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ABSTRACT. We construct DG-enhanced versions of the degenerate affine Hecke algebra and of the affine q-Hecke algebra. We extend Brundan–Kleshchev and Rouquier's isomorphism and prove that after completion DG-enhanced versions of Hecke algebras are isomorphic to completed DG-enhanced versions of KLR algebras for suitably defined quivers. As a byproduct, we deduce that these DG-algebras have homologies concentrated in degree zero. These homologies are isomorphic respectively to the degenerate cyclotomic Hecke algebra and the cyclotomic q-Hecke algebra.

CONTENTS

1. INTRODUCTION

Hecke algebras and their affine versions are fundamental objects in mathematics and have a rich representation theory (see the review [9] for an account of some of the current trends). The representation theory of finite dimensional Hecke algebras also carries interesting symmetries which occur in categorification of Fock spaces and Heisenberg algebras [5, 11]

In a series of outstanding papers, Lauda [10], Khovanov–Lauda [6, 7, 8] and independently Rouquier [20], have constructed categorifications of quantum groups. They take the form of 2-categories whose Grothendieck rings are isomorphic to the idempotent version of the quantum enveloping algebra of a Kac–Moody algebra. Both constructions were later proved to be equivalent by Brundan [1]. As a main ingredient of the constructions by Khovanov–Lauda and Rouquier there is a certain family of algebras, nowadays known as KLR algebras, that are constructed using actions of symmetric groups on polynomial spaces.

It turns out that in type A the KLR algebras are instances of affine Hecke algebras. It was proved by Rouquier [20, Section 3.2] that KLR algebras of type A become isomorphic to affine Hecke algebras after a suitable localization of both algebras. Independently, Brundan and Kleshchev [2] have proved a similar result for cyclotomic quotient algebras. This endows cyclotomic Hecke algebras with a presentation as graded idempotented algebras. In particular, in the case of KLR for the quiver of type A_∞, the isomorphism to the group algebra of the symmetric group in d letters $\Bbbk\mathfrak{S}_d$ gives the latter a graded presentation. The grading on $\Bbbk\mathfrak{S}_d$ was already known to exist (see [19]) but transporting the grading from the KLR algebras allowed to construct it explicitly. This gave rise to a new approach to the representation theory of symmetric groups and Hecke algebras [3]. These results are valid over an arbitrary field $\Bbbk$.

The BKR (Brundan–Kleshchev–Rouquier) isomorphism was later extended to isomorphisms between families of other KLR-like algebras and Hecke-like algebras. A similar isomorphism between the Dipper-James-Mathas cyclotomic q-Schur algebra and the cyclotomic quiver Schur algebra is given in [21]. The papers [12] and [22] have constructed a higher level version of the affine Hecke algebra and have proved that after completion they are isomorphic to a completion of Webster's tensor product algebras [23]. A weighted version of this isomorphism is also given in [22]. A similar relation between quiver Schur algebras and affine Schur algebras is given in [13]. The paper [12] have constructed a higher level version of the affine Schur algebra and have proved that after completion it is isomorphic to a completion of the higher level quiver Schur algebras.

The BKR isomorphism was also generalized to other algebras. For example, in [18] it is used to show that cyclotomic Yokonuma-Hecke algebras are particular cases of cyclotomic KLR algebras for certain cyclic quivers, and in [17] the BKR isomophism is extended to connect affine Hecke algebras of type B and a generalization of KLR algebras for a Weyl group of type B.

More recently, the second author and Naisse have constructed categorifications of (parabolic) Verma modules in a series of papers [14, 15, 16]. The construction in [16], motivated by the work of Khovanov–Lauda, Rouquier, and Kang–Kashiwara [6, 7, 20, 4], introduces DG-enhanced versions $\mathcal{R}(\nu)$ of KLR algebras, which are some sort of resolutions of cyclotomic KLR algebras. These can be seen as a sort of integration of cyclotomic KLR algebras into free (over the polynomial ring) algebras, where the cyclotomic condition is replaced by a differential with the property that the DG-algebras $\mathcal{R}(\nu)$ are quasi-isomorphic to cyclotomic KLR algebras, the latter seen as DG-algebras with zero differential. The algebras $\mathcal{R}(\nu)$ also provide categorification of universal Verma modules.

It seems natural to ask the following question.

Question 1. (a) Are there DG-enhanced versions of affine Hecke algebras that are quasi-isomorphic to cyclotomic Hecke algebras?

(b) In this case, does the BKR isomorphism extend to an isomorphism between (completions of) DG-enhanced versions of KLR algebras and DG-enhanced versions of Hecke algebras?

In this article we answer this question affirmatively. We construct DG-enhanced versions of the degenerate affine Hecke algebra and of the affine q-Hecke algebra.

Let us give an overview of our Hecke algebras and the main results in this article. Fix $d \in \mathbb{N}$ and a field $\Bbbk$ that for simplicity we consider to be algebraically closed (we follow the convention that $0 \in \mathbb{N}$, i.e., we have $\mathbb{N} = \{0, 1, 2, \ldots\}$). We consider the $\mathbb{Z}$-graded algebra $\bar{\mathcal{H}}_d$ generated by $T_1, \ldots, T_{d-1}$ and $X_1, \ldots, X_d$ in degree zero and θ in degree 1. The generators $T_1, \ldots, T_{d-1}$ and $X_1, \ldots, X_d$ satisfy the relations of the degenerate affine Hecke algebra $\bar{H}_d$. The generator θ commutes with the X_r's and with $T_2, \ldots, T_{d-1}$ and satisfies $\theta^2 = 0$ and $T_1\theta T_1\theta + \theta T_1\theta T_1 = 0$. This implies that the subalgebra of $\bar{\mathcal{H}}_d$ concentrated in degree zero is isomorphic to $\bar{H}_d$. For $\mathbf{Q} = (Q_1, \ldots, Q_\ell) \in \Bbbk^\ell$ introduce a differential $\partial_{\mathbf{Q}}$ by declaring that it acts as zero on $\bar{H}_d$ while $\partial_{\mathbf{Q}}(\theta) = \prod_{r=1}^{\ell}(X_1 - Q_r)$. We denote $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ the completion of the algebra $\bar{\mathcal{H}}_d$ at a sequence of ideals depending on $\mathbf{a} \in \Bbbk^d$.

In order to make the connection to DG-enhanced versions of KLR algebras we consider a quiver Γ with a vertex set $I \subseteq \Bbbk$ and with an edge $i \to j$ iff $j + 1 = i$. We assume that $Q_r \in I$ for each r. We fix $\mathbf{a} \in I^d$ and we set ν and Λ such that ν_i and Λ_i are the multiplicities of i in respectively $\mathbf{a}$ and $\mathbf{Q}$. We have $\prod_{r=1}^{\ell}(X_1 - Q_r) = \prod_{i \in I}(X_1 - i)^{\Lambda_i}$. Let $(\mathcal{R}(\nu), d_\Lambda)$ be the DG-enhanced version of the KLR algebra of type Γ with parameters ν and Λ as above and $(\widehat{\mathcal{R}}(\nu), d_\Lambda)$ its completion.

The first main result in this article is the DG-enhanced version of the BKR isomorphism for the degenerate affine Hecke algebra:

Theorem 4.5. *There is an isomorphism of DG-algebras* $(\widehat{\mathcal{R}}(\nu), d_\Lambda) \simeq (\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}, \partial_{\mathbf{Q}})$.

Next, we give a similar construction for the affine q-Hecke algebra. Fix $q \in \Bbbk$, $q \neq 0, 1$. We consider the $\mathbb{Z}$-graded algebra $\mathcal{H}_d$ generated by $T_1, \ldots, T_{d-1}$ and $X_1^{\pm 1}, \ldots, X_d^{\pm 1}$ in degree zero and θ in degree 1. The generators $T_1, \ldots, T_{d-1}$ and $X_1^{\pm 1}, \ldots, X_d^{\pm 1}$ satisfy the relations of the affine q-Hecke algebra H_d. The generator θ commutes with the X_r's and with $T_2, \ldots, T_{d-1}$ and satisfies the relations $\theta^2 = 0$ and $T_1\theta T_1\theta + \theta T_1\theta T_1 = (q-1)\theta T_1\theta$. This implies that the subalgebra of $\mathcal{H}_d$ concentrated in degree zero is isomorphic to H_d. For $\mathbf{Q} = (Q_1, \ldots, Q_\ell) \in (\Bbbk^\times)^\ell$ introduce a differential $\partial_{\mathbf{Q}}$ by declaring that it acts as zero on H_d while $\partial_{\mathbf{Q}}(\theta) = \prod_{r=1}^{\ell}(X_1 - Q_r)$. We denote $\widehat{\mathcal{H}}_{\mathbf{a}}$ the completion of the algebra $\mathcal{H}_d$ at a sequence of ideals depending on $\mathbf{a} \in (\Bbbk^\times)^d$.

To make the connection to DG-enhanced versions of KLR algebras we consider a quiver Γ with a vertex set $I \subseteq \Bbbk^\times$ with an edge $i \to j$ iff $i = qj$. We assume that $Q_r \in I$ for each r. Finally set ν and Λ such that ν_i and Λ_i are the multiplicities of i in respectively $\mathbf{a}$ and $\mathbf{Q}$. We have $\prod_{r=1}^{\ell}(X_1 - Q_r) = \prod_{i \in I}(X_1 - i)^{\Lambda_i}$. Let $(\mathcal{R}(\nu), d_\Lambda)$ be the DG-enhanced version of the KLR algebra of type Γ with ν and Λ as above and $(\widehat{\mathcal{R}}(\nu), d_\Lambda)$ its completion.

The second main result in this article is the DG-enhanced version of the BKR isomorphism for the affine q-Hecke algebra:

Theorem 4.7. *There is an isomorphism of DG-algebras* $(\widehat{\mathcal{R}}(\nu), d_\Lambda) \simeq (\widehat{\mathcal{H}}_{\mathbf{a}}, \partial_{\mathbf{Q}})$.

The two main results above imply that we have a family of isomorphisms $\widehat{\mathcal{R}}(\nu) \simeq \widehat{\mathcal{H}}_{\mathbf{a}}$ between the underlying algebras parametrized by integral dominant weights.

The DG-enhanced versions of BKR isomorphisms above allow us to compute the homology of the DG-algebras $\bar{\mathcal{H}}_d$ and $\mathcal{H}_d$ in the following way. It is already proved in [16, Proposition 4.14] that the DG-algebra $\mathcal{R}(\nu)$ is quasi-isomorphic to the cyclotomic KLR algebra. The most difficult part of this proof is to show that the DG-algebra $\mathcal{R}(\nu)$ has homology concentrated in degree zero. The proof of this fact is quite technical and there is no obvious way to rewrite it for Hecke algebras. So we use the following strategy: we deduce the statement for Hecke algebras from the statement for KLR algebras using the DG-enhanced version of the BKR isomorphism.

As a corollary of Theorem 4.5 and Theorem 4.7 and [16, Proposition 4.14], the DG-algebras $(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}})$ and $(\mathcal{H}_d, \partial_{\mathbf{Q}})$ are resolutions of the degenerate cyclotomic Hecke algebra $\bar{H}_d^{\mathbf{Q}}$ and of the cyclotomic q-Hecke algebra $H_d^{\mathbf{Q}}$ respectively.

Proposition 2.12. *The DG-algebras* $(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}})$ *and* $(\bar{H}_d^{\mathbf{Q}}, 0)$ *are quasi-isomorphic.*

Proposition 2.23. *The DG-algebras* $(\mathcal{H}_d, \partial_{\mathbf{Q}})$ *and* $(H_d^{\mathbf{Q}}, 0)$ *are quasi-isomorphic.*

Up to our knowledge, the DG-enhanced versions of Hecke algebras we introduce are new. Proposition 2.12 and Proposition 2.23 are important for a forthcoming paper.

We would also like to emphasize the fact that the algebras $\bar{\mathcal{H}}_d$ and $\mathcal{H}_d$ have triangular decompositions (see Remark 2.6 and Remark 2.20). This looks like an analogy with Cherednik algebras.

Plan of the paper. In Section 2 we introduce DG-enhanced versions of the degenerate affine Hecke algebra and of the affine q-Hecke algebra and their completions, that will be used in the BKR isomorphism. The material in this section is new.

In Section 3 we review the DG-enhanced version of the KLR algebra introduced in [16]. We give the minimal presentation of this algebra explained in [16, Remark 3.10], which is more convenient to us, and present its completion, which is involved in the BKR isomorphism.

Section 4 contains the main results. We first generalize the BKR isomorphism to a class of algebras satisfying some properties. The most important point is that to have a generalization of the BKR isomorphism we need to construct an isomorphism between the completed polynomial representation of the Hecke-like algebra and the completed polynomial representation of the KLR-like algebra, this isomorphism must intertwine the action of the symmetric group. Our main results, Theorem 4.5 and Theorem 4.7, are then proved by showing that our DG-enhanced versions of Hecke algebras $\bar{\mathcal{H}}_d$ and $\mathcal{H}_d$ on one side, and the DG-enhanced versions of KLR algebras $\mathcal{R}(\nu)$ on the other satisfy the properties that are required for them to be isomorphic (after completion). We then use the fact that the DG-algebras $\mathcal{R}(\nu)$ are quasi-isomophic to cyclotomic KLR algebras together with the DG-enhanced version of the BKR isomorphism to show

in Corollary 4.10 that the algebras $\bar{\mathcal{H}}_d$ and $\mathcal{H}_d$ are quasi-isomorphic to degenerate cyclotomic Hecke algebras and cyclotomic q-Hecke algebras respectively.

Acknowledgments. We thank Jonathan Grant for useful discussions. PV was supported by the Fonds de la Recherche Scientifique - FNRS under Grant no. MIS-F.4536.19.

2. DG-enhanced versions of Hecke algebras

2.1. **The polynomial rings Pol_d and Poll_d and the rings P_d and Pl_d.** Fix an algebraically closed field $\Bbbk$, $q \in \Bbbk$, $q \neq 0, 1$ and $d \in \mathbb{N}$ once and for all.

2.1.1. *The polynomial rings Pol_d and Poll_d.* Set $\mathrm{Pol}_d = \Bbbk[X_1, \dots, X_d]$. Let $\mathfrak{S}_d$ be the symmetric group on d letters, which we view as being generated as a Coxeter group with generators s_i. These correspond to the simple transpositions $(i\ i{+}1)$, and we use these two descriptions interchangeably throughout. It acts from the left on Pol_d by permuting the variables: for $w \in \mathfrak{S}_d$ we have

$$w(X_i) = X_{w(i)}.$$

Using the $\mathfrak{S}_d$-action above, we introduce the *Demazure operators* ∂_i on P_d for all $1 \leqslant i \leqslant d{-}1$ in the usual way, as

$$\partial_i(f) = \frac{f - s_i(f)}{X_i - X_{i+1}}. \tag{1}$$

We have $s_i\partial_i(f) = \partial_i(f)$ and $\partial_i(s_i f) = -\partial_i(f)$ for all i, so ∂_i is in fact an operator from Pol_d to the subring $\mathrm{Pol}_d^{s_i} \subseteq \mathrm{Pol}_d$ of invariants under the transposition $(i\ i{+}1)$. It is well-known that the action of the Demazure operators on Pol_d satisfy the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g), \tag{2}$$

for all $f, g \in \mathrm{Pol}_d$ and for $1 \leqslant i \leqslant n-1$, and the relations

$$\partial_i^2 = 0, \qquad \partial_i\partial_{i+1}\partial_i = \partial_{i+1}\partial_i\partial_{i+1}, \tag{3}$$

$$\partial_i\partial_j = \partial_j\partial_i \quad \text{for } |i-j| > 1, \tag{4}$$

$$X_i\partial_i - \partial_i X_{i+1} = 1, \qquad \partial_i X_i - X_{i+1}\partial_i = 1. \tag{5}$$

Set $\mathrm{Poll}_d = \Bbbk[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$. We have, $\mathrm{Poll}_d = \mathrm{Pol}_d[X_1^{-1}, \dots, X_d^{-1}]$. Moreover, the $\mathfrak{S}_d$-action on Pol_d can be obviously extended to a $\mathfrak{S}_d$-action on Poll_d. This means that the action of the Demazures on Pol_d also extends to operators on Poll_d that satisfy the relations in (2) (for f and g in Poll_d) and (3)-(5).

2.1.2. *The rings P_d and Pl_d.* Let $\underline{\theta} = (\theta_1, \dots, \theta_d)$ and form the ring $P_d = \mathrm{Pol}_d \otimes \bigwedge^\bullet(\underline{\theta})$, where $\bigwedge^\bullet(\underline{\theta})$ is the exterior ring in the variables $\underline{\theta}$ and coefficients in $\Bbbk$. Introduce a $\mathbb{Z}$-grading on P_d denoted $\lambda(\bullet)$ and defined as $\lambda(X_i) = 0$ and $\lambda(\theta_i) = 1$. This grading is of cohomological nature and (up to a sign) is half the grading $\deg_\lambda$ introduced in [14, §3.1].

As explained in [14, §8.3], the action of $\mathfrak{S}_d$ on Pol_d extends to an action on P_d by setting

$$s_i(\theta_j) = \theta_j + \delta_{i,j}(X_i - X_{i+1})\theta_{i+1}. \tag{6}$$

This action respects the grading, as one easily checks.

With this $\mathfrak{S}_d$-action above, the action of the Demazure operators ∂_i on Pol_d given by (1) extends to any $f \in P_d$ and defines Demazure operators on P_d, which we denote by the same symbol.

Similarly to the operators above, ∂_i is an operator from P_d to the subring $P_d^{s_i} \subseteq P_d$ of invariants under the transposition $(i\ i{+}1)$.

It was proved in [14, §8.2] that the action of the Demazure operators on P_d satisfies the Leibniz rule (2) (with f and g in P_d), the relations (3)-(5), and

$$\partial_i \theta_k = \theta_k \partial_i \qquad \text{for } k \neq i,$$

$$\partial_i(\theta_i - X_{i+1}\theta_{i+1}) = (\theta_i - X_{i+1}\theta_{i+1})\partial_i,$$

for all $i = 1, \dots, d-1$.

Form the supercommutative ring

$$Pl_d = \mathrm{Poll}_d \otimes \bigwedge\nolimits^{\bullet}(\underline{\theta}).$$

We have, $Pl_d = P_d[X_1^{-1}, \dots, X_d^{-1}]$. Moreover, the $\mathfrak{S}_d$-action on Pol_d can be obviously extended to a $\mathfrak{S}_d$-action on Pl_d. This means that the action of the Demazures on Pl_d also extends to operators on Pl_d that satisfy the relations in (2) (for f and g in Pl_d) and (3)-(5).

2.2. **Degenerate version.**

2.2.1. *Degenerate affine Hecke algebra.* The *degenerate affine Hecke* algebra $\bar{H}_d$ is the $\Bbbk$-algebra generated by $T_1, \dots, T_{d-1}$ and $X_1, \dots, X_d$, with relations (7) to (9) below.

$$T_i^2 = 1, \qquad T_iT_j = T_jT_i \ \text{ if } \ |i-j| > 1, \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \tag{7}$$

$$X_iX_j = X_jX_i, \tag{8}$$

$$T_iX_i - X_{i+1}T_i = -1, \qquad T_iX_j = X_jT_i \ \text{ if } \ j - i \neq 0, 1. \tag{9}$$

For $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_d$ a reduced decomposition we put $T_w = T_{i_1} \cdots T_{i_k}$. Then T_w is independent of the choice of the reduced decomposition of w and the set

$$\{X_1^{m_1} \cdots X_d^{m_d} T_w\}_{w \in \mathfrak{S}_d, m_i \in \mathbb{Z}_{\geqslant 0}}$$

is a basis of the $\Bbbk$-vector space $\bar{H}_d$.

There is a faithful representation of $\bar{H}_d$ on Pol_d, where $T_i(f) = s_i(f) - \partial_i(f)$. It is immediate that $\bar{H}_d$ contains $\Bbbk\mathfrak{S}_d$ and Pol_d as subalgebras and that for $p \in \mathrm{Pol}_d$,

$$T_ip - s_i(p)T_i = -\partial_i(p).$$

Let ℓ be a positive integer and $\mathbf{Q} = (Q_1, \dots, Q_\ell)$ be an ℓ-tuple of elements of the field $\Bbbk$.

Definition 2.1. The *degenerate cyclotomic Hecke algebra* is the quotient

$$\bar{H}_d^{\mathbf{Q}} = \bar{H}_d / \prod_{r=1}^{\ell} (X_1 - Q_r).$$

2.2.2. *The algebra $\bar{\mathcal{H}}_d$.*

Definition 2.2. Define the algebra $\bar{\mathcal{H}}_d$ as the $\Bbbk$-algebra generated by $T_1, \dots, T_{d-1}$ and $X_1, \dots, X_d$ in λ-degree zero, and an extra generator θ in λ-degree 1, with relations (7) to (9) and

$$\theta^2 = 0 \tag{10}$$

$$X_r\theta = \theta X_r \qquad \text{for } r = 1, \dots, d, \tag{11}$$

$$T_r\theta = \theta T_r \qquad \text{for } r > 1, \tag{12}$$

$$T_1\theta T_1\theta + \theta T_1\theta T_1 = 0. \tag{13}$$

The algebra $\bar{\mathcal{H}}_d$ contains the degenerate affine Hecke algebra $\bar{H}_d$ as a subalgebra concentrated in λ-degree zero.

Lemma 2.3. *The algebra $\bar{\mathcal{H}}_d$ acts faithfully on P_d by*

$$T_r(f) = s_r(f) - \partial_r(f),$$
$$X_r(f) = X_r f,$$
$$\theta(f) = \theta_1 f,$$

for all $f \in P_d$ and where $s_r(f)$ and $\partial_r(f)$ are as in (6) and (1).

Proof. The defining relations of $\bar{\mathcal{H}}_d$ can be checked by a straightforward computation. Faithfulness follows from the proof of Proposition 2.5 below. □

Define $\xi_1, \dots, \xi_d \in \bar{\mathcal{H}}_d$ by the rules $\xi_1 = \theta$, $\xi_{i+1} = T_i\xi_i T_i$. The following is straightforward.

Lemma 2.4. *The elements ξ_r satisfy for all $r = 1, \dots, d-1$ and all $\ell = 1, \dots, d$,*

$$\xi_\ell^2 = 0, \qquad \xi_r\xi_\ell + \xi_\ell\xi_r = 0, \qquad T_r\xi_\ell = \xi_{s_r(\ell)}T_r.$$

It is not hard to write a basis of $\bar{\mathcal{H}}_d$ in terms of the ξ_r's.

Proposition 2.5. *The set*

$$\{X_1^{a_1} \cdots X_d^{a_d} T_w \xi_1^{b_1} \cdots \xi_d^{b_d} | w \in \mathfrak{S}_d, (a_1, \dots, a_d) \in \mathbb{N}^d, (b_1, \dots, b_d) \in \{0,1\}^d\},$$

is a basis of the $\Bbbk$-vector space $\bar{\mathcal{H}}_d$.

Proof. First, we show that this set spans $\bar{\mathcal{H}}_d$. We have no explicit commutation relations between X's and ξ's. But this problem is easy to overcome because we know that θ commutes with X's. First, each monomial on θ, X's and T's can be rewritten as a linear combination of similar monomials with all X's on the left. After that, we replace θ by ξ_1 and we move all ξ's to the right. This shows that the set above spans $\bar{\mathcal{H}}_d$.

The linear independence follows from Lemma 2.7 and Lemma 2.9 below. □

Remark 2.6. We see from the proposition above that the algebra $\bar{\mathcal{H}}_d$ has a triangular decomposition (only as a vector space)

$$\bar{\mathcal{H}}_d = \Bbbk[X_1, \dots, X_d] \otimes \Bbbk\mathfrak{S}_d \otimes \bigwedge{}^\bullet(\xi_1, \dots, \xi_d).$$

Abusing the notation, we will write θ_r for the operator on P_d that multiplies each element of P_d by θ_r. Set $M = \{0,1\}^d$. Denote by $\mathbf{1}$ the sequence $\mathbf{1} = (1,1,\dots,1) \in M$. For each sequence $\mathbf{b} = (b_1,\dots,b_d) \in M$ we set $\theta^{\mathbf{b}} = \theta_1^{b_1}\dots\theta_d^{b_d}$. For each $\mathbf{b} \in M$ we set $\overline{\mathbf{b}} = \mathbf{1} - \mathbf{b}$. In particular we have $\theta^{\mathbf{b}} \cdot \theta^{\overline{\mathbf{b}}} = \pm\theta_1\theta_2\dots\theta_d = \theta^{\mathbf{1}}$. Set also $|\mathbf{b}| = b_1 + b_2 + \dots + b_d$.

Lemma 2.7. *The operators $\{\theta^{\mathbf{b}};\ \mathbf{b} \in M\}$ acting on P_d are linearly independent over $\bar{H}_d$. More precisely, if we have $\sum_{\mathbf{b}\in M} h_{\mathbf{b}}\theta^{\mathbf{b}} = 0$ with $h_{\mathbf{b}} \in \bar{H}_d$ then we have $h_{\mathbf{b}} = 0$ for each $\mathbf{b} \in M$.*

Proof. Let $H = \sum_{\mathbf{b}\in M} h_{\mathbf{b}}\theta^{\mathbf{b}}$ be an operator that acts by zero. Assume that H has a nonzero coefficient. Let $\mathbf{b}_0$ be such that $h_{\mathbf{b}_0} \neq 0$ and such that $|\mathbf{b}_0|$ is minimal with this property. Then for each element $P \in P_d$ we have $H(\theta^{\mathbf{b}'}P) = \pm\theta^{\mathbf{1}}h_{\mathbf{b}_0}P$. This shows that $h_{\mathbf{b}_0}$ acts by zero on $\theta^{\mathbf{1}}P_d = \theta^{\mathbf{1}}\operatorname{Pol}_d$. But this implies $h_{\mathbf{b}_0} = 0$ because the polynomial representation Pol_d of $\bar{\mathcal{H}}_d$ is faithful. □

For each, $k \in \{0,1,\dots,d\}$ we denote by $\bar{\mathcal{H}}_d^{\leqslant k}$ the subalgebra of the algebra of operators on P_d generated by X_i, θ_i for $i \leqslant k$ and T_r for $r < k$. We mean that for $k = 0$ we have $\bar{\mathcal{H}}_d^{\leqslant 0} = \Bbbk$. The λ-grading on P_d induces a grading on $\bar{\mathcal{H}}_d^{\leqslant k}$ that we also call λ-grading.

Lemma 2.8. *The set*

$$\{X_1^{a_1}\cdots X_k^{a_k}T_w\theta_1^{b_1}\cdots\theta_k^{b_k} | w \in \mathfrak{S}_k, (a_1,\dots,a_k) \in \mathbb{N}^k, (b_1,\dots,b_k) \in \{0,1\}^k\},$$

is a basis of the $\Bbbk$-vector space $\bar{\mathcal{H}}_d^{\leqslant k}$.

Proof. It is clear that the given set spans $\bar{\mathcal{H}}_d^{\leqslant k}$. The linear independence follows from Lemma 2.7. □

Similarly to the notation $\theta^{\mathbf{b}}$ above, we set $\xi^{\mathbf{b}} = \xi_1^{b_1}\dots\xi_d^{b_d}$. For two elemtns $\mathbf{b}, \mathbf{b}' \in M$ we write $\mathbf{b}' < \mathbf{b}$ if there is an index $r \in [1;d]$ such that $b_r > b'_r$ and $b_t = b'_t$ for $t > r$.

Lemma 2.9. *The element $\xi^{\mathbf{b}} \in \bar{\mathcal{H}}_d$ acts on P_d by an operator of the form $h_{\mathbf{b}}\theta^{\mathbf{b}} + \sum_{\mathbf{b}'<\mathbf{b}} h_{\mathbf{b}'}\theta^{\mathbf{b}'}$, where $h_{\mathbf{b}}, h_{\mathbf{b}'} \in \bar{H}_d$ and $h_{\mathbf{b}}$ is invertible.*

Proof. It is easy to see by induction that for each $k \in \{1,2,\dots,d\}$ the element ξ_k acts on P_d by an operator of the form $c_k + d_k\theta_k$, where $c_k, d_k \in \bar{\mathcal{H}}_d^{\leqslant k-1}$, $\lambda(c_k) = 1$, $\lambda(d_k) = 0$ and d_k is invertible.

The element $\xi^{\mathbf{b}}$ can be written up to sign in the form $\xi_{i_r}\xi_{i_{r-1}}\dots\xi_{i_1}$ with $i_r > i_{r-1} > \dots > i_1$. It acts by the operator $(c_{i_r} + d_{i_r}\theta_{i_r})(c_{i_{r-1}} + d_{i_{r-1}}\theta_{i_{r-1}})\dots(c_{i_1} + d_{i_1}\theta_{i_1})$. Since each θ_k super-commutes with $\bar{\mathcal{H}}_d^{\leqslant k-1}$, we see that this operator can be rewritten as

$$d_{i_r}d_{i_{r-1}}\dots d_{i_1}\theta_{i_r}\theta_{i_{r-1}}\dots\theta_{i_1} + \sum_{\mathbf{b}'<\mathbf{b}} h_{\mathbf{b}'}\theta^{\mathbf{b}'}$$

for some $h_{\mathbf{b}'} \in \bar{\mathcal{H}}_d$. We see that the additional terms above are indeed of the form $\sum_{\mathbf{b}'<\mathbf{b}} h_{\mathbf{b}'}\theta^{\mathbf{b}'}$ from Lemma 2.8.

□

2.2.3. *DG-enhancement of $\bar{\mathcal{H}}_d$.* Let ℓ and $\mathbf{Q}$ be as in Section 2.2.1.

Definition 2.10. Define an operator $\partial_{\mathbf{Q}}$ on $\bar{\mathcal{H}}_d$ by declaring that $\partial_{\mathbf{Q}}$ acts as zero on $\bar{H}_d$, while

$$\partial_{\mathbf{Q}}(\theta) = \prod_{r=1}^{\ell}(X_1 - Q_r),$$

and for $a, b \in \bar{\mathcal{H}}_d$, $\partial_{\mathbf{Q}}(ab) = \partial_{\mathbf{Q}}(a)b + (-1)^{\lambda(a)}a\partial_{\mathbf{Q}}(b)$.

Lemma 2.11. *The operator $\partial_{\mathbf{Q}}$ is a differential on $\bar{\mathcal{H}}_d$.*

Proof. We prove something slightly more general. Let $P \in \Bbbk[X_1, \dots, X_d]$ be a polynomial. Define $d_P \colon \bar{\mathcal{H}}_d \to \bar{\mathcal{H}}_d$ by declaring that d_P acts as zero on $\bar{\mathcal{H}}_d$, while $d_P(\theta) = P$, together with the graded Leibniz rule. Then d_P is a differential on $\bar{\mathcal{H}}_d$. To prove the claim is suffices to check that $d_P(T_1\theta T_1\theta + \theta T_1\theta T_1) = 0$.

We have $T_1P = s_1(P)T_1 - \partial_1(P)$ and $PT_1 = T_1s_1(P) - \partial_1(P)$, where ∂_1 is the Demazure operator. This also implies $T_1PT_1 = s_1(P) - \partial_1(P)T_1$. Note also that $\partial_1(P)$ is a symmetric polynomial with respect to X_1, X_2, so it commutes with T_1. So, we have

$$\begin{aligned} d_P(T_1\theta T_1\theta + \theta T_1\theta T_1) =& T_1PT_1\theta - T_1\theta T_1P + PT_1\theta T_1 - \theta T_1PT_1 \\ =& (s_1(P)\theta - \partial_1(P)T_1\theta) - (T_1\theta s_1(P)T_1 - T_1\theta\partial_1(P)) \\ &+ (T_1s_1(P)\theta T_1 - \partial_1(P)\theta T_1) - (\theta s_1(P) - \theta\partial_1(P)T_1) \\ =& 0, \end{aligned}$$

which proves the claim. □

The following is proved in Section 4.4.

Proposition 2.12. *The DG-algebras $(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}})$ and $(\bar{H}_d^{\mathbf{Q}}, 0)$ are quasi-isomorphic.*

2.2.4. *Completions of $\bar{\mathcal{H}}_d$.* Consider the algebra of symmetric polynomials $\mathrm{Sym}_d = \mathrm{Pol}_d^{\mathfrak{S}_d}$. We consider it as a (central) subalgebra of $\bar{\mathcal{H}}_d$.

For each d-tuple $\mathbf{a} = (a_1, \dots, a_d) \in \Bbbk^d$ we have a character $\chi_{\mathbf{a}} \colon \mathrm{Sym}_d \to \Bbbk$ given by the evaluation $X_r \mapsto a_r$. It is obvious from the definition that if the d-tuples $\mathbf{a}'$ is a permutation of the d-tuple $\mathbf{a}$ then the characters $\chi_{\mathbf{a}}$ and $\chi_{\mathbf{a}'}$ are the same. Denote by $\mathfrak{m}_{\mathbf{a}}$ the kernel of $\chi_{\mathbf{a}}$.

Definition 2.13. Denote by $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ the completion of the algebra $\bar{\mathcal{H}}_d$ at the sequence of ideals $\bar{\mathcal{H}}_d\mathfrak{m}_{\mathbf{a}}^j\bar{\mathcal{H}}_d$.

Set also $\widehat{P}_{\mathbf{a}} = \bigoplus_{\mathbf{b}\in\mathfrak{S}_d\mathbf{a}}(\Bbbk[[X_1 - b_1, \dots, X_d - b_d]] \otimes \bigwedge^\bullet(\underline{\theta})1_{\mathbf{b}}$. We can obviously extend the action of $\bar{\mathcal{H}}_d$ on P_d to an action of $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ on $\widehat{P}_{\mathbf{a}}$. Each finite dimensional $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$-module M decomposes into its generalized eigenspaces $M = \bigoplus_{\mathbf{b}\in\mathfrak{S}_d\mathbf{a}} M_{\mathbf{b}}$, where

$$M_{\mathbf{b}} = \{m \in M \mid \exists N \in \mathbb{Z}_{\geqslant 0} \text{ such that } (X_r - b_r)^N m = 0 \ \forall r\}.$$

For each $\mathbf{b} \in \mathfrak{S}_d\mathbf{a}$ the algebra $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ contains an idempotent $1_{\mathbf{b}}$ that project onto $M_{\mathbf{b}}$ when applied to M.

Proposition 2.14. (a) *The* $\widehat{\mathrm{Pol}}_{\mathbf{a}}$*-module* $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ *is free with basis*

$$\{T_w \xi_1^{b_1} \dots \xi_d^{b_d} |\ w \in \mathfrak{S}_d, (b_1, \dots, b_d) \in \{0,1\}^d\}.$$

(b) *The representation* $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ *of* $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$ *is faithful.*

Proof. It is clear that the elements from the statement generate the $\widehat{\mathrm{Pol}}_{\mathbf{a}}$-module $\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$. To see that they form a basis, it is enough to remark that they act by linear independent (over $\widehat{\mathrm{Pol}}_{\mathbf{a}}$) operators on the representation $\widehat{\mathrm{Pol}}_{\mathbf{a}}$. This proves (a). Then (b) also holds because a basis acts on $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ by linearly independent operators. □

The algebra $\bar{H}_d^{\mathbf{Q}}$ has a decomposition $\bar{H}_d^{\mathbf{Q}} = \oplus_{\mathbf{a}} \bar{H}_{\mathbf{a}}^{\mathbf{Q}}$ (with a finite number of nonzero terms) such that Sym_d acts on each finite dimensional $\bar{H}_{\mathbf{a}}^{\mathbf{Q}}$-module with a generalized character $\chi_{\mathbf{a}}$.

2.3. q-version.

2.3.1. *Affine* q*-Hecke algebra.* The *affine* q*-Hecke algebra* H_d is the $\Bbbk$-algebra generated by $T_1, \dots, T_{d-1}$ and $X_1^{\pm 1}, \dots, X_d^{\pm 1}$, with relations (14)-(16) below.

$$X_r X_r^{-1} = X_r^{-1} X_r = 1, \quad X_i X_j = X_j X_i, \quad X_i^{\pm 1} X_j^{\pm 1} = X_j^{\pm 1} X_i^{\pm 1}, \tag{14}$$

$$(T_i - q)(T_i + 1) = 0, \quad T_i T_j = T_j T_i \text{ if } |i-j| \neq 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}. \tag{15}$$

$$T_i X_j = X_j T_i \quad \text{for } j - i \neq 0, 1, \quad T_i X_i T_i = q X_{i+1}. \tag{16}$$

For $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_d$ a reduced decomposition we put $T_w = T_{i_1} \cdots T_{i_k}$. Then T_w is independent of the choice of the reduced decomposition of w and the set

$$\{X_1^{m_1} \cdots X_d^{m_d} T_w\}_{w \in \mathfrak{S}_d, m_i \in \mathbb{Z}}$$

is a basis of the $\Bbbk$-vector space H_d. There is a faithful representation of H_d on Poll_d, where $T_i(f) = q s_i(f) - (q-1) X_{i+1} \partial_i(f)$.

Let ℓ be a positive integer. Let $\mathbf{Q} = (Q_1, \dots, Q_\ell)$ be an ℓ-tuple of nonzero elements of the field $\Bbbk$.

Definition 2.15. The *cyclotomic* q*-Hecke algebra* is the quotient

$$H_d^{\mathbf{Q}} = H_d / \prod_{r=1}^{\ell} (X_1 - Q_r).$$

2.3.2. *The algebra* $\mathcal{H}_d$.

Definition 2.16. The algebra $\mathcal{H}_d$ is the $\Bbbk$-algebra generated by $T_1, \dots, T_{d-1}$ and $X_1^{\pm 1}, \dots, X_d^{\pm 1}$ in λ-degree zero, and an extra generator θ in λ-degree 1, with relations (14) to (16) and

$$\theta^2 = 0 \tag{17}$$

$$X_r^{\pm 1} \theta = \theta X_r^{\pm 1} \qquad \text{for } r = 1, \dots, d, \tag{18}$$

$$T_r \theta = \theta T_r \qquad \text{for } r > 1, \tag{19}$$

$$T_1 \theta T_1 \theta + \theta T_1 \theta T_1 = (q-1) \theta T_1 \theta. \tag{20}$$

The algebra $\mathcal{H}_d$ contains the affine q-Hecke algebra H_d as a subalgebra concentrated in λ-degree zero.

Lemma 2.17. *The algebra $\mathcal{H}_d$ acts faithfully on Pl_d by*

$$\begin{aligned} T_r(f) &= qs_r(f) - (q-1)X_{r+1}\partial_r(f), \\ X_r^{\pm 1}(f) &= X_r^{\pm 1} f, \\ \theta(f) &= \theta_1 f, \end{aligned}$$

for all $f \in P_d$ and where $s_r(f)$ and $\partial_r(f)$ are as in (6) and (1).

Proof. The defining relations of $\mathcal{H}_d$ can be checked by a straightforward computation. Faithfulness follows from Proposition 2.19 below. □

Define $\xi_1, \dots, \xi_d \in \mathcal{H}_d$ by the rules $\xi_1 = \theta$, $\xi_{i+1} = T_i \xi_i T_i^{-1}$. The following is straightforward.

Lemma 2.18. *The elements ξ_r satisfy for all $r = 1, \dots, d-1$ and all $\ell = 1, \dots, d$,*

$$\xi_\ell^2 = 0, \qquad \xi_r \xi_\ell + \xi_\ell \xi_r = 0$$

and

$$T_\ell \xi_r = \begin{cases} \xi_r T_\ell & \text{if } r \neq \ell, \ell+1, \\ \xi_\ell T_\ell + (q-1)(\xi_{\ell+1} - \xi_\ell) & \text{if } r = \ell+1, \\ \xi_{\ell+1} T_\ell & \text{if } r = \ell. \end{cases}$$

It is not hard to write a basis of $\mathcal{H}_d$ in terms of the ξ_r's.

Proposition 2.19. *The set*

$$\{X_1^{a_1} \cdots X_d^{a_d} T_w \xi_1^{b_1} \cdots \xi_d^{b_d} | w \in \mathfrak{S}_d, (a_1, \dots, a_d) \in \mathbb{Z}^d, (b_1, \dots, b_d) \in \{0,1\}^d\},$$

is a basis of the $\Bbbk$-vector space $\mathcal{H}_d$.

Proof. Imitate the proof of Proposition 2.5. □

Remark 2.20. We see from the proposition above that the algebra $\mathcal{H}_d$ has a triangular decomposition (only as a vector space)

$$\mathcal{H}_d = \Bbbk[X_1, \dots, X_d] \otimes H_d^{\mathrm{fin}} \otimes \bigwedge\nolimits^\bullet(\xi_1, \dots, \xi_d),$$

where H_d^{fin} is the (finite dimensional) Hecke algebra of the group $\mathfrak{S}_d$. More precisely, the algebra H_d^{fin} is defined by generators $T_1, \dots, T_{d-1}$ and relations (15).

2.3.3. *DG-enhancement of $\mathcal{H}_d$.* Let ℓ and $\mathbf{Q}$ be as in Section 2.3.1.

Definition 2.21. Define an operator $\partial_{\mathbf{Q}}$ on $\mathcal{H}_d$ by declaring that $\partial_{\mathbf{Q}}$ acts as zero on H_d, while

$$\partial_{\mathbf{Q}}(\theta) = \prod_{r=1}^{\ell} (X_1 - Q_r),$$

and for $a, b \in \bar{\mathcal{H}}_d$, $\partial_{\mathbf{Q}}(ab) = \partial_{\mathbf{Q}}(a)b + (-1)^{\lambda(a)} a \partial_{\mathbf{Q}}(b)$.

Lemma 2.22. *The operator $\partial_{\mathbf{Q}}$ is a differential on $\mathcal{H}_d$.*

Proof. Similarly to the proof of Lemma 2.11, we consider a more general differential d_P. We have to check

$$d_P(T_1\theta T_1\theta + \theta T_1\theta T_1) = d_P((q-1)\theta T_1\theta).$$

We have $T_1P = s_1(P)T_1 - (q-1)X_2\partial_1(P)$ and $PT_1 = T_1s_1(P) - (q-1)X_2\partial_1(P)$, where ∂_1 is the Demazure operator. Note also that $\partial_1(P)$ is a symmetric polynomial with respect to X_1, X_2, so it commutes with T_1. So, we have

$$\begin{aligned}
d_P(T_1\theta T_1\theta + \theta T_1\theta T_1) =& T_1PT_1\theta - T_1\theta T_1P + PT_1\theta T_1 - \theta T_1PT_1\\
=&(T_1^2s_1(P)\theta - (q-1)\partial_1(P)T_1X_2\theta) - (T_1\theta s_1(P)T_1\\
&- (q-1)T_1\theta X_2\partial_1(P)) + (T_1s_1(P)\theta T_1 - (q-1)X_2\partial_1(P)\theta T_1)\\
&- (\theta s_1(P)T_1^2 - (q-1)\theta\partial_1(P)X_2T_1)\\
=&T_1^2s_1(P)\theta - \theta s_1(P)T_1^2\\
=&(q-1)PT_1\theta - (q-1)\theta T_1P\\
=&d_P\big((q-1)\theta T_1\theta\big),
\end{aligned}$$

which proves the claim. □

The following is proved in Section 4.4.

Proposition 2.23. *The DG-algebras $(\mathcal{H}_d, \partial_{\mathbf{Q}})$ and $(H_d^{\mathbf{Q}}, 0)$ are quasi-isomorphic.*

2.3.4. *Completions of $\mathcal{H}_d$.* Similarly to Section 2.2.4, we want to define a completion of the algebra $\mathcal{H}_d$. Consider the algebra of symmetric Laurent polynomials $\mathrm{Syml}_d = \Bbbk[X_1^{\pm 1}, \dots, X_d^{\pm 1}]^{\mathfrak{S}_d}$. We consider it as a (central) subalgebra of $\mathcal{H}_d$.

For each d-tuple $\mathbf{a} = (a_1, \dots, a_d) \in (\Bbbk^\times)^n$ we have a character $\chi_{\mathbf{a}}\colon \mathrm{Syml}_d \to \Bbbk$ given by the evaluation $X_r \mapsto a_r$. Denote by $\mathfrak{m}_{\mathbf{a}}$ the kernel of $\chi_{\mathbf{a}}$.

Definition 2.24. Denote by $\widehat{\mathcal{H}}_{\mathbf{a}}$ the completion of the algebra $\mathcal{H}_d$ at the sequence of ideals $\mathcal{H}_d\mathfrak{m}_a^j\mathcal{H}_d$.

Set also $\widehat{P}_{\mathbf{a}} = \Bbbk[[X_1 - a_1, \dots, X_d - a_d]] \otimes \bigwedge^\bullet(\underline{\theta})$. We can obviously extend the action of $\mathcal{H}_d$ on P_d to an action of $\widehat{\mathcal{H}}_{\mathbf{a}}$ on $\widehat{P}_{\mathbf{a}}$. Similarly to $\widehat{\overline{\mathcal{H}}}_{\mathbf{a}}$, the algebra $\widehat{\mathcal{H}}_{\mathbf{a}}$ has idempotents $1_{\mathbf{b}}$, $\mathbf{b} \in \mathfrak{S}_d\mathbf{a}$ that are defined in the same way as in Section 2.2.4.

Similar to Proposition 2.14 we have the following.

Proposition 2.25. (a) *The $\widehat{\mathrm{Pol}}_{\mathbf{a}}$-module $\widehat{\mathcal{H}}_{\mathbf{a}}$ is free with basis*

$$\{T_w\xi_1^{b_1}\dots\xi_d^{b_d} |\ w \in \mathfrak{S}_d, (b_1, \dots, b_d) \in \{0,1\}^d\}.$$

(b) *The representation $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ of $\widehat{\mathcal{H}}_{\mathbf{a}}$ is faithful.*

The algebra $H_d^{\mathbf{Q}}$ has a decomposition $H_d^{\mathbf{Q}} = \oplus_{\mathbf{a}} H_{\mathbf{a}}^{\mathbf{Q}}$ (with a finite number of nonzero terms) such that Syml_d acts on each finite dimensional $\bar{H}_{\mathbf{a}}^{\mathbf{Q}}$-module with a generalized character $\chi_{\mathbf{a}}$.

3. DG-ENHANCED VERSIONS OF KLR ALGEBRAS

DG-enhanced versions of KLR algebras were introduced in [16] as one of the main ingredients in the categorification of Verma modules for symmetrizable quantum Kac-Moody algebras.

Let $\Gamma = (I, A)$ be a quiver without loops with set of vertices I and set of arrows A. We call elements in I *labels*. Let also $\mathbb{N}[I]$ be the set of formal $\mathbb{N}$-linear combinations of elements of I. Fix $\nu \in \mathbb{N}[I]$,

$$\nu = \sum_{i \in I} \nu_i \cdot i, \quad \nu_i \in \mathbb{N}, i \in I,$$

and set $|\nu| = \sum_i \nu_i$. We allow the quiver to have infinite number of vertices. In this case only a finite number of ν_i is nonzero.

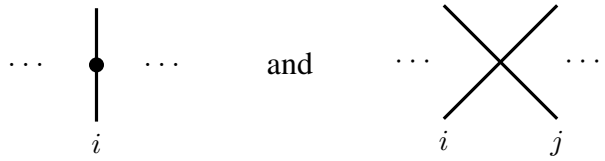
For each $i, j \in I$ we denote by $h_{i,j}$ the number of arrows in the quiver Γ going from i to j, and define for $i \neq j$ the polynomials

$$\mathcal{Q}_{i,j}(u, v) = (u - v)^{h_{i,j}}(v - u)^{h_{j,i}}.$$

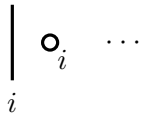
3.1. **The algebra $\mathcal{R}(\nu)$.** We give a diagrammatic definition of the algebras $\mathcal{R} = \mathcal{R}(\Gamma)$ from [16, §3], corresponding to the case of minimal parabolic $\mathfrak{p}$. The definition we give is minimal and equivalent to the one in the reference by [16, Remark 3.10].

Definition 3.1. For each $\nu \in \mathbb{N}[I]$ we define the $\Bbbk$-algebra $\mathcal{R}(\nu)$ by the data below.

- It is generated by the *KLR generators*

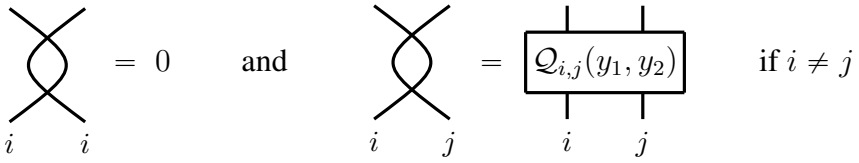


  for $i, j \in I$, where each diagram contains ν_i strands labeled i, together with *floating dots* which are labeled from elements of I and decorate the region immediately at the right of the first strand (with the same label and counted from the left),



- The multiplication is given by gluing diagrams on top of each other whenever the labels of the strands agree, and zero otherwise, subject to the local relations (21) to (27) below, for all $i, j, k \in I$.
  - ◇ The *KLR relations, for all i, j, $k \in I$*:

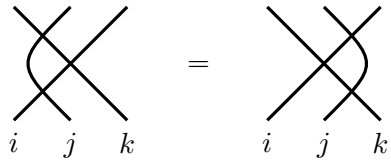
(21)



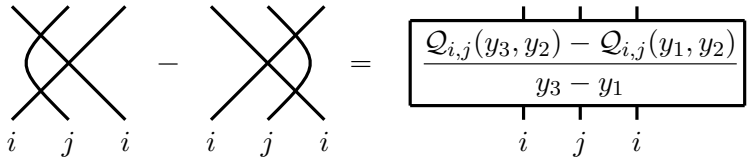
(22) 

if $i \neq j$,

(23) 

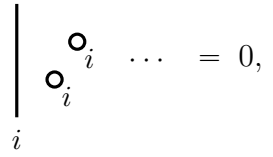
(24) 

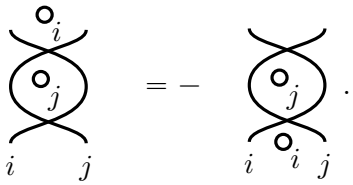
unless $i = k \neq j$,

(25) 

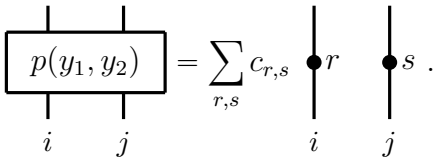
if $i \neq j$.

◇ And the *new relations, for all* $i, j \in I$:

(26) 

(27) 

Remark 3.2. A diagram with a box containing a polynomial means a polynomial in dots. The indices in the variables indicate the strands carrying the corresponding dots. For example, for $p(y_1, y_2) = \sum_{r,s} c_{r,s} y_1^r y_2^s$ with $c_{r,s} \in \Bbbk$ we have



We now define a $\mathbb{Z}\times\mathbb{Z}$-grading in $\mathcal{R}(\nu)$. Contrary to [16] we work with a single cohomological degree λ. We declare

$$\deg\left(\begin{array}{c}\bullet\\ |\\ i\end{array}\right) = (2,\ 0), \qquad \deg\left(\begin{array}{ccc} & & \\ i & \times & j\end{array}\right) = \begin{cases}(-2,0) & \text{if } i=j,\\ (-1,0) & \text{if } h_{i,j}=1,\\ (0,0) & \text{otherwise.}\end{cases}$$

and

$$\deg\left(\begin{array}{ccc}|\ \circ_i & \cdots\\ i & \end{array}\right) = \big(-2,\ 1\big),$$

where the second grading is the λ-grading, which we write $\lambda(\bullet)$. Note that the λ-grading is (up to a sign) half the grading $\deg_\lambda$ in [16] where we take the $\deg_{\lambda_i}$'s equal. The defining relations of $\mathcal{R}(\nu)$ are homogeneous with respect to this bigrading.

Remark 3.3. The algebra $\mathcal{R}(\nu)$ contains the KLR algebra $R(\nu)$ as a subalgebra concentrated in λ-degree zero.

For $\boldsymbol{i} = i_1\cdots i_d$ define the idempotent

$$1_{\boldsymbol{i}} = \begin{array}{cccccc}| & | & & & | & \\ i_1 & i_2 & \cdots & & i_d & \end{array}.$$

Let $\mathrm{Seq}(\nu)$ be the set of all ordered sequences $\boldsymbol{i} = i_1 i_2 \ldots i_d$ with each $i_k \in I$ and i appearing ν_i times in the sequence. For $\boldsymbol{i},\boldsymbol{j} \in \mathrm{Seq}(\nu)$ the idempotents $1_{\boldsymbol{i}}$ and $1_{\boldsymbol{j}}$ are orthogonal iff $\boldsymbol{i}\neq\boldsymbol{j}$, we have $1_{\mathcal{R}(\nu)} = \sum_{\boldsymbol{i}\in\mathrm{Seq}(\nu)} 1_{\boldsymbol{i}}$, where $1_{\mathcal{R}(\nu)}$ denotes the identity element in $\mathcal{R}(\nu)$, and

$$\mathcal{R}(\nu) = \bigoplus_{\boldsymbol{j},\boldsymbol{i}\in\mathrm{Seq}(\nu)} 1_{\boldsymbol{j}}\mathcal{R}(\nu)1_{\boldsymbol{i}}.$$

Finally, the algebra $\mathcal{R}$ is defined as

$$\mathcal{R} = \bigoplus_{\nu\in\mathbb{N}[I]} \mathcal{R}(\nu).$$

3.2. **Polynomial action of $\mathcal{R}(\nu)$.** We fix $\nu\in\mathbb{N}[I]$ with $|\nu| = d$. For each $i\in I$ let

$$PR_i = \Bbbk[y_{1,i},\ldots,y_{\nu_i,i}]\otimes \textstyle\bigwedge^\bullet\langle\omega_{1,i},\ldots,\omega_{\nu_i,i}\rangle.$$

Each PR_i is a bigraded superring with $\deg(y_{r,i}) = (2,0)$ and $\deg(\omega_{r,i}) = (-2r,2)$, which is isomorphic to the superring R (with the right number of variables) defined in [15, §2.1]. The symmetric group $\mathfrak{S}_{\nu_i}$ acts on PR_i by

$$\begin{aligned}\omega(y_{r,i}) &= y_{\omega(r),j},\\ s_k(\omega_{r,i}) &= \omega_{r,i} + \delta_{k,r}(y_{r,i}-y_{r+1,i})\omega_{r+1,i},\end{aligned}$$

for $\omega\in\mathfrak{S}_{\nu_i}$ and $s_k\in\mathfrak{S}_{\nu_i}$ a simple transposition.

Set $PR_I = \bigotimes_{i\in I} PR_i$ where $\otimes$ is the supertensor product, and define

$$PR_\nu = \bigoplus_{\boldsymbol{i}\in \mathrm{Seq}(\nu)} PR_I 1_{\boldsymbol{i}},$$

where $1_{\boldsymbol{i}}$ is a central idempotent.

It is sometimes convenient to use a different notation for the elements of PR_ν. For each $1 \leqslant r \leqslant |\nu|$, denote by Y_r the element of PR_ν determined by the condition that for each $\boldsymbol{i} \in \mathrm{Seq}(\nu)$ we have $Y_r 1_{\boldsymbol{i}} = y_{r',i_r} 1_{\boldsymbol{i}}$ where r' is such that the element i_r appears r' times among $i_1, i_2, \dots, i_r$. Similarly, we consider the element $\Omega_r \in PR_\nu$ given by $\Omega_r 1_{\boldsymbol{i}} = \omega_{r',i_r} 1_{\boldsymbol{i}}$, where r' is defined in the same way as above. It is clear from the definition that all Y_r commute and all Ω_r anti-commute. Then we have $PR_\nu = \bigoplus_{\boldsymbol{i}\in \mathrm{Seq}(\nu)} \Bbbk[Y_1, \dots, Y_{|\nu|}] \otimes \bigwedge^\bullet \langle \Omega_1, \dots, \Omega_{|\nu|}\rangle 1_{\boldsymbol{i}}$.

We extend the action of $\mathfrak{S}_{\nu_i}$ on PR_i to an action of $\mathfrak{S}_{|\nu|}$ on PR_ν where

$$s_k : PR_I 1_{\boldsymbol{i}} \to PR_I 1_{s_k \boldsymbol{i}},$$

sends

$$y_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} y_{p+1,i} 1_{s_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i \text{ and } p = \#\{s \leqslant k | i_s = i\}, \\ y_{p-1,i} 1_{s_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i \text{ and } p = 1 + \#\{s \leqslant k | i_s = i\}, \\ y_{p,i} 1_{s_k \boldsymbol{i}} & \text{otherwise,} \end{cases}$$

and

$$\omega_{p,i} 1_{\boldsymbol{i}} \mapsto \begin{cases} \left(\omega_{p,i} + (y_{p,i} - y_{p+1,i})\omega_{p+1,i}\right) 1_{s_k \boldsymbol{i}} & \text{if } i_k = i_{k+1} = i \text{ and } p = \#\{s \leqslant k | i_s = i\}, \\ \omega_{p,i} 1_{s_k \boldsymbol{i}} & \text{otherwise,} \end{cases}$$

with $p \in \{1, \dots, \nu_i\}$ and $\boldsymbol{i} = i_1 \dots i_d$.

For the comfort of the reader we also give the formulas of the $\mathfrak{S}_{|\nu|}$-action on PR_ν is terms of Y's and Ω's:

$$s_k : PR_I 1_{\boldsymbol{i}} \to PR_I 1_{s_k \boldsymbol{i}},$$

sends

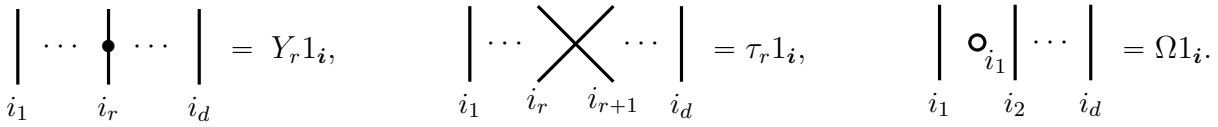
$$Y_p 1_{\boldsymbol{i}} \mapsto \begin{cases} Y_{k+1} 1_{s_k \boldsymbol{i}} & \text{if } p = k, \\ Y_k 1_{s_k \boldsymbol{i}} & \text{if } p = k+1, \\ Y_p 1_{s_k \boldsymbol{i}} & \text{otherwise,} \end{cases}$$

and

$$\Omega_p 1_{\boldsymbol{i}} \mapsto \begin{cases} \left(\Omega_k + (Y_k - Y_{k+1})\Omega_{k+1}\right) 1_{s_k \boldsymbol{i}} & \text{if } p = k \text{ and } i_k = i_{k+1}\ , \\ \Omega_{k+1} 1_{s_k \boldsymbol{i}} & \text{if } p = k \text{ and } i_k \neq i_{k+1}\ , \\ \Omega_k 1_{s_k \boldsymbol{i}} & \text{if } p = k+1 \text{ and } i_k \neq i_{k+1}\ , \\ \Omega_p 1_{s_k \boldsymbol{i}} & \text{otherwise.} \end{cases}$$

For each $i, j \in I$, $i \neq j$, we consider the polynomial $\mathcal{P}_{ij}(u, v) = (u - v)^{h_{i,j}}$. Note that we have $\mathcal{Q}_{i,j}(u, v) = \mathcal{P}_{i,j}(u, v)\mathcal{P}_{j,i}(v, u)$.

In the sequel it is useful to have an algebraic presentation of $\mathcal{R}(\nu)$ as in [2]. We set



We declare that $a \in e_{\boldsymbol{k}}\mathcal{R}(\nu)e_{\boldsymbol{j}}$ acts as zero on $PR_I 1_{\boldsymbol{i}}$ whenever $\boldsymbol{j} \neq \boldsymbol{i}$. Otherwise

$$Y_r 1_{\boldsymbol{i}} \quad \longmapsto \quad f1_{\boldsymbol{i}} \mapsto Y_r f 1_{\boldsymbol{i}},$$

$$\Omega 1_{\boldsymbol{i}} \quad \longmapsto \quad f1_{\boldsymbol{i}} \mapsto \Omega_1 f 1_{\boldsymbol{i}},$$

and

$$\tau_r 1_{\boldsymbol{i}} \quad \longmapsto \quad f1_{\boldsymbol{i}} \mapsto \begin{cases} \dfrac{f1_{\boldsymbol{i}} - s_r(f1_{\boldsymbol{i}})}{Y_r - Y_{r+1}} & \text{if } i_r = i_{r+1}, \\ \mathcal{P}_{i_r,i_{r+1}}(Y_r, Y_{r+1}) s_r(f1_{\boldsymbol{i}}) & \text{if } i_r \neq i_{r+1}. \end{cases}$$

The following is Proposition 3.8 and Theorem 3.15 in [16].

Proposition 3.4. *The rules above define a faithful action of $\mathcal{R}(\nu)$ on PR_ν.*

We now give the basis of $\mathcal{R}(\nu)$, as constructed in [16, §3.3]. Fix $\boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)$. We write ${}_{\boldsymbol{j}}\mathfrak{S}_{\boldsymbol{i}} \subseteq \mathfrak{S}_{|\nu|}$ for the subset of permutations w satisfying $w(\boldsymbol{i}) = \boldsymbol{j}$.

Recall [16, §3.3] that a reduced expression $s_{r_k} \cdots s_{r_1}$ of $w \in \mathfrak{S}_{|\nu|}$ is *left-adjusted* if $r_1 + \cdots + r_k$ is minimal among all reduced expressions for w. For each $w \in \mathfrak{S}_{|\nu|}$ we fix a left-adjusted presentation $w = s_{r_k} \cdots s_{r_1}$ of w.

For each $r \in \mathcal{I} = \{1, \cdots, d\}$ and w as above let r_m, $m \in \{1, \ldots, k\}$ be the index such that

$$s_{r_m} \cdots s_{r_1}(r) \leqslant s_{r_j} \cdots s_{r_1}(r) \qquad \forall j \in \{1, \ldots, k\},$$

and

$$s_{r_m} \cdots s_{r_1}(r) < s_{r_{m-1}} \cdots s_{r_1}(r) \quad \text{if } m > 1,$$

i.e., m is the minimal index such that $s_{r_m} \cdots s_{r_1}(r)$ is minimal.

Define $\Omega^{(r)} = \tau_{r_m - 1} \cdots \tau_2 \tau_1 \Omega \tau_1 \tau_2 \cdots \tau_{r_m - 1}$ and put

$$\tau_w(r) = \tau_{r_k} \cdots \tau_{r_{m+1}} \Omega^{(r)} \tau_{r_m} \cdots \tau_{r_1} 1_{\boldsymbol{i}} \in 1_{\boldsymbol{j}} \mathcal{R}(\nu) 1_{\boldsymbol{i}}.$$

Now, for each $I \subseteq \mathcal{I}$, define $\tau_w(I)$ by simultaneously placing all the $\Omega^{(r)}$ for $r \in I$ following the rule above. For example, for $I = \{r, r'\}$, with $s_{r_m} \cdots s_{r_1}(r)$ and $s_{r_{m'}} \cdots s_{r_1}(r')$ minimal with $m < m'$, we have

$$\tau_w(\{r, r'\}) = \tau_{r_k} \cdots \tau_{r_{m'+1}} \Omega^{(r')} \tau_{r_{m'}} \cdots \tau_{r_{m+1}} \Omega^{(r)} \tau_{r_m} \cdots \tau_{r_1}.$$

The following is Theorem 3.15 in [16].

Theorem 3.5. *The set*

$$\{\tau_w(I)Y_1^{n_1}\cdots Y_{|\nu|}^{n_{|\nu|}} \mid \omega \in {}_{\boldsymbol{j}}\mathfrak{S}_{\boldsymbol{i}},\ I \subseteq \mathcal{I},\ \underline{n} \in \mathbb{N}^{|\nu|}\}$$

is a basis of the $\Bbbk$*-vector space* $1_{\mathbf{j}}R(\nu)1_{\mathbf{i}}$.

3.3. **Completion of** $\mathcal{R}(\nu)$**.** For each $i \in I$ consider the polynomial ring $\mathrm{PolR}_i = \Bbbk[y_{1,i}, \ldots, y_{\nu_i,i}]$. Set also $\mathrm{PolR}_I = \bigotimes_{i\in I} \mathrm{PolR}_i$. We will consider PolR_I as a subalgebra of $\mathcal{R}(\nu)$. Let $\mathfrak{m}$ be the ideal of PolR_I generated by all $y_{r,i}$, $i \in I$, $1 \leqslant r \leqslant \nu_i$.

Definition 3.6. Denote by $\widehat{\mathcal{R}}(\nu)$ the completion of the algebra $\mathcal{R}(\nu)$ at the sequence of ideals $\mathcal{R}(\nu)\mathfrak{m}^j\mathcal{R}(\nu)$.

Now we construct a representation $\widehat{PR_\nu}$ of $\widehat{\mathcal{R}}(\nu)$, which is a completion of the representation PR_I of $\mathcal{R}(\nu)$. For $i \in I$, set

$$\widehat{\mathrm{PolR}}_i = \Bbbk[[y_{1,i}, \ldots, y_{\nu_i,i}]] \quad \text{and} \quad \widehat{PR}_i = \Bbbk[[y_{1,i}, \ldots, y_{\nu_i,i}]] \otimes \textstyle\bigwedge^\bullet\langle\omega_{1,i}, \ldots, \omega_{\nu_i,i}\rangle.$$

Set also

$$\widehat{\mathrm{PolR}}_I = \bigotimes_{i\in I} \widehat{\mathrm{PolR}}_i, \qquad \widehat{\mathrm{PolR}}_\nu = \bigoplus_{\boldsymbol{i}\in \mathrm{Seq}(\nu)} \widehat{\mathrm{PolR}}_I 1_{\boldsymbol{i}},$$

and

$$\widehat{PR}_I = \bigotimes_{i\in I} \widehat{PR}_i, \qquad \widehat{PR}_\nu = \bigoplus_{\boldsymbol{i}\in \mathrm{Seq}(\nu)} \widehat{PR}_I 1_{\boldsymbol{i}}.$$

The $\mathfrak{S}_{|\nu|}$-action on PR_ν extends obviously to an $\mathfrak{S}_{|\nu|}$-action on $\widehat{PR}_\nu$. Moreover, the action of $\mathcal{R}(\nu)$ on PR_ν yields an action of $\widehat{\mathcal{R}}(\nu)$ on $\widehat{PR}_\nu$.

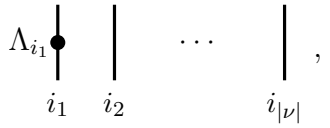
Lemma 3.7. (a) *The algebra* $\widehat{\mathcal{R}}(\nu)$ *is free over* $\widehat{\mathrm{PolR}}_\nu$ *with basis*

$$\{\tau_w(I) |\ w \in {}_{\boldsymbol{j}}\mathfrak{S}_{\boldsymbol{i}},\ I \subseteq \mathcal{I}, \boldsymbol{i}, \boldsymbol{j} \in \mathrm{Seq}(\nu)\}.$$

(b) *The representation* $\widehat{PR}_\nu$ *of* $\widehat{\mathcal{R}}(\nu)$ *is fully faithful.*

Proof. It is clear that the set in the statement generate the $\widehat{\mathrm{PolR}}_\nu$-module $\widehat{\mathcal{R}}(\nu)$. Then this set forms a basis because the elements $\tau_w(I)$ act on $\widehat{PR}_\nu$ by linearly independent (over $\widehat{\mathrm{PolR}}_\nu$) operators. This proves (a). Then (b) is also true because a basis of $\widehat{\mathcal{R}}(\nu)$ acts on $\widehat{PR}_\nu$ by linearly independent operators. □

3.4. **Cyclotomic KLR algebras.** Let Λ be a dominant integral weight of type Γ (i.e., for each vertex i of Γ we fix a nonnegative integer Λ_i). Let I^Λ be the 2-sided ideal of $R(\nu)$ generated by $Y_1^{\Lambda_{i_1}}1_{\boldsymbol{i}}$ with $\boldsymbol{i} \in \mathrm{Seq}(\nu)$. In terms of diagrams, this is the 2-sided ideal generated by all diagrams of the form



with $\boldsymbol{i} \in \mathrm{Seq}(\nu)$.

Definition 3.8. The *cyclotomic KLR algebra* is the quotient $R^\Lambda(\nu) = R(\nu)/I^\Lambda$.

3.5. **DG-enhancements of $\mathcal{R}(\nu)$.** We turn $\mathcal{R}(\nu)$ into a DG-algebra by introducing a differential d_Λ given by

$$d_\Lambda(1_{\boldsymbol{i}}) = d_\Lambda(Y_r) = d_\Lambda(\tau_k) = 0,$$

$$d_\Lambda(\Omega 1_{\boldsymbol{i}}) = (-Y_1)^{\Lambda_{i_1}} 1_{\boldsymbol{i}},$$

together with the Leibniz rule $d_\Lambda(ab) = d_\Lambda(a)b + (-1)^{\lambda(a)} d_\Lambda(b)$. This algebra is differential graded w.r.t. the homological degree given by counting the number of floating dots.

The following is [16, Proposition 4.14].

Proposition 3.9. *The DG-algebra $(\mathcal{R}(\nu), d_\Lambda)$ is quasi-isomorphic to the cyclotomic KLR algebra $R^\Lambda(\nu)$.*

4. The isomorphism theorems

4.1. **A generalization of the Brundan–Kleshchev–Rouquier isomorphisms.** Choose I, Γ and ν as in Section 3. Assume additionally that for $i, j \in I$, $i \neq j$, there is at most one arrow from i to j.

Let PolR_I be as in Section 3.3. Set $\mathrm{PolR}_\nu = \bigoplus_{\boldsymbol{i} \in \mathrm{Seq}(\nu)} \mathrm{PolR}_I\, 1_{\boldsymbol{i}}$. Let PA_ν be a PolR_ν-algebra (the most interesting examples for us are $PA_\nu = PR_\nu$ and $PA_\nu = \mathrm{PolR}_\nu$). Set also $\widehat{PA}_\nu = \widehat{\mathrm{PolR}}_\nu \otimes_{\mathrm{PolR}_\nu} PA_\nu$.

Fix an action of $\mathfrak{S}_{|\nu|}$ on $\widehat{PA}_\nu$ (by ring automorphisms) that extends the obvious $\mathfrak{S}_{|\nu|}$-action on $\widehat{\mathrm{PolR}}_\nu$. Assume additionally that for each simple generator s_r of $\mathfrak{S}_{|\nu|}$, each $\boldsymbol{i} \in \mathrm{Seq}(\nu)$ such that $i_r = i_{r+1}$ and each $f \in \widehat{PA}_\nu$, we have $(f - s_r(f))1_{\boldsymbol{i}} \in (Y_r - Y_{r+1})\widehat{PA}_\nu$. In particular, this implies that the Demazure operator $\frac{1-s_r}{Y_r - Y_{r+1}}$ is well-defined on $\widehat{PA}_\nu 1_{\boldsymbol{i}}$.

Fix a subalgebra $\widehat{PA}'_\nu$ of $\widehat{PA}_\nu$. Assume now that we have an algebra $\widehat{\mathcal{A}}(\nu)$ that has a faithful representation on $\widehat{PA}_\nu$. Assume that the action of $\widehat{\mathcal{A}}(\nu)$ on $\widehat{PA}_\nu$ is generated by multiplication by elements of $\widehat{PA}'_\nu$ and by the operators τ_r, $r \in \{1, 2, \dots, |\nu| - 1\}$ given by

- if $i_r = i_{r+1}$, then τ_r acts on $f1_{\boldsymbol{i}}$ by a (nonzero scalar) multiple of the Demazure operator, i.e., τ_r sends $f1_{\boldsymbol{i}}$ to $\dfrac{(f - s_r(f))1_{\boldsymbol{i}}}{Y_r - Y_{r+1}}$,
- $i_r \mapsto i_{r+1}$, then τ_r sends $f1_{\boldsymbol{i}}$ to a (nonzero scalar) multiple of $(Y_r - Y_{r+1})s_r(f1_{\boldsymbol{i}})$,
- in other cases, the element τ_r sends $f1_{\boldsymbol{i}}$ to a (nonzero scalar) multiple of $s_r(f1_{\boldsymbol{i}})$.

We are going to show that in some situations, an algebra satisfying some list of properties is automatically isomorphic to $\widehat{\mathcal{A}}(\nu)$.

4.1.1. *Degenerate version.* Fix $\mathbf{Q} = (Q_1, \dots, Q_\ell) \in \Bbbk^\ell$, as in Section 2.2.1. Now we fix some special choice of Γ and ν. Let I be a subset of $\Bbbk$ that contains $Q_1, \dots, Q_\ell$. We construct the quiver Γ with the vertex set I using the following rule: for $i, j \in I$ we have an edge $i \to j$ if and only if we have $j + 1 = i$. Note that this convention for Γ is opposite to [20]. Let d be a positive integer. Fix $\mathbf{a} \in I^d$ (see Section 2.2.4). Finally we consider ν such that ν_i is the multiplicity of i in $\mathbf{a}$. In particular, we see that $|\nu| = d$ is the length of $\mathbf{a}$.

For each $i \in I$ denote by Λ_i the multiplicity of i in $(Q_1, \dots, Q_r)$. In particular, this implies $\prod_{r=1}^{\ell}(X_1 - Q_r) = \prod_{i\in I}(X_1 - i)^{\Lambda_i}$.

As above, we set $\mathrm{Pol}_d = \Bbbk[X_1, \cdots, X_d]$. Let PB_d be an Pol_d-algebra. (The most interesting examples are $PB_d = P_d$ and $PB_d = \mathrm{Pol}_d$.) Set $\widehat{\mathrm{Pol}}_{\mathbf{a}} = \bigoplus_{\mathbf{b}\in\mathfrak{S}_d\mathbf{a}} \Bbbk[[X_1 - b_1, \dots, X_d - b_d]]1_{\mathbf{b}}$ and $\widehat{PB}_{\mathbf{a}} = \bigoplus_{\mathbf{b}\in\mathfrak{S}_d\mathbf{a}} (\Bbbk[[X_1 - b_1, \dots, X_d - b_d]] \otimes_{\mathrm{Pol}_d} PB_d)1_{\mathbf{b}}$. Then $\widehat{PB}_{\mathbf{a}}$ is a $\widehat{\mathrm{Pol}}_{\mathbf{a}}$-algebra.

Fix an action of $\mathfrak{S}_d$ on PB_d (by ring automorphisms) that extends the obvious $\mathfrak{S}_d$-action on Pol_d. Assume additionally that for each simple generator s_r of $\mathfrak{S}_d$ and each $f \in PB_d$, we have $f - s_r(f) \subseteq (X_r - X_{r+1})PB_d$. In particular, this implies that the Demazure operator $\partial_r = \frac{1-s_r}{X_r - X_{r+1}}$ is well-defined on PB_d. The action of $\mathfrak{S}_d$ on Pol_d and PB_d can be obviously extended to an action on $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ and $\widehat{PB}_{\mathbf{a}}$.

Fix a subalgebra $\widehat{PB}'_{\mathbf{a}}$ of $\widehat{PB}_{\mathbf{a}}$. Now, assume that there is an algebra $\hat{\hat{\mathcal{B}}}_{\mathbf{a}}$ that has a faithful representation in $\widehat{PB}_{\mathbf{a}}$ that is generated by multiplication by elements of $\widehat{PB}'_{\mathbf{a}}$ and by the operators

$$T_r = s_r - \partial_r.$$

By construction, we have the isomorphism

$$\widehat{\mathrm{PolR}}_\nu \simeq \widehat{\mathrm{Pol}}_{\mathbf{a}}, \quad Y_r 1_{\boldsymbol{i}} \mapsto (X_r - i_r)1_{\boldsymbol{i}}. \tag{28}$$

Moreover, this isomorphism commutes with the action of $\mathfrak{S}_d$. Assume that we can extend the isomorphism $\widehat{\mathrm{PolR}}_\nu \simeq \widehat{\mathrm{Pol}}_{\mathbf{a}}$ in (28) to an $\mathfrak{S}_d$-invariant isomorphism $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$. Moreover, we also assume that this extension restricts to an isomorphism $\widehat{RA}'_\nu \simeq \widehat{PB}'_{\mathbf{a}}$. Then we have the following.

Proposition 4.1. *There is an algebra isomorphism $\widehat{\mathcal{A}}(\nu) \simeq \hat{\hat{\mathcal{B}}}_{\mathbf{a}}$ that intertwines the representation in $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$.*

Proof. We only have to show that we can write the operator τ_r in terms of T_r (and multiplication by elements of $\widehat{PA}'_\nu \simeq \widehat{PB}'_{\mathbf{a}}$) and vice versa.

First, we express τ_r in terms of T_r. We can rewrite the operator T_r in the following way

$$T_r = 1 + \frac{X_r - X_{r+1} + 1}{X_r - X_{r+1}}(s_r - 1).$$

Fix $\boldsymbol{i} \in \mathrm{Seq}(\nu) = \mathfrak{S}_d\mathbf{a}$. Assume $i_r = i_{r+1}$. Then the action of the operator $(X_r - X_{r+1} + 1)^{-1}1_{\boldsymbol{i}}$ on $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$ is well-defined. The element $-(X_r - X_{r+1} + 1)^{-1}(T_r - 1)1_{\boldsymbol{i}}$ acts on $\widehat{PB}_{\mathbf{a}}$ by the same operator as $\tau_r 1_{\boldsymbol{i}}$. Now, assume that we have $i_r \neq i_{r+1}$. If additionally we have no arrow $i_r \to i_{r+1}$, we can write $s_r 1_{\boldsymbol{i}} = (\frac{X_r - X_{r+1}}{X_r - X_{r+1}+1}(T_r - 1) + 1)1_{\boldsymbol{i}}$. (We need the condition $i_{r+1} + 1 \neq i_r$ to be able to divide by $(X_r - X_{r+1} + 1)$ here.) The operator $s_r 1_{\boldsymbol{i}}$ acts on $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$ in the same way as $\tau_r 1_{\boldsymbol{i}}$. Finally, if we have $i_r \to i_{r+1}$, then the operator $(X_r - X_{r+1} + 1)s_r 1_{\boldsymbol{i}} = [(X_r - X_{r+1})(T_r - 1) + (X_r - X_{r+1} + 1)]1_{\boldsymbol{i}}$ acts on $\widehat{PB}_{\mathbf{a}}$ in the same way as $\tau_r 1_{\boldsymbol{i}}$.

Now, we express T_r in terms of τ_r. The operator $T_r 1_{\boldsymbol{i}}$ acts by $[1 + \frac{(X_r - X_{r+1}+1)}{X_r - X_{r+1}}(s_r - 1)]1_{\boldsymbol{i}}$. In the case $i_r \neq i_{r+1}$, we are allowed to divide by $X_r - X_{r+1}$ here. If we additionally have no arrow $i_r \to i_{r+1}$, then the element $s_r 1_{\boldsymbol{i}}$ acts in the same way as $\tau_r 1_{\boldsymbol{i}}$. If we have an arrow $i_r \to i_{r+1}$,

then $(X_r - X_{r+1} + 1)s_r 1_{\boldsymbol{i}}$ acts in the same way as $\tau_r 1_{\boldsymbol{i}}$. It remains to treat the case $i_r = i_{r+1}$. In this case, the element $\frac{s_r - 1}{X_r - X_{r+1}}$ acts in the same way as $-\tau_r 1_{\boldsymbol{i}}$.

□

4.1.2. *q-version.* Fix $q \in \Bbbk$, $q \neq 0, 1$. Fix also $\mathbf{Q} = (Q_1, \dots, Q_\ell) \in (\Bbbk^\times)^\ell$, as in Section 2.3.1. Now we fix some special choice of Γ and ν. Let I be a subset of $\Bbbk^\times$ that contains $Q_1, \dots, Q_\ell$. We construct the quiver Γ with the vertex set I using the following rule: for $i, j \in I$ we have an edge $i \to j$ if and only if we have $qj = i$. Note that this convention for Γ is opposite to [12] and [20]. Fix $\mathbf{a} \in I^d$ (see Section 2.3.4). Finally we consider ν such that ν_i is the multiplicity of i in $\mathbf{a}$. In particular, we see that $|\nu| = d$ is the length of $\mathbf{a}$. As in the degenerate case, for each $i \in I$ we denote by Λ_i the multiplicity of i in $(Q_1, \dots, Q_r)$.

Set $\mathrm{Poll}_d = \Bbbk[X_1^{\pm 1}, \cdots, X_d^{\pm 1}]$. Let PB_d be a Poll_d-algebra. (The most interesting examples are $PB_d = P_d$ and $PB_d = \mathrm{Poll}_d$.) Set $\widehat{\mathrm{Pol}}_{\mathbf{a}} = \bigoplus_{\mathbf{b} \in \mathfrak{S}_d \mathbf{a}} \Bbbk[[X_1 - b_1, \dots, X_d - b_d]] 1_{\mathbf{b}}$ and $\widehat{PB}_{\mathbf{a}} = \bigoplus_{\mathbf{b} \in \mathfrak{S}_d \mathbf{a}} (\Bbbk[[X_1 - b_1, \dots, X_d - b_d]] \otimes_{\mathrm{Poll}_d} PB_d) 1_{\mathbf{b}}$. Then $\widehat{PB}_{\mathbf{a}}$ is a $\widehat{\mathrm{Pol}}_{\mathbf{a}}$-algebra.

Fix an action of $\mathfrak{S}_d$ on PB_d (by ring automorphisms) that extends the obvious $\mathfrak{S}_d$-action on Poll. Assume additionally that for each simple generator s_r of $\mathfrak{S}_d$ and each $f \in PB_d$, we have $f - s_r(f) \subseteq (X_r - X_{r+1}) Pl_d$. In particular, this implies that the Demazure operator $\frac{1 - s_r}{X_r - X_{r+1}}$ is well-defined on Pl_d. The action of $\mathfrak{S}_d$ on Poll_d and Pl_d can be obviously extended to an action on $\widehat{\mathrm{Poll}}_{\mathbf{a}}$ and $\widehat{PB}_{\mathbf{a}}$.

Fix a subalgebra $\widehat{PB}'_{\mathbf{a}}$ of $\widehat{PB}_{\mathbf{a}}$. Now, assume that there is an algebra $\widehat{\mathcal{B}}_{\mathbf{a}}$ that has a faithful representation in $\widehat{PB}_{\mathbf{a}}$ that is generated by multiplication by elements of $\widehat{PB}'_{\mathbf{a}}$ and by the operators

$$T_r = q + \frac{(qX_r - X_{r+1})}{X_r - X_{r+1}}(s_r - 1).$$

By construction, we have the isomorphism

$$\widehat{\mathrm{PolR}}_\nu \simeq \widehat{\mathrm{Pol}}_{\mathbf{a}}, \quad Y_r 1_{\boldsymbol{i}} \mapsto i_r^{-1}(X_r - i_r) 1_{\boldsymbol{i}}. \tag{29}$$

Moreover, this isomorphism commutes with the action of $\mathfrak{S}_d$. Assume that we can extend the isomorphism $\widehat{\mathrm{PolR}}_\nu \simeq \widehat{\mathrm{Pol}}_{\mathbf{a}}$ in (29) to an $\mathfrak{S}_d$-invariant isomorphism $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$. Moreover, assume also that this extension restricts to an isomorphism $\widehat{PA}'_\nu \simeq \widehat{PB}'_{\mathbf{a}}$. The we have the following.

Proposition 4.2. *There is an algebra isomorphism $\widehat{\mathcal{A}}(\nu) \simeq \widehat{\mathcal{B}}_{\mathbf{a}}$ that intertwines the representation in $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$.*

Proof. We only have to show that we can write the operator τ_r in terms of T_r (and multiplication by elements of $\widehat{PA}'_\nu \simeq \widehat{PB}'_{\mathbf{a}}$) and vice versa.

First, we express τ_r in terms of T_r. Fix $\boldsymbol{i} \in \mathrm{Seq}(\nu) = \mathfrak{S}_d \mathbf{a}$. Assume $i_r = i_{r+1}$. Then the action of the operator $(qX_r - X_{r+1})^{-1} 1_{\boldsymbol{i}}$ on $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$ is well-defined. The element $-(qX_r - X_{r+1})^{-1}(T_r - q) 1_{\boldsymbol{i}}$ acts on $\widehat{PB}_{\mathbf{a}}$ by the same operator as $\tau_r 1_{\boldsymbol{i}}$. Now, assume that we have $i_r \neq i_{r+1}$. If moreover we have no arrow $i_r \to i_{r+1}$, we can write $s_r 1_{\boldsymbol{i}} = (\frac{X_r - X_{r+1}}{qX_r - X_{r+1}}(T_r - q) + 1) 1_{\boldsymbol{i}}$ (we need the condition $qi_{r+1} \neq i_r$ to be able to divide by $(qX_r - X_{r+1})$ here). The operator $s_r 1_{\boldsymbol{i}}$

acts on $\widehat{PA}_\nu \simeq \widehat{PB}_{\mathbf{a}}$ in the same way as $\tau_r 1_{\boldsymbol{i}}$. Finally, if we have $i_r \to i_{r+1}$, then the operator $(qX_r - X_{r+1})s_r 1_{\boldsymbol{i}} = [(X_r - X_{r+1})(T_r - q) + (qX_r - X_{r+1})]1_{\boldsymbol{i}}$ acts on $\widehat{PB}_{\mathbf{a}}$ in the same way as $\tau_r 1_{\boldsymbol{i}}$ up to scalar.

Now, we express T_r in terms of τ_r. The operator $T_r 1_{\boldsymbol{i}}$ acts by $[q + \frac{(qX_r - X_{r+1})}{X_r - X_{r+1}}(s_r - 1)]1_{\boldsymbol{i}}$. In the case $i_r \neq i_{r+1}$, we are allowed to divide by $X_r - X_{r+1}$ here. If we additionally have no arrow $i_r \to i_{r+1}$, then the element $s_r 1_{\boldsymbol{i}}$ acts in the same way as $\tau_r 1_{\boldsymbol{i}}$. If we have an arrow $i_r \to i_{r+1}$, then $(qX_r - X_{r+1})s_r 1_{\boldsymbol{i}}$ acts up to scalar in the same way as $\tau_r 1_{\boldsymbol{i}}$. It remains to treat the case $i_r = i_{r+1}$. In this case, the element $\frac{s_r - 1}{X_r - X_{r+1}}$ acts in the same way as $-\tau_r 1_{\boldsymbol{i}}$. □

4.2. **The DG-enhanced isomorphism theorem: the degenerate version.** In Proposition 4.1 we proved that we have an isomorphism of algebras $\widehat{\mathcal{A}}(\nu) \simeq \widehat{\mathcal{B}}_{\mathbf{a}}$ for some algebras $\widehat{\mathcal{A}}(\nu)$ and $\widehat{\mathcal{B}}_{\mathbf{a}}$ that satisfy some list of properties. Let us show that we can apply Proposition 4.2 to the special situation $\widehat{\mathcal{A}}(\nu) = \widehat{\mathcal{R}}(\nu)$ and $\widehat{\mathcal{B}}_{\mathbf{a}} = \widehat{\mathcal{H}}_{\mathbf{a}}$. (We assume that ν and $\mathbf{a}$ are related as in Section 4.1.1.) In this case we can take $\widehat{PA}_\nu = \widehat{PR}_\nu$ and $\widehat{PB}_{\mathbf{a}} = \widehat{P}_{\mathbf{a}}$. We fix the following $\widehat{PA}'_\nu \subseteq \widehat{PA}_\nu$. The subalgebra $\widehat{PA}'_\nu$ is generated by $\widehat{\mathrm{PolR}}_\nu$ and Ω_1. Similarly, we construct a subalgebra $\widehat{PB}'_{\mathbf{a}} \subseteq \widehat{PB}_{\mathbf{a}}$. The subalgebra $\widehat{PB}'_{\mathbf{a}}$ is generated by $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ and θ_1.

To be able to apply Proposition 4.1, we only have to construct a $\mathfrak{S}_d$-invariant isomorphism $\alpha\colon \widehat{P}_{\mathbf{a}} \simeq \widehat{PR}_\nu$ extending the isomorphism (28) such that α restricts to an isomorphism $\widehat{PB}'_{\mathbf{a}} \simeq \widehat{PA}'_\nu$. First, we consider the following homomorphism $\alpha'\colon \widehat{\mathrm{Pol}}_{\mathbf{a}} \to \widehat{PR}_\nu$.

$$\begin{array}{lcl} 1_{\boldsymbol{i}} & \mapsto & 1_{\boldsymbol{i}}, \\ X_r 1_{\boldsymbol{i}} & \mapsto & (Y_r + i_r)1_{\boldsymbol{i}}. \end{array}$$

This homomorphism is obviously $\mathfrak{S}_d$-invariant.

Remark 4.3. For each $1 \leqslant r < d$, the Demazure operator $\partial_r = \frac{1 - s_r}{X_r - X_{r+1}}$ is well-defined on $\widehat{P}_{\mathbf{a}}$. Now, using the isomorphism $\widehat{\mathrm{Pol}}_{\mathbf{a}} \simeq \widehat{\mathrm{PolR}}_\nu$, we can consider it as an operator on $\widehat{PR}_\nu$. The action of ∂_r on $\widehat{PR}_\nu$ can be given explicitly by

$$\partial_r(f1_{\boldsymbol{i}}) = \frac{f1_{\boldsymbol{i}} - s_r(f)1_{s_r(\boldsymbol{i})}}{Y_r - Y_{r+1} + i_r - i_{r+1}}, \qquad f \in \Bbbk[[Y_1, \dots, Y_d]].$$

Attention, the operator ∂_r on $\widehat{PR}_\nu$ should not be confused with $\frac{1 - s_r}{Y_r - Y_{r+1}}$, which is not well-defined.

The Demazure operators ∂_r on $\widehat{PR}_\nu$ satisfy the relation (3), (4), (5).

Now, we want to extend α' to a homomorphism $\alpha\colon \widehat{P}_{\mathbf{a}} \simeq \widehat{PR}_\nu$. To do this, we have to choose the images of $\theta_1, \theta_2, \dots, \theta_d$ in $\widehat{PR}_\nu$ such that this images anti-commute with each other and commute with the image of $\widehat{\mathrm{Pol}}_{\mathbf{a}}$ (i.e., with $\widehat{\mathrm{PolR}}_\nu$). Moreover, we want to make this choice in such a way that α is bijective and $\mathfrak{S}_d$-invariant.

First, we set

$$\alpha(\theta_1 1_{\boldsymbol{i}}) = \left(\prod_{i \in I, i \neq i_1} (Y_1 + i_1 - i)^{\Lambda_i} \right) (-1)^{\Lambda_{i_1}} \Omega_1 1_{\boldsymbol{i}}. \tag{30}$$

This choice is motivated by the fact that we want α to be compatible with the DG-structure. For $r > 1$, we construct the images of other θ_r inductively in the following way

$$\alpha(\theta_r 1_{\boldsymbol{i}}) = -\partial_{r-1}(\alpha(\theta_{r-1}))1_{\boldsymbol{i}} = \frac{s_{r-1}(\alpha(\theta_{r-1})) - \alpha(\theta_{r-1})}{Y_{r-1} - Y_r + i_{r-1} - i_r} 1_{\boldsymbol{i}}. \tag{31}$$

This choice is motivated by the fact that we want α to be $\mathfrak{S}_d$-invariant and we have that $\theta_r = -\partial_{r-1}(\theta_{r-1})$. Equation (31) implies immediately

$$\alpha(s_r(\theta_r)) = s_r(\alpha(\theta_r)). \tag{32}$$

Lemma 4.4. *The homomorphism $\alpha\colon \widehat{P}_{\mathbf{a}} \to \widehat{PR}_\nu$ given by* (30) *and* (31) *is an isomorphism and it is $\mathfrak{S}_d$-invariant.*

Proof. Since the homomorphism $\alpha'\colon \widehat{\mathrm{Pol}}_{\mathbf{a}} \to \widehat{PR}_\nu$ is obviously $\mathfrak{S}_d$-invariant, to show the $\mathfrak{S}_d$-invariance of α, we have to show

$$s_k(\alpha(\theta_r 1_{\boldsymbol{i}})) = \alpha(s_k(\theta_r 1_{\boldsymbol{i}})) \tag{33}$$

for each $\boldsymbol{i} \in I^\nu$, each $r \in [1; d]$ and each $k \in [1; d-1]$. We induct on r. First, we prove (33) for $r = 1$. If $k > 1$ and $r = 1$, then (33) is obvious because θ_1 and $\alpha(\theta_1)$ are s_k-invariant. The case $k = r = 1$ follows from (32).

Now, assume that $r > 1$ and that (33) is already proved for smaller values of r. The case $k = r$ follows from (32).

For $k \neq r$, the element θ_r is s_k-invariant. So (33) is equivalent to the s_k-invariance of $\alpha(\theta_r)$.

Assume that $k > r$ or $k < r - 2$. This assumption implies that s_k commutes with s_{r-1}. Moreover, we already know by induction hypothesis that $\alpha(\theta_{r-1})$ is s_k-invariant. So, the s_k-invariance of $\alpha(\theta_{r-1})$ together with (31) implies the s_k-invariance of $\alpha(\theta_r)$.

Now, assume $k = r - 1$. In this case the s_{r-1}-invariance of $\alpha(\theta_r)$ is obvious from (31).

Finally, assume $k = r - 2$. To prove the s_{r-2}-invariance of $\alpha(\theta_r)$, we have to show that $\partial_{r-2}(\alpha(\theta_r)) = 0$. We have

$$\partial_{r-2}(\alpha(\theta_r)) = \partial_{r-2}\partial_{r-1}\partial_{r-2}(\alpha(\theta_{r-2})) = \partial_{r-1}\partial_{r-2}\partial_{r-1}(\alpha(\theta_{r-2})).$$

This is equal to zero because $\partial_{r-1}(\alpha(\theta_{r-2})) = 0$ by the s_{r-1}-invariance of $\alpha(\theta_{r-2})$.

This completes the proof of the $\mathfrak{S}_d$-invariance of α.

Now, let us prove that α is an isomorphism. It is easy to see from (30) and (31) that $\alpha(\theta_r 1_{\boldsymbol{i}})$ is of the form

$$\alpha(\theta_r 1_{\boldsymbol{i}}) = \sum_{t=1}^{r} P_t \Omega_t 1_{\boldsymbol{i}}, \tag{34}$$

where $P_t \in \widehat{PR}_\nu 1_{\boldsymbol{i}}$ for $r \in \{1, 2, \ldots, r\}$ and P_r is invertible in $\widehat{PR}_\nu 1_{\boldsymbol{i}}$. Then the bijectivity is clear from (34) and from the fact that α restricts to a bijection $\widehat{\mathrm{Pol}}_{\mathbf{a}} \simeq \widehat{\mathrm{PolR}}_\nu$. □

Note that (30) implies that the isomorphism α identifies the subalgebra $\widehat{PA}'_\nu$ of $\widehat{PA}_\nu$ with the subalgebra $\widehat{PB}'_{\mathbf{a}}$ of $\widehat{PB}_{\mathbf{a}}$.

This show that Proposition 4.1 is applicable. We get the following theorem.

Theorem 4.5. *There is an isomorphism of DG-algebras* $(\widehat{\mathcal{R}}(\nu), d_\Lambda) \simeq (\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}, \partial_{\mathbf{Q}})$.

Proof. The isomorphism of algebras follows immediately from Proposition 4.1. We only have to check the DG-invariance.

Denote by γ the isomorphism of algebras $\gamma\colon \widehat{\bar{\mathcal{H}}}_{\mathbf{a}} \to \widehat{\mathcal{R}}(\nu)$. It is obvious that γ preserves the λ-grading. Let us check that for each $h \in \widehat{\bar{\mathcal{H}}}_{\mathbf{a}}$, we have

$$\gamma(\partial_{\mathbf{Q}}(h)) = d_\Lambda(\gamma(h)). \tag{35}$$

Moreover, if (35) is true for some $h = h_1$, $h = h_2$, then it is automatically true for $h = h_1 h_2$. So, it is enough to chech (35) on generators.

The algebra $\widehat{\bar{\mathcal{H}}}$ is generated by elements of λ-degree zero and by θ.

So, it is enough to check (35) for $h = \theta$. This follows directly from (30). (In fact, this is exactly the reason why we define (30) in such a way.) □

Remark 4.6. We could also take $\widehat{PA}_\nu = \widehat{PA}'_\nu = \widehat{\mathrm{PolR}}_\nu$ and $\widehat{PB}_{\mathbf{a}} = \widehat{PB}'_{\mathbf{a}} = \widehat{\mathrm{Pol}}_{\mathbf{a}}$. Then we get (the completion version of) the usual Brundan-Kleshchev-Rouquier isomorphism.

4.3. **The DG-enhanced isomorphism theorem: the q-version.** In Proposition 4.1 we proved that we have an isomorphism of algebras $\widehat{\mathcal{A}}(\nu) \simeq \widehat{\mathcal{B}}_{\mathbf{a}}$ for some algebras $\widehat{\mathcal{A}}(\nu)$ and $\widehat{\mathcal{B}}_{\mathbf{a}}$ that satisfy some list of properties. Let us show that we can apply Proposition 4.2 to the special situation $\widehat{\mathcal{A}}(\nu) = \widehat{\mathcal{R}}(\nu)$ and $\widehat{\mathcal{B}}_{\mathbf{a}} = \widehat{\mathcal{H}}_{\mathbf{a}}$. (We assume that ν and $\mathbf{a}$ are related as in Section 4.1.2.) In this case we can take $\widehat{PA}_\nu = \widehat{PR}_\nu$ and $\widehat{PB}_{\mathbf{a}} = \widehat{P}_{\mathbf{a}}$.

To be able to apply Proposition 4.2, we only have to construct a $\mathfrak{S}_d$-invariant isomorphism $\alpha\colon \widehat{PR}_\nu \simeq \widehat{P}_{\mathbf{a}}$ extending the isomorphism (29) such that α restricts to an isomorphism $\widehat{PA}'_\nu \simeq \widehat{PB}'_{\mathbf{a}}$ (we choose the subalgebras $\widehat{PA}'_\nu \subseteq \widehat{PA}_\nu$ and $\widehat{PB}'_{\mathbf{a}} \subseteq \widehat{PB}_{\mathbf{a}}$ in the same way as in Section 4.2). This can be done in the same way as in the degenerate case. However, some formulas in this case are different from the previous section because of the difference between (28) and (29). Here, we only give the modified formulas. The proofs are the same as in the previous section.

We consider the $\mathfrak{S}_d$-invariant homomorphism $\alpha'\colon \widehat{\mathrm{Pol}}_{\mathbf{a}} \to \widehat{PR}_\nu$.

$$\begin{aligned} 1_{\boldsymbol{i}} &\mapsto 1_{\boldsymbol{i}}, \\ X_r 1_{\boldsymbol{i}} &\mapsto i_r(Y_r+1)1_{\boldsymbol{i}}. \end{aligned}$$

Now, we extend α' to a homomorphism $\alpha\colon \widehat{P}_{\mathbf{a}} \simeq \widehat{PR}_\nu$ in the following way.

$$\alpha(\theta_1 1_{\boldsymbol{i}}) = \left(\prod_{i\in I, i\neq i_1} (i_1(Y_1+1) - i)^{\Lambda_i}\right)(-i_1)^{\Lambda_{i_1}}\Omega_1 1_{\boldsymbol{i}}. \tag{36}$$

$$\alpha(\theta_r 1_{\boldsymbol{i}}) = -\partial_{r-1}(\alpha(\theta_{r-1}))1_{\boldsymbol{i}} = \frac{s_{r-1}(\alpha(\theta_{r-1})) - \alpha(\theta_{r-1})}{i_{r-1}(Y_{r-1}+1) - i_r(Y_r+1)} 1_{\boldsymbol{i}}. \tag{37}$$

As in the previous section, we can show that α is a $\mathfrak{S}_d$-invariant isomorpism.

We get the following theorem.

Theorem 4.7. *There is an isomorphism of DG-algebras* $(\widehat{\mathcal{R}}(\nu), d_\Lambda) \simeq (\widehat{\mathcal{H}}_{\mathbf{a}}, \partial_{\mathbf{Q}})$.

Remark 4.8. We could also take $\widehat{PA}_\nu = \widehat{PA}'_\nu = \widehat{\mathrm{PolR}}_\nu$ and $\widehat{PB}_{\mathbf{a}} = \widehat{PB}'_{\mathbf{a}} = \widehat{\mathrm{Pol}}_{\mathbf{a}}$. Then we get (the completion version of) the usual Brundan-Kleshchev-Rouquier isomorphism.

4.4. **The homology of $\bar{\mathcal{H}}_d$ and $\mathcal{H}_d$.** We now prove Proposition 2.12 and Proposition 2.23.

Proposition 4.9. *The DG-algebras* $(\widehat{\mathcal{R}}(\nu), d_\Lambda)$ *and* $(R^\Lambda(\nu), 0)$ *are quasi-isomorphic.*

Proof. It is proved in [16, Proposition 4.14] that the DG-algebras $(\mathcal{R}(\nu), d_\Lambda)$ and $(R^\Lambda(\nu), 0)$ are quasi-isomorphic. The same proof with minor modifications applies to our case. (We just have to replace polynomials by power series.) □

Corollary 4.10. *There are quasi-isomorphisms* $(\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}, \partial_{\mathbf{Q}}) \simeq (\bar{H}^{\mathbf{Q}}_{\mathbf{a}}, 0)$ *and* $(\widehat{\mathcal{H}}_{\mathbf{a}}, \partial_{\mathbf{Q}}) \simeq (H^{\mathbf{Q}}_{\mathbf{a}}, 0)$.

Proof. Proposition 4.9, Theorem 4.5 and the usual Brundan-Kleshchev-Rouquier isomorphism imply

$$(\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}, \partial_{\mathbf{Q}}) \simeq (\widehat{\mathcal{R}}(\nu), d_\Lambda) \simeq (R^\Lambda(\nu), 0) \simeq (\bar{H}^{\mathbf{Q}}_{\mathbf{a}}, 0).$$

This proofs the first part. The second part is similar. □

Proof of Proposition 2.12 and Proposition 2.23. It is obvious that the homology group of $(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}})$ in degree zero is $\bar{H}^{\mathbf{Q}}_d$. We only have to check that the homology groups in other degrees are zero.

Assume, that for some $i > 0$, we have $H^i(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}}) \neq 0$ and consider it as a Pol_d-module. The annihilator of this Pol_d-module is contained in some maximal ideal $\mathcal{M} \subseteq \mathrm{Pol}_d$. The ideal $\mathcal{M}$ is of the form $\mathcal{M} = (X_1 - a_1, \dots, X_d - a_d)$ for some $\mathbf{a} = (a_1, \dots, a_d) \in \Bbbk^d$.

Then the completion of $H^i(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}}) \neq 0$ with respect to the ideal $\mathcal{M}$ is nonzero. This leads to a contradiction because $H^i(\widehat{\bar{\mathcal{H}}}_{\mathbf{a}}, \partial_{\mathbf{Q}}) = 0$ together with Künneth formula implies

$$\Bbbk[[X_1 - a_1, \dots, X_d - a_d]] \otimes_{\mathrm{Pol}_d} H^i(\bar{\mathcal{H}}_d, \partial_{\mathbf{Q}}) = 0.$$

Proposition 2.23 is proved in the same way. □

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