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A BESICOVITCH-MORSE FUNCTION PRESERVING THE LEBESGUE MEASURE

JOZEF BOBOK AND SERGE TROUBETZKOY

ABSTRACT. We continue the investigation of which non-differentiable maps can occur in the framework of ergodic theory started in [2]. We construct a Besicovitch-Morse function map which preserves the Lebesgue measure. We also show that the set of Besicovitch functions is of first category in the set of continuous functions which preserve the Lebesgue measure.

1. INTRODUCTION

In 1925 Besicovitch constructed a continuous function, $f : [0, 1] \rightarrow [0, 1]$, for which unilateral derivatives, finite or infinite, do not exist at any point [1]. A few years later, Pepper gave a more geometric proof of the same result [5]. Saks has shown that such functions form a set of first category in the space of all continuous functions [6]. After this, Morse constructed a continuous function with a stronger conclusion [4], not only do unilateral derivatives not exist, but additionally

$$\max\{|D^+ f(t)|, |D_+ f(t)|\} = \max\{|D^- f(t)|, |D_- f(t)|\} = \infty, \quad t \in [0, 1].$$

See [3] for a more detailed historical development.

We are interested in whether such non-differentiable maps can occur in the framework of ergodic theory, more precisely whether such nowhere differentiable functions can exist for a continuous map of $[0, 1]$ which preserves the Lebesgue measure. Our main result is the existence of a Besicovitch-Morse function in the space of continuous functions preserving the Lebesgue measure (Theorem 4), improving an earlier result of Bobok who showed the existence of a Besicovitch function in this space [2]. Furthermore, in analogy to Saks' classical theorem [6], we show that the set of Besicovitch functions is of first category in the set of continuous functions which preserve the Lebesgue measure (Corollary 3). Our construction of the Besicovitch-Morse function is inspired by Pepper's construction.

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2. NOWHERE DIFFERENTIABLE MAPS IN $C(\lambda)$

Let $I := [0, 1]$. Let λ denote the Lebesgue measure on I and \mathcal{B} the Borel sets in I . Let $C(\lambda)$ consist of all continuous λ -preserving functions from I onto I , i.e.,

$$C(\lambda) = \{f: I \rightarrow I: \forall A \in \mathcal{B}, \lambda(A) = \lambda(f^{-1}(A))\}.$$

We define the upper, lower, left and right *Dini derivatives* of f at t :

$$\begin{aligned} D^+ f(t) &:= \limsup_{\substack{x \rightarrow t^+ \\ x \in I}} \frac{f(x) - f(t)}{x - t} & D_+ f(t) &:= \liminf_{\substack{x \rightarrow t^+ \\ x \in I}} \frac{f(x) - f(t)}{x - t} \\ D^- f(t) &:= \limsup_{\substack{x \rightarrow t^- \\ x \in I}} \frac{f(x) - f(t)}{x - t} & D_- f(t) &:= \liminf_{\substack{x \rightarrow t^- \\ x \in I}} \frac{f(x) - f(t)}{x - t}. \end{aligned}$$

We say that a finite one sided derivative exists at t if $D^+ f(t) = D_+ f(t) \in \mathbb{R}$ or $D^- f(t) = D_- f(t) \in \mathbb{R}$, and that a finite or infinite one sided derivative exists at t if $D^+ f(t) = D_+ f(t) \in \mathbb{R} \cup \{\pm\infty\}$ or $D^- f(t) = D_- f(t) \in \mathbb{R} \cup \{\pm\infty\}$. We introduce the following classes of continuous nowhere differentiable functions

A *Besicovitch function* is an $f \in \mathcal{C}(I, \mathbb{R})$ such that for every $t \in I$, there is neither a finite or infinite right nor a finite or infinite left derivative at t .

A *Morse functions*, is an $f \in \mathcal{C}(I, \mathbb{R})$ such that

$$\max\{|D^+ f(t)|, |D_+ f(t)|\} = \max\{|D^- f(t)|, |D_- f(t)|\} = \infty, \quad t \in I;$$

we skip the left, resp. right term of the $\max\{\}$ if t is the right, resp. left endpoint of the interval I .

We endow $C(\lambda)$ with the uniform metric $\rho(f, g) := \sup_{x \in I} |f(x) - g(x)|$.

Proposition 1. *$C(\lambda)$, endowed by the uniform metric ρ , is a complete metric space.*

We leave the standard proof of this result to the reader.

Recall that a *knot point* of function f is a point x where $D^+ f(x) = D^- f(x) = \infty$ and $D_+ f(x) = D_- f(x) = -\infty$. The following theorem states a consequence of more general result proved in [2].

Theorem 2. *The $C(\lambda)$ -typical function has a knot point at λ -almost every point.*

The next result generalizes a classical result result of Saks [6].

Corollary 3. *The set of Besicovitch functions is a meager set in $C(\lambda)$.*

Proof. We use the following well known result (see [7, Theorem 7.3]): *if $D^+f(x) \geq 0$ for a.e. $x \in I$ and $D^+f(x) > -\infty$ for every $x \in I$, then f is non-decreasing.*

By Theorem 2 there is a residual set $K \subset C(\lambda)$ such that each element of K has a knot point at λ almost every point of I . Fix $f \in K$, we have $D^+f(x) = +\infty \geq 0$ a.e., and f can not be non-decreasing. Applying the above result, we conclude that $D^+(x_0) = -\infty$ for at least one point $x_0 \in I$; in particular f is not a Besicovitch function. \square

Now we state our main result.

Theorem 4. *There is a Besicovitch-Morse function in $C(\lambda)$.*

Proof. We begin by a sketch of our construction. The first step is to construct an irregular Cantor staircase $f_0 : [0, 1/2] \rightarrow \mathbb{R}$ then to extend by symmetry to a tent-like devils' staircase map (see Figure 1). Next we modify this map by replacing each flat segment by an affinely rescaled copy of f_0 pointing downwards, producing the map f_1 . At each stage we will have a modify the resulting map by replacing the flat segments by affinely rescaled copies of the original map, the scaling becoming more skewed at each step, and the direction alternates between tent maps pointing up and down.

Given a σ positive integer we construct a discontinuum $E_\sigma \subset [0, \frac{1}{2}]$:

$$E_\sigma = \left[0, \frac{1}{2}\right] \setminus L_\sigma, \quad \text{where } L_\sigma = \bigcup_{m=1}^{\infty} \bigcup_{p=1}^{2^{m-1}} r_{m,p},$$

the open intervals $r_{m,p} = (a_{m,p}, b_{m,p})$ are chosen as follows:

$$(m = 1) \quad d_{1,1} = \left[0, \frac{1}{2}\right], \quad r_{1,1} \subset d_{1,1},$$

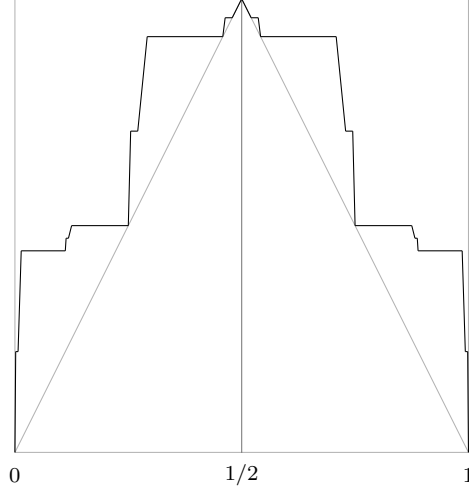
$$(K_{1,\sigma}) \quad b_{1,1} \text{ is the center of } d_{1,1}, \quad \frac{\lambda(r_{1,1})}{\lambda(d_{1,1})} = \frac{1}{2} - \frac{1}{2^{1+\sigma}};$$

($m > 1$, m even), if $d_{m,1} \cdots d_{m,2^{m-1}}$ are (from left to right) the intervals of the set $[0, \frac{1}{2}] \setminus \bigcup_{q=1}^{m-1} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$, then $r_{m,p} \subset d_{m,p}$, and for a suitable increasing sequence $(k_\sigma(m))_{m \geq 2, \text{ even}}$ of positive integers (to be determined later)

$$(K_m^{\text{even}}) \quad a_{m,p} = \min d_{m,p} + \frac{\lambda(d_{m,p})}{2^{k_\sigma(m)}}, \quad b_{m,p} = \max d_{m,p} - \frac{\lambda(d_{m,p})}{2^{k_\sigma(m)}};$$

($m > 1$, m odd), if $d_{m,1} \cdots d_{m,2^{m-1}}$ are (from left to right) the intervals of the set $[0, \frac{1}{2}] \setminus \bigcup_{q=1}^{m-1} \bigcup_{p=1}^{2^{q-1}} r_{q,p}$, then $r_{m,p} \subset d_{m,p}$,

$$(K_{m,\sigma}^{\text{odd}}) \quad b_{m,p} \text{ is the center of } d_{m,p} \text{ (we refer to this as the } \textit{center property}), \\ \frac{\lambda(r_{m,p})}{\lambda(d_{m,p})} = \frac{1}{2} - \frac{1}{2^{m+\sigma}}.$$

FIGURE 1. The map $f_{0,\sigma}$

Given a map $f : I \rightarrow I$ and $x, y \in I$, $x \neq y$, define

$$R(f, x, y) := \frac{f(x) - f(y)}{x - y}.$$

We consider a continuous nondecreasing function $f_{0,\sigma} : [0, 1/2] \rightarrow I$ satisfying $f_{0,\sigma}(0) = 0$, $f_{0,\sigma}(1/2) = 1$, $f_{0,\sigma}$ constant on every interval $r_{m,p}$ and satisfying for each $d_{m,p} := [c, d]$, $b_{m,p}$

$$(1) \quad R(f_{0,\sigma}, b_{m,p}, c) = R(f_{0,\sigma}, d, c).$$

Notice that this number is at least 2 for every $d_{m,p}$. The function $f_{0,\sigma}$ is a Cantor steplike function.

Next let

$$k_\sigma(m) := 1 + \log_2 \left(1/2 + \max_p \frac{\lambda(f_{0,\sigma}(d_{m,p}))}{\lambda(d_{m,p})} \right);$$

this definition implies that for all even m and $d_{m,p} = [c, d]$ we have

$$(2) \quad k_\sigma(m) > 2 \quad \text{and} \quad 0 < R(f_{0,\sigma}, d, a_{m,p}) \leq 1/2.$$

We extend $f_{0,\sigma}$ to the interval I by setting

$$f_{0,\sigma}(x) := f_{0,\sigma}(1 - x), \quad x \in [1/2, 1].$$

The function $f_{0,\sigma}$ and the interval I form the *basic* ($\sigma = 1$)-*step triangle* of our construction. The set $\{(x, f_{0,\sigma}(x)); x \in [0, 1/2]\}$ is the left side of the triangle, analogously the set $\{(x, f_{0,\sigma}(x)); x \in [1/2, 1]\}$ is the right side of the triangle. Now, we construct the desired function f as follows:

(c_0) Start with the basic ($\sigma = 1$)-step triangle with the base I and height 1; the sides of the basic ($\sigma = 1$)-step triangle are the graph of f_0 (see Figure

1). All E_1 -contiguous intervals, i.e., the holes in the Cantor set E_1 , and their counterparts in $[1/2, 1]$ will be called *0th L-segments* - the set of all 0th *L-segments* will be denoted by \mathcal{L}_0 .

(c_1) The flat segment corresponding to an interval $(a, b) \subset r \in \mathcal{L}_0$ (and f_0) is the set

$$\{(x, f_0(a)) : x \in [a, b]\};$$

m odd: for every element $r_{m,p}$ of \mathcal{L}_0 construct affinely rescaled ($\sigma = 1+m$)-step triangle whose base is the flat segment corresponding to $r_{m,p}$;

m even: for every element $r_{m,p} = (a, b)$ of \mathcal{L}_0 construct two affinely rescaled ($\sigma = 1+m$)-step triangles whose bases are the flat segments corresponding to

$$(3) \quad \left(a, b - \frac{1}{2^{k_1(m)} - 2}(b-a)\right), \left(b - \frac{1}{2^{k_1(m)} - 2}(b-a), b\right);$$

constructed step triangles are placed inwards the basic step triangle, the height of step triangle with the base corresponding to $r_{m,p} = (a, b) \subset d_{m,p} = [c, d]$ is equal to

$$(4) \quad f_0(a) - f_0(c).$$

The union of sides of all so far constructed step triangles defines the function f_1 . All new contiguous intervals (subintervals of some previous 0th *L-segments*) will be called *1st L-segments* - the set of all 1st *L-segments* will be denoted by \mathcal{L}_1 .

(c_n) Consider an element $r_{m_{n-1},\cdot}^{n-1} = (a, b)$ from \mathcal{L}_{n-1} satisfying

$$r_{m_0,\cdot}^0 \supset r_{m_1,\cdot}^1 \supset \dots \supset r_{m_{n-1},\cdot}^{n-1}, \quad r_{m_i,\cdot}^i \in \mathcal{L}_i;$$

m_{n-1} odd: construct affinely rescaled ($\sigma = 1 + m_0 + \dots + m_{n-1}$)-step triangle whose base is the flat segment corresponding to $r_{m_{n-1},\cdot}^{n-1}$ and f_{n-1} ;

m_{n-1} even: construct two affinely rescaled ($\sigma = 1 + m_0 + \dots + m_{n-1}$)-step triangles whose bases are the flat segments corresponding to f_{n-1} and (3) with $k_{1+m_0+\dots+m_{n-2}}(m_{n-1})$; the constructed step triangles are placed inwards the bigger step triangle on whose side has its base, the height of the step triangle corresponding to $r_{m_{n-1},\cdot}^{n-1} = (a, b) \subset d_{m_{n-1},\cdot}^{n-1} = [c, d]$ is equal to

$$|f_{n-1}(a) - f_{n-1}(c)|.$$

Realizing the construction described above for all elements from \mathcal{L}_{n-1} , the union of sides of all so far constructed step triangles define the function f_n . All new contiguous intervals (subintervals of some previous $(n-1)$ st *L-segments*)

will be called *n*th *L*-segments - the set of all *n*th *L*-segments will be denoted by \mathcal{L}_n .

Finally, put $f = \lim_{n \rightarrow \infty} f_n$ (obviously $\rho(f_{n-1}, f_n) \leq \frac{1}{2^{n/2}}$). In what follows we will repeatedly use the following easy consequence of our construction:

$$(5) \quad \forall n \in \mathbb{N} \cup \{0\} \quad \forall x \in [0, 1] \setminus \bigcup_{L \in \mathcal{L}_n} : f(x) = f_n(x).$$

In order to verify that the function f is a Besicovitch-Morse function we distinguish several cases.

I. First, we assume that $x \in [0, 1]$ is not a point of any 0th *L*-segment. Because of symmetry we only consider points from $[0, 1/2]$.

I(+) Assume that $x \in [0, 1/2)$ is not the left endpoint of any 0th *L*-segment. We show that $f'_+(x)$ does not exist and at least one of the right Dini derivatives of f at x is infinite.

Fix $h > 0$ arbitrarily small, then there is an odd m such that for some p , the 0th *L*-segment $r_{m,p} = (a, b) \subset d_{m,p} = [c, d]$ is contained in $(x, x + h)$. We choose p so that it is the left most such segment, thus any 0th *L*-segment $r_{m',p'}$ between x and a satisfies $m' > m$. If $x < c$, then since x is not a point of any 0th *L*-segment, we would have another *L*-segment $r_{m',p'}$ between x and a with $m' \leq m$, thus

$$c \leq x < a < \frac{a+b}{2}.$$

Since by (5) $f(t) = f_0(t)$ for $t \in \{c, x, a\}$, since m is odd we obtain from (4)

$$(6) \quad f\left(\frac{a+b}{2}\right) = f(c).$$

Furthermore, since f_0 is monotone on E , again using (5)

$$(7) \quad f(c) \leq f(x) < f(a).$$

The number h was chosen arbitrarily small, so (6) and (7) imply

$$D_+f(x) \leq 0 \leq D^+f(x).$$

Let us evaluate $\max\{|D^+f(x)|, |D_+f(x)|\}$. Define $\varepsilon_m \in (0, 1]$ by

$$(8) \quad f(a) - f(x) = \varepsilon_m[f(a) - f(c)]$$

Using the fact that m is odd, and b is in the middle of $[c, d]$, ($K_{m,1}^{\text{odd}}$) implies that $a - c = \lambda(d_{m,p})/2^{m+1}$, thus from (8) we have

$$(9) \quad \frac{f(a) - f(x)}{a - x} = \frac{\varepsilon_m[f(a) - f(c)]}{a - x} \geq \frac{\varepsilon_m[f(a) - f(c)]}{a - c} = \frac{\varepsilon_m[f(a) - f(c)]}{\frac{\lambda(d_{m,p})}{2^{m+1}}}.$$

Furthermore combining (1) with the fact that b is in the middle of $[c, d]$, and then the fact that $R(f_0, d, c)$ is at least 2, yields

$$(10) \quad \frac{\varepsilon_m [f(a) - f(c)]}{\frac{\lambda(d_{m,p})}{2^{m+1}}} = \frac{\varepsilon_m \frac{\lambda(f_0(d_{m,p}))}{2}}{\frac{\lambda(d_{m,p})}{2^{m+1}}} \geq \frac{\varepsilon_m \frac{2\lambda(d_{m,p})}{2}}{\frac{\lambda(d_{m,p})}{2^{m+1}}} = \varepsilon_m 2^{m+1}.$$

So if $\varepsilon_m \geq \varepsilon$ for some positive ε and a sequence of odd m 's, we immediately have $D^+ f(x) = \infty$.

To the contrary assume that the sequence $(\varepsilon_m)_{m\text{-odd}}$ converges to zero. In this case we will show $D_+ f(x) = -\infty$. Let us denote $d_{1,1}^{(1)} := [c^{(1)}, d^{(1)}] := [a, \frac{a+b}{2}]$ and $r_{1,1}^{(1)} = (a^{(1)}, b^{(1)}) \in \mathcal{L}_1$, $r_{1,1}^{(1)} \subset d_{1,1}^{(1)}$. Using (c_1) for the intervals $d_{1,1}^{(1)}, r_{1,1}^{(1)}$, and the center property we obtain

$$a^{(1)} - a = \frac{\lambda(d_{1,1}^{(1)})}{2} - \lambda(r_{1,1}^{(1)}) = \frac{\lambda(d_{1,1}^{(1)})}{2^{1+m}}.$$

Using $(K_{m,1}^{\text{odd}})$ and the inequalities $\lambda(d_{1,1}^{(1)}) = \frac{\lambda(r_{m,p})}{2} < \frac{\lambda(d_{m,p})}{2^2}$ (which both follow from the center property) we obtain

$$\frac{\lambda(d_{1,1}^{(1)})}{2^{1+m}} < \frac{1}{2^2} \frac{\lambda(d_{m,p})}{2^{m+1}} = \frac{a-c}{2^2}.$$

Thus

$$a^{(1)} - x = a^{(1)} - a + a - x < \frac{a-c}{2^2} + a - c < 2(a-c).$$

By construction, the center property implies $f(a^{(1)}) = (f(a) + f(c))/2$, hence from (8) we obtain

$$-R(f, x, a^{(1)}) = \frac{(1/2 - \varepsilon_m)[f(a) - f(c)]}{a^{(1)} - x} > (1/2 - \varepsilon_m) \frac{R(f, a, c)}{2}$$

But, (9) and (10) imply $R(f, a, c) \geq 2^m$, and we conclude

$$(11) \quad -R(f, x, a^{(1)}) \geq -(1/2 - \varepsilon_m) \cdot 2^{m-1}.$$

By our assumption ε_m converge to zero, so (11) implies

$$-D_+ f(x) = \infty = \max\{|D^+ f(x)|, |D_+ f(x)|\}.$$

We have already seen that $D_+ f(x) \leq 0 \leq D^+ f(x)$, i.e., $f'_+(x)$ does not exist.

I(-) Assume that $x \in (0, 1/2]$ is not the right endpoint of any 0th L -segment. We show that $f'_-(x)$ does not exist and $D^- f(x) = \infty$.

Fix $h > 0$ arbitrarily small, let $r_{m,p} = (a, b) \subset d_{m,p} = [c, d]$ be the 0th L -segment contained in $(x-h, x)$. W.l.o.g. we can assume that m is even and that any 0th L -segment between x and b is labeled by an $m' > m$. Then

$$b < x \leq d$$

and similarly as in (6) and (7), denoting by $e := e(m) := b - \frac{b-a}{2^{k_1(m)+1}-2}$ the middle of the interval $(b - \frac{1}{2^{k_1(m)-2}}(b-a), b)$, from (3)

$$f(e) = f(c) < f(b) < f(x) \leq f(d).$$

Using (K_m^{even}) , (2) and (5) we obtain for each even m

$$0 < R(f, x, a) \leq \frac{f(d) - f(a)}{x - a} = R(f, d, a) \frac{d - a}{x - a} < 1/2 \frac{d - a}{x - a}$$

But for even m we have

$$\frac{d - a}{x - a} < \frac{d - a}{b - a} = \frac{2^{k_1(m)} - 1}{2^{k_1(m)} - 2},$$

thus

$$(12) \quad 0 < R(f, x, a) < 1.$$

Using $f(e) = f(c)$, $f(x) \geq f(b)$ and the definition of $k_1(m)$ yields the first inequality

$$(13) \quad R(f, x, e) \geq \frac{f(b) - f(c)}{2 \frac{d-c}{2^{k_1(m)}}} = \frac{(2^{k_1(m)} - 1)2^{k_1(m)-1}}{2^{k_1(m)}} R(f, d, c) > 2^{k_1(m)} - 2;$$

while the equality follows from (1) and the last inequality follows from the fact that by our construction $\frac{\lambda(f(d_{m,p}))}{\lambda(d_{m,p})} \geq 2$ for each even m and each p .

When h approaches 0, the integer m tends to $+\infty$ and thus, (12) and (13) imply $D_- f(x) \leq 1 < D^- f(x) = \infty$.

II. Second, we assume that for some positive integer n , $x \in I$ is a point of some $n - 1$ st L -segment and does not belong to any n th L segment. Then the point $(x, f(x))$ lies on the side of a step triangle which is an affinely rescaled version of the basic $(\sigma = 1 + m_0 + \dots + m_{n-1})$ -step triangle; the facts that neither finite nor infinite $f'_+(x)$, $f'_-(x)$ exist and

$$\max\{|D^+ f(x)|, |D_+ f(x)|\} = \max\{|D^- f(x)|, |D_- f(x)|\} = \infty$$

can be proven analogously as in **I**.

III. Finally suppose that $x \in I$ belong to L -segments of all orders, i.e., $\{x\} = \bigcap_{n=1}^{\infty} s_{m_n, p_n}$, where $s_{m_n, p_n} = (a_n, b_n)$ equals to r_{m_n, p_n} for m_n odd, resp. $s_{m_n, p_n} \subset r_{m_n, p_n}$ for m_n even and r_{m_n, p_n} denotes the $(n - 1)$ st L -segment the point x belongs to. The function f_n and the flat segment on the graph of f_{n-1} corresponding to the interval (a_n, b_n) form a rescaled step triangle Δ_n which we use to estimate the derivatives $D^+ f$, $D_+ f$, $D^- f$, $D_- f$ at the point x . From the construction it follows that

- i) Δ_n is oriented upwards for n even, and downwards for n odd and $f((a_n, b_n)) \supset f((a_{n+1}, b_{n+1}))$ for each n .

Denote ℓ_n , resp. h_n the length of the base, resp. height of Δ_n . In our construction at each step on the flat segment of f_{n-1} we build a “tent” consisting of two sides of Δ_n in such a way that

$$R(f, a_n, \gamma_n) > 2R(f, a_{n-1}, \gamma_{n-1}) \text{ for each } n,$$

(here $\gamma_n := a_n + (b_n - a_n)/2$), hence

$$\text{ii) } \lim_{n \rightarrow \infty} h_n/\ell_n = \infty.$$

It follows from i) that $D^+(x) \geq 0 \geq D_+(x)$ and $D^-(x) \geq 0 \geq D_-(x)$. Thus if $f'_+(x)$ exists then it equals 0, but this is impossible if $f \in C(\lambda)$ (and similarly for $f'_-(x)$). We will prove that $f \in C(\lambda)$ below, and thus f can not be a Besicovitch function.

The set $\{(t, f_n(t)); t \in (a_n, \gamma_n)\}$ is the left side of Δ_n . By symmetry, we can suppose without loss of generality, that the point x corresponds to the left side of the step triangles for infinitely many n .

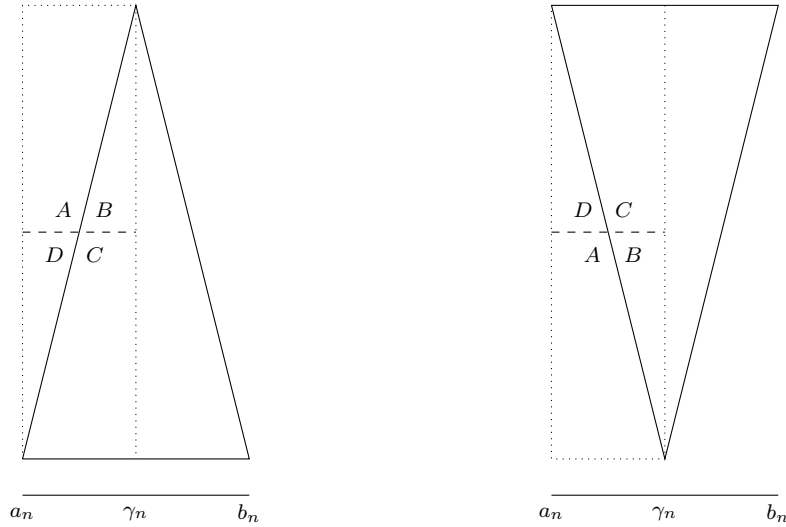


FIGURE 2. Δ_n oriented upwards (left) and downwards (right)

Consider Figure 2, the horizontal dotted line cuts the step triangle Δ_n in the middle (in height). The point $(x, f(x))$ must be in one of the sets A, B, C or D . For convenience we take A, B open, and C, D closed.

III₁. From ii) we deduce that if for infinitely many n

- (1) the step triangle Δ_n is oriented upwards, resp. downwards and $(x, f(x)) \in A \cup B$, then $D_+f(x) = -\infty$ and $D^-f(x) = \infty$, resp. $D^+f(x) = \infty$ and $D_-f(x) = -\infty$,
- (2) the step triangle Δ_n is oriented upwards, resp. downwards and $(x, f(x)) \in C \cup D$, then $D^+f(x) = \infty$, resp. $D_+f(x) = -\infty$.

III₂. The argument for $D^-f(x)$ and $D_-f(x)$ when $(x, f(x)) \in C \cup D$ is more complicated. First of all notice that the above argument works without change if we replace the dashed line in the middle of the figure with a line at any fixed percentage of the height. Thus the remain case is when this percentage tends to zero.

Denote the percentage of the height of Δ_n corresponding to the position of $(x, f(x))$ by $\alpha_n = \frac{|f(x)-f(a_n)|}{h_n} \in (0, 1)$. Our assumption is

$$\lim_{n \rightarrow \infty} \alpha_n = 0.$$

Let n be even and sufficiently large to satisfy $\alpha_m \in (0, 1/2)$ for each $m \geq n$. By our construction and the definition of α_n, α_{n+1} (see Figure 3)

$$(14) \quad f(a_{n+1}) - \alpha_{n+1}h_{n+1} = f(a_n) + \alpha_n h_n, \quad \alpha_n h_n \geq (1 - \alpha_{n+1})h_{n+1}.$$

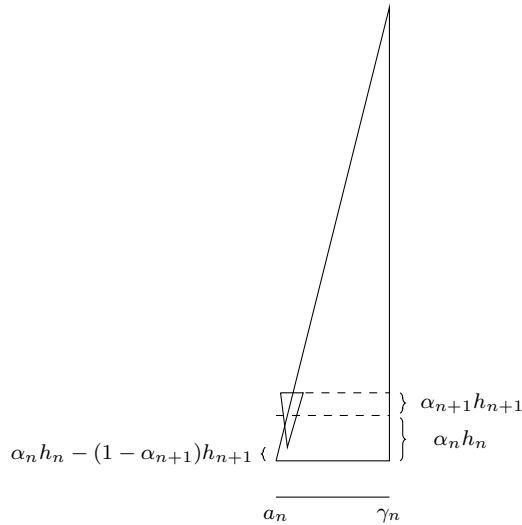


FIGURE 3. The upper left endpoint of the small triangle has coordinates $(a_{n+1}, f(a_{n+1}))$.

Assume that

$$f(a_{n+1}) > f(a_n) + 2\alpha_n h_n;$$

using (14) we get

$$(15) \quad f(a_{n+1}) - f(a_n) = \alpha_{n+1}h_{n+1} + \alpha_n h_n > 2\alpha_n h_n$$

hence again from (14)

$$\alpha_{n+1}h_{n+1} > \alpha_n h_n \geq (1 - \alpha_{n+1})h_{n+1}$$

and $\alpha_{n+1} \geq 1/2$, a contradiction with our choice of α_{n+1} . It shows that

$$f(a_{n+1}) \leq f(a_n) + 2\alpha_n h_n.$$

Choose $\kappa(n) \in \mathbb{N}$ such that $\alpha_n \in [1/2^{\kappa(n)+2}, 1/2^{\kappa(n)+1}]$; this implies

$$(16) \quad f(a_{n+1}) \leq f(a_n) + \frac{h_n}{2^{\kappa(n)}}.$$

In our basic construction for each fixed m there are finitely many 0th L -segments $r_{m,p}^{(0)}$ and (see Figure 1)

$$f_0(0) + \frac{1}{2^m} = \frac{1}{2^m} \leq f_0(r_{m,1}^{(0)}) < f_0(r_{m,2}^{(0)}) < \cdots < f_0(r_{m,2^{m-1}}^{(0)}),$$

hence analogously for each n , for f_n on (a_n, b_n) and for each fixed m we have

$$(17) \quad f_n(a_n) + \frac{h_n}{2^m} \leq f_n(r_{m,1}^{(n)}) < f_n(r_{m,2}^{(n)}) < \cdots < f_n(r_{m,2^{m-1}}^{(n)}),$$

where $r_{m,p}^{(n)}$ are n th L -segments corresponding to the flat segments on the left side of Δ_n .

Since $f(a_n) = f_n(a_n)$ and $f(a_{n+1}) = f_n(a_{n+1}) = f_n(s_{m_{n+1}, p_{n+1}}) \subset f_n(r_{m_{n+1}, p_{n+1}}^{(n)})$, choosing $m = m_{n+1}$ in (17) and combining with (16) yields

$$(18) \quad m_{n+1} \geq \kappa(n).$$

This enables us to estimate the length ℓ_{n+1} . By our construction, for each $m = 2j$ or $m = 2j + 1$, the leftmost segment $r_{m,1}^{(0)}$ of the 0th category satisfies

$$(19) \quad \lambda(r_{2^{j+1},1}^{(0)}) \leq \frac{1}{2^{2+j^2+j+\sum_{i=1}^j k(2i)}} \leq \frac{1}{2^{2j+1}}, \quad j \geq 0,$$

$$(20) \quad \lambda(r_{2^j,1}^{(0)}) \leq \frac{1}{2^{1+j^2+j+\sum_{i=1}^{j-1} k(2i)}} \leq \frac{1}{2^{2j}}, \quad j \geq 1.$$

Moreover, all segments from \mathcal{L}_0 placed to the left of $r_{m,1}^{(0)}$ are shorter than $r_{m,1}^{(0)}$. Since Δ_n and Δ_{n+1} are affinely rescaled version of the basic σ -step triangle with large σ , we conclude from (18), (19) and (20) that

$$(21) \quad \lambda(s_{m_{n+1}, p_{n+1}}) = \ell_{n+1} \leq \lambda(r_{\kappa(n),1}^{(n)}) \leq \frac{\ell_n}{2^{\kappa(n)}}.$$

With the help of (15) we can write

$$(22) \quad R(f, x, a_n) \geq \frac{f(a_{n+1}) - \alpha_{n+1}h_{n+1} - f(a_n)}{\frac{a_{n+1}+b_{n+1}}{2} - a_n} = \frac{\alpha_n h_n}{\frac{b_{n+1}-a_{n+1}}{2} + a_{n+1} - a_n} = \\ = \frac{\alpha_n h_n}{\frac{\ell_{n+1}}{2} + a_{n+1} - a_n} = \frac{h_n}{\ell_n} \cdot \frac{\ell_n}{\ell_{n+1}} \cdot \frac{\alpha_n}{\frac{1}{2} + \frac{a_{n+1}-a_n}{\ell_{n+1}}}.$$

III₂^b. If $\frac{a_{n+1}-a_n}{\ell_{n+1}}$ is bounded, since $a_{n+1} > a_n$ we conclude that $\frac{1}{\frac{1}{2} + \frac{a_{n+1}-a_n}{\ell_{n+1}}}$ is positive and bounded by a constant C . Thus using (21) and (22), ii) and the

definition of $\kappa(n)$ yields

$$R(f, x, a_n) \geq \frac{h_n 2^{\kappa(n)}}{\ell_n} \alpha_n C \geq C \frac{h_n}{4\ell_n} \rightarrow \infty.$$

We conclude that $D^-f(x) = +\infty$ and finishes the proof of the fact that f is Besicovitch-Morse when this ratio is bounded.

III₂^a. Finally assume that the ratio $\frac{a_{n+1}-a_n}{\ell_{n+1}}$ is not bounded. If the liminf of these ratios over n even is finite, we can use the corresponding subsequence and conclude in the same way. Thus we can assume that the limit over even n is infinite. Then

$$\begin{aligned} R(f, x, a_n) &= \frac{f(x) - f(a_{n+1}) + f(a_{n+1}) - f(a_n)}{x - a_{n+1} + a_{n+1} - a_n} \\ &= R(f, a_{n+1}, a_n) \frac{1 + \frac{f(x)-f(a_{n+1})}{f(a_{n+1})-f(a_n)}}{1 + \frac{x-a_{n+1}}{a_{n+1}-a_n}}. \end{aligned}$$

But $\ell_{n+1}/2 \geq x - a_{n+1} > 0$, so

$$R(f, x, a_n) \geq R(f, a_{n+1}, a_n) \frac{1 + \frac{f(x)-f(a_{n+1})}{f(a_{n+1})-f(a_n)}}{1 + \frac{\ell_{n+1}}{2(a_{n+1}-a_n)}}.$$

From the definition of α_n we have

$$0 < \frac{f(a_{n+1}) - f(x)}{f(a_{n+1}) - f(a_n)} \leq \frac{f(a_{n+1}) - f(x)}{h_{n+1}} = \alpha_{n+1},$$

by our assumption on the unboundedness of the ratios we have

$$\frac{\ell_{n+1}}{2(a_{n+1} - a_n)} \rightarrow 0,$$

and by our construction

$$R(f, a_{n+1}, a_n) \geq \frac{2h_n}{\ell_n}.$$

Combining the last four equations we conclude $D^-f(x) = \infty$, and our proof that f is Besicovitch-Morse is finished.

In order to finish our proof let us show that the function f preserves the Lebesgue measure. To this end let us define a new sequence $(g_n)_{n \geq 0}$ of functions from $C(I)$ for which $\lim_{n \rightarrow \infty} g_n = f$.

We define g_0 as the full tent map, i.e., the function $g_0(x) = 1 - |1 - 2x|$, $x \in I$. To define the function g_n we put

$$g_n := f_n \text{ on } I \setminus \bigcup \mathcal{L}_{n-1}^*,$$

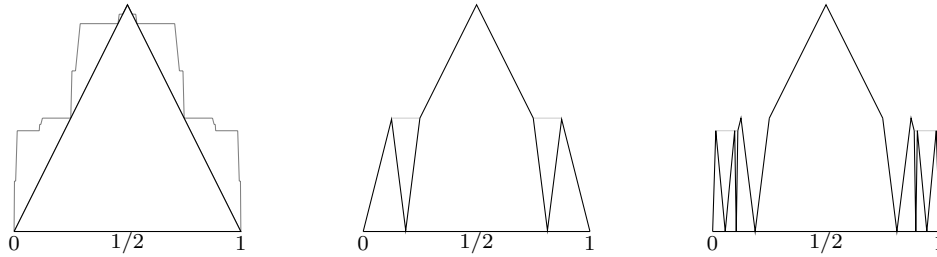


FIGURE 4. i) $g_1^{(0)}$ and f_0 , ii) $g_1^{(1)}$ (the segment $r_{1,1}$ is drawn for comparison), iii) $g_1^{(2)}$ (the segment $r_{2,1}$ is drawn for comparison)

where \mathcal{L}_{n-1}^* denotes the set of all $(n - 1)$ st L -segments and their counterparts in $[1/2, 1]$. On each element of \mathcal{L}_{n-1}^* instead of rescaled step triangle we use a rescaled tent map of the same base, height and orientation. Then

$$\lim_n g_n = \lim_n f_n = f,$$

so it is sufficient to show that each $g_n \in C(\lambda)$. It is true for g_0 . Let $g_1^{(0)} := g_0$ and using the lexicographical order on (m, p) (first m , then p) we consider the j th-interval $r_{m,p}$ and its counterpart in $[1/2, 1]$ to modify $g_1^{(j-1)}$ to a map $g_1^{(j)}$ as in the sequence of pictures. Property (1) implies that each of these modifications is in $C(\lambda)$, then Proposition 1 implies $g_1 := \lim_{j \rightarrow \infty} g_1^{(j)} \in C(\lambda)$. In order to verify that $g_n \in C(\lambda)$ we put $g_n^{(0)} = g_{n-1}$ and define the sequence $g_n^{(j)}$, $j \geq 1$, in an analogous way to (Figure 4) on each element of \mathcal{L}_{n-1}^* . \square

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