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Existence and multiplicity results for a new $p(x)$ –Kirchhoff problem with variable exponent

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Abstract

In this work, we study existence and multiplicity results for the following nonlocal $p(x)$ –Kirchhoff problem with variable exponent:

$$\begin{cases} -\left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda |u|^{p(x)-2} u + g(x, u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \quad (0.1)$$

where $a \geq b > 0$ are constants, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, λ is a real parameter and g is a continuous function. The analysis developed in this paper corresponds to propose an approach based on the idea of considering a new nonlocal term which is presents interesting difficulties.

Keywords: Variable exponents; New nonlocal Kirchhoff equations; $p(x)$ -Laplacian operator Palais-Smale condition; Mountain Pass theorem; Fountain Theorem.

PACS: Primary: 35J55, 35J65; Secondary: 35B65.

1. Introduction and statement of main result

In this paper we deal with the question of the existence of nontrivial weak solution for the following problem

$$\begin{cases} -\left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda |u|^{p(x)-2} u + g(x, u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, $a, b > 0$ are constants, g is a continuous function satisfying conditions which will be stated later, $\lambda > 0$ is a

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real parameter and $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$ -Laplacian operator, that is,

$$\Delta_{p(x)} = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \sum_{i=1}^N \left(|\nabla u|^{p(x)-2} \frac{\partial u}{\partial x_i} \right),$$

which is not homogeneous and related to variable exponent Lebesgue space $L^{p(x)}(\Omega)$ and variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. These facts imply some difficulties. For example, some classical theories and methods, including the Lagrange multiplier theorem and the theory of Sobolev space, cannot be applied. We call (1.1) a problem of Kirchhoff type because of the appearance of the term $b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx$ which makes the study of (1.1) interesting.

In the previous decades, the Kirchhoff type problem (1.1) with $p(x) \equiv 2$ has been object of intensive research as its strong relevance in applications (see [27, 26, 36]). Indeed, the study of Kirchhoff type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. More precisely, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

where ρ , p_0 , h , E , L are constants which represent some physical meanings respectively. Equation (1.2) extends the classical D'Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations.

Since the variable exponent spaces were thoroughly studied by Kováčik and Rákosník [25], they have been used in the previous decades to model various phenomena. In the studies of a class of non-standard variational problems and PDEs, variable exponent spaces play an important role such as in electrorheological fluids [34, 33, 32], thermorheological fluids [6], image processing [1, 9, 28] and so on. In recent years, there has been a great deal work dealing with problem (1.1), specially on the existence, multiplicity, uniqueness and regularity of solutions. Some important and interesting results can be found, for example, in [4, 5, 3, 2, 7, 10, 8, 11, 12, 13, 15, 18, 21, 19, 22, 23, 24, 25, 29, 30, 31, 39] and references therein.

At first, the eigenvalues of $p(x)$ -Laplacian Dirichlet problem were studied in [17], i.e., if $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, the Rayleigh quotient

$$\lambda_{p(\cdot)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx}{\int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx} \quad (1.3)$$

is zero in general, and only under some special conditions $\lambda_{p(\cdot)} > 0$ holds. For example, when $\Omega \subset \mathbb{R}(N = 1)$ is an interval, results show that $\lambda_{p(\cdot)} > 0$ if and only if $p(\cdot)$ is monotone. We all know that $\lambda_p > 0$ plays a very important role in the study of p -Laplacian problems.

Motivated by the papers mentioned above, our main purpose is to consider the perturbed problem (1.1) with a new nonlocal term $a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx$ which is presents interesting difficulties. The key argument in our main result is to prove that the energy functional J (which appeared in (2.2)) of problem (1.1) possesses a Mountain Pass energy c . To deal with the difficulty caused by the noncompactness due to the Kirchhoff function term, we should estimate precisely the value of c and give a threshold value (see Lemma 3.1 below) under which the Palais-Smale condition at the level c for J is satisfied. **So the variational technique for problem (1.1) becomes more delicate.** We will obtain a nontrivial weak solution by using the Mountain Pass theorem. To our best knowledge, the present papers results are not covered in the literature.

Next we assume that the nonlinearity $g(x, t) \in C(\overline{\Omega} \times \mathbb{R})$ satisfies the following assumptions:

(g₁) : the subcritical growth condition:

$$|g(x, s)| \leq C(1 + |s|^{q(x)-1}), \text{ for all } (x, s) \in \Omega \times \mathbb{R},$$

where $C > 0$ and $p(x) < q(x) < p^*(x)$;

$$(g_2) : \lim_{s \rightarrow 0} \frac{g(x, s)}{|s|^{p(x)-2}s} = 0;$$

(g₃) : there exist $s_A > 0$ and $\theta \in (p^+, \frac{2p^-}{p^+})$ such that

$$0 < \theta G(x, s) \leq sg(x, s), \quad \forall |s| \geq s_A, \quad \forall x \in \Omega,$$

$$\text{where } G(x, s) = \int_0^s g(x, t) dt.$$

(g₄) : $g(x, -s) = -g(x, s)$ for all $(x, s) \in \Omega \times \mathbb{R}$.

Now, we state our main result as follows.

Theorem 1.1. *Suppose that the function $q \in C(\overline{\Omega})$ satisfies*

$$1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p^*(x), \quad (1.4)$$

then for any $\lambda \in \mathbb{R}$, with (g₁)-(g₃) satisfied, (1.1) has a nontrivial weak solution.

Theorem 1.2. *Suppose that the function $q \in C(\overline{\Omega})$ satisfies*

$$1 < p^- < p(x) < p^+ < 2p^- < q^- < q(x) < p^*(x),$$

then for any $\lambda \in \mathbb{R}$, with (g₁)-(g₄) satisfied, (1.1) has infinitely many solutions $\{u_n\}$ such that $I(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

Remark 1.1. *The hypothesis (g₃) is known as the Ambrosetti-Rabinowitz's superlinear condition (see [11]). Moreover, the condition (g₃) ensures that the Euler Lagrange's functional associated to the problem (1.1) possesses the geometry of Mountain Pass Theorem and, also, it guarantees the boundedness of the Palais Smale sequences corresponding to the Euler-Lagrange's functional.*

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 4, we prove Theorem 1.1. In the following, $|\cdot|$ denotes the Lebesgue measure in Ω and C (respectively C_ϵ) denotes always a generic positive constant independent of n and ϵ (respectively independent of n), even their value could be changed from one line to another one.

2. Preliminaries on variable exponent spaces

In order to discuss problem (1.1), we need some theories on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which we call generalized Lebesgue Sobolev spaces. Let Ω be a bounded domain of \mathbb{R}^N , denote $C_+(\overline{\Omega}) = \{p(x); p(x) \in C(\overline{\Omega}), p(x) > 1, \forall x \in \overline{\Omega}\}$ and $p^- = \inf_{\Omega} p(x) \leq p(x) \leq p^+ = \sup_{\Omega} p(x) < N$.

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : u \text{ is a measurable real-valued function such that } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = |u|_{p(\cdot)} = \inf \left\{ \mu > 0; \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

which is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces we refer to [20, 25, 38].

Proposition 2.1. [38] *The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is separable, uniformly convex, reflexive and its conjugate space is $(L^{q(x)}(\Omega), |\cdot|_{q(x)})$ where $q(x)$ is the conjugate function of $p(x)$ i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For all $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder's type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)}$$

holds true.

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \leq p_2$ in Ω then there exists a continuous embedding $L^{p_2(x)}(\Omega) \rightarrow L^{p_1(x)}(\Omega)$.

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the $m(\cdot)$ -modular of the $L^{p(\cdot)}(\Omega)$ space, which is the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u|^{p(x)} dx.$$

Provide a reference to Lemma 2.1. Use for example [14]

Lemma 2.1. *If $u_n, u \in L^{p(\cdot)}$ and $p_+ < +\infty$, then the following properties hold:*

1. $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^+}$;
2. $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq |u|_{p(\cdot)}^{p^-}$;
3. $|u|_{p(\cdot)} < 1$ (respectively $= 1; > 1$) $\iff \rho_{p(\cdot)}(u) < 1$ (respectively $= 1; > 1$);
4. $|u_n|_{p(\cdot)} \rightarrow 0$ (respectively $\rightarrow +\infty$) $\iff \rho_{p(\cdot)}(u_n) \rightarrow 0$ (respectively $\rightarrow +\infty$);

$$5. \lim_{n \rightarrow \infty} \|u_n - u\|_{p(x)} = 0 \iff \lim_{n \rightarrow \infty} \rho_{p(\cdot)}(u_n - u) = 0.$$

The Sobolev space with variable exponent $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) := \left\{ u : \Omega \subset \mathbb{R}^N \rightarrow \mathbb{R} : u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

Then $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. In this way, $L^{p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ separable and reflexive Banach spaces. For more details, we refer to [14, 16, 18]. Moreover, define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

The following results are proved in [18].

Proposition 2.2. (Sobolev embedding). For $p, q \in C_+(\overline{\Omega})$ such that $1 \leq q(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we replace \leq with $<$, the embedding is compact.

Proposition 2.3. (Poincaré inequality). There is a constant $C > 0$, such that

$$\|u\|_{L^{p(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)} \quad (2.1)$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Remark 2.1. By Proposition 2.3, we know that $\|\nabla u\|_{L^{p(x)}(\Omega)}$ and $\|u\|_{W^{1,p(x)}(\Omega)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Lemma 2.2. [21] Denote

$$A(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

Then, $A(u) \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ and the derivative operator A' of A is

$$\langle A'(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx \quad \text{for all } u, v \in W_0^{1,p(x)}(\Omega),$$

and we have

1. A is a convex functional;
2. $A' : W_0^{1,p(x)}(\Omega) \rightarrow (W_0^{-1,p'(x)}(\Omega)) = (W_0^{1,p(x)}(\Omega))^*$ is a bounded homeomorphism and strictly monotone operator and the conjugate exponent satisfies $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$;

3. A' is a mapping of type S_+ , namely: $u_n \rightharpoonup u$ and $\limsup \langle A'(u_n), u_n - u \rangle \leq 0$, imply $u_n \rightarrow u$ (strongly) in $W_0^{1,p(x)}(\Omega)$.

Definition 2.1. We call that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1), if

$$\left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - \lambda \int_{\Omega} |u|^{p(x)-2} u \varphi dx = \int_{\Omega} g(x, u) \varphi dx,$$

where $\varphi \in W_0^{1,p(x)}(\Omega)$.

The energy functional $J : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$ associated with problem (1.1)

$$J(u) = a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx, \quad (2.2)$$

for all $u \in W_0^{1,p(x)}(\Omega)$ is well defined and of C^1 class on $W_0^{1,p(x)}(\Omega)$. Moreover, we have

$$\begin{aligned} \langle J'(u), \varphi \rangle &= \left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - \lambda \int_{\Omega} |u|^{p(x)-2} u \varphi dx \\ &\quad - \int_{\Omega} g(x, u) \varphi dx, \end{aligned} \quad (2.3)$$

for all $u, \varphi \in W_0^{1,p(x)}(\Omega)$. Hence, we can notice that the critical points of the functional J are the weak solutions for problem (1.1). In order to simplify the presentation we will denote the norm of $W_0^{1,p(x)}(\Omega)$ by $\|\cdot\|$ instead of $\|\cdot\|_{W_0^{1,p(x)}(\Omega)}$.

3. The Palais-Smale Compactness Condition

Recall now the definition of the Palais-Smale compactness condition.

Definition 3.1. Let $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ be a Banach space and $J \in C^1(W_0^{1,p(x)}(\Omega))$, given $c \in \mathbb{R}$, we say that J satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ (" $(PS)_c$ condition" for short) if any sequence $\{u_n\} \in W_0^{1,p(x)}(\Omega)$ satisfying

$$J(u_n) \rightarrow c \text{ and } J'(u_n) \rightarrow 0 \text{ in } W^{-1,p'(x)}(\Omega) \text{ as } n \rightarrow \infty, \quad (3.1)$$

has a convergent subsequence.

First we investigate the compactness conditions for the functional J .

Lemma 3.1. Assume (g_1) - (g_3) hold. The functional J satisfies the $(PS)_c$ condition where precisely $c < \frac{a^2}{2b}$.

Proof. We proceed by two steps.

Step 1. We prove that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$. Let $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$ be a $(PS)_c$ sequence

such that $c < \frac{a^2}{2b}$.

• For $\lambda \leq 0$. From (3.1) and (g_3) , for n large enough, we have

$$\begin{aligned}
C + \|u_n\| &\geq \theta J(u_n) - \langle J'(u_n), u_n \rangle \\
&\geq \theta \left(a \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u_n|^{p(x)} dx - \int_{\Omega} G(x, u_n) dx \right) \\
&\quad - \left(\left[a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right] \int_{\Omega} |\nabla u_n|^{p(x)} dx - \lambda \int_{\Omega} |u_n|^{p(x)} dx - \int_{\Omega} g(x, u_n) u_n dx \right) \\
&\geq a \left(\frac{\theta}{p^+} - 1 \right) \int_{\Omega} |\nabla u_n|^{p(x)} dx + b \left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+} \right) \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right)^2 - \lambda \left(\frac{\theta}{p^-} - 1 \right) \int_{\Omega} |u_n|^{p(x)} dx - C|\Omega|,
\end{aligned}$$

where $|\Omega| = \int_{\Omega} dx$. Since $\lambda \leq 0$, then we deduce that

$$C + \|u_n\| \geq a \left(\frac{\theta}{p^+} - 1 \right) \|u_n\|^{p^-} + b \left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+} \right) \|u_n\|^{2p^-} - C|\Omega|.$$

It follows from (1.5) that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

• For $\lambda > 0$. Arguing by contradiction we assume that, passing eventually to a subsequence, still denote by $\{u_n\}$, we have $\|u_n\| \rightarrow +\infty$ as $n \rightarrow +\infty$. From (3.1) and (g_3) , for n large enough, we have

$$\begin{aligned}
C + \|u_n\| &\geq \theta J(u_n) - \langle J'(u_n), u_n \rangle \\
&\geq a \left(\frac{\theta}{p^+} - 1 \right) \int_{\Omega} |\nabla u_n|^{p(x)} dx + b \left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+} \right) \left(\int_{\Omega} |\nabla u_n|^{p(x)} dx \right)^2 - \lambda \left(\frac{\theta}{p^-} - 1 \right) \int_{\Omega} |u_n|^{p(x)} dx - C|\Omega|,
\end{aligned}$$

Thus, the last inequality together with (2.1) imply

$$C + \|u_n\| + \lambda C \left(\frac{\theta}{p^-} - 1 \right) \|u_n\|^{p^+} \geq a \left(\frac{\theta}{p^+} - 1 \right) \|u_n\|^{p^-} + b \left(\frac{-\theta}{2p^{-2}} + \frac{1}{p^+} \right) \|u_n\|^{2p^-} - C|\Omega|.$$

Dividing the above inequality by $\|u_n\|^{p^+}$, taking into account (1.5) holds true and passing to the limit as $n \rightarrow \infty$, we obtain a contradiction. It follows that $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Step 2. Here, we will prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p(x)}(\Omega)$. It follows from Proposition 2.2 that the embedding

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact, where $1 \leq s(x) < p(x)^*$. Going if necessary to a subsequence, there exists $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p(x)}(\Omega), \quad u_n \rightarrow u \text{ in } L^{s(x)}(\Omega), \quad u_n(x) \rightarrow u(x), \text{ a.e. in } \Omega. \quad (3.2)$$

By Hölder's inequality and (3.2), we have

$$\begin{aligned} \left| \int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx \right| &\leq \int_{\Omega} |u_n|^{p(x)-1} |u_n - u| dx \\ &\leq \left\| |u_n|^{p(x)-1} \right\|_{\frac{p(x)}{p(x)-1}} \|u_n - u\|_{p(x)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

and thus,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx = 0. \quad (3.3)$$

By virtue of conditions (g_1) and (g_2) , one has for every $\epsilon \in (0, 1)$ there exists $C_{\epsilon} > 0$ such that

$$|g(x, u_n)| \leq \epsilon |u_n|^{p(x)-1} + C_{\epsilon} |u_n|^{q(x)-1}. \quad (3.4)$$

By (3.4) and Proposition 2.2, it follows

$$\begin{aligned} \left| \int_{\Omega} g(x, u_n) (u_n - u) dx \right| &\leq \int_{\Omega} \epsilon |u_n|^{p(x)-1} |u_n - u| + C_{\epsilon} |u_n|^{q(x)-1} |u_n - u| dx \\ &\leq \epsilon \left\| |u_n|^{p(x)-1} \right\|_{\frac{p(x)}{p(x)-1}} \|u_n - u\|_{p(x)} + C_{\epsilon} \epsilon \left\| |u_n|^{q(x)-1} \right\|_{\frac{q(x)}{q(x)-1}} \|u_n - u\|_{q(x)} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

which shows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} g(x, u_n) (u_n - u) dx = 0. \quad (3.5)$$

From (3.1), we have

$$\langle J'(u_n), u_n - u \rangle \rightarrow 0.$$

Thus,

$$\begin{aligned} \langle J'(u_n), u_n - u \rangle &= \left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \\ &\quad - \lambda \int_{\Omega} |u_n|^{p(x)-2} u_n (u_n - u) dx - \int_{\Omega} g(x, u_n) (u_n - u) dx \rightarrow 0. \end{aligned}$$

So, we deduce from (3.3) and (3.5) that

$$\left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right) \int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0. \quad (3.6)$$

Since $\{u_n\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, passing to a subsequence, if necessary, we may assume that

$$\int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow t_0 \geq 0 \text{ as } n \rightarrow \infty.$$

Case 1. If $t_0 = 0$ then $\{u_n\}$ converges strongly to $u = 0$ in $W_0^{1,p(x)}(\Omega)$ and the proof is finished.

Case 2. If $t_0 > 0$ we obtain two subcases:

Subcase 1. If $t_0 \neq \frac{a}{b}$ then $a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0$ is not true and any subsequence of $\{a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0\}$ does not converge to zero. Therefore, there exists $\delta > 0$ such that $\left| a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right| > \delta > 0$ when n is large enough. So, it is clear that

$$\{a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0\} \text{ is bounded.} \quad (3.7)$$

Subcase 2.¹ If $t_0 = \frac{a}{b}$ then $a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0$.

We define

$$\varphi(u) = \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx + \int_{\Omega} G(x, u) dx, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Then,

$$\langle \varphi'(u), v \rangle = \lambda \int_{\Omega} |u|^{p(x)-2} uv dx + \int_{\Omega} g(x, u) v dx, \quad \forall u, v \in W_0^{1,p(x)}(\Omega).$$

It follows that

$$\langle \varphi'(u_n) - \varphi'(u), v \rangle = \lambda \int_{\Omega} (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) v dx + \int_{\Omega} (g(x, u_n) - g(x, u)) v dx.$$

To complete this proof we need the following lemma.

Lemma 3.2. Let $u_n, u \in W_0^{1,p(x)}(\Omega)$ such that (3.2) holds. Passing to a subsequence if necessary, we have:

- (i) $\int_{\Omega} (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) v dx = 0$.
- (ii) $\lim_{n \rightarrow \infty} \int_{\Omega} |g(x, u_n) - g(x, u)| |v| dx = 0$.
- (iii) $\langle \varphi'(u_n) - \varphi'(u), v \rangle \rightarrow 0, v \in W_0^{1,p(x)}(\Omega)$.

Proof. From (3.2) we have $u_n \rightarrow u$ in $L^{p(x)}(\Omega)$ which implies that

$$|u_n|^{p(x)-2} u_n \rightarrow |u|^{p(x)-2} u \text{ in } L^{\frac{p(x)}{p(x)-1}}(\Omega). \quad (3.8)$$

Due to Hölder's inequality, we have

$$\begin{aligned} \left| \int_{\Omega} (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) v dx \right| &\leq \int_{\Omega} \left| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right| |v| dx \\ &\leq \left\| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right\|_{\frac{p(x)}{p(x)-1}} \|v\|_{p(x)} \\ &\leq C \left\| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right\|_{\frac{p(x)}{p(x)-1}} \|v\| \\ &\rightarrow 0. \end{aligned} \quad (3.9)$$

¹This case doesn't exist if the Kirchhoff function is given by $a + b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx$.

By a exactly slight modification of the proof above, we can prove part (ii) so we omit proof details.

$$\int_{\Omega} |g(x, u_n) - g(x, u)| |v| dx \leq \int_{\Omega} [\epsilon(|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) + C_{\epsilon}(|u_n|^{q(x)-1} - |u|^{q(x)-1})] |v| dx \rightarrow 0.$$

Therefore, part (iii) follows by combining parts (i) and (ii). Consequently, $\|\varphi'(u_n) - \varphi'(u)\|_{W^{-1, p'(x)}} \rightarrow 0$ and $\varphi'(u_n) \rightarrow \varphi'(u)$.

We now return to the proof of Subcase 2:

According to Lemma 3.2 and while $\langle J'(u), u \rangle = \left(a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi dx - \langle \varphi'(u), v \rangle$, $\langle J'(u), u \rangle \rightarrow 0$ and $a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \rightarrow 0$, hence $\varphi'(u_n) \rightarrow 0$ ($n \rightarrow \infty$), i.e.,

$$\langle \varphi'(u), v \rangle = \lambda \int_{\Omega} |u|^{p(x)-2} u v dx + \int_{\Omega} g(x, u) v dx, \quad \forall v \in W_0^{1, p(x)}(\Omega),$$

and then we have

$$\lambda |u(x)|^{p(x)-2} u(x) + g(x, u(x)) = 0 \text{ for a.e. } x \in \Omega$$

by the fundamental lemma of the variational method (see [37]). It follows that $u = 0$. So

$$\varphi(u_n) = \lambda \int_{\Omega} \frac{1}{p(x)} |u_n|^{p(x)} dx + \int_{\Omega} G(x, u_n) dx \rightarrow \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx + \int_{\Omega} G(x, u) dx = 0.$$

Hence, we see if $t_0 = \frac{a}{b}$ that

$$J(u_n) = a \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u_n|^{p(x)} dx - \int_{\Omega} G(x, u_n) dx \rightarrow \frac{a^2}{2b}.$$

This is a contradiction since $J(u_n) \rightarrow c < \frac{a^2}{2b}$, then $a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0$ is not true and similarly to subcase 1 we have

$$\left\{ a - b \int_{\Omega} \frac{1}{p(x)} |\nabla u_n|^{p(x)} dx \rightarrow 0 \right\} \text{ is bounded.} \quad (3.10)$$

So, it follows from the two cases above that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) dx \rightarrow 0.$$

Invoking now the S_+ condition (see lemma 2.2) we deduce that $\|u_n\| \rightarrow \|u\|$ as $n \rightarrow \infty$ which means that J satisfies the $(PS)_c$ condition. \square

Remark 3.1. The $(PS)_c$ condition is not satisfied for $c > \frac{a^2}{2b}$.

Indeed,

$$J(u) \leq a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 \leq \frac{a^2}{2b}$$

and so if $\{u_n\}$ is a $(PS)_c$ sequence of J , then we have $c \leq \frac{a^2}{2b}$, which is a contradiction.

4. Mountain Pass Solution

Proof of Theorem 1.1. To prove the conditions of the Mountain Pass theorem (see e.g., [37]), we need to verify the following lemmas.

Lemma 4.1. *Assume that g satisfies (g_1) and (g_2) . Then there exist $\rho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha > 0$, for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$.*

Proof.

• For $\lambda \leq 0$. By assumptions (g_1) and (g_2) , we have

$$|G(x, u)| \leq \frac{\epsilon}{p(x)}|u|^{p(x)} + \frac{C_\epsilon}{q(x)}|u|^{q(x)}. \quad (4.1)$$

Let $\epsilon = \frac{1}{8}a\lambda_{p(x)}$ and $u \in W_0^{1,p(x)}(\Omega)$ be such that $\|u\| = \rho \in (0, 1)$. By considering Lemma 2.1, Proposition 2.2 and (1.5), we deduce that

$$\begin{aligned} J(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx \\ &\geq a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \epsilon \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx - C_\epsilon \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx. \\ &\geq \left(a - \frac{\epsilon}{\lambda_{p(x)}} \right) \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \frac{CC_\epsilon}{q^-} \int_{\Omega} |\nabla u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \left(a - \frac{\epsilon}{\lambda_{p(x)}} \right) \rho_{p(x)}(\nabla u) - \frac{b}{2p^{-2}} (\rho_{p(x)}(\nabla u))^2 - \frac{CC_\epsilon}{q^-} \rho_{q(x)}(\nabla u) \\ &\geq \frac{1}{p^+} \left(a - \frac{\epsilon}{\lambda_{p(x)}} \right) \|u\|^{p^+} - \frac{b}{2p^{-2}} \|u\|^{2p^-} - \frac{CC_\epsilon}{q^-} \|u\|^{q^-} \\ &\geq \left(\frac{7a}{8p^+} - \frac{b}{2p^{-2}} \|u\|^{2p^- - p^+} - \frac{CC_\epsilon}{q^-} \|u\|^{q^- - p^+} \right) \|u\|^{p^+}. \end{aligned}$$

We can choose ρ sufficiently small (i.e. ρ such that $\frac{7a}{8p^+} - \frac{b}{2p^{-2}} \rho^{2p^- - p^+} - \frac{CC_\epsilon}{q^-} \rho^{q^- - p^+} > 0$), so that

$$I(u) \geq \rho^{p^+} \left(\frac{7a}{8p^+} - \frac{b}{2p^{-2}} \rho^{2p^- - p^+} - \frac{CC_\epsilon}{q^-} \rho^{q^- - p^+} \right) =: \alpha > 0.$$

• For $\lambda > 0$. Let $\epsilon > 0$ be small enough such that $\frac{1}{2p^+} \left(a - \frac{\lambda}{\lambda_{p(x)}} \right) = \frac{\epsilon}{\lambda_{p(x)} p^-}$.

Let $\rho \in (0, 1)$ and $u \in W_0^{1,p(x)}(\Omega)$ be such that $\|u\| = \rho$. By considering Lemma 2.1, Proposition

2.2, (1.5) and (4.1), we deduce that

$$\begin{aligned}
J(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx \\
&\geq a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \frac{\lambda}{\lambda_{p(x)}} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \\
&\quad - \epsilon \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} dx - C_{\epsilon} \int_{\Omega} \frac{|u|^{q(x)}}{q(x)} dx \\
&\geq \left(a - \frac{\lambda}{\lambda_{p(x)}} \right) \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 \\
&\quad - \frac{\epsilon}{\lambda_{p(x)}} \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{CC_{\epsilon}}{q^{-}} \int_{\Omega} |\nabla u|^{q(x)} dx \\
&\geq \left(\frac{1}{p^{+}} \left(a - \frac{\lambda}{\lambda_{p(x)}} \right) - \frac{\epsilon}{\lambda_{p(x)} p^{-}} \right) \rho_{p(x)}(\nabla u) - \frac{b}{2p^{-2}} (\rho_{p(x)}(\nabla u))^2 - \frac{CC_{\epsilon}}{q^{-}} \rho_{q(x)}(\nabla u) \\
&\geq \left(\frac{1}{p^{+}} \left(a - \frac{\lambda}{\lambda_{p(x)}} \right) - \frac{\epsilon}{\lambda_{p(x)} p^{-}} \right) \|u\|^{p^{+}} - \frac{b}{2p^{-2}} \|u\|^{2p^{-}} - \frac{CC_{\epsilon}}{q^{-}} \|u\|^{q^{-}} \\
&\geq \left(\frac{1}{2p^{+}} \left(a - \frac{\lambda}{\lambda_{p(x)}} \right) - \frac{b}{2p^{-2}} \|u\|^{2p^{-}-p^{+}} - \frac{CC_{\epsilon}}{q^{-}} \|u\|^{q^{-}-p^{+}} \right) \|u\|^{p^{+}}.
\end{aligned}$$

Set

$$\lambda^{*} = \frac{qp^{-2} \lambda_{p(x)} a - bp^{+} q^{-} \rho^{2p^{-}-p^{+}} - 2CC_{\epsilon} p^{-2} \rho^{q^{-}-p^{+}}}{q^{-} p^{-2}} \text{ and } \alpha = \lambda^{*} \rho^{p^{+}}. \quad (4.2)$$

Then, we conclude that for any $\lambda \in (0, \lambda^{*})$, there exists $\alpha > 0$ such that for any $u \in W_0^{1,p(x)}(\Omega)$ with $\|u\| = \rho$ we have $J(u) \geq \alpha > 0$. \square

Lemma 4.2. Assume that g satisfies (g_3) . Then there exists $e \in W_0^{1,p(x)}(\Omega)$ with $\|e\| > \rho$ (where ρ is given in Lemma 4.1) such that $J(e) < 0$.

Proof. In view of (g_3) we have for all $A > 0$, there is $C_A > 0$ such that

$$G(x, u) \geq A|u|^{\theta} - C_A, \quad \forall (x, u) \in \Omega \times \mathbb{R}. \quad (4.3)$$

Let $\psi \in C_0^{\infty}(\Omega)$, $\psi \geq 0$ and $\psi \neq 0$ and $t > 1$. By (4.3) we have

$$\begin{aligned}
J(t\psi) &= a \int_{\Omega} \frac{1}{p(x)} |t\nabla\psi|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |t\nabla\psi|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |t\psi|^{p(x)} dx - \int_{\Omega} G(x, t\psi) dx \\
&\leq a \int_{\Omega} \frac{1}{p(x)} |t\nabla\psi|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |t\nabla\psi|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |t\psi|^{p(x)} dx - At^{\theta} \int_{\Omega} |t\psi|^{\theta} dx + C_A |\Omega| \\
&\leq \frac{at^{p^{+}}}{p^{-}} \int_{\Omega} |\nabla\psi|^{p(x)} dx - \frac{bt^{2p^{-}}}{2p^{+2}} \left(\int_{\Omega} |\nabla\psi|^{p(x)} dx \right)^2 - \frac{\lambda}{p^{+}} t^{p^{-}} \int_{\Omega} |\psi|^{p(x)} dx - At^{\theta} \int_{\Omega} |\psi|^{\theta} dx + C_A |\Omega|.
\end{aligned}$$

Since $\theta > 2p^{-} > p^{+} > p^{-}$ we obtain $J(t\psi) \rightarrow -\infty$ ($t \rightarrow +\infty$). then for $t > 1$ large enough, we can take $e = t\psi$ such that $\|e\| > \rho$ and $J(e) < 0$. \square

Proof of Theorem 1.1.

From Lemmas 3.1-4.2 and the fact that $J(0) = 0$, J satisfies the Mountain Pass theorem (see e.g., [37]). Therefore, problem (1.1) has a nontrivial weak solution. \square

5. Infinitely Many Solutions

The proof of Theorem 1.2 rests mainly on the application of the Fountain theorem. As $X := W_0^{1,p(x)}(\Omega)$ is a separable and reflexive real Banach space, there exist $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, \dots\}}, \quad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, \dots\}}$$

and

$$\langle e_j^*, e_j \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For convenience, we write $X_j = \text{span}\{e_j\}$, $Y_k = \bigoplus_{j=1}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$.

Theorem A. (*Fountain Theorem, see [37]*) Assume that the even functional $\Phi \in C^1(X, \mathbb{R})$ satisfies the $(PS)_c$ condition, if there exists $k_0 > 0$ such that for $k \geq k_0$ there exists $\rho_k > r_k > 0$ such that

$$(i) \quad a_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0.$$

$$(ii) \quad b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \rightarrow +\infty \text{ as } k \rightarrow \infty.$$

Then Φ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \rightarrow +\infty$.

Lemma 5.1. If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for any $x \in \overline{\Omega}$, denote

$$\beta_k = \sup_{u \in Z_k, \|u\|=1} |u|_{\alpha(x)}.$$

Then $\lim_{k \rightarrow \infty} \beta_k = 0$.

Proof. Obviously, $0 < \beta_{k+1} \leq \beta_k$, so $\beta_k \rightarrow \beta \geq 0$. Let $u_k \in Z_k$ satisfy

$$\|u_k\| = 1, \quad 0 \leq \beta_k - |u_k|_{\alpha(\cdot)} < \frac{1}{k}.$$

Then there exists a subsequence of $\{u_k\}$ (which we still denote by u_k) such that $u_k \rightharpoonup u$, and

$$\langle e_j^*, u \rangle = \lim_{k \rightarrow \infty} \langle e_j^*, u_k \rangle = 0, \quad \forall e_j^*,$$

which implies that $u = 0$, and so $u_k \rightarrow 0$. Since the embedding from $W_0^{1,p(x)}(\Omega)$ to $L^{\alpha(x)}(\Omega)$ is compact, then $u_k \rightarrow 0$ in $L^{\alpha(x)}(\Omega)$. Hence, we get $\beta_k \rightarrow 0$ as $k \rightarrow \infty$. Proof of Lemma 5.1 is complete. \square

Proof of Theorem 1.2 completed. From Lemma 3.1 the functional J satisfies the $(PS)_c$ condition where precisely $c < \frac{a^2}{2b}$. Now, we shall verify that J satisfies the conditions of Theorem A

item by item.

(i) From (g_3) , there exist $C_1 > 0, M > 0$ such that

$$G(x, s) \geq C_1 |s|^\theta, \quad \forall |s| \geq M, \quad x \in \Omega. \quad (5.1)$$

Note from (g_1) that

$$\begin{aligned} |G(x, s)| &\leq \int_0^1 |g(x, zs)| s dz \\ &\leq \int_0^1 C(1 + |zs|^{q(x)-1}) |s| dz \leq C|s| + C|s|^{q(x)}, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \end{aligned} \quad (5.2)$$

Therefore, if $|s| \leq M$, there exists $C_2 > 0$ such that

$$|G(x, s)| \leq |s|(C + C|s|^{q(x)-1}) \leq C_2 |s|.$$

Combining this with (5.1), we find

$$G(x, s) \geq C_1 |s|^\theta - C_2 |s|, \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

For $u \in Y_k$, when $\|u\| > 1$,

$$\begin{aligned} J(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx \\ &\leq a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - C_1 \int_{\Omega} |u|^\theta dx + C_2 \int_{\Omega} |u| dx. \end{aligned}$$

Consequently, because all norms on the finite dimensional space Y_k , when $\|u\| > 1$ are equivalent, there is $C_W > 0$ such that

$$\int_{\Omega} |u|^{p(x)} dx \geq C_W \|u\|^{p^-}, \quad \int_{\Omega} |u|^\theta dx \geq C_W \|u\|^\theta \quad \text{and} \quad \int_{\Omega} |u| dx \geq C_W \|u\|.$$

Hence, we get

$$J(u) \leq \frac{a}{p^-} \|u\|^{p^+} - \frac{b}{2p^{-2}} \|u\|^{2p^-} - \frac{\lambda C_W}{p^-} \|u\|^{p^-} - C_1 C_W \|u\|^\theta + C_2 C_W \|u\|.$$

Since $\theta > 2p^- > p^+ > p^-$ then for some $\rho_k = \|u\| > 0$ large enough we deduce that

$$a_k = \max_{u \in Y_k, \|u\| = \rho_k} J(u) \leq 0.$$

Hence, condition (i) of Theorem A holds true.

(ii) By (g_1) and (g_2) , there exist $C_3, C_4 > 0$ such that

$$|G(x, u)| \leq \frac{C_3}{p(x)} |u|^{p(x)} + \frac{C_4}{q(x)} |u|^{q(x)}.$$

By computation, for any $u \in Z_k$ with $\|u\| \leq 1$, we obtain

$$\begin{aligned}
J(u) &= a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \lambda \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx - \int_{\Omega} G(x, u) dx \\
&\geq a \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \frac{b}{2} \left(\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right)^2 - \int_{\Omega} \frac{\lambda}{p(x)} |u|^{p(x)} dx - \int_{\Omega} \frac{C_3}{p(x)} |u|^{p(x)} dx \\
&\quad - \int_{\Omega} \frac{C_4}{q(x)} |u|^{q(x)} dx \\
&\geq \frac{a}{p^+} \|u\|^{p^+} - \frac{b}{2p^{-2}} \|u\|^{p^{2-}} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} \|u\|^{p^-} - \frac{C_4}{q^-} \beta_k^{q^-} \|u\|^{q^-}.
\end{aligned}$$

Let $\varphi \in Z_k$, $\|\varphi\| = 1$ and $0 < t < 1$, then it follows

$$\begin{aligned}
J(t\varphi) &\geq \frac{a}{p^+} t^{p^+} - \frac{b}{2p^{-2}} t^{p^{2-}} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-} - \frac{C_4}{q^-} \beta_k^{q^-} t^{q^-} \\
&\geq \left(\frac{a}{p^+} - \frac{b}{2p^{-2}} \right) t^{p^{2-}} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-} - \frac{C_4}{q^-} \beta_k^{q^-} t^{q^-}.
\end{aligned}$$

Conditions $a > b$ and $p^+ < 2p^{-2}$ implies that $\frac{a}{p^+} - \frac{b}{2p^{-2}} = \frac{2p^{-2}a - bp^+}{2p^{-2}p^+} > 0$.

Hence we get

$$\begin{aligned}
J(t\varphi) &\geq \left(\frac{2p^{-2}a - bp^+}{2p^{-2}p^+} \right) t^{p^{2-}} - \frac{C_4}{q^-} \beta_k^{q^-} t^{q^-} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-} \\
&\geq \left(\frac{2p^{-2}a - bp^+}{2p^{-2}p^+} - \frac{C_4}{q^-} \beta_k^{q^-} \right) t^{q^-} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-}
\end{aligned}$$

since $p^- < p(x) < p^+ < 2p^- < q^-$ for sufficiently large k , by choosing $\frac{C_4}{q^-} \beta_k^{q^-} < \frac{2p^{-2}a - bp^+}{4p^{-2}p^+}$, we get

$$J(t\varphi) \geq \frac{2p^{-2}a - bp^+}{4p^{-2}p^+} t^{q^-} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-}$$

Obviously, there exists a large enough k such that

$$\begin{aligned}
J(t\varphi) &\geq \frac{2p^{-2}a - bp^+}{4p^{-2}p^+} t^{q^-} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} t^{p^-} \\
&= t^{p^-} \left(\frac{2p^{-2}a - bp^+}{4p^{-2}p^+} t^{q^- - p^-} - (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} \right)
\end{aligned}$$

Put $\rho_k := \left(\frac{4p^{-2}p^+}{2p^{-2}a - bp^+} (\lambda + C_3) \frac{\beta_k^{p^-}}{p^-} \right)^{\frac{1}{q^- - p^-}}$, then, for sufficiently large k , $\rho_k < 1$. When $t = \rho_k$, $\rho_k \in Z_k$ with $\|\varphi\| = 1$, we have $J(t\varphi) \geq 0$. Therefore, condition (ii) of Theorem A holds true. This ends the proof of Theorem 1.2. \square

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