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Stabilization of infinite-dimensional linear control systems by POD reduced-order Riccati feedback

Emmanuel Trélat* Gengsheng Wang† Yashan Xu‡

Abstract

There exist many ways to stabilize an infinite-dimensional linear autonomous control systems when it is possible. Anyway, finding an exponentially stabilizing feedback control that is as simple as possible may be a challenge. The Riccati theory provides a nice feedback control but may be computationally demanding when considering a discretization scheme. Proper Orthogonal Decomposition (POD) offers a popular way to reduce large-dimensional systems. In the present paper, we establish that, under appropriate spectral assumptions, an exponentially stabilizing feedback Riccati control designed from a POD finite-dimensional approximation of the system stabilizes as well the infinite-dimensional control system.

Keywords: Feedback stabilization, Proper Orthogonal Decomposition (POD), Riccati theory, linear quadratic optimal control.

AMS subject classifications: 34H15, 35K05, 49M27, 49N10.

1 Introduction and main result

Stabilization of linear autonomous control systems is classically done in finite dimension by pole-shifting or by Riccati theory (see, e.g., [25, 29, 40, 43]). In infinite dimension, pole-shifting may be used for some appropriate classes of systems (see [5, 10, 11], see also [37, page 711] and [48, Chapter 3]), but such approaches rely on spectral considerations and in practice require the numerical computation of eigenelements, which may be hard in general. Riccati theory has also been much explored in infinite dimension (see, e.g., [12, 26, 27, 49] and provides a powerful way for stabilizing a linear control system. Anyway, in practice, computing an approximation of the Riccati operator requires to consider a numerical approximation scheme and to compute the solution of a high-dimensional algebraic Riccati equation (see, e.g., [4, 19, 30, 26, 27] for convergence results for space semi-discretizations of the Riccati procedure, see also the survey [44]), which raises also a number of numerical difficulties.

Given these facts, it appears interesting to use dimension reduction procedures. Indeed, model reduction can generate low-dimensional models for which one may expect reasonable performances for stabilization issues while keeping a computationally tractable numerical problem. Proper Orthogonal Decomposition (POD) is a popular reduction model approach and can be used to generate, from a finite number n of snapshots, a reduced-order control system in dimension n , approximating

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in the least square sense the initial infinite-dimensional system. Such an approach is completely general and does not consist of computing eigenelements (POD does not see eigenvectors). It is then natural to expect that, if n is large enough, then a linear stabilizing feedback computed from the n -dimensional reduced-order control system, stabilizes as well the whole infinite-dimensional control system. Proving that this assertion holds true under appropriate assumptions is the objective of this paper: we prove that a low-order feedback control obtained by the Riccati procedure applied to a POD reduced-order model suffices to stabilize the complete infinite-dimensional control system.

The idea of using POD as a way to efficiently stabilize infinite-dimensional control systems, such as controlled PDEs, by means of a low-order feedback control, has been implemented in [3, 23, 24], where a number of convincing numerical simulations have been provided, showing the relevance of that approach. Feasibility of this methodology is nicely illustrated in [3] for heat equations and in [23] for the Burgers equation. But, in these papers, the above theoretical issue has been left as an open problem. In this paper, we provide the first general theorem providing a positive answer.

The paper is structured as follows. In Section 1.1 we give all assumptions under which our general result will be established. We provide in Section 1.2 some elements on the POD approach. Our main result is stated in Section 1.3. An idea of the strategy of its proof is given in Section 1.4. Section 2 contains some reminders and useful results on POD, useful in the proof of the main result. Section 3 is devoted to proving the main theorem. In Section 4, we give a conclusion and some open problems and perspectives. Finally, in Appendix A, we establish an asymptotic result in Riccati theory, which is instrumental in the proof of our main result.

1.1 General setting and assumptions

Let H and U be real Hilbert spaces. Let $A : D(A) \rightarrow H$ be a densely defined, closed selfadjoint operator, such that there exists some $\alpha \in \mathbb{R}$ for which $A - \alpha \text{id}$ is dissipative. By the Lumer-Phillips theorem (see [13, 34]), A generates a quasicontraction C_0 semi-group $(S(t))_{t \geq 0}$ on H , i.e., satisfying $\|S(t)\| \leq e^{\alpha t}$ for every $t \geq 0$. Let $B \in L(U, H)$ be a bounded control operator. Consider the control system

$$\dot{y}(t) = Ay(t) + Bu(t), \quad t \in (0, +\infty) \quad (1)$$

with controls $u \in L^2(0, +\infty; U)$.

The objective of our paper is to exponentially stabilize the control system (1) with a feedback control designed from a finite-dimensional projection of (1) obtained by POD.

In what follows, we denote by $\|\cdot\|$ the norm in H and by $\langle \cdot, \cdot \rangle$ the corresponding scalar product. Throughout the paper, we make the following assumptions.

(H₁) We assume that the Hilbert space H can be written as the direct orthogonal sum

$$H = E_\ell \overset{\perp}{\oplus} F_\ell$$

where $E_\ell \subset D(A)$ is of dimension ℓ , F_ℓ is a closed subspace of H such that $F_\ell \cap D(A)$ is dense in F_ℓ (for the induced topology), satisfying $E_\ell \perp F_\ell$ and

$$AE_\ell \subset E_\ell \quad \text{and} \quad A(F_\ell \cap D(A)) \subset F_\ell$$

(invariance under A).

We denote by P_ℓ the orthogonal projection of H onto E_ℓ ; then $\text{id} - P_\ell$ is the orthogonal projection of H onto F_ℓ . By **(H₁)**, we have

$$P_\ell A(\text{id} - P_\ell)|_{D(A)} = 0, \quad (\text{id} - P_\ell)AP_\ell = 0 \quad (2)$$

and

$$A = P_\ell A P_\ell + (\text{id} - P_\ell) A (\text{id} - P_\ell). \quad (3)$$

It follows from the Hille-Yosida theorem (see, e.g., [13, 34]) that

- the (bounded) operator $P_\ell A P_\ell$ on E_ℓ (which can be identified with a matrix of size $\ell \times \ell$) generates on E_ℓ the uniformly continuous semigroup $(P_\ell S(t) P_\ell)_{t \geq 0}$, with $P_\ell S(t) P_\ell = \exp(t P_\ell A P_\ell)$ for every $t \geq 0$;
- the operator $(\text{id} - P_\ell) A (\text{id} - P_\ell)$ on F_ℓ , of domain $F_\ell \cap D(A)$, generates the (quasicontraction) C_0 semigroup $((\text{id} - P_\ell) S(t) (\text{id} - P_\ell))_{t \geq 0}$.

We make the two following assumptions on those semigroups:

- (**H₂**) We assume that the latter semigroup is exponentially stable, i.e., that there exists $\gamma > 0$ such that

$$\|(\text{id} - P_\ell) S(t) (\text{id} - P_\ell) z\| \leq e^{-\gamma t} \|(\text{id} - P_\ell) z\| \quad \forall t \geq 0 \quad \forall z \in H.$$

- (**H₃**) The operator $P_\ell A P_\ell$ (restriction of A to E_ℓ) can be identified with a selfadjoint $\ell \times \ell$ matrix, which is therefore diagonalizable with real-valued eigenvalues. We assume that all eigenvalues of $P_\ell A P_\ell$ are simple and have a positive real part. We define

$$\beta_\ell = \min\{\lambda \mid \lambda \in \text{Spec}(P_\ell A P_\ell)\} > 0. \quad (4)$$

In other words, we assume in particular that $0 \notin \text{Spec}(A)$, that E_ℓ is the finite-dimensional instable part of the system and that F_ℓ is the exponentially stable part.

- (**H₄**) We assume that the pair $(P_\ell A P_\ell, P_\ell B)$ satisfies the Kalman condition

$$\text{rank}(P_\ell B, P_\ell A P_\ell B, \dots, P_\ell A^{\ell-1} P_\ell B) = \ell.$$

This assumption is satisfied under the following much stronger assumption of unique continuation (which is equivalent, by duality, to approximate controllability for the system (1)): there exists $T > 0$ such that, given any $z \in H$, if $B^* S(t)^* z = 0$ for every $t \in [0, T]$ then $z = 0$.

The assumptions (**H₁**), (**H₂**), (**H₃**) and (**H₄**) are satisfied, for instance, for heat-like equations with internal control, i.e., when

$$A = \Delta + a \text{id} \quad \text{and} \quad B = \chi_\omega$$

where $a \in L^\infty(\Omega, \mathbb{R})$, Δ is the Dirichlet-Laplacian on a bounded C^2 domain Ω of \mathbb{R}^d , $\omega \subset \Omega$ is a nonempty open subset of Ω and χ_ω is its characteristic function. Taking $H = U = L^2(\Omega, \mathbb{R})$, the operator $A = \Delta$ on $D(A) = H_0^1(\Omega, \mathbb{R}) \cap H^2(\Omega, \mathbb{R})$ is selfadjoint and of compact inverse and thus is diagonalizable. We assume that a and Ω are such that the spectrum of A is simple (this is true under generic assumptions, see [41]) and such that 0 is not an eigenvalue. Then there exists a Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of H consisting of real-valued eigenfunctions corresponding to the real eigenvalues

$$-\infty \leftarrow \lambda_j \cdots < \lambda_{\ell+1} < 0 < \lambda_\ell < \lambda_{\ell-1} < \cdots < \lambda_1$$

(with a slight abuse of notation because the number ℓ of instable modes may be equal to 0). Taking $E_\ell = \text{Span}\{\phi_j \mid j = 1 \dots \ell\}$ and $F_\ell = \text{Span}\{\phi_j \mid j \geq \ell + 1\}$, Assumptions (**H₁**), (**H₂**) and (**H₃**) are satisfied. Assumption (**H₄**) is satisfied because of unique continuation: indeed we have $\chi_\omega \phi_j \neq 0$ for $j = 1, \dots, \ell$.

Of course, when a is such that all eigenvalues of A are negative, any solution of (1) converges exponentially to 0. We are interested in the case where there are (a finite number of) positive eigenvalues, i.e., $\ell > 0$, and then stabilization is an issue.

More generally, the assumptions **(H₁)**, **(H₂)**, **(H₃)** are satisfied when $A - \alpha \text{id}$ is of compact inverse, with A having a finite number of instable (positive) eigenvalues which are moreover simple. Our framework even allows for more general situations in which spectrum may not be discrete, but does not involve the case of wave-like equations for instance (for which A is not selfadjoint). Assumption **(H₄)** follows from unique continuation but is much weaker and may be satisfied for finite-rank control operators B .

Thanks to the assumptions **(H₁)**, **(H₂)**, **(H₃)** and **(H₄)**, to stabilize (1) it would suffice to focus on the finite-dimensional instable part E_ℓ of the infinite-dimensional system (1), as this was done for instance in [5, 10, 11] (see also [37, page 711] and [48, Chapter 3]). However, in practice eigenlements are not known in general or may be difficult to compute numerically. In particular, the integer ℓ is not known in general or may be difficult to compute although we know its existence.

Stabilizing the system from a finite-dimensional approximation of (1) that is *not* of a spectral nature but which is anyway, in some sense, *compatible* with the above spectral decomposition, is the main challenge that we address in this paper.

We address this issue by approximating the control system (1) thanks to the POD method, described hereafter, which generates a m -dimensional *reduced-order* control system, with m sufficiently large ($m \geq \ell$ will be enough).

In what follows, we consider an arbitrary element

$$y_0 \in D(A)$$

which, used as an initial condition, generates the trajectory $y(t) = S(t)y_0$, solution of (1) with $u = 0$. We will consider it to generate snapshots in the POD method as explained next.

1.2 Proper Orthogonal Decomposition (POD)

The main idea of POD is to design an orthogonal basis of reduced order (called a POD basis) from a given collection of data (called snapshots). In order to face with too costly computations of a too complex model, the rationale behind POD is to generate a reduced set of basis functions able to capture the essential information of the physical process under consideration. POD has been developed long time ago, and independently, by many authors in various contexts. POD is closely related to Karhunen-Loève decompositions and to principal component analysis (PCA) or factor analysis. It has been widely used in the context of fluid mechanics and in particular turbulence (see [6, 8, 15, 18, 22, 31]) of chemical reactions (see [32, 38, 42]) and it has become a classical approach for nonlinear model reduction (see [9, 16, 17, 21, 28, 33, 35, 39] and see [3, 20, 23, 24, 36, 45] for applications to control of PDEs). The POD method consists of designing an unstructured low-rank approximation of a matrix composed of snapshots of the state. It can roughly be thought of as a Galerkin approximation in the spatial variable, built from values $y_k = y(t_k)$ of solutions of the physical system taken at prescribed times $0 = t_1 < t_2 < \dots < t_n < +\infty$, for some $n \in \mathbb{N}^*$. These values y_k (assumed to be known) are called *snapshots*.

Here, we take n snapshots

$$y_k = y(kT; y_0, 0) = S(kT)y_0, \quad k = 1, \dots, n \tag{5}$$

of the solution $y(\cdot; y_0, 0)$ to (1) with initial condition y_0 and with the control $u = 0$, taken at times kT , for some $T > 0$. We set

$$D_n = \text{Span}(y_1, y_2, \dots, y_n) \quad \text{and} \quad d_n = \dim D_n. \tag{6}$$

Note that, since $y_0 \in D(A)$, we have $D_n \subset D(A)$.

Given some integer $m \leq d_n$, the POD method consists of determining a subspace $D_{n,m}$ of D_n , of dimension $\leq m$, such that the mean square discrepancy between all snapshots y_k and their orthogonal projection $\Pi_{D_{n,m}} y_k$ onto $D_{n,m}$ is minimal, i.e., it consists of minimizing the functional

$$\boxed{J(D_{n,m}) = \sum_{k=1}^n \left\| y_k - \Pi_{D_{n,m}} y_k \right\|^2 = \sum_{k=1}^n \left\| \Pi_{D_{n,m}^\perp} y_k \right\|^2} \quad (7)$$

over all possible subspaces $D_{n,m}$ of D_n of dimension $\leq m$ (equivalently, of dimension equal to m). Here, $\Pi_{D_{n,m}^\perp}$ is the orthogonal projection onto the orthogonal $D_{n,m}^\perp$ of $D_{n,m}$ in D_n . This minimization problem has at least one solution $\overline{D}_{n,m}$ (see, e.g., [47]) and we denote by $\overline{J}_{n,m}$ the optimal value, but the optimal approximating subspace $\overline{D}_{n,m}$ may not be unique.

The problem is often formulated as follows. Assume that $D_{n,m} = \text{Span}(\psi_1, \dots, \psi_m)$ and complete these m orthonormal vectors into an orthonormal basis $(\psi_j)_{j=1, \dots, d_n}$ of D_n ; write $y_k = \sum_{j=1}^{d_n} \langle y_k, \psi_j \rangle \psi_j$ and $\Pi_{D_{n,m}} y_k = \sum_{j=1}^m \langle y_k, \psi_j \rangle \psi_j$. Then, the POD method consists of minimizing

$$\sum_{k=1}^n \left\| y_k - \sum_{j=1}^m \langle y_k, \psi_j \rangle \psi_j \right\|^2$$

over all possible orthonormal families $(\psi_j)_{j=1, \dots, m}$ in D_n . A subspace $\overline{D}_{n,m} = \text{Span}(\overline{\psi}_1, \dots, \overline{\psi}_m)$, optimal solution of the minimization problem (7), i.e., such that $J(\overline{D}_{n,m}) = \overline{J}_{n,m}$, is then used as a best approximating subspace of D_n of dimension m . Any orthonormal basis $(\psi_j)_{j=1, \dots, m}$ of $\overline{D}_{n,m}$ is called a POD basis of rank m .

Other properties of POD, related to SVD (Singular Value Decomposition), are recalled further in Section 2.1.

1.3 Main result

POD reduced-order control system. Keeping the assumptions and notations of the previous subsections, we fix an arbitrary $T > 0$ and we consider an optimal solution $\overline{D}_{n,m}$ of the (POD) minimization problem (7), i.e., a best approximating m -dimensional subspace of the space $D_n \subset D(A)$ defined by (6).

Applying the orthogonal projection $\Pi_{\overline{D}_{n,m}}$ to the control system (1) yields

$$\frac{d}{dt} \Pi_{\overline{D}_{n,m}} y(t) = \Pi_{\overline{D}_{n,m}} A \Pi_{\overline{D}_{n,m}} y(t) + \Pi_{\overline{D}_{n,m}} B u(t) + \Pi_{\overline{D}_{n,m}} A (\text{id} - \Pi_{\overline{D}_{n,m}}) y(t).$$

The last term at the right-hand side of the above equation is seen as a perturbation term, and we are thus led to consider the following *POD reduced-order control system* in the space $\overline{D}_{n,m}$

$$\boxed{\dot{Y}(t) = A_{n,m} Y(t) + B_{n,m} v(t)} \quad (8)$$

with

$$A_{n,m} = \Pi_{\overline{D}_{n,m}} A \Pi_{\overline{D}_{n,m}} \quad \text{and} \quad B_{n,m} = \Pi_{\overline{D}_{n,m}} B.$$

Note that $A_{n,m}$ is well defined because $\overline{D}_{n,m} \subset D(A)$. The control system (8) is a linear autonomous control system in the m -dimensional space $\overline{D}_{n,m}$ with controls $v \in L^2(0, +\infty; U)$. The operator $A_{n,m}$ can be identified with a square matrix of size m and $B_{n,m}$ with a matrix of size $m \times \dim(U)$ (with U of finite or infinite dimension).

Stabilizing the reduced-order control system. In the proof of our main result (Theorem 1 hereafter), we will prove that the POD reduced-order m -dimensional control system (8) is stabilizable when $m \geq \ell$ and n is large enough. To design an exponentially stabilizing linear feedback $v = K_{n,m}Y$, we use the Riccati theory (see also Appendix A for some reminders on the Riccati theory).

Given any $\varepsilon \geq 0$, since the pair $(A_{n,m}, B_{n,m})$ is stabilizable, by [1, Corollary 2.3.7 page 55] there exists a (unique) maximal symmetric positive semidefinite solution $P_{n,m}(\varepsilon)$ (of size $m \times m$) of the algebraic Riccati equation

$$A_{n,m}^* P_{n,m}(\varepsilon) + P_{n,m}(\varepsilon) A_{n,m} - P_{n,m}(\varepsilon) B_{n,m} B_{n,m}^* P_{n,m}(\varepsilon) + \varepsilon I_m = 0. \quad (9)$$

Here and throughout, I_k denotes the identity matrix of size k . Moreover, $A_{n,m} - B_{n,m} B_{n,m}^* P_{n,m}(\varepsilon)$ is semi-stabilizing, i.e., its eigenvalues have nonpositive real part. If $\varepsilon > 0$ then $P_{n,m}(\varepsilon)$ is positive definite, and $A_{n,m} - B_{n,m} B_{n,m}^* P_{n,m}(\varepsilon)$ is Hurwitz, i.e., its eigenvalues have a negative real part (these facts are established in Appendix A). Therefore, setting

$$K_{n,m}(\varepsilon) = -B_{n,m}^* P_{n,m}(\varepsilon),$$

the linear feedback

$$v = K_{n,m}(\varepsilon)Y = -B_{n,m}^* P_{n,m}(\varepsilon)Y$$

exponentially stabilizes the reduced-order control system (8) to the origin.

Remark 1. Given any $\varepsilon \geq 0$ and any $Y_0 \in \overline{D}_{n,m}$, there exists a unique optimal control minimizing the functional

$$\int_0^{+\infty} \left(\varepsilon \|Y(t)\|_{\overline{D}_{n,m}}^2 + \|v(t)\|_U^2 \right) dt$$

over all possible controls $v \in L^2(0, +\infty; U)$, where $Y(\cdot)$ is the solution to (8) with control v and with initial condition $Y(0) = Y_0$. If $\varepsilon > 0$ then the optimal control is exactly the stabilizing feedback $v = K_{n,m}(\varepsilon)Y$.

We will prove that the closed-loop matrix $A_{n,m} + B_{n,m} K_{n,m}(\varepsilon)$, which is Hurwitz if $\varepsilon > 0$, actually remains *uniformly* Hurwitz as $\varepsilon \rightarrow 0$ (precise asymptotic results are established in Appendix A). In particular, the matrix $\exp(t(A_{n,m} + B_{n,m} K_{n,m}(\varepsilon)))$ decreases exponentially, with an exponential rate which remains uniformly bounded below by some positive constant as $\varepsilon \rightarrow 0$.

Main result. We now use the above feedback matrix $K_{n,m}(\varepsilon)$ in the original infinite-dimensional control system (1), by taking the feedback control

$$u = K_{n,m}(\varepsilon) \Pi_{\overline{D}_{n,m}} y = -B_{n,m}^* P_{n,m}(\varepsilon) \Pi_{\overline{D}_{n,m}} y. \quad (10)$$

Since B is bounded, the operator $A + BK_{n,m}(\varepsilon) \Pi_{\overline{D}_{n,m}}$ is defined on $D(A)$ and generates a C_0 semigroup. Our main result establishes that, under appropriate assumptions, this semigroup is exponentially stable. In other words, the “finite-dimensional” feedback (10), which exponentially stabilizes the finite-dimensional control system (8), also exponentially stabilizes the infinite-dimensional control system (1) if the number n of snapshots is large enough and if $\varepsilon > 0$ is small enough.

Theorem 1. *We make the assumptions (H₁), (H₂), (H₃) and (H₄), and we assume that the pair $(P_\ell A P_\ell, P_\ell y_0)$ satisfies the Kalman condition, i.e.,*

$$\text{rank}(P_\ell y_0, P_\ell A P_\ell y_0, \dots, P_\ell A^{\ell-1} P_\ell y_0) = \ell. \quad (11)$$

Let $m \geq \ell$ be arbitrary. There exist $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that, for every $\varepsilon \in [0, \varepsilon_0]$ and every $n \geq n_0$, the control system (1) in closed-loop with the feedback $u = K_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}y$,

$$\boxed{\dot{y}(t) = \left(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}} \right) y(t)} \quad (12)$$

is exponentially stable, meaning that any solution of (12) converges exponentially to 0 in H as $t \rightarrow +\infty$.

Remark 2 (On Assumption (11)). Under Assumption (11), we have that $d_n = \dim(D_n) \geq \ell$ when $n \geq \ell$. This is why we can take $m \geq \ell$ in the theorem. Since $P_\ell AP_\ell$ is selfadjoint, recalling that $y_k = S(kT)y_0 = S(T)^k y_0$ and noting that $P_\ell S(T)P_\ell = \exp(TP_\ell AP_\ell)$, we see that Assumption (11) is equivalent to the assumption that the pair $(P_\ell S(T)P_\ell, P_\ell y_0)$ satisfies the Kalman condition, i.e.,

$$\text{rank}(P_\ell y_0, P_\ell y_1, \dots, P_\ell y_{\ell-1}) = \text{rank}(P_\ell y_0, P_\ell S(T)P_\ell y_0, \dots, P_\ell S(T)^{\ell-1} P_\ell y_0) = \ell. \quad (13)$$

This is rather this condition that we will use in the proof.

A second remark is the following. Let $(\phi_1, \dots, \phi_\ell)$ be an orthonormal basis of E_ℓ , consisting of eigenvectors of $P_\ell AP_\ell$, corresponding to the (real-valued) eigenvalues λ_j , $j = 1, \dots, \ell$. Assumption (11) (equivalently, Assumption (13)) is satisfied if and only if all eigenvalues of $P_\ell AP_\ell$ are simple and

$$\langle \phi_j, P_\ell y_0 \rangle \neq 0 \quad \forall j \in \{1, \dots, \ell\} \quad (14)$$

i.e., the component of $P_\ell y_0$ in the direction ϕ_j is nonzero, for every $j \in \{1, \dots, \ell\}$. The condition (14) is generic in the sense that the set of $y_0 \in D(A)$ of which one of the ℓ first spectral modes is zero has codimension 1 (and thus has measure zero) in $D(A)$.

Remark 3. Define the best exponential decay rate γ^* of an exponentially stable C_0 semigroup $(T(t))_{t \geq 0}$ on H as the supremum of all possible $\gamma > 0$ for which there exists $M \geq 1$ such that $\|T(t)\|_{L(H)} \leq M e^{-\gamma t}$ for every $t \geq 0$, i.e., $\gamma^* = -\inf_{t > 0} \frac{1}{t} \ln \|T(t)\|_{L(H)} = -\lim_{t \rightarrow +\infty} \frac{1}{t} \ln \|T(t)\|_{L(H)}$ (see [13, 34]).

Let $\gamma > 0$ be the best decay rate of the exponentially stable quasicontraction C_0 semigroup $((\text{id} - P_\ell)S(t)(\text{id} - P_\ell))_{t \geq 0}$ (see Assumption (H₂)). Let $\gamma_\varepsilon > 0$ be the best decay rate of the matrix $\exp(t(A_{n,m} + B_{n,m}K_{n,m}(\varepsilon)))$ ($-\gamma_\varepsilon$ is the spectral abscissa of $A_{n,m} + B_{n,m}K_{n,m}(\varepsilon)$).

Then, in Theorem 1, the growth bound $\gamma^*(\varepsilon)$ of the exponentially stable C_0 semigroup generated by $A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}$ satisfies

$$\lim_{\varepsilon \rightarrow 0} \gamma^*(\varepsilon) = \min(\gamma, \gamma_\varepsilon). \quad (15)$$

1.4 Strategy of the proof

Establishing Theorem 1 is easier under the additional assumption

$$E_\ell \subset \overline{D}_{n,m} \quad (16)$$

and we first sketch the argument under this simplifying assumption. In this case, we write

$$\overline{D}_{n,m} = E_\ell \oplus^\perp F_1 \quad (17)$$

where F_1 is a subspace of F_ℓ . Since $\text{Ran}(AP_\ell) \subset E_\ell \perp F_1$, we have $P_{F_1}AP_\ell = 0$ where P_{F_1} is the orthogonal projection onto F_1 . In the decomposition (17), the control system (8) is written as

$$\dot{Y}_1 = P_\ell AP_\ell Y_1 + P_\ell B u \quad (18)$$

$$\dot{Y}_2 = P_{F_1}AP_{F_1}Y_2 + P_{F_1}B u \quad (19)$$

By **(H₄)**, the pair $(P_\ell A P_\ell, P_\ell B)$ satisfies the Kalman condition and thus the subsystem **(18)** is stabilizable. Besides, by **(H₂)**, the subsystem **(19)** is exponentially stable with control $u = 0$. It follows from Appendix **A** that the control system **(18)**-**(19)** (which is equivalent to **(17)**) is stabilizable by the Riccati procedure: the optimal feedback $u = K_{n,m}(\varepsilon)P_\ell Y_1 + K_{n,m}(\varepsilon)P_{F_1} Y_2$ exponentially stabilizes the system **(18)**-**(19)**, i.e., the closed-loop matrix

$$\begin{pmatrix} P_\ell(A + BK_{n,m}(\varepsilon))P_\ell & P_\ell BK_{n,m}(\varepsilon)P_{F_1} \\ P_{F_1}BK_{n,m}(\varepsilon)P_\ell & P_{F_1}(A + BK_{n,m}(\varepsilon))P_{F_1} \end{pmatrix} \quad (20)$$

is Hurwitz, for every $\varepsilon > 0$. Moreover, by **(49)** in Appendix **A**, we have $K_{n,m}(\varepsilon)P_{F_1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the closed-loop matrix **(20)** remains uniformly Hurwitz as $\varepsilon \rightarrow 0$. This fact is important in our analysis.

Now, plugging this finite-dimensional feedback into the initial control system **(1)**, we obtain the closed-loop system **(12)**, with $A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}$ that is written in the above decomposition as the infinite-dimensional matrix

$$\left(\begin{array}{c|cc} P_\ell(A + BK_{n,m}(\varepsilon))P_\ell & P_\ell BK_{n,m}(\varepsilon)P_{F_1} & 0 \\ \hline P_{F_1}BK_{n,m}(\varepsilon)P_\ell & P_{F_1}(A + BK_{n,m}(\varepsilon))P_{F_1} & P_{F_1}A(\text{id} - P_\ell - P_{F_1}) \\ 0 & (\text{id} - P_\ell - P_{F_1})AP_{F_1} & (\text{id} - P_\ell - P_{F_1})A(\text{id} - P_\ell - P_{F_1}) \end{array} \right). \quad (21)$$

Since $K_{n,m}(\varepsilon)P_{F_1} \rightarrow 0$ as $\varepsilon \rightarrow 0$, the matrix **(21)** is approximately lower block triangular, with the first diagonal block being exponentially stable (because **(20)** is Hurwitz) and the second diagonal block being exponentially stable as well (because it is close to $(\text{id} - P_\ell)A(\text{id} - P_\ell)$ as ε is small enough). Therefore, **(21)** is exponentially stable and Theorem **1** follows, under the simplifying assumption **(16)**.

In general, however, **(16)** is not true: there is indeed no reason that, when performing the POD reduction, the space $\overline{D}_{n,m}$ contain the spectral subspace E_ℓ . Indeed, “*POD does not see eigenmodes*”.

Anyway, our complete analysis, done in Section **3**, will reveal that this is *almost* the case: we will prove in particular that

$$\overline{D}_{n,\ell} \simeq E_\ell$$

when n is large enough, which implies that the inclusion **(16)** is almost satisfied (because $\overline{D}_{n,\ell} \subset \overline{D}_{n,m}$). Establishing such a result will require a quite fine analysis. This shows that, in some sense, our problem is a small perturbation problem of **(21)** when n is large. Theorem **1** will then be proved in Section **3.2**, using an asymptotic result in Riccati theory developed in Appendix **A**, roughly stating that, when considering a linear system having an instable part and a stable part, the Riccati stabilization procedure with weight ε on the state and weight 1 on the control yields feedbacks that essentially act on the instable part for ε small. The results established in Appendix **A** are a bit delicate and require a particular care, although all notions thereof remain quite elementary.

2 Some results on POD

2.1 Relationship with Singular Value Decomposition (SVD)

It is well known that optimal solutions of POD can be expressed thanks to SVD. Let $\{y_1, \dots, y_n\}$ be given by **(5)**. We consider the $(\infty \times n)$ -matrix

$$Y_n = (y_1, \dots, y_n)$$

expressed in an arbitrary Hilbert basis of H . We have $D_n = \text{Ran}(Y_n)$ and $d_n = \text{rank}(Y_n) = \text{dim}(D_n)$. Since the matrix Y_n is of finite rank d_n , SVD works exactly as in finite dimension (because $Y_n Y_n^*$ and $Y_n^* Y_n$ are compact and selfadjoint, see [7]). According to the SVD theorem, we have

$$Y_n = V_n \Sigma_n U_n^*$$

where V_n is an orthogonal matrix of infinite size (unitary operator in H , consisting of eigenvectors of $Y_n Y_n^*$), U_n is an orthogonal matrix of size n (consisting of eigenvectors of $Y_n^* Y_n$) and Σ_n is a matrix of size $\infty \times n$ consisting of the diagonal $\sigma_{n,1} \geq \sigma_{n,2} \geq \dots \geq \sigma_{n,n} \geq 0$ (singular values of Y_n), completed with zeros. The singular values of Y_n are nonnegative real numbers, with the d_n first ones being positive and all others being zero. Denoting by $u_{n,i} \in H$ and $v_{n,i} \in H$ the columns of U_n and V_n , we have

$$Y_n = \sum_{i=1}^{d_n} \sigma_{n,i} v_{n,i} u_{n,i}^*. \quad (22)$$

Let $m \leq d_n$ be an integer. We define the $(\infty \times m)$ -matrix $V_{n,m}$ as the submatrix of V_n consisting of the first m columns of V_n , which are $v_{n,1}, \dots, v_{n,m}$. Similarly, we define the $(n \times m)$ -matrix $U_{n,m}$ as the submatrix of U_n consisting of the first m columns of U_n , which are $u_{n,1}, \dots, u_{n,m}$. Finally, we define the square diagonal matrix $\Sigma_{n,m}$ of size m , consisting of the elements $\sigma_{n,1} \geq \dots \geq \sigma_{n,m}$. It is then well known (see, e.g., [14, 33]) that the “best” projection of rank $\leq m$ onto $D_n = \text{Ran}(Y_n)$ is $V_{n,m} V_{n,m}^*$ and that the “best” approximation of rank $\leq m$ of the matrix Y_n , over all matrices of rank $\leq m$, is the matrix

$$Y_{n,m} = V_{n,m} V_{n,m}^* Y_n = V_{n,m} \Sigma_{n,m} U_{n,m}^* = \sum_{i=1}^m \sigma_{n,i} v_{n,i} u_{n,i}^*.$$

“Best” is understood here in the sense of the Frobenius norm as well as of the subordinate 2-norm (and actually, of any norm invariant under the orthogonal group), and the Frobenius norm of $Y_n - Y_{n,m}$ is

$$\|Y_n - Y_{n,m}\|_F^2 = \sum_{i=m+1}^{d_n} \sigma_{n,i}^2.$$

Recall that the square $\|D\|_F^2$ of the Frobenius norm of a matrix M (of any size, possibly infinite) is equal to the sum of squares of all elements of M . Moreover, when considering the Frobenius norm (also called Hilbert-Schmidt norm), we have uniqueness of the minimizer $Y_{n,m}$ if and only if $\sigma_{n,m} \neq \sigma_{n,m+1}$. Note that the range of $Y_{n,m}$ is contained in the range of Y_n , and thus $Y_{n,m}$ is also the best approximation of rank $\leq m$ of Y_n over all possible matrices of rank $\leq m$ whose range is contained in the range of Y_n .

By definition, the quantity $J(D_{n,m})$ defined by (7) is exactly the Frobenius norm of $Y_n - \Pi_{D_{n,m}} Y_n$:

$$J(D_{n,m}) = \|Y_n - \Pi_{D_{n,m}} Y_n\|_F^2$$

By the above remarks, since $\text{rank}(\Pi_{D_{n,m}} Y_n) \leq m$, the POD problem is exactly equivalent to searching the best approximation of rank $\leq m$ of the matrix Y_n for the Frobenius norm. Therefore, we have

$$\bar{J}_{n,m} = J(\bar{D}_{n,m}) = \|Y_n - Y_{n,m}\|_F^2 = \sum_{i=m+1}^{d_n} \sigma_{n,i}^2 \quad (23)$$

with

$$\begin{aligned}\bar{D}_{n,m} &= \text{Ran}(\Pi_{\bar{D}_{n,m}}) = \text{Span}(v_{n,1}, \dots, v_{n,m}), \\ \Pi_{\bar{D}_{n,m}} &= V_{n,m}V_{n,m}^*, \\ Y_{n,m} &= \Pi_{\bar{D}_{n,m}}Y_n = V_{n,m}\Sigma_{n,m}U_{n,m}^*.\end{aligned}$$

2.2 Boundedness of the optimal value

Recall that $y_k = S(kT)y_0$ for every $k \in \mathbb{N}$, where $T > 0$ is fixed.

Lemma 1. *Under Assumptions **(H₁)** and **(H₂)**:*

- *There exists $C > 0$ such that*

$$\|(\text{id} - P_\ell)y_k\| + \|A(\text{id} - P_\ell)y_k\| \leq Ce^{-\gamma kT} \quad \forall k \in \mathbb{N}. \quad (24)$$

- *Given any $m \geq \ell$, the optimal value $\bar{J}_{n,m}$ (given by (23)) of the minimization problem (7) remains bounded as $n \rightarrow +\infty$.*

Proof. Since P_ℓ commutes with $S(t)$ by Assumption **(H₁)**, we have $P_\ell y_k = P_\ell S(kT)y_0 = S(kT)P_\ell y_0$, and hence

$$\|y_k - P_\ell y_k\| = \|(\text{id} - P_\ell)S(kT)(\text{id} - P_\ell)y_0\| \leq e^{-\gamma kT} \|(\text{id} - P_\ell)y_0\|$$

where we have used Assumption **(H₂)** to get the latter inequality, and similarly,

$$\|Ay_k - AP_\ell y_k\| = \|(\text{id} - P_\ell)S(kT)(\text{id} - P_\ell)Ay_0\| \leq e^{-\gamma kT} \|(\text{id} - P_\ell)Ay_0\|$$

and the first item follows because $y_0 \in D(A)$.

For the second item, using the SVD interpretation of the POD, we have in particular (since $m \geq \ell$)

$$\bar{J}_{n,m} \leq \|Y_n - P_\ell Y_n\|_F^2 = \sum_{k=1}^n \|y_k - P_\ell y_k\|^2.$$

Therefore

$$\bar{J}_{n,m} \leq \|y_0\|^2 \sum_{k=1}^n e^{-2\gamma kT} \leq \|y_0\|^2 \frac{e^{-2\gamma T}}{1 - e^{-2\gamma T}}$$

and the lemma follows. \square

Remark 4. It follows from the second item of Lemma 1 that there exists $C_1 > 0$ (which is the bound on $\bar{J}_{n,m}$) such that, for all integers n and m satisfying $n \geq m \geq \ell$, we have

$$\sum_{k=1}^n \|(\text{id} - \Pi_{\bar{D}_{n,m}})y_k\|^2 \leq C_1$$

3 Proof of Theorem 1

3.1 Several convergence results

The lemmas established in this subsection are the key results to prove Theorem 1 in the next subsection. Throughout, we make the assumptions **(H₁)**, **(H₂)**, **(H₃)**, **(H₄)** and (11) (equivalently, (13)).

Lemma 2. *If $m \geq \ell$ then there exists $C > 0$ such that*

$$\|(\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell\| \leq Ce^{-n\beta_\ell T} \quad \forall n \geq m.$$

Proof. By Remark 4, there exists $C_1 > 0$ such that, for every $n \geq m$,

$$\sum_{k=n-\ell+1}^n \left\| y_k - \Pi_{\overline{\mathcal{D}}_{n,m}} y_k \right\|^2 \leq C_1. \quad (25)$$

Now, given any $k \in \{n - \ell + 1, n - \ell + 2, \dots, n\}$, we have

$$\begin{aligned} \left\| y_k - \Pi_{\overline{\mathcal{D}}_{n,m}} y_k \right\| &= \left\| (\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell y_k + (\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})(\text{id} - P_\ell)y_k \right\| \\ &\geq \left\| (\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell y_k \right\| - \|(\text{id} - P_\ell)y_k\| \end{aligned}$$

because $\|\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}}\| \leq 1$. We infer that

$$\left\| (\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell y_k \right\|^2 \leq 2 \left\| y_k - \Pi_{\overline{\mathcal{D}}_{n,m}} y_k \right\|^2 + 2 \|(\text{id} - P_\ell)y_k\|^2. \quad (26)$$

It follows from (24), (25) and (26) that

$$\sum_{k=n-\ell+1}^n \left\| (\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell y_k \right\|^2 \leq C_2 \quad (27)$$

for some $C_2 > 0$ independent of $n \geq m$. Now, for $k \in \{n - \ell + 1, \dots, n\}$, setting $j = k - n + \ell - 1 \in \{0, \dots, \ell - 1\}$, we have

$$P_\ell y_k = P_\ell S(T)^{n-\ell+1} P_\ell P_\ell S(T)^j P_\ell y_0$$

and by Assumption (13), the elements $P_\ell S(T)^j P_\ell y_0$, $j = 0, \dots, \ell - 1$, generate the ℓ -dimensional subspace E_ℓ . We then infer from (27) that the norm of the operator $(\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell S(T)^{n-\ell+1} P_\ell$ is bounded by a constant C_3 that is independent of $n \geq \ell$. Since $P_\ell S(T)^{n-\ell+1} P_\ell = \exp((n - \ell + 1)TP_\ell A P_\ell)$ is boundedly invertible, it follows that

$$\|(\text{id} - \Pi_{\overline{\mathcal{D}}_{n,m}})P_\ell\| \leq C_3 \|\exp(-(n - \ell + 1)TP_\ell A P_\ell)\|$$

The result follows by using the definition (4) of β_ℓ and the estimate of a matrix written in its canonical Jordan form. \square

Remark 5. Thanks to Lemma 2, we have obtained that $\Pi_{\overline{\mathcal{D}}_{n,m}} P_\ell \rightarrow P_\ell$ as $n \rightarrow +\infty$ (with an exponential convergence rate).

Lemma 3. *Recalling that $\sigma_{n,i}$, $i \in \{1, \dots, n\}$ are the singular values of Y_n (see (22)), we have*

$$\lim_{n \rightarrow +\infty} \sigma_{n,i} = +\infty \quad \forall i \in \{1, \dots, \ell\} \quad (28)$$

while all other singular values remain bounded.

Proof. We already know that the singular values $\sigma_{n,i}$, $i \in \{\ell+1, \dots, n\}$ remain bounded as $n \rightarrow +\infty$ (this is because $\sum_{i=\ell+1}^n \sigma_{n,i}^2$ is bounded as $n \rightarrow +\infty$).

Let us prove that $\sigma_{n,\ell} \rightarrow +\infty$ as $n \rightarrow +\infty$. This will imply the result for $\sigma_{n,i}$ with $i \leq \ell$ because $\sigma_{n,1} \geq \sigma_{n,2} \geq \dots \geq \sigma_{n,\ell}$. By contradiction, let us assume that $\sigma_{n,\ell}$ remains bounded

as $n \rightarrow +\infty$. By (23), we have $\bar{J}_{n,\ell-1} = \sigma_{n,\ell}^2 + \bar{J}_{n,\ell}$ and hence, using Lemma 1, $\bar{J}_{n,\ell-1}$ remains bounded as well as $n \rightarrow +\infty$. But then we can repeat the reasoning done in Remark 4 and then in Lemma 2, replacing m with $\ell - 1$: we thus obtain that $\Pi_{\bar{D}_{n,\ell-1}} P_\ell \rightarrow P_\ell$ as $n \rightarrow +\infty$. Hence $\text{rank}(P_\ell) \leq \liminf \text{rank}(\Pi_{\bar{D}_{n,\ell-1}} P_\ell)$, and this raises a contradiction because $\text{rank}(P_\ell) = \ell$ while $\text{rank}(\Pi_{\bar{D}_{n,\ell-1}} P_\ell) \leq \text{rank}(\Pi_{\bar{D}_{n,\ell-1}}) \leq \dim \bar{D}_{n,\ell-1} \leq \ell - 1$. \square

Remark 6. Since the argument is by contradiction in the above lemma, we do not know the blow-up rate of $\sigma_{n,i}$, $i \in \{1, \dots, \ell\}$.

We next consider an appropriate decomposition of the vector space $\bar{D}_{n,m}$. In what follows we assume that $\ell < m \leq n$ (when $m = \ell$ we have $G_{n,m,\ell} = \{0\}$ below). We already know that $D_n = \text{Ran}(V_{n,n} V_{n,n}^*) = \text{Span}(v_{n,1}, \dots, v_{n,n})$ and we consider the decomposition

$$\bar{D}_{n,m} = \text{Ran}(V_{n,m} V_{n,m}^*) = \text{Span}(v_{n,1}, \dots, v_{n,m}) = \bar{D}_{n,\ell} \dot{\oplus} G_{n,m,\ell} \quad (29)$$

with

$$\bar{D}_{n,\ell} = \text{Span}(v_{n,1}, \dots, v_{n,\ell}) \quad \text{and} \quad G_{n,m,\ell} = \text{Span}(v_{n,\ell+1}, \dots, v_{n,m})$$

Lemma 4. *We have the following convergence properties:*

$$\lim_{n \rightarrow +\infty} (\text{id} - P_\ell) \Pi_{\bar{D}_{n,\ell}} = 0, \quad \lim_{n \rightarrow +\infty} A(\text{id} - P_\ell) \Pi_{\bar{D}_{n,\ell}} = 0 \quad (30)$$

$$\lim_{n \rightarrow +\infty} \Pi_{\bar{D}_{n,\ell}} = P_\ell \quad (31)$$

$$\lim_{n \rightarrow +\infty} \Pi_{G_{n,m,\ell}} P_\ell = 0, \quad \lim_{n \rightarrow +\infty} P_\ell \Pi_{G_{n,m,\ell}} = 0 \quad (32)$$

and

$$\begin{aligned} \lim_{n \rightarrow +\infty} \Pi_{\bar{D}_{n,\ell}} A \Pi_{\bar{D}_{n,\ell}} &= P_\ell A P_\ell, & \lim_{n \rightarrow +\infty} \Pi_{G_{n,m,\ell}} A \Pi_{\bar{D}_{n,\ell}} &= 0 \\ \lim_{n \rightarrow +\infty} \Pi_{\bar{D}_{n,\ell}} A \Pi_{G_{n,m,\ell}} &= 0, & \lim_{n \rightarrow +\infty} \Pi_{G_{n,m,\ell}} P_\ell A P_\ell \Pi_{G_{n,m,\ell}} &= 0 \end{aligned} \quad (33)$$

Remark 7. According to (31), we have $\bar{D}_{n,\ell} \simeq E_\ell$ when n is large enough, as announced at the end of Section 1.4.

Proof. Recall that $\Pi_{\bar{D}_{n,\ell}} = V_{n,\ell} V_{n,\ell}^*$ where $V_{n,\ell} = (v_{n,1}, \dots, v_{n,\ell})$ is a matrix whose columns form an orthonormal basis of $\bar{D}_{n,\ell}$ (see Section 2.1). Then it suffices to prove that $(\text{id} - P_\ell) \Pi_{\bar{D}_{n,\ell}} V_{n,\ell} \rightarrow 0$ and $A(\text{id} - P_\ell) \Pi_{\bar{D}_{n,\ell}} V_{n,\ell} \rightarrow 0$ as $n \rightarrow +\infty$. But $\Pi_{\bar{D}_{n,\ell}} V_{n,\ell} = V_{n,\ell} = Y_{n,\ell} U_{n,\ell} \Sigma_{n,\ell}^{-1}$, and (30) follows by (24) and (28).

By Lemma 2, we have $\Pi_{\bar{D}_{n,\ell}} P_\ell - P_\ell \rightarrow 0$ (exponentially). Taking the adjoint (and using there that $P_\ell = P_\ell^*$), we obtain that $P_\ell \Pi_{\bar{D}_{n,\ell}} - P_\ell \rightarrow 0$. Since $\Pi_{\bar{D}_{n,\ell}} - P_\ell \Pi_{\bar{D}_{n,\ell}} \rightarrow 0$ by (30), we infer that $\Pi_{\bar{D}_{n,\ell}} \rightarrow P_\ell$, i.e., (31) is established.

Still by Lemma 2, using that $\Pi_{\bar{D}_{n,m}} P_\ell - P_\ell \rightarrow 0$ and that $\Pi_{\bar{D}_{n,m}} = \Pi_{\bar{D}_{n,\ell}} + \Pi_{G_{n,m,\ell}}$ by definition, we obtain $\Pi_{G_{n,m,\ell}} P_\ell \rightarrow 0$, and then $P_\ell \Pi_{G_{n,m,\ell}} \rightarrow 0$ by taking the adjoint. We have proved (32).

Let us now establish (33). Using (3), we have

$$\Pi_{\bar{D}_{n,\ell}} A \Pi_{\bar{D}_{n,\ell}} = \Pi_{\bar{D}_{n,\ell}} P_\ell A P_\ell \Pi_{\bar{D}_{n,\ell}} + \Pi_{\bar{D}_{n,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{\bar{D}_{n,\ell}}$$

Using (31), the first term at the right-hand side converges to $P_\ell A P_\ell$. Using the second part of (30) and the fact that $\|\Pi_{\bar{D}_{n,\ell}} (\text{id} - P_\ell)\| \leq 1$, we infer that the second term converges to 0. Hence $\Pi_{\bar{D}_{n,\ell}} A \Pi_{\bar{D}_{n,\ell}} \rightarrow P_\ell A P_\ell$.

Using (3) again, we have

$$\Pi_{G_{n,m,\ell}} A \Pi_{\overline{\mathcal{D}}_{n,\ell}} = \Pi_{G_{n,m,\ell}} P_\ell P_\ell A P_\ell \Pi_{\overline{\mathcal{D}}_{n,\ell}} + \Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{\overline{\mathcal{D}}_{n,\ell}}$$

The first term at the right-hand side converges to 0 because the range of $P_\ell A P_\ell \Pi_{\overline{\mathcal{D}}_{n,\ell}}$ is finite-dimensional and $\Pi_{G_{n,m,\ell}} P_\ell \rightarrow 0$ by (32). The second term converges to 0 because $A(\text{id} - P_\ell) \Pi_{\overline{\mathcal{D}}_{n,\ell}} \rightarrow 0$ by (30) and $\|\Pi_{G_{n,m,\ell}}(\text{id} - P_\ell)\| \leq 1$. Hence $\Pi_{G_{n,m,\ell}} A \Pi_{\overline{\mathcal{D}}_{n,\ell}} \rightarrow 0$.

Taking the adjoint and using the fact that A is selfadjoint (this is the only place where this assumption is actually crucial), we obtain that $\Pi_{\overline{\mathcal{D}}_{n,\ell}} A \Pi_{G_{n,m,\ell}} \rightarrow 0$.

Finally, $\Pi_{G_{n,m,\ell}} P_\ell A P_\ell \Pi_{G_{n,m,\ell}} = \Pi_{G_{n,m,\ell}} P_\ell A P_\ell P_\ell \Pi_{G_{n,m,\ell}}$ converges to 0 because $P_\ell \Pi_{G_{n,m,\ell}} \rightarrow 0$ by (32) and $\|\Pi_{G_{n,m,\ell}} P_\ell A P_\ell\| \leq \|P_\ell A P_\ell\|$ is uniformly bounded. \square

Lemma 5. *If $m \geq \ell$ then the POD reduced-order control system (8) is stabilizable whenever n is large enough.*

Proof. Without loss of generality, we assume that $m > \ell$. In the decomposition (29), we have

$$A_{n,m} = \begin{pmatrix} \Pi_{\overline{\mathcal{D}}_{n,\ell}} A \Pi_{\overline{\mathcal{D}}_{n,\ell}} & \Pi_{\overline{\mathcal{D}}_{n,\ell}} A \Pi_{G_{n,m,\ell}} \\ \Pi_{G_{n,m,\ell}} A \Pi_{\overline{\mathcal{D}}_{n,\ell}} & \Pi_{G_{n,m,\ell}} A \Pi_{G_{n,m,\ell}} \end{pmatrix} \quad \text{and} \quad B_{n,m} = \begin{pmatrix} \Pi_{\overline{\mathcal{D}}_{n,\ell}} B \\ \Pi_{G_{n,m,\ell}} B \end{pmatrix}.$$

Using Lemma 4 and the fact that

$$\begin{aligned} \Pi_{G_{n,m,\ell}} A \Pi_{G_{n,m,\ell}} &= \Pi_{G_{n,m,\ell}} P_\ell A P_\ell \Pi_{G_{n,m,\ell}} + \Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}} \\ &\sim \Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}} \end{aligned}$$

we infer that

$$A_{n,m} \sim \begin{pmatrix} P_\ell A P_\ell & 0 \\ 0 & \Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}} \end{pmatrix} \quad \text{and} \quad B_{n,m} = \begin{pmatrix} P_\ell B \\ \Pi_{G_{n,m,\ell}} B \end{pmatrix} \quad (34)$$

as $n \rightarrow +\infty$. Assumption **(H₂)** implies (by a Laplace transform argument) that any (real-valued) eigenvalue λ of $(\text{id} - P_\ell) A (\text{id} - P_\ell)$ satisfies $\lambda \leq -\gamma$, hence the matrix $\Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}}$ is uniformly Hurwitz with respect to n large enough, by Lemma 6 hereafter. Since the pair $(P_\ell A P_\ell, P_\ell B)$ satisfies the Kalman condition (Assumption **(H₄)**), the conclusion is now obvious. \square

Lemma 6. *The matrix $\Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}}$ is uniformly Hurwitz with respect to n .*

Proof. It follows from Assumption **(H₂)** that the operator $(\text{id} - P_\ell)(A + \gamma \text{id})(\text{id} - P_\ell)$ generates the contraction C_0 semigroup $((\text{id} - P_\ell)e^{\gamma t} S(t)(\text{id} - P_\ell))_{t \geq 0}$. By the converse of the Lumer-Phillips theorem (see [34, Chapter 1, Section 1.4, Theorem 4.3 and Corollary 4.4]), the (selfadjoint) operator $(\text{id} - P_\ell)(A + \gamma \text{id})(\text{id} - P_\ell)$ is dissipative, which means that

$$\langle (\text{id} - P_\ell)(A + \gamma \text{id})(\text{id} - P_\ell)y, y \rangle \leq 0 \quad \forall y \in H.$$

Hence

$$\langle \Pi_{G_{n,m,\ell}} (\text{id} - P_\ell)(A + \gamma \text{id})(\text{id} - P_\ell) \Pi_{G_{n,m,\ell}} y, y \rangle \leq 0 \quad \forall y \in H.$$

By the Lumer-Phillips theorem, $\Pi_{G_{n,m,\ell}} (\text{id} - P_\ell)(A + \gamma \text{id})(\text{id} - P_\ell) \Pi_{G_{n,m,\ell}}$ generates a contraction semigroup, hence, using (32) and the fact that $\Pi_{G_{n,m,\ell}} (\text{id} - P_\ell)(\text{id} - P_\ell) \Pi_{G_{n,m,\ell}} \sim \Pi_{G_{n,m,\ell}}$ as $n \rightarrow +\infty$, we infer that the semigroup generated by $\Pi_{G_{n,m,\ell}} (\text{id} - P_\ell) A (\text{id} - P_\ell) \Pi_{G_{n,m,\ell}}$ decreases exponentially (like $e^{-\gamma t}$). The lemma is proved. \square

3.2 Proof of Theorem 1

We are now in a position to prove Theorem 1. With respect to the decomposition $H = E_\ell \oplus F_\ell$, thanks to (2), the operator $A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}$ is written as the infinite-dimensional matrix

$$\left(\begin{array}{c|c} P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell & P_\ell BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}(\text{id} - P_\ell) \\ \hline (\text{id} - P_\ell)BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}P_\ell & (\text{id} - P_\ell)(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})(\text{id} - P_\ell) \end{array} \right) \quad (35)$$

and the exponential stability of (12) is equivalent to that of (35).

The idea is the following. By Proposition 2 in Appendix A, when ε is small and n is large enough, the feedback matrix $K_{n,m}(\varepsilon)$ essentially acts on the ℓ first modes that are instable, and $K_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}(\text{id} - P_\ell)$ converges to 0 as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$. Therefore, if ε is small enough and n is large enough then (35) is almost lower block triangular, with the $\ell \times \ell$ block at the top left being uniformly Hurwitz and the infinite-dimensional block at the bottom right (close to $(\text{id} - P_\ell)A(\text{id} - P_\ell)$) generating a uniformly exponentially stable semigroup.

We now give the detail of the arguments.

Lemma 7. *Given any $m \geq \ell$, the $\ell \times \ell$ matrix $P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell$ is uniformly Hurwitz and*

$$K_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}}(\text{id} - P_\ell) = o(1) \quad (36)$$

as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$.

Proof. Without loss of generality, we assume that $m > \ell$. Let us consider the POD reduced-order control system (8). In the decomposition $\overline{D}_{n,m} = \overline{D}_{n,\ell} \oplus G_{n,m,\ell}$ (see (29)), we have

$$A_{n,m} = \begin{pmatrix} \Pi_{\overline{D}_{n,\ell}} A \Pi_{\overline{D}_{n,\ell}} & \Pi_{\overline{D}_{n,\ell}} A \Pi_{G_{n,m,\ell}} \\ \Pi_{G_{n,m,\ell}} A \Pi_{\overline{D}_{n,\ell}} & \Pi_{G_{n,m,\ell}} A \Pi_{G_{n,m,\ell}} \end{pmatrix} \quad \text{and} \quad B_{n,m} = \begin{pmatrix} \Pi_{\overline{D}_{n,\ell}} B \\ \Pi_{G_{n,m,\ell}} B \end{pmatrix}$$

and accordingly, the maximal positive semidefinite solution $P_{n,m}(\varepsilon)$ of the algebraic Riccati equation (9) is written as

$$P_{n,m}(\varepsilon) = \begin{pmatrix} P_{n,m}^1(\varepsilon) & P_{n,m}^3(\varepsilon) \\ P_{n,m}^3(\varepsilon)^* & P_{n,m}^2(\varepsilon) \end{pmatrix}$$

and

$$\begin{aligned} K_{n,m}(\varepsilon) &= -B_{n,m}^* P_{n,m}(\varepsilon) \\ &= - \left(B^* \Pi_{\overline{D}_{n,\ell}} P_{n,m}^1(\varepsilon) + B^* \Pi_{G_{n,m,\ell}} P_{n,m}^3(\varepsilon)^*, B^* \Pi_{\overline{D}_{n,\ell}} P_{n,m}^3(\varepsilon) + B^* \Pi_{G_{n,m,\ell}} P_{n,m}^2(\varepsilon) \right) \end{aligned}$$

We apply Proposition 2 in Appendix A, roughly speaking, with $\alpha = \frac{1}{n}$. More precisely, the matrices $A(\alpha)$ and $B(\alpha)$ of Appendix A are any continuous interpolations at $\alpha = \frac{1}{n}$, $n \in \mathbb{N}^*$, of the matrices $A_{n,m}$ and $B_{n,m}$. By Lemma 4, all assumptions done in Appendix A are satisfied, because we have (34) with $-P_\ell A P_\ell$ being Hurwitz and $\Pi_{G_{n,m,\ell}}(\text{id} - P_\ell)A(\text{id} - P_\ell)\Pi_{G_{n,m,\ell}}$ being uniformly Hurwitz (by Lemma 6) and the pair $(P_\ell A P_\ell, P_\ell B)$ satisfies the Kalman condition by (H4). According to Proposition 2, we have $P_{n,m}(\varepsilon) \rightarrow P_{\infty,m}(0)$ as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$, with $P_{\infty,m}^1(0)$ symmetric positive semidefinite, and $A_{n,m} - B_{n,m} B_{n,m}^* P_{n,m}(\varepsilon)$ is uniformly Hurwitz with respect to ε small enough and n large enough. Moreover, by (49), using (31), we have

$$K_{n,m}(\varepsilon) = - \left(B^* P_\ell P_{\infty,m}^1(0), 0 \right) + o(1)$$

as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$. This is more than required to obtain (36). Moreover, using (34), we have

$$A_{n,m} + B_{n,m}K_{n,m}(\varepsilon) \sim \begin{pmatrix} P_\ell(A - BB^*P_\ell P_{\infty,m}^1(0))P_\ell & 0 \\ \Pi_{G_{n,m,\ell}}BB^*P_\ell P_{\infty,m}^1(0) & \Pi_{G_{n,m,\ell}}(\text{id} - P_\ell)A(\text{id} - P_\ell)\Pi_{G_{n,m,\ell}} \end{pmatrix} \quad (37)$$

as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$. But the block at the top left is equivalent to $P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell$, indeed:

$$P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell = P_\ell(A - BB^*P_\ell P_{\infty,m}^1(0))P_\ell + o(1)$$

as $\varepsilon \rightarrow 0$ and $n \rightarrow +\infty$.

It follows that $P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell$, which is the block on the top left of (35), is uniformly Hurwitz for ε small enough and n large enough. \square

Theorem 1 is proved.

To finish, let us prove (15) in Remark 3. As said above, in approximation, the infinite-dimensional matrix (35) is lower block triangular. The block at the bottom right is approximately equal to $(\text{id} - P_\ell)A(\text{id} - P_\ell)$, which generates a semigroup that is exponentially stable, with decay rate γ (given by (H₂)). The block at the top left is treated thanks to the proof of Lemma 7. By (37), $P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell$ is approximately equal to the block at the top left of the lower block triangular matrix $A_{n,m} + B_{n,m}K_{n,m}(\varepsilon)$. Therefore, the spectral abscissa of $A_{n,m} + B_{n,m}K_{n,m}(\varepsilon)$ is approximately equal to the minimum of the spectral abscissa of $P_\ell(A + BK_{n,m}(\varepsilon)\Pi_{\overline{D}_{n,m}})P_\ell$ and of the spectral abscissa of $\Pi_{G_{n,m,\ell}}(\text{id} - P_\ell)A(\text{id} - P_\ell)\Pi_{G_{n,m,\ell}}$ (which is greater than or equal to γ). The formula (15) follows.

4 Conclusion and perspectives

Considering a linear autonomous control system in infinite dimension, the Proper Orthogonal Decomposition (POD) approach generates a finite-dimensional reduced-order model, which can be used to generate low-order controls. Applying the Riccati theory to the reduced-order control system leads to a finite-dimensional linear stabilizing feedback. In this paper, we have proved that, under appropriate assumptions, this reduced-order feedback exponentially stabilizes as well the whole infinite-dimensional control system.

Our assumptions are of spectral nature. The operator A underlying the control system is assumed to be selfadjoint, and the system with null control is assumed to have a finite-dimensional instable part and an infinite-dimensional stable part. Our assumptions involve the case of heat-like equations with internal control, but not the case of damped wave equations.

Several comments and open issues are in order.

- In the POD method, we have taken snapshots at times $t_k = kT$. The choice of such regular snapshots is compatible with the application of the Kalman condition in our proofs. Treating the case of other, more random snapshots t_k is an interesting issue.
- It is likely that, under controllability assumptions, one can obtain a result of rapid stabilization in the following sense: given any $\gamma > 0$, find a low-order feedback matrix $K_{n,m}^\gamma$, designed from a POD approximation, such that $\|y(t)\| \leq C(\gamma)e^{-\gamma t}\|y(0)\|$ for every $t \geq 0$. We let this issue for further investigations.
- The spectral assumptions (H₁), (H₂), (H₃) done at the beginning of the paper typically apply to a parabolic partial differential equation with a selfadjoint operator, like heat, anomalous

equations or Stokes equations. As mentioned above, our assumptions do not involve the case of damped wave equations, for which

$$A = \begin{pmatrix} 0 & \text{id} \\ \Delta + a \text{id} & -b \text{id} \end{pmatrix}$$

with $b > 0$ on Ω , is neither selfadjoint nor normal. Treating such operators is a major open issue.

- Throughout the paper we have considered bounded control operators B . We are thus able to treat the case of internal controls, but not, in general, the case of boundary controls. The case where $B \in L(U, D(A^*)')$ is unbounded but admissible (see [46]) is open.
- We have considered linear control systems, but, since POD can be applied as well to nonlinear systems, we expect that similar results may hold true for semilinear control systems, of the form

$$\dot{y}(t) = Ay(t) + F(y(t)) + Bu(t)$$

where $F : H \rightarrow H$ is a nonlinear mapping of class C^1 , Lipschitz on the bounded sets of H , with $F(0) = 0$ (see [2] for stabilization results and discretization issues). Adapting our approach to that framework is another interesting open issue.

A Appendix: an asymptotic result in Riccati theory

Let ℓ , m and p be integers satisfying $0 < \ell < m$. Hereafter, we always denote by $\|\cdot\|$ the Euclidean norm, may it be in \mathbb{R}^ℓ , in $\mathbb{R}^{m-\ell}$, in \mathbb{R}^m or in \mathbb{R}^p . We consider the family of autonomous linear control systems in \mathbb{R}^m , indexed by $\alpha \in (0, \alpha_0]$ for some $\alpha_0 > 0$,

$$\begin{aligned} \dot{x}_1 &= A_1(\alpha)x_1 + A_3(\alpha)x_2 + B_1(\alpha)u \\ \dot{x}_2 &= A_4(\alpha)x_1 + A_2(\alpha)x_2 + B_2(\alpha)u \end{aligned} \tag{38}$$

with matrices $A_1(\alpha)$ of size $\ell \times \ell$, $A_2(\alpha)$ of size $(m-\ell) \times (m-\ell)$, $A_3(\alpha)$ of size $\ell \times (m-\ell)$, $A_4(\alpha)$ of size $(m-\ell) \times \ell$, $B_1(\alpha)$ of size $\ell \times p$ and $B_2(\alpha)$ of size $(m-\ell) \times p$, depending continuously on α and satisfying

$$\begin{aligned} A_1(\alpha) &= A_1 + \underset{\alpha \rightarrow 0}{\mathbf{o}}(1), & A_3(\alpha) &= \underset{\alpha \rightarrow 0}{\mathbf{o}}(1), & A_4(\alpha) &= \underset{\alpha \rightarrow 0}{\mathbf{o}}(1), \\ B_1(\alpha) &= B_1 + \underset{\alpha \rightarrow 0}{\mathbf{o}}(1), \\ \|B_2(\alpha)\| &\text{ uniformly bounded with respect to } \alpha \in (0, \alpha_0], \end{aligned} \tag{39}$$

with A_1 of size $\ell \times \ell$ and B_1 of size $\ell \times p$. We assume that:

- $-A_1(\alpha)$ and $A_2(\alpha)$ are uniformly Hurwitz with respect to $\alpha \in (0, \alpha_0]$, i.e., there exists $\eta > 0$ such that, for every $\alpha \in (0, \alpha_0]$, all eigenvalues λ of $-A_1(\alpha)$ and of $A_2(\alpha)$ satisfy $\text{Re}(\lambda) \leq -\eta$. This means that, for $u = 0$, the first part of the system (38) is instable, while the second part is asymptotically stable, uniformly with respect to $\alpha \in (0, \alpha_0]$.
- The pair (A_1, B_1) satisfies the Kalman condition.

Note that this implies that the pair $(A_1(\alpha), B_1(\alpha))$ satisfies the Kalman condition for every $\alpha \in (0, \alpha_0]$, provided α_0 is small enough. We set

$$A(\alpha) = \begin{pmatrix} A_1(\alpha) & A_3(\alpha) \\ A_4(\alpha) & A_2(\alpha) \end{pmatrix} \quad \text{and} \quad B(\alpha) = \begin{pmatrix} B_1(\alpha) \\ B_2(\alpha) \end{pmatrix}.$$

Of course, it follows from (39) that, as soon as α_0 is small enough, given any $\alpha \in (0, \alpha_0]$, $A(\alpha)$ has no pure imaginary eigenvalue and the pair $(A(\alpha), B(\alpha))$ is stabilizable. Then, in order to stabilize (38), it suffices to stabilize the first part of the system, by choosing $u = K_1 x_1$ such that $A_1 + B_1 K_1$ is Hurwitz (this is possible because the pair (A_1, B_1) satisfies the Kalman condition), and then clearly this control stabilizes as well the whole system (38) for any α small enough.

But here, following the spirit of the method developed in this paper, we “forget” that we know how to split the above system in a stable part and an instable part, and hereafter we propose to stabilize the control system (38) “blindly”, i.e., ignoring the exact splitting, by using the classical Riccati theory, as follows.

Algebraic Riccati equation. Let $\varepsilon \geq 0$ and $\alpha \in (0, \alpha_0]$ be arbitrary. Since the pair $(A(\alpha), B(\alpha))$ is stabilizable, by [1, Corollary 2.3.7 page 55] there exists a (unique) maximal symmetric positive semidefinite matrix (of size $m \times m$)

$$P(\varepsilon, \alpha) = \begin{pmatrix} P_1(\varepsilon, \alpha) & P_3(\varepsilon, \alpha) \\ P_3(\varepsilon, \alpha)^* & P_2(\varepsilon, \alpha) \end{pmatrix}$$

solution of the algebraic Riccati equation

$$A(\alpha)^* P(\varepsilon, \alpha) + P(\varepsilon, \alpha) A(\alpha) - P(\varepsilon, \alpha) B(\alpha) B(\alpha)^* P(\varepsilon, \alpha) + \varepsilon I_m = 0. \quad (40)$$

Moreover:

- $A(\alpha) - B(\alpha) B(\alpha)^* P(\varepsilon, \alpha)$ is semi-stabilizing, i.e., its eigenvalues have nonpositive real part;
- if $\varepsilon > 0$ then $P(\varepsilon, \alpha)$ is positive definite, and $A(\alpha) - B(\alpha) B(\alpha)^* P(\varepsilon, \alpha)$ is Hurwitz, i.e., its eigenvalues have negative real part.¹

As a consequence, setting

$$K(\varepsilon, \alpha) = -B(\alpha)^* P(\varepsilon, \alpha), \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the linear feedback

$$\begin{aligned} \hat{u}_{\varepsilon, \alpha} &= K(\varepsilon, \alpha) x = -B(\alpha)^* P(\varepsilon, \alpha) x \\ &= -(B_1^* P_1(\varepsilon, \alpha) + B_2^* P_3(\varepsilon, \alpha)^*) x_1 - (B_1^* P_3(\varepsilon, \alpha) + B_2^* P_2(\varepsilon, \alpha)) x_2 \end{aligned} \quad (41)$$

exponentially stabilizes the system (38) to the origin.

Relationship with linear quadratic optimal control. Throughout, for every $\alpha \in (0, \alpha_0]$ we denote by

$$x(t; \alpha, u, x_1(0), x_2(0)) = \begin{pmatrix} x_1(t; \alpha, u, x_1(0), x_2(0)) \\ x_2(t; \alpha, u, x_1(0), x_2(0)) \end{pmatrix}$$

the solution of (38) with initial condition $(x_1(0), x_2(0))$ and with control u .

¹This is a consequence of [1, Chapter 2, Corollary 2.4.3 page 60 and Theorem 2.4.25 page 77]. Indeed, since $A(\alpha)$ has no pure imaginary eigenvalue, the Hamiltonian matrix

$$\begin{pmatrix} A(\alpha) & -B(\alpha)^* B(\alpha) \\ -\varepsilon I_m & -A(\alpha)^* \end{pmatrix}$$

has no pure imaginary eigenvalue.

According to the well known linear quadratic Riccati theory (see, e.g., [1, 25, 29, 40, 43]), given any $\varepsilon \geq 0$ and any $\alpha \in (0, \alpha_0]$, given any initial condition $(x_{1,0}, x_{2,0}) \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$, there exists a unique optimal control minimizing the cost functional

$$J_{\varepsilon, \alpha}(u; x_{1,0}, x_{2,0}) = \int_0^{+\infty} (\varepsilon \|x_1(t; \alpha, u, x_{1,0}, x_{2,0})\|^2 + \|x_2(t; \alpha, u, x_{1,0}, x_{2,0})\|^2 + \|u(t)\|^2) dt \quad (42)$$

over all possible controls $u \in L^2(0, +\infty; \mathbb{R}^p)$. If $\varepsilon = 0$ then the optimal control is $u = 0$. If $\varepsilon > 0$ then the optimal control coincides with the feedback control $\hat{u}_{\varepsilon, \alpha}$ defined by (41), which exponentially stabilizes the control system (38). Moreover, $\hat{u}_{\varepsilon, \alpha}$ is linear with respect to the initial condition $(x_{1,0}, x_{2,0})$, and we have

$$\begin{aligned} J_{\varepsilon, \alpha}(\hat{u}_{\varepsilon, \alpha}; x_{1,0}, x_{2,0}) &= \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix}^* P(\varepsilon, \alpha) \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} \\ &= x_{1,0}^* P_1(\varepsilon, \alpha) x_{1,0} + 2x_{1,0}^* P_3(\varepsilon, \alpha) x_{2,0} + x_{2,0}^* P_2(\varepsilon, \alpha) x_{2,0}. \end{aligned}$$

Our objective is to investigate the asymptotics of $P(\varepsilon, \alpha)$ as $\varepsilon \rightarrow 0$. We are going to prove that $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ remains uniformly Hurwitz with respect to (ε, α) small enough and that the optimal feedback $K(\varepsilon, \alpha) = -B(\alpha)^*P(\varepsilon, \alpha)$ essentially acts on the ℓ first modes that are unstable (eigenvalues of $A_1(\alpha)$), in the sense that $K(\alpha) = -(B_1(\alpha)^*P_1(0, 0), 0) + o(1)$ as $(\varepsilon, \alpha) \rightarrow 0$.

A.1 A first result in the block diagonal case

In this first subsection, we assume that

$$A_3(\alpha) = 0 \quad \text{and} \quad A_4(\alpha) = 0 \quad (43)$$

for every $\alpha \in (0, \alpha_0]$, i.e., that $A(\alpha)$ is block diagonal.

Proposition 1. *We have the following results:*

- For every $\alpha \in (0, \alpha_0]$, $P(\varepsilon, \alpha)$ is continuous with respect to $\varepsilon \geq 0$. Moreover, we have $P_2(0, \alpha) = 0$ and $P_3(0, \alpha) = 0$ for every $\alpha \in (0, \alpha_0]$ and

$$P_1(\varepsilon, \alpha) = P_1(0, \alpha) + o_{\varepsilon \rightarrow 0}(1), \quad P_2(\varepsilon, \alpha) = o_{\varepsilon \rightarrow 0}(1), \quad P_3(\varepsilon, \alpha) = o_{\varepsilon \rightarrow 0}(1) \quad (44)$$

where the remainder terms are uniform with respect to $\alpha \in (0, \alpha_0]$. In particular, the mapping $(\varepsilon, \alpha) \mapsto P(\varepsilon, \alpha)$ has a continuous extension at $(0, 0)$.

- For every $\alpha \in [0, \alpha_0]$, the matrix $P_1(0, \alpha)$ is symmetric positive semidefinite (but not necessarily definite).
- The matrix $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is uniformly Hurwitz with respect to (ε, α) small enough, meaning that there exist $\varepsilon_0 > 0$ and $\eta > 0$ such that, for every $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha_0]$, every eigenvalue λ of $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is such that $\text{Re}(\lambda) \leq -\eta$.

Proof. The proof goes in three steps.

Step 1. The fact that for α fixed $P(\varepsilon, \alpha)$ depends continuously on ε is completely general. We sketch the proof. Let $\alpha \in (0, \alpha_0]$ be fixed. By the minimization property, the symmetric positive semidefinite matrix $P(\varepsilon, \alpha)$ is monotone increasing with respect to ε , hence $P(\varepsilon, \alpha)$ is uniformly bounded with respect to $\varepsilon \in [0, 1]$. Let $\varepsilon \geq 0$ be fixed and let $(\varepsilon_k)_{k \in \mathbb{N}^*}$ be a sequence of nonnegative real numbers converging to ε . Since $P(\varepsilon_k, \alpha)$ is bounded, up to some subsequence

we have $P(\varepsilon_k, \alpha) \rightarrow \tilde{P}(\alpha)$ as $k \rightarrow +\infty$. Since $P(\varepsilon_k, \alpha)$ is the (unique) maximal symmetric positive semidefinite matrix solution of (40) (with $\varepsilon = \varepsilon_k$), we easily infer that $\tilde{P}(\alpha)$ is the maximal symmetric positive semidefinite matrix solution of (40). Hence (by uniqueness) $\tilde{P}(\alpha) = P(\varepsilon, \alpha)$, and since the argument is valid for any sequence, the continuity property follows. In particular, we have $P(\varepsilon, \alpha) = P(0, \alpha) + \underset{\varepsilon \rightarrow 0}{o}(1)$ as $\varepsilon \rightarrow 0$. Anyway this has been done for α fixed and at this step we do not know yet that $(\varepsilon, \alpha) \mapsto P(\varepsilon, \alpha)$ has a continuous extension at $(0, 0)$.

Note that $P_1(\varepsilon, \alpha)$ and $P_2(\varepsilon, \alpha)$ are symmetric positive semidefinite for every $\varepsilon \geq 0$ and every $\alpha \in (0, \alpha_0]$, and positive definite when $\varepsilon > 0$ (because $P(\varepsilon, \alpha)$ is so).

Step 2. Let us prove that $P_2(\varepsilon, \alpha) = \underset{\varepsilon \rightarrow 0}{o}(1)$ and that $P_3(\varepsilon, \alpha) = \underset{\varepsilon \rightarrow 0}{o}(1)$ uniformly with respect to $\alpha \in (0, \alpha_0]$, provided α_0 is small enough.

Given any $\varepsilon > 0$, any $\alpha \in (0, \alpha_0]$ and any $x_{2,0} \in \mathbb{R}^{m-\ell}$, we have $x_{2,0}^* P_2(\varepsilon, \alpha) x_{2,0} \leq J_{\varepsilon, \alpha}(0; 0, x_{2,0})$ by the minimization property. Besides, using (43) and the fact that $A_2(\alpha)$ is uniformly Hurwitz, we have

$$J_{\varepsilon, \alpha}(0; 0, x_{2,0}) = \varepsilon \int_0^{+\infty} \|e^{tA_2(\alpha)} x_{2,0}\|^2 dt \leq C\varepsilon \|x_{2,0}\|^2$$

for some $C > 0$ independent of (ε, α) and of $x_{2,0}$. It follows that $P_2(\varepsilon, \alpha) = \underset{\varepsilon \rightarrow 0}{o}(1)$ uniformly with respect to $\alpha \in (0, \alpha_0]$.

Now, let $x_{1,0} \in \mathbb{R}^\ell$ and $x_{2,0} \in \mathbb{R}^{m-\ell}$ be arbitrary. For every $\varepsilon > 0$ and every $\alpha \in (0, \alpha_0]$, let $u_{\varepsilon, \alpha} = u_{\varepsilon, \alpha}(x_{1,0}, 0)$ be the solution of the minimization problem $\inf_u J_{\varepsilon, \alpha}(u; x_{1,0}, 0)$. Then $J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}; x_{1,0}, 0) = x_{1,0}^* P_1(\varepsilon, \alpha) x_{1,0}$ and we have the following lemma.

Lemma 8. *We have $J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}; x_{1,0}, x_{2,0}) - J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_{1,0}, 0) = \underset{\varepsilon \rightarrow 0}{o}(1) (\|x_{1,0}\|^2 + \|x_{2,0}\|^2)$ uniformly with respect to $\alpha \in (0, \alpha_0]$.*

Proof. We have

$$\begin{aligned} J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_{1,0}, x_{2,0}) &= \int_0^{+\infty} (\varepsilon \|x_1(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, x_{2,0})\|^2 + \varepsilon \|x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, x_{2,0})\|^2 \\ &\quad + \|u_{\varepsilon, \alpha}(t)\|^2) dt \\ J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_{1,0}, 0) &= \int_0^{+\infty} (\varepsilon \|x_1(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)\|^2 + \varepsilon \|x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)\|^2 + \|u_{\varepsilon, \alpha}(t)\|^2) dt \end{aligned}$$

with the first integral being possibly equal to $+\infty$ (this is not a problem and anyway we show below that it is finite). Now, on the one part, using (43), we have $x_1(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, x_{2,0}) = x_1(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)$. On the other part, we infer from the Duhamel formula that

$$x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, x_{2,0}) = e^{tA_2(\alpha)} x_{2,0} + x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0).$$

Hence

$$\begin{aligned} &J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}; x_{1,0}, x_{2,0}) - J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_{1,0}, 0) \\ &= \varepsilon \int_0^{+\infty} (\|e^{tA_2(\alpha)} x_{2,0} + x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)\|^2 - \|x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)\|^2) dt \quad (45) \\ &= \varepsilon \int_0^{+\infty} (\|e^{tA_2(\alpha)} x_{2,0}\|^2 + 2\langle e^{tA_2(\alpha)} x_{2,0}, x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0) \rangle) dt \end{aligned}$$

As already said, as a general fact of the Riccati theory, all eigenvalues of $A(\alpha) - B(\alpha)B(\alpha)^* P(\varepsilon, \alpha)$ have nonpositive real part for every $\varepsilon \geq 0$ and every $\alpha \in (0, \alpha_0]$, and thus the solutions of $\dot{x} =$

$(A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha))x$ remain uniformly bounded by the norm of their initial condition: there exists $C > 0$ such that $\|x_2(t; \alpha, u_{\varepsilon, \alpha}, x_{1,0}, 0)\| \leq C\|x_{1,0}\|$ for every $t \geq 0$, every $\varepsilon \in [0, 1]$ and every $\alpha \in (0, \alpha_0]$. Since $A_2(\alpha)$ is uniformly Hurwitz, using (45) and the Cauchy-Schwarz inequality, the lemma follows. \square

By the minimization property, we have $(x_{1,0}^*, x_{2,0}^*)P(\varepsilon, \alpha) \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} \leq J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}; x_{1,0}, x_{2,0})$ and therefore, using Lemma 8,

$$\begin{aligned} 2x_{1,0}^*P_3(\varepsilon, \alpha)x_{2,0} &= (x_{1,0}^*, x_{2,0}^*)P(\varepsilon, \alpha) \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} - x_{1,0}^*P_1(\varepsilon, \alpha)x_{1,0} - x_{2,0}^*P_2(\varepsilon, \alpha)x_{2,0} \\ &\leq J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}; x_{1,0}, x_{2,0}) - J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_{1,0}, 0) + \underset{\varepsilon \rightarrow 0}{o}(1)\|x_{2,0}\|^2 \\ &= \underset{\varepsilon \rightarrow 0}{o}(1) (\|x_{1,0}\|^2 + \|x_{2,0}\|^2) \end{aligned}$$

uniformly with respect to $\alpha \in (0, \alpha_0]$. Since $x_{1,0}$ and $x_{2,0}$ are arbitrary, it follows that $P_3(\varepsilon, \alpha) = \underset{\varepsilon \rightarrow 0}{o}(1)$ uniformly with respect to $\alpha \in (0, \alpha_0]$.

At this step, we have also obtained that the mappings $(\varepsilon, \alpha) \mapsto P_2(\varepsilon, \alpha)$ and $(\varepsilon, \alpha) \mapsto P_3(\varepsilon, \alpha)$ have a continuous extension at $(0, 0)$.

Step 3. The Riccati equation (40) gives the equation satisfied by $P_1(\varepsilon, \alpha)$:

$$\begin{aligned} A_1(\alpha)^*P_1(\varepsilon, \alpha) + P_1(\varepsilon, \alpha)A_1(\alpha) - P_1(\varepsilon, \alpha)B_1(\alpha)B_1(\alpha)^*P_1(\varepsilon, \alpha) - P_3(\varepsilon, \alpha)B_2(\alpha)B_1(\alpha)^*P_1(\varepsilon, \alpha) \\ - P_1(\varepsilon, \alpha)B_1(\alpha)B_2(\alpha)^*P_3(\varepsilon, \alpha)^* - P_3(\varepsilon, \alpha)B_2(\alpha)B_2(\alpha)^*P_3(\varepsilon, \alpha)^* = -\varepsilon I_\ell. \end{aligned} \quad (46)$$

Taking the limit $\varepsilon \rightarrow 0$ in (46), we obtain

$$A_1(\alpha)^*P_1(0, \alpha) + P_1(0, \alpha)A_1(\alpha) - P_1(0, \alpha)B_1(\alpha)B_1(\alpha)^*P_1(0, \alpha) = 0 \quad (47)$$

which is a Riccati equation with zero weight on the state. Since the pair $(A_1(\alpha), B_1(\alpha))$ satisfies the Kalman condition and thus is stabilizable, and since the Hamiltonian matrix

$$\begin{pmatrix} A_1(\alpha) & -B_1(\alpha)^*B_1(\alpha) \\ 0 & -A_1(\alpha)^* \end{pmatrix}$$

has no pure imaginary eigenvalue (because $-A_1(\alpha)$ is Hurwitz), it follows from [1, Chapter 2, Corollary 2.4.3 page 60 and Theorem 2.4.25 page 77] that the Riccati equation (47) has a (unique) maximal symmetric positive semidefinite solution, which is moreover stabilizing, hence $P_1(0, \alpha)$ is a symmetric positive semidefinite matrix such that $A_1(\alpha) - B_1(\alpha)B_1(\alpha)^*P_1(0, \alpha)$ is Hurwitz, uniformly with respect to $\alpha \in (0, \alpha_0]$.

Moreover, for $\alpha = 0$, the above argument shows that (47) has a (unique) maximal symmetric positive semidefinite solution \tilde{P} . We claim that $P_1(\varepsilon, \alpha) = \tilde{P} + o(1)$ as $(\varepsilon, \alpha) \rightarrow (0, 0)$. By (46), using that $P_2(\varepsilon, \alpha) = o(1)$ and $P_3(\varepsilon, \alpha) = o(1)$ as $(\varepsilon, \alpha) \rightarrow (0, 0)$, we have

$$A_1(\alpha)^*P_1(\varepsilon, \alpha) + P_1(\varepsilon, \alpha)A_1(\alpha) - P_1(\varepsilon, \alpha)B_1(\alpha)B_1(\alpha)^*P_1(\varepsilon, \alpha) = o(1)$$

as $(\varepsilon, \alpha) \rightarrow (0, 0)$. Given any sequence $(\varepsilon_k, \alpha_k)$ converging to $(0, 0)$, up to some subsequence we must have $P(\varepsilon_k, \alpha_k) \rightarrow \tilde{P}$ as $k \rightarrow +\infty$ (by uniqueness and maximality in the Riccati theory). Since this argument is valid for any subsequence, it follows that the mapping $(\varepsilon, \alpha) \mapsto P_1(\varepsilon, \alpha)$ has a continuous extension at $(0, 0)$.

Finally, let us prove that the closed-loop matrix $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$, which is written (omitting the dependence in ε and α to keep a better readability) as

$$\left(\begin{array}{c|c} A_1 - B_1B_1^*P_1 - B_1B_2^*P_3 & -B_1B_1^*P_3 - B_1B_2^*P_2 \\ \hline -B_2B_1^*P_1 - B_2B_2^*P_3 & A_2 - B_2B_1^*P_3 - B_2B_2^*P_2 \end{array} \right)$$

is uniformly Hurwitz with respect to (ε, α) small enough. By the Riccati theory, we already know that, for every fixed $\varepsilon > 0$, $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is Hurwitz, uniformly with respect to $\alpha \in [0, \alpha_0]$. Now, thanks to (44),

$$A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha) = \left(\begin{array}{c|c} A_1(\alpha) - B_1(\alpha)B_1(\alpha)^*P_1(0, \alpha) + \underset{\varepsilon \rightarrow 0}{\text{o}}(1) & \underset{\varepsilon \rightarrow 0}{\text{o}}(1) \\ \hline -B_2(\alpha)B_1(\alpha)^*P_1(0, \alpha) + \underset{\varepsilon \rightarrow 0}{\text{o}}(1) & A_2(\alpha) + \underset{\varepsilon \rightarrow 0}{\text{o}}(1) \end{array} \right)$$

where the remainder terms are uniform with respect to $\alpha \in [0, \alpha_0]$. Using the fact that the determinant is differentiable, and that $d \det(X) \cdot H = \text{tr}(\text{com}(X)^*H)$, we get that

$$\begin{aligned} \det(A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha) - \lambda I_m) \\ = \det(A_1(\alpha) - B_1(\alpha)B_1(\alpha)^*P_1(0, \alpha) - \lambda I_\ell) \det(A_2(\alpha) - \lambda I_{m-\ell}) + \underset{\varepsilon \rightarrow 0}{\text{o}}(1) \end{aligned}$$

from which it follows that $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is uniformly Hurwitz with respect to $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha_0]$ for some $\varepsilon_0 > 0$ small enough. The proposition is proved. \square

A.2 Asymptotic result in the general case

We now drop the assumption (43) and we provide a generalization of Proposition 1. Since $A_3(\alpha) = \underset{\alpha \rightarrow 0}{\text{o}}(1)$ and $A_4(\alpha) = \underset{\alpha \rightarrow 0}{\text{o}}(1)$, the control system (38) can be viewed as a perturbation of the system (38) under the assumption (43), which has been studied in Appendix A.1.

Proposition 2. *We have the following results:*

- For every $\alpha \in (0, \alpha_0]$, $P(\varepsilon, \alpha)$ is continuous with respect to $\varepsilon \geq 0$. Moreover, the mapping $(\varepsilon, \alpha) \mapsto P(\varepsilon, \alpha)$ has a continuous extension at $(0, 0)$, with $P_2(0, 0) = 0$ and $P_3(0, 0) = 0$:

$$P_1(\varepsilon, \alpha) = P_1(0, 0) + \text{o}(1), \quad P_2(\varepsilon, \alpha) = \text{o}(1), \quad P_3(\varepsilon, \alpha) = \text{o}(1) \quad (48)$$

as $(\varepsilon, \alpha) \rightarrow (0, 0)$.

- For every $\alpha \in [0, \alpha_0]$, the matrix $P_1(0, \alpha)$ is symmetric positive semidefinite (but not necessarily definite).
- The matrix $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is uniformly Hurwitz with respect to (ε, α) small enough, meaning that there exist $\varepsilon_0 > 0$, $\alpha'_0 \in (0, \alpha_0]$ and $\eta > 0$ such that, for every $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha'_0]$, every eigenvalue λ of $A(\alpha) - B(\alpha)B(\alpha)^*P(\varepsilon, \alpha)$ is such that $\text{Re}(\lambda) \leq -\eta$.

Note that, as a consequence of (48), the feedback matrix $K(\varepsilon, \alpha) = -B(\alpha)^*P(\varepsilon, \alpha)$ satisfies

$$\begin{aligned} K(\varepsilon, \alpha) &= -(B_1(\alpha)^*P_1(\varepsilon, \alpha) + B_2(\alpha)^*P_3(\varepsilon, \alpha), B_1(\alpha)^*P_3(\varepsilon, \alpha) + B_2(\alpha)^*P_2(\varepsilon, \alpha)) \\ &= -(B_1(\alpha)^*P_1(0, 0), 0) + \text{o}(1) \end{aligned} \quad (49)$$

as $(\varepsilon, \alpha) \rightarrow 0$. In other words, for (ε, α) small the feedback matrix $K(\varepsilon, \alpha)$ essentially acts on the ℓ first modes that are instable (eigenvalues of $A_1(\alpha)$).

Proof. First of all, proceeding as in the first step of the proof of Proposition 1 is the same, we obtain the continuity of $P(\varepsilon, \alpha)$ with respect to ε . Anyway at this step we do not know yet that $(\varepsilon, \alpha) \mapsto P(\varepsilon, \alpha)$ has a continuous extension at $(0, 0)$.

The proof is now different from the second and third steps of the proof of Proposition 1 in which we used the specific (block diagonal) form of the system under the assumption (43), but we are going to use the result of Proposition 1.

For the block diagonal control system studied in Section A.1, which is written as

$$\dot{y} = A^{\text{diag}}(\alpha)y + B(\alpha)v \quad (50)$$

with

$$A^{\text{diag}}(\alpha) = \begin{pmatrix} A_1(\alpha) & 0 \\ 0 & A_2(\alpha) \end{pmatrix}$$

we denote in what follows by

$$Q(\varepsilon, \alpha) = \begin{pmatrix} Q_1(\varepsilon, \alpha) & Q_3(\varepsilon, \alpha) \\ Q_3(\varepsilon, \alpha)^* & Q_2(\varepsilon, \alpha) \end{pmatrix}$$

(instead of P) the Riccati matrix solution of (40). It satisfies the conclusions of Proposition 1.

We are now going to compare $P(\varepsilon, \alpha)$ (attached to the complete system (38)) with $Q(\varepsilon, \alpha)$ (attached to the block diagonal system (50)).

Lemma 9. *There exist $\varepsilon_0 > 0$ and $\alpha'_0 \in (0, \alpha_0]$ such that:*

- $A^{\text{diag}}(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$ and $A(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$ are uniformly Hurwitz with respect to $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha'_0]$.

- We have

$$e^{t(A(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha))} = e^{t(A^{\text{diag}}(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha))} + \underset{\alpha \rightarrow 0}{o}(1) \quad (51)$$

uniformly with respect to $t \geq 0$ and to $\varepsilon \in [0, \varepsilon_0]$.

Proof. The Hurwitz property of $A^{\text{diag}}(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$ has been obtained in Proposition 1. Since $A_3(\alpha) = \underset{\alpha \rightarrow 0}{o}(1)$ and $A_4(\alpha) = \underset{\alpha \rightarrow 0}{o}(1)$, we have $A(\alpha) = A^{\text{diag}}(\alpha) + \underset{\alpha \rightarrow 0}{o}(1)$ and the Hurwitz property of $A(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$ follows for every α small enough.

For the second point, we note that, by the Duhamel formula,

$$e^{t(A - BB^*Q)} = e^{t(A^{\text{diag}} - BB^*Q)} + \int_0^t e^{(t-s)(A^{\text{diag}} - BB^*Q)} (A - A^{\text{diag}}) e^{s(A - BB^*Q)} ds$$

where we have dropped the dependence in ε and α for readability. Since the exponentials in the integral involve Hurwitz matrices, and since the middle term in the integral is a $\underset{\alpha \rightarrow 0}{o}(1)$, the estimate (51) follows. \square

By Lemma 9, the matrix $A^{\text{diag}}(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$ (resp., $A(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha)$) is uniformly Hurwitz with respect to $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha'_0]$. Hence the feedback control

$$v_{\varepsilon, \alpha} = -B(\alpha)^*Q(\varepsilon, \alpha)y \quad (\text{resp., } u_{\varepsilon, \alpha} = -B(\alpha)^*Q(\varepsilon, \alpha)x)$$

stabilizes the block diagonal system (50) (resp., the system (38)) and we infer that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$\|y(t; \alpha, v_{\varepsilon, \alpha}, x_0)\|^2 + \|v_{\varepsilon, \alpha}(t)\|^2 + \|x(t; \alpha, u_{\varepsilon, \alpha}, x_0)\|^2 + \|u_{\varepsilon, \alpha}(t)\|^2 \leq C_1 e^{-C_2 t} \|x_0\|^2$$

for every $x_0 \in \mathbb{R}^m$, uniformly with respect to $t \geq 0$ and to $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha'_0]$. For every $\delta > 0$, let $T_\delta > 0$ be such that $\int_{T_\delta}^{+\infty} C_1 e^{-C_2 t} dt \leq \delta$. Then

$$\int_{T_\delta}^{+\infty} (\|y(t; \alpha, v_{\varepsilon, \alpha}, x_0)\|^2 + \|v_{\varepsilon, \alpha}(t)\|^2 + \|x(t; \alpha, u_{\varepsilon, \alpha}, x_0)\|^2 + \|u_{\varepsilon, \alpha}(t)\|^2) dt \leq \delta \|x_0\|^2 \quad (52)$$

for every $x_0 \in \mathbb{R}^m$, for every $t \geq 0$ and for all $(\varepsilon, \alpha) \in [0, \varepsilon_0] \times [0, \alpha'_0]$.

Besides, since by definition

$$\begin{aligned} y(t; \alpha, v_{\varepsilon, \alpha}, x_0) &= e^{t(A^{\text{diag}}(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha))} x_0 \\ x(t; \alpha, u_{\varepsilon, \alpha}, x_0) &= e^{t(A(\alpha) - B(\alpha)B(\alpha)^*Q(\varepsilon, \alpha))} x_0 \end{aligned}$$

we infer from Lemma 9 that

$$\|y(t; \alpha, v_{\varepsilon, \alpha}, x_0)\|^2 + \|v_{\varepsilon, \alpha}(t)\|^2 = \|x(t; \alpha, u_{\varepsilon, \alpha}, x_0)\|^2 + \|u_{\varepsilon, \alpha}(t)\|^2 + \underset{\alpha \rightarrow 0}{o}(1) \|x_0\|^2$$

uniformly with respect to $t \geq 0$ and to $\varepsilon \in [0, \varepsilon_0]$. Hence, combining with (52), we get that

$$\begin{aligned} J_{\varepsilon, \alpha}^{\text{diag}}(v_{\varepsilon, \alpha}, x_0) &= \int_0^{+\infty} (\varepsilon \|y(t; \alpha, v_{\varepsilon, \alpha}, x_0)\|^2 + \|v_{\varepsilon, \alpha}(t)\|^2) dt \\ &= \int_0^{+\infty} (\varepsilon \|x(t; \alpha, u_{\varepsilon, \alpha}, x_0)\|^2 + \|u_{\varepsilon, \alpha}(t)\|^2) dt + \underset{\alpha \rightarrow 0}{o}(1) \|x_0\|^2 \\ &= J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_0) + \underset{\alpha \rightarrow 0}{o}(1) \|x_0\|^2 \end{aligned} \quad (53)$$

uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$. Here, with evident notations, $J_{\varepsilon, \alpha}^{\text{diag}}$ stands for the cost functional (42) attached to the block diagonal control system (50). Since $Q(\varepsilon, \alpha)$ is the solution of the Riccati equation, $J_{\varepsilon, \alpha}^{\text{diag}}(v_{\varepsilon, \alpha}, x_0)$ is the optimal cost for the diagonal block system (50) and we have

$$J_{\varepsilon, \alpha}^{\text{diag}}(v_{\varepsilon, \alpha}, x_0) = x_0^* Q(\varepsilon, \alpha) x_0.$$

Besides, $J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_0)$ may not be the optimal cost for the system (38) because $u_{\varepsilon, \alpha}$ is defined with the matrix $Q(\varepsilon, \alpha)$ which may differ from $P(\varepsilon, \alpha)$. Anyway, since $x(t; \alpha, u_{\varepsilon, \alpha}, x_0)$ and $u_{\varepsilon, \alpha}$ are linear in x_0 , there exists a symmetric positive semidefinite matrix

$$R(\varepsilon, \alpha) = \begin{pmatrix} R_1(\varepsilon, \alpha) & R_3(\varepsilon, \alpha) \\ R_3(\varepsilon, \alpha)^* & R_4(\varepsilon, \alpha) \end{pmatrix}$$

such that

$$J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_0) = x_0^* R(\varepsilon, \alpha) x_0.$$

We infer from (53) that

$$x_0^* Q(\varepsilon, \alpha) x_0 = x_0^* R(\varepsilon, \alpha) x_0 + \underset{\alpha \rightarrow 0}{o}(1) \|x_0\|^2$$

for every $x_0 \in \mathbb{R}^m$, uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$. Applying Proposition 1, we get that

$$R_1(\varepsilon, \alpha) = Q_1(0, 0) + o(1), \quad R_2(\varepsilon, \alpha) = o(1), \quad R_3(\varepsilon, \alpha) = o(1)$$

as $(\varepsilon, \alpha) \rightarrow (0, 0)$, and by the arguments of Step 3 in the proof of Proposition 1, $R_1(0, 0) = Q_1(0, 0)$ is the (unique) maximal symmetric positive semidefinite matrix solution of the Riccati equation

$$A_1(0)^* X + X A_1(0) - X B_1(0) B_1(0)^* X = 0. \quad (54)$$

By the minimization property of the Riccati matrix for the control system (38), we have $x_0^* P(\varepsilon, \alpha) x_0 \leq J_{\varepsilon, \alpha}(u_{\varepsilon, \alpha}, x_0) = x_0^* R(\varepsilon, \alpha) x_0$ and therefore

$$P_2(\varepsilon, \alpha) = o(1) \quad \text{and} \quad P_3(\varepsilon, \alpha) = o(1) \quad (55)$$

as $(\varepsilon, \alpha) \rightarrow (0, 0)$ and $P_1(\varepsilon, \alpha) \leq R_1(\varepsilon, \alpha)$ in the sense of symmetric positive semidefinite matrices. In particular, $P_1(0, 0) \leq R_1(0, 0) = Q_1(0, 0)$.

It remains to prove that $P_1(\varepsilon, \alpha) = P_1(0, 0) + o(1)$ as $(\varepsilon, \alpha) \rightarrow (0, 0)$. The Riccati equation (40) gives the equation satisfied by $P_1(\varepsilon, \alpha)$:

$$A_1^* P_1 + A_4^* P_3^* + P_1 A_1 + P_3 A_4 - P_1 B_1 B_1^* P_1 - P_1 B_1 B_2^* P_3^* - P_3 B_2 B_1^* P_1 - P_3 B_2 B_2^* P_3^* = -\varepsilon I_\ell$$

where we have dropped the dependence in ε and α for readability. Using (55) and the fact that $A_4(\alpha) = \underset{\alpha \rightarrow 0}{o}(1)$, we obtain

$$A_1(\alpha)^* P_1(\varepsilon, \alpha) + P_1(\varepsilon, \alpha) A_1(\alpha) - P_1(\varepsilon, \alpha) B_1(\alpha) B_1(\alpha)^* P_1(\varepsilon, \alpha) = o(1) \quad (56)$$

as $(\varepsilon, \alpha) \rightarrow (0, 0)$. Noting that $P_1(\varepsilon, \alpha) \leq R_1(\varepsilon, \alpha)$ and thus is bounded, given any sequence $(\varepsilon_k, \alpha_k)$ converging to $(0, 0)$, up to some subsequence we have $P(\varepsilon_k, \alpha_k) \rightarrow \tilde{P}$ as $k \rightarrow +\infty$ with \tilde{P} which is, thanks to (56), the maximal symmetric positive semidefinite matrix solution of (54) (as in Step 3 in the proof of Proposition 1). By uniqueness, it follows that $\tilde{P} = P_1(0, 0)$. Since this argument is valid for any subsequence, the conclusion follows.

The proposition is proved. \square

References

- [1] H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations*, Systems & Control: Foundations & Applications, Birkhäuser Verlag, Basel, 2003.
- [2] F. Alabau-Boussouira, Y. Privat, E. Trélat, *Nonlinear damped partial differential equations and their uniform discretizations*, J. Funct. Anal. **273** (2017), no. 1, 352–403.
- [3] J. A. Atwell, B. B. King, *Proper orthogonal decomposition for reduced basis feedback controllers for parabolic equations*, Math. Comput. Modelling, **33** (2001), 1–19.
- [4] H.T. Banks, K. Kunisch, *The linear regulator problem for parabolic systems*, SIAM J. Control Optim., **22** (1984), no. 5, 684–698.
- [5] V. Barbu, G. Wang, *Internal stabilization of semilinear parabolic systems*, J. Math. Anal. Appl., **285** (2003), 387–407.
- [6] G. Berkooz, P. Holmes, J. L. Lumley, *The proper orthogonal decomposition in the analysis of turbulent flows*, Annu. Rev. Fluid Mech., **25** (1993), 539–575.
- [7] F. J. Beutler, *The operator theory of the pseudo-inverse. I. Bounded operators*, J. Math. Anal. Appl., **10** (1965), 451–470.
- [8] D. H. Chambers, R. J. Adrian, P. Moin, D. S. Stewart, H. J. Sung, *Karhunen-Loève expansion of Burgers model of turbulence*, Phys. Fluids, **31** (1988), 2573–2582.
- [9] D. Chapelle, A. Gariah, J. Sainte-Marie, *Galerkin approximation with proper orthogonal decomposition: new error estimates and illustrative examples*, ESAIM Math. Model. Numer. Anal., **46** (2012), 731–757.

- [10] J.-M. Coron, E. Trélat, *Global steady-state controllability of 1-D semilinear heat equations*, SIAM J. Control Optim., **43** (2004), 549–569.
- [11] J.-M. Coron, E. Trélat, *Global steady-state stabilization and controllability of 1-D semilinear wave equations*, Commun. Contemp. Math., **8** (2006), 535–567.
- [12] R. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, Texts in Applied Mathematics, 21. Springer-Verlag, New York, 1995.
- [13] K. J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.
- [14] G. H. Golub, C. F. Van Loan, *Matrix computations*, Fourth edition, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2013.
- [15] M. Gunzburger, N. Jiang, M. Schneier, *An Ensemble-Propor Orthogonal Decomposition method for the nonstationary Navier-Stokes equations*, SIAM J. Numer. Anal., **55** (2017), 286–304.
- [16] J.S. Hesthaven, G. Rozza, B. Stamm, *Certified reduced basis methods for parametrized partial differential equations*, SpringerBriefs in Mathematics, BCAM SpringerBriefs, Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2016.
- [17] M. Hinze, S. Volkwein, *Error estimates for abstract linearquadratic optimal control problems using proper orthogonal decomposition*, Comput. Optim. Appl., **39** (2008), 319–345.
- [18] P. Holmes, J. L. Lumley, G. Berkooz, *Turbulence, coherent structures, dynamical systems and symmetry*, Cambridge University Press, Cambridge, 1996.
- [19] F. Kappel, D. Salamon, *An approximation theorem for the algebraic Riccati equation*, SIAM J. Control Optim., **28** (1990), no. 5, 1136–1147.
- [20] K. Kunisch, S. Volkwein, *Control of the Burgers equation by a reduced-order approach using proper orthogonal decomposition*, J. Optim. Theory Appl., **102** (1999), 345–371.
- [21] K. Kunisch, S. Volkwein, *Galerkin proper orthogonal decomposition methods for parabolic problems*, Numer. Math., **90** (2001), 117–148.
- [22] K. Kunisch, S. Volkwein, *Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics*, SIAM J. Numer. Anal., **40** (2002), 492–515.
- [23] K. Kunisch, S. Volkwein, L. Xie, *HJB-POD-based feedback design for the optimal control of evolution problems*, SIAM J. Appl. Dyn. Syst., **3** (2004), 701–722.
- [24] K. Kunisch, L. Xie, *POD-based feedback control of the Burgers equation by solving the evolutionary HJB equation*, Comput. Math. Appl., **49** (2005), 1113–1126.
- [25] H. Kwarkernaak, R. Sivan, *Linear optimal control systems*, Wiley-Interscience, John Wiley & Sons, New-York-London-Sydney, 1972.
- [26] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories. I. Abstract parabolic systems*, Encyclopedia of Mathematics and its Applications, 74, Cambridge University Press, Cambridge, 2000.

- [27] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories. II. Abstract hyperbolic-like systems over a finite time horizon*, Encyclopedia of Mathematics and its Applications, 75, Cambridge University Press, Cambridge, 2000.
- [28] T. Lassila, A. Manzoni, A. Quarteroni, G. Rozza, *Model order reduction in fluid dynamics: challenges and perspectives*, Reduced order methods for modeling and computational reduction, 235–273, Model. Simul. Appl., 9, Springer, Cham, 2014.
- [29] E. B. Lee, L. Markus, *Foundations of optimal control theory*, John Wiley, New York-London-Sydney, 1967.
- [30] Z. Liu, S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC Research Notes in Mathematics, 398, Boca Raton, FL, 1999.
- [31] J. L. Lumley, *The structure of inhomogeneous turbulence flows*, in: A. M. Yaslom and V. I. Tatarski, editors, Atmospheric Turbulence and wave Propagation, 166–178, Nauka, Moscow, 1967.
- [32] H. V. Ly, H. T. Tran, *Proper orthogonal decomposition for flow calculations and optimal control in a horizontal CVD reactor*, Quart. Appl. Math., **60** (2002), 631–656.
- [33] I. Markovsky, *Low rank approximation, algorithms, implementation, applications*, Communications and Control Engineering Series, Springer, London, 2012.
- [34] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Math. Sci., 44, Springer-Verlag, 1983.
- [35] A. Quarteroni, A. Manzoni, F. Negri, *Reduced basis methods for partial differential equations, An introduction*, Unitext, 92, La Matematica per il 3+2, Springer, Cham, 2016.
- [36] S. S. Ravindran, *A reduced-order approach for optimal control of fluids using proper orthogonal decomposition*, Internat. J. Numer. Methods Fluids, **34** (2000), 425–448.
- [37] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions*, SIAM Rev., **20** (1978), 639–739.
- [38] M. A. Singer, W. H. Green, *Using adaptive proper orthogonal decomposition to solve the reaction-diffusion equation*, Appl. Numer. Math., **59** (2009), 272–279.
- [39] L. Sirovich, B. W. Knight, J. D. Rodriguez, *Optimal low-dimensional dynamical approximations*, Quart. Appl. Math., **48** (1990), 535–548.
- [40] E. D. Sontag, *Mathematical control theory, Deterministic finite-dimensional systems*, Second edition, Texts in Applied Mathematics, 6, Springer-Verlag, New York, 1998.
- [41] M. Teytel, *How rare are multiple eigenvalues?*, Comm. Pure Applied Math., **52** (1999), no. 8, 917–934.
- [42] A. Theodoropoulou, R. A. Adomaitis, E. Zafiriou, *Model reduction for optimization of rapid thermal chemical vapor deposition systems*, IEEE Trans. Semiconductor Manufacturing, **11** (1998), 85–98.
- [43] E. Trélat, *Contrôle optimal. (French) [Optimal control] Théorie & applications. [Theory and applications]*, Mathématiques Concrètes, [Concrete Mathematics] Vuibert, Paris, 2005.

- [44] E. Trélat, *Stabilization of semilinear PDEs, and uniform decay under discretization*, Evolution equations: long time behavior and control, 31–76, London Math. Soc. Lecture Note Ser., 439, Cambridge Univ. Press, Cambridge, 2018.
- [45] F. Tröltzsch, S. Volkwein, *POD a-posteriori error estimates for linear-quadratic optimal control problems*, Comput. Optim. Appl., **44** (2009), 83–115.
- [46] M. Tucsnak, G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basel Textbooks, Birkhäuser Verlag, Basel, 2009.
- [47] S. Volkwein, *Optimal and suboptimal control of partial differential equations: augmented Lagrange-SQP methods and reduced-order modeling with proper orthogonal decomposition*, Grazer Mathematische Berichte [Graz Mathematical Reports], 343, Karl-Franzens-Universität Graz, Graz, 2001.
- [48] G. Wang, Y. Xu, *Periodic feedback stabilization for linear periodic evolution equations*, SpringerBriefs in Mathematics. Springer, Cham; BCAM Basque Center for Applied Mathematics, Bilbao, 2016.
- [49] J. Zabczyk, *Mathematical control theory, An introduction*, Reprint of the 1995 edition, Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008.