Invariants of Morse complexes and persistent homology

"Homological Mirror Symmetry at NRU HSE" conference.

S.Barannikov (Paris VII, NRU HSE Moscow)

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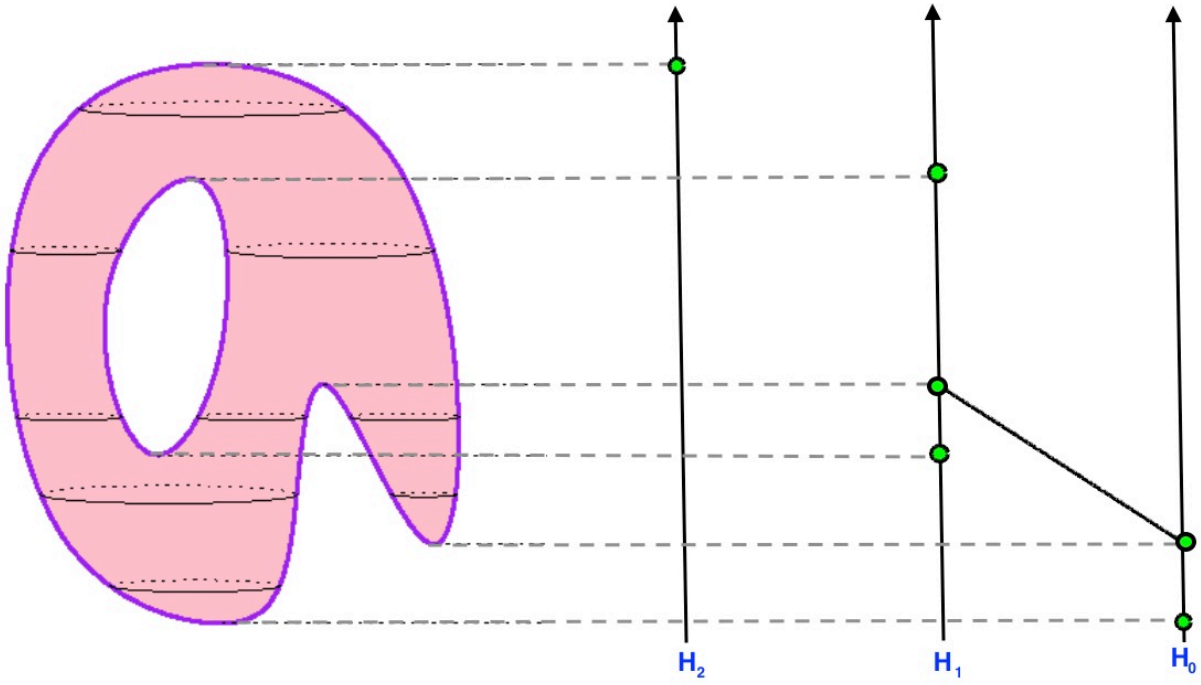
Partition of critical values into pairs "birth-death" plus homological critical values ("births" paired with $+\infty$)



"Canonical form" invariants of filtered complexes

- The decomposition arises from bringing of the Morse complex over field F, defined by gradient trajectories of the function, to what I called "canonical form", by a linear transform respecting the filtration, given by order of the critical values.
- These "canonical forms" are combinatorial invariants of filtered complexes.

Arnold's problem on extension of smooth function inside a ball

- Given $f \in C^\infty\left(\partial B^n \times [-\varepsilon, \varepsilon]\right) \to$ how many crtitical points can a generic smooth extension of f inside the ball B have?
- Example: function on closed manifold and a ball containing all critical points of the function. Then a restriction of the function to the neighborhood of the boundary of this ball must contain information on the *Betti numbers of the manifold*.

Morse complex

$f : M^n \to \mathbb{R}$, $f \in C^\infty$, generic, $\{x \mid f(x) \leq c\}$ $-$compact. Then $p_\alpha-$critical points, $df\mid_{T_{p_\alpha}} = 0$, are isolated, near p_α:$f = \sum_{l=1}^{j} -(x^l)^2 + \sum_{l=j}^{n} (x^l)^2$. Let g is a generic metric. Then

$$C_j = \oplus_{\text{index}(p_\alpha)=j} \left[p_\alpha, \text{or}(T^-_{p_\alpha})\right]$$

where $T_{p_\alpha} = T^-_{p_\alpha} \oplus T^+_{p_\alpha}$ is wrt $\partial^2 f$ and g.
The differential is

$$\partial_j \left[p_\alpha, \text{or}\right] = \sum_{\text{index}(p_\beta)=j-1} \left[p_\beta, or\right] \#\mathcal{M}(p_\alpha, p_\beta)$$

$$\mathcal{M}(p_\alpha, p_\beta) = \{\gamma : \mathbb{R} \to M^n \mid$$

$$\left.\dot\gamma = -(\text{grad}_g f)(\gamma(t)), \lim_{t\to-\infty} = p_\alpha, \lim_{t\to+\infty} = p_\beta \right\} /\mathbb{R}$$

What are the invariants of Morse complexes independent of metrics?

- The Morse complex is naturally filtered $F_{\leq s}C_* \subset F_{\leq r}C_*$, $s < r$, by the set $\{f(p_\alpha)\}$ of critical values of f: $[p_\alpha] \in F_s C_*$ if $f(p_\alpha) \leq s$
- Claim: under generic perturbation of the metrics, the anti-gradient trajectory exceptionnaly goes from the critical point p_α to the critical point $p_{\tilde{\alpha}_{\text{lower}}}$ with $\text{index}(p_\alpha) = \text{index}(p_{\tilde{\alpha}_{\text{lower}}})$
- The change of the Morse complex is described then by the change of the basis: $[p_\alpha] \to [p_\alpha] \pm [p_{\tilde{\alpha}_{\text{lower}}}]$.
- What can one do with the complex using such upper-triangular change of bases?

"Canonical form" invariants of filtered complexes [SB1994]

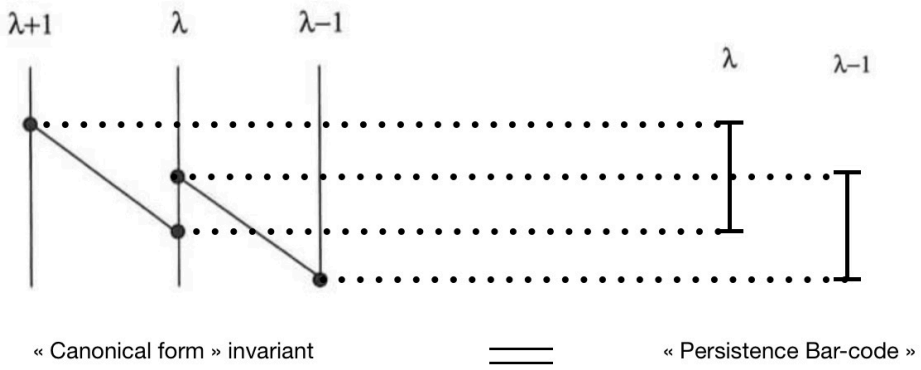
- Let C_* is a filtered chain complex, $F_{\leq s}C_* \subset F_{\leq r}C_*$, $s < r$, indexed by finite set of real numbers, $F_{\leq max}C_* = C_*$. It can come with a basis compatible with filtration so that each subspace $F_{\leq r}C_j$ is the span $\left\langle e_1^{(j)}, \ldots, e_{i_r}^{(j)} \right\rangle$
- Chain complex with bases $\{\tilde{e}_i^{(j)}\}_{i\in\{0,1,\ldots\}}^{i\in 1,\ldots,\dim_F C_j}$ is in "canonical form" if for any basis element $\tilde{e}_i^{(j)}$ either $\partial\tilde{e}_i^{(j)} = 0$ or $\partial\tilde{e}_i^{(j)} = \tilde{e}_{i'}^{(j-1)}$, so that in the latter case, $\tilde{e}_i^{(j)} \neq \tilde{e}_{i_1}^{(j)}$ $\Rightarrow \partial\tilde{e}_i^{(j)} \neq \partial\tilde{e}_{i_1}^{(j)}$.
- **Theorem (SB, 1994)** One can bring a filtered complex to the canonical form by an upper-triangular change of compatible basis preserving the filtration. The resulting pairing "birth-death" between indices of the filtration plus the filtration indices of homology generators, are **canonically determined**.

Category of filtered complexes is semi-simple

- Equivalent reformulation: any object in the category of filtered complexes over field is isomorphic to *a canonically* defined sum of simple objects: the 1-dimensional with zero differential, indexed by $s \in \mathbb{R}$, and the 2-dimensional with $\partial \tilde{e}^{(j)}_{i_2} = \tilde{e}^{(j-1)}_{i_1}$, indexed by s_1, $s_2 \in \mathbb{R}$.
- Proof : bring the complex to the required canonical form by induction, starting from the lowest generators of degrees 1, then 2 etc, the claim is that manipulating degree k generators does not destroy the canonical form in degree $k-1$.
- This is somewhat similar in spirit to the Poincare's definition of the torsion in homology groups.

"Canonical form" invariants ="Persistence Bar-codes"

There are three equivalent visualizations



of the same invariants. "Persistence Bar-codes"/"Persistence diagrams" were introduced in applied mathematics in the beginning of 2000s (H.Edelsbrunner, J.Harer, A.Zamorodian "Hierarchical Morse complexes for piecewise linear 2-manifolds" Proc. of Symp on Comput Geometry, June 2001, A.Zamorodian "Persistence and hierarchical Morse complexes, PhD Thesis, University of Illinois, 2001). There are several software packages for computing these invariants of a finite filtration. The principal algorithm is based on the bringing of the filtered complex to its canonical form by upper-triangular matrices from [SB1994].

"Canonical form" invariants of Morse complexes

- The Morse complex is naturally filtered by the set $\{f(p_\alpha)\}$ of critical values of f: $[p_\alpha, \mathrm{or}] \in F_{\leq s} C_*$ if $f(p_\alpha) \leq s$
- $\rightarrow$canonical partition of the set of critical values $\{f(p_\alpha)\}$ into pairs "birth-death", plus separate set giving a basis $H(M, F)$ or "births" paired with $+\infty$
- Claim: the "canonical form" of Morse complex does not depend on the metrics: under generic perturbation of the metrics the complex changes via series of chage of bases: $e_i^{(j)} \rightarrow e_i^{(j)} \pm e_{l_{\mathrm{lower}}}^{(j)}$
- when the function is deformed the "canonical form" invariant changes naturally in continuous way. This can be expressed in $\epsilon - \delta$ language.

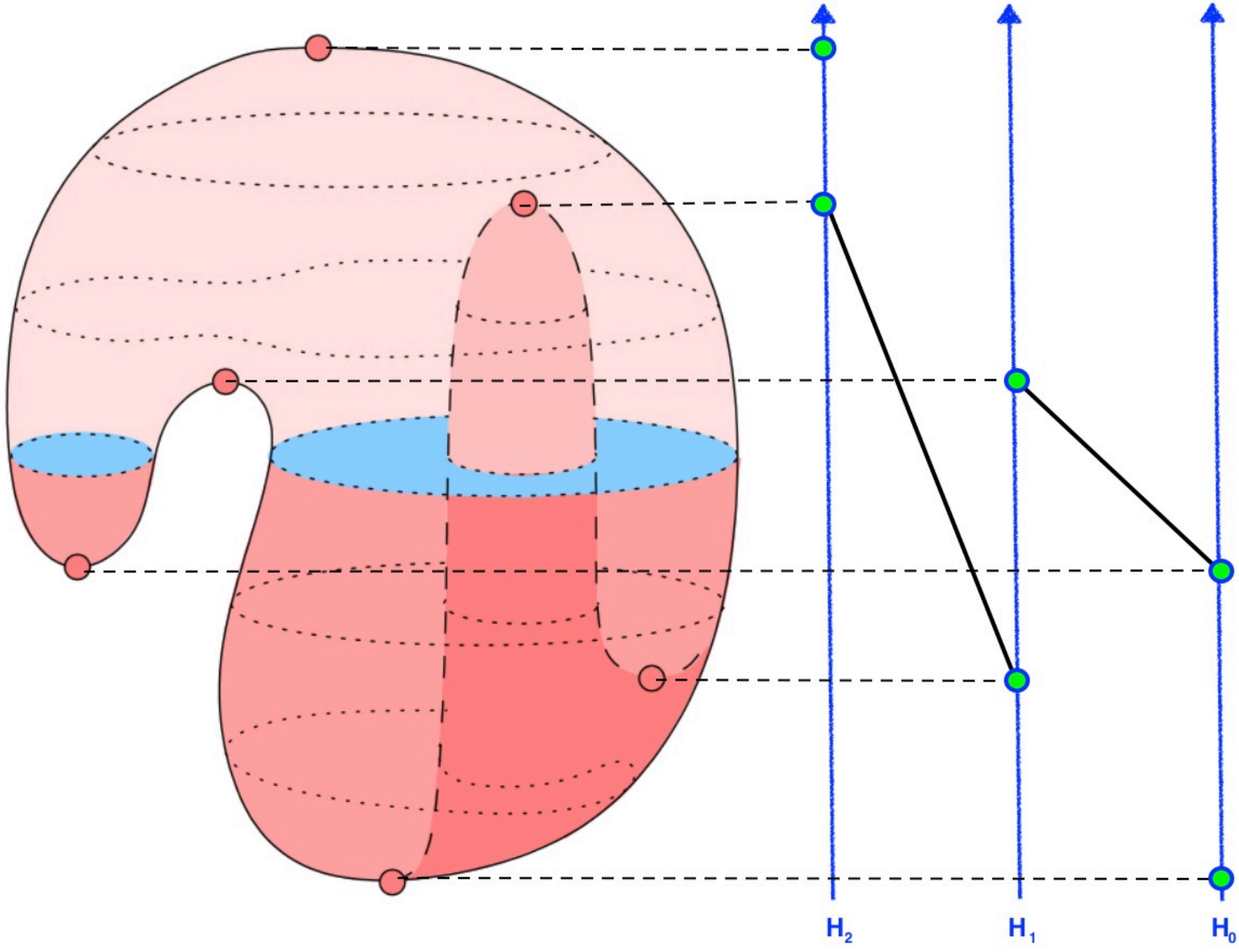
Change of $H_*(f \leq c)$

- From the "canonical form" invariant one can immediately read the homology of any subcomplex $H_*\left(F_{\leq s}C_*\right)$ as well as the images of $H_i\left(F_{\leq s_j}C_*\right) \to H_i\left(F_{\leq s_k}C_*\right)$
- For each pair $\partial\tilde{e}_{i_2}^{(j)} = \tilde{e}_{i_1}^{(j-1)}$, indexed by $s_1, s_2 \in \mathbb{R}$ a new homology classe is born at s_1 and this class is killed at s_2.

Definition of the persistent homology

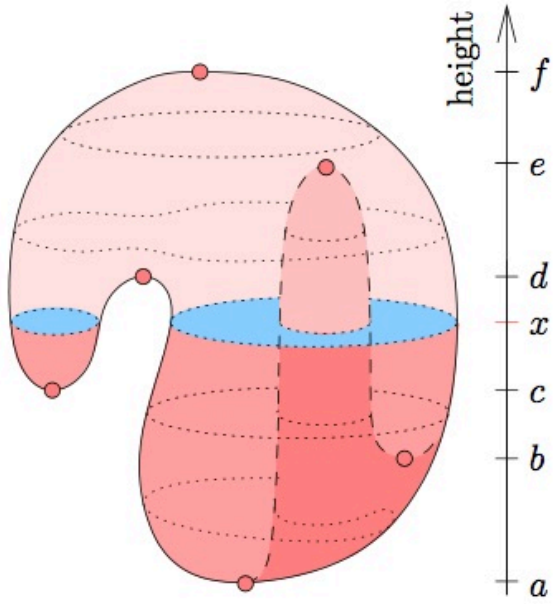
- Persistent homology by definition is the dimensions of images $H_i\left(F_{\leq s_j} C_*\right) \to H_i\left(F_{\leq s_k} C_*\right)$
- When one speaks about "persistent homology" it is these "persistence bar-codes/diagrams"="canonical form" invariants which researchers have in mind

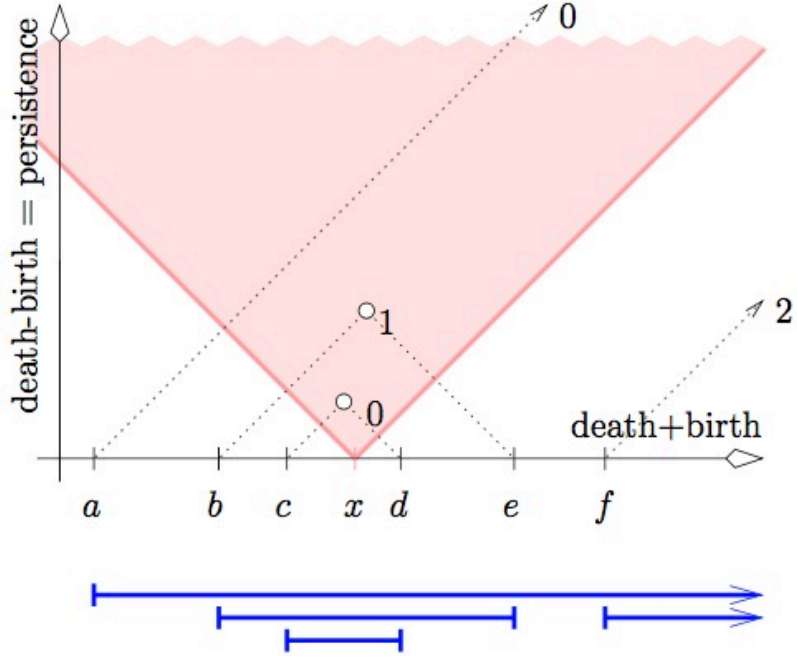
Partition of critical values into pairs "birth-death" plus homological critical values ("births" paired with $+\infty$)



Another comparison

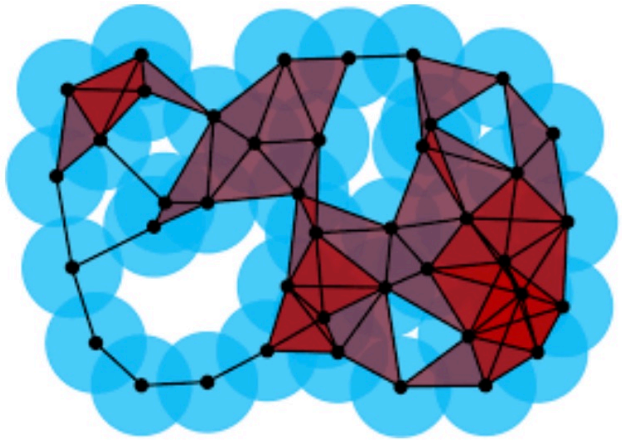
This illustration is taken from the plenary talk "Persistent Homology: Theory and Practice." H. Edelsbrunner, D. Morozov, at the *European Congress of Mathematics*, 2012:





Point clouds and Čech Complex

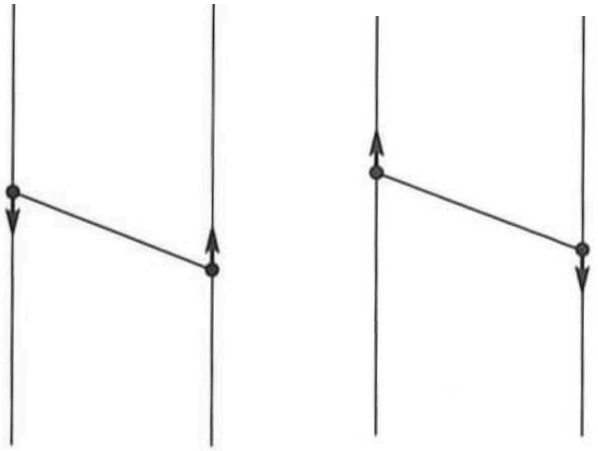
- $f =$ distance to the set of points(point cloud), $M = \mathbb{R}^n$, sublevel sets $\{x \mid f(x) \leq d\}$ are unions of balls, their intersections define the Čech complex for each d, increasing the distance $d \rightarrow$more intersections $\rightarrow$ the filtered complex.



- the "canonical form" invariants="persistence bar-codes/diagrams" of the filtered complex$\rightarrow$main tool in the *topological data analisys*.

Arnold's problem on extension of smooth function

- Given $f \in C^{\infty}\left(\partial B^{n} \times [-\varepsilon, \varepsilon]\right) \rightarrow$how many are there crtitical points of generic smooth extension of f inside the ball B?
- Theorem (SB,1994) These pairs in "canonical form" $f\mid_{\partial B^{n}}$ indicate the crititical points of certain index in B^{n} :



They can cancell each other in certain configurations, details are in [SB1994].

Small eigenvalues of twisted Laplacian

These "canonical form" invariants were applied in Le Peutrec D., Nier F., Viterbo C. *"The Witten Laplacian and Morse–Barannikov Complex"* [LNV2011] to find formulas for small eigenvalues of the Witten Laplacian of $d_{f,h} = hd + df$

$$\Delta_{f,h} = (d_{f,h} + d_{f,h}^*)^2 = d_{f,h}^* d_{f,h} + d_{f,h} d_{f,h}^* = \bigoplus_{p=0}^{\dim M} \Delta_{f,h}^{(p)}.$$

There is a one to one correspondance j_p between $\mathcal{U}^{(p)}$ and the set of eigenvalues (counted with multiplicities) of $\Delta_{f,h}^{(p)}$ lying in $[0, h^{3/2})$ such that

$$j_p(U^{(p)}) = 0 \quad \text{if} \quad U^{(p)} \in \mathcal{U}_H^{(p)}$$

$$j_p(U^{(p)}) = \kappa^2(U^{(p+1)}) \frac{h}{\pi} \frac{|\lambda_1^{(p+1)} \dots \lambda_{p+1}^{(p+1)}|}{|\lambda_1^{(p)} \dots \lambda_p^{(p)}|} \frac{|\mathrm{Hess} f(U^{(p)})|^{1/2}}{|\mathrm{Hess} f(U^{(p+1)})|^{1/2}} (1 + \mathcal{O}(h)) e^{-2\frac{f(U^{(p+1)}) - f(U^{(p)})}{h}}$$

$$\text{if } \partial_B U^{(p+1)} = U^{(p)}$$

$$j_p(U^{(p)}) = \kappa^2(U^{(p)}) \frac{h}{\pi} \frac{|\lambda_1^{(p)} \dots \lambda_p^{(p)}|}{|\lambda_1^{(p-1)} \dots \lambda_{p-1}^{(p-1)}|} \frac{|\mathrm{Hess} f(U^{(p-1)})|^{1/2}}{|\mathrm{Hess} f(U^{(p)})|^{1/2}} (1 + \mathcal{O}(h)) e^{-2\frac{f(U^{(p)}) - f(U^{(p-1)})}{h}}$$

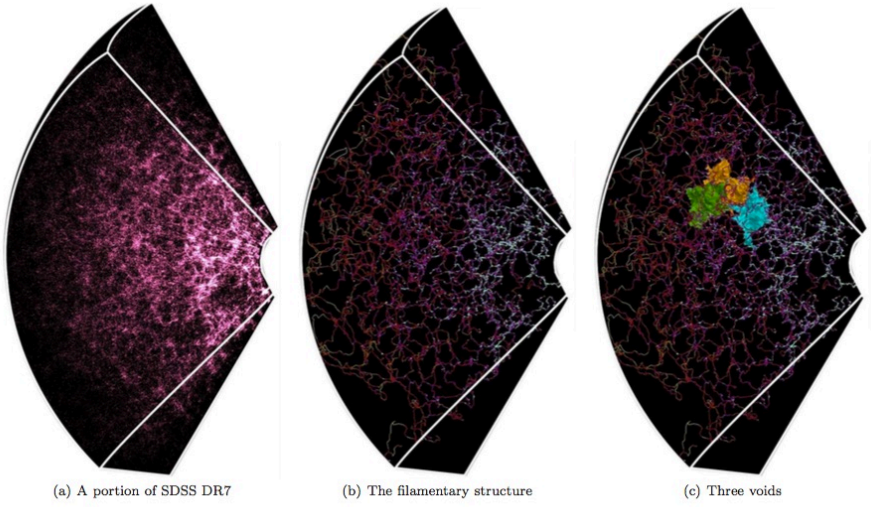
$$\text{if } \partial_B U^{(p)} = U^{(p-1)}$$

Here the λ's denote the negative eigenvalues of the $\mathrm{Hess} f$ at the corresponding points.

Arnold's 4 cusps conjecture

conjecture solved in [ChP], V.Arnold described it in 2002 in ([Arnold2002], page 79) :
"For any 1-parameter smooth family of legendrian curves (integral curves of the natural contact structure in the space of contact elements), connecting the legendrian curve of the front, which moves on the plane inside of the bounded by the curve disk, with the legendrian curve of the inside-out front, moving outside of the disk, the front of some intermediate legendrian curve from this family has at least 4 cusps. The proof of this remarkable, having a deep physical meaning topological theorem is difficult and uses, on one hand, the recent progress in the symplectic topology (Floer homology, quantum cohomology etc), and on the other - the results of S.Barannikov on the algebra of the Morse complex (which, regretably, hadn't received the proper recognition several years ago)."

Persistent cosmic web (cosmology, Sousbie & al [SPK])



(a) A portion of SDSS DR7 (b) The filamentary structure (c) Three voids

(d) A zoom on the voids and the filamentary structure

Figure 10. The detected filamentary structure at a significance level of $5-\sigma$ and three voids within a portion of SDSS DR7. Note that only the upper half of the distribution shown on figure 7 is displayed here for clarity reasons. The color of the filaments corresponds the the logarithm of the density field. The filaments of the SDSS extracted with DisPerSE is readily available online at the URL `http://www.iap.fr/users/sousbie/`.

The function $f =$ density of the matter

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[LNV2011] Le Peutrec, D., Nier, F., Viterbo, C. *"Precise Arrhenius Law for p-forms: The Witten Laplacian and Morse–Barannikov Complex"* Ann. Henri Poincaré (2013) 14: 567.

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[EM2012] H.Edelsbrunner, D. Morozov, "Persistent Homology: Theory and Practice." Proceedings of the European Congress of Mathematics, 2012.

[ChP] Yu. V. Chekanov, P. E. Pushkar', *"Combinatorics of fronts of Legendrian links and the Arnol'd 4-conjectures" Uspekhi Mat. Nauk, 2005, Vol 60, 1(361), 99–154*

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[SPK] *T. Sousbie, C. Pichon, H. Kawahara, "The persistent cosmic web and its filamentary structure II: Illustrations" MNRAS, Vol 414, Issue 1,2011, 384–403.*