiting the slicing formula and Remark 2: indeed, if $f'' \leq c$ one finds*

$$\mathcal{E}_h(u) \leq \frac{c}{2} \left(\int_{\mathbb{R}^d} K(z) |z|^2 dz \right) \int_{\mathbb{R}^d} |\nabla^2 u(x)|^2 dx.$$

We can now accomplish the proof of Theorem 2.

Proof of Theorem 2. Lemma 6 provides the compactness result of statement (1) in Theorem 2.

Turning to the lower limit inequality, for any $u \in X$ and for any family $\{u_h\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$, we may focus on the situation when there exists $M \geq 0$ such that $\mathcal{E}_h(u_h) \leq M$ for all $h > 0$. In view of Lemma 6, we have that $u \in H^2(\mathbb{R}^d)$, thus statement (2) follows by Proposition 3.

For what concerns the upper limit inequality, we reason as in the 1-dimensional case (see the proof of Proposition 1). In order to adapt the argument, we observe that, if $u \in X \cap H^2(\mathbb{R}^d)$, by mollification, we can construct a sequence $\{u_\ell\} \subset X$ of smooth functions that tend to u in $H^2(\mathbb{R}^d)$ and satisfy $\lim_{\ell \nearrow +\infty} \mathcal{E}_0(u_\ell) = \mathcal{E}_0(u)$, provided that f'' is bounded or $u \in X \cap H^2(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$. Indeed, when one of these assumptions holds, there exists $c > 0$ such that $f''(|\nabla u_\ell(x) \cdot \hat{z}|) \leq c$ for a.e. x and all z , and Lebesgue's Theorem applies. Then, we can establish the upper limit

inequality by combining the approximation by smooth functions and Proposition 3. \square

We conclude with a couple of remarks.

Remark 6. *As in Remark 1, we see that the Γ -limit of*

$$h\mathcal{E}_h(u) = \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h}$$

in $H^1(\mathbb{R}^d)$ is 0. The same Γ -limit is found if one considers the $L^2(\mathbb{R}^d)$ -topology on X , because $h\mathcal{E}_h(u) \geq 0$ for all $u \in X$ and Proposition 3 provides a constant recovery sequence for smooth functions.

Remark 7. *Statements (2), (3a), and (3b) in Theorem 2, that is, the Γ -convergence result, are not affected if we replace X with $H^1(\mathbb{R}^d)$; the proof remains essentially the same. On the other hand, if we substitute Ω with \mathbb{R}^d , the compactness provided by statement (1) of Theorem 2 may fail.*

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