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ON THE CONVERGENCE RATE OF SOME NONLOCAL ENERGIES

ANTONIN CHAMBOLLE, MATTEO NOVAGA, AND VALERIO PAGLIARI

Abstract. We study the rate of convergence of some nonlocal functionals recently considered by Bourgain, Brezis and Mironescu. In particular, we establish the Γ-convergence of the corresponding rate functionals, suitably rescaled, to a limit functional of second order.

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1. Introduction

We are interested in the rate of converge, as $h \searrow 0$, of the nonlocal functionals

$$\mathcal{F}_h(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h(z) f\left(\frac{|u(x+z)-u(x)|}{|z|}\right) dz dx,$$

to the limit functional

$$\mathcal{F}_0(u) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z) f(|\nabla u(x)\cdot \hat{z}|) dz dx.$$

Here $f\colon \mathbb{R} \to [0,+\infty)$ is a convex function of class C^2 satisfying $f(0) = f'(0) = 0$, $K : \mathbb{R}^d \to [0,+\infty)$ is a kernel such that $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^d$, and

$$\int_{\mathbb{R}^d} K(z)\left(1+|z|^2\right) dz < +\infty,$$

and we set $K_h(z) := h^{-d}K(z/h)$.

It has been proved by Bourgain, Brezis and Mironescu in [3] that $\mathcal{F}_h(u)$ tends to $\mathcal{F}_0(u)$ as $h \searrow 0$ for all $u \in H^1(\mathbb{R}^d)$, and in [8, 11] (see also [2]) it is shown that such convergence also holds in the sense of Γ-convergence [4, 6], with respect to the $L^2(\mathbb{R}^d)$-topology.

Let now

$$\mathcal{E}_h(u) := \frac{\mathcal{F}_0(u)-\mathcal{F}_h(u)}{h^2} = \frac{1}{h^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[K(z)f(|\nabla u(x)\cdot\hat{z}|) - K_h(z) f\left(\frac{|u(x+z)-u(x)|}{|z|}\right)\right] dz dx \tag{1}$$

be the functional which measures the rate of convergence of $\mathcal{F}_h$ to $\mathcal{F}_0$. In this paper, under the assumption that the function f is strongly convex (see condition (12) below), we prove that the family $\{\,\mathcal{E}_h\,\}$ Γ-converges, with respect to the $H^1(\mathbb{R}^d)$-topology, to the *second order* limit functional

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24}\displaystyle\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z)\,|z|^2\, f''(|\nabla u(x)\cdot\hat{z}|)\,\left|\nabla^2 u(x)\hat{z}\cdot\hat{z}\right|^2 dz dx & \text{if } u\in H^2(\mathbb{R}^d),\\ +\infty & \text{otherwise,}\end{cases}$$

where $\hat{z} = z/|z|$. The uniform convexity assumption on f, which is needed for the Γ-liminf inequality, excludes from our analysis some interesting cases such as $f(x) = |x|^p$ with $p \geq 1$, $p \neq 2$. In particular, when $f(x) = |x|$ and K is radially symmetric, the problem is related to a geometric problem considered in [9] in the context of a physical model for liquid drops with dipolar repulsion. We also observe that the problem we study is different from a higher order Γ-limit of $\mathcal{F}_h$ (see [5]), which would rather correspond to considering the Γ-limit of the functionals

$$\frac{\mathcal{F}_h - \min \mathcal{F}_0}{h^\alpha} \qquad \text{for some } \alpha > 0\,.$$

As a consequence of our result (see Remark 4) we also get that, if the rate of convergence of $\mathcal{F}_h(u)$ to $\mathcal{F}_0(u)$ is fast enough, more precisely if $|\mathcal{E}_h(u)| \leq M$ for all h's sufficiently small, then $u \in H^2(\mathbb{R}^d)$.

We notice that our result is reminiscent to the one obtained by Peletier, Planqué and Röger in [10], motivated by a model for bilayer membranes, where they consider the convolution functionals

$$\mathcal{G}_h(u) := \int_{\mathbb{R}^d} f\,(K_h * u)\,dx,$$

which converge to the functional $\mathcal{G}_0(u) = c\int_{\mathbb{R}^d} f\,(u)\,dx$ as $h \searrow 0$, where $c = c(K,d)$ is a positive constant, and show that the corresponding rate functionals

$$\frac{\mathcal{G}_0(u) - \mathcal{G}_h(u)}{h^2} = \frac{1}{h^2}\int_{\mathbb{R}^d}\big(c\,f\,(u) - f\,(K_h * u)\,\big)dx \tag{2}$$

converge pointwise to the limit functional

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z)|z|^2 f''(u(x))|\nabla u(x)\cdot\hat{z}|^2 dz dx \qquad \text{for } u\in H^1(\mathbb{R}^d).$$

In particular, the rate functionals are uniformly bounded if and only if $u \in H^1(\mathbb{R}^d)$.

In the proof of our convergence result, we follow a strategy similar to the one in [7, 8]: we first consider a related 1-dimensional problem, and then reduce the general case to it by a *slicing* procedure. More precisely, in Section 2 we study the functionals

$$E_h(u) := \frac{1}{h^2}\int_{\mathbb{R}}\left[f(u(x)) - f\left(\fint_x^{x+h} u(y)dy\right)\right]dx,$$

which are a particular case of (2), and we show their convergence (see Theorem 1) to the limit energy

$$E_0(u) := \frac{1}{24}\int_{\mathbb{R}} f''(u(x))\,|u'(x)|^2\,dx \qquad \text{for } u\in H^1(\mathbb{R}).$$

Then, in Section 3 we consider the general functionals in (1) and we prove the Γ-convergence to $\mathcal{E}_0$ (see Theorem 2), which is the main result of this paper. We first show the convergence for $d = 1$, using the result of Section 2, and then we reduce to the 1-dimensional case by means of a delicate slicing technique.

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2. Finite difference functionals in the 1-dimensional case

For $u \in L^2(\mathbb{R})$ and $h > 0$, we define the energy

$$E_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \left[f(u(x)) - f\left(D_h U(x)\right) \right] dx, \tag{3}$$

where

$$U(x) := \int_0^x u(y)dy \quad \text{and} \quad D_h U(x) := \frac{U(x+h) - U(x)}{h} = \fint_x^{x+h} u(y)dy. \tag{4}$$

Let us fix an open interval $I := (a, b) \subset \mathbb{R}$. We shall compute the Γ-limit of $\{ E_h \}$ regarded as a family of functionals on the closed subspace $Y \subset L^2(\mathbb{R})$ defined as

$$Y := \{ u \in L^2(\mathbb{R}) : u = 0 \text{ in } \mathbb{R} \setminus I \} \tag{5}$$

endowed with the L^2-topology. Let us set

$$E_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}} f''(u(x)) \left|u'(x)\right|^2 dx & \text{if } u \in Y \cap H^1(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

We shall prove the following:

Theorem 1. *Let us assume that there exists $\gamma > 0$ such that $2f(t) - \gamma t^2$ is convex. Then, the restriction to Y of the family $\{ E_h \}$ Γ-converges, as $h \searrow 0$, to E_0 w.r.t. the $L^2(\mathbb{R})$-topology, that is, for every $u \in Y$ the following properties hold:*

(1) *For any family $\{ u_h \} \subset Y$ that converges to u in $L^2(\mathbb{R})$ we have*

$$E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h).$$

(2) *There exists a sequence $\{ u_h \} \subset Y$ converging to u in $L^2(\mathbb{R})$ such that*

$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

The Γ-upper limit is established in Proposition 1, while Proposition 2 takes care of the lower limit. In turn, the latter is achieved by exploiting a suitable lower bound on the energy (see Lemma 1) and a compactness result (see Lemma 2).

2.1. Pointwise limit and upper bound.

We now compute the limit of $E_h(u)$, as $h \searrow 0$, for a function $u \in Y \cap C^2(\mathbb{R})$.

Proposition 1. *Let $u \in Y \cap C^2(\mathbb{R})$. Then, there exists a continuous, bounded, and increasing function $m \colon [0, +\infty) \to [0, +\infty)$ such that $m(0) = 0$ and*

$$|E_h(u) - E_0(u)| \leq c\, m(h), \tag{7}$$

where $c := c(b - a, \|u\|_{C^2(\mathbb{R})}, \|f\|_{C^2([-\|u\|_{C^2(\mathbb{R})}, \|u\|_{C^2(\mathbb{R})}])}) > 0$ is a constant. In particular,

$$\lim_{h \searrow 0} E_h(u) = E_0(u),$$

moreover for every $u \in Y$ there exists a sequence $\{ u_h \} \subset Y$ that converges to u in $L^2(\mathbb{R})$ and satisfies

$$\limsup_{h \searrow 0} E_h(u_h) \leq E_0(u).$$

Proof. Since $u \in Y \cap C^2(\mathbb{R})$ and $f \in C^2(\mathbb{R})$, it is easy to see that $F_h(u) - F_0(u)$ and $E_0(u)$ are uniformly bounded in h. Thus, there exists a constant $c_\infty > 0$ such that

$$|E_h(u) - E_0(u)| \le c_\infty \qquad \text{for } h > 1. \tag{8}$$

Next, we focus on the case $h \in (0,1]$. If $x \notin (a-h, b)$, then $D_h U(x) = 0$, and hence

$$F_0(u) - F_h(u) = \int_a^b \left[f(u(x)) - f\left(D_h U(x)\right)\right] dx - \int_{a-h}^a f(D_h U(x)) dx.$$

Being u regular, for any $x \in (a-h, b)$ we have the Taylor's expansion

$$D_h U(x) = u(x) + \frac{h}{2} u'(x) + \frac{h^2}{6} u''(x_h), \qquad \text{with } x_h \in (x, x+h),$$

which we rewrite as

$$D_h U(x) = u(x) + h v_h(x), \qquad \text{with } \ v_h(x) := \frac{u'(x)}{2} + \frac{h}{6} u''(x_h); \tag{9}$$

note that v_h converges uniformly to $u'/2$ as $h \searrow 0$.

Plugging (9) into the definition of F_h, we get

$$\begin{aligned} F_0(u) - F_h(u) =& -\int_a^b \left[f\big(u(x) + h v_h(x)\big) - f(u(x))\right] dx - \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \\ =& -h \int_a^b f'(u(x)) v_h(x) dx - \frac{h^2}{2} \int_a^b f''(w_h(x)) v_h(x)^2 dx \\ & - \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx, \end{aligned}$$

where w_h fulfils $w_h(x) \in (u(x), u(x) + h v_h(x_h))$ for all $x \in (a,b)$.

It easy to see that

$$\left| \int_{a-h}^a f\left(\frac{h^2}{6} u''(x_h)\right) dx \right| \le c_1 h^5,$$

for a constant $c_1 > 0$ that depends only on $N := \|u\|_{C^2(\mathbb{R})}$ and on $\|f''\|_{L^\infty([-N,N])}$. Moreover, recalling the definition of v_h, we have

$$\int_a^b f'(u(x)) v_h(x) dx = \frac{h}{6} \int_a^b f'(u(x)) u''(x_h) dx,$$

and therefore

$$\begin{aligned} |E_h(u) - E_0(u)| \le& \frac{1}{6} \left| -\int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| \\ & + \frac{1}{2} \left| \frac{1}{4} \int_a^b f''(u(x)) u'(x)^2 dx - \int_a^b f''(w_h(x)) v_h(x)^2 dx \right| + c_1 h^5. \end{aligned} \tag{10}$$

Since $u \in Y \cap C^2(\mathbb{R})$, u'' admits a uniform modulus of continuity $m_{u''} \colon [0, +\infty) \to [0, \infty)$. An integration by parts gives that

$$\begin{aligned} \left| -\int_a^b f'(u(x)) u''(x_h) dx - \int_a^b f''(u(x)) u'(x)^2 dx \right| \le& \int_a^b |f'(u(x))| \, |u''(x) - u''(x_h)| \, dx \\ \le& \, c_2 m_{u''}(h), \end{aligned}$$

where $c_2 := (b-a) \|f'\|_{L^\infty([-N,N])}$.

In a similar manner, denoting by $m_{f''}$ the modulus of continuity of the restriction of f'' to the interval $[-N,N]$, we also find

$$\left|\frac{1}{4}\int_a^b f''(u(x))u'(x)^2dx - \int_a^b f''(w_h(x))v_h(x)^2dx\right|$$
$$\leq \int_a^b |f''(u(x))|\left|\frac{1}{4}u'(x)^2 - v_h(x)^2\right|dx + \int_a^b |f''(u(x)) - f''(w_h(x))|\, v_h(x)^2dx$$
$$\leq c_3(h + m_{f''}(h)),$$

with c_3 depending on $b-a$, N, and $\|f''\|_{L^\infty([-N,N])}$.

By combining (10) with the inequalities above, we obtain

$$|E_h(u) - E_0(u)| \leq c_0\big(m_{u''}(h) + m_{f''}(h) + h + h^5\big) \qquad \text{for } h \in (0,1], \tag{11}$$

for a suitable constant $c_0 > 0$.

The conclusion now follows by (8) and (11). □

Notice that, by standard density arguments, the second statement of Theorem 1 follows directly by Proposition 1.

Remark 1. Notice that, as a consequence of Proposition 1, the Γ-limit of the rate functionals

$$hE_h(u) = \frac{1}{h}\int_{\mathbb{R}} \left[f(u(x)) - f\left(D_hU(x)\right)\right]dx$$

is equal to zero.

2.2. **Lower bound in the strongly convex case.** In view of Proposition 1, to complete the proof of the Theorem 1, it only remains to establish statement 1, that is, for any $u \in Y$ and for any family $\{\, u_h \,\} \subset Y$ converging to u in $L^2(\mathbb{R})$ it holds

$$E_0(u) \leq \liminf_{h\searrow 0} E_h(u_h).$$

We prove the inequality under the hypothesis that the function f is strongly convex, i.e., we assume that

$$\text{there exists } \gamma > 0 \text{ such that } 2f(t) - \gamma t^2 \text{ is convex.} \tag{12}$$

Thanks to this additional assumption on f, we are able to provide a lower bound on the energy E_h, and we use it to prove that sequences with equibounded energy are relatively compact w.r.t. the L^2-topology.

Lemma 1 (Lower bound on the energy)**.** *Assume that f fulfils* (12)*. Then, for any $u \in Y$, it holds*

$$E_h(u) \geq \sup_{\varphi\in C_c^\infty(\mathbb{R}^2)} \left\{\int_{\mathbb{R}} \fint_x^{x+h} \left(\frac{u(y) - D_hU(x)}{h}\varphi(x,y) - \frac{\varphi(x,y)^2}{4\lambda_h(x,y)}\right)dydx\right\}, \tag{13}$$

with

$$\lambda_h(x,y) := \int_0^1 (1-\vartheta)f''\big((1-\vartheta)D_hU(x) + \vartheta u(y)\big)d\vartheta. \tag{14}$$

Moreover,

$$E_h(u) \geq \frac{\gamma}{4}\int_{\mathbb{R}}\int_{-h}^{h} J_h(r)\left(\frac{u(y+r) - u(y)}{h}\right)^2 drdy, \tag{15}$$

where

$$J(r) := (1-|r|)^+ \quad \textit{and} \quad J_h(r) := \frac{1}{h}J\left(\frac{r}{h}\right).$$

Proof. For a given $h > 0$, let us consider $u \in Y$ such that $E_h(u)$ is finite. We write

$$E_h(u) = \frac{1}{h^2}\int_{\mathbb{R}} e_h(x)dx, \quad \text{where } e_h(x) := \fint_x^{x+h} [f(u(y)) - f(D_hU(x))]dy.$$

Thanks to the identity

$$f(s) - f(t) = f'(t)(s-t) + (s-t)^2\int_0^1 (1-\vartheta)f''((1-\vartheta)t + \vartheta s))d\vartheta,$$

we find

$$e_h(x) = \fint_x^{x+h} \lambda_h(x,y)\left(u(y) - D_hU(x)\right)^2 dy, \tag{16}$$

where $\lambda_h(x,y)$ is as in (14). Observe that, by the strong convexity of f, $\lambda_h(x,y) \geq \gamma/2$ for all $(x,y) \in \mathbb{R}^2$ and $h > 0$, so that (13) holds.

By the same bound on λ_h, we also deduce that

$$e_h(x) \geq \frac{\gamma}{2}\fint_x^{x+h} \left(u(y) - D_hU(x)\right)^2.$$

Hence, we get

$$\begin{aligned} E_h(u) \geq & \frac{\gamma}{2}\int_{\mathbb{R}}\fint_x^{x+h}\left(\frac{u(y) - D_hU(x)}{h}\right)^2 dydx \\ \geq & \frac{\gamma}{4}\int_{\mathbb{R}}\fint_x^{x+h}\fint_x^{x+h}\left(\frac{u(z) - u(y)}{h}\right)^2 dzdydx, \end{aligned}$$

where the last inequality follows from the identity

$$\int |\varphi(y)|^2 d\mu(y) = \left|\int \varphi(y)d\mu(y)\right|^2 + \frac{1}{2}\int\int |\varphi(z) - \varphi(y)|^2 d\mu(z)d\mu(y),$$

which holds whenever μ is a probability measure and $\varphi \in L^2(\mu)$. By Fubini's Theorem and neglecting contributions near the boundary, we find the lower bound on the energy:

$$\begin{aligned} E_h(u) \geq & \frac{\gamma}{4}\int_{\mathbb{R}}\fint_{y-h}^{y}\fint_x^{x+h}\left(\frac{u(z) - u(y)}{h}\right)^2 dzdxdy \\ = & \frac{\gamma}{4h}\int_{\mathbb{R}}\int_{y-h}^{y+h}\left(1 - \frac{|z-y|}{h}\right)\left(\frac{u(z) - u(y)}{h}\right)^2 dzdy. \end{aligned}$$

The conclusion (13) is now achieved by the change of variables $r = z - y$. □

Lemma 2 (Compactness)**.** *Assume that f fulfils* (12)*. Let $\{u_h\} \subset Y$ be a sequence of functions such that $E_h(u_h) \leq M$ for some $M \geq 0$. Then, there exist a subsequence $\{u_{h_\ell}\}$ and a function $u \in Y \cap H^1(\mathbb{R})$ such that $u_{h_\ell} \to u$ in $L^2(\mathbb{R})$.*

Proof. We adapt the strategy of [1, Theorem 3.1].

By Lemma 1, we infer that

$$\frac{\gamma}{4}\int_{\mathbb{R}}\int_{-h}^{h} J_h(r)\left(\frac{u_h(y+r) - u_h(y)}{h}\right)^2 drdy \leq M. \tag{17}$$

Observe that $J_h(r)dr$ is a probability measure on $[-h,h]$.

We now introduce the mollified functions $v_h := \rho_h * u_h$, where $\{\rho_h\}$ is the family

$$\rho_h(r) := \frac{1}{ch}\rho\left(\frac{r}{h}\right), \qquad \text{with } c := \int_{\mathbb{R}} \rho(r)dr.$$

Here, $\rho \in C_c^\infty(\mathbb{R})$ is an even kernel, and it is chosen in such a way that its support is contained in $[-1,1]$,

$$0 \le \rho \le J, \quad \text{and} \quad |\rho'| \le J.$$

Note that, for all $h > 0$, $v_h : \mathbb{R} \to \mathbb{R}$ is a smooth function whose support is a subset of $(a-h, b+h)$. Moreover, the family of derivatives $\{ v_h' \}_{h\in(0,1)}$ is uniformly bounded in $L^2(\mathbb{R})$; indeed, since $\int_{\mathbb{R}} \rho'(r)dr = 0$, it holds

$$\begin{aligned}
\int_{\mathbb{R}} |v_h'(y)|^2 \, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h'(r)[u_h(y+r) - u_h(y)]dr \right|^2 dy \\
&\le \int_{\mathbb{R}} \left(\int_{-h}^{h} |\rho_h'(r)| \, |u_h(y+r) - u_h(y)| \, dr \right)^2 dy \\
&\le \frac{1}{c^2} \int_{\mathbb{R}} \left(\int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right| dr \right)^2 dy \\
&\le \frac{1}{c^2} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \left| \frac{u_h(y+r) - u_h(y)}{h} \right|^2 dr dy,
\end{aligned}$$

and thus

$$\int_{\mathbb{R}} |v_h'(y)|^2 \, dy \le \frac{4M}{c^2\gamma}. \tag{18}$$

For all $h \in (0,1)$, let $\tilde{v}_h$ be the restriction of v_h to the interval $(a-1, b+1)$. By Poincaré inequality, (18) entails boundedness in $H_0^1((a-1, b+1))$ of the family $\{ \tilde{v}_h \}_{h\in(0,1)}$, and, in view of Sobolev's Embedding Theorem, this grants in turn that there exists a subsequence $\{ \tilde{v}_{h_\ell} \}$ uniformly converging to some $\tilde{u} \in H_0^1([a-1, b+1])$. Since each $\tilde{v}_{h_\ell}$ is supported in $(a - h_\ell, b + h_\ell)$, we see that $\tilde{u} \in H_0^1(\bar{I})$; therefore, if we set

$$u(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \bar{I}, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that $\{ v_{h_\ell} \}$ converges uniformly to $u \in Y \cap H^1(\mathbb{R})$.

Lastly, to achieve the conclusion, we provide a bound on the L^2-distance between u_h and v_h. Similarly to the previous computations, we have

$$\begin{aligned}
\int_{\mathbb{R}} |v_h(y) - u_h(y)|^2 \, dy &= \int_{\mathbb{R}} \left| \int_{-h}^{h} \rho_h(r)[u_h(y+r) - u_h(y)]dr \right|^2 dy \\
&\le \int_{\mathbb{R}} \int_{-h}^{h} \rho_h(r) \, |u_h(y+r) - u_h(y)|^2 \, dr dy \\
&\le \frac{1}{c} \int_{\mathbb{R}} \int_{-h}^{h} J_h(r) \, |u_h(y+r) - u_h(y)|^2 \, dr dy,
\end{aligned}$$

and, by (17), we get

$$\int_{\mathbb{R}} |v_h(y) - u_h(y)|^2 dy \le \frac{4M}{c\gamma} h^2. \tag{19}$$

Since there exists a subsequence $\{ v_{h_\ell} \}$ uniformly converging to a function $u \in Y \cap H^1(\mathbb{R})$, (19) gives the conclusion. □

Now we can prove statement (1) of Theorem 1.

Proposition 2. *Let f satisfy* (12)*. Then, for any $u \in Y$ and for any family $\{ u_h \} \subset Y$ that converges to u in $L^2(\mathbb{R})$, it holds*

$$E_0(u) \leq \liminf_{h \searrow 0} E_h(u_h). \tag{20}$$

Proof. Fix $u, u_h \in Y$ in such a way that $u_h \to u$ in $L^2(\mathbb{R})$. We can suppose that the inferior limit in (20) is finite, otherwise the conclusion holds trivially. Consequently, up to extracting a subsequence, which we do not relabel, there exists $\lim_{h \searrow 0} E_h(u_h)$ and it is finite. In particular, there exists $M \geq 0$ such that $E_h(u_h) \leq M$ for all $h > 0$, and, by Lemma 2, this yields that $u \in Y \cap H^1(\mathbb{R})$.

We use formula (13) for each u_h, choosing, for $(x, y) \in \mathbb{R}^2$,

$$\varphi(x, y) = \psi\left(x, \frac{y - x}{h}\right), \qquad \text{with } \psi \in C_c^\infty(\mathbb{R}^2).$$

We get

$$\begin{aligned} E_h(u) \geq & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u_h(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y - x}{h}\right) dy dx \\ & - \frac{1}{4} \int_{\mathbb{R}} \fint_x^{x+h} \frac{\psi\left(x, \frac{y-x}{h}\right)^2}{\lambda_h(x, y)} dy dx, \end{aligned} \tag{21}$$

where, coherently with (14),

$$\lambda_h(x, y) := \int_0^1 (1 - \vartheta) f''\left((1 - \vartheta) \fint_x^{x+h} u_h(z) dz + \vartheta u_h(y)\right) d\vartheta \geq \frac{\gamma}{2}.$$

Let us focus on the first quantity on the right-hand side of (21). We have

$$\begin{aligned} & \frac{1}{h} \int_{\mathbb{R}} \fint_x^{x+h} \left(\fint_x^{x+h} u_h(z) dx\right) \psi\left(x, \frac{y - x}{h}\right) dy dx \\ & = \frac{1}{h^3} \int_{\mathbb{R}} \int_x^{x+h} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y - x}{h}\right) dy dz dx \\ & = \frac{1}{h^3} \int_{\mathbb{R}} \int_{z-h}^{z} \int_x^{x+h} u_h(z) \psi\left(x, \frac{y - x}{h}\right) dy dx dz, \end{aligned}$$

and, by similar computations, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \fint_x^{x+h} \frac{u(y) - \fint_x^{x+h} u_h}{h} \psi\left(x, \frac{y - x}{h}\right) dy dx \\ & = \frac{1}{h} \int_{\mathbb{R}} \fint_{y-h}^{y} \fint_x^{x+h} u_h(y) \left[\psi\left(x, \frac{y - x}{h}\right) - \psi\left(x, \frac{z - x}{h}\right)\right] dz dx dy. \end{aligned} \tag{22}$$

By a simple change of variable, we get

$$\begin{aligned} \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{y - x}{h}\right) dz dx &= \int_0^1 \int_0^1 \psi(y - hr, r) dq dr, \\ \fint_{y-h}^{y} \fint_x^{x+h} \psi\left(x, \frac{z - x}{h}\right) dz dx &= \fint_{y-h}^{y} \int_0^1 \psi(x, r) dr dx \\ &= \int_0^1 \int_0^1 \psi(y - hq, r) dq dr, \end{aligned}$$

hence

$$\begin{aligned}\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx\\ &=\int_0^1\int_0^1\frac{\psi(y-hr,r)-\psi(y-hq,r)}{h}dqdr\\ &=-\int_0^1\int_0^1\int_q^r\partial_1\psi(y-hs,r)dsdqdr\\ &=-\int_0^1\int_0^1(r-q)\fint_q^r\partial_1\psi(y-hs,r)dsdqdr.\end{aligned}$$

Being ψ smooth, we have that $\partial_1\psi(y-hs,r)=\partial_1\psi(y,r)+O(h)$ as $h\searrow 0$, uniformly for $s\in[0,1]$. Consequently,

$$\frac{1}{h}\fint_{y-h}^{y}\fint_{x}^{x+h}\left[\psi\left(x,\frac{y-x}{h}\right)-\psi\left(x,\frac{z-x}{h}\right)\right]dzdx=-\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

Plugging this equality in (22) yields

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{u(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}u_h(y)\int_0^1\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)dr+O(h).$$

It is possible to take the limit $h\searrow 0$ in the previous formula, since $u_h\to u$ in $L^2(\mathbb{R})$. We then get

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{u(y)-\fint_x^{x+h}u_h}{h}\psi\left(x,\frac{y-x}{h}\right)dydx=-\int_{\mathbb{R}}\int_0^1u(y)(r-\tfrac{1}{2})\partial_1\psi(y,r)drdy. \tag{23}$$

Now, we turn to the second addendum on the right-hand side of (21). By Fubini's Theorem and a change of variables, we have

$$\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=\int_{\mathbb{R}}\int_0^1\frac{\psi\left(y-hr,r\right)^2}{\lambda_h(y-hr,y)}drdy.$$

The function ψ has compact support and $\lambda_h\geq\gamma/2$ for all $h>0$, therefore we can apply Lebesgue's Convergence Theorem to let $h\searrow 0$ in the previous expression, and we get

$$\begin{aligned}\lim_{h\searrow 0}\int_{\mathbb{R}}\int_0^1\frac{\psi\left(y-hr,r\right)^2}{\int_0^1(1-\vartheta)f''\left((1-\vartheta)\fint_{y-hr}^{y+(1-r)h}u_h(z)dz+\vartheta u_h(y)\right)d\vartheta}drdy\\ =\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{\int_0^1(1-\vartheta)f''(u(y))d\vartheta}drdy,\end{aligned}$$

thus

$$\lim_{h\searrow 0}\int_{\mathbb{R}}\fint_x^{x+h}\frac{\psi(x,\frac{y-x}{h})^2}{\lambda_h(x,y)}dydx=2\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}drdy. \tag{24}$$

Summing up, by (23) and (24), we deduce

$$\liminf_{h\searrow 0}E_h(u_h)\geq-\int_{\mathbb{R}}\int_0^1\left[u(y)\left(r-\frac{1}{2}\right)\partial_1\psi(y,r)drdy+\frac{1}{2}\int_{\mathbb{R}}\int_0^1\frac{\psi(y,r)^2}{f''(u(y))}\right]drdy, \tag{25}$$

for all $\psi\in C_c^\infty(\mathbb{R}^2)$.

We can reach the conclusion from the last inequality by a suitable choice of the test function ψ. To see this, we let $\eta \in C_c^\infty(\mathbb{R})$ and we choose a standard sequence of mollifiers $\{\rho_k\}$. We then set

$$\psi(x,y) = \psi_k(x,y) := \eta(x)\left(\zeta_k(y) - \frac{1}{2}\right), \quad \text{with } \zeta_k(y) := \int_{\mathbb{R}} \rho_k(z-y) z dz,$$

so that (25) reads

$$\begin{aligned}\liminf_{h\searrow 0} E_h(u_h) \geq &- \int_0^1 \left(r - \frac{1}{2}\right)\left(\zeta_k(r) - \frac{1}{2}\right) dr \int_{\mathbb{R}} u(y)\eta'(y)dy \\ &- \frac{1}{2}\int_0^1 \left(\zeta_k(r) - \frac{1}{2}\right)^2 dr \int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy.\end{aligned}$$

Because of the identity $\int_0^1 (r-1/2)^2 dr = 1/12$, letting $k \to +\infty$ yields

$$\begin{aligned}\liminf_{h\searrow 0} E_h(u_h) \geq &- \frac{1}{12}\left[\int_{\mathbb{R}} u(y)\eta'(y)dy + \frac{1}{2}\int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy\right] \\ =&\frac{1}{12}\left[\int_{\mathbb{R}} u'(y)\eta(y)dy - \frac{1}{2}\int_{\mathbb{R}} \frac{\eta(y)^2}{f''(u(y))} dy\right],\end{aligned}$$

where $u' \in L^2(\mathbb{R})$ is the distributional derivative of u, which exists since $u \in H^1(\mathbb{R})$. Recall that, in the previous formula, the test function η is arbitrary, thus, to recover (20), it suffices to take the supremum w.r.t. $\eta \in C_c^\infty(\mathbb{R})$. □

3. Γ-limit in arbitrary dimension

Let us fix an open, bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a function $K\colon \mathbb{R}^d \to [0,+\infty)$ such that

$$\int_{\mathbb{R}^d} K(z)\left(1 + |z|^2\right) dz < +\infty. \tag{26}$$

We require that $K(z) = K(-z)$ for a.e. $z \in \mathbb{R}^d$ and that the support of K contains a sufficiently large annulus centered at the origin. More precisely, let us set

$$\sigma_d := \begin{cases} 1 & \text{when } d = 2, \\ \dfrac{d-2}{d-1} & \text{when } d > 2; \end{cases} \tag{27}$$

we suppose that there exist $r_0 \geq 0$ and $r_1 > 0$ such that $r_0 < \sigma_d r_1$ and

$$\operatorname{ess\,inf}\{\, K(z) : z \in B(0,r_1) \setminus B(0,r_0) \,\} > 0. \tag{28}$$

The simplest case for which (28) holds is when there exists $k > 0$ such that $K(z) \geq k$ for all $z \in B(0,r_1)$.

Let $f\colon [0,+\infty) \to [0,+\infty)$ be a C^2 function such that $f(0) = f'(0) = 0$, and (12) is satisfied. For $u \in H^1(\mathbb{R}^d)$, we define the functionals

$$\begin{aligned}\mathcal{F}_0(u) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(z) f(|\nabla u(x)\cdot \hat{z}|) dz dx, \\ \mathcal{F}_h(u) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_h(y-x) f\left(\frac{|u(y)-u(x)|}{|y-x|}\right) dy dx,\end{aligned}$$

where $K_h(z) := h^{-d}K(z/h)$.

By results in [11], we know that $\mathcal{F}_h(u)$ tends to $\mathcal{F}_0$ as $h \searrow 0$ when $u \in H^1(\mathbb{R}^d)$, and also that $\mathcal{F}_0$ is the Γ-limit of the family $\{\mathcal{F}_h\}$. As before, we are interested in the asymptotics of

$$\frac{\mathcal{F}_0 - \mathcal{F}_h}{h} \quad \text{and} \quad \frac{\mathcal{F}_0 - \mathcal{F}_h}{h^2}$$

Analogously to the 1-dimensional case, we define

$$\mathcal{E}_h(u) := \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h^2}.$$

Let us set

$$X := \{\, u \in H^1(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega \,\} \tag{29}$$

and

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24} \displaystyle\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z)\, |z|^2\, f''(|\nabla u(x) \cdot \hat{z}|)\, \left|\nabla^2 u(x) \hat{z} \cdot \hat{z}\right|^2 dz dx & \text{if } u \in X \cap H^2(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{30}$$

Remark 2 (Radial case)**.** When K is radial, that is $K(z) = \bar{K}(|z|)$ for some $\bar{K} \colon [0, +\infty) \to [0, +\infty)$, we have

$$\mathcal{F}_0(u) = \|K\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f(|\nabla u(x) \cdot e|) d\mathcal{H}^{d-1}(e) dx,$$

$$\mathcal{E}_0(u) = \frac{1}{24} \left(\int_{\mathbb{R}^d} K(z)\, |z|^2\, dz \right) \int_{\mathbb{R}^d} \fint_{\mathbb{S}^{d-1}} f''(|\nabla u(x) \cdot e|)\, \left|\nabla^2 u(x) e \cdot e\right|^2 d\mathcal{H}^{d-1}(e) dx.$$

This Section is devoted to the proof of the following Γ-convergence result:

Theorem 2. *Under the previous assumptions on Ω, K, and f, there holds:*

(1) *For any family $\{\, u_h \,\} \subset X$ such that $\mathcal{E}_h(u_h) \le M$ for some $M > 0$, there exists a subsequence $\{\, u_{h_\ell} \,\}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $\nabla u_{h_\ell} \to \nabla u$ in $L^2(\mathbb{R}^d)$.*

(2) *For any family $\{\, u_h \,\} \subset X$ that converges to $u \in X$ in $H^1(\mathbb{R}^d)$*

$$\mathcal{E}_0(u) \le \liminf_{h \searrow 0} \mathcal{E}_h(u_h).$$

(3) *For any $u \in X$, there exists a family $\{\, u_h \,\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$ with the property that*

$$\limsup_{h \searrow 0} \mathcal{E}_h(u_h) \le \mathcal{E}_0(u).$$

3.1. **Slicing and upper bound.** When the dimension is 1, in virtue of the analysis in Section 2, it is not difficult to derive the Γ-convergence of the functionals $\mathcal{E}_h$.

Corollary 1. *Let $K \colon \mathbb{R} \to [0, +\infty)$ be an even function such that* (26) *holds. For $h > 0$ and $u \in H^1(\mathbb{R})$, let us define the family*

$$\mathcal{E}_h(u) := \frac{1}{h^2} \int_{\mathbb{R}} \int_{\mathbb{R}} K_h(z) \left[f(|u'(x)|) - f\left(\left| \frac{u(x+z) - u(x)}{z} \right| \right) \right] dz dx.$$

Let also $\Omega = (a, b)$ be an open interval, and let $X \subset\in H^1(\mathbb{R})$ be defined as in (29)*. Then, the restrictions of the functionals $\mathcal{E}_h$ to X Γ-converge w.r.t. the $H^1(\mathbb{R})$-topology to*

$$\mathcal{E}_0(u) := \begin{cases} \dfrac{1}{24} \left(\displaystyle\int_{\mathbb{R}} K(z) z^2 dz \right) \int_{\mathbb{R}} f''(u'(x))\, |u''(x)|^2\, dx & \text{if } u \in X \cap H^2(\mathbb{R}), \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. A change of variables gives

$$\mathcal{E}_h(u) = \int_{\mathbb{R}} K(z) z^2 \left[\frac{1}{(hz)^2} \int_{\mathbb{R}} f(|u'(x)|) - f\left(\left| \frac{u(x+hz) - u(x)}{hz} \right| \right) dx \right] dz.$$

Recalling (3), we notice that the quantity between square brackets is equal to $E_{hz}(u')$, therefore the conclusion follows by a straightforward adaptation of the proof of Theorem 1 (see also the proof of Proposition 3). □

Corollary 1 concludes the analysis when $d = 1$, so we may henceforth assume that $d \geq 2$. Our aim is proving that the restrictions to X of the functionals $\mathcal{E}_h$ Γ-converge w.r.t. the $H^1(\mathbb{R}^d)$-topology to $\mathcal{E}_0$. The gist of our proof is a slicing procedure, which amounts to express the d-dimensional energies $\mathcal{E}_h$ as superpositions of the 1-dimensional energies E_h, regarded as functionals on each line of $\mathbb{R}^d$.

Lemma 3 (Slicing)**.** *For $u \in X$, $z \in \mathbb{R}^d \setminus \{0\}$, and $\xi \in \hat{z}^\perp$, we define $w_{\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ as $w_{\hat{z},\xi}(t) := u(\xi + t\hat{z})$. Then, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z}$ and*

$$\mathcal{E}_h(u) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2\, E_{h|z|}(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz, \tag{31}$$

where $E_{h|z|}$ is as in (3) (note that the function f in (3) must be replaced here by $f(|t|)$).

Proof. The formula (31) is an easy consequence of Fubini's Theorem. Indeed, once the direction $\hat{z} \in \mathbb{S}^{d-1}$ is fixed, we can write $x \in \mathbb{R}^d$ as $x = \xi + t\hat{z}$ for some $\xi \in \mathbb{R}^d$ such that $\xi \cdot z = 0$ and $t \in \mathbb{R}$. Using this decomposition, we have

$$\begin{aligned}\mathcal{F}_h(u) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(z) f\left(\frac{|u(x+hz) - u(x)|}{h\,|z|}\right) dz dx \\ &= \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) f\left(\frac{|w_{\hat{z},\xi}(t + h\,|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|}\right) dt d\mathcal{H}^{d-1}(\xi) dz,\end{aligned}$$

whence

$$\mathcal{E}_h(u) = \frac{1}{h^2} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} K(z) \left[f\left(\left|w'_{\hat{z},\xi}(t)\right|\right) - f\left(\frac{|w_{\hat{z},\xi}(t + h|z|) - w_{\hat{z},\xi}(t)|}{h\,|z|}\right)\right] dt d\mathcal{H}^{d-1}(\xi) dz.$$

To obtain (31), it now suffices to multiply and divide the integrands by $|z|^2$. □

The connection with the 1-dimensional case provided by Lemma 3 suggests that the Γ-convergence of the functionals E_h might be exploited to prove Theorem 2. Though, to be able to apply the results of Section 2, we need the functions $w_{\hat{z},\xi}$ in (31) to admit a second order weak derivative for a.e. z and ξ. This poses no real problem for the proof of the upper limit inequality, because we may reason on regular functions; as for the lower limit one, we shall tackle the difficulty in the next subsection by means of a compactness criterion, see Lemma 6 below.

We now establish the following:

Proposition 3. *Let $u \in X \cap H^2(\mathbb{R}^d)$. Then:*

(1) *For any family $\{u_h\} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$, there holds*

$$\mathcal{E}_0(u) \leq \liminf_{h \searrow 0} \mathcal{E}_h(u).$$

(2) *If $u \in X \cap C^3(\mathbb{R}^d)$, then*

$$\mathcal{E}_0(u) = \lim_{h \searrow 0} \mathcal{E}_h(u).$$

Proof. We prove both the assertions by using the slicing formula (31).

(1) For all $h > 0$, $z \in \mathbb{R}^d \setminus \{0\}$, and $\xi \in \hat{z}^\perp$, we let $w_{h;\hat{z},\xi} \colon \mathbb{R} \to \mathbb{R}$ be defined as $w_{h;\hat{z},\xi}(t) := u_h(\xi + t\hat{z})$. Then,

$$\mathcal{E}_h(u_h) = \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2\, E_{h|z|}(w'_{h;\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz,$$

and, by Fatou's Lemma,

$$\liminf_{h \searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2 \left[\liminf_{h \searrow 0} E_{h|z|}(w'_{h;\hat{z},\xi})\right] d\mathcal{H}^{d-1}(\xi) dz. \tag{32}$$

Let us set $w_{\hat{z},\xi}(t) := u(\xi+t\hat{z})$ and note that if $\rho\colon \mathbb{R}^d \to [0,+\infty)$ is any kernel such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$, we may write

$$\int_{\mathbb{R}^d} |\nabla u_h - \nabla u|^2 = \int_{\mathbb{R}^d} \rho(z) \int_{\hat{z}^\perp} \int_{\mathbb{R}} \left| w'_{h;\hat{z},\xi}(t) - w'_{\hat{z},\xi}(t) \right|^2 dt d\mathcal{H}^{d-1}(\xi) dz.$$

Since the left-hand side vanishes as $h \searrow 0$, it follows that there exists a subsequence of $\{ w'_{h;\hat{z},\xi} \}$, which we do not relabel, that converges in $L^2(\mathbb{R})$ to $w'_{\hat{z},\xi}$ for $\mathcal{L}^d$-a.e. $z \in \mathbb{R}^d$ and $\mathcal{H}^{d-1}$-a.e. $\xi \in \hat{z}^\perp$. In particular, by assumption, $w'_{\hat{z},\xi} \in H^1(\mathbb{R})$ for a.e. (z,ξ) and it equals 0 on the complement of some open interval $I_{\hat{z},\xi}$.

From the previous considerations, we see that Proposition 2 can be applied on the right-hand side of (32), yielding

$$\liminf_{h\searrow 0} \mathcal{E}_h(u_h) \geq \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2\, E_0(w'_{\hat{z},\xi}) d\mathcal{H}^{d-1}(\xi) dz = \mathcal{E}_0(u).$$

(2) As above, for any fixed $z \in \mathbb{R}^d \setminus \{0\}$ and $\xi \in \hat{z}^\perp$, we define the function $w_{\hat{z},\xi} \in C^3(\mathbb{R})$ setting $w(t) := u(\xi + t\hat{z})$. Since Ω is bounded, there exists $r > 0$ such that, for any choice of z, $w'_{\hat{z},\xi}(t) = \nabla u(\xi + t\hat{z}) \cdot \hat{z} = 0$ whenever $\xi \in z^\perp$ satisfies $|\xi| \geq r$, while $w'_{\hat{z},\xi}(t)$ is supported in an open interval $I_{\hat{z},\xi}$ if $|\xi| < r$.

By virtue of the slicing formula (31), we obtain

$$\begin{aligned} |\mathcal{E}_h(u) - \mathcal{E}_0(u)| &\leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2 \left| E_{h|z|}(w'_{\hat{z},\xi}) - E_0(w'_{\hat{z},\xi}) \right| d\mathcal{H}^{d-1}(\xi) dz \\ &\leq \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2 \left| E_{h|z|}(w'_{\hat{z},\xi}) - E_0(w'_{\hat{z},\xi}) \right| d\mathcal{H}^{d-1}(\xi) dz \end{aligned}$$

Proposition 1 gets the existence of a constant $c > 0$ and of a continuous, bounded, and increasing function $m\colon [0,+\infty) \to [0,+\infty)$ such that $m(0) = 0$ and

$$|\mathcal{E}_h(u) - \mathcal{E}_0(u)| \leq c \int_{\mathbb{R}^d} \int_{z^\perp} K(z)\, |z|^2\, m(h\,|z|) d\mathcal{H}^{d-1}(\xi) dz.$$

Note here that m can be chosen depending only on ∇u, and not on $\hat{z}$ and ξ.

Recalling (26), to achieve the conclusion it now suffices to appeal to Lebesgue's Convergence Theorem.

□

Remark 3. As observed in Remark 1, from Proposition 3 it follows that the Γ-limit of the rate functionals

$$h\mathcal{E}_h(u) = \frac{\mathcal{F}_0(u) - \mathcal{F}_h(u)}{h}$$

is equal to zero (even with respect to the $L^2(\mathbb{R}^d)$-topology on X).

3.2. **Lower bound and compactness.** Similarly to the 1-dimensional case, we shall prove the compactness of functions with equibounded energy by first establishing a lower bound on the functionals $\mathcal{E}_h$. More precisely, Lemma 4 below shows that, when f is strongly convex, $\mathcal{E}_h(u)$ is greater than a double integral which takes into account, for each $z \in \mathbb{R}^d \setminus \{0\}$, the squared projection of the difference quotients of ∇u in the direction of z. Thanks to the slicing formula, the inequality follows with no effort by applying Lemma 1 on each line of $\mathbb{R}^d$.

We point out that our approach results in the appearance of an effective kernel $\tilde{K}$ in front of the difference quotients. This function stands as a multidimensional counterpart of the kernel J in Lemma 1; actually, $\tilde{K}$ depends both on K and on J (see (33) for the precise definition). In Lemma 5, we shall collect some properties of the effective kernel that will be useful in the proof of Lemma 6.

Lemma 4 (Lower bound on the energy)**.** *Let Ω, K, and f be as above, and suppose that*

$$\tilde{K}(z) := \int_{-1}^{1} J(r) K_{|r|}(z) dr \qquad \text{for a.e. } z \in \mathbb{R}^d, \tag{33}$$

with J as in Lemma 1. Then, it holds

$$\mathcal{E}_h(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{\big(\nabla u(x+hz) - \nabla u(x)\big) \cdot \hat{z}}{h} \right]^2 dx dz. \tag{34}$$

Proof. Thanks to Lemma 3, we can reduce to the 1-dimensional case, and we take advantage of the lower bound provided by Lemma 1. Keeping the notation of Lemma 3, we find

$$\begin{aligned}
\mathcal{E}_h(u) \geq & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \, |z|^2 \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h\,|z|} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
= & \frac{\gamma}{4} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt \mathcal{H}^{d-1}(\xi) dz.
\end{aligned}$$

To cast this bound in the form of (34), we change variables and use Fubini's Theorem:

$$\begin{aligned}
I := & \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-h|z|}^{h|z|} J_{h|z|}(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t+r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
= & \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \int_{-1}^{1} J(r) K(z) \left(\frac{w'_{\hat{z},\xi}(t + h\,|z|\, r) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dr dt d\mathcal{H}^{d-1}(\xi) dz \\
= & \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}(t - h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
& \qquad + \int_{0}^{1} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{r}(z) \left(\frac{w'_{\hat{z},\xi}(t + h\,|z|) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr
\end{aligned}$$

Note that

$$\begin{aligned}
& \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}\big(t + h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
= & \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{-\hat{z},\xi}\big(-(t + h\,|z|)\big) - w'_{-\hat{z},\xi}(-t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr \\
= & \int_{-1}^{0} \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} J(r) K_{-r}(z) \left(\frac{w'_{\hat{z},\xi}\big(t + h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz dr,
\end{aligned}$$

because $w'_{-\hat{z},\xi}(-s) = -w'_{\hat{z},\xi}(s)$ for all $s \in \mathbb{R}$. Thus, we conclude that

$$\begin{aligned}
I = & \int_{\mathbb{R}^d} \int_{z^\perp} \int_{\mathbb{R}} \left(\int_{-1}^{1} J(r) K_{|r|}(z) dr \right) \left(\frac{w'_{\hat{z},\xi}\big(t + h\,|z|\,\big) - w'_{\hat{z},\xi}(t)}{h} \right)^2 dt d\mathcal{H}^{d-1}(\xi) dz \\
& = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{K}(z) \left[\frac{\big(\nabla u(x+hz) - \nabla u(x)\big) \cdot \hat{z}}{h} \right]^2 dx dz,
\end{aligned}$$

which concludes the proof. □

Let us remind that, by assumption, the kernel K is bounded away from 0 in a suitable annulus. The next lemma shows that the effective kernel appearing $\tilde{K}$ in (34) inherits a similar property.

Lemma 5. *Let $\tilde{K}\colon \mathbb{R}^d \to [0,+\infty)$ be as in (33). Then,*

$$\int_{\mathbb{R}^d} \tilde{K}(z)\left(1+|z|^2\right)dz < +\infty. \tag{35}$$

Moreover, if σ_d and r_1 are the constants in (27) and (28), then,

$$\operatorname{ess\,inf}\left\{\tilde{K}(z) : z \in B(0,\sigma_d r_1)\right\} > 0. \tag{36}$$

Proof. The convergence of the integral in (35) follows easily from (26). Indeed, by the definition of $\tilde{K}$, we see that

$$\int_{\mathbb{R}^d} \tilde{K}(z)dz = \int_{-1}^{1}\int_{\mathbb{R}^d} J(r)K_{|r|}(z)dzdr = \int_{\mathbb{R}^d} K(z)dz;$$

analogously, one finds that

$$\int_{\mathbb{R}^d} \tilde{K}(z)\,|z|^2\,dz = c\int_{\mathbb{R}^d} K(z)\,|z|^2\,dz,$$

for some $c>0$.

For what concerns (36), let us set $k \coloneqq \operatorname{ess\,inf}\{\,K(z) : z \in B(0,r_1)\setminus B(0,r_0)\,\}$. In view of (28), $k>0$.

We distinguish between the case $z \in B(0,r_0)$ and the case $z \in B(0,r_1)\setminus B(0,r_0)$. In the first situation, for a.e. $z \in \mathbb{R}^d$,

$$\begin{aligned}\tilde{K}(z) \geq& 2\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} J(r)K_r(z)dr \geq 2k\int_{\frac{|z|}{r_1}}^{\frac{|z|}{r_0}} \frac{1}{r^d}J(r)dr\\ =&\frac{2k}{|z|^{d-1}}\int_{r_0}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds.\end{aligned}$$

When $z \in B(0,r_1)\setminus B(0,r_0)$, instead, similar computations get

$$\tilde{K}(z) \geq 2\int_{\frac{|z|}{r_1}}^{1} J(r)K_r(z)dr = \frac{2k}{|z|^{d-1}}\int_{|z|}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z \in \mathbb{R}^d,$$

so that we obtain

$$\tilde{K}(z) \geq \frac{2k}{|z|^{d-1}}\int_{\max(r_0,|z|)}^{r_1} s^{d-2}\left(1-\frac{|z|}{s}\right)ds \qquad \text{for a.e. } z \in \mathbb{R}^d. \tag{37}$$

When $d=2$, the estimate above becomes

$$\tilde{K}(z) \geq 2k\left[\frac{r_1-\max(r_0,|z|)}{|z|} - \log\left(\frac{r_1}{\max(r_0,|z|)}\right)\right] \qquad \text{for a.e. } z \in \mathbb{R}^d.$$

Exploiting the concavity of the logarithm, we see that the lower bound that we have obtained is strictly positive if $|z| < r_1 = \sigma_2 r_1$.

On the other hand, putting $M \coloneqq \max(r_0,|z|)$ for shortness, if $d \geq 3$, the right-hand side in (37) equals

$$\frac{2k}{(d-1)(d-2)\,|z|^{d-1}}\left[(d-2)\left(r_1^{d-1}-M^{d-1}\right)-(d-1)\,|z|\left(r_1^{d-2}-M^{d-2}\right)\right],$$

and therefore

$$\begin{aligned}\tilde{K}(z) \geq& \frac{2kM^{d-2}}{(d-1)(d-2)\,|z|^{d-1}}\\ &\cdot\left\{\left(\frac{r_1}{M}\right)^{d-2}[(d-2)r_1-(d-1)\,|z|]-[(d-2)M-(d-1)\,|z|]\right\}\end{aligned}$$

for a.e. $z \in \mathbb{R}^d$. When $|z| < \frac{d-2}{d-1} r_1 = \sigma_d r_1$, the quantity between braces is strictly positive if

$$\frac{(M-|z|)d-(2M-|z|)}{(r_1-|z|)d-(2r_1-|z|)} < \left(\frac{r_1}{M}\right)^{d-2}.$$

Observe that both the left-hand side and the right-hand one are strictly increasing in d; also, the left-hand side is bounded above by $(M-|z|)/(r_1-|z|)$, so the last inequality holds if

$$\frac{M-|z|}{r_1-|z|} < \frac{r_1}{M},$$

which, in turn, is true for all $z \in B(0, r_1)$. □

We are now in the position to prove that families with equibounded energy are compact in $H^1(\mathbb{R}^d)$, and that their accumulation points admit second order weak derivatives.

Lemma 6 (Compactness)**.** *Assume that Ω, K, and f are as above. Then, if $\{ u_h \} \subset X$ satisfies $\mathcal{E}_h(u_h) \le M$ for some $M \ge 0$, there exist a subsequence $\{ u_{h_\ell} \}$ and a function $u \in X \cap H^2(\mathbb{R}^d)$ such that $u_{h_\ell} \to u$ in $H^1(\mathbb{R}^d)$.*

Proof. Let $\tilde{k} := \operatorname{ess\,inf} \{ \tilde{K}(z) : z \in B(0, \sigma_d r_1) \}$; Lemma 5 ensures that $\tilde{k} > 0$. We consider a function $\rho \in C_c^\infty([0, +\infty))$ such that

$$\rho(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right),$$

and we further require that

$$0 \le \rho(r) \le \tilde{k} \quad \text{and} \quad |\rho'(r)| \le \tilde{k}.$$

For $h > 0$ and $y \in \mathbb{R}^d$, we set

$$\rho_h(y) := \frac{1}{ch^d} \rho\left(\frac{|y|}{h}\right), \qquad \text{with } c := \int_{\mathbb{R}^d} \rho(|y|) dy,$$

and we introduce the functions $v_h := \rho_h * u_h$, as above.

Each function v_h is a smooth function and, for all $\tilde{h} \in (0, 1)$, its support is contained in

$$\Omega_{\tilde{h}} := \{ x : \operatorname{dist}(x, \Omega) \le 2^{-1/2} \tilde{h} \sigma_d r_1 \}$$

if $h \in (0, \tilde{h})$. In particular, we can choose $\tilde{h}$ so small that $\partial\Omega_{\tilde{h}}$ is still Lipschitz. For such an $\tilde{h}$, we assert that the family $\{ v_h \}_{h \in (0, \tilde{h})}$ is relatively compact in $H_0^1(\Omega_{\tilde{h}})$. In order to prove this, we first remark that

$$\int_{\Omega_{\tilde{h}}} \left|\nabla^2 v_h\right|^2 = \int_{\Omega_{\tilde{h}}} \left|\Delta v_h\right|^2, \tag{38}$$

and next we show that the right-hand side is uniformly bounded.

We observe that $\int_{\mathbb{R}^d} \nabla \rho_h(y) dy = 0$ for all $h > 0$, because ρ is compactly supported. Hence,

$$\begin{aligned}
\|\Delta v_h\|^2_{L^2(\Omega_{\tilde{h}})} &= \int_{\mathbb{R}^d} |\Delta v_h|^2 \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla \rho_h(y) \cdot \big(\nabla u_h(x+y) - \nabla u_h(x)\big) dy \right|^2 dx \\
&\le \int_{\mathbb{R}^d} \left[\frac{1}{ch^{d+1}} \int_{\mathbb{R}^d} \left| \rho'\left(\frac{|y|}{h}\right) \right| \left| \big(\nabla u_h(x+y) - \nabla u_h(x)\big) \cdot \hat{y} \right| dy \right]^2 dx.
\end{aligned}$$

By our choice of ρ and (36), we find

$$\begin{aligned}\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})} &\le \int_{\mathbb{R}^d}\left[\frac{1}{ch}\int_{\mathbb{R}^d}\tilde K_h(y)\left|\big(\nabla u_h(x+y)-\nabla u_h(x)\big)\cdot\hat y\right|dy\right]^2 dx\\ &\le \int_{\mathbb{R}^d}\left[\frac{1}{ch}\int_{\mathbb{R}^d}\tilde K(z)\left|\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot\hat z\right|dz\right]^2 dx\end{aligned}$$

Further, since $\tilde K\in L^1(\mathbb{R}^d)$, Jensen's inequality and Fubini's Theorem yield

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})}\le\frac{\|\tilde K\|_{L^1(\mathbb{R}^d)}}{c^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde K(z)\left[\frac{\big(\nabla u_h(x+hz)-\nabla u_h(x)\big)\cdot\hat z}{h}\right]^2 dxdz.$$

The lower bound (34) entails

$$\|\Delta v_h\|^2_{L^2(\Omega_{\tilde h})}\le\frac{4}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}\mathcal{E}_h(u_h),$$

so that, in view of the assumption $\mathcal{E}_h(u_h)\le M$ and of (38), we get

$$\|\nabla^2 v_h\|^2_{L^2(\Omega_{\tilde h})}\le\frac{4M}{c^2\gamma}\|\tilde K\|_{L^1(\mathbb{R}^d)}. \tag{39}$$

We argue as in the proof of Lemma 2. We recall that, for $h\in(0,\tilde h)$, each v_h vanishes on the complement of $\Omega_{\tilde h}$, and thus, by Poincaré inequality, (39) implies a uniform bound on the norms $\|v_h\|_{H^2_0(\Omega_{\tilde h})}$. As a consequence, by Rellich-Kondrachov Theorem, the family $\{\,\tilde v_h\,\}_{h\in(0,\tilde h)}$ of the restrictions of the functions v_h to $\Omega_{\tilde h}$ admits a subsequence $\{\,\tilde v_{h_\ell}\,\}$ that converges in $H^1_0(\Omega_{\tilde h})$ to a function $\tilde u\in H^2_0(\Omega_{\tilde h})$. Actually, the support of $\tilde u$ is contained in $\bar\Omega$, and, if we put,

$$u(x):=\begin{cases}\tilde u(x) & \text{if } x\in\bar\Omega,\\ 0 & \text{otherwise,}\end{cases}$$

we infer that $\{\,v_{h_\ell}\,\}$ converges in $H^1(\mathbb{R}^d)$ to $u\in X\cap H^2(\mathbb{R}^d)$.

To accomplish the proof, it suffices to show that the L^2 distance between ∇u_h and ∇v_h vanishes when $h\searrow 0$. Since ρ_h has unit $L^1(\mathbb{R}^d)$-norm and it is radial, we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla v_h(x)-\nabla u_h(x)|^2dx &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)-\nabla u_h(x)\big)dy\right|^2 dx\\ &\le\frac14\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\rho_h(y)\big(\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\big)dy\right|^2 dx\\ &\le\frac14\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_h(y)\left|\nabla u_h(x+y)+\nabla u_h(x-y)-2\nabla u_h(x)\right|^2 dydx.\end{aligned}$$

We remark that for any fixed $y\in\mathbb{R}^d\setminus\{\,0\,\}$ and for all $p\in\mathbb{R}^d$, the identity $|p|^2=|p\cdot y|^2+|(\mathrm{Id}-y\otimes y)p|^2$ can be reformulated as

$$\begin{aligned}|p|^2 &= |p\cdot y|^2+\int_{\hat y^\perp}\pi(|\eta|)\,|p\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta)\\ &= |p\cdot y|^2+\frac{1}{h^2}\int_{\hat y^\perp}\pi_h(\eta)\,|p\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta),\end{aligned} \tag{40}$$

where $\pi\colon[0,+\infty)\to[0,+\infty)$ is a continuous function such that

$$\int_{e_d^\perp}\pi(|\eta|)\,|\eta|^2\,d\mathcal{H}^{d-1}(\eta)=1,$$

and $\pi_h(\eta) \coloneqq h^{-d+1}\pi(|\eta|/h)$. We further prescribe that

$$\pi(r) = 0 \qquad \text{if } r \in \left[\frac{\sigma_d r_1}{\sqrt{2}}, +\infty\right)$$

and that $\lim_{r\searrow 0} \eta(r)/r \in \mathbb{R}$.

We apply the formula (40) to $p_h(x,y) \coloneqq \nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)$ and we find that

$$\int_{\mathbb{R}^d} |\nabla v_h(x) - \nabla u_h(x)|^2 \, dx \le \frac{1}{4}\left(I_1 + I_2\right), \tag{41}$$

where

$$I_1 \coloneqq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_h(y)\,|y|^2\,|p_h(x,y)\cdot\hat{y}|^2\,dydx,$$
$$I_2 \coloneqq \frac{1}{h^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho_h(y)\pi_h(\eta)\,|p_h(x,y)\cdot\eta|^2\,d\mathcal{H}^{d-1}(\eta)dydx.$$

We first consider I_1. Keeping in mind that ρ is compactly supported and $\rho(|y|) \le \tilde{k} \le \tilde{K}(y)$ for a.e. $y \in B(0, 2^{-1/2}\sigma_d r_1)$, we get

$$\begin{aligned} I_1 \le \frac{(\sigma_d r_1)^2}{c}\Bigg[&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{K}_h(y)\left|\big(\nabla u_h(x+y) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx \\ &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \tilde{K}_h(y)\left|\big(\nabla u_h(x-y) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\Bigg], \end{aligned}$$

and, by (34),

$$I_1 \le \frac{8(\sigma_d r_1)^2 M}{c\gamma}h^2. \tag{42}$$

As for I_2, we assert that there exist a constant $L > 0$, depending on d, σ_d, r_1, $\tilde{k}$, and c, such that

$$I_1 \le \frac{LM}{\gamma}h^2. \tag{43}$$

To prove the claim, we write the integrand appearing in I_2 as follows:

$$\begin{aligned} p_h(x,y)\cdot\eta =& \big(\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x)\big)\cdot\eta \\ =& \big(\nabla u_h(x+y) + \nabla u_h(x-y) - 2\nabla u_h(x-\eta)\big)\cdot\eta \\ &+ 2\big(\nabla u_h(x-\eta) - \nabla u_h(x)\big)\cdot\eta \\ =& \big(\nabla u_h(x+y) - \nabla u_h(x-\eta)\big)\cdot(\eta+y) \\ &+ \big(\nabla u_h(x-y) - \nabla u_h(x-\eta)\big)\cdot(\eta-y) \\ &- \big(\nabla u_h(x+y) - \nabla u_h(x-y)\big)\cdot y + 2\big(\nabla u_h(x-\eta) - \nabla u_h(x)\big)\cdot\eta. \end{aligned}$$

We plug this identity in the definition of I_2 and we find that

$$\begin{aligned} I_2 \le& \frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy) - \nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\ &+ \frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy) - \nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\ &+ \frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy) - \nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx \\ &+ \frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta) - \nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx. \end{aligned}$$

We estimate separately each of the contributions on the right-hand side.

Let us set $\mathbb{S}^{d-1}_+ := \{\, e \in \mathbb{S}^{d-1} : e \cdot e_d > 0 \,\}$ and $\mathbb{S}^{d-1}_- := \{\, e \in \mathbb{S}^{d-1} : e \cdot e_d < 0 \,\}$. Hereafter, we denote by L any strictly positive constant depending only on d, σ_d, r_1, and on the norms of ρ and π.

Taking advantage of the Coarea Formula, we rewrite the first addendum as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&\qquad = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+y))-\nabla u_h(x)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dxdy \\
&= \int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\int_{e^\perp} r^{d-1}\rho(r)\pi(|\eta|)\left|\big(\nabla u_h(x+h(\eta+re))-\nabla u_h(x)\big)\cdot(\eta+re)\right|^2 d\mathcal{H}^{d-1}(\eta)drdxd\mathcal{H}^{d-1}(e) \\
&= \int_{\mathbb{S}^{d-1}_+}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^2\,|y\cdot e|^{d-1}\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&= \int_{\mathbb{S}^{d-1}_-}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^2\,|y\cdot e|^{d-1}\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e),
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp} \rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx \\
&= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y\cdot e|^{d-1}\,|y|^2\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e).
\end{aligned}$$

Let us recall that $\rho(r)=\eta(r)=0$ if $r\notin[0,2^{-1/2}\sigma_d r_1)$, whence, for any $e\in\mathbb{S}^{d-1}$, the product $\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)$ vanishes outside the cylinder

$$C_e := \{\, y\in\mathbb{R}^d : |y\cdot e|\,,|(\mathrm{Id}-e\otimes e)y| \in [0,2^{-1/2}\sigma_d r_1)\,\} \subset B(0,\sigma_d r_1).$$

We therefore see that the last multiple integral equals

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e} |y\cdot e|^{d-1}\,|y|^2\,\rho(|y\cdot e|)\pi\big(\,|(\mathrm{Id}-e\otimes e)y|\,\big)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e) \\
&\qquad \le L\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}\int_{C_e} \tilde{K}(y)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydxd\mathcal{H}^{d-1}(e) \\
&\qquad\qquad \le \frac{LM}{\gamma}h^2.
\end{aligned}$$

We then obtain

$$\begin{aligned}(44)\quad &\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta+y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &+\frac{4}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x-hy)-\nabla u_h(x-h\eta)\big)\cdot(\eta-y)\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &\leq\frac{LM}{\gamma}h^2.\end{aligned}$$

Next, we have

$$\frac{8}{c}\|\pi\|_{L^1(e_d^\perp)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|y|)\,|y|^2\left|\big(\nabla u_h(x+hy)-\nabla u_h(x)\big)\cdot\hat{y}\right|^2 dydx\leq\frac{LM}{\gamma}h^2, \tag{45}$$

$$\frac{16}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\leq\frac{LM}{\gamma}h^2. \tag{46}$$

The bound in (45) may be deduced as the one in (42), so, to establish (43), we are only left to prove (46). To this aim, let $\psi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ be a test function. By a standard argument and Fubini's Theorem we have that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\mathcal{H}^{d-1}(\eta)dy\\ &=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|y|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\pi(|\eta|)\psi(y,\eta)d\eta dy\\ &=\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d}\frac{\pi(|\eta|)}{|\eta|}\left(\int_{\mathbb{R}^d}\frac{|\eta|}{2\varepsilon}\chi_{\{\,t<\varepsilon\,\}}(|\eta\cdot y|)\rho(|y|)\,|y|\,\psi(y,\eta)dy\right)d\eta\\ &=\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\,\psi(y,\eta)d\mathcal{H}^{d-1}(y)d\eta.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{y}^\perp}\rho(|y|)\pi(|\eta|)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(\eta)dydx\\ &=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\hat{\eta}^\perp}\frac{\pi(|\eta|)}{|\eta|}\rho(|y|)\,|y|\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\mathcal{H}^{d-1}(y)d\eta dx\\ &\leq L\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\tilde{K}(\eta)\left|\big(\nabla u_h(x+h\eta)-\nabla u_h(x)\big)\cdot\eta\right|^2 d\eta dx\end{aligned}$$

(recall that we assume $\lim_{r\searrow 0}\eta(r)/r$ to be finite). In view of the bound on the energy, we retrieve (46).

In conclusion, from (41), (42), and (43), we obtain

$$\int_{\mathbb{R}^d}\left|\nabla v_h(x)-\nabla u_h(x)\right|^2 dx\leq\frac{LM}{\gamma}h^2,$$

as desired. This concludes the proof. □

Remark 4. By choosing $u_h=u$ in Lemma 6 we obtain a criterion for an H^1-function to belong to $H^2(\mathbb{R}^d)$, namely a function $u\in X$ is in $H^2(\mathbb{R}^d)$ if and only if $\mathcal{E}_h(u)\leq M$ for some $M>0$ and for all h small enough.

We can now conclude the proof of Theorem 2.

Proof of Theorem 2. The compactness result in statement (1) of Theorem 2 is contained in Lemma 6.

The upper limit inequality (3) follows by Proposition 3 and a standard density argument, as in the 1-dimensional case.

As for the lower limit inequality (2), for any $u \in X$ and for any family $\{ u_h \} \subset X$ that converges to u in $H^1(\mathbb{R}^d)$, we may focus on the situation when there exists $M \geq 0$ such that $|\mathcal{E}_h(u_h)| \leq M$ for all $h > 0$. By Lemma 6, we have that $u \in H^2(\mathbb{R}^d)$. Then, (2) follows by Proposition 3. □

Remark 5. We conclude by noticing that the Γ-convergence result in Theorem 2, statements (2) and (3), still holds if we replace X with $H^1(\mathbb{R}^d)$, with essentially the same proof. On the other hand, being $\mathbb{R}^d$ non-compact, the compactness result in statement (1) of Theorem 2 does not hold in this case.

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