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# Numerical Approximation of Optimal Strategies for Impulse Control of Piecewise Deterministic Markov Processes Application to Maintenance Optimisation

Benoîte de Saporta, François Dufour, Huilong Zhang  
Univ. Montpellier, Bordeaux INP, Univ. Bordeaux



# Outline

Introduction

Motivation

Piecewise deterministic Markov processes

Impulse control for PDMPs

Numerical implementation

Conclusion

# Maintenance optimization

## Equipments

- ▶ with several components
- ▶ subject to **random** degradation and failures

**Maintenance optimization problem:** find some optimal balance between

- ▶ repairing/changing components too often
- ▶ do nothing and wait for the total failure of the system

Optimize some criterion

- ▶ minimize a **cost**: repair, maintenance, unavailability penalty, failure penalty, . . .
- ▶ maximize a **reward**: availability, production, . . .

# Maintenance optimization

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# Impulse control problem

## Impulse control

Select

- ▶ **intervention dates**
- ▶ new **starting point** for the process at interventions

to **minimize** a cost function

## Piecewise deterministic Markov processes

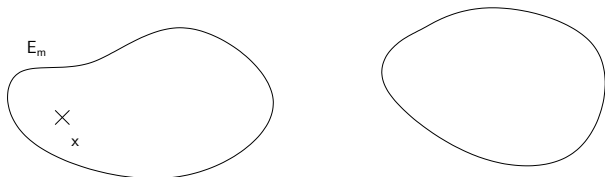
General class of **non-diffusion** dynamic stochastic **hybrid** models:  
**deterministic** motion punctuated by **random** jumps.

[CD 89], [Davis 93], [dSDZ 14], ...

# Piecewise deterministic Markov processes

Starting point

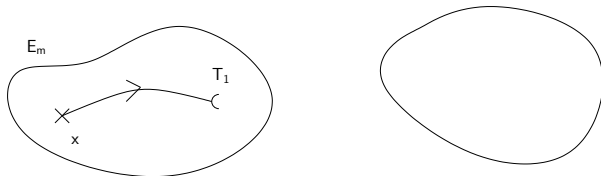
$$X_0 = (m, x)$$



# Piecewise deterministic Markov processes

$X_t$  follows the deterministic flow until the first jump time  $T_1 = S_1$

$$X_t = (m, \phi_m(x, t)), \quad \mathbb{P}_{(m,x)}(T_1 > t) = e^{-\int_0^t \lambda_m(\phi_m(x,s)) ds}$$

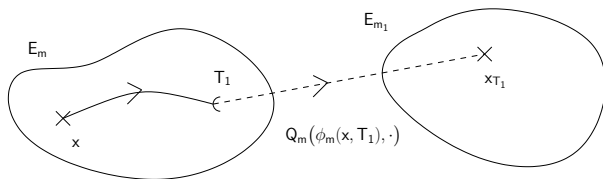




# Piecewise deterministic Markov processes

Post-jump location  $(m_1, x_{T_1})$  selected by the Markov kernel

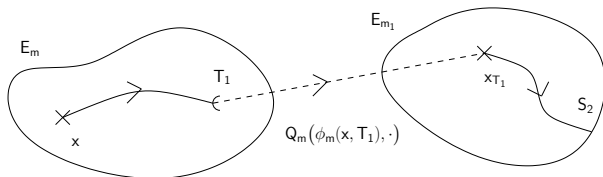
$$Q_m(\phi_m(x, T_1), \cdot)$$



# Piecewise deterministic Markov processes

$X_t$  follows the flow until the next jump time  $T_2 = T_1 + S_2$

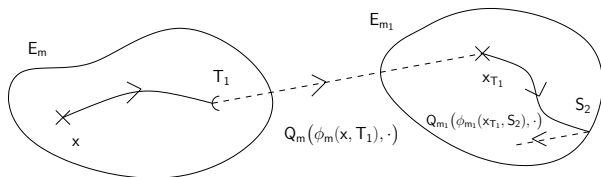
$$X_{T_1+t} = (m_1, \phi_{m_1}(x_{T_1}, t)), \quad t < S_2$$



# Piecewise deterministic Markov processes

Post-jump location  $(m_2, x_{T_2})$  selected by Markov kernel

$$Q_{m_1}(\phi_{m_1}(x_{T_1}, S_2), \cdot) \dots$$



# Embedded Markov chain

$\{X_t\}$  strong Markov process [Davis 93]

Natural embedded Markov chain

- ▶  $Z_0$  starting point,  $S_0 = 0$ ,  $S_1 = T_1$
- ▶  $Z_n$  new mode and location after  $n$ -th jump,  $S_n = T_n - T_{n-1}$ ,  
time between two jumps

## Proposition

$(Z_n, S_n)$  is a discrete-time Markov chain

Only source of randomness of the PDMP

# Mathematical definition of impulse control

Strategy  $\mathcal{S} = (\tau_n, R_n)_{n \geq 1}$

- ▶  $\tau_n$  intervention times
- ▶  $R_n$  new positions after intervention

## Value function

$$\mathcal{J}^{\mathcal{S}}(x) = E_x^{\mathcal{S}} \left[ \int_0^{\infty} e^{-\alpha s} f(Y_s) ds + \sum_{i=1}^{\infty} e^{-\alpha \tau_i} c(Y_{\tau_i}, Y_{\tau_i}^+) \right]$$

$$\mathcal{V}(x) = \inf_{\mathcal{S} \in \mathbb{S}} \mathcal{J}^{\mathcal{S}}(x)$$

- ▶  $f, c$  cost functions,  $\alpha$  discount factor
- ▶  $Y_t$  controlled process,  $\mathbb{S}$  set of admissible strategies

# Dynamic programming

Costa, Davis, 1988

For any function  $g \geq$  cost of the no-impulse strategy

- ▶  $v_0 = g$
- ▶  $v_n = L(v_{n-1})$

$$v_n(x) \xrightarrow[n \rightarrow \infty]{} \mathcal{V}(x)$$

dS, Dufour, Geeraert, 2017

Construction of  $\epsilon$ -optimal strategies based on the dynamic programming operator

# Dynamic programming

## Jump-or-intervention operator

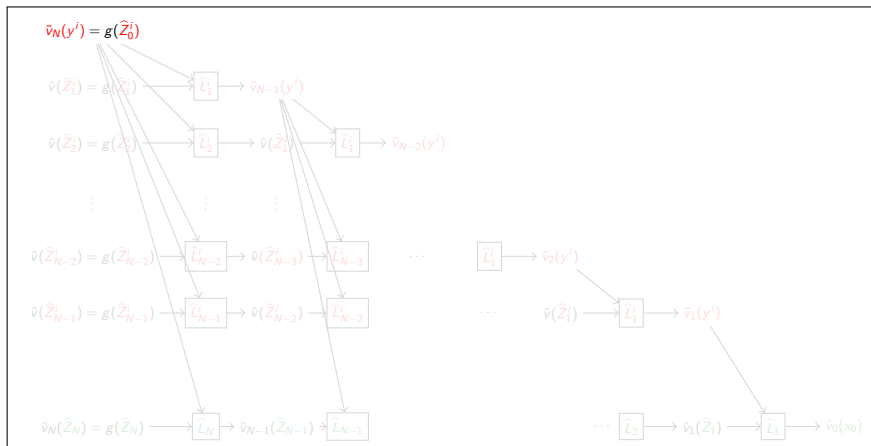
$$\begin{aligned}
 v_n(Z_n) &= L(Mv_{n+1}, v_{n+1})(Z_n) \\
 &= \left( \inf_{t \leq t^*(Z_n)} \mathbb{E} \left[ F(Z_n, t) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) \mathbb{1}_{\{S_{n+1} < t \wedge t^*(Z_n)\}} \right. \right. \\
 &\quad \left. \left. + e^{-\alpha t \wedge t^*(Z_n)} Mv_{n+1}(\phi(Z_n, t \wedge t^*(Z_n))) \mathbb{1}_{\{S_{n+1} \geq t \wedge t^*(Z_n)\}} \mid Z_n \right] \right) \\
 &\quad \wedge \mathbb{E} \left[ F(Z_n, t^*(Z_n)) + e^{-\alpha S_{n+1}} v_{n+1}(Z_{n+1}) \mid Z_n \right]
 \end{aligned}$$

with

$$\begin{aligned}
 F(x, t) &= \int_0^{t \wedge t^*(x)} e^{-\alpha s - \int_0^s \lambda(\phi(x, u)) du} f(\phi(x, s)) ds \\
 Mv_{n+1}(x) &= \inf_{y \in \mathbb{U}} \{c(x, y) + v_{n+1}(y)\}
 \end{aligned}$$

# Approximation scheme - Value function

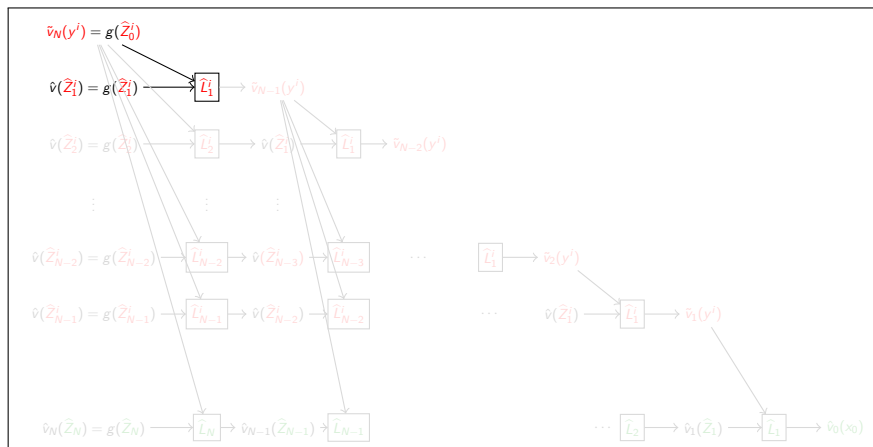
Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$





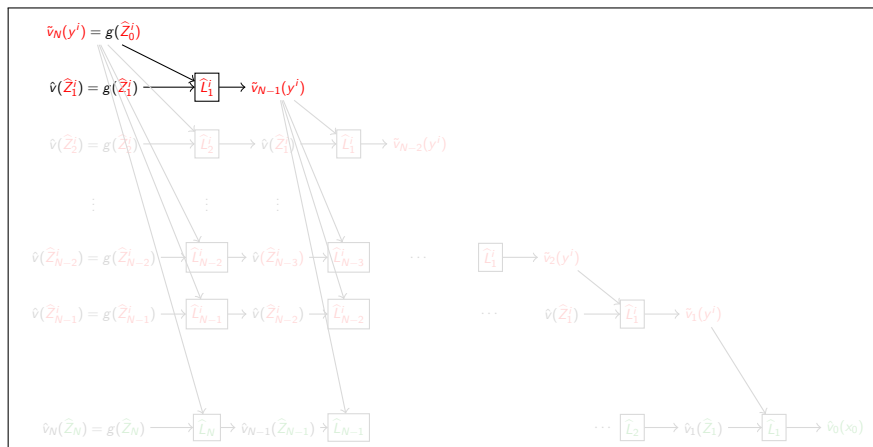
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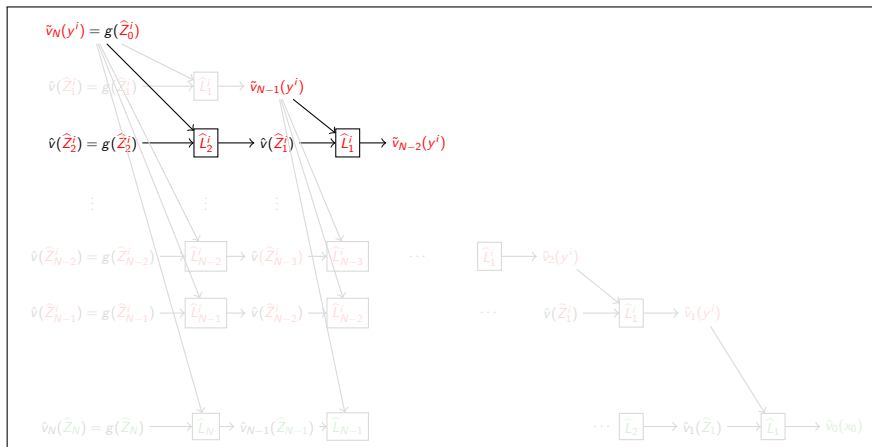
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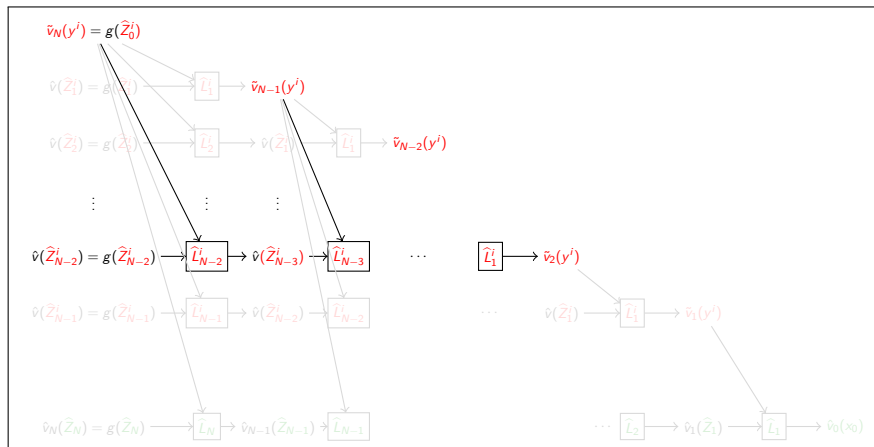
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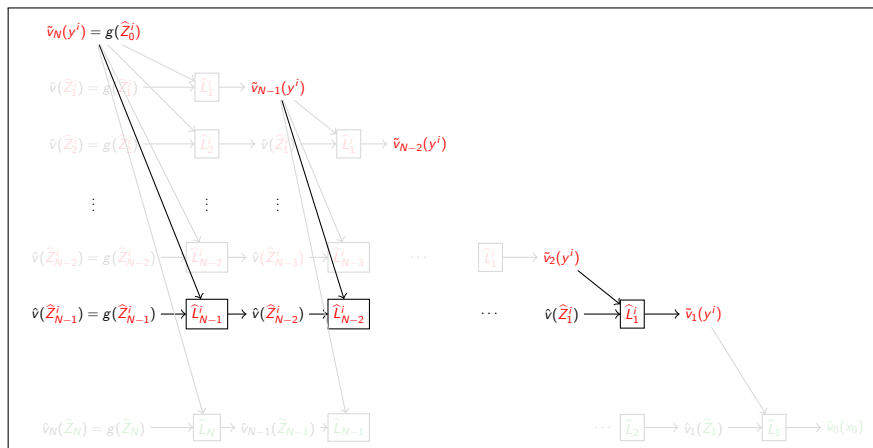
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# Approximation scheme - Value function

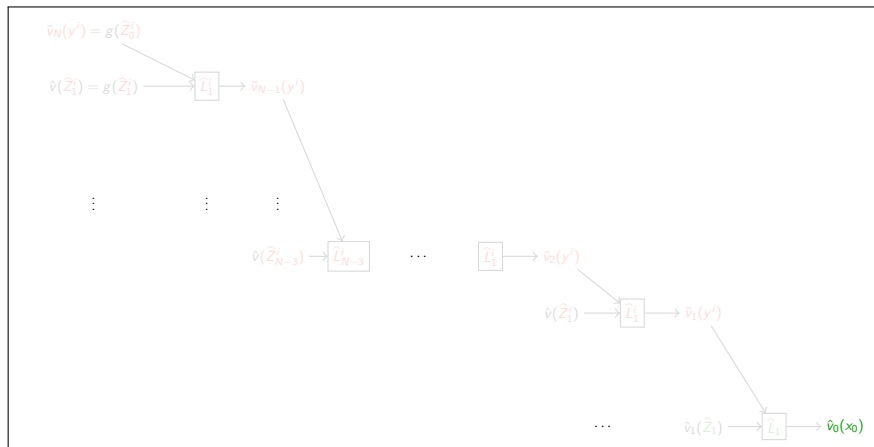
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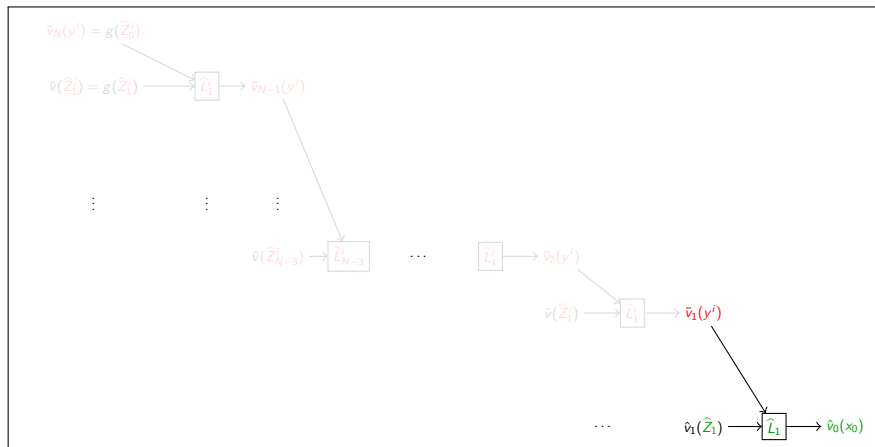
# Approximation scheme - $\epsilon$ -optimal strategy

Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$



# Approximation scheme - $\epsilon$ -optimal strategy

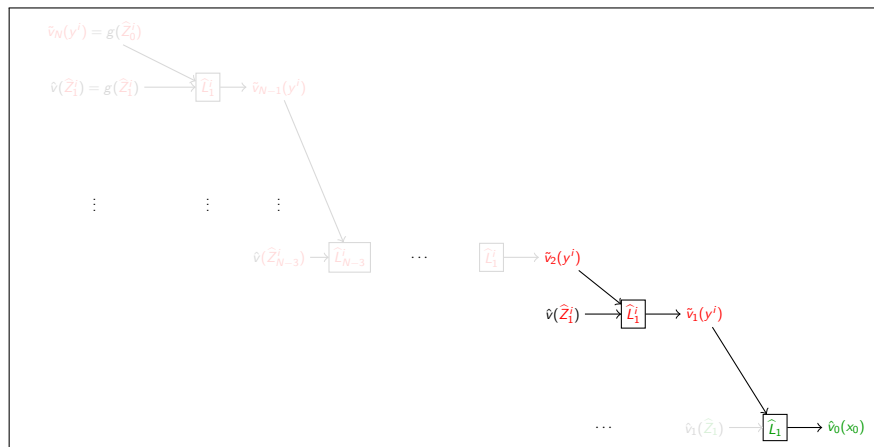
Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$





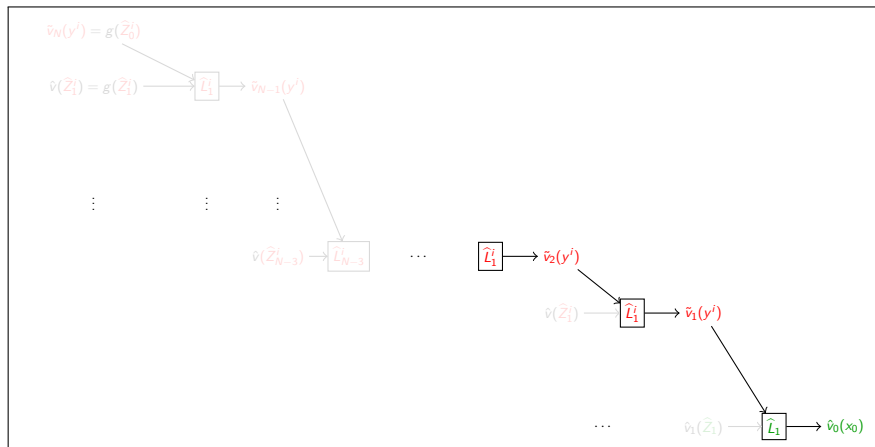
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Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$



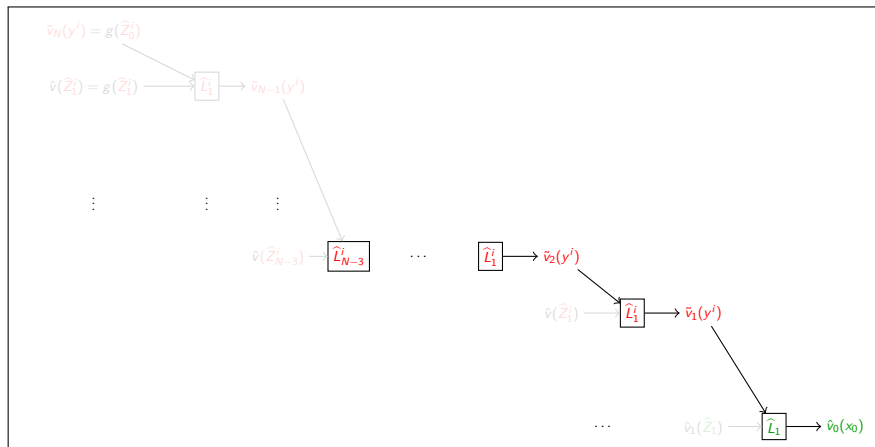
# Approximation scheme - $\epsilon$ -optimal strategy

Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$



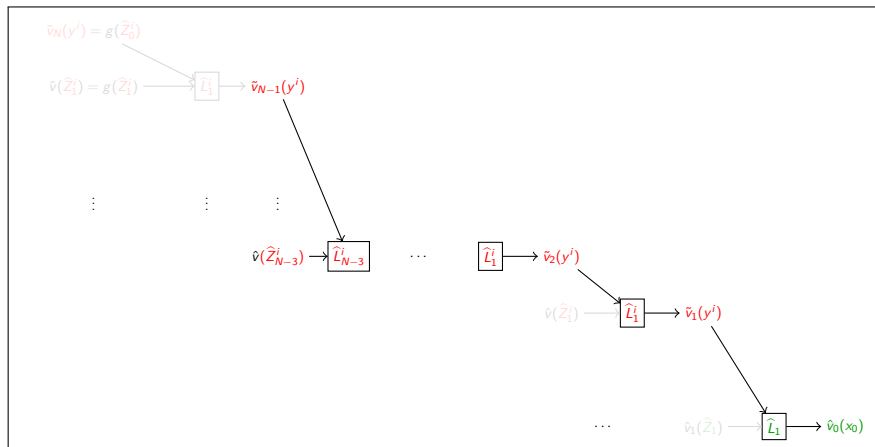
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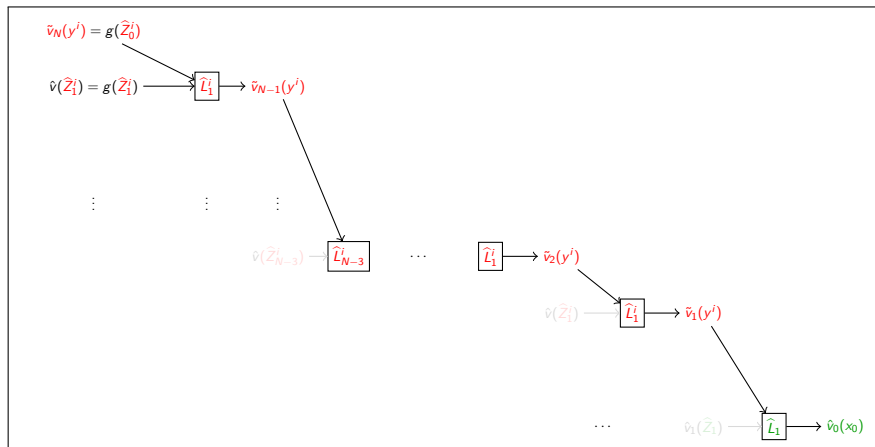
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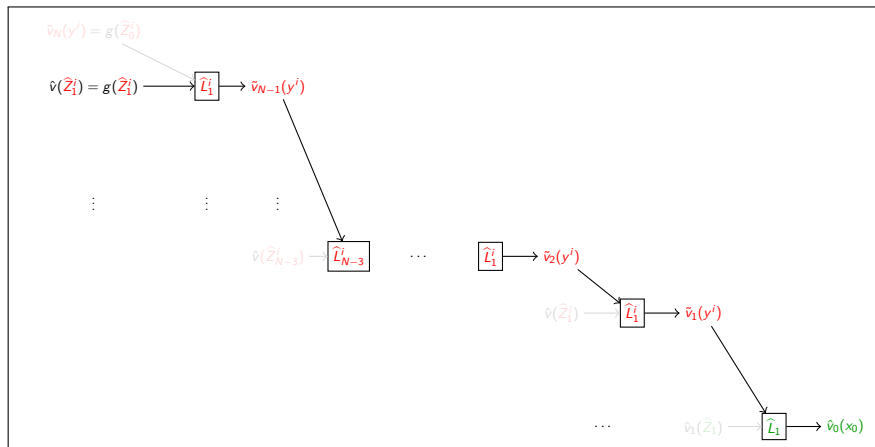
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# Approximation scheme - $\epsilon$ -optimal strategy

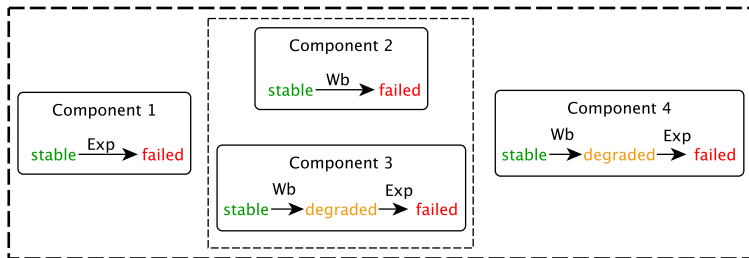
Based on time-dependent discretizations of the state space of  $(Z_n, S_n)$



# Equipment model

Typical model with 4 components

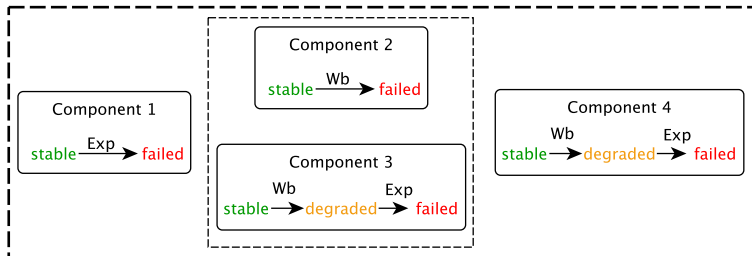
- ▶ Component 1: 2 states – **stable**  $\xrightarrow{\text{Exponential}}$  **failed**
- ▶ Component 2: 2 states – **stable**  $\xrightarrow{\text{Weibull}}$  **failed**
- ▶ Components 3 and 4: 3 states  
**stable**  $\xrightarrow{\text{Weibull}}$  **degraded**  $\xrightarrow{\text{Exponential}}$  **failed**



# Maintenance operations

Possible maintenance operations

- ▶ All components, all states: do nothing
- ▶ Components 1 and 2, all states: change
- ▶ Components 3 and 4: change in all states, repair only in stable or degraded states

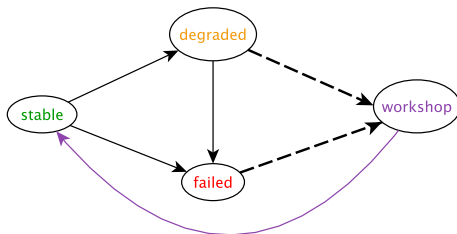




# Criterion to optimize

Minimize the maintenance + unavailability costs

- ▶ **unavailability** cost proportional to time spend in **failed** state
- ▶ fixed cost for going to the workshop + **repair** < **change** costs



# PDMP model of the equipment

- ▶ **Euclidean variables:** 5 time variables
  - ▶ functioning time of components 2, 3 and 4
  - ▶ calendar time
  - ▶ time spent in the workshop
- ▶ **Discrete variables:** 225 modes
  - ▶ state of the components / maintenance operations

# Parameters to tune

- ▶ Number of points in the **control grid** (underlying continuous model)
- ▶ Number of point in the **quantization grids** for  $(Z_n, S_n)$
- ▶ **Approximation horizon**  $N$  such that  $v_N(x) - \mathcal{V}(x)$  small enough  $\simeq$  allowed number of jumps + interventions
- ▶ bounding function  $g$
- ▶ **Time discretization step** for inf

## Step 1: Exact simulation of the PDMP

Implementation of an exact simulator for **reference strategies** to serve as benchmark

- ▶ **Strategy 1**: do nothing
- ▶ **Strategy 2**: send equipment to **workshop** 1 day after **failure**, **change** all degraded components, **change** all **failed** ones
- ▶ **Strategy 3**: send equipment to **workshop** 1 day after **degradation**, **change** all degraded components, **change** all **failed** ones

Strategy	1	2	3
Mean cost	19952	11389	8477

## Step 1: Exact simulation of the PDMP

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## Step 2 and 3: Discretisation of the control set $\mathbb{U}$ and the embedded Markov chain

### Tests on strategy 3

Finite control set  $\mathbb{U}$

$\implies$  discretize the functioning times at interventions

$\implies$  project the real times on the grid feasibly

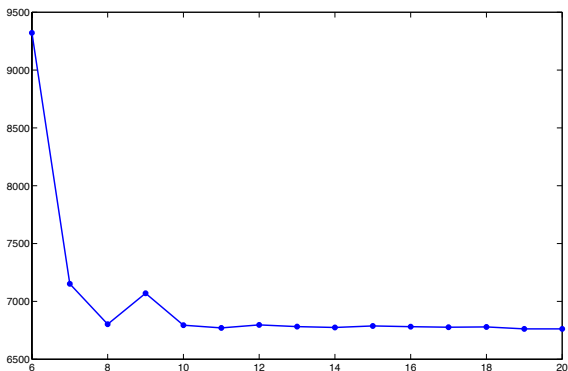
Compromise between precision and computation time

Grid	Number of points	relative error
$3 \times 3 \times 3 \times 5$	246	0.10344
$4 \times 4 \times 4 \times 5$	331	0.0241
$5 \times 5 \times 5 \times 5$	592	0.0062
$3 \times 3 \times 3 \times 11$	615	0.0341
$4 \times 4 \times 4 \times 11$	923	0.0819
$5 \times 5 \times 5 \times 11$	1855	0.0186
$6 \times 6 \times 6 \times 11$	2110	0.0066
$7 \times 7 \times 7 \times 11$	2617	0.0071
$8 \times 8 \times 8 \times 11$	3359	0.0066
$3 \times 3 \times 3 \times 21$	1230	0.0034
$4 \times 4 \times 4 \times 21$	1899	0.0170
$5 \times 5 \times 5 \times 21$	2960	0.0095
$6 \times 6 \times 6 \times 21$	4220	0.0065
$7 \times 7 \times 7 \times 21$	5536	0.0059
$8 \times 8 \times 8 \times 21$	7111	0.0047

## Step 4: Calibrating $N$ the number of allowed jumps + interventions

Horizon  $N$  (number of iterations)

- ▶ 5 for Strategy 1
- ▶ up to 30 for Strategy 2 (mean 6)
- ▶ up to 25 for Strategy 3 (mean 6)



## Step 5: Approximation of the value function

Strategy 1	Strategy 2	Strategy 3	Approx. Value function
19952	11389	8477	7076

- ▶ relative gain of 19.8% vs Strategy 5



## Step 6: Optimally controlled trajectories

Strategy 1	Strategy 2	Strategy 3	Approx. Value function	Optimally controlled traj.
19952	11389	8477	7076	6733

- ▶ numerical [validation](#) of the algorithm with various starting points: consistent results

# Conclusion

Numerical method to derive a feasible  $\epsilon$ -optimal strategy

- ▶ rigorously validated [dSD 12, dSDG 17]
- ▶ with general error bounds for the approximation of the value function
- ▶ numerically demanding but viable in low dimensional examples

## References

- [CD 89] O. COSTA, M. DAVIS *Impulse control of piecewise-deterministic processes*
- [Davis 93] M. DAVIS, *Markov models and optimization*
- [dSD 12] B. DE SAPORTA, F. DUFOUR *Numerical method for impulse control of piecewise deterministic Markov processes*
- [dSDG 17] B. DE SAPORTA, F. DUFOUR, A. GEERAERT *Optimal strategies for impulse control of piecewise deterministic Markov processes*
- [dSDZ 14] B. DE SAPORTA, F. DUFOUR, H. ZHANG *Numerical methods for simulation and optimization of PDMPs: application to reliability*
- [P 98] G. PAGÈS *A space quantization method for numerical integration*
- [PPP 04] G. PAGÈS, H. PHAM, J. PRINTEMPS *An optimal Markovian quantization algorithm for multi-dimensional stochastic control problems*