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# Nonautonomous fractional Hamiltonian system with critical exponential growth

João Marcos do Ó, Jacques Giacomoni and Pawan Kumar Mishra

**Abstract.** In this paper, we study the following nonlocal nonautonomous Hamiltonian system on whole  $\mathbb{R}$

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + u = Q(x)g(v) & \text{in } \mathbb{R}, \\ (-\Delta)^{\frac{1}{2}} v + v = P(x)f(u) & \text{in } \mathbb{R}, \end{cases}$$

where  $(-\Delta)^{\frac{1}{2}}$  is the square root Laplacian operator. We assume that the nonlinearities  $f, g$  have critical growth at  $+\infty$  in the sense of Trudinger-Moser inequality and the nonnegative weights  $P(x)$  and  $Q(x)$  vanish at  $+\infty$ . Using suitable variational method combined with the generalized linking theorem, we obtain the existence of at least one positive solution for the above system.

**Mathematics Subject Classification (2010).** Primary 35J50, 35R11, 35A15.

**Keywords.** elliptic systems involving square root of the Laplacian, critical growth nonlinearities of Trudinger-Moser type, linking theorem.

## 1. Introduction and main results

In this paper, we study the following system

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + u = Q(x)g(v) & \text{in } \mathbb{R}, \\ (-\Delta)^{\frac{1}{2}} v + v = P(x)f(u) & \text{in } \mathbb{R}, \end{cases} \quad (1.1)$$

where  $(-\Delta)^{\frac{1}{2}}$  is the square root Laplacian operator defined as

$$(-\Delta)^{\frac{1}{2}} u(x) = -\frac{1}{2\pi} \int_{-1}^1 \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2} dy.$$

One difficulty in studying Hamiltonian elliptic systems via variational methods is that the energy functional is strongly indefinite, that is, its quadratic part is respectively coercive and anti-coercive in infinite dimensional subspaces of the energy space. To deal with such difficulty we use a Galerkin

method, introduced by Rabinowitz in [25, 26]. Another important obstacle is to handle the lack of compactness, which roughly speaking, originates from the non-compactness of the Trudinger-Moser embedding. To add on, we face a lack of compactness of Sobolev embedding because the problem is posed in whole  $\mathbb{R}$ .

In the local case *i.e.* in the case of the standard Laplacian operator, the existence of solution for Hamiltonian elliptic systems has been extensively studied in the literature mostly in higher dimensions involving Sobolev critical growth in bounded as well as unbounded domain of  $\mathbb{R}^N$ . For the case of a bounded domain, see for instance, [3, 6, 8, 10, 18, 20]. On the other hand, Hamiltonian elliptic systems in whole  $\mathbb{R}^N$  have been explored to a lesser extend, see for example, [11, 22, 29, 30]. We refer the reader to [4] for a recent survey on this subject.

For  $N = 2$ , Hamiltonian systems in a bounded domain in  $\mathbb{R}^2$  have been studied by D. G. de Figueiredo, J. M. do Ó and B. Ruf [7] in the critical growth range. We cite N. Lam and G. Lu [21] for a similar result without Ambrosetti-Rabinowitz condition (see (H3)). In whole  $\mathbb{R}^2$ , D.G. de Figueiredo, J. M. do Ó and J. Zhang [9] studied the ground state solution for the system (1.1) with  $P = Q = 1$  using the idea of generalized Nehari manifold under a monotonicity assumption on nonlinearity *i.e.*  $f(t)/|t|$  and  $g(t)/|t|$  are strictly increasing in  $(-\infty, 0)$  and  $(0, +\infty)$ . Another reference in unbounded domain is of M. de Souza and J. M. do Ó [13], where authors have considered a system of the type

$$\begin{cases} -\Delta u + V(x)u = g(v) & \text{in } \mathbb{R}^2, \\ -\Delta v + V(x)v = f(u) & \text{in } \mathbb{R}^2. \end{cases}$$

Under some suitable conditions on  $V$ , the loss of compactness in critical case was recovered.

We know from classical fractional Sobolev embedding that  $H^{s,2}(\mathbb{R}^N)$  is continuously embedded in  $L^q(\mathbb{R}^N)$  for all  $q \in [2, 2_s^*]$ , where  $2_s^* = 2N/(N-2s)$ . Note that formally,  $2_s^* = \infty$  if  $N = 2s$ . The only choice for this fact to be true is  $N = 1$  and  $s = 1/2$ , since  $s \in (0, 1)$ . At this point a natural question arises: What is the optimal space where  $H^{1/2,2}(\mathbb{R})$  is embedded? This answer was first given by Ozawa [24] and later improved by Iula, Maalaoui and Martinazzi [23] which we have stated in Theorem A below.

In case of  $u = v$ ,  $f = g$  and  $P = Q$ , the system (1.1) converts into the following scalar equation

$$(-\Delta)^{\frac{1}{2}}u + u = P(x)f(u) \text{ in } \mathbb{R}. \quad (1.2)$$

Motivated from fractional Trudinger Moser inequality as in Theorem A, J. M. do Ó, Miyagaki and Squassina [15] studied the existence of positive solutions to (1.2) with a class of weights  $P$  containing the Lebesgue integrable functions and nonlinearities  $f$  having subcritical and critical exponential growth (see also [16]). In the case of an open and bounded interval we cite an earlier work of Iannizzotto and Squassina [19], where authors have

proved existence and multiplicity of positive solutions with critical exponential growth (see [17] also). We cite [1, 2, 5, 32, 33] for more results in the case of Sobolev critical growth for dimension  $N \geq 2$  involving the square root of Laplacian.

Inspired from the above literature, we consider in the present paper an Hamiltonian system involving the square root Laplacian operator posed in whole space  $\mathbb{R}$  and investigate the effect of critical exponential growth of nonlinearities. We use a generalized linking theorem as compared to other results in the literature related to Hamiltonian systems which are heavily dependent on finite dimension Galerkin approximation to prove existence of at least one positive solution. For that, facing of the lack of compactness due to the critical growth of nonlinearities and unboundedness of the domain, we need to investigate accurately the behavior of suitable Palais Smale sequences. As far as we know there is no result regarding such class of fractional Hamiltonian systems. Our results appear to be the first of its kind for fractional Hamiltonian systems and we expect that the insights and the methods used in the present paper apply to a wider class of semilinear elliptic operators.

### 1.1. Critical exponential nonlinearity

It is usual in the literature to say that  $h$  has critical growth of Trudinger-Moser type at  $+\infty$  if there exists  $\alpha_0 > 0$  such that

$$\lim_{s \rightarrow +\infty} \frac{h(s)}{e^{\alpha s^2} - 1} = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ +\infty & \text{if } \alpha < \alpha_0. \end{cases}$$

We note that such notion is motivated by a fractional version of Trudinger-Moser inequality in the whole space  $\mathbb{R}$  as follows.

**Theorem A.** *(A fractional Trudinger-Moser inequality) It holds*

$$\sup_{u \in H^{1/2,2}(\mathbb{R}), \|u\|_{1/2} \leq 1} \int_{\mathbb{R}} (e^{\alpha|u|^2} - 1) dx \begin{cases} < \infty, & \alpha \leq \pi, \\ = \infty, & \alpha > \pi, \end{cases}$$

where  $H^{1/2,2}(\mathbb{R})$  is the fractional order Sobolev space equipped with  $\|\cdot\|_{1/2}$  norm which is defined in Section 2.

### 1.2. Assumptions on the weights $P(x)$ and $Q(x)$

We assume  $P, Q \not\equiv 0$  belonging to  $C_0(\mathbb{R}, \mathbb{R}^+)$  that is,  $P, Q$  are continuous, nonnegative and

$$\lim_{|x| \rightarrow +\infty} P(x) = 0 = \lim_{|x| \rightarrow +\infty} Q(x). \quad (1.3)$$

Without loss of generality, for the ease of reference, we assume:

$$\sqrt{P(0)Q(0)} = 1 \quad (1.4)$$

**Remark 1.1.** *We recall that the assumption (1.3) does not imply that  $P(x)$  and  $Q(x)$  are Lebesgue integrable. For example take  $P(x) = 1$  if  $|x| \leq 1$  otherwise  $1/|x|$ .*

### 1.3. Assumptions on the nonlinearities $f$ and $g$

We consider the following assumptions on nonlinearities  $f$  and  $g$  which are  $\alpha_0$ -critical at  $+\infty$ .

(H1) (continuity)  $f, g : \mathbb{R} \rightarrow [0, \infty)$  are continuous functions and  $f(0) = g(0) = 0$ .

(H2) (behavior near the origin)  $f(t) = o(t)$  and  $g(t) = o(t)$  near the origin.

(H3) (Ambrosetti-Rabinowitz type condition) there exist constants  $\theta > 2$  such that, for all  $t > 0$ , one has

$$0 < \theta F(t) := \theta \int_0^t f(s) ds \leq tf(t) \quad \text{and} \quad 0 < \theta G(t) := \theta \int_0^t g(s) ds \leq tg(t).$$

(H4) (asymptotic behavior)  $\lim_{t \rightarrow +\infty} tf(t)e^{-\alpha_0 t^2} = +\infty$  and  $\lim_{t \rightarrow +\infty} tg(t)e^{-\alpha_0 t^2} = +\infty$ .

### 1.4. Main result

Before stating the main result of the paper, we define the following class of functions. We say that a function  $f$  belongs to the class  $\mathbb{C}_P$ , if for any sequence  $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$  satisfying for some positive constant  $C$

$$u_n \rightharpoonup 0 \quad \text{and} \quad \int_{\mathbb{R}} P(x)f(u_n)u_n dx < C \quad \text{implies} \quad \int_{\mathbb{R}} P(x)F(u_n) dx \rightarrow 0.$$

**Theorem 1.1.** *Assume  $f$  and  $g$  belong to the class  $\mathbb{C}_P$  and  $\mathbb{C}_Q$  respectively and have  $\alpha_0$ -critical growth at  $+\infty$ . Let  $f$  and  $g$  satisfy (H1) – (H4) and  $P, Q$  satisfy (1.3) then the system (1.1) possesses a positive solution.*

**Remark 1.2.** *The assumption (H5) given below together with (H2) implies that  $f \in \mathbb{C}_P$  and  $g \in \mathbb{C}_Q$ .*

$$(H5) \quad \lim_{t \rightarrow \infty} \frac{F(t)}{tf(t)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{G(t)}{tg(t)} = 0.$$

*We highlight that when  $f$  and  $g$  are of  $O(t^2)$  near origin,  $\mathbb{C}_P$  implies the assumption (L6) adopted in [21, Theorem 4] in the critical case. However if  $f$  and  $g$  decay slowly at 0, say are of  $O(t^{1+\ell})$  with  $\ell < 1$ ,  $u_n \rightharpoonup 0$  and  $\{\int_{\mathbb{R}} P(x)f(u_n)u_n dx\}$  is bounded do not imply that  $\int_{\mathbb{R}} P(x)f(u_n) dx \rightarrow 0$ . Precisely, vanishing behavior may occur for the sequence  $\{P(x)f(u_n)\}$  in this case (see Remark 5.1). To validate our remark, we have added a proof in the appendix of this paper.*

**Remark 1.3.** *The result of Theorem 1.1 holds good even for the class of positive weights satisfying  $P \in L^\gamma(\mathbb{R})$  and  $Q \in L^{\gamma'}(\mathbb{R})$  with  $\gamma, \gamma' > 1$  or  $P, Q \in L^1(\mathbb{R}) \cap C(\mathbb{R}, \mathbb{R})$ .*

**Remark 1.4.** *We have the following observations regarding the assumptions above.*

- (i) *The assumption (H3) is a global Ambrosetti-Rabinowitz condition which is used to prove the boundedness of the Palais-Smale sequence.*

- (ii) Instead of (H4) one can take even a slightly weaker assumption (H4)' :  $\lim_{t \rightarrow +\infty} t f(t) e^{-\alpha_0 t^2} \geq \eta$  and  $\lim_{t \rightarrow +\infty} t g(t) e^{-\alpha_0 t^2} \geq \eta$  where  $\eta$  is a sufficiently large positive real number.
- (iii) Examples of functions satisfying (H1)-(H5) are  $f(t) = t^{\ell_1} e^{t^{\beta_1}} e^{\alpha_0 t^2}$ ,  $g(t) = t^{\ell_2} e^{t^{\beta_2}} e^{\alpha_0 t^2}$  with  $\ell_1, \ell_2 > 1$  and  $0 \leq \beta_1, \beta_2 < 2$ .

**Remark 1.5.** We can consider different critical growth for  $f$  and  $g$ , say  $\alpha_0-$  and  $\beta_0-$  critical growth respectively. Precisely, we can use the scaling  $\bar{v} = \sqrt{\frac{\beta_0}{\alpha_0}} v$  in (1.1) to get

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u + u = Q(x) \bar{g}(\bar{v}) & \text{in } \mathbb{R}, \\ (-\Delta)^{\frac{1}{2}} \bar{v} + \bar{v} = \bar{P}(x) f(u) & \text{in } \mathbb{R}, \end{cases}$$

where  $\bar{P}(x) = \sqrt{\frac{\beta_0}{\alpha_0}} P(x)$  and  $\bar{g}(\bar{v}) = g\left(\sqrt{\frac{\alpha_0}{\beta_0}} \bar{v}\right)$ . It is easy to see that  $\bar{g}$  has the  $\alpha_0-$  critical growth as  $g$  and  $\bar{P} \in C_0(\mathbb{R}, \mathbb{R})$ .

The paper is organized as follows. In section 2, we discuss the abstract framework related to the problem (1.1). A generalized version of Linking geometry and related estimates are shown in section 3. In particular, in Proposition 3.3, we give an upper estimate of the energy level that guarantees the compactness of suitable Palais Smale sequences. Here the assumption (H4) plays a crucial role. This estimate is subsequently used in Section 4 where the behavior of Palais Smale sequences is described. Finally, Section 5 contains the proof of the main result of Theorem 1.1.

## 2. Abstract framework

In order to apply variational methods, we recall the Bessel potential space  $H^{s,p}(\mathbb{R}^N)$  as

$$H^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : (-\Delta)^{s/2} u \in L^p(\mathbb{R}^N) \right\}.$$

On the other hand, the Sobolev-Slobodeckij space  $W^{s,p}(\mathbb{R}^N)$  is defined as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p,\mathbb{R}^N} < \infty \right\},$$

where

$$[u]_{s,p,\mathbb{R}^N}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy.$$

In general,  $H^{s,p}(\mathbb{R}^N) \neq W^{s,p}(\mathbb{R}^N)$  for  $p \neq 2$ . In this paper, we are interested in the limiting Sobolev embedding case *i.e.* when  $2_s^* = 2N/(N - 2s) = \infty$  which, as observed before, corresponds to  $s = 1/2$  and  $p = 2$  in dimension  $N = 1$ . In this case, the space  $H^{1/2,2}(\mathbb{R})$  coincides with the Sobolev-Slobodeckij space  $W^{1/2,2}(\mathbb{R})$  and both seminorms are related as

$$\|(-\Delta)^{1/4} u\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} [u]_{W^{1/2,2}(\mathbb{R})}^2,$$

see Proposition 3.6. in [14]. The space  $H^{1/2,2}(\mathbb{R})$  is the Hilbert space with the norm defined as

$$\|u\|_{1/2}^2 = \|u\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx$$

induced from the inner product given as

$$\langle u, v \rangle_{1/2} = \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} v dx + \int_{\mathbb{R}} uv dx.$$

We recall that  $H^{1/2,2}(\mathbb{R})$  is continuously embedded in  $L^q(\mathbb{R})$ , for any  $q \in [2, \infty)$ .

Consider the following weighted Banach space

$$L^r(\mathbb{R}; P) = \left\{ u : \mathbb{R} \rightarrow \mathbb{R} \text{ measurable: } \int_{\mathbb{R}} P(x) |u|^r dx < \infty \right\}$$

and similarly we define

$$L^r(\mathbb{R}; Q) = \left\{ u : \mathbb{R} \rightarrow \mathbb{R} \text{ measurable: } \int_{\mathbb{R}} Q(x) |u|^r dx < \infty \right\}.$$

We state our first result related to the compactness of the embedding of  $H^{1/2,2}(\mathbb{R})$  into weighted integrable spaces defined above as follows.

**Lemma 2.1.** *The space  $H^{1/2,2}(\mathbb{R})$  is compactly embedded in  $L^q(\mathbb{R}; P)$  and  $L^q(\mathbb{R}; Q)$  respectively for  $q \in [2, \infty)$ .*

*Proof.* We prove only the compact embedding of  $H^{1/2,2}(\mathbb{R})$  into  $L^q(\mathbb{R}; P)$ . The other one is similar. Let  $q \in [2, \infty)$  and  $\epsilon > 0$ . Then from the assumption (1.3), there exists  $L = L(\epsilon) > 0$  large enough, such that

$$P(x) < \epsilon, \text{ for all } |x| > L. \quad (2.1)$$

If  $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$  is such that  $u_n \rightharpoonup u$  weakly in  $H^{1/2,2}(\mathbb{R})$  for some  $u \in H^{1/2,2}(\mathbb{R})$ , then using the continuous injection of  $H^{1/2,2}(\mathbb{R})$  in an arbitrary  $L^r(\mathbb{R})$  space with  $r \in [2, \infty)$  there exist  $M, M_r > 0$  such that

$$\int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u_n|^2 dx + \int_{\mathbb{R}} |u_n|^2 dx \leq M, \int_{\mathbb{R}} |u_n|^r dx \leq M_r. \quad (2.2)$$

Now from (2.1) and (2.2), we get

$$\begin{aligned} \int_{\mathbb{R} \setminus [-L, L]} P(x) |u_n|^q dx &\leq \epsilon \int_{\mathbb{R} \setminus [-L, L]} |u_n|^q dx \\ &\leq \epsilon \int_{\mathbb{R}} |u_n|^q dx \leq \epsilon \|u_n\|_{1/2}^q \leq \epsilon M^{q/2}. \end{aligned}$$

Using in addition using the compact injection of  $H^{1/2,2}(\mathbb{R})$  in an arbitrary  $L^r_{loc}(\mathbb{R})$  space with  $r \geq 2$  together with the boundedness of  $P(x)$  in the compact subset of  $\mathbb{R}$  as  $P(x)$  is continuous, we have compactness of the embedding of  $H^{1/2,2}(\mathbb{R})$  in  $L^q(\mathbb{R}; P)$  for all  $q \in [2, \infty)$ .  $\square$

Next, associated with our system we consider the Hilbert space  $\mathbb{H} = H^{1/2,2}(\mathbb{R}) \times H^{1/2,2}(\mathbb{R})$  with the inner product and norm

$$\langle (u, v), (\varphi, \psi) \rangle := \langle u, \varphi \rangle_{1/2} + \langle v, \psi \rangle_{1/2}, \quad \|(u, v)\| := \left( \|u\|_{1/2}^2 + \|v\|_{1/2}^2 \right)^{1/2}.$$

Consider the natural associated functional with system (1.1), defined in  $\mathbb{H}$  by

$$I(u, v) := \Phi(u, v) - \int_{\mathbb{R}} P(x)F(u) dx - \int_{\mathbb{R}} Q(x)G(v) dx, \quad (2.3)$$

where the associated quadratic part is defined by

$$\Phi(u, v) := \int_{\mathbb{R}} \left( (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} v + uv \right) dx.$$

Using standard arguments it is possible to verify that  $I$  is well defined and is of class  $C^1$  with

$$\begin{aligned} I'(u, v)(\phi, \psi) &= \int_{\mathbb{R}} \left( (-\Delta)^{\frac{1}{4}} u (-\Delta)^{\frac{1}{4}} \psi + (-\Delta)^{\frac{1}{4}} v (-\Delta)^{\frac{1}{4}} \phi + u\psi + v\phi \right) dx \\ &\quad - \int_{\mathbb{R}} (P(x)f(u)\phi + Q(x)g(v)\psi) dx \quad \forall (\phi, \psi) \in \mathbb{H}. \end{aligned}$$

Consequently, critical points of the functional  $I$  are precisely the weak solutions to (1.1).

Note that the functional  $I$  is strongly indefinite, since  $\Phi(u, v) > 0$  when  $0 \neq (u, v) \in \mathbb{H}^+$  and  $\Phi(u, v) < 0$  whenever  $0 \neq (u, v) \in \mathbb{H}^-$ , where  $\mathbb{H}^+ := \{(u, u) : u \in H^{1/2,2}(\mathbb{R})\}$ ,  $\mathbb{H}^- := \{(u, -u) : u \in H^{1/2,2}(\mathbb{R})\}$  are infinite dimensional subspaces of  $\mathbb{H}$ .

In order to deal with the fact that the functional  $I$  is strongly indefinite, we shall use a version of the Palais-Smale condition inspired by the Galerkin method which we describe next.

### 2.1. Palais-Smale condition and Generalized Mountain Pass Theorem

Let  $\mathbb{W}$  be a real separable Banach space, and suppose  $\{\phi_i : i \in J \subset \mathbb{N}\}$  is a basis of  $\mathbb{W}$ . Given a family  $\{J_n\}_{n \in \mathbb{N}}$  such that  $J_1 \subset J_2 \subset \dots \subset J_n \subset \dots \subset J$ , and  $\bigcup_{n \in \mathbb{N}} J_n = J$ , we set for every  $n \in \mathbb{N}$ ,

$$\mathbb{W}_n = \overline{\text{span}\{\phi_i : i \in J_n\}} \quad \text{and} \quad I_n = I|_{\mathbb{W}_n}$$

Now we shall consider the  $(PS)^*$  condition with respect to the family  $(\mathbb{W}_n)$  of  $\mathbb{W}$  in the following sense:

**Definition 2.1.** *Given  $c \in \mathbb{R}$ , we say that  $(u_n) \subset \mathbb{W}$  is a  $(PS)_c^*$  sequence for the functional  $I \in C^1(\mathbb{W}, \mathbb{R})$  if*

- (i). *There exists a sequence  $(n_j) \subset \mathbb{N}$ ,  $n_j \rightarrow \infty$  as  $j \rightarrow \infty$  such that  $u_{n_j} \in \mathbb{W}_{n_j}$ , for every  $j \in \mathbb{N}$ ;*
- (ii).  *$I(u_{n_j}) \rightarrow c$  as  $j \rightarrow \infty$ ;*
- (iii).  *$\|I'_{n_j}(u_{n_j})\|_{\mathbb{W}_{n_j}^*} \rightarrow 0$ , as  $j \rightarrow \infty$ .*

**Definition 2.2.** *Given  $c \in \mathbb{R}$ , we say that the functional  $I \in C^1(\mathbb{W}, \mathbb{R})$  satisfies the  $(PS)_c^*$  condition (with respect to the family  $(\mathbb{W}_n)$ ) if every  $(PS)_c^*$  sequence for the functional  $I$  possesses a subsequence converging to a critical point of  $I$ .*

We apply the following version of Generalized Mountain Pass Theorem (see [27] and [28]). Considering  $\mathbb{W} = X_1 \oplus X_2$ , we shall suppose  $I \in C^1(\mathbb{W}, \mathbb{R})$  satisfies the  $(PS)_c^*$  condition with respect to the family  $(\mathbb{W}_n) \subset \mathbb{W}$  with  $\mathbb{W}_n = X_1^n \oplus X_2^n$ ,  $X_i^n \subset X_i$ ,  $i = 1, 2$ ,  $\dim(X_1^n) < \infty$  for any  $n \in \mathbb{N}$ .

**Theorem 2.2.** (*Generalized Mountain Pass Theorem*) *Let  $\mathbb{W} = X_1 \oplus X_2$  be a real Banach space. Suppose  $I \in C^1(\mathbb{W}, \mathbb{R})$  satisfies*

( $I_0$ ). *For every  $u \in X_1$ , it holds  $I(u) \leq 0$ .*

( $I_1$ ). *There exist  $\rho > 0$  and  $\sigma > 0$  such that*

$$I(u) \geq \sigma, \quad \text{for every } u \in \partial B_\rho(0) \cap X_2.$$

( $I_2$ ). *For each  $n \in \mathbb{N}$ , there exist  $e_n \in \partial B_1(0) \cap X_2^n$  and  $\beta \in \mathbb{R}$  such that*

$$I(u) \leq \beta, \quad \text{for every } u \in X_1 \oplus \mathbb{R}^+ e_n,$$

where

$$X_1 \oplus \mathbb{R}^+ e_n = \{u = v + te_n \in X_1 \oplus \mathbb{R}e_n : v \in X_1, t \geq 0\}.$$

*If  $I$  satisfies  $(PS)_c^*$  for every  $c \in [\sigma, \beta]$ , then  $I$  possesses a critical point  $u \in \mathbb{W}$  such that  $I(u) \in [\sigma, \beta]$ .*

### 3. The linking geometry and estimates for the critical level

In this section we verify that the functional  $I$ , defined in (2.3), satisfies the geometrical properties of Theorem 2.2. We start by verifying the hypothesis ( $I_0$ ) and ( $I_1$ ).

**Lemma 3.1.** *There exist  $\rho, \sigma > 0$  such that  $I(z) \geq \sigma$ , for all  $z \in S := \partial B_\rho \cap \mathbb{H}^+$ . Moreover,  $I(z) \leq 0$  if  $z \in \mathbb{H}^-$ .*

*Proof.* Assumption ( $H_2$ ) implies that, for a given  $\epsilon_0 > 0$ , there exists  $t_0 > 0$  such that

$$f(t) \leq 2\epsilon_0 t \quad \text{and} \quad F(t) \leq \epsilon_0 t^2, \quad \text{for all } t \leq t_0.$$

On the other hand, given  $\alpha > \alpha_0$  and  $q > 2$  there exists a positive constant  $C_1$  such that,

$$F(t) \leq C_1 t^q (\exp \alpha t^2 - 1), \quad \text{for all } t \geq t_0.$$

Given  $u \in H^1(\mathbb{R})$  and define  $\Omega_{t_0} := \{x \in \mathbb{R} : 0 \leq u(x) \leq t_0\}$ . Thus,

$$\begin{aligned} \int_{\mathbb{R}} P(x)F(u) dx &= \int_{\Omega_{t_0}} P(x)F(u) dx + \int_{\mathbb{R} \setminus \Omega_{t_0}} P(x)F(u) dx \\ &\leq \epsilon_0 \int_{\mathbb{R}} P(x)u^2 dx + C_1 \int_{\mathbb{R} \setminus \Omega_{t_0}} P(x)u^q (\exp \alpha u^2 - 1) dx. \end{aligned}$$

Now using the Hölder's inequality with  $r^{-1} + s^{-1} = 1$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}} P(x)F(u) dx &\leq \epsilon_0 \|u\|_{L^2(\mathbb{R}; P)}^2 \\ &+ C_1 \|u\|_{L^{qr}(\mathbb{R}; P)}^q \left( \int_{\mathbb{R} \setminus \Omega_{t_0}} P(x) (\exp \alpha u^2 - 1)^s dx \right)^{\frac{1}{s}}. \end{aligned} \quad (3.1)$$

We use the following inequality to estimate the second term in (3.1). There exists a constant  $C_2 = C_2(t_0)$  such that

$$(\exp \alpha t^2 - 1)^s \leq C_2(\exp \alpha s t^2 - 1) \text{ for all } t \geq t_0$$

which implies

$$\int_{\mathbb{R} \setminus \Omega_{t_0}} P(x)(\exp(\alpha u^2) - 1)^s dx \leq C_2 \int_{\mathbb{R} \setminus \Omega_{t_0}} P(x)(\exp(\alpha s u^2) - 1) dx.$$

From (1.3), we have

$$\int_{\mathbb{R} \setminus \Omega_{t_0}} P(x)(\exp \alpha s u^2 - 1) dx \leq C \int_{\mathbb{R}} (\exp \alpha s u^2 - 1) dx$$

which applied to (3.1) together with Lemma 2.1 gives

$$\int_{\mathbb{R}} P(x)F(u) dx \leq \epsilon_0 C \|u\|_{1/2}^2 + C_3 \|u\|_{1/2}^q \left( \int_{\mathbb{R}} (\exp \alpha s u^2 - 1) dx \right)^{\frac{1}{s}}.$$

Analogously, possibly for different positive constants  $C$  and  $C_3$ , we have

$$\int_{\mathbb{R}} Q(x)G(u) dx \leq \epsilon_0 C \|u\|_{1/2}^2 + C_3 \|u\|_{1/2}^q \left( \int_{\mathbb{R}} (\exp \alpha s u^2 - 1) dx \right)^{\frac{1}{s}}.$$

The previous inequalities used in (2.3) imply

$$I(u, u) \geq \|u\|_{1/2}^2 - C_1 \epsilon_0 \|u\|_{1/2}^2 - C_4 \|u\|_{1/2}^q \left( \int_{\mathbb{R}} (\exp \alpha s u^2 - 1) dx \right)^{\frac{1}{s}}.$$

The Trudinger-Moser inequality as in Theorem A reads

$$\int_{\mathbb{R}} (\exp \alpha s u^2 - 1) dx = \int_{\mathbb{R}} \left\{ \exp \left[ \alpha s \|u\|_{1/2}^2 \left( \frac{u}{\|u\|_{1/2}} \right)^2 \right] - 1 \right\} dx \leq C$$

whenever  $\|u\|_{1/2}^2 = \rho$  and  $\alpha s \rho < \pi$  (we can choose  $\rho$  sufficiently small for this to hold good). Hence

$$I(u, u) \geq \|u\|_{1/2}^2 - C_1 \epsilon_0 \|u\|_{1/2}^2 - C_5 \|u\|_{1/2}^q.$$

Therefore, since  $q > 2$ , we can find  $\sigma > 0$  sufficiently small, such that  $I(u, u) \geq \sigma > 0$  for  $\|u\|_{1/2} = \rho$ . To complete the proof, one can see that for any  $z = (u, -u) \in \mathbb{H}^-$ ,

$$I(u, -u) = -\|u\|_{1/2}^2 - \int_{\mathbb{R}} [P(x)F(u) + Q(x)G(u)] dx \leq 0,$$

by using assumption (H1).  $\square$

Now we obtain an upper bound for the minimax level, and consequently  $I$  satisfies  $(I_2)$ . The argument strongly relies on the Trudinger-Moser inequality and the growth condition (H4). This estimate depends on an intricate reasoning involving the Moser sequence to be introduced. The idea is similar

to that found in the celebrated work due to H. Brezis and L. Nirenberg. However here we replace the Talenti's functions by the Moser's functions, defined as truncations and dilations of the fundamental solution: given  $k \in \mathbb{N}$ ,

$$m_k(x) = \frac{\bar{\omega}_k(x)}{\|\bar{\omega}_k\|_{1/2}},$$

where

$$\bar{\omega}_k(x) = \frac{1}{\sqrt{\pi}} \begin{cases} (\log k)^{1/2}, & |x| \leq 1/k, \\ \frac{\log \frac{1}{|x|}}{(\log k)^{1/2}}, & 1/k \leq |x| \leq 1, \\ 0, & |x| \geq 1. \end{cases} \quad (3.2)$$

**Lemma 3.2.** *Defining  $\rho_k := \frac{\log k}{\pi} - m_k^2$ , we have that there exists  $C_o > 0$  such that*

$$0 \leq \rho_k \leq C_o \quad \text{for every } |x| \leq \frac{1}{k}.$$

*Proof.* Indeed, we have

$$\|\bar{\omega}_k\|_{1/2}^2 = \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} \bar{\omega}_k(x)|^2 dx + \int_{B(0,1)} |\bar{\omega}_k(x)|^2 dx = 1 + r_k.$$

Here  $r_k \geq 0$  and direct calculations yield  $r_k = O\left(\frac{1}{\log k}\right)$  as  $k \rightarrow \infty$  (see estimates (2.3) – (2.5) in [31]). Hence,  $|r_k \log k| \leq C$  for  $k > k_0$  leading to  $\lim_{k \rightarrow \infty} r_k = 0$ . Therefore, if  $|x| \leq 1/k$ , from the definition of  $m_k$  and (3.2), we get

$$m_k^2 := \frac{\log k}{\pi \|\bar{\omega}_k\|_{1/2}^2} = \frac{\log k}{\pi(1 + r_k)} = \frac{\log k}{\pi} - \frac{(\log k)r_k}{\pi(1 + r_k)}.$$

Hence

$$\rho_k = \frac{(\log k)r_k}{\pi(1 + r_k)}$$

which together with the logarithmic decay estimate on  $r_k$ , as above, completes the proof.  $\square$

**Proposition 3.3.** *There exists  $l_0 \in \mathbb{N}$  such that for all  $l \geq l_0$  the corresponding Moser's function  $m_l$  satisfies*

$$\sup_{\mathbb{R}^+(m_l, m_l) \oplus \mathbb{H}^-} I < \frac{\pi}{\alpha_0}.$$

*Proof.* Suppose by contradiction that there exists a sequence  $(l_k) \subset \mathbb{N}$  such that  $l_k \nearrow +\infty$  and for every  $k \in \mathbb{N}$ ,

$$\sup_{\mathbb{R}^+(m_{l_k}, m_{l_k}) \oplus \mathbb{H}^-} I \geq \frac{\pi}{\alpha_0}.$$

So, for every  $k \in \mathbb{N}$ ,  $j \in \mathbb{N}$ , there exists  $u_{j,k} \in H^{1/2,2}(\mathbb{R})$  and  $\tau_{j,k} > 0$  such that

$$I(\eta_{j,k}) \geq \frac{\pi}{\alpha_0} - \frac{1}{j}, \quad (3.3)$$

where

$$\eta_{j,k} = \tau_{j,k}(m_{l_k}, m_{l_k}) + (u_{j,k}, -u_{j,k}) \in \mathbb{R}^+(m_{l_k}, m_{l_k}) \oplus \mathbb{H}^-.$$

Let  $h : [0, \infty) \rightarrow \mathbb{R}$  be defined by  $h(t) := I(t\eta_{j,k})$ . Since  $h(0) = 0$  and  $\lim_{t \rightarrow +\infty} h(t) = -\infty$ , there exists a maximum point  $t_o \in (0, \infty)$  such that

$$h(t_o) = I(t_o\eta_{j,k}) \geq \frac{\pi}{\alpha_0} - \frac{1}{j}.$$

Without loss of generality we may assume that  $t_o = 1$ . Hence

$$I'(\eta_{j,k})\eta_{j,k} = 0. \quad (3.4)$$

Using (3.3) and (3.4), we can write

$$\begin{aligned} \tau_{j,k}^2 &\geq \frac{\pi}{\alpha_0} - \frac{1}{j} + \|u_{j,k}\|_{1/2}^2 + \int_{\mathbb{R}} P(x)F(\tau_{j,k}m_{l_k} + u_{j,k}) \, dx \\ &\quad + \int_{\mathbb{R}} Q(x)G(\tau_{j,k}m_{l_k} - u_{j,k}) \, dx \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} 2\tau_{j,k}^2 &= 2\|u_{j,k}\|_{1/2}^2 + \int_{\mathbb{R}} P(x)f(\tau_{j,k}m_{l_k} + u_{j,k})(\tau_{j,k}m_{l_k} + u_{j,k}) \, dx \\ &\quad + \int_{\mathbb{R}} Q(x)g(\tau_{j,k}m_{l_k} - u_{j,k})(\tau_{j,k}m_{l_k} - u_{j,k}) \, dx. \end{aligned} \quad (3.6)$$

From (3.5) it follows

$$\frac{\pi}{\alpha_0} + s_{j,k} := \tau_{j,k}^2 \geq \frac{\pi}{\alpha_0} - \frac{1}{j}. \quad (3.7)$$

Therefore  $\liminf_{j \rightarrow \infty} s_{j,k} \geq 0$ .

Consider  $C_o > 0$  given by Lemma 3.2 and take  $\beta_0 > 0$  be such that

$$\beta_0 > \frac{\pi}{2\alpha_0} \exp(\pi C_o).$$

By (H4), there exists  $R_0 = R_0(\beta_0) > 0$  such that

$$tf(t)e^{-\alpha_0 t^2} \geq \beta_0 \quad \text{and} \quad tg(t)e^{-\alpha_0 t^2} \geq \beta_0, \quad \text{for all } t \geq R_0. \quad (3.8)$$

Therefore there exists  $C > 0$  such that  $tf(t) \geq \beta_0 e^{\alpha_0 t^2} - C$  and  $tg(t) \geq \beta_0 e^{\alpha_0 t^2} - C$  for  $t \geq 0$ . We also take  $\bar{s} > 0$  and  $k_1 \in \mathbb{N}$  such that for all  $s \geq \bar{s}$  and  $k \geq k_1$

$$\begin{aligned} &2 \left( \frac{\pi}{\alpha_0} + s \right) \exp(\pi C_o) \\ &< \beta_0 l_k \exp \left( \alpha_0 s \left( \frac{\log l_k}{\pi} - \rho_{l_k} \right) \right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx. \end{aligned} \quad (3.9)$$

Next we use the inequality  $e^x + e^y \geq 2e^{\frac{x+y}{2}}$  to get the following estimate

$$\begin{aligned}
 & \int_{B_{1/l_k}(0)} (P(x)f(\tau_{j,k}m_{l_k} + u_{j,k}) + Q(x)g(\tau_{j,k}m_{l_k} - u_{j,k}))(\tau_{j,k}m_{l_k} - u_{j,k}) \, dx \\
 & \geq \beta_0 \int_{B_{1/l_k}(0)} P(x) \exp(\alpha_0(\tau_{j,k}m_{l_k} + u_{j,k})^2) \, dx \\
 & \quad + \beta_0 \int_{B_{1/l_k}(0)} Q(x) \exp(\alpha_0(\tau_{j,k}m_{l_k} - u_{j,k})^2) \, dx - O(1/l_k) \\
 & \geq 2\beta_0 \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \exp(\alpha_0(\tau_{j,k}m_{l_k})^2) \, dx - O(1/l_k). \tag{3.10}
 \end{aligned}$$

From (3.6), (3.7), (3.8), (3.10) and Lemma 3.2,

$$\begin{aligned}
 & \frac{\pi}{\alpha_0} + s_{j,k} \\
 & = \tau_{j,k}^2 \geq \beta_0 \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \exp\left(\alpha_0 \tau_{j,k}^2 \left(\frac{\log l_k}{\pi} - \rho_{l_k}\right)\right) - O(1/l_k) \\
 & \geq \beta_0 l_k \exp(-\pi \rho_{l_k}) \exp\left(\alpha_0 s_{j,k} \left(\frac{\log l_k}{\pi} - \rho_{l_k}\right)\right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx \\
 & \quad - O(1/l_k)
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 & \left(\frac{\pi}{\alpha_0} + s_{j,k}\right) \exp(\pi \rho_{l_k}) \\
 & \geq \beta_0 l_k \exp\left(\alpha_0 s_{j,k} \left(\frac{\log l_k}{\pi} - \rho_{l_k}\right)\right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx - O(1/l_k). \tag{3.11}
 \end{aligned}$$

Without loss of generality, we can assume that for  $k \geq k_1$  one has  $\log l_k > 2C_o\pi$ . Then we claim that for every  $k \geq k_1$ , we have  $s_{j,k} \leq \bar{s}$ . Indeed, otherwise from Lemma 3.2,

$$\begin{aligned}
 & 2 \left(\frac{\pi}{\alpha_0} + s_{j,k}\right) \exp(\pi C_o) \\
 & \geq \beta_0 l_k \exp\left(\alpha_0 s_{j,k} \left(\frac{\log l_k}{\pi} - \rho_{l_k}\right)\right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx \\
 & \geq \beta_0 l_k \exp(\alpha_0 s_{j,k} C_o) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx.
 \end{aligned}$$

However, this contradicts (3.9). The claim is proved.

In view of the above claim and (3.7), we have

$$-\frac{1}{j} \leq s_{j,k} \leq \bar{s} \quad \text{for every } k \geq k_1.$$

Thus, taking a subsequence if necessary, we may suppose that

$$\lim_{j \rightarrow \infty} s_{j,k} = s_{o,k} \in [0, \bar{s}] \quad \text{for every } k \geq k_1.$$

Consequently, from (3.11), for every  $k \geq k_1$ ,

$$\begin{aligned} & \left( \frac{\pi}{\alpha_0} + s_{o,k} \right) \exp(\pi \rho_{l_k}) \\ & \geq \beta_0 l_k \exp \left( \alpha_0 s_{o,k} \left( \frac{\log l_k}{\pi} - \rho_{l_k} \right) \right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx - O(1/l_k). \end{aligned}$$

Hence, since  $s_{o,k} \geq 0$  and from Lemma 3.2, we get

$$\begin{aligned} & \left( \frac{\pi}{\alpha_0} + s_{o,k} \right) \exp(\pi C_o) \exp(\alpha_0 s_{o,k} C_o) \\ & \geq \beta_0 l_k \exp \left( \alpha_0 s_{o,k} \left( \frac{\log l_k}{\pi} \right) \right) \int_{B_{1/l_k}(0)} \sqrt{P(x)Q(x)} \, dx - O(1/l_k). \end{aligned} \quad (3.12)$$

Note that, from (3.12) and  $s_{o,k} \in [0, \bar{s}]$  we have that  $s_{o,k} \rightarrow 0$  as  $k \rightarrow \infty$ . Consequently, taking  $k \rightarrow \infty$  in (3.12) and using (1.4), we get

$$\frac{\pi}{2\alpha_0} \exp(\pi C_o) \geq \beta_0.$$

However, this contradicts our choice of  $\beta_0$ . Thus Proposition 3.3 is proved.  $\square$

#### 4. The $(PS)_c^*$ condition

In this section we shall verify that the functional  $I$  satisfies the  $(PS)_c^*$  condition with respect to an appropriate family of subspaces  $(\mathbb{H}_n)_n$  of  $\mathbb{H}$ . As a consequence of compact embedding results as in Lemma 2.1, there exists an orthonormal basis  $\{\varphi_1, \varphi_2, \dots\}$  in  $H^{1/2,2}(\mathbb{R})$  of normalized eigenfunctions associated to the eigenvalues of the compact operator  $((-\Delta)^{-1/2}, P)$  in the weighted  $L^2(\mathbb{R}; P)$  space. Set,

$$\begin{aligned} \mathbb{E}_n^+ &= \text{span}\{(\varphi_i, \varphi_i) : i = 1, \dots, n\} \\ \mathbb{E}_n^- &= \text{span}\{(\varphi_i, -\varphi_i) : i = 1, \dots, n\} \end{aligned}$$

Considering  $e = m_k$  given by Proposition 3.3, we define

$$\begin{aligned} \mathbb{H}_n &= \mathbb{H}_n^+ \oplus \mathbb{H}_n^- \quad \text{and} \quad I_n = I|_{\mathbb{H}_n}, \quad \text{where} \\ \mathbb{H}_n^+ &= \mathbb{R}(e, e) \oplus \mathbb{E}_n^+, \\ \mathbb{H}_n^- &= \mathbb{R}(e, -e) \oplus \mathbb{E}_n^-. \end{aligned}$$

**Remark 4.1.** We observe that it is easy to see that if  $z := (u, v) \in \mathbb{H}_n$  then  $(v, 0) \in \mathbb{H}_n$  and  $(0, u) \in \mathbb{H}_n$ .

Next result establishes that  $(PS)^*$  sequences are bounded.

**Proposition 4.1.** *Given  $c \in \mathbb{R}$ , let  $\{z_n\} := \{(u_n, v_n)\} \subset \mathbb{H}$  be a  $(PS)_c^*$  sequence for the functional  $I$  with respect to the family  $(\mathbb{H}_n)_n$  defined above. Then  $\{z_n\}$  is bounded in  $\mathbb{H}$ . Furthermore, there exists  $C_1 > 0$  such that*

$$\begin{aligned} \int_{\mathbb{R}} P(x) f(u_n) u_n \, dx &\leq C_1, & \int_{\mathbb{R}} Q(x) g(v_n) v_n \, dx &\leq C_1, \\ \int_{\mathbb{R}} P(x) F(u_n) \, dx &\leq C_1, & \int_{\mathbb{R}} Q(x) G(v_n) \, dx &\leq C_1. \end{aligned}$$

*Proof.* The sequence  $\{z_n\} \subset \mathbb{H}$  being a  $(PS)_c^*$  sequence for  $I$  satisfies the following

$$(a) \quad I(u_n, v_n) \rightarrow c \quad \text{and} \quad (b) \quad \|I'_n(u_n, v_n)\|_{\mathbb{H}_n^*} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Taking  $(\varphi, \psi) = (u_n, v_n)/\|(u_n, v_n)\| \in \mathbb{H}_n$  as testing functions according to **(b)**, we have for some sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  tending to 0:

$$\left| 2 \int_{\mathbb{R}} ((-\Delta)^{\frac{1}{4}} u_n (-\Delta)^{\frac{1}{4}} v_n + u_n v_n) \, dx - \int_{\mathbb{R}} (P(x)f(u_n)u_n + Q(x)g(v_n)v_n) \, dx \right| \leq \varepsilon_n \|(u_n, v_n)\|$$

which together with **(b)** and  $(H3)$  imply for some sequence  $(\delta_n)_{n \in \mathbb{N}}$  tending to 0 and  $\theta > 2$

$$\begin{aligned} & \int_{\mathbb{R}} P(x)f(u_n)u_n \, dx + \int_{\mathbb{R}} Q(x)g(v_n)v_n \, dx \\ & \leq 2 \int_{\mathbb{R}} P(x)F(u_n) \, dx + \int_{\mathbb{R}} Q(x)G(v_n) \, dx + 2c + 2\delta_n + \varepsilon_n \|(u_n, v_n)\| \\ & \leq \frac{2}{\theta} \int_{\mathbb{R}} P(x)f(u_n)u_n \, dx + \int_{\mathbb{R}} Q(x)g(v_n)v_n \, dx + 2c + 2\delta_n + \varepsilon_n \|(u_n, v_n)\|. \end{aligned}$$

Thus

$$\int_{\mathbb{R}} P(x)f(u_n)u_n \, dx + \int_{\mathbb{R}} Q(x)g(v_n)v_n \, dx \leq \frac{\theta}{\theta - 2} (1 + 2\delta_n + \varepsilon_n \|(u_n, v_n)\|). \tag{4.1}$$

Next taking  $(\varphi, \psi) = (v_n, 0)/\|v_n\|_{1/2}$  and  $(\varphi, \psi) = (0, u_n)/\|u_n\|_{1/2}$  in **(b)** we have

$$\begin{aligned} \|v_n\|_{1/2}^2 - \varepsilon_n \|v_n\|_{1/2} & \leq \int_{\mathbb{R}} P(x)f(u_n)v_n \, dx, \\ \|u_n\|_{1/2}^2 - \varepsilon_n \|u_n\|_{1/2} & \leq \int_{\mathbb{R}} Q(x)g(v_n)u_n \, dx. \end{aligned}$$

Setting  $U_n = u_n/\|u_n\|_{1/2}$  and  $V_n = v_n/\|v_n\|_{1/2}$  we have

$$\begin{aligned} \|v_n\|_{1/2} & \leq \int_{\mathbb{R}} P(x)f(u_n)V_n \, dx + \varepsilon_n, \\ \|u_n\|_{1/2} & \leq \int_{\mathbb{R}} Q(x)g(v_n)U_n \, dx + \varepsilon_n. \end{aligned} \tag{4.2}$$

We now rely on the following Young type inequality ( see [12], Lemma 2.4):

$$s t \leq \begin{cases} (e^{t^2} - 1) + s(\log^+ s)^{1/2}, & \text{for all } t \geq 0 \text{ and } s \geq e^{1/4}, \\ (e^{t^2} - 1) + \frac{1}{2}s^2, & \text{for all } t \geq 0 \text{ and } 0 \leq s \leq e^{1/4}. \end{cases} \tag{4.3}$$

Using the critical growth of  $f$  and  $g$ , for any  $\epsilon > 0$ , there exists a constant  $C_1 = C_1(\epsilon) > 0$  such that

$$f(s) \leq C_1 e^{\alpha_0(1+\epsilon)s^2} \quad \text{and} \quad g(s) \leq C_1 e^{\alpha_0(1+\epsilon)s^2} \quad \text{for all } s \in \mathbb{R}. \tag{4.4}$$

Let  $A_n = \left\{x \in \mathbb{R} : \frac{1}{C_1} f(u_n)(x) \geq e^{1/4}\right\}$ ,  $B_n = \left\{x \in \mathbb{R} : \frac{1}{C_1} f(u_n)(x) \leq e^{1/4}\right\}$ . Then

$$\begin{aligned} \int_{\mathbb{R}} P(x) f(u_n) V_n \, dx &\leq C_1 \int_{\mathbb{R}} P(x) (e^{V_n^2} - 1) \, dx + \\ &\int_{A_n} P(x) f(u_n) \left[ \log \frac{1}{C_1} f(u_n) \right]^{1/2} \, dx + \frac{1}{2C_1} \int_{B_n} P(x) (f(u_n))^2 \, dx. \end{aligned} \quad (4.5)$$

By (4.4), the second integral on the right hand side yields

$$\begin{aligned} \int_{A_n} P(x) f(u_n) \left[ \log \frac{1}{C_1} f(u_n) \right]^{1/2} \, dx &\leq \int_{A_n} P(x) f(u_n) \left[ \log e^{\alpha_0(1+\epsilon)u_n^2} \right]^{1/2} \, dx \\ &\leq \sqrt{\alpha_0(1+\epsilon)} \int_{\mathbb{R}} P(x) f(u_n) u_n \, dx. \end{aligned}$$

Note that by using condition (H2), for some fixed  $s_o > 0$  there exists a positive constant  $C = C(s_o)$  we have  $f(s) \leq Cs$  for  $0 < s < s_o$ . Therefore,

$$f(s)^2 \leq Cf(s)s \quad \text{for all } 0 \leq s \leq s_o$$

and

$$f(s)^2 \leq \frac{C_1 e^{1/4}}{s_o} f(s)s \quad \text{for all } s \geq s_o \quad \text{when } f(s) \leq C_1 e^{1/4}$$

which implies

$$\begin{aligned} \int_{B_n} P(x) [f(u_n)]^2 \, dx &= \int_{\{x \in B_n : u_n(x) \in [0, s_o]\}} P(x) (f(u_n))^2 \, dx \\ &\quad + \int_{\{x \in B_n : u_n(x) \geq s_o\}} P(x) (f(u_n))^2 \, dx \\ &\leq C \int_{\mathbb{R}} P(x) f(u_n) u_n \, dx. \end{aligned}$$

Now, in the light of Theorem A, the first integral in the right hand side of (4.5) is bounded *i.e.*

$$C_1 \int_{\mathbb{R}} P(x) (e^{V_n^2} - 1) \, dx \leq C.$$

Substituting the above estimates in (4.5), we obtain for some positive constant  $C$

$$\int_{\mathbb{R}} P(x) f(u_n) V_n \, dx \leq C \left( 1 + \int_{\mathbb{R}} P(x) f(u_n) u_n \, dx \right).$$

This estimate together with (4.1)-(4.2) imply

$$\|v_n\|_{1/2} \leq C(1 + 2\delta_n + \varepsilon_n \cdot \|(u_n, v_n)\|). \quad (4.6)$$

Repeating the argument above it follows

$$\|u_n\|_{1/2} \leq C(1 + 2\delta_n + \varepsilon_n \cdot \|(u_n, v_n)\|). \quad (4.7)$$

Now considering the estimates (4.6) and (4.7) we finally obtain

$$\|(u_n, v_n)\| \leq C$$

which completes the proof.  $\square$

**Proposition 4.2.** *Let  $c \in (0, \pi/\alpha_0)$  and  $\{z_n\} \subset \mathbb{H}$  be a  $(PS)_c^*$  sequence for the functional  $I$ , with respect to the family  $(\mathbb{H}_n)_n$  of  $\mathbb{H}$ , then  $\{z_n\}$  possesses a subsequence which converges weakly in  $\mathbb{H}$  to a critical point of  $I$ .*

*Proof.* By Proposition 4.1,  $\{z_n\}$  is bounded sequence in  $\mathbb{H}$ . Hence, invoking Lemma 2.1 and the fact that  $\mathbb{H}$  is a Hilbert space,  $\{z_n\}$  has a subsequence (still denote by  $\{z_n\}$ ) such that

$$\begin{aligned} z_n &:= (u_n, v_n) \rightharpoonup z_o := (u_o, v_o) \text{ in } \mathbb{H}, \\ u_n \rightarrow u_o &\text{ and } v_n \rightarrow v_o \text{ in } L^q(\mathbb{R}; P) \text{ and } L^q(\mathbb{R}; Q), \forall q \in [2, \infty), \\ u_n(x) \rightarrow u_o(x) &\text{ and } v_n(x) \rightarrow v_o(x) \text{ a.e. in } \mathbb{R}. \end{aligned} \quad (4.8)$$

We also note that, in view of Proposition 4.1, there exists  $C > 0$  such that, for every  $n \in \mathbb{N}$ ,

$$\int_{\mathbb{R}} P(x)f(u_n)u_n \, dx \leq C, \quad \int_{\mathbb{R}} Q(x)g(v_n)v_n \, dx \leq C. \quad (4.9)$$

We claim that for any  $\phi \in C_0^\infty(\mathbb{R})$  with  $\text{supp } \phi = K \subset \subset \mathbb{R}$ , we have

$$\int_{\mathbb{R}} P(x)f(u_n)\phi \, dx \rightarrow \int_{\mathbb{R}} P(x)f(u_o)\phi \, dx. \quad (4.10)$$

Indeed, let  $M > 0$  and  $\psi_M$  a smooth cut-off function such that

$$\psi_M(t) = \begin{cases} 1 & \text{for } |t| \leq M \\ 0 & \text{for } |t| \geq M + 1. \end{cases}$$

Then we have

$$\begin{aligned} &\int_{\mathbb{R}} P(x)f(u_n)\phi \, dx \\ &= \int_{\mathbb{R}} P(x)(1 - \psi_M(u_n))f(u_n)\phi \, dx + \int_{\mathbb{R}} P(x)\psi_M(u_n)f(u_n)\phi \, dx \\ &= \int_{\mathbb{R} \cap \{u_n > M\}} P(x)(1 - \psi_M(u_n))f(u_n)\phi \, dx \\ &\quad + \int_{\mathbb{R} \cap \{u_n < M+1\}} P(x)\psi_M(u_n)f(u_n)\phi \, dx \\ &\leq \frac{\|\phi\|_\infty}{M} \int_{\mathbb{R}} P(x)f(u_n)u_n \, dx + \int_{\mathbb{R} \cap \{u_n < M+1\}} P(x)\psi_M(u_n)f(u_n)\phi \, dx. \end{aligned}$$

In the above inequality, using the Lebesgue dominated convergence theorem in the second integral and (4.9) in the first integral, we get for large  $n$

$$\int_{\mathbb{R}} P(x)f(u_n)\phi \, dx \leq \frac{C\|\phi\|_\infty}{M} + \int_{\mathbb{R} \cap \{u_o \leq M+1\}} P(x)\psi_M(u_o)f(u_o)\phi \, dx. \quad (4.11)$$

Now we estimate the second integral, keeping in mind the fact that  $\psi_M(u_n) \rightarrow 1$  a.e. in  $\mathbb{R}$  for  $M$  large enough as

$$\begin{aligned} & \left| \int_{\mathbb{R}} P(x)f(u_0)\phi \, dx - \int_{\mathbb{R}} P(x)\psi_M(u_0)f(u_0)\phi \, dx \right| \\ & \leq \frac{\|\phi\|_{\infty}}{M} \int_{\mathbb{R}} P(x)f(u_0)u_0 \, dx = O(1/M). \end{aligned} \quad (4.12)$$

Now for large  $M > 0$ , we combine (4.12) with (4.11) to get our required result. Similarly one can show that

$$\int_{\mathbb{R}} Q(x)g(v_n)\psi \, dx \rightarrow \int_{\mathbb{R}} Q(x)g(v_o)\psi \, dx. \quad (4.13)$$

Finally using (4.8), (4.10) and (4.13) we conclude that

$$I'(u_o, v_o)(\phi, \psi) = 0 \quad \text{for all } (\phi, \psi) \in C_0^{\infty}(\mathbb{R}) \times C_0^{\infty}(\mathbb{R}). \quad (4.14)$$

Using the fact that  $\cup \mathbb{H}_n$  is dense in  $\mathbb{H}$  it follows that (4.14) holds for any  $(\phi, \psi) \in \mathbb{H}$ . In other words,  $z_o$  is a critical point of  $I$ . Proposition 4.2 is now proved.  $\square$

## 5. Proof of Theorem 1.1

Arguing by contradiction, we suppose that the origin is the only critical point of the functional  $I$ . Note that the positivity of nontrivial weak solutions follows from the classical regularity theory and strong maximum principle for fractional Laplacian problems.

We shall verify that, under this assumption,  $I$  satisfies  $(PS)_c^*$  for every  $c \in (0, \pi/\alpha_0)$ .

Let  $\{z_n\} = \{(u_n, v_n)\} \in \mathbb{H}_n$  be such that

$$I(z_n) \rightarrow c \in (0, \pi/\alpha_0), \quad (5.1)$$

$$\|I'_n(z_n)\|_{\mathbb{H}_n^*} \rightarrow 0. \quad (5.2)$$

By Lemma 2.1, Propositions 3.3, 4.1 and the fact that the origin is the only possible critical point of  $I$ , we may assume that there exists  $C > 0$  such that

$$\|z_n\| \leq C, \quad \text{in } \mathbb{H} \quad (5.3)$$

$$z_n = (u_n, v_n) \rightharpoonup (0, 0) \quad \text{weakly in } \mathbb{H},$$

$$u_n \rightarrow 0 \quad \text{strongly in } L^q(\mathbb{R}; P), \quad \forall q \in [2, \infty),$$

$$v_n \rightarrow 0 \quad \text{strongly in } L^q(\mathbb{R}; Q), \quad \forall q \in [2, \infty),$$

$$z_n(x) = (u_n(x), v_n(x)) \rightarrow (0, 0) \quad \text{a. e. in } \mathbb{R},$$

$$\int_{\mathbb{R}} P(x)f(u_n)u_n \, dx \leq C, \quad \int_{\mathbb{R}} Q(x)g(v_n)v_n \, dx \leq C. \quad (5.4)$$

Since  $f$  and  $g$  belong to the class  $\mathbb{C}_P$  and  $\mathbb{C}_Q$  respectively, we get

$$\int_{\mathbb{R}} P(x)F(u_n) \, dx \rightarrow 0, \quad \int_{\mathbb{R}} Q(x)G(v_n) \, dx \rightarrow 0. \quad (5.5)$$

From (5.5) and (5.1), we may find  $\delta > 0$  such that, for  $n$  sufficiently large,

$$\int_{\mathbb{R}} \left( (-\Delta)^{\frac{1}{4}} u_n (-\Delta)^{\frac{1}{4}} v_n + u_n v_n \right) dx < \frac{\pi}{\alpha_0} - 2\delta.$$

This inequality, combined with (5.2) and (5.3), implies, for  $n$  sufficiently large,

$$\int_{\mathbb{R}} P(x) f(u_n) u_n dx + \int_{\mathbb{R}} Q(x) g(v_n) v_n dx \leq \frac{\pi}{\alpha_0} - \delta. \quad (5.6)$$

We claim that, given  $\beta > \alpha_0$ , for  $n$  sufficiently large,

$$\|u_n\|_{1/2} + \|v_n\|_{1/2} \leq \left( \frac{\beta}{\alpha_0} \left( \frac{\pi}{\alpha_0} - \delta \right) \right)^{1/2}. \quad (5.7)$$

In order to prove this claim we note that  $\|u_n\|_{1/2} \not\rightarrow 0$ . Indeed, assuming otherwise, from (5.3) we get that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left( (-\Delta)^{\frac{1}{4}} u_n (-\Delta)^{\frac{1}{4}} v_n + u_n v_n \right) dx = 0$$

which together with (5.1) and (5.5), leads to a contradiction. Thus, we assume that  $\|u_n\|_{1/2} \geq b > 0$  and  $\|v_n\|_{1/2} \geq b > 0$  for all  $n$ . Now, set  $\tilde{u}_n = \left( \frac{\pi}{\alpha_0} - \delta \right)^{1/2} \frac{u_n}{\|u_n\|_{1/2}}$ ,  $s = \frac{g(v_n)}{\sqrt{\alpha_0}}$  and  $t = \sqrt{\alpha_0} \tilde{u}_n$ . Then, using (5.2) and (4.3) with  $s$  and  $t$  as above results into

$$\begin{aligned} & \left( \frac{\pi}{\alpha_0} - \delta \right)^{1/2} \|u_n\|_{1/2} \\ &= \int_{\mathbb{R}} Q(x) g(v_n) \tilde{u}_n dx + o_n(1) \leq \int_{\mathbb{R}} Q(x) (e^{\alpha_0 \tilde{u}_n^2} - 1) dx \\ & \quad + \int_{\{x \in \mathbb{R}: g(v_n)(x) \geq \sqrt{\alpha_0} e^{1/4}\}} Q(x) \frac{g(v_n)}{\sqrt{\alpha_0}} \left[ \log \left( \frac{g(v_n)}{\sqrt{\alpha_0}} \right) \right]^{1/2} dx \\ & \quad + \frac{1}{2} \int_{\{x \in \mathbb{R}: g(v_n)(x) \leq \sqrt{\alpha_0} e^{1/4}\}} Q(x) \frac{(g(v_n))^2}{\alpha_0} dx + o_n(1). \end{aligned} \quad (5.8)$$

Let us evaluate each integral on the right hand side of (5.8). Taking  $p > 1$  and using the elementary inequalities for real numbers

$$\begin{aligned} e^{\alpha_0 t^2} - 1 - \alpha_0 t^2 &\leq \alpha_0 t^2 (e^{\alpha_0 t^2} - 1) \\ (e^{\alpha_0 t^2} - 1)^p &\leq C (e^{p\alpha_0 t^2} - 1), \quad \text{for } t \geq 0, \end{aligned}$$

together with the Hölder inequality, we obtain

$$\begin{aligned} & \int_{\mathbb{R}} Q(x) (e^{\alpha_0 \tilde{u}_n^2} - 1 - \alpha_0 \tilde{u}_n^2) dx \\ & \leq \alpha_0 \|\tilde{u}_n\|_{L^{2s}(\mathbb{R}; Q)}^2 \left[ \int_{\mathbb{R}} Q(x) (e^{\alpha_0 \tilde{u}_n^2} - 1)^{s'} dx \right]^{1/s'} \\ & \leq C \alpha_0 \|\tilde{u}_n\|_{L^{2s}(\mathbb{R}; Q)}^2 \left[ \int_{\mathbb{R}} (e^{\alpha_0 s' \tilde{u}_n^2} - 1) dx \right]^{1/s'} \end{aligned}$$

where  $s' = s/(s-1)$ . If  $s'(\pi - \delta\alpha_0) < \pi$  we may apply Theorem A, to find a constant  $C > 0$  such that

$$\int_{\mathbb{R}} Q(x) \left( e^{\alpha_0 \tilde{u}_n^2} - 1 \right) dx \leq \alpha_0 \|\tilde{u}_n\|_{L^2(\mathbb{R}; Q)}^2 + C \|\tilde{u}_n\|_{L^{2s}(\mathbb{R}; Q)}^2.$$

Now using compactness result from Lemma 2.1 in the above inequality, we get

$$\int_{\mathbb{R}} Q(x) \left( e^{\alpha_0 \tilde{u}_n^2} - 1 \right) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.9)$$

The second integral in (5.8) may be estimated as follows: Considering  $\beta_1 > \alpha_0$ , we may find  $C = C(\beta_1) > 0$ , such that for any  $t \geq 0$

$$g(t) \leq C e^{\beta_1 t^2}.$$

Hence

$$\log \left( \frac{g(t)}{\sqrt{\alpha_0}} \right) \leq \log \left( \frac{C}{\sqrt{\alpha_0}} \right) + \beta_1 t^2. \quad (5.10)$$

Using condition (H2), there exists a positive constant  $C$  independent of  $n$  such that

$$v_n(x) \geq C \quad \text{for all } x \in B_n \quad (5.11)$$

where

$$B_n := \left\{ x \in \mathbb{R} : g(v_n)(x) \geq \sqrt{\alpha_0} e^{1/4} \right\}.$$

Thus using (5.10)-(5.11) we get

$$\begin{aligned} & \int_{B_n} Q(x) \frac{g(v_n)}{\sqrt{\alpha_0}} \left[ \log \left( \frac{g(v_n)}{\sqrt{\alpha_0}} \right) \right]^{1/2} dx \\ & \leq \frac{1}{\sqrt{\alpha_0}} \sqrt{\log \left( \frac{C}{\sqrt{\alpha_0}} \right)} \int_{B_n} Q(x) g(v_n) dx + \frac{\sqrt{\beta_1}}{\sqrt{\alpha_0}} \int_{\mathbb{R}} Q(x) g(v_n) v_n dx. \end{aligned} \quad (5.12)$$

We shall prove that

$$\lim_{n \rightarrow \infty} \int_{B_n} Q(x) g(v_n) dx = 0. \quad (5.13)$$

Indeed, given  $M > 1$ , we consider  $A_M^n := \{x \in \mathbb{R} : v_n(x) \geq M\}$ . By (H1) and (H2), there exists  $C > 0$ , independent of  $n \in \mathbb{N}$ , such that  $g(v_n(x))v_n(x) \leq C v_n^2(x)$ , for every  $x \in B_n \setminus A_M^n$ . Consequently, by (5.4),

$$\begin{aligned} & \int_{B_n} Q(x) g(v_n) dx = \int_{B_n \setminus A_M^n} Q(x) g(v_n) dx + \int_{B_n \cap A_M^n} Q(x) g(v_n) dx \\ & \leq \int_{\{B_n \setminus A_M^n\} \cap B_r^c(0)} Q(x) g(v_n) dx + \int_{\{B_n \setminus A_M^n\} \cap B_r(0)} Q(x) g(v_n) dx \\ & \quad + \frac{1}{M} \int_{\mathbb{R}} Q(x) g(v_n) v_n dx. \end{aligned}$$

Let  $\epsilon > 0$ . For  $L$  large enough and using (5.11), (H2), the Hölder inequality together with  $Q(x) < \epsilon$  for  $|x| > L$ , we have for some constant  $\tilde{C} > 0$

$$\int_{\{B_n \setminus A_M^n\} \cap B_L^c(0)} Q(x) g(v_n) dx \leq \epsilon C \int_{\{B_n \setminus A_M^n\} \cap B_L(0)} |v_n|^2 dx \leq \tilde{C} \epsilon.$$

From compact embeddings, we have also for a fixed  $L > 0$

$$\int_{\{B_n \setminus A_n^M\} \cap B_L(0)} Q(x)g(v_n) dx = o_n(1).$$

Using above estimates and noting that  $M > 1$  may be chosen properly large, the above inequality implies that (5.13) must hold.

From (5.12) and (5.13) we obtain

$$\int_{B_n} Q(x) \frac{g(v_n)}{\sqrt{\alpha_0}} \left[ \log \left( \frac{g(v_n)}{\sqrt{\alpha_0}} \right) \right]^{1/2} dx \leq \sqrt{\frac{\beta_1}{\alpha_0}} \int_{\mathbb{R}} Q(x)g(v_n)v_n dx + o_n(1). \tag{5.14}$$

Next, we use (H1) – (H2) and Lemma 2.1 to obtain

$$\frac{1}{2} \int_{\{x \in \mathbb{R}: g(v_n)(x) \leq \sqrt{\alpha_0}e^{1/4}\}} Q(x) \frac{(g(v_n))^2}{\alpha_0} dx \leq \frac{C}{2\alpha_0} \int_{\mathbb{R}} Q(x)v_n^2 dx = o_n(1). \tag{5.15}$$

Again using Lemma 2.1, (5.9), (5.14), (5.15), we may write

$$\left( \frac{\pi}{\alpha_0} - \delta \right)^{1/2} \|u_n\|_{1/2} \leq o_n(1) + \sqrt{\frac{\beta_1}{\alpha_0}} \int_{\mathbb{R}} Q(x)g(v_n)v_n dx. \tag{5.16}$$

Repeating the same argument for  $v_n$ , we also obtain

$$\left( \frac{\pi}{\alpha_0} - \delta \right)^{1/2} \|v_n\|_{1/2} \leq o_n(1) + \sqrt{\frac{\beta_1}{\alpha_0}} \int_{\mathbb{R}} P(x)f(u_n)u_n dx. \tag{5.17}$$

Choosing  $\beta_1 \in (\alpha_0, \beta)$ , the estimate (5.7) is a direct consequence of (5.6), (5.16), (5.17). The claim is proved.

Using (5.7) we can choose  $\beta$  close enough to  $\alpha_0$  and  $\varepsilon > 0$  such that  $(\alpha_0 + \varepsilon) \max\{\|u_n\|_{1/2}^2, \|v_n\|_{1/2}^2\} < \pi$ . Now using (H1) – (H2) together with the exponential critical growth of  $f$  and Theorem A we obtain

$$\begin{aligned} \int_{\mathbb{R}} P(x)f(u_n)u_n dx &\leq \varepsilon \int_{\mathbb{R}} P(x)|u_n|^2 dx + C_\varepsilon \int_{\mathbb{R}} P(x)u_n(e^{(\alpha_0+\varepsilon)u_n^2} - 1) dx \\ &\leq \varepsilon \|u_n\|_{L^2(\mathbb{R};P)}^2 + C_\varepsilon \|u_n\|_{L^s(\mathbb{R};P)} \left( \int_{\mathbb{R}} P(x)(e^{s'(\alpha_0+\varepsilon)u_n^2} - 1) dx \right)^{1/s'} \end{aligned}$$

where  $1/s + 1/s' = 1$ . Now on choosing  $s > 2$  with  $s'$  sufficiently close to 1 such that  $(\alpha_0 + \varepsilon) \max\{\|u_n\|_{1/2}^2, \|v_n\|_{1/2}^2\}s' < \pi$ , we can do the similar calculations as before to control the integral in the second term of the above inequality. Hence using compactness from Lemma 2.1, we obtain

$$\int_{\mathbb{R}} P(x)f(u_n)u_n dx \rightarrow 0. \tag{5.18}$$

Analogously,

$$\int_{\mathbb{R}} Q(x)g(v_n)v_n dx \rightarrow 0. \tag{5.19}$$

Finally, using (5.18), (5.19) and (5.2) we obtain

$$\int_{\mathbb{R}} \left( (-\Delta)^{\frac{1}{4}} u_n (-\Delta)^{\frac{1}{4}} v_n + u_n v_n \right) dx \rightarrow 0$$

which together with (5.5) gives a contradiction with (5.1). The proof of the theorem is complete.

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## Appendix: Validation of Remark 1.2

In this appendix, we show the claim that under the assumptions (H2) and (H5),  $f$  (respectively  $g$ ) belongs to the class  $\mathbb{C}_P$  (respectively  $\mathbb{C}_Q$ ) and that if  $f(t) = O(t^2)$  then  $u_n \rightarrow 0$  and  $\{\int_{\mathbb{R}} P(x)f(u_n)u_n dx\}$  bounded imply that  $\int_{\mathbb{R}} P(x)f(u_n) dx \rightarrow 0$ .

*Proof.* Let  $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$  be a sequence such that

$$u_n \rightarrow 0, \quad \text{and} \quad \int_{\mathbb{R}} P(x)f(u_n)u_n dx < C. \quad (5.20)$$

For any given  $\epsilon > 0$ , using (H2) and (H5), there exists  $c_o = c_o(\epsilon) > 0$  sufficiently small and  $M > 0$  sufficiently large such that

$$F(t) \leq \epsilon|t|^2 \quad \text{for} \quad |t| < c_o \quad \text{and} \quad F(t) \leq \epsilon f(t)t \quad \text{for} \quad t > M.$$

Hence from (5.20), we have

$$\begin{aligned} \int_{\mathbb{R}} P(x)F(u_n) dx &\leq \epsilon \int_{\{u_n \leq c_o\}} P(x)|u_n|^2 dx + \int_{\{c_o \leq u_n \leq M\}} P(x)F(u_n) dx \\ &\quad + \epsilon \int_{\{u_n \geq M\}} P(x)f(u_n)u_n dx \\ &\leq \epsilon C + \int_{\{c_o \leq u_n \leq M\}} P(x)F(u_n) dx. \end{aligned} \quad (5.21)$$

Further for  $L = L(\epsilon) > 0$  large enough such that  $P(x) < \epsilon$  for  $x \in B_L^c(0)$ , we have

$$\int_{\{c_o \leq u_n \leq M\} \cap B_L^c(0)} P(x)F(u_n) dx \leq \epsilon C, \quad (5.22)$$

where  $C$  is independent of  $n$ . Indeed,

$$c_o^2 |\{c_o \leq u_n \leq M\}| \leq \int_{\{c_o \leq u_n \leq M\}} |u_n|^2 dx \leq \int_{\mathbb{R}} |u_n|^2 dx \leq C.$$

Now by Lebesgue theorem, for a such fixed  $L > 0$ , we have also

$$\int_{\{c_o \leq u_n \leq M\} \cap [-L, L]} P(x)F(u_n) dx \rightarrow 0. \quad (5.23)$$

Gathering (5.21)-(5.23), we get

$$\int_{\mathbb{R}} P(x)F(u_n) dx \rightarrow 0.$$

It finishes the proof of the first part of the claim. Next, we show the second statement of the claim. From  $f(t) = O(t^2)$  and for  $c_o, M > 0$  respectively small and large enough, we have

$$\begin{aligned} \int_{\mathbb{R}} P(x)f(u_n) dx &= \int_{\{u_n < c_o\}} P(x)f(u_n) dx + \int_{\{c_o \leq u_n \leq M\}} P(x)f(u_n) dx \\ &\quad + 1/M \int_{\{u_n > M\}} P(x)f(u_n)u_n dx \\ &\leq C \int_{\mathbb{R}} P(x)u_n^2 dx + \int_{\{c_o \leq u_n \leq M\}} P(x)f(u_n) dx + C/M. \end{aligned}$$

Using Lemma 2.1 and estimating the second integral in the above inequality in a similar way as in (5.22) and (5.23), we get the required result.  $\square$

**Remark 5.1.** Let the function  $P$  be defined as  $P(x) = \frac{1}{(|x|+1)^\epsilon}$ , for  $\epsilon > 0$  sufficiently small. Consider the sequence  $\{u_n\} \subset H^{1/2,2}(\mathbb{R})$  such that

$$u_n(x) = \begin{cases} \frac{1}{n^\alpha}, & \text{for } x \in [n, 2n], \\ \frac{(x-(n-1))}{n^\alpha}, & \text{for } x \in [n-1, n], \\ \frac{((2n+1)-x)}{n^\alpha}, & \text{for } x \in [2n, 2n+1], \\ 0 & \text{elsewhere} \end{cases}$$

with  $\alpha \in [1/2, 1)$ . Then, by straightforward calculations, we can prove  $\{u_n\}$  is bounded in  $H^{1/2,2}(\mathbb{R})$ . Furthermore, as  $n \rightarrow \infty$

$$\int_{\mathbb{R}} P(x)u_n^q dx \rightarrow 0$$

if and only if  $q$  satisfies  $\alpha q + \epsilon > 1$ . Therefore if  $f$  is of  $O(t^{1+\ell})$  near 0 with  $0 < \ell \leq \frac{1-\alpha-\epsilon}{\alpha}$ , we easily get that  $u_n \rightarrow 0$  weakly in  $H^{1/2,2}(\mathbb{R})$  and

$$\int_{\mathbb{R}} P(x)f(u_n)u_n dx \rightarrow 0$$

as  $n \rightarrow \infty$ . However,  $\int_{\mathbb{R}} P(x)f(u_n) dx \rightarrow 0$  is not verified.

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