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# ON LISBON INTEGRALS

DANIEL BARLET AND TERESA MONTEIRO FERNANDES

ABSTRACT. We introduce a family of integral transforms, the Lisbon Integrals, which naturally arise in the study of the affine space of unitary polynomials  $P_s(z)$  where  $s \in \mathbb{C}^k$  and  $z \in \mathbb{C}$ ,  $s_i$  identified to the  $i$ -th symmetric function of the roots of  $P_s(z)$ . We completely determine the  $\mathcal{D}$ -module (or system of partial differential equations) the Lisbon Integrals satisfy and prove that they are their unique global solutions of this  $\mathcal{D}$ -module.

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## 1. INTRODUCTION

The purpose of this article is to show that the completely explicit differential system satisfied by the “Lisbon integrals” (defined in formulas (3) and (4) below) is in fact a  $\mathcal{D}$ -module obtained from a very simple one by the usual functorial operations on  $\mathcal{D}$ -modules (inverse image and direct image).

More precisely, the  $\mathcal{D}$ -module approach shows that the Lisbon integrals are holomorphic solutions of modules obtained as integral transforms associated to the affine space of unitary polynomials of degree  $k$  in the variable  $z \in \mathbb{C}$  with complex coefficients. Writing its elements in the form  $P(s, z) = \sum_{i=0}^k (-1)^i s_i z^{k-i}$  with  $s = (s_1, \dots, s_k) \in \mathbb{C}^k$  with the convention  $s_0 = 1$ , this affine space (see [2, Ch. I]) is naturally identified to the hypersurface  $H$  of  $\mathbb{C}^{k+1}$  endowed with the coordinates  $(s, z)$  of equation  $s_k = (-1)^{k-1} \sum_{i=1}^k (-1)^i s_i z^{k-i}$ . Let  $j : H \subset \mathbb{C}^{k+1}$  denote the closed embedding and let  $B_{H|\mathbb{C}^{k+1}}$  denote the regular holonomic  $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module of holomorphic distributions supported by  $H$ . Since the restriction of the projection

$$\pi : \mathbb{C}^{k+1} \longrightarrow \mathbb{C}^k, (s, z) \mapsto s$$

to  $H$  is proper (with finite fibers), given a  $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module  $\mathcal{L}$  non characteristic for  $H$ , following [8] we obtain a complex in  $\mathbf{D}_{\text{coh}}^b(\mathcal{D}_{\mathbb{C}^k})$ , the integral transformed of  $\mathcal{L}$ , given by the composition of the usual derived functors of direct image and inverse image for  $\mathcal{D}$ -modules,

$$D\pi_*(B_{H|\mathbb{C}^{k+1}} \overset{L}{\otimes}_{\mathcal{O}_{\mathbb{C}^{k+1}}} \mathcal{L}) \simeq D\pi_*(\mathcal{H}_{[H]}^1(\mathcal{L})) \simeq \pi_* j^* \mathcal{L}$$

Note that  $B_{H|\mathbb{C}^{k+1}} = \mathcal{H}_{[H]}^1(\mathcal{O}_{\mathbb{C}^{k+1}})$  and that for such a module  $\mathcal{L}$ , we have, thanks to Kashiwara’s equivalence theorem (cf Th 4.1 and Prop 4.2, [4]),  $\mathcal{H}_{[H]}^1(\mathcal{L}) \simeq \mathcal{H}^0 D j_* D j^* \mathcal{L} \simeq j_* j^* \mathcal{L}$ . In this note  $\mathcal{L}$  is the quotient of  $\mathcal{D}_{\mathbb{C}^{k+1}}$  by the ideal generated by the partial derivatives in  $s_i$ ,  $i = 1, \dots, k$ , hence the sheaf of solutions of  $\mathcal{L}$  is  $p^{-1}\mathcal{O}_{\mathbb{C}}$  where  $p(s, z) = z$ .

As a consequence, we show that the Lisbon Integrals are exactly the global solutions of  $\pi_* j^* \mathcal{L}$ .

These computations illustrate the fact that it is not so easy, even in a rather simple situation, to follow explicitly the computations hidden in the “yoga” of  $\mathcal{D}$ -module theory.

## 2. LISBON INTEGRALS AND THE DIFFERENTIAL SYSTEM THEY SATISFY

**2.a. Lisbon integrals.** For  $(z_1, \dots, z_k) \in \mathbb{C}^k$  denote  $s_1, \dots, s_k$  the elementary symmetric functions of  $z_1, \dots, z_k$ . We shall consider in the sequel  $s_1, \dots, s_k$  as coordinates on  $\mathbb{C}^k \simeq \mathbb{C}^k / \mathfrak{S}_k$ , isomorphism given by the standard symmetric function theorem.

We shall denote  $P_s(z) := \prod_{j=1}^k (z - z_j) = \sum_{h=0}^k (-1)^h s_h z^{k-h}$  with the convention  $s_0 \equiv 1$ .

**We shall often write  $P(s, z)$  instead of  $P_s(z)$  with no risk of ambiguity.**

The discriminant  $\Delta(s)$  of  $P_s$  is the polynomial in  $s$  corresponding to the symmetric polynomial  $\prod_{1 \leq i < j \leq k} (z_i - z_j)^2$  via the symmetric function theorem.

**Lemma 2.1.** For  $h \in \mathbb{N}$  and  $f \in \mathcal{O}(\mathbb{C})$  any entire holomorphic function, let us define, for  $R \gg \|s\|$ ,

$$(1) \quad \varphi_h(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)\zeta^h d\zeta}{P_s(\zeta)}.$$

Then  $\varphi_h(s)$  is independent of the choice of  $R$  large enough and defines a holomorphic function on  $\mathbb{C}^k$ . For  $\Delta(s) \neq 0$  we have

$$(2) \quad \varphi_h(s) = \sum_{j=1}^k \frac{z_j^h f(z_j)}{P'_s(z_j)}$$

where  $z_1, \dots, z_k$  are the roots of  $P_s(z)$ .

PROOF. The independence on  $R$  large enough when  $s$  stays in a compact set of  $\mathbb{C}^n$  is clear. For  $s$  in the interior of a compact set,  $P_s(\zeta)$  does not vanish on  $\{|\zeta| = R\}$  for  $R$  large enough, so we obtain the holomorphy of  $\varphi_h$  near any point in  $\mathbb{C}^k$ . The formula (2) is given by a direct application of the Residue's theorem.  $\blacksquare$

In fact, it will be convenient to consider the  $k$  functions  $\varphi_0, \dots, \varphi_{k-1}$  as the com-

ponent of a vector valued function  $\Phi := \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \dots \\ \varphi_{k-1} \end{pmatrix}$ . Defining  $E(z) := \begin{pmatrix} 1 \\ z \\ \dots \\ z^{k-1} \end{pmatrix}$  we

obtain

$$(3) \quad \Phi(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta)d\zeta}{P_s(\zeta)}.$$

**Definition 2.2.** We call  $\Phi$  (sometimes also denoted by  $\Phi_f$  when precision is required) the *Lisbon Integral associated to  $f$*

It will be also interesting to introduce another type of Lisbon integrals:

$$(4) \quad \Psi(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{f(\zeta)E(\zeta) \cdot P'_s(\zeta) d\zeta}{P_s(\zeta)}$$

$\Psi$  will also be noted below by  $\Psi_f$  when precision is required.

It is easy to see that this is again a vector valued holomorphic function on  $\mathbb{C}^k$  and the Residue's theorem gives that, for  $\Delta(s) \neq 0$ , the component  $\psi_h$  of  $\Psi$  is given by:

$$(5) \quad \psi_h(s) = \sum_{j=1}^k z_j^h f(z_j).$$

**Proposition 2.3.** If  $f$  is not identically zero then  $\Phi$  and  $\Psi$  are non zero vector-valued holomorphic functions on  $\mathbb{C}^k$ .

PROOF. Suppose that  $f$  is non identically zero. Then the statement follows as an immediate consequence of the non vanishing of the Van der Monde determinant of  $z_1, \dots, z_k$  when these complex numbers are pair-wise distinct.  $\blacksquare$

AN EXAMPLE. Take  $f \equiv 1$ . Then formula (5) shows that  $\psi_h(s)$  is the  $h$ -th Newton symmetric functions of the roots of the polynomial  $P_s$ . So it is a quasi-homogeneous polynomial in  $s$  of weight  $h$  (the weight of  $s_j$  is  $j$  by definition).

Let us show that we have  $\varphi_h(s) \equiv 0$  for  $h \in [0, k-2]$  and  $\varphi_{k-1}(s) \equiv 1$  in this case. For  $h \in [0, k-2]$  the formula (1) gives the estimate (with  $f \equiv 1$ )

$$|\varphi_h| \leq \frac{R^{h+1}}{(R-a)^{k-1}}$$

if each root of  $P_s$  is in the disc  $\{|z| \leq a\}$  when  $R > a > 0$ . When  $R \rightarrow +\infty$  this gives  $\varphi_h(s) \equiv 0$  for  $h \in [0, k-2]$ .

For  $h = k-1$  write

$$kz^{k-1} = P'_s(z) - \sum_{h=1}^{k-1} (-1)^h k(k-h) s_h z^{k-h-1}.$$

This gives, using the previous case and formula (2), that  $\varphi_{k-1}(s) \equiv 1$ .

**2.b. The partial differential system.** Let us introduce the  $(k, k)$  matrix  $A$  associated to the polynomial  $P_s$  by

$$(6) \quad A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & 0 & & \cdots & & 0 \\ & & & & & & 0 & 1 \\ (-1)^{k-1} \cdot s_k & \cdots & (-1)^{h-1} \cdot s_h & \cdots & \cdots & s_1 \end{pmatrix}$$

**Theorem 2.b.1.** *The vector value holomorphic function  $\Phi$  on  $\mathbb{C}^k$  satisfies the following differential system*

$$(@) \quad (-1)^{k+h} \cdot \frac{\partial \Phi}{\partial s_h}(s) = \frac{\partial (A^{k-h} \Phi)}{\partial s_k}(s) \quad \forall s \in \mathbb{C}^k \quad \text{and} \quad \forall h \in [1, k-1].$$

Moreover, this system is integrable<sup>1</sup> and if  $\Phi$  is a solution of this system, so is  $A\Phi$ .

The proof of this result will use several lemmas.

**Lemma 2.4.** *Let  $A$  be a  $(k, k)$  matrix with entries in  $\mathbb{C}[x]$  and put  $B := \lambda \frac{\partial A}{\partial x}$  where  $\lambda$  is a complex number. Let  $M$  be the  $(2k, 2k)$  matrix given by*

$$M := \begin{pmatrix} A & B \\ 0 & A \end{pmatrix}.$$

Then for each  $p \in \mathbb{N}$  we have

$$(*) \quad M^p = \begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix}$$

<sup>1</sup>We shall explain in the proof what we mean here.

where  $B_p := \lambda \frac{\partial(A^p)}{\partial x}$ .

PROOF. As the relation (\*) is clear for  $p = 0, 1$  let us assume that it has been proved for  $p$  and let us prove it for  $p + 1$ . We have:

$$\begin{pmatrix} A^p & B_p \\ 0 & A^p \end{pmatrix} \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} = \begin{pmatrix} A^{p+1} & A^p B + B_p A \\ 0 & A^{p+1} \end{pmatrix}$$

which allows to conclude. ■

**Corollary 2.b.2.** *For each integer  $p \in [0, k - 1]$  the following equality holds in the module  $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]/(P^2)$  over the  $\mathbb{C}$ -algebra  $\mathbb{C}[s_1, \dots, s_k, z]/(P^2)$*

$$(7) \quad z^p E(z) = A^p E(z) + (-1)^{k-1} P_s(z) \frac{\partial(A^p)}{\partial s_k} E(z)$$

In particular for any entire function on  $z$ ,

$$\Phi_{zf} = A(s) \Phi_f$$

Moreover the following identity in the module  $\mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}[s_1, \dots, s_k, z]/(P^2)$  holds

$$(7\text{-bis}) \quad P'_s(z) E(z) = P'_s(A) E(z) + (-1)^{k-1} P_s(z) \frac{\partial(P'_s(A))}{\partial s_k} E(z).$$

PROOF. In the basis  $1, z, \dots, z^{k-1}, P_s(z), zP_s(z), \dots, z^{k-1}P_s(z)$  of this algebra which is a free rank  $2k$  module on  $\mathbb{C}[s_1, \dots, s_k]$ , the multiplication by  $z$  is given by the matrix  $M$  of the previous lemma with  $A$  as in (6) and with  $B := (-1)^{k-1} \frac{\partial A}{\partial s_k}$ . This proves formula (7).

As  $P'_s(z) = \sum_{h=0}^{k-1} (-1)^h (k-h) z^{k-h-1}$  does not depend on  $s_k$  it is enough to sum up the previous equalities with  $p = k - h - 1$  with the convenient coefficients to obtain the equality (7 - bis). ■

**Lemma 2.5.** *Let  $A$  be a  $(k, k)$  matrix with entries in  $\mathbb{C}[x, y]$ . Assume that for some integer  $q$  we have*

$$\frac{\partial A}{\partial x} A^q = \frac{\partial A}{\partial y}$$

then we have for any integer  $h$

$$\frac{\partial A^h}{\partial x} A^q = \frac{\partial A^h}{\partial y}$$

PROOF. The assertion is our hypothesis for  $h = 1$ . Assume the assertion proved for some  $h \geq 1$ . Then Leibnitz rule gives:

$$\frac{\partial A^{h+1}}{\partial x} A^q = \frac{\partial A^h}{\partial x} A^{q+1} + A^h \frac{\partial A}{\partial x} A^q$$

which implies:

$$\frac{\partial A^{h+1}}{\partial x} A^q = \left( \frac{\partial A^h}{\partial y} A + A^h \frac{\partial A}{\partial y} \right) = \frac{\partial A^{h+1}}{\partial y}$$

which is our assertion for  $h + 1$ . ■

**Corollary 2.b.3.** *For any  $h \in [1, k]$  and any  $p \in \mathbb{N}$  the matrix  $A$  in (6) satisfies the relation:*

$$(8) \quad (-1)^{k-h} \frac{\partial A^p}{\partial s_h} = \frac{\partial A^p}{\partial s_k} A^{k-h}$$

PROOF. The case  $p = 1$  of (7) is an easy direct computation on the matrix  $A$ . Assume that the assertion is proved for  $p \geq 1$ . Then Leibnitz's rule gives:

$$\begin{aligned} (-1)^{k-h} \frac{\partial A^{p+1}}{\partial s_h} &= (-1)^{k-h} \frac{\partial A^p}{\partial s_h} A + A^p (-1)^{k-h} \frac{\partial A}{\partial s_h} \\ &= \frac{\partial A^p}{\partial s_k} A^{k-h+1} + A^p \frac{\partial A}{\partial s_k} A^{k-h} = \frac{\partial A^{p+1}}{\partial s_k} A^{k-h} \end{aligned}$$

concluding the proof of (8). ■

PROOF OF THE THEOREM 2.b.1. By derivation inside the integral in (3) we obtain:

$$\begin{aligned} \frac{\partial \Phi}{\partial s_h}(s) &= \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) (-1)^{h+1} \zeta^{k-h} \frac{d\zeta}{P_s(\zeta)^2} \quad \text{and in particular} \\ \frac{\partial \Phi}{\partial s_k}(s) &= \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) (-1)^{k+1} \frac{d\zeta}{P_s(\zeta)^2} \end{aligned}$$

Now for  $h \in [1, k-1]$  we use the formula of corollary 2.b.2 to obtain:

$$\frac{\partial \Phi}{\partial s_h} = (-1)^{h+1} A^{k-h} (-1)^{k-1} \frac{\partial \Phi}{\partial s_k} + (-1)^{h+1} (-1)^{k-1} \frac{\partial A^{k-h}}{\partial s_k} \Phi$$

that is to say:

$$(\textcircled{a}) \quad (-1)^{k+h} \frac{\partial \Phi}{\partial s_h} = \frac{\partial (A^{k-h} \Phi)}{\partial s_k} \quad \forall h \in [1, k]$$

By the integrability of the system  $(\textcircled{a})$  we mean that for any  $\Phi$  such that  $(\textcircled{a})$  holds, then the computation of the partial derivatives  $\frac{\partial^2 \Phi}{\partial s_h \partial s_j}$  using the system  $(\textcircled{a})$  gives a symmetric result in  $(h, j)$  for any pair  $(h, j)$  in  $[1, k]$ . Note that if  $h$  or  $j$  is equal to  $k$  the assertion is trivial.

So consider a couple  $(h, j) \in [1, k-1]^2$ . Thanks to corollary 2.b.3 we have :

$$\begin{aligned} (-1)^{h+j} \frac{\partial^2 \Phi}{\partial s_j \partial s_h} &= (-1)^{k-j} \frac{\partial}{\partial s_k} \left[ \frac{\partial (A^{k-h} \Phi)}{\partial s_j} \right] \\ &= (-1)^{k-j} \frac{\partial}{\partial s_k} \left[ \frac{\partial A^{k-h}}{\partial s_j} \Phi + A^{k-h} \frac{\partial \Phi}{\partial s_j} \right] \\ &= \frac{\partial}{\partial s_k} \left[ \frac{\partial A^{k-h}}{\partial s_j} A^{k-j} \Phi + A^{k-h} \frac{\partial (A^{k-j} \Phi)}{\partial s_k} \right] \\ &= \frac{\partial^2}{\partial s_k^2} [A^{2k-h-j} \Phi] \end{aligned}$$

which is symmetric in  $(h, j)$ .

To finish the proof of the theorem we have to show that  $A\Phi$  is a solution of  $(\textcircled{a})$  when  $\Phi$  is a solution of  $(\textcircled{a})$ . This is given by the following computation

$$\begin{aligned}
(-1)^{k-h} \frac{\partial(A\Phi)}{\partial s_h} &= (-1)^{k-h} \frac{\partial A}{\partial s_h} \Phi + (-1)^{k-h} A \frac{\partial \Phi}{\partial s_h} \\
&= \frac{\partial A}{\partial s_k} A^{k-h} \Phi + A \frac{\partial A^{k-h} \Phi}{\partial s_k} = \frac{\partial(A^{k-h}(A\Phi))}{\partial s_k}
\end{aligned}$$

which also uses corollary 2.b.3. ■

REMARK. A consequence of our computation on the integrability of the system (@) is the fact that for any solution  $\Phi$  and any pair  $(h, j) \in [1, k]$  the second order partial derivative  $\frac{\partial^2 \Phi}{\partial s_j \partial s_h}$  only depends on  $h + j$ . This implies that any scalar Lisbon integral  $\varphi_h$  satisfies

$$(9) \quad \frac{\partial^2 \varphi_h}{\partial s_p \partial s_{q+1}} = \frac{\partial^2 \varphi_h}{\partial s_{p+1} \partial s_q} \quad \forall p, q \text{ such that } 1 \leq p < q \leq k - 1$$

The following corollary of theorem 2.b.1 describes the analogous system for the the vector functions  $\Psi$  defined in (4) which is singular along the discriminant subset  $\{\Delta(s) = 0\}$  in  $\mathbb{C}^k$ .

**Corollary 2.b.4.** *The vector value holomorphic function  $\Psi$  on  $\mathbb{C}^k$  satisfies the following differential system:*

$$(@@) \quad (-1)^{k+h} \frac{\partial \Psi}{\partial s_h}(s) = \frac{\partial(A^{k-h} \Psi)}{\partial s_k}(s) + (-1)^k (k-h) A^{k-h-1} P'_s(A)^{-1} \Psi(s)$$

for all  $h \in [1, k-1]$ , which is singular along the discriminant sub-set

$$\Delta := \{s \in \mathbb{C}^k / \Delta(s) = 0\}.$$

Again, if  $\Psi$  is any solution of (@@) then  $A\Psi$  is also a solution of (@@).

PROOF. Define  $F(z) := P'(z)E(z)$ . Then using formula (7) we obtain  $\Psi(s) = P'_s(A)\Phi(s)$  and this gives for each  $h \in [1, k-1]$

$$(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-h} A^{k-h-1} \Phi(s) + \sum_{p=0}^{k-1} (-1)^p (k-p) (-1)^{k-h} s_p \cdot \frac{\partial(A^{k-p-1} \Phi)}{\partial s_h}(s)$$

and using now the fact that  $A^{k-p-1} \Phi$  is solution of (@) we obtain

$$(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-h} A^{k-h-1} \Phi(s) + \sum_{p=0}^{k-1} (-1)^p (k-p) s_p \cdot \frac{\partial(A^{2k-p-h-1} \cdot \Phi)}{\partial s_k}(s)$$

$$(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-h} A^{k-h-1} \Phi(s) + \frac{\partial(A^{k-h} P'_s(A) \Phi)}{\partial s_k}(s)$$

$$(-1)^{k-h} \frac{\partial \Psi}{\partial s_h}(s) = (-1)^{k-h} A^{k-h-1} P'_s(A)^{-1} \Psi(s) + \frac{\partial(A^{k-h} \Psi)}{\partial s_k}(s)$$

Since  $P'_s(A)$  commutes with  $A$ , the last assertion is easy. ■



2.c. **Example: The cases**  $k = 2, 3$ . The left ideal of partial differential operators in the Weyl algebra  $\mathbb{C}[s_1, \dots, s_k] \langle \partial_{s_1}, \dots, \partial_{s_k} \rangle$  which annihilate Lisbon integrals is described in the article [1].

Below we will explicit the cases  $k = 2$  and  $k = 3$ .

Case  $k = 2$ :

Here we use the notations  $s := s_1$  and  $p := s_2$ . In that case, the differential system (@) becomes:

$$(10) \quad -\frac{\partial \Phi_0}{\partial s} = \frac{\partial \Phi_1}{\partial p}$$

$$(11) \quad -\frac{\partial \Phi_1}{\partial s} = -\Phi_0 - p \frac{\partial \Phi_0}{\partial p} + s \frac{\partial \Phi_1}{\partial p}$$

corresponding to the matrix  $A := \begin{pmatrix} 0 & 1 \\ -p & s \end{pmatrix}$ . Differentiating (9) with respect to  $p$  we obtain, after substituting via (10)

$$\begin{aligned} \frac{\partial^2 \Phi_0}{\partial s^2} &= -2 \frac{\partial \Phi_0}{\partial p} - p \frac{\partial^2 \Phi_0}{\partial p^2} - s \frac{\partial^2 \Phi_0}{\partial s \partial p} \\ (\#) \quad \text{and so} \quad \frac{\partial^2 \Phi_0}{\partial s^2} + s \frac{\partial^2 \Phi_0}{\partial s \partial p} + p \frac{\partial^2 \Phi_0}{\partial p^2} + 2 \frac{\partial \Phi_0}{\partial p} &= 0 \end{aligned}$$

Differentiating (10) with respect to  $s$  we obtain, after substituting via (9)

$$\begin{aligned} -\frac{\partial^2 \Phi_1}{\partial s^2} &= \frac{\partial \Phi_1}{\partial p} + p \frac{\partial^2 \Phi_1}{\partial p^2} + \frac{\partial \Phi_1}{\partial p} + s \frac{\partial^2 \Phi_1}{\partial s \partial p} \\ (\#\#) \quad \text{and so} \quad \frac{\partial^2 \Phi_1}{\partial s^2} + s \frac{\partial^2 \Phi_1}{\partial s \partial p} + p \frac{\partial^2 \Phi_1}{\partial p^2} + 2 \frac{\partial \Phi_1}{\partial p} &= 0 \end{aligned}$$

Then the second order differential operator of weight  $-2$

$$(\#\#\#) \quad \Theta := \frac{\partial^2}{\partial s^2} + s \frac{\partial^2}{\partial s \partial p} + p \frac{\partial^2}{\partial p^2} + 2 \frac{\partial}{\partial p}$$

annihilates  $\Phi_0$  and  $\Phi_1$  for any solution  $\Phi$  of the system (10), (11).

A DIRECT PROOF THAT  $\Theta$  ANNIHILATES  $\varphi_m$  FOR ALL  $m \in \mathbb{N}$ . We have, for any entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$  and for  $R \gg \max\{|s|, |p|\}$ :

$$(a) \quad \varphi_m(s, p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{\zeta^2 - s\zeta + p}$$

This gives:

$$\begin{aligned}
(b) \quad & \frac{\partial \varphi_m}{\partial s}(s, p) = \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^2} \\
(c) \quad & \frac{\partial \varphi_m}{\partial p}(s, p) = -\frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^2} \\
(d) \quad & \frac{\partial^2 \varphi_m}{\partial s^2}(s, p) = 2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+2} d\zeta}{(\zeta^2 - s\zeta + p)^3} \\
(e) \quad & \frac{\partial^2 \varphi_m}{\partial s \partial p}(s, p) = -2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^{m+1} d\zeta}{(\zeta^2 - s\zeta + p)^3} \\
(f) \quad & \frac{\partial^2 \varphi_m}{\partial p^2}(s, p) = 2 \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) \frac{\zeta^m d\zeta}{(\zeta^2 - s\zeta + p)^3}
\end{aligned}$$

Now it is easy to check that  $(d) + s(e) + p(f) + 2(c) = 0$ .

Case  $k = 3$ :

The differential system system (@) becomes now:

$$(@) \quad (-1)^{k-h} \partial_h \Phi = \partial_k (A^{k-h} \Phi) \quad \text{for } h \in [1, k-1]$$

with  $k = 3$ . Here  $\Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix}$  and  $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s_3 & -s_2 & s_1 \end{pmatrix}$ . Explicitly this gives:

$$\begin{aligned}
\partial_1 \Phi_0 &= \partial_3 \Phi_2 \\
\partial_1 \Phi_1 &= \partial_3 (s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2) \\
\partial_1 \Phi_2 &= \partial_3 (s_1 s_3 \Phi_0 - (s_3 - s_1 s_2) \Phi_1 + (s_1^2 - s_2) \Phi_2) \\
- \partial_2 \Phi_0 &= \partial_3 \Phi_1 \\
- \partial_2 \Phi_1 &= \partial_3 \Phi_2 \\
- \partial_2 \Phi_2 &= \partial_3 (s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2)
\end{aligned}$$

which implies the equalities

$$\partial_1 \Phi_0 = -\partial_2 \Phi_1 = \partial_3 \Phi_2 \quad \partial_1 \Phi_1 = -\partial_2 \Phi_2$$

Then we deduce that:

$$(\partial_2^2 - \partial_1 \partial_3)(\Phi_j) = 0 \quad \text{for } j = 0, 1, 2$$

because

$$\begin{aligned}
\partial_2^2 \Phi_0 &= -\partial_2 \partial_3 \Phi_1 = \partial_3 \partial_3 \Phi_2 = \partial_3 \partial_1 \Phi_0, \\
\partial_1 \partial_3 \Phi_1 &= -\partial_1 \partial_2 \Phi_0 = \partial_2 \partial_3 \Phi_2 = \partial_2^2 \Phi_1, \\
\text{and } \partial_2^2 \Phi_2 &= -\partial_2 \partial_1 \Phi_1 = \partial_3 \partial_2 \Phi_2.
\end{aligned}$$

**Lemma 2.6.** *Let us define:*

$$\begin{aligned}\mathcal{L} &:= \partial_1^2 + s_1 \partial_1 \partial_2 + s_2 \partial_2^2 + s_3 \partial_2 \partial_3 + 2\partial_2 \\ \mathcal{L}' &:= \partial_1 \partial_2 + s_1 \partial_2^2 + s_2 \partial_2 \partial_3 + s_3 \partial_3^2 + 2\partial_3\end{aligned}$$

*Then we have:*

$$\mathcal{L}\Phi_0 = 0 \quad \mathcal{L}'\Phi_1 = 0 \quad \text{et} \quad \mathcal{L}\Phi_2 = 0$$

*and also:*

$$(r) \quad \partial_3 \mathcal{L} - \partial_2 \mathcal{L}' \in \mathcal{D}_{\mathbb{C}^k}(\partial_1 \partial_3 - \partial_2^2)$$

*This entails that  $\partial_3 \mathcal{L}$  and  $\partial_2 \mathcal{L}'$  annihilate  $\Phi_j$  for  $j = 0, 1, 2$ .*

PROOF. We have

$$\begin{aligned}\partial_1 \Phi_1 &= \Phi_0 + s_3 \partial_3 \Phi_0 - s_2 \partial_3 \Phi_1 + s_1 \partial_3 \Phi_2 \\ &= \Phi_0 + s_3 \partial_3 \Phi_0 + s_2 \partial_2 \Phi_0 + s_1 \partial_1 \Phi_0 \\ \text{and so} \quad \partial_2 \partial_1 \Phi_1 &= -\partial_1^2 \Phi_0 = \\ &= \partial_2 \Phi_0 + s_3 \partial_2 \partial_3 \Phi_0 + \partial_2 \Phi_0 + s_2 \partial_2^2 \Phi_0 + s_1 \partial_1 \partial_2 \Phi_0\end{aligned}$$

and so  $\Phi_0$  is killed by  $\mathcal{L}$  as  $(\partial_1 \partial_3 - \partial_2^2)\Phi_0 = 0$ .

In a similar way:

$$\begin{aligned}\partial_1 \Phi_1 &= -\partial_2 \Phi_2 = \partial_3 (s_3 \Phi_0 - s_2 \Phi_1 + s_1 \Phi_2) \\ \partial_2 \partial_1 \Phi_1 &= -\partial_3 (s_3 \partial_3 \Phi_1) - \partial_2 \partial_3 (s_2 \Phi_1) - s_1 \partial_2^2 \Phi_1 \\ &= -\partial_3 \Phi_1 - s_3 \partial_3^2 \Phi_1 - \partial_3 \Phi_1 - s_2 \partial_2 \partial_3 \Phi_1 - s_1 \partial_2^2 \Phi_1\end{aligned}$$

showing that  $\mathcal{L}'$  annihilates  $\Phi_1$ .

In a similar way:

$$\begin{aligned}\partial_1 \Phi_2 &= s_1 \Phi_0 + s_1 s_3 \partial_3 \Phi_0 + \Phi_1 + (s_3 - s_1 s_2) \partial_3 \Phi_1 + (s_1^2 - s_2) \partial_3 \Phi_2 \\ &= s_1 \partial_1 \Phi_1 + \Phi_1 - s_2 \partial_3 \Phi_2 + s_3 \partial_3 \Phi_1 \\ &= s_1 \partial_1 \Phi_1 + \Phi_1 + s_2 \partial_2 \Phi_1 + s_3 \partial_3 \Phi_1 \\ \partial_1^2 \Phi_2 &= -\partial_2 \Phi_2 - s_1 \partial_1 \partial_2 \Phi_2 - \partial_2 \Phi_2 + s_2 \partial_2 (-\partial_2 \Phi_2) + s_3 \partial_3 (\partial_1 \Phi_1) \\ &= -2\partial_2 \Phi_2 - s_1 \partial_1 \partial_2 \Phi_2 - s_2 \partial_2^2 \Phi_2 - s_3 \partial_2 \partial_3 \Phi_2\end{aligned}$$

and  $\Phi_2$  is annihilated by  $\mathcal{L}$ .

Let us prove now the relation (r):

$$\begin{aligned}\partial_3 \mathcal{L} &= \partial_3 \partial_1^2 + s_1 \partial_1 \partial_2 \partial_3 + s_2 \partial_1 \partial_3^2 + s_3 \partial_2 \partial_3^2 + 3\partial_2 \partial_3 \\ \partial_2 \mathcal{L}' &= \partial_1 \partial_2^2 + s_3 \partial_2 \partial_3^2 + s_2 \partial_2^2 \partial_3 + s_1 \partial_2^3 + 3\partial_2 \partial_3\end{aligned}$$

this allows to conclude, as

$$\begin{aligned}\partial_3 \partial_1^2 - \partial_1 \partial_2^2 &= \partial_1 (\partial_1 \partial_3 - \partial_2^2) \\ \partial_1 \partial_2 \partial_3 - \partial_2^3 &= \partial_2 (\partial_1 \partial_3 - \partial_2^2)\end{aligned}$$

For the study of differential operators annihilating all the scalar components of the Lisbon integrals (for any value of  $k$ ) see [1].

3. THE LEFT ACTION OF  $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$  ON LISBON INTEGRALS

For each entire function  $f$  of the variable  $z$  we shall henceforward denote by  $\Phi_f$  the associated Lisbon integral (previously generically denoted by  $\Phi$ ). Clearly the assignment

$$f \mapsto \Phi_f$$

is  $\mathbb{C}$ -linear and, according to Proposition 2.3, it is injective.

**Lemma 3.1.** *Let  $f$  be an entire function on  $\mathbb{C}$  and let  $\Phi_f$  the corresponding Lisbon integral.*

- (1) *Let  $h$  be an entire function of  $z$ . Then  $\Phi_{hf} = h(A(s))\Phi_f$ . In particular  $\Phi_f = f(A(s))\Phi_1$ .*  
(2) *We have the identity*

$$\Phi_{\partial_z(f)}(s) = -\nabla\Phi_f(s) + \left( \sum_{h=0}^{k-1} (k-h)s_h \partial_{s_{h+1}} \right) (\Phi_f)(s)$$

$$\text{where } \nabla \text{ is the constant } (k, k) \text{ matrix } \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & k-1 & 0 \end{pmatrix}.$$

PROOF. When  $h$  is a polynomial on  $z$ , statement (1) follows easily in Corollary 2.b.2. For an arbitrary entire function  $h$ , it is a consequence of [2, Lem 3.1.8].

Let us prove (2): Consider the Lisbon integral

$$\Phi_{\partial_z f}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} f'(\zeta) E(\zeta) \frac{d\zeta}{P_s(\zeta)}$$

Integration by parts gives, as  $\partial_z(E)(z) = \nabla E(z)$  :

$$(*) \quad \Phi_{\partial_z(f)}(s) = -\nabla\phi_f(s) + \frac{1}{2i\pi} \int_{|\zeta|=R} f(\zeta) E(\zeta) \frac{P'_s(\zeta)}{P_s(\zeta)^2} d\zeta$$

Now, using the equalities  $P'_s(\zeta) = \sum_{h=0}^{k-1} (-1)^h (k-h)s_h \zeta^{k-h-1}$  and

$$\frac{\partial\Phi_f}{\partial s_h}(s) = - \int_{|\zeta|=R} f(\zeta) E(\zeta) \frac{(-1)^h \zeta^{k-h} d\zeta}{P_s(\zeta)^2}$$

we obtain the formula of the lemma. ■

REMARK. From the formula (\*) and using the fact that  $P'_s(\zeta)E(\zeta) = P'_s(A)E(\zeta)$  we obtain also the formula

$$(**) \quad \Phi_{\partial_z(f)}(s) = -\nabla\Phi_f(s) + (-1)^k \frac{\partial(P'_s(A)\Phi_f)}{\partial s_k}(s)$$

It is not obvious that when  $\Phi$  is solution of the system (@), then  $(\Phi\partial_z)(s) := -\nabla\Phi(s) + (-1)^k \frac{\partial(P'_s(A)\Phi)}{\partial s_k}(s)$  (given by the formula (\*\*)) is also a solution of the same system. A direct verification of this fact is consequence of the formula given in the lemma below.

In order to verify that we have  $\Phi_{(z,f)'} = \Phi_f + \Phi_{zf'}$  corresponding to the usual relation  $\partial_z z - z \partial_z = 1$  the following lemma is useful.

**Lemma 3.2.** *We have the following identity:*

$$\nabla A - A \nabla + Id = (-1)^{k-1} \partial_k(A) P'_s(A), \quad \forall s \in \mathbb{C}^k.$$

PROOF. It is an easy computation to obtain that  $\nabla A - A \nabla + Id$  is the matrix which have all lines equal to 0 excepted its last one which is given by  $(x_1, \dots, x_k)$  with  $x_h = (-1)^{k-h} h s_{k-h}$  for  $h \in [1, k]$  with  $s_0 \equiv 0$ . On the other hand, the matrix  $(-1)^{k-1} \partial_k A$  has only a non zero term at the place  $(k, 1)$  which equal to 1, so it is quite easy to see that  $(-1)^{k-1} (\partial_k A) A^p$  has only a non zero term at the place  $(k, p+1)$  with value  $(-1)^{k-1}$ . According to the computation of  $(-1)^{k-1} \partial_k(A) P'_s(A)$  we conclude the desired formula.  $\blacksquare$

#### 4. THE $\mathcal{D}_{\mathbb{C}^k}$ -MODULE ASSOCIATED TO LISBON INTEGRALS

We shall begin by recalling some basic facts on  $\mathcal{D}$ -modules and fixing our notations.

For a morphism of manifolds  $f : Y \rightarrow X$ , we use the notation of [7]

$$f_d := {}^t f' : T^*X \times_X \times Y \longrightarrow T^*Y$$

and

$$f_\pi : T^*X \times_X \times Y \longrightarrow T^*X$$

the associated canonical morphisms of vector bundles.

We recall that a conic involutive submanifold  $V$  of the cotangent bundle  $T^*Z$  of a manifold  $Z$  (real or complex) is *regular* if the restriction  $\omega|_V$  of the canonical 1-form  $\omega$  on  $T^*Z$  never vanishes outside the 0-section. Recall also that if  $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$  are canonical symplectic coordinates on  $T^*Z$ , then  $\omega(x, \xi) = \sum_{i=1}^n \xi_i dx_i$ .

Let us fix some  $k \in \mathbb{N}$ ,  $k \geq 2$ . In  $\mathbb{C}^{k+1} = \mathbb{C}^k \times \mathbb{C}$  we consider the coordinates  $(s_1, \dots, s_k, z)$ . As in the previous sections we set

$$P(s, z) = z^k + \sum_{h=1}^{k-1} (-1)^h s_h z^{k-h}$$

Obviously

$$P(s_1, \dots, s_k, z) = 0 \iff s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h},$$

where  $s_0 = 1$ . We note  $s = (s_1, \dots, s_k)$  and  $s' := (s_1, \dots, s_{k-1})$ .

Let  $H$  be the smooth hypersurface of  $\mathbb{C}^{k+1}$  given by the zeros of  $P(s, z)$ .

Let us denote by  $\mathcal{L}$  the  $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module with one generator  $u$  defined by the equations  $\partial u / \partial s_1 = \dots = \partial u / \partial s_k = 0$ . Such module is an example of a so called *partial de Rham systems*, which have the feature, among others, that their characteristic varieties are non singular regular involutive. In our case we have

$$\text{Char } \mathcal{L} = \{(s, z); (\eta, \tau) \in \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \text{ such that } \eta = 0\}.$$

Since  $H \subset \mathbb{C}^{k+1}$  is defined by the equation

$$P(s, z) = (-1)^k s_k + \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h} = 0,$$

$T_H^* \mathbb{C}^{k+1}$  is the subbundle of  $T^* \mathbb{C}^{k+1}|_H$  described by

$$\{(s, z; \eta, \tau), (s, z) \in H, \exists \lambda \in \mathbb{C} \text{ such that } (\eta, \tau) = \lambda dP(s, z)\}.$$

This means that  $P(s, z) = 0$  and that there exists  $\lambda \in \mathbb{C}$  with  $\eta_h = \lambda \cdot (-1)^h \cdot z^{k-h}$  for each  $h \in [1, k]$  and that  $\tau = \lambda \cdot P'(s, z)$ . Hence as  $\eta_k = \lambda(-1)^k$  and this implies:

$$\text{Char } \mathcal{L} \cap T_H^* \mathbb{C}^{k+1} \subset T_{\mathbb{C}^{k+1}}^* \mathbb{C}^{k+1}$$

(as usual,  $T_{\mathbb{C}^{k+1}}^* \mathbb{C}^{k+1}$  denotes the zero section of  $T^* \mathbb{C}^{k+1}$ ), in other words  $H$  is non-characteristic for  $\mathcal{L}$ . Let us denote by  $j : H \hookrightarrow \mathbb{C}^{k+1}$  the closed embedding. By Kashiwara's classical results (which can be found in [5]) it follows that the induced system  $Dj^* \mathcal{L}$  by  $\mathcal{L}$  on  $H$  is concentrated in degree zero and  $\mathcal{N} := \mathcal{H}^0 Dj^* \mathcal{L}$  is a  $\mathcal{D}_{\mathbb{C}^k}$ -coherent module whose characteristic variety is exactly

$$j_* j^* \text{Char}(\mathcal{L}).$$

Recall that

$$Dj^* \mathcal{L} := \mathcal{O}_H \otimes_{j^{-1} \mathcal{O}_{\mathbb{C}^{k+1}}}^L j^{-1} \mathcal{L}$$

and in this non-characteristic case

$$\simeq j^{-1} \frac{\mathcal{O}_{\mathbb{C}^{k+1}}}{P(s', s_k, z) \mathcal{O}_{\mathbb{C}^{k+1}}} \otimes_{j^{-1} \mathcal{O}_{\mathbb{C}^{k+1}}} j^{-1} \mathcal{L}$$

We have

$$\mathcal{N} := j^{-1} \left( \frac{\mathcal{D}_{\mathbb{C}^{k+1}}}{P \mathcal{D}_{\mathbb{C}^{k+1}} + \mathcal{D}_{\mathbb{C}^{k+1}} \partial_{s_1} + \cdots + \mathcal{D}_{\mathbb{C}^{k+1}} \partial_{s_k}} \right) \simeq j^{-1} (\mathcal{O}_{\mathbb{C}^{k+1}} / P \mathcal{O}_{\mathbb{C}^{k+1}}) \langle \partial_z \rangle$$

which is isomorphic as a  $\mathcal{D}_H$ -module to

$$\mathcal{O}_H \langle \partial_z \rangle \simeq \frac{\mathcal{D}_H}{\mathcal{D}_H \partial_{s_1} + \cdots + \mathcal{D}_H \partial_{s_{k-1}}}$$

where  $\partial_{s_i}$  stands for the derivation  $\partial/\partial s_i$  on  $\mathcal{O}_H$  and  $\partial_z$  as a derivation on  $\mathcal{O}_H$  is the class of  $\partial_z$  in the quotient above.

In particular  $\mathcal{N}$  is sub-holonomic and it is a partial de Rham system similarly to  $\mathcal{L}$ . Let us now determine the image of  $\mathcal{N}$  under the morphism  $\pi : \mathbb{C}^k \simeq H \rightarrow \mathbb{C}^k$  given by  $(s', z) \mapsto (s', s_k(s', z))$ . Clearly  $\pi$  is proper surjective with finite fibers.

Recall that one denotes by  $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$  the transfer  $(\pi^{-1} \mathcal{D}_{\mathbb{C}^k}, \mathcal{D}_H)$ -bimodule

$$(\pi^{-1} \mathcal{D}_{\mathbb{C}^k} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1}) \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H$$

Recall also that, according to the properness and the fiber finiteness of  $\pi$ , we have

$$D\pi_* \mathcal{N} \simeq \mathcal{H}^0 D\pi_* \mathcal{N} = \pi_* (\mathcal{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathcal{D}_H} \mathcal{M})$$

where we abusively use the notation  $\pi_*$  for the direct image functor in the category of  $\mathcal{D}$ -modules in the two left terms and for the direct image functor for sheaves in the

right term. According to [5, Th. 4.25 and 4.27] (see also the comments in loc.cit. before Theorem 4.27), one knows that  $D\pi_*\mathcal{N}$  is concentrated in degree zero and that

$$\text{Char } \mathcal{H}^0 D\pi_*\mathcal{N} = \pi_\pi \pi_d^{-1} \text{Char } \mathcal{N}.$$

So we may henceforward denote for short  $\pi_*\mathcal{N} := D\pi_*\mathcal{N}$  without ambiguity. Let  $\Delta$  as above be the zero set of the discriminant of  $P$ , which can also be defined as the image by  $\pi$  of the subset of  $H$  defined by  $\{P'(s, z) = 0\}$ . Since  $\pi_d$  is given by the  $k \times k$  matrix

$$\begin{pmatrix} Id & 0 \\ \partial s_k / \partial s' & \partial s_k / \partial z \end{pmatrix}^T$$

we conclude that  $\text{Char } \pi_*\mathcal{N}$  is the image by  $\pi_\pi$  of the set

$$\{(s', z); (\eta', \tau) \in \mathbb{C}^k \times \mathbb{C}^k / \eta_j = -(-z)^{k-j} \cdot \tau \quad \forall j \in [1, k-1]\}$$

so it is given by the set

$$\{(s, \eta) \in \mathbb{C}^k \times \mathbb{C}^k / \exists z \in \mathbb{C} \text{ such that } P_s(z) = 0 \text{ and with } \eta_j = (-z)^{k-j} \eta_k \quad \forall j \in [1, k-1]\}$$

Then  $\text{Char } \pi_*\mathcal{N}$  is an involutif analytic subset of  $T^*\mathbb{C}^k$  with codimension  $k-1$ , hence  $\pi_*\mathcal{N}$  is a subholonomic  $\mathcal{D}_{\mathbb{C}^{k+1}}$ -module. Denoting by  $p$  the projection:  $T^*\mathbb{C}^k \rightarrow \mathbb{C}^k$ , for any  $s$  such that  $\Delta(s) \neq 0$ , the Implicit Function Theorem implies that

$$p^{-1}(s) \cap \text{Char } \pi_*\mathcal{N} \text{ is a cone of } l \text{ distinct complex lines } l_j = \mathbb{C} \begin{pmatrix} (-z_j)^{k-1} \\ (-z_j)^{k-2} \\ \dots \\ (-z_j) \\ 1 \end{pmatrix}, \text{ where}$$

$z_j$  runs over the set of  $l$  distinct roots of  $P(s, z)$

**Proposition 4.1.** *The  $\mathcal{D}_{\mathbb{C}^k}$ -module  $\pi_*\mathcal{N}$  is the quotient of  $\mathcal{D}_{\mathbb{C}^k}^k \simeq \mathcal{D}_{\mathbb{C}^k} \otimes_{\mathbb{C}} \mathbb{C}^k$  by the action of*

$$\mathcal{A}_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1} \partial_{s_k} \otimes A(s)^{k-p} \quad \text{for } j \in [1, k-1].$$

Moreover the action of  $z$  and  $\partial_z$  on  $\pi_*\mathcal{N}$  deduced from the action of  $\mathcal{D}_H$  on  $\mathcal{N}^2$  are given respectively by

$$\mathcal{A}_0 := 1 \otimes A(s) \quad \text{and} \quad \mathcal{B} := 1 \otimes \nabla + (-1)^{k-1} \partial_{s_k} \otimes P'_s(A(s))$$

$$\text{where we put } P'_s(z) := (\partial_z(P_s(z))) \text{ and } \nabla := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & k-1 & 0 \end{pmatrix}$$

---

<sup>2</sup>Note that  $z$  and  $\partial_z$  commute with  $\partial_{s_h}$  for  $h \in [1, k-1]$  in  $\mathcal{D}_H$ .

PROOF. Our goal is to explicit  $\pi_*\mathcal{N}$  and to check that it coincides with the  $\mathcal{D}_{\mathbb{C}^k}$ -module associated to the system  $(\textcircled{a})$  in Theorem 2.b.1.

In a first step we explicit the transfer-module

$$\mathcal{D}_{\mathbb{C}^k \leftarrow H} := \pi^{-1} \mathcal{D}_{\mathbb{C}^k} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} (\pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H)$$

as a  $(\pi^{-1} \mathcal{D}_{\mathbb{C}^k}, \mathcal{D}_H)$ -bimodule.

The next step is to determine the cokernel of

$$\begin{aligned} \alpha : (\mathcal{D}_{\mathbb{C}^k \leftarrow H})^{k-1} &\longrightarrow \mathcal{D}_{\mathbb{C}^k \leftarrow H} \\ (u_1, \dots, u_{k-1}) &\mapsto \sum_{i=1}^{k-1} u_i \partial_{s_i} \end{aligned}$$

The last step is to apply  $\pi_*$ .

Let us denote for short

$$\sigma := \omega_H \otimes \omega_{\mathbb{C}^k}^{\otimes -1} := ds_1 \wedge \dots \wedge ds_{k-1} \wedge d_z \otimes (ds_1 \wedge \dots \wedge ds_{k-1} \wedge ds_k)^{\otimes -1}$$

the generator of the line bundle  $\pi^{-1} \Omega_{\mathbb{C}^k}^{\otimes -1} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{C}^k}} \Omega_H$ .

Recall that  $\mathcal{O}_H = \mathcal{O}_{\mathbb{C}^{k+1}}/\mathcal{J}$ , where  $\mathcal{J}$  is the ideal generated by  $P(s, z)$ . Hence  $\mathcal{O}_H$  is a  $\pi^{-1} \mathcal{O}_{\mathbb{C}^k}$ -free module with rank  $k$  since each section  $a(s', z)$  of  $\mathcal{O}_H$  is equivalent, by Weierstrass Division Theorem, to a unique polynomial  $\sum_{j=0}^{k-1} a_j(s', s_k) z^j$  modulo  $P(s, z)$ , for some sections  $a_j$  of  $\mathcal{O}_{\mathbb{C}^k}$ .

Hence  $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$  is a left  $\pi^{-1} \mathcal{D}_{\mathbb{C}^k}$ -free module of rank  $k$  generated by the  $k$ -sections  $(1 \otimes z^j \sigma)_{j=0, \dots, k-1}$ . Since the right action of each operator in  $\mathcal{D}_H$  is  $\pi^{-1} \mathcal{D}_{\mathbb{C}^k}$ -linear, it is sufficient to calculate each  $(1 \otimes z^j \sigma) \partial_{s_i}, i = 1, \dots, k-1, j = 0, \dots, k-1$ .

Now recall that  $H$  is defined in  $\mathbb{C}^{k+1}$  by the equation  $s_k = (-1)^{k-1} \sum_{h=0}^{k-1} (-1)^h s_h z^{k-h}$  with the convention  $s_0 = 1$  and so  $s_1, \dots, s_{k-1}, z$  are global coordinates on  $H$ . Then we have in  $H$

$$\frac{\partial s_k}{\partial s_h} = (-1)^{k-h-1} z^{k-h} \quad \text{and} \quad \frac{\partial s_k}{\partial z} = (-1)^{k-1} P'_s(z)$$

where  $P'_s(z)$  does not depend on  $s_k$ .

Let  $F := 1 \otimes E(z) \sigma$  denote the basis  $(1 \otimes z^j \sigma), j \in [0, k-1]$  of the free rank  $k$  left  $\pi^{-1}(\mathcal{D}_{\mathbb{C}^k})$ -module  $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$ . Recall that, according to [5, Rem. 4.18], in view of the generators described above, the action of  $\mathcal{D}_H$  in  $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$  is defined by the following formulas, where we consider  $F$  as a  $k$ -vector and use the usual matrix product

- (0)  $F\theta(s') = \theta(s')F$  where  $\theta \in \mathcal{O}_H$  does not depend on  $z$
- (1)  $Fz = A(s)F$
- (2)  $-F\partial_{s_h} = \partial_{s_h}F + (-1)^{k-h-1} \partial_{s_k}(A(s)^{k-h}F) \quad \forall h \in [1, k-1]$
- (3)  $-F\partial_z = \nabla F + (-1)^{k-1} \partial_{s_k}(P'_s(A(s))F)$

where we have used the equalities  $zE(z) = A(s)E(z)$  and  $\partial_z(E(z)) = \nabla E(z)$ .

Summing up:



- For  $g \in \mathcal{O}_H$  represented by  $\sum_{r=0}^{k-1} g_r(s)z^r$ , the  $(k \times k)$  matrix  $G$  of the  $\pi^{-1}\mathcal{D}_{\mathbb{C}^k}$ -linear morphism defined by  $g$  on  $\mathcal{D}_{\mathbb{C}^k \leftarrow H}$  is given by  $G := \sum_{r=0}^{k-1} g_r(s)A(s)^r$
- Let us consider the  $\mathcal{D}_{\mathbb{C}^k}$ -linear morphism  $\alpha : (\mathcal{D}_{\mathbb{C}^k}^k)^{(k-1)} \rightarrow \mathcal{D}_{\mathbb{C}^k}^k$  defined by the following  $k-1$   $(k, k)$ -matrices

$$A_h := \partial_{s_h} \otimes Id_{\mathbb{C}^k} + (-1)^{k-h-1} \partial_{s_k} \otimes A(s)^{k-h}$$

Let  $\Phi$  be in  $\mathcal{O}_{\mathbb{C}^k}^k$ . In view of the relation (2), the map  $(1 \otimes z^h \sigma) \mapsto \Phi_h$ , for  $h \in [0, k-1]$  will induce an element of  $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ , that is to say a solution of  $\pi_*\mathcal{N}$ , if and only if we have  $\partial_{s_h}(\Phi) = (-1)^{k-h} \partial_{s_k}(A^{k-h}\Phi)$ , since the generator of  $\pi_*\mathcal{N}$  is annihilated by the action of  $\partial_{s_h}$  for each  $h \in [1, k-1]$ .

In conclusion:

$$\pi_*\mathcal{N} \simeq \text{coker } \alpha$$

by the finiteness of the fibers of  $\pi$ . ■

**Remark 4.2.**  $\mathcal{N}$  is naturally endowed with a structure of right  $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ -module. By functoriality  $\pi_*\mathcal{N}$  is also a right  $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$ -module and its structure coincides with the induced by the right  $\mathcal{D}_{\mathbb{C}}$  action defined by (1) and (3) on  $\mathcal{D}_{\mathbb{C}^k \leftarrow H} \otimes_{\mathcal{D}_H} \mathcal{N}$ , since it commutes with each  $\partial_{s_i}$ , for  $i = 1, \dots, k-1$ . Therefore we obtain a natural left action of  $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$  on  $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ .

We also conclude, according to Lemma 3.1:

**Proposition 4.3.** *The left action of  $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$  defined by the above Remark 4.2 on  $\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$  coincides with the left action of  $\Gamma(\mathbb{C}; \mathcal{D}_{\mathbb{C}})$  on Lisbon integrals.*

If  $\Phi$  is a solution of  $\pi_*\mathcal{N}$ , replacing in the formula (3) the second term thanks to the equality obtained for  $\phi$  after applying (2), we also derive a right action of  $\partial_z$  which is given by the formula

$$-\Phi \partial_z = \nabla \Phi + \sum_{h=0}^{k-1} (k-h) s_h \partial_{h+1} \Phi$$

Our next goal is to conclude in Proposition 4.5 below that there are no global holomorphic solutions of  $\pi_*\mathcal{N}$  other than those of the form  $\Phi_f$ , for some holomorphic function  $f$  only depending on  $z$ . Since  $j$  is non-characteristic we have an isomorphism

$$j^{-1} R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^{k+1}}}(\mathcal{L}, \mathcal{O}_{\mathbb{C}^{k+1}}) \simeq R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)$$

According to Theorem 4.33 (2) of [5], making  $X = H, Y = \mathbb{C}^k, f = \pi, \mathcal{N} = \mathcal{O}_{\mathbb{C}^k}$  in loc.cit, we obtain

**Theorem 4.4.** *For any open subset  $\Omega$  of  $\mathbb{C}^k$  we have an isomorphism functorial on  $\mathcal{N}$  compatible with restrictions to open subsets*

$$R\Gamma(\pi^{-1}(\Omega); R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)) \simeq R\Gamma(\Omega; R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))$$

Recall that this isomorphism uses as a tool the "trace morphism":  $\pi_*\mathcal{O}_H \rightarrow \mathcal{O}_{\mathbb{C}^k}$  constructed in [5, Prop. 4.34].

Since for any open subset  $\Omega$  and any  $\mathcal{D}_H$ -module  $\mathcal{P}$ ,  $\Gamma(\Omega; \cdot)$  and  $\mathcal{H}om_{\mathcal{D}_H}(\mathcal{P}, \cdot)$  are left exact functors, since if  $\Omega$  is a Stein open set and if  $\mathcal{P}$  admits a global resolution by free  $\mathcal{D}_H$ -modules of finite rank, then  $R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{P}, \mathcal{O})$  is represented by a complex in degrees  $\geq 0$  with  $\Gamma(\Omega, \cdot)$ -acyclic entries, we conclude that

$$H^0(R\Gamma(H; R\mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H))) = \Gamma(H; \mathcal{H}om_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H)) \quad \text{and}$$

$$H^0(R\Gamma(\mathbb{C}^k; R\mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))) = \Gamma(\mathbb{C}^k; \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k}))$$

therefore Theorem 4.4 entails a  $\mathbb{C}$ -linear isomorphism

$$T : \text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) \simeq \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$$

**Proposition 4.5.** *The correspondence*

$$f \mapsto \Phi(f) := \Phi_f$$

*defines a  $\mathbb{C}$ -linear isomorphism*

$$\Phi : \Gamma(\mathbb{C}; \mathcal{O}_{\mathbb{C}}) \longrightarrow \Gamma(\mathbb{C}^k; \mathcal{H}om_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})) = \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$$

*Moreover, this isomorphism is also  $\Gamma(\mathbb{C}, \mathcal{D}_{\mathbb{C}})$ -left linear.*

PROOF. The last statement is clear from Proposition 4.3 and Lemma 3.1.

The remaining of the statement is equivalent to prove that  $\Phi : f \mapsto \Phi_f$  defines an isomorphism  $\text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) \rightarrow \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$ .

We already know that  $\Phi$  is injective. It remains to prove that  $\Phi$  is surjective. For each  $f \in \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$ , we introduce the regular holonomic  $\mathcal{D}_{\mathbb{C}}$ -module  $\mathcal{M}_f$  (a regular flat holomorphic connection on  $\mathbb{C}$ ) of which the constant sheaf  $\mathbb{C}f$  in degree zero is the complex of holomorphic solutions. We denote by  $\mathcal{N}_f$  the regular holonomic  $\mathcal{D}_H$ -module (a regular flat holomorphic connection on  $H$ )

$$\mathcal{N}_f := \mathcal{N} \boxtimes \mathcal{M}_f$$

It is clear that  $\mathcal{N}_f$  is a quotient of  $\mathcal{N}$ , and, by the left exactness of  $\pi_*$ ,  $\pi_*\mathcal{N}_f$  is a quotient of  $\pi_*\mathcal{N}$ . Moreover, according to Proposition 4.3 and Lemma 3.1,  $\Phi_f$  belongs to  $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$ . According to Theorem 4.4, for each  $f$  we have a  $\mathbb{C}$ -linear isomorphism  $T_f : \text{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H) \simeq \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$  which makes this last one a one dimensional  $\mathbb{C}$ -vector space. Moreover, by left exactness of  $\text{Hom}$  and the exactness of  $\pi_*$ , we have monomorphisms  $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k}) \subset \text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}, \mathcal{O}_{\mathbb{C}^k})$  and, by functoriality, we have  $T(f) = T_f(f)$ .

We shall use the following result:

**Lemma 4.6.** *Suppose that  $f \neq 0$ . Then  $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$  is a one dimensional  $\mathbb{C}$ -vector space generated by  $\Phi_f$ .*

PROOF. The result follows by Proposition 2.3 since  $\Phi_f$  is a non zero element of  $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$  hence it is a generator as a  $\mathbb{C}$ -vector space. ■

Let us now end the proof of Proposition 4.5. Clearly  $\text{Hom}_{\mathcal{D}_H}(\mathcal{N}, \mathcal{O}_H) = \sum_f \text{Hom}_{\mathcal{D}_H}(\mathcal{N}_f, \mathcal{O}_H)$  and, according to Lemma 4.6, for each  $f$ ,  $\text{Hom}_{\mathcal{D}_{\mathbb{C}^k}}(\pi_*\mathcal{N}_f, \mathcal{O}_{\mathbb{C}^k})$  is the  $\mathbb{C}$ -vector space spanned by  $\Phi_f$ ; hence  $T(f) = \lambda\Phi_f$  for some  $\lambda \in \mathbb{C}^*$ . Since  $\phi_{\lambda f} = \lambda\Phi_f$  we conclude that  $\Phi$  is surjective which gives the desired result. ■

As a consequence, isomorphism  $\Phi$  explicit isomorphism of Theorem 4.4 since they coincide up to the multiplication by a constant  $\lambda \neq 0$ .

4.a. **An example.** To conclude this article, let us give an interesting example of choice of the entire function  $f$  on  $\mathbb{C}$  for which we explicit the regular holonomic system on  $\mathbb{C}^k$  associated to the corresponding Lisbon integrals.

THE CASE  $f_t(z) := e^{tz}$ . Let us fix a parameter  $t \in \mathbb{C}^*$  and consider the entire function  $f_t(z) := e^{tz}$ .

First remark that according to Lemma 3.1, we have  $\frac{\partial E(z)}{\partial z} = \nabla E(z)$  where  $\nabla$  is the  $(k, k)$  matrix given by

$$\nabla := \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & k-1 & 0 \end{pmatrix}$$

Integration by parts in the formula<sup>3</sup>

$$\Phi_{f_t}(s) := \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{t\zeta} \cdot E(\zeta) d\zeta}{P_s(\zeta)}$$

gives, using formula (7 – bis):

$$\begin{aligned} t\Phi_{f_t}(s) &= -\frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{\partial}{\partial \zeta} \left[ \frac{E(\zeta)}{P_s(\zeta)} \right] d\zeta \\ &= -\frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{\nabla E(\zeta) P_s(\zeta) - P'_s(\zeta) E(\zeta)}{P_s^2(\zeta)} d\zeta \\ &= -\nabla \Phi_t(s) + \frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{P'_s(\zeta) E(\zeta)}{P_s^2(\zeta)} d\zeta \\ (tId + \nabla)\Phi_{f_t}(s) &= P'_s(A) \frac{1}{2i\pi} \int_{|\zeta|=R} e^{t\zeta} \frac{E(\zeta)}{P_s^2(\zeta)} d\zeta + (-1)^k \frac{\partial P'_s(A)}{\partial s_k} \Phi_{f_t}(s) \\ &= P'_s(A) (-1)^{k-1} \frac{\partial \Phi_{f_t}}{\partial s_k}(s) + (-1)^{k-1} \frac{\partial P'_s(A)}{\partial s_k} \Phi_{f_t}(s) \\ &= (-1)^{k-1} \frac{\partial (P'_s(A) \Phi_{f_t})}{\partial s_k}(s) \end{aligned}$$

<sup>3</sup>Remember that  $t$  is a fixed non zero parameter

This implies the following equation

$$(12) \quad (t.Id + \nabla)P'_s(A)^{-1}\Psi_{f_t}(s) = (-1)^{k-1} \frac{\partial(\Psi_{f_t})}{\partial s_k}(s)$$

for

$$\Psi_{f_t}(s) = P'_s(A)\Phi_{f_t}(s) = \frac{1}{2i\pi} \int_{|\zeta|=R} \frac{e^{t\zeta} P'_s(\zeta) E(\zeta) d\zeta}{P_s(\zeta)}.$$

Combining with the system (@@) it is easy to see that we obtain a meromorphic integrable connexion on the trivial vector bundle of rank  $k$  on  $\mathbb{C}^k$  with a pole along the discriminant hypersurface  $\{\Delta(s) = 0\}$ . The regularity of this meromorphic connexion is then consequence of the regularity of the  $\mathcal{D}_H$ -module  $\mathcal{N}_{etz}$  which is given by the equations

$$\partial_{s_h} = 0, \quad \forall h \in [1, k-1] \quad \text{and} \quad \partial_z - t = 0$$

which is clearly regular locally on  $H$ . So its direct image by  $\pi$  (as a  $\mathcal{D}_{\mathbb{C}^k}$ -module) is regular holonomic on  $\mathbb{C}^k$  (see [6, Th. 8.1]).

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