



Nonlinearly preconditioned FETI

P. Gosselet (1,2), C. Negrello (1,3), C. Rey (4)

1 LMT – ENS Paris-Saclay/CNRS/Université Paris-Saclay


2 now at LaMcube – Univ. Lille/CNRS/Centrale Lille

3 now at CEA

4 Safran Tech

MAFELAP 2019 – London

Work supported by French ANR-14-CE07-0037 program “SEMAFOR”

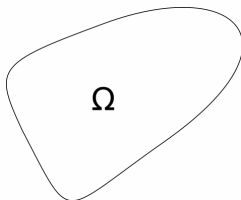
- 
- 1 Context
 - 2 Problem and notations
 - 3 Nonlinear FETI
 - 4 Nonlinearly preconditioned FETI
 - 5 Assessments

Context: domain decomposition for nonlinear problems

- (NKS) Newton solver in outer loop, linear DD (Krylov) in inner loop
 - Schwarz DD (Cai et al., 1994);
 - Schur DD (FETI) (Bhardwaj et al., 1999). . . .
 - Coupling with inexact Newton technique (Biros and Ghattas, 2003).
- (SNK) Solve the nonlinearity at the scale of the subdomains:
 - Schwarz DD: ASPIN (Xiao-Chuan Cai and Keyes, 2002), RASPEN (Dolean et al., 2016), LATIN (Ladevèze, Néron, and Gosselet, 2007), (Badea, 2009), global/local non-invasive coupling (Gendre et al., 2009; Gosselet, Blanchard, et al., 2018).
 - Schur DD, “nonlinear relocalization”: primal (Dirichlet) and mixed (Robin) approach (Cresta et al., 2007), dual approach (Pebrel, Rey, and Gosselet, 2008) and nonlinear FETIDP and BDDC (Axel Klawonn, Lanser, and Rheinbach, 2014) with large scale assessments in (A. Klawonn et al., 2017; Axel Klawonn, Lanser, Rheinbach, and Uran, 2017).

Global framework for primal/dual/mixed approaches (Negrello, Gosselet, Rey, and Pebrel, 2016);

Improvement of the impedance of the mixed approach (Negrello, Gosselet, and Rey, 2017).



Nonlinear PDE representative of quasi-static structure mechanics or thermal problem on domain Ω + Finite Element discretization (at a given time step):

$$g_{int}(u) + g_{ext} = 0$$

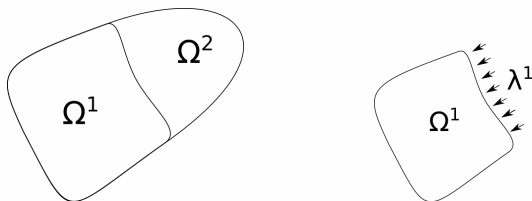
Assumptions:

- g_{int} is differentiable, tangent matrices are SPsD
- Global problem is well posed
- Classical Newton-Raphson would converge to the solution.

In the linear case $g_{int}(u) = -Ku$ where K is the stiffness matrix.

Substructuring and local equilibriums

Ω is partitioned into N_s subdomains ($\Omega^{(s)}$).



We have local equilibriums:

$$g_{int}^{(s)}(u^{(s)}) + g_{ext}^{(s)} + T^{(s)T} \lambda_b^{(s)} = 0$$

- $\lambda_b^{(s)}$ is the unknown local nodal reaction on the boundary,
- $T^{(s)T}$ is the extension-by-zero operator (transpose of the trace operator)

For a given boundary condition, all the fields involved can be computed.

$$\mathbf{g}_{int}^{(s)}(u^{(s)}) + \mathbf{g}_{ext}^{(s)} + \mathbf{T}^{(s)T} \lambda_b^{(s)} = 0$$

Dirichlet-to-Neumann operator

Given $u_b^{(s)}$, we compute $\lambda_b^{(s)} = s_{nl}^{(s)} \left(u_b^{(s)}; \mathbf{g}_{ext}^{(s)} \right)$.

- $s_{nl}^{(s)}$ is NL primal Schur complement;
- The tangent S_t can be evaluated:

$$\forall s \in \{0, \dots, N_s\}, \quad S_t^{(s)} = \frac{\partial s_{nl}^{(s)}}{\partial u_b^{(s)}} = K_{t_{bb}}^{(s)} - K_{t_{bi}}^{(s)} K_{t_{ii}}^{(s)-1} K_{t_{ib}}^{(s)}$$

- In the linear case $s_l^{(s)}$ is affine: $s_l^{(s)} \left(u_b^{(s)}; \mathbf{g}^{(s)} \right) = S_t^{(s)} u_b^{(s)} - b_p^{(s)}$
with $b_p^{(s)} = \mathbf{g}_b^{(s)} - K_{t_{bi}}^{(s)} K_{t_{bi}}^{(s)-1} \mathbf{g}_i^{(s)}$

$$\mathbf{g}_{int}^{(s)}(\mathbf{u}^{(s)}) + \mathbf{g}_{ext}^{(s)} + \mathbf{T}^{(s)T} \lambda_b^{(s)} = 0$$

Equivalent to small strain hypothesis

There exists a basis $R^{(s)}$ of rigid body motions s.t.

$$\forall \mathbf{u}^{(s)}, \alpha^{(s)}, \mathbf{g}_{int}^{(s)}(\mathbf{u}^{(s)} + R^{(s)}\alpha^{(s)}) = \mathbf{g}_{int}^{(s)}(\mathbf{u}^{(s)}), \text{ and } R^{(s)T} \mathbf{g}_{int}^{(s)}(\mathbf{u}^{(s)}) = 0$$

Compatibility condition:

$$R^{(s)T} \left(\mathbf{g}_{ext}^{(s)} + \mathbf{T}^{(s)T} \lambda_b^{(s)} \right) = 0$$

Neumann-to-Dirichlet

Given $\lambda_b^{(s)}$, we compute $\mathbf{u}_b^{(s)} = \mathbf{f}_{nl} \left(\lambda_b^{(s)}; \mathbf{g}^{(s)} \right) + \mathbf{T}^{(s)} R^{(s)} \alpha^{(s)}$.

- $\mathbf{f}_{nl}^{(s)}$ is NL dual Schur complement;
- The tangent F_t can be evaluated:

$$F_t^{(s)} = \frac{\partial \mathbf{f}_{nl}^{(s)}}{\partial \lambda_b^{(s)}} = \mathbf{T}^{(s)} K_t^{s\dagger} \mathbf{T}^{(s)T}$$

- In the linear case $\mathbf{f}_l^{(s)}$ is affine: $\mathbf{F}_l^{(s)} \left(\lambda_b^{(s)}; \mathbf{g}^{(s)} \right) = \mathbf{F}_t^{(s)} \lambda_b^{(s)} + \mathbf{b}_d^{(s)}$

Assembly and scaling operators, block notations

$$\text{vector: } \mathbf{x} = \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(N_s)} \end{pmatrix}, \text{ matrix: } \mathbf{M} = \begin{pmatrix} M^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M^{(N_s)} \end{pmatrix}$$

$$\begin{aligned} \text{assembly: } \mathbf{A} &= (A^{(1)}, \dots, A^{(N_s)}) \\ \mathbf{B} &= (B^{(1)}, \dots, B^{(N_s)}) \end{aligned}$$

\mathbf{A} is interface sum operator ($\mathbf{Ax} = x^{(1)} + x^{(2)}$),

\mathbf{B} is interface difference operator ($\mathbf{Bx} = x^{(1)} - x^{(2)}$).

Orthogonality: $\text{Range}(\mathbf{B}) = \text{Ker}(\mathbf{A}^T)$.

Scalings (Axel Klawonn and Widlund, 2001; Daniel J Rixen and Charbel Farhat, 1999):

$$\mathbf{A} \tilde{\mathbf{A}}^T = \tilde{\mathbf{A}} \mathbf{A}^T = \mathbf{I}$$

$$\mathbf{B} \tilde{\mathbf{B}}^T \mathbf{B} = \mathbf{B}, \quad \tilde{\mathbf{B}} \mathbf{B}^T \tilde{\mathbf{B}} = \tilde{\mathbf{B}}$$

Primal and dual scalings can be built such that (Gosselet, Rey, and D. Rixen, 2003):

$$\mathbf{A}^T \tilde{\mathbf{A}} + \tilde{\mathbf{B}}^T \mathbf{B} = \mathbf{I}$$

Global nonlinear interface problem

Local equilibriums + interface conditions

$$\mathbf{g}_{\text{int}}(\mathbf{u}) + \mathbf{g}_{\text{ext}} + \mathbf{T}^T \lambda_{\mathbf{b}} = 0 \quad \text{with} \quad \begin{cases} \mathbf{B} \mathbf{u}_{\mathbf{b}} = 0 \\ \mathbf{A} \lambda_{\mathbf{b}} = 0 \end{cases}$$

Primal formulation

Find $\mathbf{u}_A \in \mathbb{R}^{n_A}$ such that :

$$\mathbf{A} \mathbf{s}_{\text{nl}}(\mathbf{A}^T \mathbf{u}_A; \mathbf{g}_{\text{ext}}) = 0$$

Dual formulation

Find $\lambda_B \in \mathbb{R}^{n_B}, \alpha \in \mathbb{R}^r$ such that :

$$\mathbf{B}(\mathbf{f}_{\text{nl}}(\mathbf{B}^T \lambda_B; \mathbf{g}_{\text{ext}}) + \mathbf{TR}\alpha) = 0$$

$$\mathbf{R}^T(\mathbf{g}_{\text{ext}} + \mathbf{T}^T \mathbf{B}^T \lambda_B) = 0$$

Mixed (Robin) formulation in (Negrello, Gosselet, Rey, and Pebrel, 2016).

Find $\lambda_B \in \mathbb{R}^{n_B}$, $\alpha \in \mathbb{R}^r$ such that :

$$\mathbf{B} \left(\mathbf{f}_{nl} \left(\mathbf{B}^T \lambda_B; \mathbf{g}_{ext} \right) + \mathbf{TR} \alpha \right) = 0$$

$$\mathbf{R}^T \left(\mathbf{g}_{ext} + \mathbf{T}^T \mathbf{B}^T \lambda_B \right) = 0$$

We write $\mathbf{R}_b = \mathbf{TR}$ and $\mathbf{G}_B = \mathbf{BR}_b$.

The admissibility defines a **affine space**, we use initialisation/projection:

$$\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B \text{ with: } \begin{cases} \mathbf{P}_B = \mathbf{I} - \mathbf{Q}_B \mathbf{G}_B \left(\mathbf{G}_B^T \mathbf{Q}_B \mathbf{G}_B \right)^{-1} \mathbf{G}_B^T \\ \lambda_{B_0} = -\mathbf{Q}_B \mathbf{G}_B \left(\mathbf{G}_B^T \mathbf{Q}_B \mathbf{G}_B \right)^{-1} \mathbf{R}^T \mathbf{g}_{ext} \end{cases}$$

\mathbf{Q}_B is a SPD matrix which **does not need to be updated**.

α can be computed by minimizing the \mathbf{Q}_B -norm of the nonlinear interface condensed dual residual (=displacement jump).

Finally the system to be solved can be written as:

$$\text{Find } \tilde{\lambda}_B \in \mathbb{R}^{n_B} \text{ such that: } \rho_B(\tilde{\lambda}_B) \equiv \mathbf{P}_B^T \mathbf{B} \mathbf{f}_{nl} \left(\mathbf{B}^T (\lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B); \mathbf{g}_{ext} \right) = 0$$

Find $\tilde{\lambda}_B \in \mathbb{R}^{n_B}$ such that: $\rho_B(\tilde{\lambda}_B) \equiv \mathbf{P}_B^T \mathbf{B} \mathbf{f}_{nl}(\mathbf{B}^T(\lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B); \mathbf{g}_{ext}) = 0$

$$\begin{cases} \text{Solve for } d\tilde{\lambda}_B : \left[\frac{\partial \rho_B}{\partial \tilde{\lambda}_B}(\tilde{\lambda}_{B_k}) \right] d\tilde{\lambda}_B = -\rho_B(\tilde{\lambda}_{B_k}) \\ \text{Update } \tilde{\lambda}_{B_{k+1}} = \tilde{\lambda}_{B_k} + d\tilde{\lambda}_B \end{cases}$$

Two steps:

- Computation of the **right-hand side** $\rho_B(\tilde{\lambda}_{B_k})$: independent solutions to **nonlinear Neumann problems** per subdomains. These are computed by applying **local Newton** algorithms, and the resulting displacement are assembled to build the nonlinear projected interface residual $\rho_B(\tilde{\lambda}_{B_k})$.
- Solution to the **interface tangent system**. It has exactly the structure of a FETI system:

$$\frac{\partial \rho_B}{\partial \tilde{\lambda}_B}(\tilde{\lambda}_{B_k}) = \mathbf{P}_B^T \mathbf{B} \mathbf{F}_{t_k} \mathbf{B}^T \mathbf{P}_B$$

It can be solved with preconditioned conjugate gradients.

Classical **Dirichlet preconditioner** for the tangent system:

$$(\tilde{\mathbf{B}} \mathbf{S}_{t_k} \tilde{\mathbf{B}}^T) \mathbf{P}_B^T \mathbf{B} \mathbf{F}_{t_k} \mathbf{B}^T \mathbf{P}_B d\tilde{\lambda}_B = -(\tilde{\mathbf{B}} \mathbf{S}_{t_k} \tilde{\mathbf{B}}^T) \rho_B(\tilde{\lambda}_{B_k})$$

where \mathbf{S}_{t_k} is the primal Schur complement, $\mathbf{F}_{t_k} = \mathbf{S}_{t_k}^\dagger$.

Algorithm 1: FETI-NL

define: $\mathbf{r}_{nl}(\mathbf{u}, \lambda_b) = \mathbf{g}_{int}(\mathbf{u}) + \mathbf{g}_{ext} + \mathbf{T}^T \lambda_b$

initialization: \mathbf{u}_0 (with $\mathbf{B}\mathbf{T}\mathbf{u}_0 = 0$) and $\tilde{\lambda}_{B_0}$ given

admissibility condition: $\lambda_{B_0} = -\mathbf{Q}_B \mathbf{G}_B (\mathbf{G}_B^T \mathbf{Q}_B \mathbf{G}_B)^{-1} \mathbf{R}^T \mathbf{r}_{nl}(\mathbf{u}_0, \mathbf{P}_B \tilde{\lambda}_{B_0})$

Set $k = 0$ (Global Newton index)

while $\|\mathbf{r}_{nl}(\mathbf{u}_k, \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_{B_k})\| + \|\mathbf{P}_B^T \mathbf{B}\mathbf{T}\mathbf{u}_k\| > \varepsilon_{GN}$ **do** Global Newton

Set $j = 0$ (Local Newton index), $\mathbf{u}_{k,0} = \mathbf{u}_k$

while $\|\mathbf{r}_{nl}(\mathbf{u}_{k,j}, \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_{B_k})\| > \varepsilon_{LN}$ **do** Local Newton

$\mathbf{u}_{k,j+1} = \mathbf{u}_{k,j} + \mathbf{K}_{t_{k,j}}^\dagger \mathbf{r}_{nl}(\mathbf{u}_{k,j}, \lambda_{b_k})$

$j += 1$

end

Assemble interface displacement jump updated with rigid body motions:

$\mathbf{P}_B^T \mathbf{B}\mathbf{T}\mathbf{u}_{k,j}$

Keep tangent operator $\mathbf{K}_{t_{k,j}}$

Set $m = 0$ (Krylov index), $\mathbf{u}_{k,0} = \mathbf{u}_{k,j}$, $d\tilde{\lambda}_{B_k,0} = 0$

while $\|\mathbf{B}\mathbf{T}\mathbf{u}_{k,m}\| > \varepsilon_K$ **do** tangent FETI iteration

 Use Dirichlet preconditioner associated with $\mathbf{K}_{t_{k,j}}$

 Conjugate gradient iteration gives $d\tilde{\lambda}_{B_k,m+1}$ and $\mathbf{u}_{k,m+1}$

$m += 1$

end

Update: $\mathbf{u}_{k+1} = \mathbf{u}_{k,m}$ and $\tilde{\lambda}_{B_{k+1}} = \tilde{\lambda}_{B_k} + d\tilde{\lambda}_{B_k,m}$

$k += 1$

end

- FETI-NL adds local nonlinear solves per subdomains.
- This may lead to better approximation of the local displacement without adding communications.
- This may lead to bad load balancing (localized nonlinearities).
- Depending on the nonlinearity, local nonlinear problems may be a bad idea.
- The primal approach (BDD-NL) is easy to derive.
 - The handling of RBM is made during the preconditioning of the linear system.
 - It involves the current matrix \mathbf{S}_{t_k} instead of the constant matrix \mathbf{Q}_B .

Nonlinearly preconditioned FETI tries to intensify even more the local nonlinear computations. It may improve the convergence, but it does not solve load balancing problems.

Notations: we remove the dependence of \mathbf{f}_{nl} and \mathbf{s}_{nl} wrt \mathbf{g}_{ext} , and the current iteration number k .

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

- Local Neumann problems:

$$\mathbf{u}_b = \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

\mathbf{u}_b is not continuous ($\mathbf{B}\mathbf{u}_b \neq 0$).

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

- Local Neumann problems:

$$\mathbf{u}_b = \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

\mathbf{u}_b is not continuous ($\mathbf{B} \mathbf{u}_b \neq 0$).

- Building of a continuous displacement (with optimal α):

$$\hat{\mathbf{u}}_b \equiv [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{B}] \mathbf{u}_b = [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

- Local Neumann problems:

$$\mathbf{u}_b = \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

\mathbf{u}_b is not continuous ($\mathbf{B} \mathbf{u}_b \neq 0$).

- Building of a continuous displacement (with optimal α):

$$\hat{\mathbf{u}}_b \equiv [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{B}] \mathbf{u}_b = [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

- Local nonlinear Dirichlet problems:

$$\hat{\lambda}_b = \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$$

$\hat{\lambda}_b$ is not balanced ($\mathbf{A} \hat{\lambda}_b \neq 0$).

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

- Local Neumann problems:

$$\mathbf{u}_b = \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

\mathbf{u}_b is not continuous ($\mathbf{B}\mathbf{u}_b \neq 0$).

- Building of a continuous displacement (with optimal α):

$$\hat{\mathbf{u}}_b \equiv [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{B}] \mathbf{u}_b = [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

- Local nonlinear Dirichlet problems:

$$\hat{\lambda}_b = \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$$

$\hat{\lambda}_b$ is not balanced ($\mathbf{A}\hat{\lambda}_b \neq 0$).

- Balancing of the traction:

$$\hat{\hat{\lambda}}_b = [\mathbf{I} - \tilde{\mathbf{A}}^T \mathbf{A}] \hat{\lambda}_b = \mathbf{B}^T \tilde{\mathbf{B}} \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$$

Nonlinearly preconditioned FETI

Let $\lambda_B = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B$ be a RBM-admissible interface reaction.

- Local Neumann problems:

$$\mathbf{u}_b = \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

\mathbf{u}_b is not continuous ($\mathbf{B}\mathbf{u}_b \neq 0$).

- Building of a continuous displacement (with optimal α):

$$\hat{\mathbf{u}}_b \equiv [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{B}] \mathbf{u}_b = [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B) + \mathbf{R}_b \alpha$$

- Local nonlinear Dirichlet problems:

$$\hat{\lambda}_b = \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$$

$\hat{\lambda}_b$ is not balanced ($\mathbf{A}\hat{\lambda}_b \neq 0$).

- Balancing of the traction:

$$\hat{\hat{\lambda}}_b = [\mathbf{I} - \tilde{\mathbf{A}}^T \mathbf{A}] \hat{\lambda}_b = \mathbf{B}^T \tilde{\mathbf{B}} \mathbf{s}_{nl}(\hat{\mathbf{u}}_b)$$

In the end:

$$\hat{\hat{\lambda}}_b = \mathbf{B}^T \tilde{\mathbf{B}} \mathbf{s}_{nl}([\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_B))$$

If λ_B is the solution to the FETI-NL problem, then $\hat{\hat{\lambda}}_b = \mathbf{B}^T \lambda_B$.

New nonlinear interface condensed problem

associated with the fixed-point equation:

$$\text{Let } \mathbf{h}_{\text{nl}}(\lambda_{\mathbf{b}}) \equiv \mathbf{B}^T \tilde{\mathbf{B}} \mathbf{s}_{\text{nl}} \left(\left[\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B} \right] \mathbf{f}_{\text{nl}}(\lambda_{\mathbf{b}}) \right) - \lambda_{\mathbf{b}}$$

$$\text{Find } \tilde{\lambda}_B \text{ such that } \mathbf{h}_{\text{nl}} \left(\mathbf{B}^T \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B \right) = 0$$

Inspiration from: (Daniel J Rixen and Charbel Farhat, 1999), (Parret-Fréaud et al., 2010).

Remark: linear case

$$\begin{aligned} \mathbf{h}_l \left(\mathbf{B}^T \lambda_B \right) &= \mathbf{B}^T \tilde{\mathbf{B}} \mathbf{s}_l \left(\left[\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B} \right] \mathbf{f}_l \left(\mathbf{B}^T \lambda_B \right) \right) - \mathbf{B}^T \lambda_B \\ &= -\mathbf{B}^T \tilde{\mathbf{B}} \mathbf{S}_t \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B} \left(\mathbf{F}_t \mathbf{B}^T \lambda_B + \mathbf{b}_d \right) \end{aligned}$$

This is exactly the preconditioned FETI system.

- The spectrum is bounded from below by 1, **this is not a contraction**.
- Non-stationary iteration is mandatory (Krylov in linear, Newton in nonlinear).

Remark: other approaches

With FETI we are lucky $\tilde{\lambda}_B$ belongs to a vector space.

With BDD(C) and FETI-DP, the coarse problem bends the search space into a manifold making it more complex to manage.

Left-hand side

$$\frac{\partial \mathbf{h}_{nl}(\mathbf{B}^T(\lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B))}{\partial \tilde{\lambda}_B} = \mathbf{B}^T (\tilde{\mathbf{B}} \mathbf{S}_t [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{F}_t \mathbf{B}^T - \mathbf{I}) \mathbf{P}_B$$

⚠ \mathbf{S}_t and \mathbf{F}_t are not computed in the same configuration.
If we use modified Newton, we can simplify:

$$\frac{\partial \mathbf{h}_{nl}(\mathbf{B}^T(\lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_B))}{\partial \tilde{\lambda}_B} \simeq -\mathbf{B}^T (\tilde{\mathbf{B}} \mathbf{S}_t \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B} \mathbf{F}_t \mathbf{B}^T \mathbf{P}_B)$$

Let $\lambda_{B_k} = \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_{B_k}$, modified Newton leads to:

$$\underbrace{\mathbf{B}^T (\tilde{\mathbf{B}} \mathbf{S}_{t_k} \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B} \mathbf{F}_{t_k} \mathbf{B}^T \mathbf{P}_B)}_{-\mathbf{H}_{B, t_k}} d\tilde{\lambda}_B = \mathbf{B}^T \underbrace{[\tilde{\mathbf{B}} \mathbf{S}_{nl} ([\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{f}_{nl}(\mathbf{B}^T \lambda_{B_k})) - \lambda_{B_k}]}_{\mathbf{h}_{B, nl}(\lambda_{B_k})}$$

We can directly consider the system:

$$\mathbf{H}_{B, t_k} d\tilde{\lambda}_B = -\mathbf{h}_{B, nl}(\lambda_{B_k})$$

Newton method applied to FETI prec-NL (2)

$$\mathbf{H}_{B, t_k} d\tilde{\lambda}_B = -\mathbf{h}_{B, nl}(\lambda_{B_k})$$

- The mechanical quantity $\mathbf{B}^T \lambda_{B_{k+1}}$ is uniquely defined.
- The left-hand side is a typical linear preconditioned projected FETI operator.
- The right-hand side $\mathbf{h}_{B, nl}$ is the **composition of two nonlinear local solves**:
Neumann then assembly then Dirichlet then assembly.

The computation is more locally intensive, just one NeighborAlltoAll exchange is needed.

- The tangent system can not be solved using (projected) preconditioned conjugate gradient.

Algorithm 2: FETI-precNL

Set $k = 0$ (Global Newton index)

while $\|r_{nl}(\mathbf{u}_k, \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_{B_k})\| + \|\mathbf{P}_B^T \mathbf{B} \mathbf{T} \mathbf{u}_k\| > \varepsilon_{GN}$ **do** Global Newton

 Set $j = 0$ (Local Newton [Neumann] index), $\mathbf{u}_{k,0} = \mathbf{u}_k$

while $\|r_{nl}(\mathbf{u}_{k,j}, \lambda_{B_0} + \mathbf{P}_B \tilde{\lambda}_{B_k})\| > \varepsilon_{LN}$ **do** Local Newton [Neumann]

$\mathbf{u}_{k,j+1} = \mathbf{u}_{k,j} + \mathbf{K}_{t_{k,j}}^\dagger r_{nl}(\mathbf{u}_{k,j}, \lambda_{B_k})$
 $j += 1$

end

 Assemble interface displacement jump with updated rigid body motions $\mathbf{P}_B^T \mathbf{B} \mathbf{T} \mathbf{u}_{k,j}$

 Compute continuous displacement $\hat{\mathbf{u}}_{b_k} \equiv [\mathbf{I} - \tilde{\mathbf{B}}^T \mathbf{P}_B^T \mathbf{B}] \mathbf{T} \mathbf{u}_{k,j}$

 Set $q = 0$ (Local Newton [Dirichlet] index), $\mathbf{u}_{i_{k,0}} = \mathbf{u}_{i_{k,j}}$

while $\|r_{nl_i}(\mathbf{u}_{i_{k,q}}, \hat{\mathbf{u}}_{b_k})\| > \varepsilon_{LN}$ **do** Local Newton [Dirichlet]

$\mathbf{u}_{i_{k,q+1}} = \mathbf{u}_{i_{k,q}} + \mathbf{K}_{t_{i_{k,q}}}^{-1} r_{nl_i}(\mathbf{u}_{i_{k,q}}, \hat{\mathbf{u}}_{b_k})$
 $q += 1$

end

 Post-process associated nodal reaction $\hat{\lambda}_{b_k} = -\mathbf{T} [\mathbf{g}_{int}([\mathbf{u}_{i_{k,q}}, \hat{\mathbf{u}}_{b_k}]) + \mathbf{g}_{ext}]$

 Keep local tangent operator $\mathbf{K}_{t_{k,q}}$

 Set $m = 0$ (Krylov index), $\mathbf{u}_{k,0} = \mathbf{u}_{k,j}$, $d\tilde{\lambda}_{B_{k,0}} = 0$

while $\|\tilde{\mathbf{B}} \hat{\lambda}_{b_k} - \mathbf{H}_{B, t_k} d\tilde{\lambda}_{B_{k,m}}\| > \varepsilon_K$ **do** tangent iteration

 with \mathbf{H}_{B, t_k} computed from $\mathbf{K}_{t_{k,q}}$
 Krylov iteration gives $d\tilde{\lambda}_{B_{k,m+1}}$ and $\mathbf{u}_{k,m+1}$
 $m += 1$

end

 Update: $\mathbf{u}_{k+1} = \mathbf{u}_{k,m}$ and $\tilde{\lambda}_{B_{k+1}} = \tilde{\lambda}_{B_k} + d\tilde{\lambda}_{B_{k,m}}$

$k += 1$

end Gosselet

FETI precNL

23

- Python code: FEniCS + mpi4py
- Meshes by Gmsh
- Piecewise linear triangle FE (P1 Lagrange)
- Nonlinearity without internal variables (no Gauss points)
- Tangent operators computed by FEniCS

Bad implementation choices + departure of C. Negrello → not enough assessments

Indicators:

- Global Newton iterations (outer loop)
- Krylov iterations (which imply exchanges and reductions)
- Local Newton iterations (inner loops)

Richards equation (Water diffusion in soils)

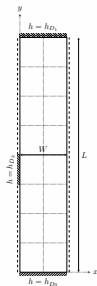
Test-case inspired from the standard Polmann case.

Find pressure field $h(x, t)$ such that:
$$\frac{\partial \theta(h)}{\partial t} - \nabla \cdot (K(h) \nabla (h - z))$$

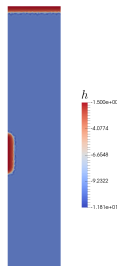
- $\theta(h) = \theta_r + (\theta_s - \theta_r) S(h)$ is the volumic water content:
- $K(h)$ is the soil hydraulic conductivity: $K(h) = K_s K_r(h)$

Van Genuchten model:

$$\begin{cases} S(h) = (1 + \varepsilon|h|^n)^{-m} \\ K_r(h) = (1 - (\varepsilon|h|)^{n-1} (1 + (\varepsilon|h|)^n)^{-m})^2 (S(h))^{1/2} \end{cases}$$



Soil column



Soil column with
water diffusion
at $t = 1000s$

Richards equation (Water diffusion in soils)

About the transient nature of the problem

Implicit Euler is used to discretize.

There is no rigid body motion.

There is no natural coarse space.

In the absence of far/fast-propagating effects, the intensification of local computations makes sense.

We do not know yet how to add extra RBM coarse space (C. Farhat and Chen, 1995; Fragakis and Papadarakakis, 2007), GENE modes or multipreconditioning (Leistner, Gosselet, and Daniel J. Rixen, 2018).

Discrete problem

5244 degrees of freedom.

16 subdomains, (333 interface degrees of freedom).

Relative precision of all solvers: $\varepsilon_{rel} = 10^{-6}$; absolute precision: $\varepsilon_{abs} = 10^{-20}$

Richards equation (Water diffusion in soils)

Reduction of global Newton iteration and Krylov iterations

# Global Newton cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
20	8	9	4	50	56
80	35	39	15	57	62
400	147	159	73	50	54
1000	311	325	179	42	45

Table: Richards equation - Global iterations

# Krylov cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
20	32	37	16	50	57
80	157	169	65	59	62
400	715	714	338	53	53
1000	1535	1535	835	46	46

Table: Richards equation - Krylov iterations

Richards equation (Water diffusion in soils)

Extra local computations

# Local Newton cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Ratios of FETI-precNL	
				over BDD-NL	over FETI-NL
20	224	249	241	1.08	0.97
80	595	714	618	1.04	0.87
400	1576	2020	1981	1.26	0.98
1000	2559	3173	3608	1.41	1.14

Table: Richards equation - Local iterations

Richards equation (Water diffusion in soils)

Comparison wrt classical Newton Krylov Schur method

# Krylov cumulated iterations			
Time	NKS	FETI-precNL	Gains of FETI-precNL (%)
20	84	16	81
100	321	89	72
200	546	189	65
400	906	338	63
700	1356	573	58
1000	1801	835	54

# Local Newton cumulated iterations			
Time	NKS	FETI-precNL	Ratio of FETI-precNL over NKS
20	22	241	10.95
100	77	769	9.99
200	127	1354	10.66
400	209	1981	9.48
700	314	2779	8.85
1000	418	3608	8.63

Table: Richards equation - Comparison with NKS method

Nonlinear thermal problem

Symmetrized stationary nonlinear thermal behavior:

$$r - \nabla \cdot (-K(T) \nabla T) = 0$$

with T the temperature, r a source of heat, and $K(T)$ the thermal conductivity.

Special (non-classical) law to ensure equivalent to “small strain hypothesis”:

$$K(T) = (1 + \nabla T \cdot \nabla T)^\alpha$$

→ there is potentially one rigid body mode per subdomain.

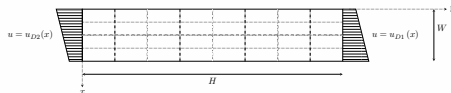


Figure: Nonlinear thermal problem



Figure: Temperature map for the nonlinear thermal problem with $\alpha = 0.6$

The global mesh is composed of 21 342 nodes,
Xe compare performance with 8/16/32 subdomains.

Nonlinear thermal problem

# Global Newton iterations															
#SD	BDD-NL			FETI-NL			FETI-precNL			Gains of FETI-precNL (%)					
	8	16	32	8	16	32	8	16	32	vs. BDD-NL			vs. FETI-NL		
$\alpha = 0$	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
$\alpha = 0.2$	6	6	7	5	6	5	3	3	4	50	50	43	40	50	20
$\alpha = 0.4$	7	6	8	7	8	9	3	3	4	57	50	50	57	63	56
$\alpha = 0.6$	7	6	8	8	7	9	4	4	5	43	33	38	50	43	44

# Krylov iterations															
#SD	BDD-NL			FETI-NL			FETI-precNL			Gains of FETI-precNL (%)					
	8	16	32	8	16	32	8	16	32	vs. BDD-NL			vs. FETI-NL		
$\alpha = 0$	8	8	14	8	8	15	8	8	15	0	0	-6	0	0	0
$\alpha = 0.2$	20	21	30	17	24	32	17	17	29	15	19	3	0	29	9
$\alpha = 0.4$	22	22	31	20	25	40	16	15	27	27	32	13	20	40	33
$\alpha = 0.6$	21	21	38	25	23	43	22	20	33	-5	5	13	12	13	23

Table: Nonlinear thermal behavior: global Newton and Krylov iterations

max # Local Newton iterations															
#SD	BDD-NL			FETI-NL			FETI-precNL			Ratio of FETI-precNL					
	8	16	32	8	16	32	8	16	32	vs. BDD-NL			vs. FETI-NL		
$\alpha = 0$	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
$\alpha = 0.2$	11	13	17	11	14	14	18	14	17	1.6	1.1	1	1.6	1	1.2
$\alpha = 0.4$	15	15	29	18	20	33	22	18	20	1.5	1.2	1.1	1.2	0.9	0.6
$\alpha = 0.6$	18	17	20	24	24	34	29	24	29	1.6	1.4	1.5	1.2	1	0.9

Table: Nonlinear thermal behavior: local Newton iterations

- New technique of nonlinear preconditioning for FETI method.
 - Formulation of the interface problem as a fixed point.
 - Use of Newton solver because stationary iteration can not work.
 - FETI natural coarse space let us work in a vector space.
 - The method doubles the intensity of local computations compared to FETI-NL.
- Assessments are too limited to draw firm conclusions.
- Problems:
 - Potentially poor load balancing.
 - For now, there are no robustness-enhancing techniques.

Paper is available on <https://hal.archives-ouvertes.fr, #hal-01774589>