A Nonlinear Quantum Adiabatic Approximation

Clotilde Fermanian Kammerer, Alain Joye

Nonlinear Quantum Adiabatic Approximation

Clotilde Fermanian-Kammerer * Alain Joye†

Abstract: This paper is devoted to a generalisation of the quantum adiabatic theorem to a nonlinear setting. We consider a Hamiltonian operator which depends on the time variable and on a finite number of parameters, defined on a separable Hilbert space with a fixed basis. The right hand side of the nonlinear evolution equation we study is given by the action of the Hamiltonian on the unknown vector, with its parameters replaced by the moduli of the first coordinates of the vector. We prove existence of solutions to this equation and consider their asymptotics in the adiabatic regime, *i.e.* when the Hamiltonian is slowly varying in time. Under natural spectral hypotheses, we prove the existence of instantaneous nonlinear eigenvectors for the Hamiltonian, and show the existence of solutions which remain close to these time-dependent nonlinear eigenvectors, up to a rapidly oscillating phase, in the adiabatic regime. We first investigate the case of bounded operators and then exhibit a set of spectral assumptions under which the result extends to unbounded Hamiltonians.

Keywords: Adiabatic approximation, nonlinear adiabatic theorem.

Acknowledgments: This work is partially supported by the ANR grant NONSTOPS (ANR-17-CE40-0006-01) and by the Von Neumann visiting professorship of the Technische Universität München. It has been partially written in the Mathematics Department of the Technische Universität München and during a visit of AJ at the Institut Mittag-Leffler. The authors thank these two institutions for their warm hospitality, with a special thought for the stimulating discussions with Simone Warzel and Herbert Spohn.

1 Introduction

We consider a time dependent Hamiltonian on a separable Hilbert space $\mathcal{H}$ that depends on a finite number of real parameters:

$$\mathbb{R}\times\mathbb{R}^p\ni(t,x)\to H(t,x)\in\mathcal{L}(\mathcal{H}), \tag{1.1}$$

with $H(t,x)$ self-adjoint and smooth. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$ and for $f\in\mathcal{H}$, we denote by f_j its coordinate along e_j, i.e. $f_j=\langle e_j,f\rangle$. We are interested in the following nonlinear evolution equation

$$i\varepsilon\partial_t v^\varepsilon(t)=H(t,|v_1^\varepsilon(t)|^2,\dots,|v_p^\varepsilon(t)|^2)v^\varepsilon(t),\quad v^\varepsilon(0)=v_0\in\mathcal{H},\ t\in\mathcal{T}, \tag{1.2}$$

in the adiabatic limit where the small parameter ε tends to zero, and $\mathcal{T}$ is an interval of $\mathbb{R}$ containing 0 and independent of ε. Our aim is to provide an approximation of the solution to (1.2) that bears some

*Université Paris Est, CNRS, LAMA, UMR CNRS 8050, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France,
†Univ. Grenoble Alpes, CNRS, Institut Fourier, F-38000 Grenoble, France

similarities with that provided by the adiabatic theorem of quantum mechanics for linear equations, see *e.g.* [K1].

More precisely, we prove that for the systems we consider, there exist an interval of times $\mathcal{T}$ (containing 0) and a family of smooth nonlinear eigenvectors, *i.e.* two smooth maps $t \mapsto \omega(t) \in \mathcal{H}$ and $t \mapsto \lambda(t) \in \mathbb{R}$, such that $\|\omega(t)\| = 1$ and

$$H(t, |\omega_1(t)|^2, \dots, |\omega_p(t)|^2)\omega(t) = \lambda(t)\omega(t), \quad \forall t \in \mathcal{T}$$

and we provide conditions under which the deviations of $v^\varepsilon(t)$ from $\omega(t)\mathrm{e}^{-\frac{i}{\varepsilon}\int_0^t \lambda(s)ds}$ are small as $\varepsilon \to 0$, in the case where the initial data is taken along $\omega(0)$ ($v^\varepsilon(0) = \omega(0)$). We stress that the evolution equation (1.2) depends on the choice of basis $\{e_j\}_{j\in\mathbb{N}}$.

The adiabatic theorem of quantum mechanics has found numerous extensions since its first formulations [BF, K1] for self-adjoint time dependent Hamiltonians with an isolated eigenvalue. It was extended to accommodate isolated parts of spectrum [N1, ASY] and it was shown to be exponentially accurate for analytic time dependence [JKP, JP, N2, J1]. Then, it was extended to deal with gapless situations where the eigenvalue of interest is not isolated in the rest of the spectrum, [AHS, AE, Te]. Generalisations to non-self-adjoint generators were provided in [A-SF, J3, AFGG], leading to extensions to gapless, non self-adjoint generators provided in [Sc]. Also, formulations of the adiabatic approximation have been shown to hold true for unitary and non unitary discrete time evolutions, [DKS, Ta, HJPR1, HJPR2], and for extended many body systems [BDR]. From this perspective, we prove a generalisation of the adiabatic theorem to nonlinear non-autonomous evolution equations in a Hilbert space defined by (1.2) and (1.1).

Such nonlinear evolution equations occur for example in condensed matter Physics or nonlinear Optics within certain parameter regimes. In particular, the analysis of Landau-Zener tuneling of a Bose-Einstein condensate between Bloch bands in an optical lattice or in double well potentials, as in [BQ] , [J-L et al., Kh, KhRu] or the study of optical waveguides known as nonlinear coherent couplers [Je, A], lead to systems of this form. Indeed, within a certain regime, the relevant Hamiltonians take the explicit form (1.2) for $p = 2$ with an explicit two by two matrix $H(t, x_1, x_2)$, see the book [LLFY] for examples and more references.

Adiabatic issues have been already addressed in the PDE literature in a nonlinear setting with different perspectives. With a scattering point of view, the long time behaviour of nonlinear two by two problems with generators similar to those mentioned above was analysed by [CFK2]. In a PDE setting, [CFK1] studies the adiabatic propagation of coherent states for systems of Schrödinger equations with a non linearity and [S] considers the adiabatic regime of the nonlinear Schrödinger equation for small data. A common feature of these works is that the effective nonlinearity is weak in the sense that it decays with ε. This is not the case in [GG] where a PDE with a nonlinearity of order one as $\varepsilon \to 0$ is studied, for small initial data, but of order one in ε. The authors consider therein the time dependent Gross-Pitaevskii equation in a potential which varies slowly in time. Under suitable conditions on the potential, a unique ground state exists for the stationary linear equation parametrized by the time variable, playing the role of a nonlinear eigenvector in the sense of the previous paragraphs, and the solution to the Gross-Pitaevskii equation is shown to follow the instantaneous ground state, for large times.

Our aim here is to provide a general functional framework for nonlinear adiabatic evolution equations (1.2) and (1.1), characterised by non linearities of order one as $\varepsilon \to 0$ and admitting solutions of norm

one, in contrast to the PDE results mentionned above. We then discuss a set of reasonable spectral hypotheses on $H(t,x)$ allowing us to provide an approximation of the solutions to (1.2) as $\varepsilon \to 0$, for times t of order one. Our main result is first proven for bounded Hamiltonians, and then extended to unbounded $H(t,x)$, under suitable spectral assumptions. In particular, the latter case applies to a certain type of nonlinear Schrödinger equation on $L^2(\mathbb{R})$ that we discuss.

Note that the matrix cases considered in [CFK2] or [LLFY] and in the references therein, appear as special cases of those that we consider, whereas our hypotheses excludes the PDE setup considered in [CFK1, S, GG]. This is due to the fact that the nonlinearity in (1.2) depends on the norm of the projections of the wave function on some subset of the basis vectors of the Hilbert space, and not of the modulus of the wave function itself as in the Gross-Pitaievski equation or in Hartree equation. In this sense, the nonlinearity that we consider is weaker.

1.1 Setup and main result

To ease notations, we will write from now on

$$H(t,|v_1|^2,\dots,|v_p|^2) =: H(t,[v]), \tag{1.3}$$

for any vector $v \in \mathcal{H}$, where H depends on $p < \infty$ components of v only. The form of the nonlinearity we choose, depending on the modulus of (certain components of) the solution, is reminiscent of that of the nonlinear Schrödinger equation. It entails in particular the fact that H actually depends on $\{v_1, \bar{v}_1, v_2, \bar{v}_2, \dots v_p, \bar{v}_p\}$. This motivates the introduction of the anti-unitary complex conjugation C on $\mathcal{H}$ defined by

$$\forall\ v = \sum_j v_j e_j \in \mathcal{H}, \quad Cv = \sum_j \bar{v_j} e_j \tag{1.4}$$

to be used later on. For any $A \in \mathcal{L}(\mathcal{H})$, we define the operator $\bar{A} = CAC \in \mathcal{L}(\mathcal{H})$ and will call operators such that $\bar{A} = A$, real operators. We will work under the following general hypotheses.

- $\mathbf{H}_0$ The map $\mathcal{T} \times \mathcal{X}^p \ni (t,x) \mapsto H(t,x) \in \mathcal{L}(\mathcal{H})$ is C^∞ in the strong sense, where $\mathcal{T}$ and $\mathcal{X}$ are open neighbourhoods of $[0,1]$. For all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$, $H(t,x) = H^*(t,x)$.
- $\mathbf{H}_1$ There exists $\delta > 0$ such that $\|\partial_{x_j} H(x,t)\| \leq \delta$, for all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$ and $j \in \{1,\dots,p\}$.
- $\mathbf{H}_2$ For all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$, the spectrum $\sigma(H(x,t))$ consists in N eigenvalues $\{\lambda_j(t,x)\}_{j=1}^N$ that are separated from one another by a gap $g > 0$, uniform in (t,x).
- $\mathbf{H}_3$ There exists $1 \leq j_0 \leq N$ such that $\lambda_{j_0}(x,t)$ is simple.

Consequently, the corresponding spectral decomposition of $H(t,x)$ reads

$$H(t,x) = \sum_{j=1}^N \lambda_j(t,x) P_j(t,x), \tag{1.5}$$

where the orthogonal spectral projectors $P_j(t,x)$ satisfy $1 \leq \dim(P_j(t,x)) \leq \infty$ is constant, while $\dim(P_{j_0}(t,x)) \equiv 1$. We shall make use of the following facts: the projectors $P_j(t,x)$ are as smooth as

$H(t,x)$ and so are the eigenvalues $\lambda_j(t,x)$. Moreover, for $j = j_0$, there exists a global smooth map $\mathcal{T} \times \mathcal{X}^p \ni (t,x) \mapsto \varphi_{j_0}(t,x) \in \mathcal{H}$ such that

$$\forall (t,x) \in \mathcal{T} \times \mathcal{X}^d, \;\; H(t,x)\varphi_{j_0}(t,x) = \lambda_{j_0}(t,x)\varphi_{j_0}(t,x).$$

These facts are briefly discussed in Section 2 below.

The form of the nonlinearity immediately implies a gauge invariance, which will turn out to be crucial later on. Under $\mathbf{H}_0$, we have for any $\theta \in \mathbb{R}$, any $v \in \mathcal{H}$,

$$H(t,[\mathrm{e}^{i\theta}v]) = H(t,[v]). \tag{1.6}$$

If $\mathbf{H}_2$ and $\mathbf{H}_3$ hold as well, this implies

$$P_j(t,[\mathrm{e}^{i\theta}v]) = P_j(t,[v]), \quad \lambda_j(t,[\mathrm{e}^{i\theta}v]) = \lambda_j(t,[v]), \quad \varphi_{j_0}(t,[\mathrm{e}^{i\theta}v]) = \varphi_{j_0}(t,[v]). \tag{1.7}$$

We first note that $H(t,x)$ self-adjoint ensures the existence of global solutions to (1.2) via Cauchy-Lipschitz Theorem. Moreover, gauge invariance (1.6) implies symmetries that we exploit below. These elementary properties are stated in the next Lemma with the convention (1.3).

Lemma 1.1 *Under assumption $\mathbf{H}_0$, the equation*

$$i\varepsilon\partial_t v^\varepsilon(t) = H(t,[v^\varepsilon(t)])v^\varepsilon(t), \quad v^\varepsilon(0) = v_0 \in \mathcal{H}, \; t \in \mathcal{T}, \tag{1.8}$$

admits a unique global solution for any initial condition $v_0 \in \mathcal{H}$. Moreover, $\|v^\varepsilon(t)\| \equiv \|v_0\|$.
Besides, given a C^0 map $\mathcal{T} \times \mathcal{X}^p \ni (t,x) \to \chi(t,x) \in \mathbb{R}$, and $v^\varepsilon(t)$ a solution to (1.8), the solution to

$$i\varepsilon\partial_t s^\varepsilon(t) = (H(t,[s^\varepsilon(t)]) + \chi(t,[s^\varepsilon(t)]\mathrm{Id})s^\varepsilon(t), \quad s^\varepsilon(0) = v_0 \in \mathcal{H}, \; t \in \mathcal{T}$$

reads

$$s^\varepsilon(t) = \mathrm{e}^{-i\int_0^t \chi(u,[v^\varepsilon(u)])du/\varepsilon} v^\varepsilon(t), \quad \forall \; t \in \mathcal{T}.$$

Our analysis focuses on solutions to (1.2) that are tightly related to the simple eigenvalue $\lambda_{j_0}(t,x)$ and associated eigenvector $\varphi_{j_0}(t,x)$. Therefore, to simplify the notation, we drop the index j_0 for these spectral data from now on. We start by introducing a vector $\omega(t) \in \mathcal{H}$ that we call a nonlinear eigenvector, and which is defined in a neighbourhood of $0 \in \mathcal{T}_0$ by

$$H(t,[\omega(t)])\omega(t) = \lambda(t,[\omega(t)])\omega(t), \quad \forall \; t \in \mathcal{T}_0.$$

As discussed in Section 2, this algebraic nonlinear equation turns out to always have a local nontrivial solution when $\lambda(t,x)$ is a simple eigenvalue of $H(t,x)$.

Proposition 1.2 *Assume $\mathbf{H}_0$, $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_3$. Then, for any $t_0 \in \mathcal{T}$, there exists a neighbourhood $\mathcal{T}_0 \subset \mathcal{T}$ of t_0 such that for all $t \in \mathcal{T}_0$, a solution $\omega(t) \in \mathcal{H}$ of norm one to the algebraic equation*

$$P(t,[\omega(t)])\omega(t) = \omega(t) \tag{1.9}$$

exists. Moreover $\mathcal{T}_0 \ni t \mapsto \omega(t)$ is smooth and can be chosen to satisfy $\langle \omega(t) | \dot{\omega}(t) \rangle \equiv 0$.

We can now give our main statements which establish nonlinear adiabatic theorems in the considered framework.

Theorem 1.3 *Assume* $\mathbf{H}_0$, $\mathbf{H}_1$ *with* δ *small enough, and suppose* $\mathbf{H}_2$ *holds with all eigenvalues being simple. Moreover, assume that* $H(t,x)$ *is real, that is* $\overline{H}(t,x) = H(t,x)$*, and generic in the sense that* $\sigma(H(t,x) - \lambda(t,x)) \cap \sigma(-H(t,x) + \lambda(t,x)) = \{0\}$*. Let* $\omega(t)$ *be a smooth solution to (1.9) in a neighbourhood* $\mathcal{T}_0$ *of* $t_0 = 0$*. Then the solution* $v^\varepsilon(t)$ *to (1.2) with* $v^\varepsilon(0) = \omega(0)$ *satisfies for all* $t \in \mathcal{T}_0$

$$v^\varepsilon(t) = \mathrm{e}^{-\frac{i}{\varepsilon}\int_0^t \lambda(s,[\omega(s)])ds}\omega(t) + O_t(\varepsilon).$$

Remark 1.4 *i) The simplicity of the spectrum of* $H(t,x)$ *implies that we are actually dealing with the matrix case.*
ii) The genericity condition always holds if $\lambda(t,x)$ *is the ground state or the largest eigenvalue of* $H(t,x)$*.*

To consider genuinely infinite dimensional situations, we need another spectral assumption on a non-selfadjoint operator appearing naturally in this context, as stated in the result below. Actually, the previous theorem is a special case of the following theorem.

Theorem 1.5 *Assume* $\mathbf{H}_0$, $\mathbf{H}_1$ *with* δ *small enough,* $\mathbf{H}_2$ *and* $\mathbf{H}_3$*. Moreover, suppose that* $H(t,x)$ *is real, that is* $\overline{H}(t,x) = H(t,x)$*. Let* $\omega(t)$ *be a smooth solution to (1.9) in a neighbourhood* $\mathcal{T}_0$ *of* $t_0 = 0$*. Provided the operator* $F(t)$ *defined by (3.5) below is semisimple with real eigenvalues of constant multiplicity for all* $t \in \mathcal{T}_0$*, the solution* $v^\varepsilon(t)$ *to (1.2) with* $v^\varepsilon(0) = \omega(0)$ *satisfies for all* $t \in \mathcal{T}_0$

$$v^\varepsilon(t) = \mathrm{e}^{-i\int_0^t \lambda(s,[\omega(s)])ds/\varepsilon}\omega(t) + O_t(\varepsilon).$$

Remark 1.6 *As we will see in the proof, there exist positive constants* c_0, c_1 *such that the norm of the remainder satisfies*

$$\|O_t(\varepsilon)\| \leq \min(c_0 t, c_1\varepsilon), \ \forall t \in \mathcal{T}_0. \tag{1.10}$$

As already mentioned, the assumptions of Theorem 1.3 guarantee the adequate spectral behavior of the operator $F(t)$ defined by (3.5) to get the conclusion of Theorem 1.5. In other words, assuming in $\mathbf{H}_2$ that all eigenvalues of the real operator $H(t,x)$ are of multiplicity one is enough to obtain the assumption on the spectral decomposition of $F(t)$. In Section 3 we describe another set of assumptions which are sufficient to satisfy the hypothesis of Theorem 1.5 in the case $p = 1$ (see Lemma 3.1 and the remarks that follow it) and which does not reduce to finite dimension as it is the case for Theorem 1.3 (see i) of Remark 1.4).

1.2 Extension of the result to unbounded operators

We now extend our results to the case where the operator $H(t,x)$ on the separable Hilbert space $\mathcal{H}$ is unbounded and takes the form $H(t,x) = H_0 + W(t,x)$, with $W(t,x) \in \mathcal{L}(\mathcal{H})$. We make the following regularity hypothesis:

- $\mathbf{R}_0$ The self-adjoint operator H_0 is defined on a dense domain $\mathcal{D} \subset \mathcal{H}$, and the family of bounded operator $W(t,x)$ is self-adjoint for all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$. Moreover, H_0, and $W(t,x)$ are real operators.
- $\mathbf{R}_1$ The map $\mathcal{T} \times \mathcal{X}^p \ni (t,x) \mapsto W(t,x) \in \mathcal{L}(\mathcal{H})$ is strongly C^∞.
- $\mathbf{R}_2$ There exist $\delta > 0$ such that $\|W(t,x)\| \leq \delta$, $\|\partial_{x_j} W(t,x)\| \leq \delta$, for all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$ and $j \in \{1,\ldots,p\}$.

We also assume the spectral hypothesis

S$_1$ The spectrum of H_0 consists in an infinite increasing sequence of simple eigenvalues $\lambda_j \geq 0$, $j \in \mathbb{N}$, and there exists $c_0 > 0$ and $\alpha > 1/2$ such that the gaps satisfy

$$\forall j \in \mathbb{N}, \;\; \lambda_{j+1} - \lambda_j \geq c_0\, j^\alpha.$$

Example 1.7 *Consider $\mathcal{H} = L^2(\mathbb{R}_y)$ and the operator*

$$H_0 = -\frac{1}{2}\Delta_y + V_0(y)$$

with domain $\mathcal{D} \subset L^2(\mathbb{R}_y)$. Assume that V_0 grows as $\langle y\rangle^\beta$ for $\beta > 6$ (see [RS]), then H_0 satisfies the assumptions **R**$_0$ *and* **S**$_1$ *above. Consider x-dependent self-adjoint perturbations of this operator ($x \in \mathcal{X}^p$)*

$$H(t,x) = -\frac{1}{2}\Delta_y + V_0(y) + W(t,y,x)$$

where W is such that the map $(t,y,x) \mapsto W(t,y,x)$ is a bounded function from $\mathcal{C}^\infty(\mathcal{T} \times \mathbb{R}_y \times \mathcal{X}^p, \mathbb{R})$ and there exists $\delta > 0$ such that

$$\forall (t,y,x) \in \mathcal{T} \times \mathbb{R}_y \times \mathcal{X}^p, \;\; \forall j \in \{1,\dots,p\}, \;\; |W(t,x,y)| + |\partial_{x_j} W(t,y,x)| \leq \delta.$$

Then, $H(t,x)$ satisfies assumptions **R**$_1$ *and* **R**$_2$ *above.*

The operator $W(t,x)$ being bounded, if δ is small enough, perturbation theory implies that for all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$, the self-adjoint operator $H(t,x) = H_0 + W(t,x)$ defined on $\mathcal{D}$ has spectrum $\sigma(H(t,x)) = \{\lambda_j(t,x)\}_{j\in\mathbb{N}}$ consisting in simple eigenvalues $\lambda_j(t,x)$ only, and there exists $c_1 > 0$ such that the gaps satisfy for $\alpha > 1/2$

$$\forall (t,x) \in \mathcal{T} \times \mathcal{X}^p, \;\; \forall j \in \mathbb{N}, \;\; \lambda_{j+1}(t,x) - \lambda_j(t,x) \geq c_1\, j^\alpha.$$

We pick some $j_0 \in \mathbb{N}$ and assume the generic property:

S$_2$ For all $(t,x) \in \mathcal{T} \times \mathcal{X}^p$, and $j \in \mathbb{N}$, $\{\lambda_j(t,x) - \lambda_{j_0}(t,x)\} \cap \{-\lambda_j(t,x) + \lambda_{j_0}(t,x)\} = \{0\}$.

Note that, since H_0 is bounded from below, this assumption concerns only a finite number of eigenvalues. Besides, this property can be inherited from a similar assumption on the eigenvalue λ_{j_0} of H_0.

We consider for all $(t_0,x_0) \in \mathcal{T} \times \mathcal{X}^p$, the smooth map $(t,x) \mapsto \varphi(t,x)$ from $\mathcal{T} \times \mathcal{X}^p$ to $\mathcal{D} \subset \mathcal{H}$ such that

$$H(t,x)\varphi(t,x) = \lambda_{j_0}(t,x)\varphi(t,x).$$

We drop the index j_0 as before. Provided with these properties, we can develop the same analysis as in the situation addressed above, namely, the existence of a nonlinear eigenvector and an adiabatic approximation for the nonlinear evolution equation associated with $H(t,x)$: We consider p orthonormal vectors $\{e_1,\dots,e_p\}$ in $\mathcal{D}$ and set

$$\forall \psi \in \mathcal{H}, \;\; [\psi] = (|\langle e_1|\psi\rangle|^2, \cdots, |\langle e_p|\psi\rangle|^2).$$

Proposition 1.2 ensures that for any $t_0 \in \mathcal{T}$, there exists a neighbourhood $\mathcal{T}_0 \subset \mathcal{T}$ of t_0 such that for all $t \in \mathcal{T}_0$, a solution $\omega(t) \in \mathcal{D}$ of norm one to the algebraic equation (1.9) exists, see Remark 2.1. Moreover

$\mathcal{T}_0 \ni t \mapsto \omega(t)$ is smooth and can be chosen to satisfy $\langle \omega(t)|\dot{\omega}(t)\rangle \equiv 0$. Taking initial data $\omega(0)$ in (1.2) gives the equation in which we are interested, namely

$$i\varepsilon\partial_t\psi^\varepsilon(t) = (H_0 + W(t,[\psi^\varepsilon(t)])\psi^\varepsilon(t), \quad \psi^\varepsilon(0) = \omega(0), \tag{1.11}$$

in the weak sense on $\mathcal{D}$. By solution in the weak sense on $\mathcal{D}$ we mean the following, see [RS], vol. II, p. 284 for the linear case: For any $\chi \in \mathcal{D}$,

$$\mathrm{i}\varepsilon\partial_t\langle\chi|\psi^\varepsilon(t)\rangle = \langle(H_0 + W(t,[\psi^\varepsilon(t)]))\chi|\psi^\varepsilon(t)\rangle, \quad \psi^\varepsilon(0) = \omega(0). \tag{1.12}$$

Theorem 1.8 *Assume $\mathbf{R}_0$ and $\mathbf{R}_1$, then equation (1.11) admits a unique global solution in the weak sense of norm one. Assume moreover $\mathbf{R}_2$ with δ small enough, $\mathbf{S}_1$ and $\mathbf{S}_2$ and let $\omega(t)$ be a smooth solution to (1.9) in a neighbourhood $\mathcal{T}_0$ of $t_0 = 0$. then the solution $v^\varepsilon(t)$ to (1.2) with $v^\varepsilon(0) = \omega(0)$ satisfies for all $t \in \mathcal{T}_0$*

$$\psi^\varepsilon(t) = \mathrm{e}^{-\frac{i}{\varepsilon}\int_0^t \lambda(t,[\omega(t)])dt}\omega(t) + O_t(\varepsilon).$$

1.3 Energy content of the solutions

We close this introduction by discussing briefly an important feature of the solutions provided by Theorems 1.3 and 1.5. A physically relevant quantity for the nonlinear equation (1.2) we consider is the instantaneous energy content of a solution $v^\varepsilon(t)$, defined for all $t \in \mathcal{T}_0$ by

$$E_{v^\varepsilon}(t) = \langle v^\varepsilon(t)|H(t,[v^\varepsilon(t)])v^\varepsilon(t)\rangle.$$

For bounded operators $\mathcal{T} \times \mathcal{X}^p \ni (t,x) \mapsto H(t,x) \in \mathcal{L}(\mathcal{H})$, and $\varepsilon-$independent initial conditions $v^\varepsilon(0) = v(0)$ the energy content satisfies the uniform bound

$$|E_{v^\varepsilon}(t)| \leq \sup_{(t,x)\mathcal{T}\times\mathcal{X}^p} \|H(t,x)\|\|v(0)\|^2.$$

For a solution of the form $v^\varepsilon(t) = \mathrm{e}^{-i\int_0^t \lambda(s,[\omega(s)])ds/\varepsilon}\omega(t) + O_t(\varepsilon)$, the energy content simply coincides, to leading order, with the corresponding instantaneous nonlinear eigenvalue

$$E_{v^\varepsilon}(t) = \langle\omega(t)|H(t,[\omega(t)])\omega(t)\rangle + O_t(\varepsilon) = \lambda(t,[\omega(t)]) + O_t(\varepsilon).$$

In general, the behaviour in time of the energy content of a solution does not necessarily admit such a regular behaviour in the limit $\varepsilon \to 0$.

Let us illustrate this point on the following simple example. Let $\mathbb{R} \ni t \mapsto \gamma(t) \geq \gamma_0 > 0$ and consider

$$H(t,x) = \begin{pmatrix} 0 & \gamma(t)x \\ \gamma(t)x & 0 \end{pmatrix}$$

on $\mathcal{H} = \mathbb{C}^2$. The evolution equation (1.2) reads

$$i\varepsilon\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = H(t,|v_1|^2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \gamma(t)|v_1|^2\begin{pmatrix} v_2 \\ v_1 \end{pmatrix}, \tag{1.13}$$

with initial conditions $\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}$, and the energy content of the solutions reads

$$E_v(t) = \gamma(t)|v_1|^2 2\Re(v_1\overline{v_2})(t).$$

The corresponding real normalised nonlinear eigenvectors $\omega_\pm(t)$ are time-independent,

$$\omega_\pm(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

and associated to the eigenvalues $\lambda_\pm(t, [\omega_\pm]) = \pm\gamma(t)/2$. Hence, the approximate solutions provided by Theorem 1.3 read

$$v_\pm(t) = e^{\mp \frac{i}{2\varepsilon}\int_0^t \gamma(u)du}\omega_\pm(t), \tag{1.14}$$

which turn out to be exact solutions for all $t \in \mathbb{R}$, since $\omega_\pm$ are time-independent. Their energy contents are thus given by $E_{v_\pm}(t) = \pm\gamma(t)/2$, which is ε-independent. However, for general solutions $v^\varepsilon(t)$ the situation is different, as stated in the next Lemma which is proved in Appendix B.

Lemma 1.9 *Let $v^\varepsilon(t)$ be a solution of equation (1.13) with real-valued initial data such that $v_1(0) > 0$, $v_2(0) \neq 0$. Then the energy content reads*

$$E_{v^\varepsilon}(t) = 2\,\gamma(t)v_1(0)^3v_2(0)\left[\cos^2\left(\frac{\aleph(t)}{\varepsilon}\right) + \left(\frac{v_1(0)}{v_2(0)}\right)^2 \sin^2\left(\frac{\aleph(t)}{\varepsilon}\right)\right]^{-1}$$

with $\aleph(t) = -v_1(0)v_2(0)\displaystyle\int_0^t \gamma(u)du$. Hence, $E_{v^\varepsilon}(t)/\gamma(t)$ is actually a function of $\int_0^t \gamma(u)du$, which oscillates between the extremal values $2v_1(0)^3v_2(0)$ and $2v_1(0)v_2(0)^3$ with a period of order ε, unless $v_1(0)/v_2(0) = \pm 1$ in which case it is a constant.

Remark 1.10 *By contrast, the linear quantum adiabatic theorem implies that the energy content of any solution is given by an ε-independent weighted sum of instantaneous eigenvalues of the Hamiltonian, to leading order.*

1.4 Organisation of the paper

We prove Proposition 1.2 and discuss the limitation that may occur to its validity in Section 2. Then Section 3 is devoted to the proof of the main results, Theorems 1.3 and 1.5. The proof that is provided adapts to the unbounded setting of Theorem 1.8, which we explain in Section 4. Finally, two Appendices are devoted to the discussion of examples.

2 The existence of nonlinear eigenvectors

We focus in this section on the existence of the generalized nonlinear eigenvector $\omega(t)$ defined in Proposition 1.2. We first recall well-known facts in the linear setting, mainly to introduce notations. Then, we explain why a similar result remains true locally in the nonlinear regime we consider and why the obtained eigenvectors may not exist globally.

2.1 Existence of smooth eigenvectors in the linear adiabatic setting

The question of local (and global) existence of smooth eigenvectors is simple in the linear context. Indeed, with the notations of Assumption $\mathbf{H}_2$ and using Riesz formula on $C_j(g/2)$, a circle of radius $g/2$ and center $\lambda_j(t,x)$,

$$P_j(t,x) = -\frac{1}{2\pi i}\int_{C_j(g/2)} (H(t,x)-z)^{-1}dz,$$

one gets that the projectors $P_j(t,x)$'s are as smooth as $H(t,x)$. Moreover, $\|\partial_{x_j} P_j(t,x)\| \leq 2\delta/g$. The finitely degenerate eigenvalues $\lambda_j(t,x) = \mathrm{Tr}(P_j(t,x)H(t,x))$ are thus as smooth as $H(t,x)$, and the same is true if $\dim P_j(t,x) = \infty$.

Considering $j = j_0$, for any $(t_0,x_0) \in \mathcal{T}\times\mathcal{X}^p$, there exists an open neighbourhood of (t_0,x_0) in which a smooth normalised eigenvector $\varphi_{j_0}(t,x) \in \mathcal{H}$ exists such that

$$P_{j_0}(t,x) = |\varphi_{j_0}(t,x)\rangle\langle\varphi_{j_0}(t,x)|.$$

More specifically, given $\varphi_{j_0}(t_0,x_0)$ an eigenvector of $H(t_0,x_0)$, the vector

$$\varphi_{j_0}(t,x) := P_{j_0}(t,x)\varphi_{j_0}(t_0,x_0)/\langle\varphi_{j_0}(t_0,x_0)|P_{j_0}(t,x)\varphi_{j_0}(t_0,x_0)\rangle \tag{2.1}$$

satisfies these conditions for all (x,t) such that $P_{j_0}(t,x)\varphi_{j_0}(t_0,x_0) \neq 0$.

Actually, there exists a global smooth map $\mathcal{T}\times\mathcal{X}^p \ni (t,x) \mapsto \varphi(t,x)$, which can be viewed as follows. Using the shorthand $p = (t,x)$, set $E = \cup_{p\in\mathcal{T}\times\mathcal{X}^p}(p,\varphi(p))$, and $\pi : E \ni (p,\varphi(p)) \mapsto p \in \mathcal{T}\times\mathcal{X}^p$, so that $\pi : E \to \mathcal{T}\times\mathcal{X}^p$ defines a rank one vector bundle over the base $\mathcal{T}\times\mathcal{X}^p$. The base being contractible, it is known that the vector bundle is trivial, which is equivalent to the existence of a global smooth frame on the fibers of E, see *e.g.* [LeP, Sp]. An alternative approach is by explicit construction, making use of the parallel transport operator defined by (3.10) below. Passing to spherical coordinates $(t,x) \mapsto (r,\theta) \in \mathbb{R}^+\times\mathbb{S}^p$ and integrating the parallel transport operator along r, keeping θ as parameters, we get a smooth unit eigenvector for each $(t,x) \in \mathcal{T}\times\mathcal{X}^p$, by the smoothness of the eigenprojector. This property holds for $\dim\mathcal{H} = \infty$.

2.2 Existence of nonlinear eigenvector

We prove here Proposition 1.2.

Proof: For $t_0 \in \mathcal{T}$ fixed, dropped from the notation, $\mathbf{H}_3$ yields,

$$|\varphi([\omega])\rangle\langle\varphi([\omega])|\omega\rangle = \omega.$$

This requires ω to be parallel to $\varphi([\omega])$ where the latter is normalised. We use Schauder's fixed point Theorem in a Banach space to actually prove that, locally, there exists ω such that $\omega = \varphi([\omega])$, and thus $\|\omega\| = 1$. Set $B_1(\mathcal{H}) = \{v \in \mathcal{H} \mid \|v\| \leq 1\}$ and $F : B_1(\mathcal{H}) \mapsto B_1(\mathcal{H})$ by $F(v) = \varphi([v])$. This map is well defined, continuous and $B_1(\mathcal{H})$ is closed, convex and nonempty. Thus F will have a fixed point if $\overline{F(B_1(\mathcal{H}))}$ is compact. Let $K_\varphi = \{\varphi([x])|x \in [0,1]^p\}$. By continuity of φ in the variable x, and compactness of $[0,1]^p$, K_φ is compact. Thus the closed subset $\overline{F(B_1(\mathcal{H}))}$ of K_φ is compact and Schauder Theorem (see [E] for example) implies the existence of a fixed point for F, for each given value of t_0. Since $\|\varphi([v])\| \equiv 1$, the normalization of the fixed point $\omega(t_0)$ holds.

In order to prove the smoothness of the map $\mathcal{T}_0 \ni t \mapsto \omega(t)$, we use the implicit function theorem on the smooth map $J : \mathcal{T} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ defined by

$$J(t, v, \bar{v}) = \begin{pmatrix} J_1(t, v, \bar{v}) \\ J_2(t, v, \bar{v}) \end{pmatrix} = \begin{pmatrix} v - \varphi(t, [v]) \\ \bar{v} - \bar{\varphi}(t, [v]) \end{pmatrix}.$$

The zeros of J define $\omega(t)$, in a neighbourhood of $(t_0, \omega(t_0))$. Note that by a smooth change of phase we can consider locally the continuous vector $\varphi(t, x)$ defined by (2.1). For $1 \leq j \leq p$, we compute, with $\{e_j\}_{j\in\mathbb{N}}$ the chosen orthonormal basis of $\mathcal{H}$,

$$\begin{aligned} &\partial_{v_j} J_1(t, v, \bar{v}) = e_j - \partial_{x_j}\varphi(t, [v])\bar{v}_j, \quad \partial_{\bar{v}_j} J_1(t, v, \bar{v}) = -\partial_{x_j}\varphi(t, [v])v_j \\ &\partial_{v_j} J_2(t, v, \bar{v}) = -\partial_{x_j}\bar{\varphi}(t, [v])\bar{v}_j, \qquad \partial_{\bar{v}_j} J_2(t, v, \bar{v}) = e_j - \partial_{x_j}\bar{\varphi}(t, [v])v_j. \end{aligned} \tag{2.2}$$

Therefore, using the notation $D_{v,\bar{v}}J(t, v, \bar{v}) \in \mathcal{L}(\mathcal{H} \times \mathcal{H})$ for the derivative with respect to the variables $(v, \bar{v}) \in \mathcal{H} \times \mathcal{H}$, we get

$$D_{v,\bar{v}}J(t, v, \bar{v}) \begin{pmatrix} h \\ \bar{h} \end{pmatrix} = \begin{pmatrix} h \\ \bar{h} \end{pmatrix} - \sum_{j=1}^{p} \begin{pmatrix} \partial_{x_j}\varphi(t, [v])\langle v_j e_j | h\rangle + \partial_{x_j}\varphi(t, [v])\langle \bar{v}_j e_j | \bar{h}\rangle \\ \partial_{x_j}\bar{\varphi}(t, [v])\langle v_j e_j | h\rangle + \partial_{x_j}\bar{\varphi}(t, [v])\langle \bar{v}_j e_j | \bar{h}\rangle \end{pmatrix}. \tag{2.3}$$

We recall the notation in the scalar case

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$$

such that if $f(x, y) \equiv g(z, \bar{z})$ with $z = x + iy \in \mathbb{C}$ and $t = h + ik \in \mathbb{C}$, then

$$Df(x, y)(h, k) = \partial_x f(x, y)h + \partial_y f(x, y)k \equiv D_{z,\bar{z}}g(z, \bar{z})(h + ik, h - ik) = \partial_z g(z, \bar{z})t + \partial_{\bar{z}} g(z, \bar{z})\bar{t}.$$

With these notations in mind, we obtain equivalently

$$D_{v,\bar{v}}J(t, v, \bar{v}) = \mathrm{Id} - \sum_{j=1}^{p} \left| \begin{pmatrix} \partial_{x_j}\varphi(t, [v]) \\ \partial_{x_j}\bar{\varphi}(t, [v]) \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} v_j e_j \\ \bar{v}_j e_j \end{pmatrix} \right|.$$

Therefore, for $v \in B_1(\mathcal{H})$, it is enough to show that $\|\partial_{x_j}\varphi(t, x)\| < 1/4$, say, to satisfy the assumptions of the implicit function theorem. We compute

$$\partial_{x_j}\varphi(t, x) = \frac{\partial_{x_j}P(t, x)\varphi(t_0, x_0) - P(t, x)\varphi(t_0, x_0)\langle\varphi(t_0, x_0)|\partial_{x_j}P(t, x)\varphi(t_0, x_0)\rangle}{\langle\varphi(t_0, x_0)|P(t, x)\varphi(t_0, x_0)\rangle^2}, \tag{2.4}$$

the norm of which is bounded above by $8\delta/g$, in a neighbourhood of (t_0, x_0) characterised by

$$\|P(t, x)\varphi(t_0, x_0)\| \geq 2^{-1/4}.$$

Hence, $\mathbf{H}_1$ with δ small enough yields the existence of an open neighbourhood $\mathcal{T}_0 \ni t_0$ and of a smooth map $t \mapsto \tilde{\omega}(t)$ with $\tilde{\omega}(t_0) = \omega(t_0)$ that is solution to (1.9) for all $t \in \mathcal{T}_0$.

To conclude, the proof, we observe that the argument above ensures $\|\tilde{\omega}(t)\| \equiv 1$, so that the phase adjustment $\omega(t) = \tilde{\omega}(t)e^{-\int_{t_0}^{t}\langle\tilde{\omega}(s)|\dot{\tilde{\omega}}(s)\rangle ds}$ implies that $\omega(t)$ satisfies $\langle\omega(t)|\dot{\omega}(t)\rangle \equiv 0$. □

Remark 2.1 *Note that in the proof above, we have not used the assumption $H(t, x) \in \mathcal{L}(\mathcal{H})$ so that the result of Proposition 1.2 extends to unbounded families of operators $H(t, x)$.*

2.3 Failure of global nonlinear eigenvectors

We illustrate with an example the fact that the eigenvector constructed in Proposition 1.2 may only exists locally. For this, we consider the matrix-valued case where $H(t,x)$ is the real, symmetric, traceless two by two matrix

$$H(t,x) = \begin{pmatrix} \cos(t\theta(x)) & \sin(t\theta(x)) \\ \sin(t\theta(x)) & -\cos(t\theta(x)) \end{pmatrix},$$

where $\mathbb{R} \ni x \mapsto \theta(x)$ is a smooth map that we choose later. The eigenvalues of $H(t,x)$ are $+1$ and -1 with associated eigenvectors

$$V_+(t,x) = \begin{pmatrix} \cos\left(\frac{t\theta(x)}{2}\right) \\ \sin\left(\frac{t\theta(x)}{2}\right) \end{pmatrix}, \quad V_-(t,x) = \begin{pmatrix} -\sin\left(\frac{t\theta(x)}{2}\right) \\ \cos\left(t\frac{\theta(x)}{2}\right) \end{pmatrix},$$

respectively. We denote by $P(t,x)$ the eigenprojectors for the eigenvalue $+1$. Then a real normalised vector $\omega(t) = \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix}$ satisfies

$$P(t,[\omega(t)])\omega(t) = \omega(t)$$

if and only if

$$\omega_1(t) = \cos\left(\frac{t}{2}\,\theta(|\omega_1(t)|^2)\right), \quad \omega_2(t) = \sin\left(\frac{t}{2}\,\theta(|\omega_1(t)|^2)\right),$$

up to a global sign. It is then enough to find the function $t \mapsto \omega_1(t)$. For fixed t, it reduces to finding $Y \in [0,1]$ such that

$$Y = \cos\left(\frac{t}{2}\,\theta(Y^2)\right). \tag{2.5}$$

Let us restrict to $t \in [0,1]$ and choose the function θ according to the following picture

y	0	$y_{\max}$	$y_{\min}$	1
$\theta(y^2)$	0	$\theta_{\max}$	θ_{min}	π

We fix $\theta_{\max} < \pi$ so that $\cos\frac{t}{2}\theta_{\max} > 0$ for all $t \in [0,1]$ and cos is decreasing on the set of values of $\frac{t}{2}\theta$. For $t = 0$, the uniqueness of the solution of the equation (2.5) is guaranteed and for $t \in (0,1]$, it depends on whether $\cos\frac{t}{2}\theta_{\max} < y_{\max}$ or not. Therefore, if we choose $y_{\max}$ and θ_{max} such that

$$\cos\frac{1}{2}\theta_{\max} < y_{\max},$$

we know that there exists $\tau \in]0,1[$ such that the equation (2.5) has a unique solution for $t \in [0,\tau)$ and exactly three solutions for times $t \in (\tau,1]$. Figure 1 illustrates that fact.

3 Proof of the main results

In this section, we prove Theorems 1.3 and 1.5 the proofs of which both follow the same scheme. We first give the plan, spelling out the main steps and lemmas that we then successively prove in the next sections.

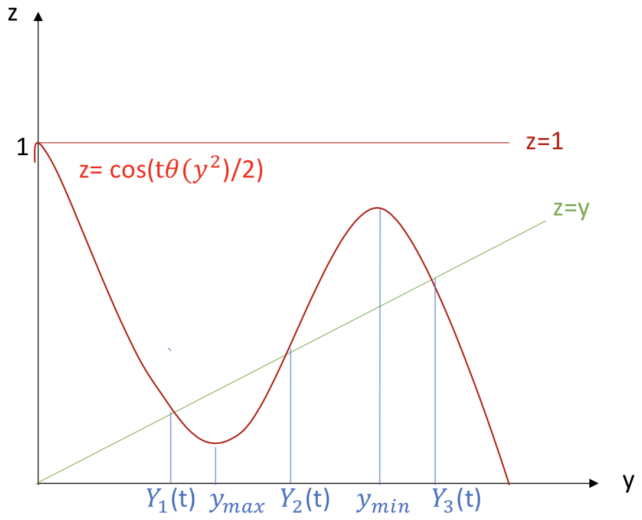

Figure 1: Non uniqueness and degeneracy of the solutions of Equation (2.5)

3.1 Proof of Theorems 1.3 and 1.5

Thanks to Lemma 1.1 with $\chi(t,x) = -\lambda(t,x)$, we can reduce the analysis to the case $\lambda(t,x) = 0$ without loss of generality, by considering the shift

$$H(t,x) \mapsto H(t,x) - \lambda(t,x). \tag{3.1}$$

The eigenvalues of the operator $H(t,x)$ are then all shifted by $\lambda(t,x)$ and we denote them by

$$0 \text{ and } \lambda_j(t,x),\ j \in \mathbb{N}^*,$$

where the functions $\lambda_j(t,x)$ may have changed compared to what they were in the introduction. We set $\Delta(t) = v^\varepsilon(t) - \omega(t)$. Then, the map $t \mapsto \Delta(t)$ (which also depends on ε) satisfies the system

$$i\varepsilon\dot{\Delta}(t) = H(t,[v^\varepsilon(t)])v^\varepsilon(t) - i\varepsilon\dot{\omega}(t),\quad \Delta(0) = 0,$$

using a dot to express derivatives with respect to time. For all $t \in \mathcal{T}_0$, the interval $\mathcal{T}_0$ is the set of times around $t_0 = 0$ where $\omega(t)$ given in Proposition 1.2 exists, we have

$$H(t,[v^\varepsilon(t)]) = H(t,[\omega(t)+\Delta(t)]) = H(t,[\omega(t)]) + 2\sum_{j=1}^{p} \partial_{x_j} H(t,[\omega(t)])\Re(\overline{\omega}_j\Delta_j) + O(\|\Delta\|^2),$$

and using $H(t,[\omega(t)])\omega(t) \equiv 0$, we obtain

$$i\varepsilon\dot{\Delta}(t) = -i\varepsilon\dot{\omega}(t) + H(t,[\omega(t)])\Delta(t) + 2\sum_{j=1}^{p} \partial_{x_j} H(t,[\omega(t)])\Re(\overline{\omega}_j\Delta_j)\omega(t) + O(\|\Delta\|^2).$$

The equation involves a source term, $-i\varepsilon\dot{\omega}(t)$, and its linear part depends on $\Delta(t)$ and $\overline{\Delta}(t)$. We write it as a system for these two vectors:

$$\left\{\begin{array}{rcl} i\varepsilon\dot{\Delta}(t) & = & -i\varepsilon\dot{\omega}(t) + H(t,[\omega(t)])\Delta(t) + \sum_{j=1}^{p}\partial_{x_j}H(t,[\omega(t)])(\overline{\omega}_j\Delta_j + \omega_j\overline{\Delta}_j)\omega(t) + O(\|\Delta\|^2), \\ i\varepsilon\dot{\overline{\Delta}}(t) & = & -i\varepsilon\dot{\overline{\omega}}(t) - \overline{H}(t,[\omega(t)])\overline{\Delta}(t) - \sum_{j=1}^{p}\partial_{x_j}\overline{H}(t,[\omega(t)])(\overline{\omega}_j\Delta_j + \omega_j\overline{\Delta}_j)\overline{\omega}(t) + O(\|\Delta\|^2). \end{array}\right. \tag{3.2}$$

We set for $j \in \{1, \cdots, p\}$,

$$v_j(t) = \partial_{x_j}H(t,[\omega(t)])\omega(t), \tag{3.3}$$

and, for later purposes, we notice that it follows from $P(t,x)H(t,x) \equiv 0$ that

$$v_j(t) = (\mathrm{Id} - P(t,[\omega(t)]))v_j(t). \tag{3.4}$$

We also set, for $j \in \{1, \cdots, p\}$,

$$\mu_j = \begin{pmatrix} v_j \\ -\overline{v}_j \end{pmatrix} \text{ and } \nu_j = \begin{pmatrix} \omega_j e_j \\ \overline{\omega}_j e_j \end{pmatrix}$$

and rewrite the system as

$$i\varepsilon\begin{pmatrix}\dot{\Delta}\\ \dot{\overline{\Delta}}\end{pmatrix} = -i\varepsilon\begin{pmatrix}\dot{\omega}\\ \dot{\overline{\omega}}\end{pmatrix} + F(t)\begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} + \begin{pmatrix} r^\varepsilon \\ -\overline{r}^\varepsilon\end{pmatrix}, \quad \Delta(0) = 0,$$

with $r^\varepsilon(t) = O(\|\Delta(t)\|^2)$ and

$$F(t) = F_0(t) + G(t),$$

$$F_0(t) = \begin{pmatrix} H(t,[\omega(t)]) & 0 \\ 0 & -\overline{H}(t,[\omega(t)]) \end{pmatrix}, \tag{3.5}$$

$$\begin{aligned} G(t) &= \sum_{j=1}^{p}\begin{pmatrix} \overline{\omega}_j(t)|v_j(t)\rangle\langle e_j| & \omega_j(t)v_j(t)\rangle\langle e_j| \\ -\overline{\omega}_j(t)|\overline{v}_j(t)\rangle\langle e_j| & -\omega_j(t)|\overline{v}_j(t)\rangle\langle e_j| \end{pmatrix} = \sum_{j=1}^{p}\left|\begin{array}{c} v_j(t) \\ -\overline{v}_j(t)\end{array}\right\rangle\left\langle\begin{array}{c}\omega_j(t)e_j\\ \overline{\omega}_j(t)e_j\end{array}\right| \\ &= \sum_{j=1}^{p}|\mu_j(t)\rangle\langle\nu_j(t)|. \end{aligned} \tag{3.6}$$

Note that $F(t), G(t) \in \mathcal{L}(\mathcal{H}\times\mathcal{H})$, $F(t)$ is non self-adjoint and $G(t)$ is of finite rank. Besides, because of the assumption $\mathbf{H}_2$, $G(t)$ can be treated as a perturbation of the self-adjoint operator $F_0(t)$. One then observes that two classical consequences of Weinstein-Aronszajn formula are that $\sigma_{ess}(F(t)) = \sigma_{ess}(F_0(t))$, and that $\sigma_d(F(t))$ consists in finitely many of eigenvalues (see e.g. [K2], Chap. IV, § 6).

The structure of the spectrum of $F(t)$ is crucial for our analysis. As we shall see in the following, the proof of Theorems 1.3 and 1.5 works out when the spectrum of $F(t)$ is semisimple with real eigenvalues of constant multiplicity for all $t \in \mathcal{T}_0$. Moreover, there are situations where this can be proved and the next lemma describes such cases. According to the assumptions of Theorems 1.3 and 1.5, we focus on the case where $H(t,[\omega(t)])$ is real.

Lemma 3.1 *a) There exists $\delta_0 > 0$ such that if, for all $t \in \mathcal{T}_0$, we have $\mathbf{H}_0$, $\mathbf{H}_1$ for some $\delta < \delta_0$, $\mathbf{H}_2$ and $\mathbf{H}_3$, then $0 \in \sigma(F(t))$ as a doubly degenerate isolated eigenvalue, with corresponding eigennilpotent $\mathbb{N}_0(t) \equiv 0$.*
b) Moreover, if $H(t,x) = \overline{H}(t,x)$, $\sigma(H(t,x))$ is simple and $\sigma(H(t,x)) \cap \sigma(-H(t,x)) = \{0\}$ for all $(t,x) \in \mathcal{T}_0 \times \mathcal{X}^p$, then δ_0 can be chosen so that the spectrum of $F(t)$ is real-valued for all $t \in \mathcal{T}_0$ and takes the form

$$-\ell_{N-1}(t) < \cdots < -\ell_1(t) < 0 < \ell_1(t) < \cdots < \ell_{N-1}(t), \tag{3.7}$$

where $\ell_0(t) \equiv 0$ is of multiplicity two, and each eigenvalue $\pm\ell_k(t)$, $1 \leq k \leq N-1$ is simple.
c) Finally, in the special case $p = 1$, we have a) and if moreover $H(t,x) = \overline{H}(t,x)$ and $\sigma(F(t)) \setminus \sigma(F_0(t))$ consists in exactly $2(N-1)$ perturbed eigenvalues, then $\sigma(F(t)) \subset \mathbb{R}$ and all corresponding eigennilpotents are zero.

The points a) and b) imply that under the assumptions of Theorem 1.3, the spectrum of $F(t)$ is semisimple with real eigenvalues of constant multiplicity for all $t \in \mathcal{T}_0$, and thus that the assumptions of Theorem 1.5 are satisfied. The point c) gives another situation with possibly degenerate eigenvalues where the assumptions of Theorem 1.5 hold.

Remark 3.2 *i) Note that for b), it is enough to assume $\sigma(H(0,x)) \cap \sigma(-H(0,x)) = \{0\}$, at the cost of making $|\mathcal{T}_0|$ smaller. This is a generic hypothesis which automatically satisfied whenevr λ_{j_0} is the ground state or the upper eigenvalue.*
ii) The condition $\#\{\sigma(F(t)) \setminus \sigma(F_0(t))\} = 2(N-1)$ states that the spectral effect of the rank one perturbation $G(t)$ is maximal, which is a genericity assumption.The multiplicities of the eigenvalues of $F_0(t)$ are arbitrary, possibly infinite, so that case c) does not necessary reduce to finite dimension, in contrast to the situation dealt with in Theorem 1.3.
iii) Besides, if the spectral effect of the rank one perturbation is maximal on $\mathcal{T}_0$, then $\sigma(F(t))$ takes the form (3.7) for all $t \in \mathcal{T}_0$, with $4(N-1)$ non zero distinct eigenvalues instead of $2(N-1)$, $2(N-1)$ of which are simple.
iv) The condition $H(t,x)$ real does not seem strong enough to ensure $\sigma(F(t)) \subset \mathbb{R}$ for $p \geq 2$; see the example of the Hamiltonian given by equation (5.1) in Appendix A.

We postpone the proof of Lemma 3.1 and we go to the next step of the proof which consists in controlling the adiabatic limit of the two-parameter evolution operator $T^\varepsilon(t,s)$ generated by $F(t)$ (see (3.14) below), and using it to estimate $\left\| \begin{pmatrix} \Delta \\ \overline{\Delta} \end{pmatrix} \right\|$ via Duhamel formula. Since $F(t)$ is not self-adjoint, this requires some care because the possible occurence of nilpotent operators in its spectral decomposition leads to subexponential divergence of the semigroup as $\varepsilon \to 0$ (see [J3]), that we cannot accommodate. However, Lemma 3.1 ensures that under the assumptions of Theorem 1.3, and by hypothesis in Theorem 1.5, for all $t \in \mathcal{T}_0$, $F(t)$ is semi-simple, with spectral decomposition

$$F(t) = \sum_{j=-N'}^{N'} \ell_j(t)\mathbb{P}_j(t), \text{ with the convention } \ell_{-|j|}(t) = -\ell_{|j|}(t), \tag{3.8}$$

where we have set $N' = N-1$ for convenience and where $\mathbb{P}_j(t)$ are smooth eigenprojectors corresponding to real eigenvalues $\ell_j(t)$. We now work under these assumptions.

Despite the eigenprojectors $\mathbb{P}_j(t)$ not being orthogonal, with norms possibly larger than 1, we prove in the next lemma that any operator $F(t)$ with real spectrum satisfying (3.8) generates an evolution operator which is uniformly bounded in ε and almost intertwines its eigenprojectors in the adiabatic limit. In line with Kato's approach ([K1] and *e.g.* [HJ]), we introduce the dynamical phase operator $\Phi^\varepsilon(t,s)$ defined by

$$\Phi^\varepsilon(t,s) = \sum_{j=-N'}^{N'} \mathbb{P}_j(0) \mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(\sigma)d\sigma}, \text{ s.t. } \Phi^\varepsilon(t,s)^{-1} = \Phi^\varepsilon(s,t), \tag{3.9}$$

and the intertwining operator $W(t)$ given by

$$i\partial_t W(t) = K(t)W(t), \;\; W(0) = \mathrm{Id}, \;\; \text{with } \; K(t) = i\sum_{j=-N'}^{N'} \dot{\mathbb{P}}_j(t)\mathbb{P}_j(t). \tag{3.10}$$

As is well known (see [K2]), for all $t \in \mathbb{R}$, we have

$$\mathbb{P}_j(t)W(t) = W(t)\mathbb{P}_j(0), \tag{3.11}$$

and thanks to Lemma 3.1, $\|\Phi^\varepsilon(t,s)\|$ is uniformly bounded in ε. Moreover, we check that

$$i\varepsilon\partial_t\Phi^\varepsilon(t,s) = W(t)^{-1}F(t)W(t)\Phi^\varepsilon(t,s) \equiv \tilde{F}(t)\Phi^\varepsilon(t,s). \tag{3.12}$$

We then introduce the bounded family of operators

$$V^\varepsilon(t,s) = W(t)\Phi^\varepsilon(t,s)W(s)^{-1}, \tag{3.13}$$

which satisfy $V^\varepsilon(t,s)^{-1} = V^\varepsilon(s,t)$ and

$$V^\varepsilon(t,s)\mathbb{P}_j(s) = \mathbb{P}_j(t)V^\varepsilon(t,s) = W(t)\mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(\sigma)d\sigma}W(s)^{-1}.$$

Moreover, because $F(t)$ is semi-simple, $V^\varepsilon(t,s)$ approximates the evolution operator generated by $F(t)$, as described by the next lemma which applies in a quite general setting.

Lemma 3.3 *Let $\mathcal{T}$ be an open bounded interval of $\mathbb{R}$ containing 0 and consider the operator defined on a Hilbert space $\mathcal{K}$ for all $(t,s) \in \mathcal{T}\times\mathcal{T}$ by the strong differential equation*

$$i\varepsilon\partial_t T^\varepsilon(t,s) = F(t)T^\varepsilon(t,s), \;\; T^\varepsilon(s,s) = \mathrm{Id}. \tag{3.14}$$

If $F \in C^2(\mathcal{T}, \mathcal{L}(\mathcal{K}))$ with continuous derivatives at $\partial\mathcal{T}$ and if $F(t)$ is semi-simple and satisfies (3.8) for all $t \in \overline{\mathcal{T}}$, then we have in $\mathcal{L}(\mathcal{K})$,

$$T^\varepsilon(t,s) = V^\varepsilon(t,s) + O_{t,s}(\varepsilon),$$

which implies the uniform boundedness of the family of operators $(T^\varepsilon(t,s))_{\varepsilon>0}$.

Remark 3.4 *i) As a consequence, $T^\varepsilon(t,s)\mathbb{P}_j(s) - \mathbb{P}_j(t)T^\varepsilon(t,s) = O_{t,s}(\varepsilon)$.*
ii) Note that N' in (3.8) is independent of $t \in \mathcal{T}$, the multiplicities of the eigenvalues of $F(t)$ are arbitrary, possibly infinite.

We postpone again the proof of this lemma and conclude the proof of Theorem 1.5. As already mentioned, Lemma 3.1 ensures we can apply Lemma 3.3 to $\mathcal{K} = \mathcal{H} \times \mathcal{H}$ and $\mathcal{T} = \mathcal{T}_0$ under the assumptions of Theorems 1.3 and 1.5. We write

$$\begin{aligned} \begin{pmatrix} \Delta(t) \\ \overline{\Delta}(t) \end{pmatrix} &= -\int_0^t T^\varepsilon(t,s) \begin{pmatrix} \dot{\omega}(s) \\ \dot{\overline{\omega}}(s) \end{pmatrix} ds - \frac{i}{\varepsilon} \int_0^t T^\varepsilon(t,s) \begin{pmatrix} r^\varepsilon(s) \\ -\overline{r}^\varepsilon(s) \end{pmatrix} ds \\ &= -\int_0^t V^\varepsilon(t,s) \begin{pmatrix} \dot{\omega}(s) \\ \dot{\overline{\omega}}(s) \end{pmatrix} ds - \frac{i}{\varepsilon} \int_0^t T^\varepsilon(t,s) \begin{pmatrix} r^\varepsilon(s) \\ -\overline{r}^\varepsilon(s) \end{pmatrix} ds + O_t(\varepsilon). \end{aligned} \tag{3.15}$$

It follows from the definition of $\omega(t)$ that $\mathbb{P}_0(t) \begin{pmatrix} \dot{\omega}(t) \\ \dot{\overline{\omega}}(t) \end{pmatrix} = 0$ for all time $t \in \mathcal{T}_0$ and a classical adiabatic argument (that we spell out in Section 3.3 below) yields the lemma below.

Lemma 3.5 *For all $t \in \mathcal{T}_0$, we have*

$$\int_0^t V^\varepsilon(t,s) \begin{pmatrix} \dot{\omega}(s) \\ \dot{\overline{\omega}}(s) \end{pmatrix} ds = O_t(\varepsilon).$$

Therefore, focusing on the first component of (3.15) and setting

$$\delta_\tau^\varepsilon = \sup_{t \in [0,\tau]} \|\Delta(t)\|,$$

with $\tau \in \mathcal{T}_0$, we deduce from the above that there exist $a, b > 0$ such that

$$\delta_\tau^\varepsilon \leq \varepsilon a + \frac{b}{\varepsilon} \tau {\delta_\tau^\varepsilon}^2.$$

Setting $X_\varepsilon(\tau) = \varepsilon^{-1} \delta_\tau^\varepsilon$, we are led to study of the second order equation

$$b\tau X^2 - X + a \geq 0. \tag{3.16}$$

Since $X_\varepsilon(0) = 0$, we deduce that $X_\varepsilon(\tau) \leq \frac{1}{2b\tau} \left(1 - \sqrt{1 - 4ab\tau}\right) = 2a/(1 + \sqrt{1 - 4ab\tau})$, as long as $4ab\tau \leq 1$. Finally, we obtain

$$\forall \tau \in [0, (4ab)^{-1}] \cap \mathcal{T}_0, \;\; \delta_\tau^\varepsilon \leq 2a\varepsilon.$$

To justify the estimate (1.10) for t small, we start from (3.15) to get the existence of $\alpha, \beta > 0$ such that

$$\delta_\tau^\varepsilon \leq \alpha\tau + \frac{\beta}{\varepsilon} \tau {\delta_\tau^\varepsilon}^2.$$

Focusing on times $\tau \leq \varepsilon$, we consider $\delta_\tau^\varepsilon \leq \alpha\tau + \beta {\delta_\tau^\varepsilon}^2$, which, by a similar argument using $\delta_0^\varepsilon = 0$, implies, as long as $4\alpha\beta\tau \leq 1$, $\delta_\tau^\varepsilon \leq 2\alpha\tau$. Increasing the constant α if necessary, we get (1.10). □

3.2 Spectral analysis of $F(t)$

We proceed with the proof of Lemma 3.1, which relies on a careful analysis of the eigenvalues of $F(t)$ and of their multiplicity.

Recall that C denotes the anti-unitary involution defined on $\mathcal{H}$ by $C\psi = \overline{\psi}$ for all $\psi \in \mathcal{H}$. It is at this stage of the proof that we shall use the assumption $H = CHC = \overline{H}$, which implies $\omega = C\omega = \overline{\omega}$ and $v_j = Cv_j = \overline{v}_j$ for all $1 \leq j \leq p$. Due to assumption $\mathbf{H}_1$, we consider the operator $F(t)$ as a perturbation of the bloc diagonal operator $F_0(t)$. Hence, since $\sigma(\overline{H}(t,[\omega(t)])) = \sigma(H(t,[\omega(t)]))$,

$$\sigma(F_0(t)) = \sigma(H(t,[\omega(t)])) \cup \sigma(-H(t,[\omega(t)])).$$

By our genericity assumption, and due to the reduction we have made to the case where $\lambda(t,[\omega(t)]) \equiv 0$, the spectrum of $F_0(t)$ consists of $2N-1 = 2N'+1$ isolated eigenvalues

$$-|\lambda_{N'}(t,[\omega(t)])| < \cdots < -|\lambda_1(t,[\omega(t)])| < 0 < |\lambda_1(t,[\omega(t)])| < \cdots < |\lambda_{N'}(t,[\omega(t)])|.$$

Since the operator $G(t)$ is of small norm by assumption $\mathbf{H}_1$ and its definition (equations (3.3) and 3.6), the spectrum of $F(t)$ can be inferred from that of $F_0(t)$ by perturbation theory. Hence $F(t)$ has eigenvalues located in small discs $\mathcal{B}_j^{\pm}$ centered at $\pm\lambda_j(t,[\omega(t)])$ and in a disk $\mathcal{B}_0$ with center 0. One can assume that these disks are of same radius $r > 0$ and that they do not intersect. Besides

- in $\mathcal{B}_j^{\pm}$, $F(t)$ has as many eigenvalues (counted with multiplicity) as the multiplicity of $\lambda_j(t,[\omega(t)])$ as an eigenvalue of $F_0(t)$, and in case the multiplicity is infinite, there are only finitely many eigenvalues of $F(t)$ in $\mathcal{B}_j^{\pm}$ that differ from $\lambda_j(t,[\omega(t)])$,
- in $\mathcal{B}_0$, $F(t)$ has at most two eigenvalues (counted with multiplicity).

We are going to use symmetry considerations to prove that these eigenvalues are real-valued and have the same symmetry properties as those of $F_0(t)$.

Remark 3.6 *We develop in Appendix A an argument showing that the spectrum of $F(t)$ is not necessarily real if $H(t,x)$ is real, in order to motivate the assumptions that its eigenvalues are simple.*

Proof: a) We start by considering the spectrum of $F(t)$ in a neighbourhood of zero. For any $z \in \mathcal{B}_0 \backslash \{0\}$, we can write

$$F(t) - z = (F_0(t) - z)\left[\mathrm{Id} + (F_0(t) - z)^{-1} G(t)\right]. \tag{3.17}$$

Introducing the spectral projector $\tilde{P}_0(t)$ associated with the doubly degenerate eigenvalue zero of $F_0(t)$ and the corresponding reduced resolvent acting on $\tilde{Q}_0(t)(\mathcal{H}\times\mathcal{H})$, $\tilde{Q}_0(t) = \mathrm{Id} - \tilde{P}_0(t)$, we have for $z \in \mathcal{B}_0 \setminus \{0\}$,

$$(F_0(t) - z)^{-1} = -\frac{\tilde{P}_0(t)}{z} + (F_0(t) - z)^{-1}_{\tilde{Q}_0(t)}, \tag{3.18}$$

where we denote by $A_{\tilde{Q}_0}$ the restriction of the operator A to the range of $\tilde{Q}_0$. Since

$$\tilde{P}_0(t) = \begin{pmatrix} |\omega(t)\rangle\langle\omega(t)| & 0 \\ 0 & |\overline{\omega}(t)\rangle\langle\overline{\omega}(t)| \end{pmatrix}$$

and $\langle \omega(t)|v_j(t)\rangle \equiv 0$ for all $1 \leq j \leq p$, see (3.4), we get $\tilde{P}_0(t)G(t) \equiv 0$ so that,

$$(F(t)-z)^{-1} = \Big[\mathrm{Id} + (F_0(t)-z)^{-1}_{\tilde{Q}_0(t)} G(t)\Big]^{-1} (F_0(t)-z)^{-1}. \tag{3.19}$$

Indeed, the reduced resolvent is analytic in $z \in \mathbb{B}_0$ and $\|G(t)\| = 2\delta$, so for δ_0 small enough, the square bracket is invertible. Therefore, the only singularity of the resolvent of $F(t)$ lies at $z = 0$, which remains a doubly degenerate eigenvalue after perturbation. The corresponding spectral projector is

$$\mathbb{P}_0(t) = -\frac{1}{2i\pi}\int_{\partial\mathbb{B}_0} (F(t)-z)^{-1} dz = \Big[\mathrm{Id} + F_0(t)^{-1}_{\tilde{Q}_0(t)} G(t)\Big]^{-1} \tilde{P}_0(t). \tag{3.20}$$

and, in view of (3.18) and (3.19), the corresponding eigennilpotent $\mathbb{N}_0(t) = F(t)\mathbb{P}_0(t)$ writes, (see [K2] Chapt. III, §5)

$$\begin{aligned} \mathbb{N}_0(t) &= -\frac{1}{2i\pi}\int_{\partial\mathbb{B}_0} z(F(t)-z)^{-1} dz \\ &= -\frac{1}{2i\pi}\int_{\partial\mathbb{B}_0} \Big[\mathrm{Id} + (F_0(t)-z)^{-1}_{\tilde{Q}_0(t)} G(t)\Big]^{-1} \Big[-\tilde{P}_0(t) + (F_0(t)-z)^{-1}_{\tilde{Q}_0(t)} z\Big]\, dz. \end{aligned} \tag{3.21}$$

Since the integrand is analytic in $\mathbb{B}_0$, we get that $\mathbb{N}_0(t) \equiv 0$, which ends the proof of a) of Lemma 3.1.

b) The perturbation $G(t)$ being of finite rank, we compute the Aronszajn-Weinstein determinant ([K2], p. 245) which reads in our case for all $z \in \rho(F_0(t))$, the resolvent set of $F_0(t)$,

$$\begin{aligned} w(z) &= \det(\delta_{j,k} + \langle \nu_k(t)|(F_0(t)-z)^{-1}\mu_j(t)\rangle)_{1\leq j,k\leq p} \\ &= \det(\delta_{j,k} + \bar{\omega}_k\langle e_k(t)|(H(t)-z)^{-1}v_j(t)\rangle + \omega_k\langle (H(t)+\bar{z})^{-1}v_j(t)|e_k(t)\rangle)_{1\leq j,k\leq p}. \end{aligned} \tag{3.22}$$

It follows that $w(z) = \overline{w(-\bar{z})}$ for all $z \in \mathbb{B}_0$. Since the zeros of $w(z)$ yield the eigenvalues of $F(t)$ in $\rho(F_0(t))$, we obtain

$$z \in \sigma(F(t)) \cap \rho(F_0(t)) \Leftrightarrow -\bar{z} \in \sigma(F(t)) \cap \rho(F_0(t)).$$

Since $\overline{H}(t,x) = H(t,x)$, we deduce

$$w(z) = \det(\delta_{j,k} + \omega_k\langle e_k(t)|((H(t)-z)^{-1} + (H(t)+z)^{-1})v_j(t)\rangle)_{1\leq j,k\leq p} = w(-z). \tag{3.23}$$

It follows then that

$$z \in \sigma(F(t)) \cap \rho(F_0(t)) \Rightarrow \{z, \bar{z}, -z, -\bar{z}\} \in \sigma(F(t)) \cap \rho(F_0(t)). \tag{3.24}$$

The nonzero eigenvalues of $F_0(t)$ being simple by assumption, the same is true by perturbation theory for those of $F(t)$ and (3.24) shows they must be real. Moreover, these conclusions hold for any $t \in \mathcal{T}_0$ under the stated hypotheses.

c) We now assume $p = 1$. Let $t \in \mathcal{T}_0$ fixed and let us drop the time variable. We make use of (1.5), with a possible relabelling of the eigenvalues due to the shift (3.1), to write with $N' = N - 1$

$$(H-z)^{-1} = \frac{P_0}{-z} + \sum_{j=1}^{N'} \frac{P_j}{\lambda_j - z}, \quad \text{where} \quad \lambda_j \neq 0 \text{ if } j \geq 1, \text{ and } \lambda_0 = 0.$$

Thus, with $p = 1$, $z \in \rho(F_0)$, and $P_0 v_1 = 0$,

$$w(z) = 1 + 2\omega_1 \sum_{j=1}^{N'} \frac{\lambda_j \langle e_1 | P_j v_1 \rangle}{\lambda_j^2 - z^2} = \frac{\Pi_{k=1}^{N'}(\lambda_k^2 - z^2) + 2\omega_1 \sum_{j=1}^{N'} \Pi_{\substack{k=1 \\ k\neq j}}^{N'}(\lambda_k^2 - z^2)\lambda_j \langle e_1 | P_j v_1 \rangle}{\Pi_{k=1}^{N'}(\lambda_k^2 - z^2)}. \tag{3.25}$$

The numerator is a polynomial of degree $2N'$ which, by assumption, possesses $2N'$ distinct simple roots in $\rho(F_0)$. These roots being in the neighbourhood of $\sigma(F_0) \setminus \{0\}$ for δ small, (3.24) implies that they are real. This proves $\sigma(F) \subset \mathbb{R}$.

We now consider the eigennilpotents. The potentially nonzero eigennilpotents $\mathbb{N}_{\pm\lambda_j}$ are thus attached to the unperturbed eigenvalues $\pm\lambda_j$ with sufficient multiplicity, *i.e.* with $\dim \tilde{P}_j \geq 3$ only. For $p = 1$ and $z \notin \mathbb{R}$, the resolvent takes the explicit form

$$\begin{aligned}(F - z)^{-1} &= \left\{ \mathrm{Id} - \frac{(F_0 - z)^{-1}|\mu\rangle\langle\nu|}{1 + \langle\nu|(F_0 - z)^{-1}\mu\rangle} \right\} (F_0 - z)^{-1} \\ &= \left\{ \mathrm{Id} - \frac{\omega_1}{w(z)} \sum_{j=1}^{N'} \left| \begin{matrix} \frac{P_j v_1}{\lambda_j - z} \\ \frac{P_j v_1}{\lambda_j + z} \end{matrix} \right\rangle \left\langle \begin{matrix} e_1 \\ e_1 \end{matrix} \right| \right\} (F_0(t) - z)^{-1}.\end{aligned}$$

The eigennilpotents are the coefficients, up to a sign, of the poles of order two of the resolvent at the eigenvalues. We consider the nonzero eigenvalue λ_k only, $-\lambda_k$ being similar. Using the fact that the numerator $\tilde{w}(z)$ of $w(z)$ in (3.25) is nonzero at λ_k by assumption, we have in a neighbourhood of λ_k

$$w(z)^{-1} = (\lambda_k - z)(\lambda_k + z)\Pi_{j\neq k}(\lambda_j^2 - z^2)/\tilde{w}(z) := (\lambda_k - z)s_k(1 + O(\lambda_k - z)),$$

with $s_k = 2\lambda_k \Pi_{j\neq k}(\lambda_j^2 - \lambda_k^2)/\tilde{w}(\lambda_k) \neq 0$. Hence, for z close to λ_k,

$$\begin{aligned}(F - z)^{-1} &= \left\{ \mathrm{Id} - \omega_1 s_k \left| \begin{matrix} P_k v_1 \\ 0 \end{matrix} \right\rangle \left\langle \begin{matrix} e_1 \\ e_1 \end{matrix} \right| + O(z - \lambda_k) \right\} \begin{pmatrix} \frac{P_k}{\lambda_k - z} & 0 \\ 0 & 0 \end{pmatrix} + O(1) \\ &= \frac{1}{\lambda_k - z} \begin{pmatrix} P_k(\mathrm{Id} - \omega_1 s_k |v_1\rangle\langle e_1|)P_k & 0 \\ 0 & 0 \end{pmatrix} + O(1).\end{aligned}$$

The absence of pole of order two shows that $\mathbb{N}_k = 0$, and the computation above further yields

$$\mathbb{P}_k = \begin{pmatrix} P_k(\mathrm{Id} - \omega_1 s_k |v_1\rangle\langle e_1|)P_k & 0 \\ 0 & 0 \end{pmatrix},$$

which concludes the proof.

We end the argument by briefly checking that $\mathbb{P}_k$ is a projector on $\mathcal{H} \times \mathcal{H}$, or equivalently that $P_k - \omega_1 s_k P_k |v_1\rangle\langle e_1| P_k$ is a projector on $\mathcal{H}$. Since $P_k^2 = P_k$

$$(P_k - \omega_1 s_k P_k |v_1\rangle\langle e_1|)P_k)^2 = P_k - 2\omega_1 s_k P_k |v_1\rangle\langle e_1|)P_k + \omega_1^2 s_k^2 P_k |v_1\rangle\langle e_1|P_k v_1\rangle\langle e_1|P_k.$$

The right hand side equals $P_k - \omega_1 s_k P_k |v_1\rangle\langle e_1|)P_k$ if $\omega_1 s_k \langle e_1|P_k v_1\rangle = 1$. With the definition of s_k, this is equivalent to $2\omega_1 \lambda_k \Pi_{j\neq k}(\lambda_j^2 - \lambda_k^2)\langle e_1|P_k v_1\rangle = \tilde{w}(\lambda_k)$. Now, (3.25) gives

$$\tilde{w}(\lambda_k) = \Pi_{l=1}^{N'}(\lambda_l^2 - \lambda_k^2) + 2\omega_1 \sum_{j=1}^{N'} \Pi_{\substack{l=1 \\ l\neq j}}^{N'}(\lambda_l^2 - \lambda_k^2)\lambda_j \langle e_1|P_j v_1\rangle,$$

where the first term equals zero, while the only non zero term in the sum corresponds to $j = k$. Altogether, $\tilde{w}(\lambda_k) = 2\omega_1 \Pi_{\substack{l=1 \\ l\neq k}}^{N'}(\lambda_l^2 - \lambda_k^2)\lambda_k \langle e_1|P_k v_1\rangle$ which yields the result. □

3.3 Non-selfadjoint adiabatic estimates

We prove here Lemma 3.3 in a way that naturally adapts to the unbounded setting that we shall consider in Section 4.

Proof: We first note that by the definition of V^ε and K (see (3.13) and (3.10)), we have

$$i\varepsilon\partial_t V^\varepsilon(t,s) = W(t)\sum_{j=-N'}^{N'} \ell_j(t)\mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(s')ds'}W(s)^{-1} + \varepsilon K(t)V^\varepsilon(t,s).$$

Using (3.11) and $\mathbb{P}_j(0)^2 = \mathbb{P}_j(0)$, we obtain

$$\begin{aligned} i\varepsilon\partial_t V^\varepsilon(t,s) &= \sum_{j=-N'}^{N'} \ell_j(t)\mathbb{P}_j(t)W(t)\mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(s')ds'}W(s)^{-1} + \varepsilon K(t)V^\varepsilon(t,s) \\ &= F(t)\sum_{j=-N'}^{N'} \mathbb{P}_j(t)W(t)\mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(s')ds'}W(s)^{-1} + \varepsilon K(t)V^\varepsilon(t,s), \end{aligned}$$

whence

$$i\varepsilon\partial_t V^\varepsilon(t,s) = F(t)V^\varepsilon(t,s) + \varepsilon K(t)V^\varepsilon(t,s). \tag{3.26}$$

We can now compare $T^\varepsilon(t,s)$ and $V^\varepsilon(t,s)$. Let $\Omega^\varepsilon(t,s) = V^\varepsilon(t,s)^{-1}T^\varepsilon(t,s)$, we have

$$i\partial_t\Omega^\varepsilon(t,s) = -V^\varepsilon(t,s)^{-1}K(t)T^\varepsilon(t,s) = -(V^\varepsilon(t,s)^{-1}K(t)V^\varepsilon(t,s))\,\Omega^\varepsilon(t,s), \tag{3.27}$$

or, equivalently

$$\Omega^\varepsilon(t,s) = \mathrm{Id} + i\int_s^t V^\varepsilon(t',s)^{-1}K(t')V^\varepsilon(t',s)\Omega^\varepsilon(t',s)dt'. \tag{3.28}$$

With the shorthand $\tilde{K}(t') = W^{-1}(t')K(t')W(t')$, we have

$$V^\varepsilon(t',s)^{-1}K(t')V^\varepsilon(t',s) = W(s)\Phi^\varepsilon(s,t')\tilde{K}(t')\Phi^\varepsilon(t',s)W^{-1}(s)$$

and

$$\mathbb{P}_j(0)\tilde{K}(t')\mathbb{P}_k(0) = i(1-\delta_{j,k})\mathbb{P}_j(0)\tilde{K}(t')\mathbb{P}_k(0).$$

Therefore, for any j,

$$\mathbb{P}_j(s)\Omega(t,s) = \mathbb{P}_j(s) + iW(s)\int_s^t \mathbb{P}_j(0)e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\tilde{K}(t')(\mathrm{Id}-\mathbb{P}_j(0))\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s)dt'. \tag{3.29}$$

Now, observe that,

$$i\varepsilon\partial_{t'}e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s) = \tilde{F}_j(t')e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s) \tag{3.30}$$

where

$$\tilde{F}_j(t) = \sum_k \mathbb{P}_k(0)(\ell_k(t)-\ell_j(t)) = \tilde{F}(t) - \ell_j(t)\mathrm{Id} \tag{3.31}$$

is invertible on $(\mathrm{Id}-\mathbb{P}_j(0)\mathcal{H}\times\mathcal{H}$, with reduced resolvent we denote by

$$\tilde{R}_j(t) := \tilde{F}_j^{-1}(t)|_{\mathrm{Id}-\mathbb{P}_j(0)} = \sum_{\substack{k\\k\neq j}} \mathbb{P}_k(0)/(\ell_k(t)-\ell_j(t)).$$

Thus the integrand in (3.29) reads, using (3.27) in the last step,

$$\begin{aligned} I: &= e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du/}\tilde{K}(t')(\mathrm{Id}-\mathbb{P}_j(0))\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s) \\ &= \tilde{K}(t')\tilde{R}_j(t')\tilde{F}_j(t')e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s) \\ &= \tilde{K}(t')\tilde{R}_j(t')\{i\varepsilon\partial_{t'}e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s)\}W^{-1}(s)\Omega^\varepsilon(t',s). \end{aligned}$$

We deduce

$$\begin{aligned} I &= i\varepsilon\partial_{t'}\left\{\tilde{K}(t')\tilde{R}_j(t')e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s)\right\} \\ &\quad -i\varepsilon\partial_{t'}\{\tilde{K}(t')\tilde{R}_j(t')\}e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s) \\ &\quad +\varepsilon\tilde{K}(t')\tilde{R}_j(t')e^{\frac{i}{\varepsilon}\int_s^{t'}\ell_j(u)du}\tilde{K}(t')\Phi^\varepsilon(t',s)W^{-1}(s)\Omega^\varepsilon(t',s). \end{aligned} \tag{3.32}$$

Note that thanks to our spectral hypothesis, we have

$$\sup_{t\in\overline{\mathcal{T}}}\{\|\tilde{R}_j(t)\|,\|\partial_t\tilde{R}_j(t)\|\}<c_0$$

for some constant c_0. We can thus integrate (3.29) by parts to get the existence of a constant $c>0$ (that may change from line to line below) such that for all $t,s\in\overline{\mathcal{T}}$

$$\|\mathbb{P}_j(s)\Omega(t,s)-\mathbb{P}_j(s)\|\le c\,c_0\,\varepsilon|||\Omega|||, \tag{3.33}$$

where $|||\Omega||| = \sup_{(s,t)\in\overline{\mathcal{T}}}\|\Omega(t,s)\|$. Therefore,

$$\sup_{(s,t)\in\overline{\mathcal{T}}}\|\Omega(t,s)-\mathrm{Id}\|^2\le c\,c_0^2\,\varepsilon^2|||\Omega|||^2\le c\,\varepsilon^2\Big(|||\Omega-\mathrm{Id}|||^2+|||\mathrm{Id}|||^2\Big),$$

from which we get the existence of $\varepsilon_0>0$, independent of t, such that $\varepsilon<\varepsilon_0$ implies

$$|||\Omega-\mathrm{Id}|||=O(\varepsilon).$$

Hence we infer the sought for bounds

$$\Omega^\varepsilon(t,s)=V^\varepsilon(t,s)^{-1}T^\varepsilon(t,s)=\mathrm{Id}+O_{t,s}(\varepsilon),\quad\text{and}\quad T^\varepsilon(t,s)=O_{t,s}(1).$$

□

Let us now prove Lemma 3.5.

Proof: Set $\chi_\omega(s)=\begin{pmatrix}\dot{\omega}(s)\\ \dot{\overline{\omega}}(s)\end{pmatrix}$ and recall that

$$\tilde{P}_0(s)\chi_\omega(s)=\tilde{P}_0(s)\begin{pmatrix}\dot{\omega}(s)\\ \dot{\overline{\omega}}(s)\end{pmatrix}\equiv 0.$$

Therefore, the perturbed projector $\mathbb{P}_0(s)$ associated to the kernel of $F(s)$ given by (3.20) satisfies

$$\mathbb{P}_0(s)\chi_\omega(s)=\Big[\mathrm{Id}+F_0(s)^{-1}_{\tilde{Q}_0(s)}G(s)\Big]^{-1}\tilde{P}_0(s)\chi_\omega(s)\equiv 0.$$

Hence, writing $\tilde{F}(s) = W^{-1}(s)F(s)W(s)$, we have

$$V^\varepsilon(t,s)\chi_\omega(s) = W(t)\Phi^\varepsilon(t,s)(\mathrm{Id}-\mathbb{P}_0(0))W^{-1}(s)\chi_\omega(s) = W(t)\Phi^\varepsilon(t,s)\tilde{F}(s)(\tilde{F}(s)^{-1}W^{-1}(s)\chi_\omega(s)),$$

where $\tilde{F}(s)^{-1}$ is to be understood as the reduced resolvent of $\tilde{F}(s)$ acting on $(\mathrm{Id}-\mathbb{P}_0(0))\mathcal{H}\times\mathcal{H}$. Thanks to (3.12) we can rewrite

$$\begin{aligned}\int_0^t V^\varepsilon(t,s)\chi_\omega(s)ds &= -\mathrm{i}\varepsilon W(t)\int_0^t \partial_s\{\Phi^\varepsilon(t,s)(\tilde{F}(s)^{-1}W^{-1}(s)\chi_\omega(s))\}ds \\ &= -\mathrm{i}\varepsilon W(t)\{\Phi^\varepsilon(t,s)(\tilde{F}(s)^{-1}W^{-1}(s)\chi_\omega(s))\}|_0^t = O_t(\varepsilon).\end{aligned} \tag{3.34}$$

□

4 Generalization to unbounded operators

In this section, we prove Theorem 1.8. To start with, we focus on the existence of solutions, then we deal with the adiabatic result, that we spell out step by step. The latter proof follows the same line as in Section 3. However, due to the unboundedness of the operator $H(t,x)$, several technical points have to be taken care of; we emphasise these points in different subsections.

4.1 Global weak solution to (1.11)

We prove the existence of a unique global solution to the nonlinear Schrödinger equation (1.11) in the weak sense, *i.e.* for any $\chi \in \mathcal{D}$, we have equation (1.12), that is

$$i\varepsilon\partial_t\langle\chi|\psi^\varepsilon(t)\rangle = \langle(H_0 + W(t,[\psi^\varepsilon(t)])\chi|\psi^\varepsilon(t)\rangle, \quad \psi^\varepsilon(0) = \omega(0).$$

We denote by e^{-itH_0} the evolution group associated with H_0 which maps $\mathcal{D}$ into $\mathcal{D}$ and is differentiable on $\mathcal{D}$ only. We first consider a solution of (1.11) as an integral solution, *i.e.* a continuous function $t \mapsto \psi^\varepsilon(t) \in \mathcal{H}$ such that

$$\forall t \in \mathcal{T}, \ \ \psi^\varepsilon(t) = \mathrm{e}^{-\frac{i}{\varepsilon}tH_0}\omega(0) + \frac{1}{i\varepsilon}\int_0^t \mathrm{e}^{-\frac{i}{\varepsilon}(t-s)H_0}W(s,[\psi^\varepsilon(s)])\psi^\varepsilon(s)ds. \tag{4.1}$$

Indeed, such a $\psi^\varepsilon(t)$ satisfies (1.12) for all $\chi \in \mathcal{D}$. Besides, if it does exist, we will show that the solution satisfies $\|\psi^\varepsilon(t)\| = \|\omega(0)\| = 1$.

To construct $\psi^\varepsilon(t)$, we consider $M \geq 1$, $\tau > 0$ such that

$$1 + \tau M \sup\|W\| \leq M, \ \text{ and } \ \sup(\|W\| + 4M\|d_xW\|)\tau < 1,$$

the ball $B(0,M)$ of $\mathcal{H}$ and the map $\Phi : \mathcal{C}([0,\tau\varepsilon],\mathcal{H}) \to \mathcal{C}([0,\tau\varepsilon],\mathcal{H})$

$$\Phi : v(t) \mapsto \mathrm{e}^{-\frac{i}{\varepsilon}tH_0}\omega(0) + \frac{1}{i\varepsilon}\int_0^t \mathrm{e}^{-\frac{i}{\varepsilon}(t-s)H_0}W(s,[v(s)])v(s)ds.$$

By the choice of τ, Φ maps $\mathcal{C}([0,\tau\varepsilon],B(0,M))$ into itself. Besides, Φ is a contraction:

$$\begin{aligned}\Phi(v)(t)-\Phi(w)(t) &= \frac{1}{i\varepsilon}\int_0^t \mathrm{e}^{-\frac{i}{\varepsilon}(t-s)H_0}(W(s,[v(s)])v(s)-W(s,[w(s)])w(s))ds \\ &= \frac{1}{i\varepsilon}\int_0^t \mathrm{e}^{-\frac{i}{\varepsilon}(t-s)H_0}(W(s,[v(s)])(v(s)-w(s))-(W(s,[v(s)])-W(s,[w(s)])w(s))ds,\end{aligned} \tag{4.2}$$

hence, uniformly in $t\in\overline{\mathcal{T}}$, $\|\Phi(v)-\Phi(w)\|\leq \sup(\|W\|+4M\|d_xW\|)\tau\|v-w\|$ with $\sup(\|W\|+4M\|d_xW\|)\tau<1$. Therefore, Φ has a unique fixed point $\psi^\varepsilon(t)\in C([0,\tau\varepsilon],B(0,M))$, which is the unique integral solution of the equation (1.11) on $[0,\tau\varepsilon]$.

Now, the vector $\varphi^\varepsilon(t)=\mathrm{e}^{\frac{i}{\varepsilon}tH_0}\psi^\varepsilon(t)$ satisfies $\forall t\in[0,\tau\varepsilon]$,

$$\varphi^\varepsilon(t)=\omega(0)+\frac{1}{i\varepsilon}\int_0^t \mathrm{e}^{\frac{i}{\varepsilon}sH_0}W(s,[\psi^\varepsilon(s)])\mathrm{e}^{-\frac{i}{\varepsilon}sH_0}\varphi^\varepsilon(s)ds, \tag{4.3}$$

where the integrand is continuous, so that strong differentiation with respect to time is allowed. Since the operator $\mathrm{e}^{\frac{i}{\varepsilon}sH_0}W(s,[\psi^\varepsilon(s)])\mathrm{e}^{\frac{-i}{\varepsilon}sH_0}$ is self-adjoint, one gets in the usual way that,

$$\forall t\in[0,\tau\varepsilon],\ \ \|\varphi^\varepsilon(t)\|=\|\psi^\varepsilon(t)\|\equiv 1.$$

Observe that the choice of τ only depends on $\|W\|,\|d_xW\|$ and M, and since $\|\psi^\varepsilon(\tau\varepsilon)\|=1$, we can reiterate the same argument on $[\tau\varepsilon,2\tau\varepsilon]$ starting from the initial data $\psi^\varepsilon(\tau\varepsilon)$ instead of $\omega(0)$. One then constructs the unique normalised integral solution of (1.11) on $[\tau\varepsilon,2\tau\varepsilon]$, so that $\|\psi^\varepsilon(2\tau\varepsilon)\|=1$. Iterating the process, we see that we have a unique global integral solution of the form (4.1) to the equation (1.11).

4.2 Adiabatic statement of Theorem 1.8

Again, the gauge invariance manifested in the conclusions of Lemma 1.1 holds in this case as well. This allows us to consider the replacement $H(t,x)\mapsto H(t,x)-\lambda(t,x)\mathrm{Id}$, keeping the notation $H(t,x)$ for the shifted Hamiltonian, which admits 0 in its spectrum and finitely many negative eigenvalues.

At this point, we follow the same strategy as in Section 3. We set $\Delta(t)=\psi^\varepsilon(t)-\omega(t)$, which solves a system similar to (3.2), as we now check. With the definitions

$$|e_j\rangle\langle e_j|\omega(t)=\omega_j(t)e_j,\ \ v_j(t)=\partial_{x_j}H(t,[\omega(t)])\omega(t), \tag{4.4}$$

and for all normalized $\chi\in\mathcal{D}$, we have, using the smoothness of the bounded operator $W(t,x)$,

$$\begin{aligned}i\varepsilon\partial_t\langle\chi|\Delta(t)\rangle =&\langle(H_0+W(t,[\psi^\varepsilon(t)]))\chi|\psi^\varepsilon(t)\rangle-i\varepsilon\langle\chi|\dot\omega(t)\rangle \\ &=\langle(H_0+W(t,[\omega(t)]))\chi|\Delta\rangle+2\sum_{j=1}^p\langle\chi|\big(\partial_{x_j}W(t,[\omega(t)])\Re(\omega_j\overline{\Delta_j(t)})\omega(t)\rangle \\ &\quad -i\varepsilon\langle\chi|\dot\omega(t)\rangle+\langle\chi|r^\varepsilon(t)\rangle.\end{aligned}$$

where $r^\varepsilon(t)$ is of order $\|\Delta(t)\|^2$. Indeed, it takes the form

$$\begin{aligned} r^\varepsilon(t) &= \sum_{1\le j,k\le p} (\Delta_j(t)\overline{\Delta}_k(t)b_{j,k}(t) + \Delta_j(t)\Delta_k(t)\tilde{b}_{j,k}(t) + \overline{\Delta}_j(t)\overline{\Delta}_k(t)\underline{b}_{j,k}(t)) \\ &\quad + \sum_{1\le j\le p} (\Delta_j(t)B_j(t)\Delta(t) + \overline{\Delta}_j(t)\tilde{B}_j(t)\Delta(t) \end{aligned}$$

for some uniformly bounded vectors $\tilde{b}_{j,k}(t), \underline{b}_{j,k}(t), b_{j,k}(t) \in \mathcal{H}$ and uniformly bounded operators $B_j(t),\ \tilde{B}_j(t)$ are bounded operators (which may also depend on $\Delta(t)$ and $\overline{\Delta}(t)$):

$$\begin{aligned} r^\varepsilon(t) &= \sum_{1\le j\le p} \int_0^1 \partial_{x_j} W(t,[\omega+s\Delta])(2\Re(\omega_j\overline{\Delta}_j)\Delta + |\Delta_j|^2(\Delta+\omega))ds \\ &\quad + \sum_{1\le j,k\le p} \int_0^1 (1-s)(2\Re(\omega_j\overline{\Delta}_j))(2\Re(\omega_k\overline{\Delta}_k) + |\Delta_k|^2)ds\ \partial^2_{x_j,x_k} W(t,[\omega+s\Delta])\omega ds. \end{aligned} \tag{4.5}$$

Besides, $i\varepsilon\partial_t\langle\chi|\overline{\Delta(t)}\rangle$ satisfies a similar equation corresponding to (3.2). Thus for the nonlinear problem, we need to consider weak solutions on $\mathcal{D}\times\mathcal{D}$ of the coupled equations, see in Section 3: For any $(\chi_1,\chi_2)\in\mathcal{D}\times\mathcal{D}$,

$$i\varepsilon\partial_t \left\langle \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \right\rangle = -i\varepsilon \left\langle \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\dot{\omega}\\ \dot{\omega}\end{pmatrix} \right\rangle + \left\langle F^*(t) \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \right\rangle + \left\langle \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}r^\varepsilon(t)\\ -\overline{r}^\varepsilon(t)\end{pmatrix} \right\rangle,\ \ \Delta(0)=0, \tag{4.6}$$

with $r^\varepsilon(t) = O(\|\Delta(t)\|^2)$ and

$$F(t) = F_0(t) + G(t) \text{ with } F_0(t) = \begin{pmatrix} H(t,[\omega(t)]) & 0 \\ 0 & -H(t,[\omega(t)]) \end{pmatrix},$$

$$G(t) = \sum_{j=1}^p \omega_j(t) \begin{pmatrix} |v_j(t)\rangle\langle e_j| & |v_j(t)\rangle\langle e_j| \\ -|v_j(t)\rangle\langle e_j| & -|v_j(t)\rangle\langle e_j| \end{pmatrix} = \sum_{j=1}^p \omega_j(t) \left| \begin{matrix} v_j(t) \\ -v_j(t) \end{matrix} \right\rangle \left\langle \begin{matrix} e_j \\ e_j \end{matrix} \right|.$$

The conjugates do not appear in the definition of F_0, F and G since assumption $\mathbf{R}_0$ entails the fact that $H(t,x)$ is real.

Note that $F_0(t) = F_0 + B(t)$, where $F_0 = \begin{pmatrix} H_0 & 0 \\ 0 & -H_0 \end{pmatrix}$ is self-adjoint on $\tilde{\mathcal{D}} := \mathcal{D}\times\mathcal{D}$ and $B(t) = \begin{pmatrix} W(t,[\omega(t)]) & 0 \\ 0 & -W(t,[\omega(t)]) \end{pmatrix}$ is bounded, self-adjoint and smooth in t. Therefore, the domain of $F_0(t)$ is $\tilde{\mathcal{D}}$, and the same is true for $F(t)$ since $G(t)$ is also bounded. In the next three paragraphs, we develop the arguments of the proof paying attention to the difficulties induced by the fact that H_0, and thus $F(t)$ are unbounded. We shall focus in particular on the existence of the propagator associated with the operator $F(t)$, on the analysis of the (unbounded) spectrum of $F(t)$, on the construction of the associated adiabatic approximate propagator and on its properties.

4.3 Existence of the propagator generated by $F(t)$

Using the latter remark, we get the following regularity result on the solutions to the linear part of the equation for $(\Delta(t), \overline{\Delta(t)})$ in $\mathcal{H}\times\mathcal{H}$.

Lemma 4.1 *Let $\mathcal{T}$ be an interval such that $0 \in \mathcal{T}$ and let $F(t) = F_0 + B(t) + G(t)$ such that F_0 is self-adjoint on $\tilde{\mathcal{D}} = \mathcal{D} \times \mathcal{D}$ and $B(t) + G(t)$ defined for all $t \in \mathcal{T}$ is smooth and bounded. Then, the equation*

$$i\varepsilon \partial_t T^\varepsilon(t,s) = F(t) T^\varepsilon(t,s), \;\; T^\varepsilon(s,s) = \mathrm{Id},$$

admits a unique strong solution with values in $\tilde{\mathcal{D}}$, that is C^1 in time. Moreover, the same is true for the equation

$$i\varepsilon \partial_t T^\varepsilon(t,s)^* = -T^\varepsilon(t,s)^* F^*(t), \;\; T^\varepsilon(s,s)^* = \mathrm{Id}. \tag{4.7}$$

Proof: The first statement follows from Thm X.70 in [RS], see also [Kr]: the regularity assumption in time of $F(t)$ is satisfied thanks to $\mathbf{R}_1$ so we need to show that for all fixed $t \in \mathcal{T}_0$, $F(t)$ generates a contraction semigroup on $\mathcal{H} \times \mathcal{H}$. The operator $F_0(t)$ being self-adjoint on $\tilde{\mathcal{D}}$, it generates a unitary group on $\mathcal{H} \times \mathcal{H}$. Since $G(t)$ is bounded, $F(t) = F_0(t) + G(t)$ generates a strongly continuous semigroup $S(s)_{s \geq t}$ (see Thm III.1.3 in [EN]) which satisfies $\|S(s)\| \leq \mathrm{e}^{\|G(t)\| s}$ in the operator norm of $\mathcal{H} \times \mathcal{H}$. By rescaling, $F(t) - \|G(t)\|\mathrm{Id}$, defined on $\tilde{\mathcal{D}}$, generates a contraction semigroup, so that Thm X.70 in [RS] applies and the first statement follows.

Since the existence of a strong derivative of $T(t,s)$ on $\tilde{\mathcal{D}}$ does not imply directly the same for $T(t,s)^*$, we resort to the following decomposition: we set $F(t) = F_0 + A(t)$, where $A(t) = B(t) + G(t)$ and define the bounded operator Θ^ε by

$$\Theta^\varepsilon(t,s) = e^{\frac{i}{\varepsilon} t F_0} T^\varepsilon(t,s) e^{-\frac{i}{\varepsilon} s F_0}, \;\; \text{s.t.} \;\; \Theta^\varepsilon(t,s)^{-1} = \Theta^\varepsilon(s,t).$$

It satisfies the strong differential equation on $\mathcal{H} \times \mathcal{H}$

$$i\varepsilon \partial_t \Theta^\varepsilon(t,s) = \tilde{A}^\varepsilon(t) \Theta^\varepsilon(t,s), \;\; \Theta^\varepsilon(s,s) = \mathrm{Id}, \;\; \text{with} \;\; \tilde{A}^\varepsilon(t) = e^{\frac{i}{\varepsilon} t F_0} A(t) e^{-\frac{i}{\varepsilon} t F_0}.$$

The generator $\tilde{A}^\varepsilon(t)$ is strongly continuous on $\tilde{\mathcal{D}}$ and satisfies $\|\tilde{A}^\varepsilon(t)\| = \|A(t)\|$ for all $t \in \bar{\mathcal{T}}$. Hence we can write $\Theta^\varepsilon(t,s)$ as a norm convergent Dyson series, uniformly in $t \in \bar{\mathcal{T}}$, where the integrals are understood in the strong sense

$$\Theta^\varepsilon(t,s) = \sum_{j \in \mathbb{N}} \Theta^\varepsilon_j(t,s), \;\; \Theta^\varepsilon_j(t,s) = \left(-\frac{i}{\varepsilon}\right)^j \int_s^t \int_s^{u_j} \cdots \int_s^{u_2} \tilde{A}^\varepsilon(u_j) \tilde{A}^\varepsilon(u_{j-1}) \ldots \tilde{A}^\varepsilon(u_1) du_1 \ldots du_{j-1} du_j.$$

The relation for $j \geq 1$,

$$\Theta^\varepsilon_j(t,s) = -\frac{i}{\varepsilon} \int_s^t \tilde{A}^\varepsilon(u) \Theta^\varepsilon_{j-1}(u,s) du$$

allows to prove by induction that $t \mapsto \Theta^\varepsilon_j(t,s)$ is continuous in norm and, for all $\varphi \in \mathcal{H} \times \mathcal{H}$

$$i\varepsilon \partial_t \Theta^\varepsilon_j(t,s)\varphi = \tilde{A}^\varepsilon(t) \Theta^\varepsilon_{j-1}(t,s)\varphi.$$

Hence $t \mapsto \Theta^\varepsilon_j(t,s)^*$ is norm continuous as well, and the same is true for $\Theta^\varepsilon(t,s)^* = \sum_{j \in \mathbb{N}} \Theta^\varepsilon_j(t,s)^*$. Moreover, $\Theta^\varepsilon_j(t,s)^* \psi$, for any $\psi \in \mathcal{H} \times \mathcal{H}$, satisfies for any $\varphi \in \mathcal{H} \times \mathcal{H}$

$$\begin{aligned} \langle \varphi | \Theta^\varepsilon_j(t,s)^* \psi \rangle &= \left\langle -\frac{i}{\varepsilon} \int_s^t \tilde{A}^\varepsilon(u) \Theta^\varepsilon_{j-1}(u,s) \varphi du \Big| \psi \right\rangle = \frac{i}{\varepsilon} \int_s^t \langle \tilde{A}^\varepsilon(u) \Theta^\varepsilon_{j-1}(u,s) \varphi | \psi \rangle du \\ &= \frac{i}{\varepsilon} \int_s^t \langle \varphi | \Theta^\varepsilon_{j-1}(u,s)^* \tilde{A}^\varepsilon(u)^* \psi \rangle du. \end{aligned} \tag{4.8}$$

Since $\tilde{A}^\varepsilon(t)^* = e^{itF_0/\varepsilon}A^*(t)e^{-itF_0/\varepsilon}$, where $t \mapsto A(t)$ is norm continuous, we get that $t \mapsto \tilde{A}^\varepsilon(t)^*$ is strongly continuous, see e.g. [Kr], and so is $t \mapsto \Theta^\varepsilon_{j-1}(t,s)^*\tilde{A}^\varepsilon(t)^*$. Hence we deduce from (4.8) that for any $\psi \in \mathcal{H} \times \mathcal{H}$,

$$\Theta^\varepsilon_j(t,s)^*\psi = \frac{i}{\varepsilon}\int_s^t \Theta^\varepsilon_{j-1}(u,s)^*\tilde{A}^\varepsilon(u)^*\psi du,$$

which, as above, implies for all $j \geq 1$ and all $\psi \in \mathcal{H} \times \mathcal{H}$,

$$i\varepsilon\partial_t\Theta^\varepsilon_j(t,s)^*\psi = -\Theta^\varepsilon_{j-1}(t,s)^*\tilde{A}^\varepsilon(t)^*\psi.$$

This differential identity allows then to get the key property

$$i\varepsilon\partial_t\Theta^\varepsilon(t,s)^*\psi = -\Theta^\varepsilon(t,s)^*\tilde{A}^\varepsilon(t)^*\psi,$$

which derives from the Dyson representation for $\Theta^\varepsilon(t,s)^*$. Therefore, $T^\varepsilon(t,s)^* = e^{-isF_0/\varepsilon}\Theta^\varepsilon(t,s)^*e^{itF_0/\varepsilon}$ is strongly continuously differentiable in t on $\tilde{\mathcal{D}}$, since all operators in the composition are, and (4.7) holds. □

4.4 The spectrum of $F(t)$

In order to be able to describe the spectrum of $F(t)$ in the same way as in the second statement of Lemma 3.1 for δ_0 small enough, we use that, as a consequence of the hypothesis $\mathbf{S}_2$:

$$\forall (t,x) \in \mathcal{T}_0 \times \mathcal{X}^p, \;\; \sigma(H(t,x)) \cap \sigma(-H(t,x)) = \{0\}.$$

Note that the operator $F_0(t)$ satisfies the assumptions of Theorem 4.15a in [K2], with the generalization stated in b) of Remark 4.16a. We deduce that the spectrum of $F(t)$ consists in a sequence of eigenvalues

$$\cdots < -\ell_j(t) < \cdots < -\ell_2(t) < -\ell_1(t) < 0 < \ell_1(t) < \ell_2(t) < \cdots < \ell_j(t) < \ldots, \tag{4.9}$$

where $\pm\ell_j(t)$ are simple eigenvalues, while $\ell_0(t) \equiv 0$ has multiplicity 2, with zero eigennilpotent. Each $\ell_j(t)$ corresponds to a unique eigenvalue of the unperturbed operator $F_0(t)$ determined by $H(t,[\omega(t)])$. We denote those corresponding eigenvalues of $F_0(t)$ by $\pm\lambda_j(t)$, $j \in \mathbb{N}$; recall that the labelling of the λ_js may differ from that of the eigenvalues of H. Besides, there exists a constant c such that $\forall t \in \mathcal{T}_0, \forall j \in \mathbb{Z}$,

$$|\ell_{j+1}(t) - \ell_j(t)| \geq c|j|^\alpha, \;\; \ell_j(t) = \ell_j(0) + (\ell_j(t) - \ell_j(0)), \text{ and } \sup_{j\in\mathbb{Z}}\sup_{t\in\overline{\mathcal{T}_0}}|\ell_j(t) - \ell_j(0)| < \infty. \tag{4.10}$$

Moreover,

$$\forall t \in \mathcal{T}_0, \;\; \forall j \in \mathbb{Z}, \;\; |\dot{\ell}_j(t)| \leq c, \tag{4.11}$$

which derives from the observation $F(t)\mathbb{P}_j(t) = \ell_j(t)\mathbb{P}_j(t)$: By differentiation,

$$\dot{A}(t)\mathbb{P}_j(t) + F(t)\dot{\mathbb{P}}_j(t) = \dot{\ell}_j(t)\mathbb{P}_j(t) + \ell_j(t)\dot{\mathbb{P}}_j(t)$$

whence, using $\mathbb{P}_j(t)\dot{\mathbb{P}}_j(t)\mathbb{P}_j(t) = 0$, one gets for the rank one projector $\mathbb{P}_j(t)$, $j \neq 0$,

$$\dot{\ell}_j(t)\mathbb{P}_j(t) = \mathbb{P}_j(t)\dot{B}(t)\mathbb{P}_j(t).$$

The fact that $F(t)$ is a slightly non-selfadjoint operator in the sense of Section V.5 in [K2] allows us to apply Theorem 4.16 in [K2] and Remark 4.17 following it, to get the following spectral decomposition, under our assumption $\alpha > 1/2$ in $\mathbf{S}_1$, and for δ_0 small enough:

$$F(t) = \sum_{j=-\infty}^{\infty} \ell_j(t)\mathbb{P}_j(t), \text{ with the convention } \ell_{-|j|}(t) = -\ell_{|j|}(t), \text{ where} \tag{4.12}$$

$$\mathbb{P}_j(t) = |\Psi_j(t)\rangle\langle\Phi_j(t)|, \ j \neq 0, \quad \mathbb{P}_0(t) = \sum_{\sigma=1,2} |\Psi_0^\sigma(t)\rangle\langle\Phi_0^\sigma(t)|, \tag{4.13}$$

with $\{\Psi_j(t), \Phi_j(t)\}_{j\neq 0} \cup \{\Psi_0^\sigma(t), \Phi_0^\sigma(t)\}_{\sigma\in\{1,2\}}$ a biorthogonal family of vectors, with $\|\Psi_j\| = \|\Psi_0^\sigma\| = 1$. The sum (4.12) is understood in the strong convergence sense on the time independent domain

$$\tilde{\mathcal{D}} = \{\chi = \sum_{j\in\mathbb{Z}} \alpha_j \Psi_j(0) \text{ s.t. } \sum_{j\in\mathbb{Z}} |\alpha_j \ell_j(0)|^2 < \infty\} \subset \mathcal{H}\times\mathcal{H}. \tag{4.14}$$

Indeed, Theorem 4.16 in [K2] states that the normalised basis $\{\Psi_j(t)\}_{j\in\mathbb{Z}}$ is a Riesz basis, and Theorem 3.4.5 in [D], giving a characterisation of Riesz basis, allows for the explicit description of the domain $\tilde{\mathcal{D}}$. In particular, there exist $0 < C, M < \infty$ such that for all $t \in \mathcal{T}$,

$$\big\| \sum_{j\in I} \mathbb{P}_j(t) \big\| \leq M, \quad \forall I \in \mathbb{Z}, \tag{4.15}$$

$$C^{-1}\|\chi\|^2 \leq \sum_{j\in\mathbb{Z}} \|\mathbb{P}_j(t)\chi\|^2 \leq C\|\chi\|^2, \quad \forall \chi \in \mathcal{H}\times\mathcal{H}, \tag{4.16}$$

where

$$\chi = \sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \alpha_j(t)\Psi_j(t) + \sum_{\sigma=1}^{2} \Psi_0^\sigma(t)\alpha_0^\sigma(t),$$

$$\mathbb{P}_j(t)\chi = \langle\Phi_j(t)|\chi\rangle\Psi_j(t), \quad \forall j \neq 0,$$

$$\mathbb{P}_0(t)\chi = \sum_{\sigma=1}^{2} \langle\Phi_0^\sigma(t)|\chi\rangle\Psi_0^\sigma(t).$$

Note that the domain of H_0 is

$$\mathcal{D} = \{\varphi = \sum_{k\in\mathbb{N}} \beta_k\varphi_k \text{ s.t. } \sum_{k\in\mathbb{N}} |\beta_j\lambda_k|^2 < \infty\},$$

where (λ_k, φ_k) are the eigenvectors and eigenvalues of H_0. The reader can refer to the paper [GZ], for example, in which Riesz spectral systems are studied.

4.5 The adiabatic propagator and its properties

At this stage of the proof, we can define, as in the bounded case, the dynamical phase operator $\Phi^\varepsilon(t,s)$ (see (3.9) and (3.10))

$$\Phi^\varepsilon(t,s) = \sum_{j=-\infty}^{\infty} \mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(\sigma)d\sigma}, \text{ s.t. } \Phi^\varepsilon(t,s)^{-1} = \Phi^\varepsilon(s,t), \tag{4.17}$$

which is a family of uniformly bounded operators that map $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{D}}$, thanks to (4.16). At this point, further making use of (4.10) and of the fact that $|(e^{ix}-1)/x|$ is uniformly bounded in $x \in \mathbb{R}$, one sees by a dominated convergence argument that $t \mapsto \Phi^\varepsilon(t,s)$ is also a strongly continuously differentiable two-parameter evolution operator on $\tilde{\mathcal{D}}$, where (3.12) holds.

We also define the intertwining operator $W(t)$ given by

$$i\partial_t W(t) = K(t)W(t), \;\; W(0) = \mathrm{Id}, \;\; \text{with} \;\; K(t) = i\sum_{j=-\infty}^{\infty} \dot{\mathbb{P}}_j(t)\mathbb{P}_j(t). \tag{4.18}$$

It is shown in Proposition 3.1 and Lemma 3.2 of [J3], that as soon as $\alpha > 0$, $K(t)$ is well defined, smooth, and $W(t)$ satisfies the intertwining property (3.11) with each of the projectors.

Actually, theses properties of W are shown in [J3] for orthogonal projectors $\mathbb{P}_j(t)$. However, as a routine inspection reveals, the proofs hold *mutatis mutandis* in the non selfadjoint case, provided the growing gap assumption **S** holds, and the resolvent $(F(t)-z)^{-1}$ can be bounded in an approximate way by the inverse of the distance to the spectrum. Our perturbative framework, characterised by δ_0 small ensures that this is the case.

We then introduce the bounded family of operators

$$V^\varepsilon(t,s) = W(t)\Phi^\varepsilon(t,s)W(s)^{-1}, \tag{4.19}$$

which map $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{D}}$ and satisfy $V^\varepsilon(t,s)^{-1} = V^\varepsilon(s,t)$, together with

$$V^\varepsilon(t,s)\mathbb{P}_j(s) = \mathbb{P}_j(t)V^\varepsilon(t,s) = W(t)\mathbb{P}_j(0)\mathrm{e}^{-\frac{i}{\varepsilon}\int_s^t \ell_j(\sigma)d\sigma}W(s)^{-1}.$$

The latter intertwining property implies that $W(t)$ maps $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{D}}$: From (4.12) and the definition of $\Phi_j(t)$ and $\Psi_j(t)$ in (4.13), for $j \neq 0$,

$$W(t)|\Psi_j(0)\rangle\langle\Phi_j(0)| = |\Psi_j(t)\rangle\langle\Phi_j(t)|W(t) \;\Rightarrow\; W(t)\Psi_j(0) = \Psi_j(t)\langle\Phi_j(t)|W(t)\Psi_j(0)\rangle,$$

so that we have the following property: if $\chi \in \tilde{D}$, see (4.14), with coefficient $\alpha_j = \langle\Phi_j(0)|\chi\rangle$, $j \neq 0$, in the basis at time 0, then $W(t)\chi$ has an expansion in the basis at time t with coefficients $\alpha_j(t) = \langle\Phi_j(t)|W(t)\Psi_j(0)\rangle\alpha_j$, $j \neq 0$, where $|\langle\Phi_j(t)|W(t)\Psi_j(0)\rangle|$ is uniformly bounded in $j \neq 0$, thanks to (4.15).

We now describe the adjustments requested to argue as in Section 3.3 to prove the analogue of Lemma 3.3, that is

$$T^\varepsilon(t,s) = V^\varepsilon(t,s) + O_{t,s}(\varepsilon).$$

We recall that the differential equation (3.14) has to be understood in the strong sense on $\tilde{\mathcal{D}}$, and $T^\varepsilon(t,s)$ is $\mathcal{C}^1$ on $\tilde{\mathcal{D}}$ and maps $\tilde{\mathcal{D}}$ on $\tilde{\mathcal{D}}$, according to Lemma 4.1. Analogously, $V^\varepsilon(t,s)$ satisfies (3.26) in the strong sense on $\tilde{D}$, and the same holds for $\Omega^\varepsilon(t,s)$ defined by (3.27). Then, integration by parts on the integrand of (3.27) is to be understood in the strong sense, on vectors of $\tilde{D}$. To deal with (3.29), one notes that (3.30) holds in the strong sense on $\tilde{\mathcal{D}}$, with $\tilde{F}_j(t) = \tilde{F}(t) - \ell_j(t)\mathrm{Id}$ the closed operator on $\tilde{D}$ obtained by extending the summation to $k \in \mathbb{Z}$ in (3.31). Similarly, its reduced resolvent on $(\mathrm{Id}-\mathbb{P}_j(0))\mathcal{H}\times\mathcal{H}$ simply reads $\tilde{R}_j(t) = \sum_{\substack{k\in\mathbb{Z}\\ k\neq j}} \mathbb{P}_k(0)/(\ell_k(t)-\ell_j(t))$. Note that thanks to (4.16) and the spectral behaviours (4.10) and (4.11), we have with the notation $\langle j\rangle = (1+j^2)^{1/2}$

$$\max\{\|\tilde{R}_j(t)\|, \|\partial_t\tilde{R}_j(t)\|\} \leq c\langle j\rangle^{-\alpha},$$

for some constant c uniform in $t \in \overline{\mathcal{T}}$, that may change from line to line below. Using this estimate in the integration by parts formula (3.32) we now get

$$\|\mathbb{P}_j(s)\Omega(t,s) - \mathbb{P}_j(s)\| \leq c\,\varepsilon\,\langle j\rangle^{-\alpha}|||\Omega|||, \tag{4.20}$$

where $|||\Omega||| = \sup_{(s,t)\in\overline{\mathcal{T}}} \|\Omega(t,s)\|$. Therefore, since $2\alpha > 1$,

$$\sup_{(s,t)\in\overline{\mathcal{T}}} \|\Omega(t,s) - \mathrm{Id}\|^2 \leq c\,\varepsilon^2|||\Omega|||^2,$$

from which we get, as in Section 3.3, that for $\varepsilon < \varepsilon_0$, ε_0 independent of t,

$$|||\Omega - \mathrm{Id}||| = O(\varepsilon).$$

In turn, this proves Lemma 3.3 in our current unbounded context.

Given the observations above, we also note that the arguments used in proof of Lemma 3.5 are valid in the unbounded case as well.

4.6 Conclusion of the proof

We set

$$\delta_\tau^\varepsilon = \sup_{t\in[0,\tau]} \sup_{\chi\in\mathcal{D},\,\|\chi\|=1} |\langle\chi|\Delta(t)\rangle| = \sup_{t\in[0,\tau]} \|\Delta(t)\|.$$

In particular, since $e_j \in \mathcal{D}$ for all $j \in \{1,\cdots,p\}$, we have

$$\forall t \in [0,\tau],\ \ |\Delta_j(t)| \leq \delta_\tau^\varepsilon.$$

Besides, for any family of bounded operators $C(t)$ on $\mathcal{H}\times\mathcal{H}$, for $\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \in \tilde{\mathcal{D}}$ normalized and for $0 \leq s \leq t \leq \tau$, using (3.3)

$$\left|\left\langle T^\varepsilon(t,s)^* \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| C(s)\begin{pmatrix}\Delta(t)\\ \bar\Delta(t)\end{pmatrix}\right\rangle\right| \leq \sup_{s\in[0,\tau]} \|T^\varepsilon(t,s)C(s)\|\sqrt{2}\delta_\tau^\varepsilon \equiv \theta \sup_{s\in[0,\tau]} \|C(s)\|\delta_\tau^\varepsilon.$$

We then deduce from (4.5) that there exists a constant $b > 0$ such that for any $\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \in \tilde{\mathcal{D}}$ and $0 \leq s \leq t \leq \tau$,

$$|\langle\chi_1|r^\varepsilon(s)\rangle| + |\langle\chi_2|\overline{r}^\varepsilon(s)\rangle| \leq b(\delta_\tau^\varepsilon)^2. \tag{4.21}$$

We observe that $T^\varepsilon(s,t)^* = (T^\varepsilon(t,s)^{-1})^*$ satisfies in the strong sense on $\tilde{\mathcal{D}}$

$$i\varepsilon\partial_t(T^\varepsilon(s,t)^*) = F(t)^*T^\varepsilon(s,t)^*,\ \ T^\varepsilon(s,s)^* = \mathrm{Id}.$$

In view of (4.6), for any $\chi_1,\chi_2 \in \mathcal{D}$,

$$\begin{aligned} i\varepsilon\partial_t \left\langle T^\varepsilon(s,t)^*\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\Delta(t)\\ \bar\Delta(t)\end{pmatrix}\right\rangle &= -\left\langle F(t)^*T^\varepsilon(s,t)^*\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\Delta(t)\\ \bar\Delta(t)\end{pmatrix}\right\rangle \\ &+ \left\langle F(t)^*T^\varepsilon(s,t)^*\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\Delta(t)\\ \bar\Delta(t)\end{pmatrix}\right\rangle + \left\langle T^\varepsilon(s,t)^*\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}r^\varepsilon(t)\\ -\overline{r}^\varepsilon(t)\end{pmatrix}\right\rangle - i\varepsilon\left\langle T^\varepsilon(s,t)^*\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \middle| \begin{pmatrix}\dot\omega(t)\\ \dot\omega(t)\end{pmatrix}\right\rangle, \end{aligned}$$

where the first term of the right hand side comes from the equation of $T^\varepsilon(s,t)^*$, and the second term comes from the fact that Δ satisfies the equation in the weak sense, making use of $T^\varepsilon(s,t)^* : \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$. Therefore, integrating between 0 and s, we obtain

$$\left\langle \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| \begin{pmatrix} \Delta(s) \\ \bar{\Delta}(s) \end{pmatrix} \right\rangle = \frac{1}{i\varepsilon} \int_0^s \left\langle T^\varepsilon(s,t)^* \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| \begin{pmatrix} r^\varepsilon(t) \\ -\overline{r}^\varepsilon(t) \end{pmatrix} \right\rangle dt - \int_0^s \left\langle T^\varepsilon(s,t)^* \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| \begin{pmatrix} \dot{\omega}(t) \\ \dot{\omega}(t) \end{pmatrix} \right\rangle dt.$$

Since $T^\varepsilon(s,t)^* \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathcal{D} \times \mathcal{D}$, we can use estimate (4.21) for normalised $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ and there exists $\widetilde{b}$ such that

$$\left| \int_0^s \left\langle T^\varepsilon(s,t)^* \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| \begin{pmatrix} r^\varepsilon(t) \\ -\overline{r}^\varepsilon(t) \end{pmatrix} \right\rangle dt \right| \leq \tilde{b}|s|(\delta_\varepsilon^s)^2.$$

Besides,

$$\begin{aligned} \int_0^s \left\langle T^\varepsilon(s,t)^* \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| \begin{pmatrix} \dot{\omega}(t) \\ \dot{\omega}(t) \end{pmatrix} \right\rangle dt &= \int_0^s \left\langle \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| T^\varepsilon(s,t) \begin{pmatrix} \dot{\omega}(t) \\ \dot{\omega}(t) \end{pmatrix} \right\rangle dt \\ &= \int_0^s \left\langle \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \Big| V^\varepsilon(s,t) \begin{pmatrix} \dot{\omega}(t) \\ \dot{\omega}(t) \end{pmatrix} \right\rangle dt + O_s(\varepsilon) = O_s(\varepsilon) \end{aligned}$$

by Lemma 3.3 and 3.5. Finally, by choosing $\chi_1 = \chi$, $\chi_2 = 0$, we obtain that there exists constants $a, \widetilde{b} > 0$, uniform in $0 \leq s \leq \tau$

$$|\langle \chi, \Delta(s) \rangle| \leq a\varepsilon + \frac{\tilde{b}}{\varepsilon}|s|(\delta_\varepsilon^s)^2,$$

whence

$$\delta_\varepsilon^\tau \leq a\varepsilon + \frac{b}{\varepsilon}\tau(\delta_\varepsilon^\tau)^2,$$

which allows to conclude the proof. □

5 Appendix A

According to Remark 3.6, we provide here an argument showing the spectrum of $F(t)$ is not necessarily real if $H(t,x)$ is real. We consider a smooth Hamiltonian $\mathbb{R} \times \mathbb{R}^2 \ni (t,x) \mapsto H(t,x)$ on a Hilbert space $\mathcal{H}$, and of the form

$$H(t,x) = \lambda_1(t,x)P_1(t,x) + \lambda_2(t,x)P_2(t,x) \quad \text{with} \quad \sum_{j=1}^{3} P_j(t,x) \equiv \mathrm{Id}, \tag{5.1}$$

with the assumption that the eigenvalue 0 is simple and that the $\lambda_j(t,x)$ are of arbitrary multiplicities ($j = 1, 2$). With the assumptions of Section 3.2, that means $N' = p = 2$ and, dropping the arguments $(t, [\omega(t)])$ in the variables, the Aronszajn-Weinstein determinant (3.22) takes the form

$$w(z) = \det \left(\delta_{jk} + \sum_{l=1}^{2} \frac{2\lambda_l \omega_j \langle e_j | P_l v_k \rangle}{(\lambda_l - z)(\lambda_l + z)} \right)_{1 \leq j,k \leq 2}.$$

Introducing $q_{12}(z) = (\lambda_1 - z)(\lambda_1 + z)(\lambda_2 - z)(\lambda_2 + z)$ and $q_j(z) = (\lambda_j - z)(\lambda_j + z)$, $j = 1, 2$, we have

$$w(z) = \frac{1}{q_{12}(z)} \det \left(q_{12}(z)\mathrm{Id} + 2 \begin{pmatrix} \omega_1 \langle e_1 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z)) v_1 \rangle & \omega_1 \langle e_1 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z)) v_2 \rangle \\ \omega_2 \langle e_2 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z)) v_1 \rangle & \omega_2 \langle e_2 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z)) v_2 \rangle \end{pmatrix} \right),$$

where $\bar{1} = 2$ and $\bar{2} = 1$. By assumption, all matrix elements are real-valued. If $z_0 \in \mathbb{R} \setminus \{\lambda_1, \lambda_2\}$ is a zero of $w(z)$, that is a real eigenvalue of F, that means $-q_{12}(z_0)/2$ is a real nonzero eigenvalue of the matrix

$$b(z_0) = \begin{pmatrix} \omega_1 \langle e_1 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z_0)) v_1 \rangle & \omega_1 \langle e_1 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z_0)) v_2 \rangle \\ \omega_2 \langle e_2 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z_0)) v_1 \rangle & \omega_2 \langle e_2 | (\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z_0)) v_2 \rangle \end{pmatrix} \in M_2(\mathbb{R}).$$

This requires $(\mathrm{Tr}\ b(z_0))^2 - 4 \det b(z_0) > 0$, which is not granted for a generic matrix in $M_2(\mathbb{R})$. While $b(z_0)$ is not completely arbitrary, it doesn't necessarily possess the symmetries that enforce this, as we argue below. Hence, the existence of nonzero real eigenvalues for F cannot be inferred from the sole requirement that H is real.

To be more quantitative, assume the eigenvalue λ_2 of $H(t, x)$ is independent of (t, x). Thus $\omega(t)$ is independent of λ_2 that we will consider as a large parameter. Consider t fixed and z_0 in the vicinity of $\lambda_1(t, [\omega(t)])$, assumed to be of order one. Then, for $\lambda_2 > 0$ large, we have $q_{12}(z_0) = \lambda_2^2(\lambda_1^2 - z_0^2) + O(1)$, $q_2(z_0) = \lambda_2^2 + O(1)$, $q_1(z_0) = O(1)$ so that $\sum_{l=1}^2 P_l \lambda_l q_{\bar{l}}(z_0) = P_1 \lambda_1 \lambda_2^2 + O(\lambda_2)$ and

$$b(z_0) = \lambda_1 \lambda_2^2 \begin{pmatrix} \omega_1 \langle e_1 | P_1 v_1 \rangle & \omega_1 \langle e_1 | P_1 v_2 \rangle \\ \omega_2 \langle e_2 | P_1 v_1 \rangle & \omega_2 \langle e_2 | P_1 v_2 \rangle \end{pmatrix} + O(\lambda_2). \tag{5.2}$$

The condition $(\mathrm{Tr}\ b(z_0))^2 - 4 \det b(z_0) > 0$ for λ_2 large, is equivalent to saying the z_0 independent leading order matrix in (5.2) has real eigenvalues, *i.e.* to having

$$(\omega_1 \langle e_1 | P_1 v_1 \rangle - \omega_2 \langle e_2 | P_1 v_2 \rangle)^2 + 4\omega_1 \omega_2 \langle e_1 | P_1 v_2 \rangle \langle e_2 | P_1 v_1 \rangle) > 0. \tag{5.3}$$

Recall that given $(\omega_1, \omega_2) = [\omega]$, the operators $H([\omega])$, $P_1([\omega])$ and $\partial_{x_j} H([\omega])$, $j = 1, 2$ are fixed, as is $\omega = \varphi([\omega])$. Hence, the same is true for

$$u_j = P_1([\omega]) v_j = P_1([\omega]) \partial_{x_j} H([\omega]) \omega \equiv K_j([\omega]) \omega, \quad \text{with} \quad \langle u_j | \omega \rangle = 0, \quad j = 1, 2, \tag{5.4}$$

so that (5.3) reads

$$(\omega_1 \langle e_1 | u_1 \rangle - \omega_2 \langle e_2 | u_2 \rangle)^2 + 4\omega_1 \omega_2 \langle e_1 | u_2 \rangle \langle e_2 | u_1 \rangle) > 0. \tag{5.5}$$

For generic vectors $\{e_1, e_2, \omega, u_1, u_2\}$ satisfying (5.4), the above condition needs not be true. Actually, for any real unitary operator R such that $R\omega = \omega$, we have $\omega_j = \langle \omega | e_j \rangle = \langle \omega | R e_j \rangle$, so that $\{f_1, f_2\} = \{Re_1, Re_2\}$ forms another orthonormal family defining the nonlinearity of the problem, keeping ω_j, $j = 1, 2$ fixed. It can be shown that if (5.5) holds for $\{e_1, e_2, \omega, u_1, u_2\}$, with $\dim(\mathbb{C}\omega)^\perp \geq 3$, $\omega_j \neq 0$, and $0 < |\langle u_1 | u_2 \rangle| < \|u_1\| \|u_2\|$, a real unitary R leaving ω invariant can be chosen to that (5.5) is false for $\{f_1, f_2, \omega, u_1, u_2\}$. The idea consists in discussing the restriction of R to $(\mathbb{C}\omega)^\perp$ so that the orthonormal vectors $\{f_1, f_2\}$ have scalar products with $\{u_1, u_2\}$ which make (5.5) false.

6 Appendix B

Let us look for more general solutions to (1.13) and prove Lemma 1.9. Reparametrising the time variable $t \mapsto s(t) = \int_0^t \gamma(u)du$ and writing $w(s(t)) = v(t)$ allows us to get rid of the factor $\gamma(t)$,

$$i\varepsilon\partial_s \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = |w_1|^2 \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}.$$

Writing out $w_1(s) = x(s) + iy(s), w_2(s) = z(s) + it(s)$, we get the equivalent system

$$\begin{cases} \varepsilon\dot{x} = (x^2+y^2)t \\ \varepsilon\dot{y} = -(x^2+y^2)z \\ \varepsilon\dot{z} = (x^2+y^2)y \\ \varepsilon\dot{t} = -(x^2+y^2)x \end{cases}$$

It is readily checked that the three following expressions are constants of the motion

$$x^2+t^2, \quad y^2+z^2, \quad xz+yt,$$

so that the system can be solved by quadratures. Refraining from spelling out the solution in full generality, we consider solutions corresponding to the initial conditions

$$y(0) = t(0) = 0, \quad x(0) > 0, z(0) \neq 0.$$

We get for all $s \in \mathbb{R}$ with $\alpha_\varepsilon(s) = -x(0)z(0)s/\varepsilon$

$$\begin{cases} x(s) = \frac{x(0)\cos(\alpha_\varepsilon(s))}{\left(\cos^2(\alpha_\varepsilon(s)) + \left(\frac{x(0)}{z(0)}\right)^2 \sin^2(\alpha_\varepsilon(s))\right)^{1/2}} \\ y(s) = \frac{x(0)\sin(\alpha_\varepsilon(s))}{\left(\cos^2(\alpha_\varepsilon(s)) + \left(\frac{x(0)}{z(0)}\right)^2 \sin^2(\alpha_\varepsilon(s))\right)^{1/2}} \end{cases}, \quad \begin{cases} z(s) = \frac{z(0)\cos(\alpha_\varepsilon(s))}{\left(\cos^2(\alpha_\varepsilon(s)) + \left(\frac{x(0)}{z(0)}\right)^2 \sin^2(\alpha_\varepsilon(s))\right)^{1/2}} \\ t(s) = \frac{x^2(0)\sin(\alpha_\varepsilon(s))/z(0)}{\left(\cos^2(\alpha_\varepsilon(s)) + \left(\frac{x(0)}{z(0)}\right)^2 \sin^2(\alpha_\varepsilon(s))\right)^{1/2}}. \end{cases}$$

In case $x(0) = 1 = \pm z(0)$, we recover (1.14), modulo the reparametrization of the time variable. In all other cases, noting that $\Re(w_1\overline{w_2})$ is conserved, we compute in the s variable

$$E_w(s) = 2\frac{x(0)^3 z(0)}{\cos^2(\alpha_\varepsilon(s)) + \left(\frac{x(0)}{z(0)}\right)^2 \sin^2(\alpha_\varepsilon(s))},$$

which gives the result of the Lemma 1.9 with $\aleph(t) = \varepsilon\alpha_\varepsilon(s)$, $x(0) = v_1(0)$ and $z(0) = v_2(0)$.

References

[A-SF] Abou Salem, W., Fröhlich, J., Adiabatic theorems and reversible isothermal processes. *Lett. Math. Phys.* 72 (2005), p. 153–163.

[A] Agrawal,G.P., Applications of Nonlinear Fiber Optics; Academic, SanDiego (2001)

[AE] Avron, J.E., Elgart,A., Adiabatic theorem without a gap condition, *Commun. Math. Phys.*, 203 (1999), p. 445–463.

[AFGG] Avron, J.E., Fraas, M., Graf, G.M., Grech, P., Adiabatic theorems for generators of contracting evolutions, *Commun. Math. Phys.*, 314 (2012), p. 163–191.

[AHS] Avron, J. E., Howland, J. S., Simon, B., Adiabatic theorems for dense point spectra, *Commun. Math. Phys.*, 128 (1990), p. 497–507.

[ASY] Avron, J.E., Seiler,R., Yaffe, L.G., Adiabatic theorems and applications to the quantum Hall effect, *Commun. Math. Phys.*, 110(1987), p. 33–49.

[BDR] Bachmann, S., De Roeck, W., Fraas, M., The adiabatic theorem and linear response theory for extended quantum systems, *Commun. Math. Phys.*, 361(2018), p. 997–1027.

[BQ] Biao, W., and Qian, N., Nonlinear Landau-Zener tunneling, *Phys. Rev. A*, 61 (2000), p. 023402.

[BF] Born, M., Fock, V., Beweis des Adiabatensatzes. *Z. Phys.*, 51 (1928), p. 165–180.

[CFK1] Carles, R. and Fermanian-Kammerer, C., A nonlinear adiabatic theorem for coherent states, *Nonlinearity*, 24 (2011) p. 1–22.

[CFK2] Carles, R. and Fermanian-Kammerer, C., A Nonlinear Landau-Zener formula, *J. Stat. Phys.*, 152 (2012), p. 619–656.

[D] Davies, E. B., Linear operators and their spectra, Cambridge studies in advanced mathematics 106, CUP, 2007.

[DKS] Dranov, A., Kellendonk, J. a,nd Seiler, R., Discrete time adiabatic theorems for quantum mechanical systems, *J. Math. Phys.*, 39 (1998), p. 1340–1349.

[E] Evans, L.C., Partial Differential Equations, Graduate Studies in Mathematics 10, American Mathematical Society, Providence, RI, 1998.

[EN] Engel, K.-J., Nagel, R., One-parameter semigroups for linear evolution equations, Springer, 2000.

[GG] Gang, Z. and Grech, P., Adiabatic theorem for the Gross-Pitaevskii equation, *Commun. in PDE*, 42 (2017), p. 731–756.

[GZ] Guo, B. Z., Zwart, H. J. Riesz spectral systems, Memorandum, Department of Applied Mathematics, University of Twente No. 1594 ISSN 0169-2690 (2001).

[HJ] Joye, A. and Hagedorn, G., Recent Results on Non–Adiabatic Transitions in Quantum Mechanics, *AMS Contemporary Mathematics Series*, 412 (2006) p. 183–198.

[HJPR1] Hanson, E. P., Joye, A., Pautrat, Y., Raquépas, R., Landauer's Principle in Repeated Interaction Systems *Commun. Math. Phys.*, 349 (2017), p. 285–327.

[HJPR2] Hanson, E. P., Joye, A., Pautrat, Y., Raquépas, R., Landauer's Principle for Trajectories of Repeated Interaction Systems, *Ann. H. Poincaré*, 19 (2018), p. 1939–1991.

[Je] Jensen,S.M., The nonlinear coherent coupler, *IEEE J. Quantum Electron.* **18**, 1580-1583, (1982)

[J1] Joye, A., Proof of the Landau-Zener formula, *Asymp. Analysis*, 9 (1994), p. 209–258.

[J2] Joye, A., Absence of Absolutely Continuous Spectrum of Floquet Operators, *J. Stat. Phys.*, 75 (1994), p. 929–952.

[J3] Joye, A., General Adiabatic Evolution with a Gap Condition, *Commun. Math. Phys.*, 275 (2007), p. 139–162.

[JKP] Joye, A., Kunz, H., Pfister, C.-E, Exponential Decay and Geometric Aspect of Transition Probabilities in the Adiabatic Limit, *Ann. Phys.*, 208 (1991), p. 299–332.

[JP] Joye, A., Pfister, C.-E., Exponentially Small Adiabatic Invariant for the Schrödinger Equation, *Commun. Math. Phys.* 140, (1991), p. 15–41.

[K1] Kato, T., On the Adiabatic Theorem of Quantum Mechanics, *J. Phys. Soc. Japan*, 5 (1950), p. 435–439.

[K2] Kato, T., Perturbation Theory for Linear Operators (Springer-Verlag Berlin Heidelberg New York 1980).

[Kh] Khomeriki, R, Nonlinear Landau-Zener tunneling in coupled waveguide arrays, *Phys. Rev. A*, 82 (2010), p. 013839.

[KhRu] Khomeriki, R. and Ruffo, S., Nonadiabatic Landau-Zener tunneling in waveguide arrays with a step in the refractive index, *Phys. Rev. Lett.*, 94 (2005), p. 113904.

[Kr] Krein, S.G., Linear Differential Equations in Banach Space, *Translations of Mathematical Monographs, vol. 29*, AMS, 1971.

[LeP] Le Potier, J., Lectures on Vector Bundles, CUP, 1997.

[J-L et al.] Jona-Lasinio, M., Morsch, O., Cristiani, M., Malossi, N., Müller, J. H., Courtade, E., Anderlini, M. and Arimondo, E., Asymmetric Landau-Zener tunneling in a periodic potential, *Phys. Rev. Lett.*, 91 (2003), p. 230406.

[LLFY] Liu, J., Li, S.-C., Fu, L.-B., Ye,D.-F., Nonlinear Adiabatic Evolution of Quantum Systems Springer Singapore, 2018.

[N1] Nenciu, G., On the adiabatic theorem of quantum mechanics, *J. Phys. A*, Math. Gen., 13 (1980), p. 15–18.

[N2] Nenciu, G., Linear adiabatic theory. Exponential estimates, *Commun. Math. Phys.*, 152 (1993), p. 479–496.

[RS] Reed, M., Simon, B., Methods of Modern Mathematical Physics, Academic Press, 1972.

[Sc] Schmid, J., Adiabatic theorems with and without spectral gap condition for non- semisimple spectral values. In: Exner, P., König, W., Neidhardt, H. (eds.) Mathematical Results in Quantum Mechanics: Pro- ceedings of the QMath12 Conference. World Scientific Publishing, Singapore, 2014.

[Sp] Spanier, E., Algebraic Topology, Springer, 1994.

[S] Sparber, C., Weakly Nonlinear Time-Adiabatic Theory, *Ann. H. Poincaré*, 17 (2016), p. 913–936.

[Ta] Tanaka, A., Adiabatic theorem for discrete time evolution, *J. Phys. Soc. Japan*, 80 (2011), p. 12.

[Te] Teufel, S., A note on the adiabatic theorem without gap condition, *Lett. Math. Phys.*, 58 (2001), p. 261–266.