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Daniel Barlet, Teresa Monteiro Fernandes

► To cite this version:

Daniel Barlet, Teresa Monteiro Fernandes. On Lisbon integrals. Mathematische Zeitschrift, 2020, 10.1007/s00209-020-02540-6 . hal-02161740

HAL Id: hal-02161740

<https://hal.science/hal-02161740>

Submitted on 21 Jun 2019

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ON LISBON INTEGRALS

DANIEL BARLET AND TERESA MONTEIRO FERNANDES

ABSTRACT. We introduce a family of integral transforms, the Lisbon Integrals, which naturally arise in the study of the affine space of unitary polynomials $P_s(z)$ where $s \in \mathbb{C}^k$ and $z \in \mathbb{C}$, s_i identified to the i -th symmetric function of the roots of $P_s(z)$. We completely determine the \mathcal{D} -module (or system of partial differential equations) the Lisbon Integrals satisfy and prove that they are their unique global solutions of this \mathcal{D} -module.

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1. INTRODUCTION

The purpose of this article is to show that the completely explicit differential system satisfied by the “Lisbon integrals” (defined in formulas (3) and (4) below) is in fact a \mathcal{D} -module obtained from a very simple one by the usual functorial operations on \mathcal{D} -modules (inverse image and direct image).

More precisely, the \mathcal{D} -module approach shows that the Lisbon integrals are holomorphic solutions of modules obtained as integral transforms associated to the affine space of unitary polynomials of degree k in the variable $z \in \mathbb{C}$ with complex coefficients. Writing its elements in the form $P(s, z) = \sum_{i=0}^k (-1)^i s_i z^{k-i}$ with $s = (s_1, \dots, s_k) \in \mathbb{C}^k$ with the convention $s_0 = 1$, this affine space (see [2, Ch. I]) is naturally identified to the hypersurface H of \mathbb{C}^{k+1} endowed with the coordinates (s, z) of equation $s_k = (-1)^{k-1} \sum_{i=1}^k (-1)^i s_i z^{k-i}$. Let $j : H \subset \mathbb{C}^{k+1}$ denote the closed embedding and let $B_{H|\mathbb{C}^{k+1}}$ denote the regular holonomic $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module of holomorphic distributions supported by H . Since the restriction of the projection

$$\pi : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k, (s, z) \mapsto s$$

to H is proper (with finite fibers), given a $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module \mathcal{L} non characteristic for H , following [8] we obtain a complex in $D_{\text{coh}}^b(\mathcal{D}_{\mathbb{C}^k})$, the integral transformed of \mathcal{L} , given by the composition of the usual derived functors of direct image and inverse image for \mathcal{D} -modules,

$$D\pi_*(B_{H|\mathbb{C}^{k+1}} \overset{L}{\otimes}_{\mathcal{O}_{\mathbb{C}^{k+1}}} \mathcal{L}) \simeq D\pi_*(\mathcal{H}_{[H]}^1(\mathcal{L})) \simeq \pi_* j^* \mathcal{L}$$

Note that $B_{H|\mathbb{C}^{k+1}} = \mathcal{H}_{[H]}^1(\mathcal{O}_{\mathbb{C}^{k+1}})$ and that for such a module \mathcal{L} , we have, thanks to Kashiwara’s equivalence theorem (cf Th 4.1 and Prop 4.2, [4]), $\mathcal{H}_{[H]}^1(\mathcal{L}) \simeq \mathcal{H}^0 D j_* D j^* \mathcal{L} \simeq j_* j^* \mathcal{L}$. In this note \mathcal{L} is the quotient of $\mathcal{D}_{\mathbb{C}^{k+1}}$ by the ideal generated by the partial derivatives in s_i , $i = 1, \dots, k$, hence the sheaf of solutions of \mathcal{L} is $p^{-1}\mathcal{O}_{\mathbb{C}}$ where $p(s, z) = z$.

As a consequence, we show that the Lisbon Integrals are exactly the global solutions of $\pi_* j^* \mathcal{L}$.

These computations illustrate the fact that it is not so easy, even in a rather simple situation, to follow explicitly the computations hidden in the “yoga” of \mathcal{D} -module theory.

2. LISBON INTEGRALS AND THE DIFFERENTIAL SYSTEM THEY SATISFY

2.a. Lisbon integrals. For $(z_1, \dots, z_k) \in \mathbb{C}^k$ denote s_1, \dots, s_k the elementary symmetric functions of z_1, \dots, z_k . We shall consider in the sequel s_1, \dots, s_k as coordinates on $\mathbb{C}^k \simeq \mathbb{C}^k / \mathfrak{S}_k$, isomorphism given by the standard symmetric function theorem.

We shall denote $P_s(z) := \prod_{j=1}^k (z - z_j) = \sum_{h=0}^k (-1)^h s_h z^{k-h}$ with the convention $s_0 \equiv 1$.

We shall often write $P(s, z)$ instead of $P_s(z)$ with no risk of ambiguity.

The discriminant $\Delta(s)$ of P_s is the polynomial in s corresponding to the symmetric polynomial $\prod_{1 \leq i < j \leq k} (z_i - z_j)^2$ via the symmetric function theorem.

Lemma 2.1. For $h \in \mathbb{N}$ and $f \in \mathcal{O}(\mathbb{C})$ any entire holomorphic function, let us define, for $R \gg \|s\|$,

$$(1) \quad \varphi_h(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)\zeta^h d\zeta}{P_s(\zeta)}.$$

Then $\varphi_h(s)$ is independent of the choice of R large enough and defines a holomorphic function on \mathbb{C}^k . For $\Delta(s) \neq 0$ we have

$$(2) \quad \varphi_h(s) = \sum_{j=1}^k \frac{z_j^h f(z_j)}{P'_s(z_j)}$$

where z_1, \dots, z_k are the roots of $P_s(z)$.

PROOF. The independence on R large enough when s stays in a compact set of \mathbb{C}^n is clear. For s in the interior of a compact set, $P_s(\zeta)$ does not vanish on $\{|\zeta| = R\}$ for R large enough, so we obtain the holomorphy of φ_h near any point in \mathbb{C}^k . The formula (2) is given by a direct application of the Residue's theorem. ■

In fact, it will be convenient to consider the k functions $\varphi_0, \dots, \varphi_{k-1}$ as the component of a vector valued function $\Phi := \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \dots \\ \varphi_{k-1} \end{pmatrix}$. Defining $E(z) := \begin{pmatrix} 1 \\ z \\ \dots \\ z^{k-1} \end{pmatrix}$ we obtain

$$(3) \quad \Phi(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.$$

Definition 2.2. We call Φ (sometimes also denoted by Φ_f when precision is required) the *Lisbon Integral associated to f*

It will be also interesting to introduce another type of Lisbon integrals:

$$(4) \quad \Psi(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta).P'_s(\zeta)d\zeta}{P_s(\zeta)}$$

Ψ will also be noted below by Ψ_f when precision is required.

It is easy to see that this is again a vector valued holomorphic function on \mathbb{C}^k and the Residue's theorem gives that, for $\Delta(s) \neq 0$, the component ψ_h of Ψ is given by:

$$(5) \quad \psi_h(s) = \sum_{j=1}^k z_j^h f(z_j).$$

Proposition 2.3. If f is not identically zero then Φ and Ψ are non zero vector-valued holomorphic functions on \mathbb{C}^k .

PROOF. Suppose that f is non identically zero. Then the statement follows as an immediate consequence of the non vanishing of the Van der Monde determinant of z_1, \dots, z_k when these complex numbers are pair-wise distinct. ■

AN EXAMPLE. Take $f \equiv 1$. Then formula (5) shows that $\psi_h(s)$ is the h -th Newton symmetric functions of the roots of the polynomial P_s . So it is a quasi-homogeneous polynomial in s of weight h (the weight of s_j is j by definition).

Let us show that we have $\varphi_h(s) \equiv 0$ for $h \in [0, k-2]$ and $\varphi_{k-1}(s) \equiv 1$ in this case. For $h \in [0, k-2]$ the formula (1) gives the estimate (with $f \equiv 1$)

$$|\varphi_h| \leq \frac{R^{h+1}}{(R-a)^{k-1}}$$

if each root of P_s is in the disc $\{|z| \leq a\}$ when $R > a > 0$. When $R \rightarrow +\infty$ this gives $\varphi_h(s) \equiv 0$ for $h \in [0, k-2]$.

For $h = k-1$ write

$$kz^{k-1} = P'_s(z) - \sum_{h=1}^{k-1} (-1)^h k(k-h)s_h z^{k-h-1}.$$

This gives, using the previous case and formula (2), that $\varphi_{k-1}(s) \equiv 1$.

2.b. The partial differential system. Let us introduce the (k, k) matrix A associated to the polynomial P_s by

$$(6) \quad A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & 0 & & \cdots & & 0 \\ & & & & & \\ 0 & & & \cdots & & 0 & 1 \\ (-1)^{k-1}.s_k & \cdots & (-1)^{h-1}.s_h & \cdots & \cdots & s_1 \end{pmatrix}$$

Theorem 2.b.1. *The vector value holomorphic function Φ on \mathbb{C}^k satisfies the following differential system*

$$(@) \quad (-1)^{k+h} \cdot \frac{\partial \Phi}{\partial s_h}(s) = \frac{\partial (A^{k-h} \Phi)}{\partial s_k}(s) \quad \forall s \in \mathbb{C}^k \quad \text{and} \quad \forall h \in [1, k-1].$$

Moreover, this system is integrable¹ and if Φ is a solution of this system, so is $A\Phi$.

The proof of this result will use several lemmas.

Lemma 2.4. *Let A be a (k, k) matrix with entries in $\mathbb{C}[x]$ and put $B := \lambda \frac{\partial A}{\partial x}$ where λ is a complex number. Let M be the $(2k, 2k)$ matrix given by*

$$M := \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

Then for each $p \in \mathbb{N}$ we have

$$(*) \quad M^p = \begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix}$$

¹We shall explain in the proof what we mean here.

where $B_p := \lambda \frac{\partial(A^p)}{\partial x}$.

PROOF. As the relation (*) is clear for $p = 0, 1$ let us assume that it has been proved for p and let us prove it for $p + 1$. We have:

$$\begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} A^{p+1} & A^p B + B_p A \\ 0 & A^{p+1} \end{pmatrix}$$

which allows to conclude. ■

Corollary 2.b.2. *For each integer $p \in [0, k - 1]$ the following equality holds in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]/(P^2)$ over the \mathbb{C} -algebra $\mathbb{C}[s_1, \dots, s_k, z]/(P^2)$*

$$(7) \quad z^p E(z) = A^p E(z) + (-1)^{k-1} P_s(z) \frac{\partial(A^p)}{\partial s_k} E(z)$$

In particular for any entire function on z ,

$$\Phi_{zf} = A(s) \Phi_f$$

Moreover the following identity in the module $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]/(P^2)$ holds

$$(7\text{-bis}) \quad P'_s(z) E(z) = P'_s(A) E(z) + (-1)^{k-1} P_s(z) \frac{\partial(P'_s(A))}{\partial s_k} E(z).$$

PROOF. In the basis $1, z, \dots, z^{k-1}, P_s(z), zP_s(z), \dots, z^{k-1}P_s(z)$ of this algebra which is a free rank $2k$ module on $\mathbb{C}[s_1, \dots, s_k]$, the multiplication by z is given by the matrix M of the previous lemma with A as in (6) and with $B := (-1)^{k-1} \frac{\partial A}{\partial s_k}$. This proves formula (7).

As $P'_s(z) = \sum_{h=0}^{k-1} (-1)^h (k-h) z^{k-h-1}$ does not depend on s_k it is enough to sum up the previous equalities with $p = k - h - 1$ with the convenient coefficients to obtain the equality (7 - bis). ■

Lemma 2.5. *Let A be a (k, k) matrix with entries in $\mathbb{C}[x, y]$. Assume that for some integer q we have*

$$\frac{\partial A}{\partial x} A^q = \frac{\partial A}{\partial y}$$

then we have for any integer h

$$\frac{\partial A^h}{\partial x} A^q = \frac{\partial A^h}{\partial y}.$$

PROOF. The assertion is our hypothesis for $h = 1$. Assume the assertion proved for some $h \geq 1$. Then Leibnitz rule gives:

$$\frac{\partial A^{h+1}}{\partial x} A^q = \frac{\partial A^h}{\partial x} A^{q+1} + A^h \frac{\partial A}{\partial x} A^q$$

which implies:

$$\frac{\partial A^{h+1}}{\partial x} A^q = \left(\frac{\partial A^h}{\partial y} A + A^h \frac{\partial A}{\partial y} \right) = \frac{\partial A^{h+1}}{\partial y}$$

which is our assertion for $h + 1$. ■

Corollary 2.b.3. *For any $h \in [1, k]$ and any $p \in \mathbb{N}$ the matrix A in (6) satisfies the relation:*

$$(8) \quad (-1)^{k-h} \frac{\partial A^p}{\partial s_h} = \frac{\partial A^p}{\partial s_k} A^{k-h}$$

PROOF. The case $p = 1$ of (7) is an easy direct computation on the matrix A . Assume that the assertion is proved for $p \geq 1$. Then Leibnitz's rule gives:

$$\begin{aligned} (-1)^{k-h} \frac{\partial A^{p+1}}{\partial s_h} &= (-1)^{k-h} \frac{\partial A^p}{\partial s_h} A + A^p (-1)^{k-h} \frac{\partial A}{\partial s_h} \\ &= \frac{\partial A^p}{\partial s_k} A^{k-h+1} + A^p \frac{\partial A}{\partial s_k} A^{k-h} = \frac{\partial A^{p+1}}{\partial s_k} A^{k-h} \end{aligned}$$

concluding the proof of (8). ■

PROOF OF THE THEOREM 2.b.1. By derivation inside the integral in (3) we obtain:

$$\begin{aligned} \frac{\partial \Phi}{\partial s_h}(s) &= \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) (-1)^{h+1} \zeta^{k-h} \frac{d\zeta}{P_s(\zeta)^2} \quad \text{and in particular} \\ \frac{\partial \Phi}{\partial s_k}(s) &= \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) (-1)^{k+1} \frac{d\zeta}{P_s(\zeta)^2} \end{aligned}$$

Now for $h \in [1, k-1]$ we use the formula of corollary 2.b.2 to obtain:

$$\frac{\partial \Phi}{\partial s_h} = (-1)^{h+1} A^{k-h} (-1)^{k-1} \frac{\partial \Phi}{\partial s_k} + (-1)^{h+1} (-1)^{k-1} \frac{\partial A^{k-h}}{\partial s_k} \Phi$$

that is to say:

$$(@) \quad (-1)^{k+h} \frac{\partial \Phi}{\partial s_h} = \frac{\partial (A^{k-h} \Phi)}{\partial s_k} \quad \forall h \in [1, k]$$

By the integrability of the system (@) we mean that for any Φ such that (@) holds, then the computation of the partial derivatives $\frac{\partial^2 \Phi}{\partial s_h \partial s_j}$ using the system (@) gives a symmetric result in (h, j) for any pair (h, j) in $[1, k]$. Note that if h or j is equal to k the assertion is trivial.

So consider a couple $(h, j) \in [1, k-1]^2$. Thanks to corollary 2.b.3 we have :

$$\begin{aligned} (-1)^{h+j} \frac{\partial^2 \Phi}{\partial s_j \partial s_h} &= (-1)^{k-j} \frac{\partial}{\partial s_k} \left[\frac{\partial (A^{k-h} \Phi)}{\partial s_j} \right] \\ &= (-1)^{k-j} \frac{\partial}{\partial s_k} \left[\frac{\partial A^{k-h}}{\partial s_j} \Phi + A^{k-h} \frac{\partial \Phi}{\partial s_j} \right] \\ &= \frac{\partial}{\partial s_k} \left[\frac{\partial A^{k-h}}{\partial s_j} A^{k-j} \Phi + A^{k-h} \frac{\partial (A^{k-j} \Phi)}{\partial s_k} \right] \\ &= \frac{\partial^2}{\partial s_k^2} [A^{2k-h-j} \Phi] \end{aligned}$$

which is symmetric in (h, j) .

To finish the proof of the theorem we have to show that $A\Phi$ is a solution of (@) when Φ is a solution of (@). This is given by the following computation

$$\begin{aligned}
(-1)^{k-h} \frac{\partial(A\Phi)}{\partial s_h} &= (-1)^{k-h} \frac{\partial A}{\partial s_h} \Phi + (-1)^{k-h} A \frac{\partial \Phi}{\partial s_h} \\
&= \frac{\partial A}{\partial s_k} A^{k-h} \Phi + A \frac{\partial A^{k-h} \Phi}{\partial s_k} = \frac{\partial(A^{k-h}(A\Phi))}{\partial s_k}
\end{aligned}$$

which also uses corollary 2.b.3. ■

REMARK. A consequence of our computation on the integrability of the system (@) is the fact that for any solution Φ and any pair $(h, j) \in [1, k]$ the second order partial derivative $\frac{\partial^2 \Phi}{\partial s_j \partial s_h}$ only depends on $h + j$. This implies that any scalar Lisbon integral φ_h satisfies

$$(9) \quad \frac{\partial^2 \varphi_h}{\partial s_p \partial s_{q+1}} = \frac{\partial^2 \varphi_h}{\partial s_{p+1} \partial s_q} \quad \forall p, q \text{ such that } 1 \leq p < q \leq k-1$$

The following corollary of theorem 2.b.1 describes the analogous system for the vector functions Ψ defined in (4) which is singular along the discriminant subset $\{\Delta(s) = 0\}$ in \mathbb{C}^k .

Corollary 2.b.4. *The vector value holomorphic function Ψ on \mathbb{C}^k satisfies the following differential system:*

$$(@@) \quad (-1)^{k+h} \frac{\partial \Psi}{\partial s_h}(s) = \frac{\partial(A^{k-h} \Psi)}{\partial s_k}(s) + (-1)^k (k-h) A^{k-h-1} P'_s(A)^{-1} \Psi(s)$$

for all $h \in [1, k-1]$, which is singular along the discriminant sub-set

$$\Delta := \{s \in \mathbb{C}^k \mid \Delta(s) = 0\}.$$

Again, if Ψ is any solution of (@@) then $A\Psi$ is also a solution of (@@).

PROOF. Define $F(z) := P'(z)E(z)$. Then using formula (7) we obtain $\Psi(s) = P'_s(A)\Phi(s)$ and this gives for each $h \in [1, k-1]$

$$(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-h} A^{k-h-1} \Phi(s) + \sum_{p=0}^{k-1} (-1)^p (k-p) (-1)^{k-h} s_p \cdot \frac{\partial(A^{k-p-1} \Phi)}{\partial s_h}(s)$$

and using now the fact that $A^{k-p-1} \Phi$ is solution of (@) we obtain

$$\begin{aligned}
(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) &= (-1)^{k-h} A^{k-h-1} \Phi(s) + \sum_{p=0}^{k-1} (-1)^p (k-p) s_p \cdot \frac{\partial(A^{2k-p-h-1} \cdot \Phi)}{\partial s_k}(s) \\
(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) &= (-1)^{k-h} A^{k-h-1} \Phi(s) + \frac{\partial(A^{k-h} P'_s(A) \Phi)}{\partial s_k}(s) \\
(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) &= (-1)^{k-h} A^{k-h-1} P'_s(A)^{-1} \Psi(s) + \frac{\partial(A^{k-h} \Psi)}{\partial s_k}(s)
\end{aligned}$$

Since $P'_s(A)$ commutes with A , the last assertion is easy. ■

2.c. **Example: The cases $k = 2, 3$.** The left ideal of partial differential operators in the Weyl algebra $\mathbb{C}[s_1, \dots, s_k] \langle \partial_{s_1}, \dots, \partial_{s_k} \rangle$ which annihilate Lisbon integrals is described in the article [1].

Below we will explicit the cases $k = 2$ and $k = 3$.

Case $k = 2$:

Here we use the notations $s := s_1$ and $p := s_2$. In that case, the differential system (@) becomes:

$$(10) \quad -\frac{\partial \Phi_0}{\partial s} = \frac{\partial \Phi_1}{\partial p}$$

$$(11) \quad -\frac{\partial \Phi_1}{\partial s} = -\Phi_0 - p \frac{\partial \Phi_0}{\partial p} + s \frac{\partial \Phi_1}{\partial p}$$

corresponding to the matrix $A := \begin{pmatrix} 0 & 1 \\ -p & s \end{pmatrix}$. Differentiating (9) with respect to p we obtain, after substituting via (10)

$$\begin{aligned} \frac{\partial^2 \Phi_0}{\partial s^2} &= -2 \frac{\partial \Phi_0}{\partial p} - p \frac{\partial^2 \Phi_0}{\partial p^2} - s \frac{\partial^2 \Phi_0}{\partial s \partial p} \\ (\#) \quad \text{and so} \quad \frac{\partial^2 \Phi_0}{\partial s^2} + s \frac{\partial^2 \Phi_0}{\partial s \partial p} + p \frac{\partial^2 \Phi_0}{\partial p^2} + 2 \frac{\partial \Phi_0}{\partial p} &= 0 \end{aligned}$$

Differentiating (10) with respect to s we obtain, after substituting via (9)

$$\begin{aligned} -\frac{\partial^2 \Phi_1}{\partial s^2} &= \frac{\partial \Phi_1}{\partial p} + p \frac{\partial^2 \Phi_1}{\partial p^2} + \frac{\partial \Phi_1}{\partial p} + s \frac{\partial^2 \Phi_1}{\partial s \partial p} \\ (\#\#) \quad \text{and so} \quad \frac{\partial^2 \Phi_1}{\partial s^2} + s \frac{\partial^2 \Phi_1}{\partial s \partial p} + p \frac{\partial^2 \Phi_1}{\partial p^2} + 2 \frac{\partial \Phi_1}{\partial p} &= 0 \end{aligned}$$

Then the second order differential operator of weight -2

$$(\#\#\#) \quad \Theta := \frac{\partial^2}{\partial s^2} + s \frac{\partial^2}{\partial s \partial p} + p \frac{\partial^2}{\partial p^2} + 2 \frac{\partial}{\partial p}$$

annihilates Φ_0 and Φ_1 for any solution Φ of the system (10), (11).

A DIRECT PROOF THAT Θ ANNIHILATES φ_m FOR ALL $m \in \mathbb{N}$. We have, for any entire function $f : \mathbb{C} \rightarrow \mathbb{C}$ and for $R \gg \max\{|s|, |p|\}$:

$$(a) \quad \varphi_m(s, p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{\zeta^2 - s\zeta + p}$$

This gives:

$$\begin{aligned}
(b) \quad & \frac{\partial \varphi_m}{\partial s}(s, p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^2} \\
(c) \quad & \frac{\partial \varphi_m}{\partial p}(s, p) = -\frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^2} \\
(d) \quad & \frac{\partial^2 \varphi_m}{\partial s^2}(s, p) = 2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+2} d\zeta}{(\zeta^2 - s\zeta + p)^3} \\
(e) \quad & \frac{\partial^2 \varphi_m}{\partial s \partial p}(s, p) = -2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^3} \\
(f) \quad & \frac{\partial^2 \varphi_m}{\partial p^2}(s, p) = 2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^3}
\end{aligned}$$

Now it is easy to check that $(d) + s(e) + p(f) + 2(c) = 0$.

Case $k = 3$:

The differential system system (@) becomes now:

$$(@) \quad (-1)^{k-h} \partial_h \Phi = \partial_k (A^{k-h} \Phi) \quad \text{for } h \in [1, k-1]$$

with $k = 3$. Here $\Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s_3 & -s_2 & s_1 \end{pmatrix}$. Explicitely this gives:

$$\begin{aligned}
\partial_1 \Phi_0 &= \partial_3 \Phi_2 \\
\partial_1 \Phi_1 &= \partial_3 (s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2) \\
\partial_1 \Phi_2 &= \partial_3 (s_1 s_3 \Phi_0 - (s_3 - s_1 s_2) \Phi_1 + (s_1^2 - s_2) \Phi_2) \\
-\partial_2 \Phi_0 &= \partial_3 \Phi_1 \\
-\partial_2 \Phi_1 &= \partial_3 \Phi_2 \\
-\partial_2 \Phi_2 &= \partial_3 (s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2)
\end{aligned}$$

which implies the equalities

$$\partial_1 \Phi_0 = -\partial_2 \Phi_1 = \partial_3 \Phi_2 \quad \partial_1 \Phi_1 = -\partial_2 \Phi_2$$

Then we deduce that:

$$(\partial_2^2 - \partial_1 \partial_3)(\Phi_j) = 0 \quad \text{for } j = 0, 1, 2$$

because

$$\begin{aligned}
\partial_2^2 \Phi_0 &= -\partial_2 \partial_3 \Phi_1 = \partial_3 \partial_3 \Phi_2 = \partial_3 \partial_1 \Phi_0, \\
\partial_1 \partial_3 \Phi_1 &= -\partial_1 \partial_2 \Phi_0 = \partial_2 \partial_3 \Phi_2 = \partial_2^2 \Phi_1, \\
\text{and } \partial_2^2 \Phi_2 &= -\partial_2 \partial_1 \Phi_1 = \partial_3 \partial_2 \Phi_2.
\end{aligned}$$

Lemma 2.6. *Let us define:*

$$\begin{aligned}\mathcal{L} &:= \partial_1^2 + s_1 \partial_1 \partial_2 + s_2 \partial_2^2 + s_3 \partial_2 \partial_3 + 2\partial_2 \\ \mathcal{L}' &:= \partial_1 \partial_2 + s_1 \partial_2^2 + s_2 \partial_2 \partial_3 + s_3 \partial_3^2 + 2\partial_3\end{aligned}$$

Then we have:

$$\mathcal{L}\Phi_0 = 0 \quad \mathcal{L}'\Phi_1 = 0 \quad \text{et} \quad \mathcal{L}\Phi_2 = 0$$

and also:

$$(r) \quad \partial_3 \mathcal{L} - \partial_2 \mathcal{L}' \in \mathcal{D}_{\mathbb{C}^k}(\partial_1 \partial_3 - \partial_2^2)$$

This entails that $\partial_3 \mathcal{L}$ and $\partial_2 \mathcal{L}'$ annihilate Φ_j for $j = 0, 1, 2$.

PROOF. We have

$$\begin{aligned}\partial_1 \Phi_1 &= \Phi_0 + s_3 \partial_3 \Phi_0 - s_2 \partial_3 \Phi_1 + s_1 \partial_3 \Phi_2 \\ &= \Phi_0 + s_3 \partial_3 \Phi_0 + s_2 \partial_2 \Phi_0 + s_1 \partial_1 \Phi_0 \\ \text{and so } \partial_2 \partial_1 \Phi_1 &= -\partial_1^2 \Phi_0 = \\ &\partial_2 \Phi_0 + s_3 \partial_2 \partial_3 \Phi_0 + \partial_2 \Phi_0 + s_2 \partial_2^2 \Phi_0 + s_1 \partial_1 \partial_2 \Phi_0\end{aligned}$$

and so Φ_0 is killed by \mathcal{L} as $(\partial_1 \partial_3 - \partial_2^2)\Phi_0 = 0$.

In a similar way:

$$\begin{aligned}\partial_1 \Phi_1 &= -\partial_2 \Phi_2 = \partial_3(s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2) \\ \partial_2 \partial_1 \Phi_1 &= -\partial_3(s_3 \partial_3 \Phi_1) - \partial_2 \partial_3(s_2 \Phi_1) - s_1 \partial_2^2 \Phi_1 \\ &= -\partial_3 \Phi_1 - s_3 \partial_3^2 \Phi_1 - \partial_3 \Phi_1 - s_2 \partial_2 \partial_3 \Phi_1 - s_1 \partial_2^2 \Phi_1\end{aligned}$$

showing that \mathcal{L}' annihilates Φ_1 .

In a similar way:

$$\begin{aligned}\partial_1 \Phi_2 &= s_1 \Phi_0 + s_1 s_3 \partial_3 \Phi_0 + \Phi_1 + (s_3 - s_1 s_2) \partial_3 \Phi_1 + (s_1^2 - s_2) \partial_3 \Phi_2 \\ &= s_1 \partial_1 \Phi_1 + \Phi_1 - s_2 \partial_3 \Phi_2 + s_3 \partial_3 \Phi_1 \\ &= s_1 \partial_1 \Phi_1 + \Phi_1 + s_2 \partial_2 \Phi_1 + s_3 \partial_3 \Phi_1 \\ \partial_1^2 \Phi_2 &= -\partial_2 \Phi_2 - s_1 \partial_1 \partial_2 \Phi_2 - \partial_2 \Phi_2 + s_2 \partial_2(-\partial_2 \Phi_2) + s_3 \partial_3(\partial_1 \Phi_1) \\ &= -2\partial_2 \Phi_2 - s_1 \partial_1 \partial_2 \Phi_2 - s_2 \partial_2^2 \Phi_2 - s_3 \partial_2 \partial_3 \Phi_2\end{aligned}$$

and Φ_2 is annihilated by \mathcal{L} .

Let us prove now the relation (r):

$$\begin{aligned}\partial_3 \mathcal{L} &= \partial_3 \partial_1^2 + s_1 \partial_1 \partial_2 \partial_3 + s_2 \partial_1 \partial_3^2 + s_3 \partial_2 \partial_3^2 + 3\partial_2 \partial_3 \\ \partial_2 \mathcal{L}' &= \partial_1 \partial_2^2 + s_3 \partial_2 \partial_3^2 + s_2 \partial_2^2 \partial_3 + s_1 \partial_2^3 + 3\partial_2 \partial_3\end{aligned}$$

this allows to conclude, as

$$\begin{aligned}\partial_3 \partial_1^2 - \partial_1 \partial_2^2 &= \partial_1(\partial_1 \partial_3 - \partial_2^2) \\ \partial_1 \partial_2 \partial_3 - \partial_2^3 &= \partial_2(\partial_1 \partial_3 - \partial_2^2)\end{aligned}$$

For the study of differential operators annihilating all the scalar components of the Lisbon integrals (for any value of k) see [1].

3. THE LEFT ACTION OF $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ ON LISBON INTEGRALS

For each entire function f of the variable z we shall henceforward denote by Φ_f the associated Lisbon integral (previously generically denoted by Φ). Clearly the assignement

$$f \mapsto \Phi_f$$

is \mathbb{C} -linear and, according to Proposition 2.3, it is injective.

Lemma 3.1. *Let f be an entire function on \mathbb{C} and let Φ_f the corresponding Lisbon integral.*

- (1) *Let h be an entire function of z . Then $\Phi_{hf} = h(A(s))\Phi_f$. In particular $\Phi_f = f(A(s))\Phi_1$.*
 (2) *We have the identity*

$$\Phi_{\partial_z(f)}(s) = -\nabla \Phi_f(s) + \left(\sum_{h=0}^{k-1} (k-h)s_h \partial_{s_{h+1}} \right) (\Phi_f)(s)$$

$$\text{where } \nabla \text{ is the constant } (k, k) \text{ matrix } \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & k-1 & 0 \end{pmatrix}.$$

PROOF. When h is a polynomial on z , statement (1) follows easily in Corollary 2.b.2. For an arbitrary entire function h , it is a consequence of [2, Lem 3.1.8].

Let us prove (2): Consider the Lisbon integral

$$\Phi_{\partial_z f}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} f'(\zeta) E(\zeta) \frac{d\zeta}{P_s(\zeta)}$$

Integration by parts gives, as $\partial_z(E)(z) = \nabla E(z)$:

$$(*) \quad \Phi_{\partial_z(f)}(s) = -\nabla \phi_f(s) + \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) \frac{P'_s(\zeta)}{P_s(\zeta)^2} d\zeta$$

Now, using the equalities $P'_s(\zeta) = \sum_{h=0}^{k-1} (-1)^h (k-h)s_h \zeta^{k-h-1}$ and

$$\frac{\partial \Phi_f}{\partial s_h}(s) = - \int_{|\zeta|=R} f(\zeta) E(\zeta) \frac{(-1)^h \zeta^{k-h} d\zeta}{P_s(\zeta)^2}$$

we obtain the formula of the lemma. ■

REMARK. From the formula (*) and using the fact that $P'_s(\zeta)E(\zeta) = P'_s(A)E(\zeta)$ we obtain also the formula

$$(**) \quad \Phi_{\partial_z(f)}(s) = -\nabla \Phi_f(s) + (-1)^k \frac{\partial(P'_s(A)\Phi_f)}{\partial_{s_k}}(s)$$

It is not obvious that when Φ is solution of the system (@), then $(\Phi \partial_z)(s) := -\nabla \Phi(s) + (-1)^k \frac{\partial(P'_s(A)\Phi)}{\partial_{s_k}}(s)$ (given by the formula (**)) is also a solution of the same system. A direct verification of this fact is consequence of the formula given in the lemma below.

In order to verify that we have $\Phi_{(z,f)'} = \Phi_f + \Phi_{zf'}$ corresponding to the usual relation $\partial_z z - z \partial_z = 1$ the following lemma is useful.

Lemma 3.2. *We have the following identity:*

$$\nabla A - A \nabla + Id = (-1)^{k-1} \partial_k(A) P'_s(A), \quad \forall s \in \mathbb{C}^k.$$

PROOF. It is an easy computation to obtain that $\nabla A - A \nabla + Id$ is the matrix which have all lines equal to 0 excepted its last one which is given by (x_1, \dots, x_k) with $x_h = (-1)^{k-h} h s_{k-h}$ for $h \in [1, k]$ with $s_0 \equiv 0$. On the other hand, the matrix $(-1)^{k-1} \partial_k A$ has only a non zero term at the place $(k, 1)$ which equal to 1, so it is quite easy to see that $(-1)^{k-1} (\partial_k A) A^p$ has only a non zero term at the place $(k, p+1)$ with value $(-1)^{k-1}$. According to the computation of $(-1)^{k-1} \partial_k(A) P'_s(A)$ we conclude the desired formula. \blacksquare

4. THE $\mathcal{D}_{\mathbb{C}^k}$ -MODULE ASSOCIATED TO LISBON INTEGRALS

We shall begin by recalling some basic facts on \mathcal{D} -modules and fixing our notations.

For a morphism of manifolds $f : Y \rightarrow X$, we use the notation of [7]

$$f_d := {}^t f' : T^*X \times_X \times Y \longrightarrow T^*Y$$

and

$$f_\pi : T^*X \times_X \times Y \longrightarrow T^*X$$

the associated canonical morphisms of vector bundles.

We recall that a conic involutive submanifold V of the cotangent bundle T^*Z of a manifold Z (real or complex) is *regular* if the restriction $\omega|_V$ of the canonical 1-form ω on T^*Z never vanishes outside the 0-section. Recall also that if $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ are canonical symplectic coordinates on T^*Z , then $\omega(x, \xi) = \sum_{i=1}^n \xi_i dx_i$.

Let us fix some $k \in \mathbb{N}$, $k \geq 2$. In $\mathbb{C}^{k+1} = \mathbb{C}^k \times \mathbb{C}$ we consider the coordinates (s_1, \dots, s_k, z) . As in the previous sections we set

$$P(s, z) = z^k + \sum_{h=1}^{k-1} (-1)^h s_h z^{k-h}$$

Obviously

$$P(s_1, \dots, s_k, z) = 0 \iff s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h},$$

where $s_0 = 1$. We note $s = (s_1, \dots, s_k)$ and $s' := (s_1, \dots, s_{k-1})$.

Let H be the smooth hypersurface of \mathbb{C}^{k+1} given by the zeros of $P(s, z)$.

Let us denote by \mathcal{L} the $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module with one generator u defined by the equations $\partial u / \partial s_1 = \dots = \partial u / \partial s_k = 0$. Such module is an example of a so called *partial de Rham systems*, which have the feature, among others, that their characteristic varieties are non singular regular involutive. In our case we have

$$\text{Char } \mathcal{L} = \{(s, z); (\eta, \tau) \in \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \text{ such that } \eta = 0\}.$$

Since $H \subset \mathbb{C}^{k+1}$ is defined by the equation

$$P(s, z) = (-1)^k s_k + \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h} = 0,$$

$T_H^* \mathbb{C}^{k+1}$ is the subbundle of $T^* \mathbb{C}^{k+1}|_H$ described by

$$\{(s, z; \eta, \tau), (s, z) \in H, \exists \lambda \in \mathbb{C} \text{ such that } (\eta, \tau) = \lambda dP(s, z)\}.$$

This means that $P(s, z) = 0$ and that there exists $\lambda \in \mathbb{C}$ with $\eta_h = \lambda(-1)^h \cdot z^{k-h}$ for each $h \in [1, k]$ and that $\tau = \lambda P'(s, z)$. Hence as $\eta_k = \lambda(-1)^k$ and this implies:

$$\text{Char } \mathcal{L} \cap T_H^* \mathbb{C}^{k+1} \subset T_{\mathbb{C}^{k+1}}^* \mathbb{C}^{k+1}$$

(as usual, $T_{\mathbb{C}^{k+1}}^* \mathbb{C}^{k+1}$ denotes the zero section of $T^* \mathbb{C}^{k+1}$), in other words H is non characteristic for \mathcal{L} . Let us denote by $j : H \hookrightarrow \mathbb{C}^{k+1}$ the closed embedding. By Kashiwara's classical results (which can be found in [5]) it follows that the induced system $Dj^* \mathcal{L}$ by \mathcal{L} on H is concentrated in degree zero and $\mathcal{N} := \mathcal{H}^0 Dj^* \mathcal{L}$ is a $\mathcal{D}_{\mathbb{C}^k}$ -coherent module whose characteristic variety is exactly

$$j_d j_\pi^{-1} \text{Char}(\mathcal{L}).$$

Recall that

$$Dj^* \mathcal{L} := \mathcal{O}_H \otimes_{j^{-1} \mathcal{O}_{\mathbb{C}^{k+1}}}^L j^{-1} \mathcal{L}$$

and in this non-characteristic case

$$\simeq j^{-1} \frac{\mathcal{O}_{\mathbb{C}^{k+1}}}{P(s', s_k, z) \mathcal{O}_{\mathbb{C}^{k+1}}} \otimes_{j^{-1} \mathcal{O}_{\mathbb{C}^{k+1}}} j^{-1} \mathcal{L}$$

We have

$$\mathcal{N} := j^{-1} \left(\frac{\mathcal{D}_{\mathbb{C}^{k+1}}}{P \mathcal{D}_{\mathbb{C}^{k+1}} + \mathcal{D}_{\mathbb{C}^{k+1}} \partial_{s_1} + \cdots + \mathcal{D}_{\mathbb{C}^{k+1}} \partial_{s_k}} \right) \simeq j^{-1} (\mathcal{O}_{\mathbb{C}^{k+1}} / P \mathcal{O}_{\mathbb{C}^{k+1}}) < \partial_z >$$

which is isomorphic as a \mathcal{D}_H -module to

$$\mathcal{O}_H < \partial_z > \simeq \frac{\mathcal{D}_H}{\mathcal{D}_H \partial_{s_1} + \cdots + \mathcal{D}_H \partial_{s_{k-1}}}$$

where ∂_{s_i} stands for the derivation $\partial/\partial s_i$ on \mathcal{O}_H and ∂_z as a derivation on \mathcal{O}_H is the class of ∂_z in the quotient above.

In particular \mathcal{N} is sub-holonomic and it is a partial de Rham system similarly to \mathcal{L} . Let us now determine the image of \mathcal{N} under the morphism $\pi : \mathbb{C}^k \simeq H \rightarrow \mathbb{C}^k$ given by $(s', z) \mapsto (s', s_k(s', z))$. Clearly π is proper surjective with finite fibers.

Recall that one denotes by $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$ the transfer $(\pi^{-1} \mathcal{D}_{\mathbb{C}^k}, \mathcal{D}_H)$ -bimodule

$$(\pi^{-1} \mathcal{D}_{\mathbb{C}^k} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1}) \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H$$

Recall also that, according to the properness and the fiber finiteness of π , we have

$$D\pi_* \mathcal{N} \simeq \mathcal{H}^0 D\pi_* \mathcal{N} = \pi_* (\mathcal{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathcal{D}_H} \mathcal{M})$$

where we abusively use the notation π_* for the direct image functor in the category of \mathcal{D} -modules in the two left terms and for the direct image functor for sheaves in the

right term. According to [5, Th. 4.25 and 4.27] (see also the comments in loc.cit. before Theorem 4.27), one knows that $D\pi_*\mathcal{N}$ is concentrated in degree zero and that

$$\text{Char } \mathcal{H}^0 D\pi_*\mathcal{N} = \pi_\pi \pi_d^{-1} \text{Char } \mathcal{N}.$$

So we may henceforward denote for short $\pi_*\mathcal{N} := D\pi_*\mathcal{N}$ without ambiguity.

Let Δ as above be the zero set of the discriminant of P , which can also be defined as the image by π of the subset of H defined by $\{P'(s, z) = 0\}$.

Since π_d is given by the $k \times k$ matrix

$$\begin{pmatrix} Id & 0 \\ \partial s_k / \partial s' & \partial s_k / \partial z \end{pmatrix}^T$$

we conclude that $\text{Char } \pi_*\mathcal{N}$ is the image by π_π of the set

$$\{(s', z); (\eta', \tau) \in \mathbb{C}^k \times \mathbb{C}^k / \eta_j = -(-z)^{k-j} \cdot \tau \quad \forall j \in [1, k-1]\}$$

so it is given by the set

$$\{(s, \eta) \in \mathbb{C}^k \times \mathbb{C}^k / \exists z \in \mathbb{C} \text{ such that } P_s(z) = 0 \text{ and with } \eta_j = (-z)^{k-j} \eta_k \quad \forall j \in [1, k-1]\}$$

Then $\text{Char } \pi_*\mathcal{N}$ is an involutif analytic subset of $T^*\mathbb{C}^k$ with codimension $k-1$, hence $\pi_*\mathcal{N}$ is a subholonomic $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module. Denoting by p the projection: $T^*\mathbb{C}^k \rightarrow \mathbb{C}^k$, for any s such that $\Delta(s) \neq 0$, the Implicit Function Theorem implies that

$$p^{-1}(s) \cap \text{Char } \pi_*\mathcal{N} \text{ is a cone of } l \text{ distinct complex lines } l_j = \mathbb{C} \begin{pmatrix} (-z_j)^{k-1} \\ (-z_j)^{k-2} \\ \vdots \\ (-z_j) \\ 1 \end{pmatrix}, \text{ where}$$

z_j runs over the set of l distinct roots of $P(s, z)$

Proposition 4.1. *The $\mathcal{D}_{\mathbb{C}^k}$ -module $\pi_*\mathcal{N}$ is the quotient of $\mathcal{D}_{\mathbb{C}^k}^k \simeq \mathcal{D}_{\mathbb{C}^k} \otimes_{\mathbb{C}} \mathbb{C}^k$ by the action of*

$$\mathcal{A}_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1} \partial_{s_k} \otimes A(s)^{k-p} \quad \text{for } j \in [1, k-1].$$

Moreover the action of z and ∂_z on $\pi_*\mathcal{N}$ deduced from the action of \mathcal{D}_H on \mathcal{N}^2 are given respectively by

$$\mathcal{A}_0 := 1 \otimes A(s) \quad \text{and} \quad \mathcal{B} := 1 \otimes \nabla + (-1)^{k-1} \partial_{s_k} \otimes P'_s(A(s))$$

$$\text{where we put } P'_s(z) := (\partial_z(P_s(z))) \text{ and } \nabla := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & k-1 & 0 \end{pmatrix}$$

²Note that z and ∂_z commute with ∂_{s_h} for $h \in [1, k-1]$ in \mathcal{D}_H .

PROOF. Our goal is to explicit $\pi_*\mathcal{N}$ and to check that it coincides with the $\mathcal{D}_{\mathbb{C}^k}$ -module associated to the system (\textcircled{a}) in Theorem 2.b.1.

In a first step we explicit the transfer-module

$$\mathcal{D}_{\mathbb{C}^k \leftarrow H} := \pi^{-1} \mathcal{D}_{\mathbb{C}^k} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} (\pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H)$$

as a $(\pi^{-1} \mathcal{D}_{\mathbb{C}^k}, \mathcal{D}_H)$ -bimodule.

The next step is to determine the cokernel of

$$\alpha : (\mathcal{D}_{\mathbb{C}^k \leftarrow H})^{k-1} \longrightarrow \mathcal{D}_{\mathbb{C}^k \leftarrow H}$$

$$(u_1, \dots, u_{k-1}) \mapsto \sum_{i=1}^{k-1} u_i \partial_{s_i}$$

The last step is to apply π_* .

Let us denote for short

$$\sigma := \omega_H \otimes \omega_{\mathbb{C}^k}^{\otimes -1} := ds_1 \wedge \dots \wedge ds_{k-1} \wedge dz \otimes (ds_1 \wedge \dots \wedge ds_{k-1} \wedge ds_k)^{\otimes -1}$$

the generator of the line bundle $\pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H$.

Recall that $\mathcal{O}_H = \mathcal{O}_{\mathbb{C}^{k+1}}/\mathcal{J}$, where \mathcal{J} is the ideal generated by $P(s, z)$. Hence \mathcal{O}_H is a $\pi^{-1} \mathcal{O}_{\mathbb{C}^k}$ -free module with rank k since each section $a(s', z)$ of \mathcal{O}_H is equivalent, by Weierstrass Division Theorem, to a unique polynomial $\sum_{j=0}^{k-1} a_j(s', s_k) z^j$ modulo $P(s, z)$, for some sections a_j of $\mathcal{O}_{\mathbb{C}^k}$.

Hence $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$ is a left $\pi^{-1} \mathcal{D}_{\mathbb{C}^k}$ -free module of rank k generated by the k -sections $(1 \otimes z^j \sigma)_{j=0, \dots, k-1}$. Since the right action of each operator in \mathcal{D}_H is $\pi^{-1} \mathcal{D}_{\mathbb{C}^k}$ -linear, it is sufficient to calculate each $(1 \otimes z^j \sigma) \partial_{s_i}, i = 1, \dots, k-1, j = 0, \dots, k-1$.

Now recall that H is defined in \mathbb{C}^{k+1} by the equation $s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h}$ with the convention $s_0 = 1$ and so s_1, \dots, s_{k-1}, z are global coordinates on H . Then we have in H

$$\frac{\partial s_k}{\partial s_h} = (-1)^{k-h-1} z^{k-h} \quad \text{and} \quad \frac{\partial s_k}{\partial z} = (-1)^{k-1} P'_s(z)$$

where $P'_s(z)$ does not depend on s_k .

Let $F := 1 \otimes E(z) \sigma$ denote the basis $(1 \otimes z^j \sigma), j \in [0, k-1]$ of the free rank k left $\pi^{-1}(\mathcal{D}_{\mathbb{C}^k})$ -module $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$. Recall that, according to [5, Rem. 4.18], in view of the generators described above, the action of \mathcal{D}_H in $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$ is defined by the following formulas, where we consider F as a k -vector and use the usual matrix product

- (0) $F\theta(s') = \theta(s')F$ where $\theta \in \mathcal{O}_H$ does not depend on z
- (1) $Fz = A(s)F$
- (2) $-F\partial_{s_h} = \partial_{s_h}F + (-1)^{k-h-1}\partial_{s_k}(A(s)^{k-h}F) \quad \forall h \in [1, k-1]$
- (3) $-F\partial_z = \nabla F + (-1)^{k-1}\partial_{s_k}(P'_s(A(s))F)$

where we have used the equalities $zE(z) = A(s)E(z)$ and $\partial_z(E(z)) = \nabla E(z)$.

Summing up:

- For $g \in \mathcal{O}_H$ represented by $\sum_{r=0}^{k-1} g_r(s)z^r$, the $(k \times k)$ matrix G of the $\pi^{-1}\mathcal{D}_{\mathbb{C}^k}$ -linear morphism defined by g on $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$ is given by $G := \sum_{r=0}^{k-1} g_r(s)A(s)^r$
- Let us consider the $\mathcal{D}_{\mathbb{C}^k}$ -linear morphism $\alpha : (\mathcal{D}_{\mathbb{C}^k}^k)^{(k-1)} \rightarrow \mathcal{D}_{\mathbb{C}^k}^k$ defined by the following $k-1$ (k, k) -matrices

$$\mathcal{A}_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1} \partial_{s_k} \otimes A(s)^{k-h}$$

Let Φ be in $\mathcal{O}_{\mathbb{C}^k}^k$. In view of the relation (2), the map $(1 \otimes z^h \sigma) \mapsto \Phi_h$, for $h \in [0, k-1]$ will induce an element of $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$, that is to say a solution of $\pi_*\mathcal{N}$, if and only if we have $\partial_{s_h}(\Phi) = (-1)^{k-h} \partial_{s_k}(A^{k-h}\Phi)$, since the generator of $\pi_*\mathcal{N}$ is annihilated by the action of ∂_{s_h} for each $h \in [1, k-1]$.

In conclusion:

$$\pi_*\mathcal{N} \simeq \text{coker } \alpha$$

by the finiteness of the fibers of π . ■

Remark 4.2. \mathcal{N} is naturally endowed with a structure of right $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ -module. By functoriality $\pi_*\mathcal{N}$ is also a right $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ -module and its structure coincides with the induced by the right $\mathcal{D}_{\mathbb{C}}$ action defined by (1) and (3) on $\mathcal{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathcal{D}_H} \mathcal{N}$, since it commutes with each ∂_{s_i} , for $i = 1, \dots, k-1$. Therefore we obtain a natural left action of $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ on $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$.

We also conclude, according to Lemma 3.1:

Proposition 4.3. *The left action of $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ defined by the above Remark 4.2 on $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ coincides with the left action of $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ on Lisbon integrals.*

If Φ is a solution of $\pi_*\mathcal{N}$, replacing in the formula (3) the second term thanks to the equality obtained for ϕ after applying (2), we also derive a right action of ∂_z which is given by the formula

$$-\Phi \partial_z = \nabla \Phi + \sum_{h=0}^{k-1} (k-h) s_h \partial_{h+1} \Phi$$

Our next goal is to conclude in Proposition 4.5 below that there are no global holomorphic solutions of $\pi_*\mathcal{N}$ other than those of the form Φ_f , for some holomorphic function f only depending on z . Since j is non-characteristic we have an isomorphism

$$j^{-1} R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^{k+1}}}(\mathcal{L}, \mathcal{O}_{\mathbb{C}^{k+1}}) \simeq R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)$$

According to Theorem 4.33 (2) of [5], making $X = H, Y = \mathbb{C}^k, f = \pi, \mathcal{N} = \mathcal{O}_{\mathbb{C}^k}$ in loc.cit, we obtain

Theorem 4.4. *For any open subset Ω of \mathbb{C}^k we have an isomorphism functorial on \mathcal{N} compatible with restrictions to open subsets*

$$R\Gamma(\pi^{-1}(\Omega); R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)) \simeq R\Gamma(\Omega; R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))$$

Recall that this isomorphism uses as a tool the "trace morphism": $\pi_*\mathcal{O}_H \rightarrow \mathcal{O}_{\mathbb{C}^k}$ constructed in [5, Prop. 4.34].

Since for any open subset Ω and any \mathcal{D}_H -module \mathcal{P} , $\Gamma(\Omega; \cdot)$ and $\mathcal{H}om_{\mathcal{D}_H}(\mathcal{P}, \cdot)$ are left exact functors, since if Ω is a Stein open set and if \mathcal{P} admits a global resolution by free \mathcal{D}_H -modules of finite rank, then $R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{P}, \mathcal{O})$ is represented by a complex in degrees ≥ 0 with $\Gamma(\Omega, \cdot)$ -acyclic entries, we conclude that

$$H^0(R\Gamma(H; R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H))) = \Gamma(H; \mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)) \quad \text{and}$$

$$H^0(R\Gamma(\mathbb{C}^k; R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))) = \Gamma(\mathbb{C}^k; \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))$$

therefore Theorem 4.4 entails a \mathbb{C} -linear isomorphism

$$T : \text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) \simeq \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$$

Proposition 4.5. *The correspondence*

$$f \mapsto \Phi(f) := \Phi_f$$

defines a \mathbb{C} -linear isomorphism

$$\Phi : \Gamma(\mathbb{C}; \mathcal{O}_{\mathbb{C}}) \longrightarrow \Gamma(\mathbb{C}^k; \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})) = \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$$

Moreover, this isomorphism is also $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ -left linear.

PROOF. The last statement is clear from Proposition 4.3 and Lemma 3.1.

The remaining of the statement is equivalent to prove that $\Phi : f \mapsto \Phi_f$ defines an isomorphism $\text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) \rightarrow \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$.

We already know that Φ is injective. It remains to prove that Φ is surjective. For each $f \in \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$, we introduce the regular holonomic $\mathcal{D}_{\mathbb{C}}$ -module \mathcal{M}_f (a regular flat holomorphic connection on \mathbb{C}) of which the constant sheaf $\mathbb{C}f$ in degree zero is the complex of holomorphic solutions. We denote by \mathcal{N}_f the regular holonomic \mathcal{D}_H -module (a regular flat holomorphic connection on H)

$$\mathcal{N}_f := \mathcal{N} \boxtimes \mathcal{M}_f$$

It is clear that \mathcal{N}_f is a quotient of \mathcal{N} , and, by the left exactness of π_* , $\pi_*\mathcal{N}_f$ is a quotient of $\pi_*\mathcal{N}$. Moreover, according to Proposition 4.3 and Lemma 3.1, Φ_f belongs to $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$. According to Theorem 4.4, for each f we have a \mathbb{C} -linear isomorphism $T_f : \text{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H) \simeq \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ which makes this last one a one dimensional \mathbb{C} -vector space. Moreover, by left exactness of Hom and the exactness of π_* , we have monomorphisms $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k}) \subset \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ and, by functoriality, we have $T(f) = T_f(f)$.

We shall use the following result:

Lemma 4.6. *Suppose that $f \neq 0$. Then $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ is a one dimensional \mathbb{C} -vector space generated by Φ_f .*

PROOF. The result follows by Proposition 2.3 since Φ_f is a non zero element of $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ hence it is a generator as a \mathbb{C} -vector space. ■

Let us now end the proof of Proposition 4.5. Clearly $\text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) = \sum_f \text{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H)$ and, according to Lemma 4.6, for each f , $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ is the \mathbb{C} -vector space spanned by Φ_f ; hence $T(f) = \lambda\Phi_f$ for some $\lambda \in \mathbb{C}^*$. Since $\phi_{\lambda f} = \lambda\Phi_f$ we conclude that Φ is surjective which gives the desired result. ■

As a consequence, isomorphism Φ explicit isomorphism of Theorem 4.4 since they coincide up to the multiplication by a constant $\lambda \neq 0$.

4.a. An example. To conclude this article, let us give an interesting example of choice of the entire function f on \mathbb{C} for which we explicit the regular holonomic system on \mathbb{C}^k associated to the corresponding Lisbon integrals.

THE CASE $f_t(z) := e^{tz}$. Let us fix a parameter $t \in \mathbb{C}^*$ and consider the entire function $f_t(z) := e^{tz}$.

First remark that according to Lemma 3.1, we have $\frac{\partial E(z)}{\partial z} = \nabla E(z)$ where ∇ is the (k, k) matrix given by

$$\nabla := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & k-1 & 0 \end{pmatrix}$$

Integration by parts in the formula³

$$\Phi_{f_t}(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{t\zeta} \cdot E(\zeta) d\zeta}{P_s(\zeta)}$$

gives, using formula (7 – bis):

$$\begin{aligned} t\Phi_{f_t}(s) &= -\frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{\partial}{\partial \zeta} \left[\frac{E(\zeta)}{P_s(\zeta)} \right] d\zeta \\ &= -\frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{\nabla E(\zeta) P_s(\zeta) - P'_s(\zeta) E(\zeta)}{P_s^2(\zeta)} d\zeta \\ &= -\nabla \Phi_t(s) + \frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{P'_s(\zeta) E(\zeta)}{P_s^2(\zeta)} d\zeta \\ (tId + \nabla) \Phi_{f_t}(s) &= P'_s(A) \frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{E(\zeta)}{P_s^2(\zeta)} d\zeta + (-1)^k \frac{\partial P'_s(A)}{\partial s_k} \Phi_{f_t}(s) \\ &= P'_s(A) (-1)^{k-1} \frac{\partial \Phi_{f_t}}{\partial s_k}(s) + (-1)^{k-1} \frac{\partial P'_s(A)}{\partial s_k} \Phi_{f_t}(s) \\ &= (-1)^{k-1} \frac{\partial (P'_s(A) \Phi_{f_t})}{\partial s_k}(s) \end{aligned}$$

³Remember that t is a fixed non zero parameter

This implies the following equation

$$(12) \quad (t.Id + \nabla)P'_s(A)^{-1}\Psi_{f_t}(s) = (-1)^{k-1}\frac{\partial(\Psi_{f_t})}{\partial s_k}(s)$$

for

$$\Psi_{f_t}(s) = P'_s(A)\Phi_{f_t}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{t\zeta}P'_s(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.$$

Combining with the system (@@) it is easy to see that we obtain a meromorphic integrable connexion on the trivial vector bundle of rank k on \mathbb{C}^k with a pole along the discriminant hypersurface $\{\Delta(s) = 0\}$. The regularity of this meromorphic connexion is then consequence of the regularity of the \mathcal{D}_H -module \mathcal{N}_{etz} which is given by the equations

$$\partial_{s_h} = 0, \quad \forall h \in [1, k-1] \quad \text{and} \quad \partial_z - t = 0$$

which is clearly regular locally on H . So its direct image by π (as a $\mathcal{D}_{\mathbb{C}^k}$ -module) is regular holonomic on \mathbb{C}^k (see [6, Th. 8.1]).

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The research of T.Monteiro Fernandes was supported by Fundação para a Ciência e a Tecnologia, UID/MAT/04561/2019.

Daniel Barlet, Institut Elie Cartan, Géométrie, Université de Lorraine, Nancy, France

CNRS UMR 7502 and Institut Universitaire de France

email: *daniel.barlet@univ-lorraine.fr*

Teresa Monteiro Fernandes

Centro de Matemática e Aplicações Fundamentais-CIO

and

Departamento de Matemática da Faculdade de Ciências da Universidade de Lisboa, Bloco C6, Piso 2, Campo Grande, 1749-016, Lisboa Portugal

email : *mtfernandes@fc.ul.pt*