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Constructive and destructive interferences in nonlinear hyperbolic equations

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TURBULENT EFFECTS THROUGH QUASI-RECTIFICATION

RÉMI CARLES AND CHRISTOPHE CHEVERRY

ABSTRACT. This article introduces a physically realistic model for explaining how electromagnetic waves can be internally generated, propagate and interact in strongly magnetized plasmas or in nuclear magnetic resonance experiments. It studies high frequency solutions of nonlinear hyperbolic equations for time scales at which dispersive and nonlinear effects can be present in the leading term of the solutions. It explains how the produced waves can accumulate during long times to produce constructive and destructive interferences which, in the above contexts, are part of turbulent effects.

Keywords. Nonlinear geometrical optics; oscillatory integrals; dispersion; interferences; turbulence; magnetized plasmas; nuclear magnetic resonance.

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1. INTRODUCTION

In this introduction, we present the main aspects of our text. In Subsection 1.1, we introduce a simple ODE model that is intended to serve as a guideline. In Subsection 1.2, we extend this model to better incorporate important specificities of two realistic situations which are related to strongly magnetized plasmas (SMP) and nuclear magnetic resonance (NMR). In Subsection 1.3, we state under simplified assumptions our two main results, Theorems 1.3 and 1.4. We also give an overview of our article.

1.1. **A toy model.** Introduce the *phase* $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$(1.1) \quad \varphi(t) := t + \gamma(\cos t - 1), \quad \gamma \in]0, 1/4[.$$

Let $\varepsilon \in]0, 1]$ be a small parameter, and $\lambda \in \mathbb{C}$. Fix numbers $(j_1, j_2, \nu) \in \mathbb{N}^2 \times \mathbb{R}$ such that $j_1 + j_2 \geq 2$. Select $n \in \mathbb{Z}$ and $\omega \in \mathbb{R}$. Then, define

$$(1.2) \quad F_L(\varepsilon, t) := \varepsilon^{3/2} e^{in\varphi(t)/\varepsilon}, \quad F_{NL}(\varepsilon, t, u) := \lambda e^\nu e^{i\omega t/\varepsilon} u^{j_1} \bar{u}^{j_2}.$$

Definition 1.1. *The number $\mathfrak{g} := \omega + j_1 - j_2 \in \mathbb{R}$ is called the gauge parameter associated with F_{NL} .*

Consider the ordinary differential equation on the complex plane \mathbb{C} given by

$$(1.3) \quad \frac{d}{dt} u - \frac{i}{\varepsilon} u = F(\varepsilon, t, u) := F_L(\varepsilon, t) + F_{NL}(\varepsilon, t, u), \quad u|_{t=0} = 0.$$

We can study the equation (1.3) on three different time scales:

- *Fast*, when $t \sim \varepsilon$, that is when F undergoes a few number of oscillations;
- *Normal*, when $t \sim 1$, that is when F generates $\mathcal{O}(\varepsilon^{-1})$ oscillations, whereas the periodic part ($\cos t$) inside φ sees a few number of oscillations;
- *Slow*, when $t \sim \varepsilon^{-1}$ or $T := \varepsilon t \sim 1$, that is when F involves $\mathcal{O}(\varepsilon^{-2})$ oscillations.

In this subsection, we analyze (1.3) during long times $t \sim \varepsilon^{-1}$ or $T \sim 1$. With this in mind, we can change u according to

$$(1.4) \quad u(t) = \varepsilon e^{it/\varepsilon} \mathcal{U}(\varepsilon t), \quad \mathcal{U}(T) := \varepsilon^{-1} e^{-iT/\varepsilon^2} u(\varepsilon^{-1} T).$$

Expressed in terms of \mathcal{U} , the equation (1.3) becomes

$$(1.5) \quad \frac{d}{dT} \mathcal{U} = \frac{1}{\sqrt{\varepsilon}} e^{i(n-1)T/\varepsilon^2 + in\gamma(\cos(T/\varepsilon)-1)/\varepsilon} + \lambda \varepsilon^{\nu+j_1+j_2-2} e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}.$$

The initial data is still zero. Denote by \mathcal{U}_{lin} the solution corresponding to the linear evolution, that is the solution obtained from (1.5) when $\lambda = 0$. When $\lambda \neq 0$ and when $\nu + j_1 + j_2 > 2$, the solution to (1.5) looks like \mathcal{U}_{lin} . Our aim is to first study the expression \mathcal{U}_{lin} . Then, we incorporate nonlinear effects by looking at a critical size for the nonlinearity, corresponding to the special case $\lambda \neq 0$ and $\nu + j_1 + j_2 = 2$. This means to single out the following equation

$$(1.6) \quad \frac{d}{dT} \mathcal{U} = \frac{1}{\sqrt{\varepsilon}} e^{i(n-1)T/\varepsilon^2 + in\gamma(\cos(T/\varepsilon)-1)/\varepsilon} + \lambda e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}, \quad \mathcal{U}|_{T=0} = 0.$$

The integral formulation of (1.6) reads

$$(1.7) \quad \mathcal{U}(T) = \mathcal{U}_{\text{lin}}(T) + \lambda \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} \mathcal{U}(s)^{j_1} \bar{\mathcal{U}}(s)^{j_2} ds.$$

In Paragraph 1.1.1, we first show that $\mathcal{U}_{\text{lin}}(T) = \mathcal{O}(1)$, an estimate which is sharp when $n = 1$. As a consequence, the nonlinear contribution brought by the integral term inside (1.7) is likely to be of the same order of magnitude as the linear one. It can be expected that $\mathcal{U}(T) \neq \mathcal{U}_{\text{lin}}(T) + o(1)$. In Paragraph 1.1.2, we prove that this is indeed the case if and only if $\mathfrak{g} = 1$.

1.1.1. *The linear case.* By construction, we have

$$(1.8) \quad u_{\text{lin}}(t) := \varepsilon e^{it/\varepsilon} \mathcal{U}_{\text{lin}}(\varepsilon t) = \varepsilon^{3/2} e^{it/\varepsilon} \int_0^t e^{i[n\varphi(s)-s]/\varepsilon} ds.$$

We start the analysis of (1.7) by looking at the part \mathcal{U}_{lin} through the expression u_{lin} of (1.8). Examine the right hand side of (1.8). For harmonics $n \in \mathbb{Z}$ with $n \neq 1$, since $0 < \gamma < 1/4$, remark that

$$(1.9) \quad \forall s \in \mathbb{R}, \quad 1/2 \leq |n\varphi'(s) - 1| = |n - 1 - \gamma n \sin s|.$$

Exploiting (1.9), a single integration by parts yields

$$\forall t \geq 0, \quad u_{\text{lin}}(t) = \mathcal{O}(\varepsilon^{5/2}(1+t)).$$

In other words, assuming that $n \neq 1$, we find

$$(1.10) \quad \forall T \geq 0, \quad \mathcal{U}_{\text{lin}}(T) = \mathcal{O}(\varepsilon^{3/2} + \sqrt{\varepsilon}T).$$

The situation is completely different when $n = 1$. Fix an integer $K \geq 1$. The solution u_{lin} computed at the time $t = 2K\pi$ can be viewed as a sum of contributions produced over time by the source term, namely

$$(1.11) \quad u_{\text{lin}}(2K\pi) = \sum_{k=0}^{K-1} u_k, \quad u_k := \varepsilon^{3/2} e^{i2K\pi/\varepsilon} \int_{2k\pi}^{2(k+1)\pi} e^{i[\varphi(s)-s]/\varepsilon} ds.$$

Since the function $s \mapsto \varphi(s) - s = \gamma(\cos s - 1)$ is periodic of period 2π , the wave packets u_k can be interpreted according to $u_k = \varepsilon^{3/2} e^{i2K\pi/\varepsilon} v_k$ with

$$(1.12) \quad v_k = \int_{2k\pi - \pi/2}^{2k\pi + 3\pi/2} e^{i\gamma(\cos s - 1)/\varepsilon} ds = v := \int_{-\pi/2}^{3\pi/2} e^{i\gamma(\cos s - 1)/\varepsilon} ds.$$

The function $s \mapsto \gamma(\cos s - 1)$ has exactly two non-degenerate stationary points in the interval $[2k\pi - \pi/2, 2k\pi + 3\pi/2]$, at the positions $s = 2k\pi$ and $s = 2k\pi + \pi$. Using the periodicity to get rid of the boundary terms and applying stationary phase formula, it follows that

$$(1.13) \quad v = \sqrt{\frac{2\pi\varepsilon}{\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \left(e^{i(\frac{\gamma}{\varepsilon} - \frac{\pi}{4})} + e^{-i(\frac{\gamma}{\varepsilon} - \frac{\pi}{4})} \right) + \mathcal{O}(\varepsilon^{3/2}).$$

Let $A_\varepsilon \in \mathbb{C}$ be such that

$$(1.14) \quad A_\varepsilon^2 = \sqrt{\frac{2}{\pi\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \cos\left(\frac{\gamma}{\varepsilon} - \frac{\pi}{4}\right), \quad \limsup_{\varepsilon \rightarrow 0} |A_\varepsilon^2| = \sqrt{\frac{2}{\pi\gamma}} \neq 0.$$

Observe that

$$(1.15) \quad v = 2\pi A_\varepsilon^2 \sqrt{\varepsilon} + \mathcal{O}(\varepsilon^{3/2}), \quad |u_k| = 2\pi |A_\varepsilon^2| \varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

The combination of (1.11), (1.14) and (1.15) indicates that, when $n = 1$, wave packets u_k of amplitude ε^2 are repeatedly created over time when solving (1.3) in the case $\lambda = 0$.

Look at (1.11). The emitted signals u_k (one per period 2π) have cumulative effects up to the stopping time $2K\pi$. They give rise to a growth rate with respect to the time variable t . For long times $T \sim 1$, assuming that $n = 1$, we can assert that

$$(1.16) \quad \mathcal{U}_{\text{lin}}(T) = A_\varepsilon^2 T + \mathcal{O}(\varepsilon) = A_\varepsilon^2 \int_0^{+\infty} \mathbb{1}_{[0, T]}(s) ds + \mathcal{O}(\varepsilon) = \mathcal{O}(1).$$

This short discussion about the linear situation ($\lambda = 0$) highlights a difference between the cases $n \neq 1$ - see (1.10) - and $n = 1$ - see (1.16). This observation is important in the perspective of nonlinear effects. As a matter of fact, it allows a first selection between the different modes $n \in \mathbb{Z}$.

1.1.2. *Nonlinear effects.* Here, we consider the nonlinear framework, when $\lambda \neq 0$ and $\nu + j_1 + j_2 = 2$. The difference $\mathcal{W} := \mathcal{U} - \mathcal{U}_{\text{lin}}$ is subject to

$$(1.17) \quad \mathcal{W}(T) = \lambda \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} (\mathcal{U}_{\text{lin}} + \mathcal{W})(s)^{j_1} (\bar{\mathcal{U}}_{\text{lin}} + \bar{\mathcal{W}})(s)^{j_2} ds.$$

Using a Picard scheme, it is easy to infer that the life span of the solution \mathcal{W} to the integral equation (1.17), and therefore of the solution \mathcal{U} to (1.6), can be bounded below by a positive constant not depending on $\varepsilon \in]0, 1]$. Knowing (1.10) and (1.16), it is also possible to deduce that $\mathcal{W}(T)$ is of size $\mathcal{O}(\varepsilon^{(j_1+j_2)/2}) = \mathcal{O}(\varepsilon)$ when $n \neq 1$, and of size $\mathcal{O}(1)$ when $n = 1$. This means that the preceding dichotomy between the two cases $n \neq 1$ and $n = 1$ remains when $\lambda \neq 0$.

Fact 1. *When solving (1.6), the harmonic $n = 1$ stands out from the others. Given $T > 0$, we find $\mathcal{U}(T) = \mathcal{O}(\sqrt{\varepsilon})$ when $n \neq 1$, and $\mathcal{U}(T) = \mathcal{O}(1)$ when $n = 1$.*

Assume that $\mathfrak{g} \neq 1$. The identity (1.7) becomes after an integration by parts

$$(1.18) \quad \begin{aligned} \mathcal{U}(T) = \mathcal{U}_{\text{lin}}(T) - \frac{i\lambda\varepsilon^2}{\mathfrak{g}-1} e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{U}(T)^{j_1} \bar{\mathcal{U}}(T)^{j_2} \\ + \frac{i\lambda\varepsilon^2}{\mathfrak{g}-1} \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} \partial_s (\mathcal{U}(s)^{j_1} \bar{\mathcal{U}}(s)^{j_2}) ds. \end{aligned}$$

From the equation (1.6), since we have seen that the solution \mathcal{U} is (at least) bounded, we know that $\partial_s \mathcal{U}(s) = \mathcal{O}(\varepsilon^{-1/2})$. From (1.18), it follows that

$$\forall T \in \mathbb{R}, \quad \mathcal{U}(T) = \mathcal{U}_{\text{lin}}(T) + \mathcal{O}(\varepsilon^{3/2}).$$

Now, assume that $n = 1$ and moreover that $\mathfrak{g} = 1$. To show that, in this situation, nonlinear effects actually occur, it suffices to produce an example. To this end, take $(j_1, j_2, \nu) = (2, 0, 0)$ and $\omega = -1$, so that $\mathfrak{g} = 1$. Choose $\lambda = 1$. Then, using (1.16), the identity (1.7) becomes

$$(1.19) \quad \mathcal{U}(T) = A_\varepsilon^2 T + \mathcal{O}(\varepsilon) + \int_0^T \mathcal{U}(s)^2 ds.$$

This implies that $\mathcal{U}(T) = A_\varepsilon \tan(A_\varepsilon T) + \mathcal{O}(\varepsilon)$, and therefore

$$\mathcal{U}(T) - \mathcal{U}_{\text{lin}}(T) = A_\varepsilon \tan(A_\varepsilon T) - A_\varepsilon^2 T + \mathcal{O}(\varepsilon) \neq o(1).$$

In view of the above formula, the asymptotic behavior of the nonlinear solution \mathcal{U} can strongly differ from the one of the linear solution \mathcal{U}_{lin} .

Fact 2. *When solving (1.6), the gauge parameter $\mathfrak{g} = 1$ stands out from the others. When $\mathfrak{g} \neq 1$, the asymptotic behaviors of \mathcal{U} and \mathcal{U}_{lin} when ε goes to 0 are the same. On the contrary, when $\mathfrak{g} = 1$, nonlinear effects can be expected at leading order.*

1.2. A more realistic model. The preceding features, Facts 1 and 2, which have been emphasized in the case of ODEs, are still present when dealing with partial differential equations arising in strongly magnetized plasmas (SMP) or in nuclear magnetic resonance experiments (NMR). But, there are two emerging issues: the first is due to dispersive effects which are completely absent in the ODE case; the second comes from the occurrence of non-trivial spatial variations when dealing with the phase φ . At all events, the discussion becomes much more subtle, and new important phenomena can and do occur.

In order to investigate SMP or NMR, we must consider the PDE counterpart of (1.3), which is

$$(1.20) \quad \partial_t u - \frac{i}{\varepsilon} p(\varepsilon D_x) u = F = F_L + F_{NL}, \quad u|_{t=0} = 0, \quad 0 < \varepsilon \ll 1,$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}$. The state variable is $u \in \mathbb{R}$ and $D_x := -i \partial_x$. The action of the pseudo-differential operator $p(\varepsilon D_x)$ is given on the Fourier side by the multiplier $p(\varepsilon \xi)$.

In what follows, we will focus on the scalar wave equation (1.20). The origin of equation (1.20), its physical significances and the reasons why it may be seen as a universal problem (when dealing with systems of hyperbolic equations) will be clearly explained in Sections 2 and 3. We will work in space dimension one. The possible multidimensional effects will not be investigated here.

We now fix some notations and we introduce simplified assumptions intended to facilitate the presentation of our main results. We suppose that the symbol p is smooth, say $p \in C^\infty(\mathbb{R})$. The function p is even. It is such that $p|_{[-\xi_c, \xi_c]} \equiv 0$ for some $\xi_c \geq 0$. It is strictly increasing on (ξ_c, ∞) . Moreover, for large values of ξ , it is subject to

$$(1.21) \quad \lim_{\xi \rightarrow +\infty} p(\xi) = 1, \quad \lim_{\xi \rightarrow +\infty} p'(\xi) = 0, \quad \exists \ell < 0, \quad \lim_{\xi \rightarrow +\infty} \xi^4 p''(\xi) = \ell,$$

as well as

$$(1.22) \quad \exists D \geq 4; \quad \forall n \in \{2, \dots, D\}, \quad \limsup_{\xi \rightarrow +\infty} \frac{|p^{(n)}(\xi)|}{p'(\xi)} < +\infty.$$

Fix some $M \in \mathbb{N}^*$. The source term F_L is defined by

$$(1.23) \quad F_L(\varepsilon, t, x) = -\varepsilon^{3/2} \sum_{m \in [-M, M] \setminus \{0\}} a_m(\varepsilon t, t, x) e^{im\varphi(t, x)/\varepsilon}.$$

In (1.23), the amplitudes $a_m(T, t, x)$ are chosen in the set $\mathcal{C}_b^\infty(\mathbb{R}^3)$ of smooth functions whose all derivatives are bounded. They are selected in such a way that, for some $\mathcal{T} > 0$ and some $r \in \mathbb{R}_+^*$ with $r < \gamma/2$, we have

$$(1.24) \quad \forall m \in [-M, M] \setminus \{0\}, \quad \text{supp } a_m \subset]-\infty, \mathcal{T}] \times [1, +\infty[\times [-r, r].$$

The amplitude $a_1(T, t, x)$ is chosen periodic for large times in the second variable. In other words, there exists $t_s \in \mathbb{R}_+^*$ and a smooth function $\underline{a}(T, t, x)$ such that

$$(1.25) \quad \forall t \geq t_s, \quad \forall n \in \mathbb{N}, \quad a_1(\cdot, t + n\pi, \cdot) \equiv \underline{a}(\cdot, t + n\pi, \cdot) \equiv \underline{a}(\cdot, t, \cdot).$$

The phase φ arising in (1.23) is more general than in (1.1). It does depend on the spatial variable $x \in \mathbb{R}$. It is the sum of a quadratic part (in t and x) and a periodic part (in t).

Assumption 1.2 (Selection of a relevant phase φ). *The function φ is*

$$(1.26) \quad \varphi(t, x) = t - x t + \gamma (\cos t - 1), \quad 0 < \gamma < 1/4.$$

In Section 2, the above assumptions on p and φ will be motivated by the study of two realistic situations which are related to strongly magnetized plasmas (SMP) and nuclear magnetic resonance (NMR). In Section 3, to better incorporate important specificities of SMP and NMR, they will be somewhat generalized.

In the right hand side of (1.20), the nonlinear part F_{NL} is, up to some localization in time and space, of the same form as in the previous subsection. Select a nonnegative cut-off function χ which is equal to 1 in a neighborhood of the origin and which is such that $\text{supp } \chi \subset [-1, 1]$. Fix some parameter $\iota \in [0, 1]$ which is aimed to measure the strength of the spatial localization. We impose

$$(1.27) \quad F_{NL}(\varepsilon, t, x, u) = \lambda \varepsilon^\nu \chi\left(3 - 2\frac{\varepsilon t}{\mathcal{T}}\right) \chi\left(\frac{x}{r\varepsilon^\iota}\right) e^{i\omega t/\varepsilon} u^{j_1} \bar{u}^{j_2}.$$

Taking into account the conditions on the support of the a_m 's and χ , the term F_{NL} becomes effective only for $t \geq \mathcal{T}/\varepsilon$, that is after the term F_L has played its part. So we observe successively two distinct phenomena: a possible linear amplification, and then nonlinear interactions.

The solution u to (1.20) exists on a time interval $[0, \tilde{\mathcal{T}}/\varepsilon]$ with $\mathcal{T} < \tilde{\mathcal{T}}$. The argument is similar to the one given for the toy model. Through the change (1.4), we can reformulate the equation (1.20) in terms of $\mathcal{W} = \mathcal{U} - \mathcal{U}_{\text{lin}}$, see (5.3) and (5.4). Since $\nu + j_1 + j_2 > 2$, the lifespan expressed in terms of $T = \varepsilon t$ does not shrink to \mathcal{T} when ε goes to zero. Note however that, due to the quadratic nonlinearity, the global-in-time existence is not at all guaranteed concerning (1.20).

We still denote by u_{lin} the linear solution obtained from (1.20) when $\lambda = 0$. One point should be underlined here. Our discussion of the linear situation is based on the analysis in L^∞ of oscillatory integrals appearing in a suitable wave packet decomposition of u_{lin} . The precise structure of these wave packets is lost under the influence of nonlinearities. It follows that our key argument cannot be iterated to obtain the existence and the asymptotic behavior of the solution to the full nonlinear equation (1.20). For this reason, we do not work with (1.20). Instead, we look at the first two iterates of an associated Picard iterative scheme, which are

$$(1.28a) \quad \partial_t u^{(0)} - \frac{i}{\varepsilon} p(\varepsilon D_x) u^{(0)} = F_L, \quad u_{|t=0}^{(0)} = 0,$$

$$(1.28b) \quad \partial_t u^{(1)} - \frac{i}{\varepsilon} p(\varepsilon D_x) u^{(1)} = F_L + F_{NL}(u^{(0)}), \quad u_{|t=0}^{(1)} = 0.$$

Generalizing (1.4), we can define

$$(1.29) \quad \mathcal{U}^{(j)}(T, z) := \frac{1}{\varepsilon} e^{-iT/\varepsilon^2} u^{(j)}\left(\frac{T}{\varepsilon}, \varepsilon z\right), \quad u^{(j)}(t, x) := \varepsilon e^{it/\varepsilon} \mathcal{U}^{(j)}\left(\varepsilon t, \frac{x}{\varepsilon}\right).$$

The expression $\mathcal{U}^{(0)}$ is the solution to the linear equation ($\lambda = 0$). Thus, we have

$$\mathcal{U}^{(0)}(T, z) = \mathcal{U}_{\text{lin}}(T, z) := \frac{1}{\varepsilon} e^{-iT/\varepsilon^2} u_{\text{lin}}\left(\frac{T}{\varepsilon}, \varepsilon z\right).$$

Symbols like p appear when looking at special branches \mathcal{V} of *characteristic varieties* describing the propagation of electromagnetic waves

$$(1.30) \quad \mathcal{V} := \{(t, x, \tau, \xi); \tau = p(\xi), (t, x, \xi) \in \mathbb{R}^3\} \subset T^*(\mathbb{R}^2) \equiv \mathbb{R}^2 \times \mathbb{R}^2.$$

On the other hand, the phase φ may reflect the transport properties of particles. The graph \mathcal{G} of the gradient of φ is associated with the *Lagrangian manifold*

$$(1.31) \quad \mathcal{G} := \{(t, x, \partial_t \varphi(t, x), \partial_x \varphi(t, x)); (t, x) \in \mathbb{R}^2\} \subset T^*(\mathbb{R}^2) \equiv \mathbb{R}^2 \times \mathbb{R}^2.$$

In the ODE framework of Paragraph 1.1, we simply find

$$\mathcal{V}_{\text{ode}} = \{(t, x, 1, \xi); (t, x, \xi) \in \mathbb{R}^3\}, \quad \mathcal{G}_{\text{ode}} = \{(t, x, 1 - \gamma \sin t, 0); (t, x) \in \mathbb{R}^2\},$$

so that

$$(1.32) \quad \mathcal{V}_{\text{ode}} \cap \mathcal{G}_{\text{ode}} = \{(k\pi, x, 1, 0); (k, x) \in \mathbb{Z} \times \mathbb{R}\}.$$

Thus, the production at the successive times $k\pi$ with $k \in \mathbb{N}$ of the wave packets u_k which appear at the level of (1.11) can be interpreted as coming from positions which are inside $\mathcal{V}_{\text{ode}} \cap \mathcal{G}_{\text{ode}}$. This principle is illustrated in Figure 1.2 below, given at x fixed and $\xi = 0$, with t in abscissa and the time frequency τ in ordinate.

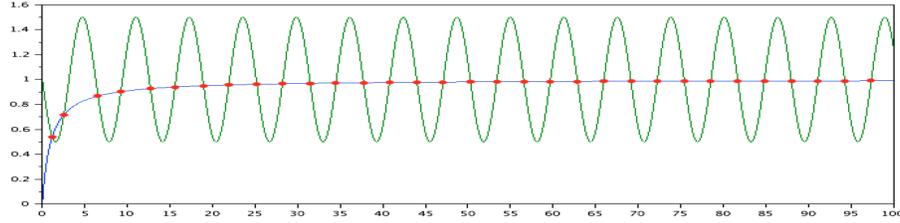


FIGURE 1. Intersection (in red) of \mathcal{V}_{ode} (in blue) and \mathcal{G}_{ode} (in green)

Similarly, in the general framework (1.20), two-dimensional oscillating waves u_k can emanate from the more complicated intersection

$$\mathcal{V} \cap \mathcal{G} = \{(t, x, p(-t), -t); (t, x) \in \mathbb{R}^2 \text{ and } p(-t) = p(t) = 1 - x - \gamma \sin t\}.$$

In view of (1.21), for large values of $|\xi|$, the dispersion relation $p(\xi) = \tau$ mimics the choice $p \equiv 1$ of (1.3). As in (1.32), the set $\mathcal{V} \cap \mathcal{G}$ contains (near $x = 0$ and for t large enough) an infinite number of curve portions (in \mathbb{R}^2) which appear repeatedly in time, and from which oscillating waves u_k may be triggered.

In the framework of SMP and NMR, the symbol p and the phase φ are issued from different physical laws. They are originally unrelated, see Section 2. But they are connected when solving the equation (1.20). The interactions between “waves” (associated with p) and “particles” (described by φ) may be revealed through the intersection between the two geometrical objects \mathcal{V} and \mathcal{G} , from which waves u_k can be emitted.

The amplification mechanism that may arise after summing the u_k 's can be viewed as a *resonance*. But now, the waves u_k are no more sure to overlap. In contrast to the toy model, since $\partial_x \varphi \neq 0$ and $p' \neq 0$, the waves u_k do propagate in \mathbb{R}^2 . They propagate in different directions and with various group velocities. They can mix before reaching the long times $t \sim \varepsilon^{-1}$.

Fact 3. *In the the PDE framework of equation (1.28), the accumulation of the emitted oscillating waves u_k can produce during long times $T \sim 1$ both *constructive and destructive interferences*.*

1.3. Statement of main results. The analysis of the creation, the propagation, the linear superposition, and the nonlinear interaction of the u_k 's is a manner to approach some kind of *turbulence*. We start with situations where the linear aspects are predominant. A standard Picard scheme can be used to approximate the nonlinear equation (1.20). The corresponding first two iterates yield the Cauchy problems (1.28a) and (1.28b).

Theorem 1.3 (Linear behavior). *Select a source term F_L as indicated in (1.23) with a phase φ depending on γ according to (1.26). Take profiles a_m satisfying both (1.24) and (1.25).*

Consider the equation (1.20) with a symbol p subject to (1.21) and (1.22). Let $\mathcal{U}^{(j)}$, with $j \in \{0, 1\}$, issued from (1.29) after solving (1.28). Select some $\mathcal{T} > 0$.

- (1) Linear case. *Concerning $\mathcal{U}^{(0)} \equiv \mathcal{U}_{\text{lin}}$, we can produce the following distinct asymptotic behaviors when ε goes to zero.*

- Constructive interferences. *For all $j \in \mathbb{Z}$ and $T \in [\mathcal{T}, 2\mathcal{T}]$,*

$$(1.33) \quad \mathcal{U}_{\text{lin}}(T, 2j) = \mathcal{O}(1) = A_\varepsilon^2 \int_0^{+\infty} e^{-i\frac{\ell}{\varepsilon}(\frac{1}{s} - \frac{T}{s^2})} \underline{a}(s, 0, 0) ds + o(1),$$

where $A_\varepsilon^2 = \sqrt{\frac{2}{\pi\gamma}} e^{-i\frac{\gamma}{\varepsilon}} \cos\left(\frac{\gamma}{\varepsilon} - \frac{\pi}{4}\right)$ is as in (1.14).

- Destructive interferences. *By contrast, for all $z \in \mathbb{R} \setminus 2\mathbb{Z}$ and for all $T \in [\mathcal{T}, 2\mathcal{T}]$, we find that*

$$(1.34) \quad |\mathcal{U}_{\text{lin}}(T, z)| = o(1).$$

- (2) Nonlinear case. *Adjust the nonlinearity F_{NL} as in (1.27), with real parameters ν , j_1 , j_2 , ω and ι . Assume that either $\nu + j_1 + j_2 > 2$, or $\nu + j_1 + j_2 = 2$ with $\omega + j_1 - j_2 \neq 1$. Fix some $\iota \in [0, 1]$. In the case $\nu + j_1 + j_2 - 2 = \omega + j_1 - j_2 = 0$, set $\iota = 1$. Then the nonlinearity plays no role at leading order in the sense that*

$$(1.35) \quad \forall (T, z) \in [0, 2\mathcal{T}] \times \mathbb{R}, \quad \mathcal{U}^{(1)}(T, z) = \mathcal{U}_{\text{lin}}(T, z) + o(1).$$

Interpreted in the setting of SMP, Theorem 1.3 shows, as forecast in [8], that small plasma waves (the u_k 's) driven by microscopic instabilities can accumulate over long times to furnish nontrivial effects. In turn, this phenomenon participate in some anomalous transport [7] and can trigger instabilities which may act as obstructions to the confinement of magnetized plasmas [11]. Applied in the context of NMR, our result investigates the processes whereby human tissues could be heated during [magnetic resonance imaging](#) [21].

It is worth noting that the turbulent aspects which are revealed by Theorem 1.3 are inherently linked to spatial heterogeneity. They are caused by the impact of the inhomogeneous source term F_L , which involves special oscillating wave front sets. Both in SMP and NMR, the input of energy is due to a strong external magnetic field \mathbf{B} , whose directions vary with the spatial positions, see Section 2.

Theorem 1.3 indicates that Facts 1, 2 and 3 indeed prevail. We still have two notions of criticality as far as nonlinear effects are concerned: the size of the nonlinearity (through the choice of $\nu + j_1 + j_2$) and the nature of oscillations (involving the gauge parameter $\mathfrak{g} = \omega + j_1 - j_2$).

The case $\nu + j_1 + j_2 > 2$ corresponds to a nonlinearity whose amplitude is too weak to have effects at leading order, regardless of the gauge. The case $\nu + j_1 + j_2 = 2$

corresponds to a nonlinearity with a critical size, for which we have to further investigate the content of the oscillations.

It turns out that, for $g \neq 1$, that is for $\omega + j_1 - j_2 \neq 1$, the oscillations in the nonlinear term are not *resonant*. They prevent the nonlinearity from having a leading order contribution. This is why we have (1.35).

In practice, the expression (1.33) is built as a sum of wave packets, which may be viewed as corresponding to the terms u_k of (1.12). But now, the wave packets accumulate only at special positions which, in the space variable x , are located on a moving lattice of size ε . The complete statement is Proposition 4.16, which takes into account the general choices of p and F_L introduced in Section 2.

By contrast, at all other positions, as indicated in (1.34), the wave packets u_k compensate to furnish asymptotic disappearance. This is due to mixing properties induced by the variations of the phase ($\partial_x \varphi \neq 0$) and dispersive effects ($p' \neq 0$), mixing properties which are recorded in the arithmetic properties of a phase shift. This is a feature of the PDE (1.20), which is completely absent from the ODE (1.3). The full statement can be found in Proposition 4.18.

Compare (1.16) and (1.33). The characteristic function $\mathbb{1}_{[0,T]}(s)$ of (1.16) plays the role of $\underline{a}(s, 0, 0)$ inside (1.33). Observe however that the formula (1.33) differs from (1.16), due to the factor $\exp(-i\frac{\ell}{6}(\frac{1}{s} - \frac{T}{s^2}))$ in front of \underline{a} . This additional factor is induced by the rate of convergence of $p^{\eta}(\xi)$ towards 1, which appears at the end of line (1.21). It is absent when $p \equiv 1$. In comparison to (1.16), due to the presence of an oscillating factor, it can reduce the amplification phenomenon which is revealed by (1.33). It reflects some microlocal effect, which is encoded in the behavior of p , on the asymptotic behavior of the solution \mathcal{U}_{lin} .

Remark that the constructive interferences (1.33) would be very difficult to detect in Lebesgue norms other than L^∞ , like L^2 . This is because the asymptotic profile of \mathcal{U}_{lin} is nontrivial only on a set of Lebesgue measure zero (the lattice \mathbb{Z}). To some extent, we can say that the underlying mechanisms rely on the recombination of small scales (rapid oscillations) into larger scales, which produces (asymptotically) a very weak solution.

As already explained, the linear part (1) of Theorem 1.3 is a direct consequence of Propositions 4.16 and 4.18. The proof relies basically on classical stationary and non-stationary phase arguments to precisely describe the infinite number ($k \in \mathbb{N}$) of emitted signals u_k . But the linear superposition of the u_k is a quite complicated mechanism. This requires to sort between dispersive and almost stationary waves, and this means to carefully examine the phase compensation phenomena that occur in the summation process. The integral inside (1.33) appears ultimately as the limit of a Riemann sum indexed by k .

The comparison between the linear solution $\mathcal{U}^{(0)} \equiv \mathcal{U}_{\text{lin}}$ and the expression $\mathcal{U}^{(1)}$ is a nontrivial test to measure whether or not nonlinear effects can alter the solution at leading order. The content of (1.35) is proved in Section 5.2. According to the choice of g or $\iota \in [0, 1]$, the size of the $o(1)$ inside (1.35) may be improved, see Propositions 5.18, 5.19 and 5.20. In view of Theorem 1.3, nonlinear phenomena can be expected only under critical nonlinearities ($\nu + j_1 + j_2 = 2$) and resonant oscillations ($g = 1$).

General nonlinear source terms will be investigated in Subsections 5.1 and 5.2. But, because it is simpler and already quite illustrative, in Subsection 5.3, we only

examine the case of u^2 . Other quadratic nonlinearities may be more difficult to resolve. Retain also that, higher-order nonlinearities, like the cubic choice $|u|^2u$, appear to be not directly manageable through our approach, see Remark 5.26.

Recall that F_L has been defined at the level of (1.23). The implementation of u^2 corresponds at the level of (1.27) to the selection of $\lambda = 1$ and $(\nu, j_1, j_2) = (0, 2, 0)$, so that $\omega = -1$ (since we want to impose $\mathfrak{g} = 1$). Thus, we consider the solution $u^{(0)} = u_{\text{lin}}$ to (1.28a), as well as the solution $u^{(1)}$ to $u|_{t=0}^{(1)} = 0$ together with

$$(1.36) \quad \partial_t u^{(1)} - \frac{i}{\varepsilon} p(-i\varepsilon \partial_x) u^{(1)} = F_L + \chi \left(3 - 2 \frac{\varepsilon t}{T} \right) \chi \left(\frac{x}{r\varepsilon^t} \right) e^{-it/\varepsilon} (u^{(0)})^2.$$

Theorem 1.4 (Nontrivial nonlinear effects in the presence of resonances). *The general context is as in Theorem 1.3. We fix $\nu = 1$, $j_1 = 2$, $j_2 = 0$ and $\omega = -1$ to deal with the quadratic source term u^2 of (1.36). It follows that the gauge parameter $\mathfrak{g} = \omega + j_1 - j_2 = 1$ is resonant. Select some $\iota \in]\iota_-, 1[$ with $\iota_- := (13 - \sqrt{89})/8$. Then, for all time $T \in [T, 2T]$ and for all position $z \in \mathbb{R}$, the expressions $\mathcal{U}^{(0)}(T, \cdot)$ and $\mathcal{U}^{(1)}(T, \cdot)$ which are issued from (1.29) after solving (1.28a) and (1.36) have the following asymptotic behaviors when ε goes to zero.*

- Constructive interferences. *When $z = 2j$ for some $j \in \mathbb{Z}$, the nonlinear interactions have some effect at leading order. As a matter of fact, we find*

$$(1.37) \quad \begin{aligned} \mathcal{W}^{(1)}(T, 2j) &:= \mathcal{U}^{(1)}(T, 2j) - \mathcal{U}^{(0)}(T, 2j) \\ &= o(1) + A_\varepsilon^4 \int_0^T \chi \left(3 - 2 \frac{s}{T} \right) \\ &\quad \times \left(\int_0^{+\infty} \int_0^{+\infty} e^{-i\frac{\ell}{6} \frac{T-s}{(\sigma_1+\sigma_2)^2}} b(\sigma_1, s) b(\sigma_2, s) d\sigma_1 d\sigma_2 \right) ds, \end{aligned}$$

where A_ε^2 is as in (1.14) and $b(\sigma, s) := e^{-i\frac{\ell}{6} \left(\frac{1}{\sigma} - \frac{s}{\sigma^2} \right)} \underline{a}(\sigma, 0, 0)$.

- Destructive interferences. *By contrast, when $z \in \mathbb{R} \setminus 2\mathbb{Z}$, the nonlinear interactions are still negligible at leading order in the sense that*

$$(1.38) \quad \forall z \in \mathbb{R} \setminus 2\mathbb{Z}, \quad |\mathcal{W}^{(1)}(T, z)| = o(1).$$

Theorem 1.4 means that both constructive and destructive interferences persist in the nonlinear framework.

The different wave packets u_k composing $\mathcal{U}^{(0)}$ interact through the quadratic term of equation (1.36). There are consequently additional nonlinear effects which are reflected in the triple integral appearing in the right hand side of (1.37). The nonlinear impact is not obtained, as could be expected by extrapolating (1.19), that is by just multiplying the linear profiles b inherited from (1.33). It also involves the correlation coefficient $\exp \left(\frac{-i\ell(T-s)}{6(\sigma_1+\sigma_2)^2} \right)$.

It should be emphasized that Theorem 1.4 cannot be inferred from Theorem 1.3, even on a formal level, due to the fact that nonlinear effects are quite strong. We will discuss more specifically these aspects at the end of Section 5, in Subsection 5.3, where Theorem 1.4 is proved.

It may seem that the assumptions made to state Theorem 1.4 are quite restrictive, for instance: the space and time localization of the nonlinearity (through the cut-off function χ), a rather strange lower bound on the parameter ι related to the spatial

scale, and the fact that we consider only the first two iterates of a Picard's scheme (this last point was already motivated above). Nevertheless, to obtain Theorem 1.4, we need already a rather involved analysis and careful estimates to deal with the oscillatory integrals coming from Duhamel's formula.

Pursuing the analysis in order to examine the “complete” nonlinear situation (1.20) is beyond the scope of this article, see Remark 5.27.

In conclusion, the key innovation of the present article is, in the context of SMP and NMR, a refined analysis of resonances, as well as a subsequent study of related interferences and nonlinear interactions. This will be done first in a linear setting (Section 4) and then in a nonlinear framework (Section 5).

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2. THE ORIGIN OF THE MODEL

The equation (1.3) with φ as in (1.1) first appears in [8] as a textbook case when it comes to studying plasma turbulence. It is a very elementary model aimed at explaining wave-particle interaction [32]. In (1.3), the “wave” is represented by u while the influence of “particles” is incorporated at the level of the source term, through the special structure of the phase φ inside F_L as well as the choice of the nonlinearity F_{NL} .

The content of φ , of equation (1.3), of F_L and of F_{NL} must be adjusted in connection with physics. In this section, we examine two frameworks. The first one deals with strongly magnetized plasmas (SMP); the second is about nuclear magnetic resonance (NMR). From these perspectives, the properties of φ , (1.3), F_L and F_{NL} selected in Subsection 1.1 are far from sufficient.

Both SMP and NMR involve a strong varying external magnetic field $\mathbf{B}(\cdot)$, and both imply rapid oscillations around the field lines generated by $\mathbf{B}(\cdot)$ at a Larmor frequency which, in the time variable t , is ε^{-1} with $\varepsilon \ll 1$. In SMP, the gyroscopic motion refers to the dynamics of charged particles, and it is governed by the [Vlasov equation](#). In NMR, this motion concerns the magnetic moment \mathbf{M} that is induced by the spin of particles, and it is handled by [Bloch equations](#).

These two applications share another remarkable feature. They both entail some secondary slower periodic motion.

- In SMP like coronas, planetary magnetospheres or fusion devices, the latter comes from the bouncing back and forth of charged particles between two mirror points [6, 7].
- In NMR, it is generated by the repeated action of many radio frequency excitations [21].

This second time periodic motion emerges at the level of the phase φ through the presence of the periodic function “ $\cos t$ ” inside (1.1). It also appears through the two time scales T/ε and T/ε^2 in the right hand side of (1.6). But there is more: the spatial inhomogeneities of the field \mathbf{B} generate variations of the phase φ with respect to the variable x . The graph \mathcal{G} of the gradient of φ , which is defined by (1.31), is associated with special *Lagrangian manifolds*, whose geometries reflect the peculiarities of \mathbf{B} .

In SMP, classical choices of \mathbf{B} are the dipole model [6] and the axisymmetric field [7] which are respectively adapted to the description of magnetospheres and tokamaks. In both situations, the condition $\nabla_x \varphi \neq 0$ results from some spreading of the characteristics. The level surfaces of φ involve very specific patterns. They give rise to wavefronts that are isolated and studied in [6, 7], where they are associated with a self-organization into coherent structures.

In NMR, the applied field \mathbf{B} is the sum of a background field \mathbf{B}_0 , plus a gradient field \mathbf{G} of the form $\beta \cdot x$ with $\beta \in \mathbb{R}^3$ and $x \in \mathbb{R}^3$, plus a time dependent periodic field \mathbf{B}_1 . In other words

$$\mathbf{B}(t, x) = \mathbf{B}_0 + \mathbf{G}(x) + \mathbf{B}_1(t), \quad \mathbf{G}(x) = \beta \cdot x.$$

In the course of an experiment, the static field \mathbf{G} is turned on and off by selecting a collection of data $\beta \in \mathbb{R}^3$ in view of signal processing. On the other hand, the radio frequency excitation \mathbf{B}_1 is triggered again and again to counterbalance the effects of noise in the measurements. The property $\nabla_x \varphi \neq 0$ is due to the gradient fields \mathbf{G} . The corresponding structure of φ is identified (without exploitation) in the text [21]. It will be more highlighted in what follows, see Paragraph 2.2.

Whether for SMP or NMR, the function φ is the sum of a linear function in t , plus (locally near the origin) a quadratic function in (t, x) , plus a periodic function in t . A representative selection of φ is the one given in (1.26). More details are given in the course of this section. Subsection 2.1 is devoted to SMP, while Subsection 2.2 deals with NMR.

2.1. Resonant wave-particle interactions. What happens inside collisionless plasmas is basically described by the Vlasov-Maxwell system, see [11] for a specific study concerning the strongly magnetized case. Simplified models (of fluid type) are also available through magnetohydrodynamics, see for instance the PhD thesis [22, Appendix A.2] and the numerous references therein. In the latter case, the equations take the form

$$(2.1) \quad \partial_t u + \frac{1}{\varepsilon} L(\varepsilon D_x) u + F = 0, \quad u|_{t=0} = 0.$$

In Paragraph 2.1.1, we exhibit some specificities of the differential operator $L(\varepsilon D_x)$, which acts on the *wave* u . In Paragraph 2.1.2, we explain the features of the source term F , which result from the motion of charged *particles* (electrons or protons). The coupling between u and F through (2.1) is a way to investigate phenomena related to wave-particle interactions [32].

2.1.1. Plasma dispersion relations. Here, the spatial dimension is $d = 3$. The state variable is $u = {}^t(B, E, \mathcal{J}) \in \mathbb{R}^9$. It involves the magnetic field $B \in \mathbb{R}^3$, the electric field $E \in \mathbb{R}^3$, and the electric current $\mathcal{J} \in \mathbb{R}^3$. Unlike the external fixed magnetic field \mathbf{B} , the electromagnetic field (E, B) is self-consistent, and unknown. The wave propagation in strongly magnetized plasmas (SMP) is studied in detail in the articles [9, 10]. It can be undertaken through the asymptotic analysis (when ε goes to zero) of

$$\partial_t u + \sum_{j=1}^d S_j \partial_{x_j} u + \frac{1}{\varepsilon} A u + F = 0, \quad u|_{t=0} = 0.$$

In practice, the number ε^{-1} is a large parameter ($\varepsilon^{-1} \simeq 10^5$) that is issued from a [gyrofrequency](#). Now, to recover the formulation (2.1), it suffices to define

$$(2.2) \quad L(\varepsilon D_x) := \sum_{j=1}^3 S_j \varepsilon \partial_{x_j} + A = \varepsilon \begin{pmatrix} 0 & +\nabla_x \times & 0 \\ -\nabla_x \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + A.$$

In (2.2), the differential operators $\pm \nabla_x$ come from Maxwell's equations in vacuum. The matrix A can be decomposed into 9 blocks of size 3×3 given by (for some constant b_e proportional to the strength of the external magnetic field)

$$(2.3) \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +\text{Id} \\ 0 & -\text{Id} & b_e \Lambda \end{pmatrix}, \quad \Lambda = e_3 \times = \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The skew-symmetric matrix A can be split into two distinct parts involving $\pm \text{Id}$ and $b_e \Lambda$. The two components $\pm \text{Id}$ are due to the coupling between the charged particles and E . They take into account one aspect of wave-particle interactions, arising in the electron cyclotron regime when computing the electric current in the Vlasov-Maxwell system. On the other hand, the skew-symmetric matrix $b_e \Lambda$ captures the influence of the Lorentz force. It corresponds to the effects of a strong external magnetic field having (rescaled) amplitude b_e and fixed direction e_3 . To underline the dependence of the semi-classical operator $L(\varepsilon D_x)$ upon S and A , we will sometimes denote by $L(S, A, \xi)$ the symbol of this operator. Thus

$$L(\xi) \equiv L(S, A, \xi) := i\xi_1 S_1 + i\xi_2 S_2 + i\xi_3 S_3 + A.$$

In vacuum, when $A = 0$, the kernel of $L(S, 0, \xi)$ is (for $\xi \neq 0$) of dimension 5. The situation is different in magnetized plasmas, when $A \neq 0$. When A is as in (2.3), the dimension of $\ker L(S, A, \xi)$ may be 2 or 3. In any case, it is strictly less than 5. This means that some nonzero eigenvalue $\tau_j(S, A, \xi)$ of $iL(S, A, \xi)$ is connected to 0 when A goes to 0, while the corresponding dispersion relation $\tau_j(S, A, \xi)$ remains bounded for large values of ξ . Emphasis will be placed on such eigenvalue.

The characteristic variety associated with (2.1) is

$$\text{Char } L = \mathbb{R}_t \times \mathbb{R}_x^3 \times \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^3; \det(i\tau \text{Id} + L(\xi)) = 0\}.$$

The analysis of $\text{Char } L$ in the context of (2.2)-(2.3) is achieved in the article [9], with explicit algebraic formulas. The general situation is rather complicated. But, for parallel propagation, meaning that $\xi = {}^t(0, 0, \xi_3) \parallel e_3$, the computations are simplified. With this in mind, we consider solutions u which depend only on the third coordinate $x_3 \in \mathbb{R}$ so that $x \equiv x_3 \in \mathbb{R}$ (we work in space dimension $d = 1$) and $\xi \equiv \xi_3 \in \mathbb{R}$. Then, the dispersion relations issued from (2.2) are displayed in this [link](#) [37], which presents basic features of electron waves. As usual in physics, in [37], the functions τ_j are available through implicit relations involving the index of refraction ξ/τ . In particular, one can distinguish the right circular polarization corresponding to R-waves (which are sometimes also called [whistler modes](#))

$$(2.4) \quad \frac{c_0^2 \xi^2}{\tau^2} = 1 - \frac{\omega_p^2/\tau^2}{1 - (\omega_c/\tau)}.$$

There is also the left circular polarization corresponding to L-waves

$$(2.5) \quad \frac{c_0^2 \xi^2}{\tau^2} = 1 - \frac{\omega_p^2/\tau^2}{1 + (\omega_c/\tau)}.$$

In (2.4) and (2.5), the three constants c_0 , ω_p and ω_c represent respectively the speed of light in vacuum, the **plasma frequency**, and the **electron cyclotron resonance frequency**. The two conditions (2.4) and (2.5) correspond to the selection of two important branches inside Char L . The first is issued from (2.4); it is valid only for $0 < \tau < \omega_c$; and it becomes physically relevant when τ becomes close to the resonance frequency ω_c . The second branch comes from (2.5); it operates when τ is above a cutoff frequency.

The two conditions (2.4) and (2.5) can be written in dimensionless form (implying that $c_0 = 1$ and $\omega_p = \omega_c = 1$). Concerning the relation (2.4), this yields

$$(2.6) \quad \frac{1}{\xi^2} = G_-(\tau), \quad G_-(\tau) := \frac{\tau - 1}{\tau^2(\tau - 1) - \tau}, \quad 0 < \tau < 1.$$

From (2.5), we can extract

$$(2.7) \quad \frac{1}{\xi^2} = G_+(\tau), \quad G_+(\tau) := \frac{\tau + 1}{\tau^2(\tau + 1) - \tau}, \quad \frac{\sqrt{5} - 1}{2} < \tau.$$

A simple calculation shows that

$$(2.8) \quad \forall \tau \in]0, 1[, \quad G'_-(\tau) = \left(\frac{-\tau + 2}{\tau^2 - \tau - 1} + \frac{1}{\tau} \right)' = \frac{(\tau - 1)(\tau - 3)}{(\tau^2 - \tau - 1)^2} - \frac{1}{\tau^2} \leq -1,$$

and that

$$(2.9) \quad \lim_{\tau \rightarrow 0^+} G_-(\tau) = +\infty, \quad G_-(1) = 0, \quad G'_-(1) = -1.$$

The function G_- is continuous and strictly decreasing from $]0, 1]$ onto $[0, +\infty[$. Therefore, it gives rise to a diffeomorphism between these two intervals, with inverse function G_-^{-1} . The whistler dispersion relation expresses τ as a function of ξ , through $\tau \equiv \tau_w(\xi) := G_-^{-1}(\xi^{-2})$. This function $\tau_w(\cdot)$ is even. This property does not come from the general condition (3.3), but from other specificities related to (2.2). By construction, we have

$$(2.10) \quad \lim_{\xi \rightarrow 0^\pm} G_-^{-1}(\xi^{-2}) = \tau_w(0) = 0, \quad \lim_{\xi \rightarrow \pm\infty} G_-^{-1}(\xi^{-2}) = \lim_{\xi \rightarrow \pm\infty} \tau_w(\xi) = 1.$$

In (2.10), the first limit means that the whistler dispersion relation is linked to some zero eigenvalue of $L(S, A, 0) \equiv A$. The second limit indicates, as noted before, that it appears as a perturbation (in terms of A) of some zero eigenvalue of $L(S, 0, \xi)$. This is consistent with a bounded behavior of τ when $|\xi|$ goes to infinity.

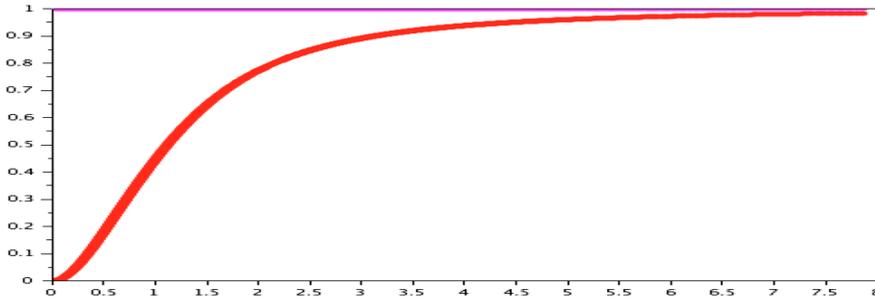


FIGURE 2. Graph of the function $\xi \mapsto \tau_w(\xi)$ on \mathbb{R}_+ in red. Asymptotic direction of the dispersion relation in magenta.

The function τ_w connects 0 (for $\xi = 0$) to $\omega_c \equiv 1$ (for $\xi = \pm\infty$), in a one-to-one smooth relation. Moreover, we can see on Figure 2 that the value τ becomes close to $\omega_c \equiv 1$ on condition that $|\xi|$ goes to infinity. In view of applications (see e.g. [20, 35]), the whistler dispersion relation τ_w has more impact near the resonance, that is when $|\xi|$ becomes large enough.

On the other hand, the regime is semiclassical. This means that the value $\xi = 0$ corresponds to a transition zone between spatial frequencies of size 1 and ε^{-1} . What happens near $\xi = 0$ is therefore physically less significant. For this reason and also to avoid a possible singularity at $\xi = 0$, we can skip what occurs near $\xi = 0$. Thus, we can multiply τ_w by $1 - \chi$ where χ is an even, smooth cut-off function which, for instance, is such that

$$(2.11a) \quad \forall |s| < 5/8, \quad \chi(s) = 1, \quad \forall s \in]5/8, 1[, \quad 0 < \chi(s) = \chi(-s),$$

$$(2.11b) \quad \forall |s| > 1, \quad \chi(s) = 0, \quad \forall s \in]5/8, 1[, \quad \chi'(s) < 0.$$

Example 2.1 (The physical model of R-waves). *Take $p \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ with*

$$(2.12) \quad p(\xi) := (1 - \chi(\xi)) G_-^{-1}(\xi^{-2}) = (1 - \chi(\xi)) \tau_w(\xi).$$

The function p inside (2.12) is even; it is equal to 0 in a neighborhood of $\xi = 0$; it coincides with the function τ_w for $1 \leq |\xi|$. Applying Faà di Bruno's formula, we can also see that

$$(2.13) \quad \forall n \in \mathbb{N}^*, \quad \lim_{\xi \rightarrow +\infty} \xi^{2+n} p^{(n)}(\xi) = (-1)^{n+1} (n+1)!$$

This article is a first mathematical approach of the subject. Thus, we will only consider a scalar wave equation in one space dimension ($d = 1$), like (1.20). In what follows, the special choice (2.12) of p will serve to guide the discussion.

2.1.2. *The impact of charged particles.* The source term F inside (2.1) is aimed to collect extra contributions appearing when passing from the Vlasov-Maxwell system to MHD equations. Typically, the function F is built with moments

$$\mathcal{M}_n(f) := \int \underbrace{v \otimes \cdots \otimes v}_{n \text{ times}} f(t, x, v) dv, \quad n \in \mathbb{N}^*$$

of the distribution function $f(t, x, v)$ satisfying the Vlasov equation. As explained in [22, Appendix A.2], the content of F must take into account the underlying physics. In the context of confined magnetized plasmas, the function F inherits from the computation of $\mathcal{M}_n(f)$ a special set of characteristics.

We consider as a first approximation that the expression F takes the following form

$$(2.14) \quad F \equiv F\left(\varepsilon, \varepsilon t, t, x, \frac{\varphi(t, x)}{\varepsilon}, u\right).$$

We must specify briefly the three-scale, oscillating and nonlinear structure of F . The function $F(\varepsilon, T, t, x, \theta, u)$ depends on the parameter $\varepsilon \in [0, 1]$, on the long time variable $T := \varepsilon t$ with $T \in [0, \mathcal{T}]$ for some $\mathcal{T} > 0$, on the time variable $t \geq 0$, on the spatial position $x \in \mathbb{R}$, on the periodic variable $\theta \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, and on the state variable $u \in \mathbb{R}$. It is a smooth function of class \mathcal{C}^∞ of all these variables, on the domain $[0, 1] \times [0, \mathcal{T}] \times \mathbb{R}^2 \times \mathbb{T} \times \mathbb{R}$. In Paragraph 2.1.2.1, we explain the origin of φ . In Paragraph 2.1.2.2, we describe the dependence of $F(\cdot)$ on θ and u .

2.1.2.1. The monophasic context. Under the influence of a strong external magnetic field, the collective motion of charged particles creates *coherent structures* which involve mesoscopic oscillations [6, 7]. Through a procedure detailed in [8], when computing the moments $\mathcal{M}_n(f)$, this furnishes macroscopic oscillations involving a specific phase $\varphi(t, x)$. As outlined in [8], see the lines (2.7) and (3.7) there, the relevant function φ is issued from a mesoscopic gyrophase after freezing the momentum v at mirror points. It can be determined through

$$(2.15) \quad \varphi(t, x) = \int_0^t b_e(X_r(s, x)) ds,$$

where the function X_r can be deduced from gyrokinetic equations or, as in [6, 7], from a notion of *reduced* Hamiltonian (the subscript r in X_r stands for *reduced*). The function X_r is smooth, and it can be viewed as a flow on \mathbb{R}^d , with $d = 3$ in the case of applications. We refer to the articles [6, 7] for more details concerning the properties of X_r in connection with plasma physics, and to [8] for a short presentation.

In what follows, we will just retain the basic representative features of X_r , and therefore of φ . There is a remarkable fact concerning X_r , which is due to underlying integrability conditions. For all x , the function $X_r(\cdot, x)$ is periodic with respect to the first variable s . To simplify the discussion (or after reductions), we can even suppose that the period of $X_r(\cdot, x)$ is uniform with respect to all positions x , say equal to 2π , so that

$$(2.16) \quad \forall (s, x) \in \mathbb{R} \times \mathbb{R}^3, \quad X_r(0, x) = x, \quad X_r(s + 2\pi, x) = X_r(s, x).$$

The periodic function $X_r(\cdot, x)$ produces a spatial periodic trajectory, starting from x at time $s = 0$. From (2.15), we can deduce a decomposition of φ separating average and oscillatory parts. The average part is

$$\langle b_e \circ X_r \rangle(x) := \frac{1}{2\pi} \int_0^{2\pi} b_e(X_r(s, x)) ds.$$

The oscillatory part $(b_e \circ X_r)^*(t, x)$ is 2π -periodic with zero average. It may be defined according to

$$(2.17) \quad \begin{aligned} (b_e \circ X_r)^*(t, x) &= \int_0^t [b_e \circ X_r(s, x) - \langle b_e \circ X_r \rangle(x)] ds \\ &= \varphi(t, x) - \langle b_e \circ X_r \rangle(x) t. \end{aligned}$$

Recall that the quantity $b_e(x)$ represents the strictly positive amplitude of the external magnetic field computed at x . In (2.17), the linear part $\langle b_e \circ X_r \rangle(x) t$ is produced by the mean effect of the bouncing back and forth of charged particles between the mirror points, whereas the oscillating part $(b_e \circ X_r)^*(t, x) \not\equiv 0$ takes into account the variations around this mean value. By definition, given $x \in \mathbb{R}^3$, the latter term is of mean value zero with respect to $t \in \mathbb{T}$. Observe that

$$(2.18) \quad \partial_t \varphi(t, x) = b_e \circ X_r(t, x) = \langle b_e \circ X_r \rangle(x) + \partial_t (b_e \circ X_r)^*(t, x) > 0.$$

Working in the vicinity of a fixed position, say near the origin $x = 0$, we can roughly replace the part $\langle b_e \circ X_r \rangle(\cdot)$ by

$$(2.19) \quad \langle b_e \circ X_r \rangle(x) \simeq \alpha + \beta \cdot x, \quad (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^3,$$

with the following identifications

$$(2.20) \quad \alpha = \langle b_e \circ X_r \rangle(0) > 0, \quad \beta = \nabla_x \langle b_e \circ X_r \rangle(0) \in \mathbb{R}^3 \setminus \{0\}.$$

In fact, for the purpose of our analysis, the choice of φ or $-\varphi$ does not matter. The remark (2.18) just says that $\partial_t \varphi$ has a constant sign which is positive under the convention (2.15) since the function b_e is positive. As a consequence, we have to retain that $\alpha > 0$ at the level of (2.20).

The dimensionless quantity α has a physical meaning. It is a measure of the ratio between the size of the magnetic field and the cyclotron resonance frequency ω_c . Thus, when $\omega_c = 1$, as will be assumed later after rescaling (the aim of this arbitrary choice is just to simplify notations), the number α indicates the average amplitude of the (rescaled) external magnetic field. Now, resonances arise when $\alpha \sim \omega_c$. For this reason, we select the value $\alpha = 1$.

In practice, both functions b_e and X_r are nontrivial functions of x . This is why we set $\beta \neq 0$ at the level of (2.20). In fact, the inhomogeneities of the external magnetic field induce some spreading of the integral curves which are associated to the Vlasov equation. This is reflected in the term $\beta \cdot x \neq 0$ of (2.19). Without loss of generality, after spatial rotations and rescalings, we can always adjust β so that $\beta = (0, 0, -1)$. For solutions which depend only on the direction x_3 , as it was supposed before, we just find $\beta = -1$. Finally, as a prototype of a nontrivial periodic function with zero mean, we can take

$$(2.21) \quad \partial_t (b_e \circ X_r)^*(t, x) = -\gamma \sin t.$$

Combining (2.19), (2.20) and (2.21), the positivity condition (2.18) is satisfied, at least for small enough positions x , on condition that $0 < \gamma < \alpha = 1$. To work on a spatial domain where the amplitude $b_e(\cdot)$ is expected to remain of magnitude comparable to the mean value α , we fix γ in the interval $]0, 1/4[$. This assumption turns out to be rather convenient for the forthcoming computations. It ensures that only one harmonic is resonant. The more general case $\gamma > 0$ would be more complicated. It may lead to supplementary dynamics compared to the one described in this paper. Since $\varphi(0, \cdot) \equiv 0$, the preceding discussion indicates that a choice of φ which should be relevant from the viewpoint of applications is given by (1.26).

2.1.2.2. Nonlinear aspects. In the articles [6, 7], the function $f(t, x, v)$ is obtained as the composition of a localized initial data $f_0(x, v)$ with the oscillatory flow that is issued from the Vlasov equation. It follows that all harmonics $m\varphi$ with $m \in \mathbb{Z}$ are necessarily involved. Accordingly, the periodic function $F(\varepsilon, T, t, x, \cdot, u)$ can be decomposed in Fourier series

$$F(\varepsilon, T, t, x, \theta, u) = \sum_{m \in \mathbb{Z}} F_m(\varepsilon, T, t, x, u) e^{im\theta}.$$

The MHD equations resulting from the Vlasov-Maxwell system are not closed. Some approximations are needed to recover self-contained equations. They usually are made in the form of nonlinearities. In the scalar setting (1.3), this means to consider that the source term F is semilinear in u and \bar{u} . The function F is made up of a part F_L which is affine with respect to (u, \bar{u}) , plus some nonlinear part F_{NL} . We can decompose F_L into $F_L(\cdot, u) = F_L^0 + F_L^1 u + F_L^1 \bar{u}$ with

$$(2.22) \quad F_L^0(\cdot) := F(\cdot, 0), \quad F_L^1(\cdot) := \partial_u F(\cdot, 0), \quad F_L^1(\cdot) := \partial_{\bar{u}} F(\cdot, 0).$$

Remark 2.2 (About the elimination of F_L^1). *The influence of F_L^1 can sometimes be removed. This can be achieved for instance by modifying the dependence on t inside F_L^0 , \underline{F}_L^1 and F_{NL} . To simplify, assume that F_L^1 does not depend on (x, θ) but only on (T, t) . Then, define*

$$(2.23) \quad v(t, x) := e^{-i\theta(t)/\varepsilon} u(t, x), \quad \theta(t) := -i\varepsilon \int_0^t F_L^1(\varepsilon, \varepsilon s, s) ds.$$

The new function v solves

$$\partial_t v - \frac{i}{\varepsilon} p(\varepsilon D_x) v + e^{-i\theta(t)/\varepsilon} F_L^0 + e^{-2i\theta(t)/\varepsilon} \underline{F}_L^1 \bar{v} + e^{-i\theta(t)/\varepsilon} F_{NL}(\varepsilon, \varepsilon t, t, e^{+i\theta(t)/\varepsilon} v).$$

In particular, when $F_L^1(\varepsilon, T, t) = \varepsilon^{-1} \tilde{F}_L^1(t)$ with a function \tilde{F}_L^1 purely imaginary and periodic in t with mean zero, the above expression θ becomes a periodic real valued function. This means that the gauge transformation (2.23) can introduce in the source term of (1.3) oscillations with a phase similar to (1.1). In other words, the oscillations of (1.1) can appear after a procedure aimed to absorb the “potential” F_L^1 , even if such oscillations are not visible at first sight. This provides another motivation for implementing phases φ like in (1.26).

The nonlinear part F_{NL} is chosen of the same form as in the introduction. As will be seen, an oscillatory source term such as (2.14) does generate oscillations of the solution u . By a mechanism similar to (1.11)-(1.13), the function u can be viewed as a sum of oscillating waves u_k . Note that, in the present context, nonlinear aspects can be revealed at the level of the source term F (coming from Vlasov) through the harmonics of φ but also inside the wave u itself (related to Maxwell) through the harmonics of the phases involved by the u_k 's.

In the case of the Earth's magnetosphere, the emission of whistler waves u_k is a long-standing experimental evidence, coming back to works of H. Barkhausen in 1917, T.L. Eckersley in 1935, and L.R.O. Storey in 1953 [36]. Thanks to progress in satellite means, like Van Allen Probes of NASA or Cluster of ESA, whistler waves u_k can be today observed in detail. The internal mechanisms underlying the production of the u_k 's are clarified in [6].

In practice, the whistler waves u_k accumulate and form a chorus. Theorems 1.3 and 1.4 describe intermittency phenomena that can occur during this process. It should be stressed that our results deal with a plasma turbulence which by nature is *anisotropic*. The energizing external field \mathbf{B} points in special directions, which undergo variations according to specific geometries (revealed by the Lagrangian manifold \mathcal{G}). We will not further investigate here the potential implications of our analysis in terms of plasma physics. We just refer to the text [8] for preliminary comparisons between mathematical previsions and concrete observations.

2.2. Nuclear magnetic resonance. The general framework concerning Nuclear Magnetic Resonance (NMR) is well explained in a paper of C.L. Epstein, see [21] and the references therein. The NMR experiments are intended (for instance) to determine the distribution of water molecules in an extended domain. To this end, an external magnetic field $\mathbf{B}(t, x)$ is repeatedly applied to the object under examination. At a microscopic level, the spins of particles react to \mathbf{B} by producing a magnetization field $\mathbf{M}(t, x)$.

The time evolution of \mathbf{M} is described by Bloch equations which, in the absence of relaxation terms, take the following form

$$(2.24) \quad \frac{d\mathbf{M}}{dt} = g\mathbf{M} \times \mathbf{B}.$$

The function \mathbf{B} is usually viewed as a sum $\mathbf{B} = \mathbf{B}_0 + \mathbf{G}(x) + \mathbf{B}_1(t)$, where \mathbf{B}_0 is a constant background field, \mathbf{G} is a gradient field which is collinear with \mathbf{B}_0 , whereas \mathbf{B}_1 is a time dependent radio frequency field which is orthogonal to \mathbf{B}_0 and $\mathbf{G}(x)$. Typically [21], the components \mathbf{B}_0 and \mathbf{G} can be adjusted according to

$$\mathbf{B}_0 = {}^t(0, 0, b_0), \quad \mathbf{G}(x) = {}^t(0, 0, \beta \cdot x),$$

where $b_0 > 0$ is the strength of the external magnet, and where the vector $\beta \in \mathbb{R}^3$ comes from a collection of static fields that are turned on and off. In hospital magnetic resonance imaging devices, acceptable orders of magnitude [21] are

$$(2.25) \quad g \simeq 10^7 - 10^8 \text{ rad/Tesla}, \quad b_0 \simeq 1 - 10 \text{ Teslas}.$$

In the absence of \mathbf{B}_1 , at the position x , the magnetization $\mathbf{M}(\cdot, x)$ undergoes a (counterclockwise) precession about the z -axis with the angular frequency

$$\omega(x) := \omega_0 + g\beta \cdot x, \quad \omega_0 := gb_0 \simeq 10^8.$$

This underscores the importance of the small parameter $\varepsilon := \omega_0^{-1} \ll 1$, which is the inverse of the Larmor frequency ω_0 . The field \mathbf{B}_1 represents the repeated action during the experiments of RF-excitations, which are all adjusted near the resonant frequency $\omega(x)$. As explained in [21], this can be modeled by

$$(2.26) \quad \mathbf{B}_1(t) = b_1(t, x) \begin{pmatrix} +\cos(\omega(x)t + c_1(t, x)) \\ -\sin(\omega(x)t + c_1(t, x)) \\ 0 \end{pmatrix}.$$

The amplitude b_1 of the field \mathbf{B}_1 is a scalar function. To model the repetition of measurements, which is aimed to reduce noise effects, the function $b_1(\cdot, x)$ is chosen periodic in t (say of period 2π). Define

$$(2.27) \quad d_1(t, x) := g \int_0^t b_1(s, x) ds = gb_1(x) + gb_1^*(t, x),$$

where

$$\bar{b}_1(x) := \frac{1}{2\pi} \int_0^{2\pi} b_1(s, x) ds, \quad b_1^*(t, x) := b_1(t, x) - \bar{b}_1(x) = b_1^*(t + 2\pi, x).$$

In (2.26), the phase shift c_1 takes into account the small variations occurring when calibrating the frequency of the RF-pulse. The equation (2.24) can be interpreted by following the motion of $\mathbf{M}(\cdot, x)$ in a frame rotating at the frequency $\omega(x)$. This amounts to replacing \mathbf{M} by the new unknown

$$\mathbf{N}(t, x) := e^{-\omega(x)t\Lambda} \mathbf{M}(t, x), \quad \Lambda := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The equation satisfied by \mathbf{N} is simply

$$\frac{d\mathbf{N}}{dt} = gb_1(t, x) \begin{pmatrix} 0 & 0 & \sin c_1(t, x) \\ 0 & 0 & \cos c_1(t, x) \\ -\sin c_1(t, x) & -\cos c_1(t, x) & 0 \end{pmatrix} \mathbf{N}.$$

When c_1 does not depend on t , for instance when $c_1 \equiv 0$, the solution is explicit. With d_1 as in (2.27), the solution operator \mathbf{U} associated with (2.24) is given by

$$(2.28) \quad \mathbf{U}(t, x) = \begin{pmatrix} +\cos(\omega t) & \sin(\omega t) & 0 \\ -\sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & +\cos d_1 & \sin d_1 \\ 0 & -\sin d_1 & \cos d_1 \end{pmatrix}.$$

The formula (2.28) reveals the role of the phase $\omega(x)t$ and also, after linearization, the presence in the description of $\mathbf{U}(t, x)$ of the two extra phases $\omega(x)t \pm d_1(t, x)$. In coherence with (2.25), we can take $b_0 = 1$, so that $\omega_0 \equiv g \equiv \varepsilon^{-1}$. In one space dimension (when the vectors β have a fixed direction), there remains $\beta \cdot x = \beta x$ with $\beta \in \mathbb{R}$ and $x \in \mathbb{R}$. Then, for the special choices $\beta = -1$, $\bar{b}_1 \equiv 0$ and $b_1^* \equiv \gamma(1 - \cos t)$, we just find $\omega t - d_1 = \varphi/\varepsilon$ with φ exactly as in (1.26). The solution \mathbf{M} to (2.24) does oscillate at the frequency ε^{-1} according to the phase φ . Thus, by plugging \mathbf{M} into Maxwell's equations through the magnetization current $\mathbf{J}_m := \nabla \times \mathbf{M}$, we end up with a model similar to (2.1). Note also that the influence of such oscillating function \mathbf{M} in the source term of Maxwell's equations can also appear in the context of Maxwell-Landau-Lifshitz equations [19].

Remark 2.3 (Dispersion relations in human tissues). *In the context of NMR, the relevant functions p do not appear to have been modeled precisely. But, like in SMP, the NMR experiments involve a magnetized medium. As a consequence, the corresponding dispersion relations should share common characteristics. Be that as it may, both situations involve hyperbolic systems to depict wave propagation. And, as will be seen in the next section, the properties of p which have been introduced in Subsection 2.1 are fairly general in such a framework.*

The production of the u_k 's corresponds to some electromagnetic radiation. When dealing with magnetic resonance imaging, this may contribute to the heating of human tissues in a way which could be a consequence of Theorems 1.3 and 1.4.

3. GENERAL SETTING AND ASSUMPTIONS

In this section, we present a general framework which includes the previous two examples: strongly magnetized plasmas and nuclear magnetic resonance. We also gather the assumptions that will be retained in the rest of the analysis.

3.1. The evolution equation. In this subsection, we start with $N \in \mathbb{N}^*$ state variables, so that $u \in \mathbb{R}^N$. The time variable is $t \in \mathbb{R}$. The spatial dimension is $d \geq 1$, so that $x \in \mathbb{R}^d$. The general context is based on dispersive nonlinear geometric optics [17, 33]. Consider a system of equations having the form

$$(3.1) \quad \partial_t u + \frac{1}{\varepsilon} L(\varepsilon D_x) u + F = 0, \quad u|_{t=0} = 0.$$

The real number $\varepsilon \in]0, 1]$ is a small parameter ($\varepsilon \ll 1$). The dual variables of t and x are denoted by $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$, respectively. In Paragraph 3.1.1, we describe precisely the content of $L(\varepsilon D_x)$. In Paragraph 3.1.2, we decompose (3.1) into a diagonal system of transport equations, which are coupled through semilinear terms. Then, we make a strong decoupling assumption to work with $N = 1$, and we restrict our attention to one dimensional effects so that $d = 1$. This reduction process leads to (1.20). At the end of this subsection, in Paragraph 3.1.3, we explain how to express the solution to (1.20) as an oscillatory integral.

3.1.1. *The pseudo-differential operator $L(\varepsilon D_x)$.* The semiclassical symbol that is associated to $L(\varepsilon D_x)$ is a matrix $L(\xi)$. We suppose that (3.1) is symmetrizable, that is, $L(\xi)$ is antihermitian. Typically, we work with systems that can be reduced to the following symmetric form

$$(3.2) \quad \partial_t u + \sum_{j=1}^d S_j \partial_{x_j} u + \frac{1}{\varepsilon} A u + F = 0, \quad u|_{t=0} = 0.$$

In (3.2), the letters S_j represent real-valued symmetric matrices. On the other hand, the matrix A may be complex-valued and is antihermitian. In other words

$$(3.3) \quad \forall j \in \{1, \dots, d\}, \quad S_j = {}^t S_j, \quad A^* = {}^t \bar{A} = -A.$$

The symbol associated to (3.2) is

$$L(S, A, \xi) := \sum_{j=1}^d i \xi_j S_j + A = -L(S, A, \xi)^*.$$

The matrix-valued symbol $iL(S, A, \xi)$ is hermitian. It is therefore diagonalizable, with real eigenvalues $\tau_j(S, A, \xi)$ satisfying

$$(3.4) \quad \tau_j(S, A, \xi) = \tau_j(S, 0, \xi) + |\xi| \left\{ \tau_j\left(S, \frac{A}{|\xi|}, \frac{\xi}{|\xi|}\right) - \tau_j\left(S, 0, \frac{\xi}{|\xi|}\right) \right\}.$$

The function τ_j is Lipschitz continuous on compact sets, including the compact neighborhoods of $\{(S, 0)\} \times \mathbb{S}^{d-1}$. With this in mind, exploiting (3.4) and assuming a little more regularity in the variable A near the position $(S, 0, \sigma)$ with $\sigma = \xi/|\xi|$, we can infer that

$$(3.5) \quad \tau_j(S, A, \xi) = \tau_j(S, 0, \xi) + (A \cdot \nabla_A) \tau_j(S, 0, \sigma) + \mathcal{O}(|\xi|^{-1}).$$

By this way, the expression $\tau_j(S, A, \xi)$ appears for large values of $|\xi|$ as a bounded perturbation of the eigenvalue $\tau_j(S, 0, \xi)$. Since $\tau_j(S, 0, \cdot)$ is homogeneous of degree one in ξ , the directions $\sigma \in \mathbb{S}^{d-1}$ with $\tau_j(S, 0, \sigma) \neq 0$ give rise to symbols $\tau_j(S, A, \xi)$ which tend to $\pm\infty$ when $\xi = \lambda\sigma$ goes to infinity (when $\lambda \rightarrow +\infty$). On the contrary, the directions $\sigma \in \mathbb{S}^{d-1}$ such that

$$(3.6) \quad \tau_j(S, 0, \sigma) = 0$$

furnish eigenvalues $\tau_j(S, A, \xi)$ of $iL(S, A, \xi)$ satisfying

$$(3.7) \quad \lim_{\lambda \rightarrow +\infty} \tau_j(S, A, \lambda\sigma) = \tau_j^\infty(S, A, \sigma) := (A \cdot \nabla_A) \tau_j(S, 0, \sigma).$$

The condition (3.7) appears already in Paragraph 2.1.1 at the level of (2.10) when studying wave propagation in magnetized plasmas. It is reflected at the level of Figure 2 by the horizontal asymptotic line. In what follows, the focus will be on such situations, which in fact have a general scope.

Remark 3.1 (Omnipresence of a finite limit). *Zero eigenvalues $\tau_j(S, 0, \xi) = 0$ of $iL(S, 0, \xi)$ are nearly always present in the evolution equations of mathematical physics. In the most favorable cases, there is a number of indices j such that (3.6) is verified for all directions $\sigma \in \mathbb{S}^{d-1}$. Otherwise, fix any $\sigma \in \mathbb{S}^{d-1}$. Then, change x into $x - t\tau_j(S, 0, \sigma)\sigma$, and modify the solution u accordingly. By this way, it can be ensured that $\tau_j(S, 0, \sigma) \equiv 0$. Consequently, given $\sigma \in \mathbb{S}^{d-1}$, the existence of a finite limit as in (3.7) is (modulo adequate transformations) systematic.*

From now on, the matrices S_j and A are fixed, with $A \neq 0$. The symbol $L(S, A, \xi)$ is simply denoted by $L \equiv L(\xi)$. We assume that, for $\xi \neq 0$, the matrix $L(\xi)$ has exactly $\tilde{N} \geq 1$ (with $\tilde{N} \leq N$) distinct eigenvalues which are of constant multiplicity, denoted by $-i\tau_j(\xi)$ with $j = 1, \dots, \tilde{N}$. The characteristic variety which is issued from $L(\varepsilon D_x)$ is $\text{Char}(L) := \mathbb{R}_t \times \mathbb{R}_x^d \times \mathcal{V}$ with

$$(3.8) \quad \mathcal{V} := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d; \det(i\tau \text{Id} + L(\xi)) = 0\}.$$

By construction, the set \mathcal{V} consists of a finite number \tilde{N} of smooth sheets, which correspond to different branches of $\text{Char}(L)$, and which are nonintersecting except possibly at the position $\xi = 0$. We have $(t, x, \tau, \xi) \in \text{Char}(L)$ if and only if $\tau = \tau_j(\xi)$ for some j . The functions τ_j are called *dispersion relations*. They are smooth away from $\xi = 0$. Retain that $\tau_j \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$.

3.1.2. Reduction to a scalar equation. The aim here is to explain how to pass from the system (3.1) to a finite number of scalar equations.

Lemma 3.2. *For all $\xi \in \mathbb{R}^d \setminus \{0\}$, the normal matrix $L(\xi)$ is unitarily similar to a diagonal matrix $D(\xi)$. In other words*

$$\exists U(\xi) \in \mathcal{U}(N); \quad U(\xi)^{-1} = U(\xi)^*, \quad U(\xi) L(\xi) U(\xi)^* = D(\xi).$$

In addition, the function U can be chosen smooth away from $\{\xi = 0\}$, bounded as well as all its derivatives, and it has a non-zero limit in each direction

$$(3.9) \quad \forall \sigma \in \mathbb{S}^{d-1}, \quad \exists U_\sigma \in \mathcal{U}(N); \quad \lim_{\lambda \rightarrow +\infty} U(\lambda\sigma) = U_\sigma.$$

Proof. The diagonalisation is straightforward since $L(\xi)$ is antihermitian. The smoothness of $\xi \mapsto U(\xi)$ follows from the constant multiplicity assumption. It remains to focus on (3.9). In the case $A = 0$, the function U is homogeneous of degree zero, and we have (3.9) with $U_\sigma = U(\sigma)$. The lemma then follows from perturbative arguments. For large values of $|\xi|$, the contribution issued from introducing A yields only $\mathcal{O}(1/|\xi|)$ terms, as in the previous paragraph. \square

To avoid a possible singularity of p at $\xi = 0$, we introduce a cut-off function $\underline{\chi}_c$ near the zero frequency. In the absence of singularity at $\xi = 0$, just take $\xi_c = 0$ and $\underline{\chi}_c \equiv 0$. In the presence of a singularity at $\xi = 0$, fix some $\xi_c > 0$, and select a smooth even cut-off function $\underline{\chi}_c$ satisfying

$$(3.10) \quad \mathbf{1}_{[-\xi_c, \xi_c]} \leq \underline{\chi}_c \leq \mathbf{1}_{[-2\xi_c, 2\xi_c]}.$$

The diagonal entries of the matrix $D(\xi)$ are the imaginary numbers $-i\tau_j(\xi)$. Apply the operator $[1 - \underline{\chi}_c(\varepsilon D_x)]U(\varepsilon D_x)$ on the left side of (3.2). Accordingly, define the modal decomposition

$${}^t(u_1, \dots, u_N) := [1 - \underline{\chi}_c(\varepsilon D_x)]U(\varepsilon D_x) u \in \mathbb{R}^N.$$

We emphasize again that this (possible) cut-off near $\xi = 0$ is consistent with physical approaches. In practice, the dispersion relation near the zero frequency requires often a distinct treatment. Examples include the Alfvén wave regime as opposed to the Whistler wave regime, see e.g. [20, 35]. By this way, the PDE (3.2) is reduced to a coupled system of N scalar equations

$$(3.11) \quad \partial_t u_n - \frac{i}{\varepsilon} p_n(\varepsilon D_x) u_n + [1 - \underline{\chi}_c(\varepsilon D_x)] [U(\varepsilon D_x) F]_n = 0, \quad 1 \leq n \leq N.$$

In (3.11), the pseudo-differential operator $p_n(\varepsilon D_x)$ involves a symbol p_n which is a smooth function that comes from a dispersion relation τ_j . More precisely

$$(3.12) \quad \exists j \in \{1, \dots, \tilde{N}\}; \quad p_n(\xi) = [1 - \chi_c(\xi)] \tau_j(\xi).$$

The multiplication by $1 - \chi_c(\xi)$ inside (3.12) allows a smooth connection of the values $\tau_j(\xi)$ for $\xi \neq 0$ to the value $p_n(0) = 0$ for $\xi = 0$. Now, we can remove the index n . The above polarization and microlocalization procedure does highlight the important role of scalar equations of the type

$$(3.13) \quad \partial_t u - \frac{i}{\varepsilon} p(\varepsilon D_x) u + \zeta(\varepsilon D_x) F = 0, \quad u|_{t=0} = 0,$$

where $\zeta(\xi) := (1 - \chi_c(\xi))U(\xi)$. In (3.13), there is only one mode of propagation. We have $u \in \mathbb{R}$ and $\tilde{N} = 1$. Moreover, assuming that the function F depends only on u , there is no more coupling between the different modes u_j 's.

Of course, the passage from (3.11) to (3.13) is a great simplification. Nonetheless, the equation (3.13) is very interesting. It is a simplified model giving a good indication of many mechanisms occurring at the level of systems like (3.2). Of special interest are the symbols p which, like in Section 2, stem from realistic models. Indeed, they allow to identify key physical phenomena.

3.1.3. *Solving the scalar equation.* In a first stage, we consider (3.13) when the source term F reduces to

$$(3.14) \quad F(\varepsilon, t, x) \equiv F_L^0(\varepsilon, t, x) = -\varepsilon^{3/2} a_m(\varepsilon t, t, x) e^{im\varphi(t,x)/\varepsilon}.$$

The function a_m is a smooth profile, which is compactly supported with respect to the spatial variable x ; the function φ is defined in (1.26), and $m \in \mathbb{Z}$. More general choices of F will be presented in Subsection 3.2. By interpreting the equation (3.13) on the Fourier side, we can extract

$$\hat{u}(t, \xi) = - \int_0^t \int e^{i[-(s-t)p(\varepsilon\xi) - \varepsilon y\xi]/\varepsilon} \zeta(-\varepsilon\xi) F(\varepsilon, s, y) ds dy,$$

where the Fourier transform is defined as

$$(3.15) \quad \hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \mathcal{F}^{-1}f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) d\xi.$$

Since ζ is smooth and bounded, and $a_m(\varepsilon s, s, \cdot)$ is in the Schwartz space, the expression $\hat{u}(t, \cdot)$ is rapidly decreasing and therefore integrable (in ξ). The inverse Fourier transform of $\hat{u}(t, \cdot)$ furnishes

$$u(t, x) = - \frac{1}{2\pi} \int e^{ix\xi} \left(\int_0^t \int e^{i[-(s-t)p(\varepsilon\xi) - \varepsilon y\xi]/\varepsilon} \zeta(-\varepsilon\xi) F(\varepsilon, s, y) ds dy \right) d\xi.$$

Replacing F as indicated in (3.14), and changing the variable ξ into $-\xi/\varepsilon$ in the resulting integral yields $u(t, x) = \mathcal{I}(\varepsilon, t, x; m\varphi, \zeta, a)$ where the oscillatory integral \mathcal{I} is given by

$$(3.16) \quad \mathcal{I} := \frac{\sqrt{\varepsilon}}{2\pi} \int \left(\int_0^t \int e^{-i\Phi(t,x;s,y,\xi)/\varepsilon} \zeta(\xi) a(\varepsilon s, s, y) ds dy \right) d\xi,$$

and where, assuming that p is even, the function Φ stands for the phase

$$(3.17) \quad \Phi(t, x; s, y, \xi) := (s - t)p(\xi) + (x - y)\xi - m\varphi(s, y).$$

The notation $\Phi(t, x; s, y, \xi)$ emphasizes the dependence of the function of (s, y, ξ) upon the parameters (t, x) . A large part of our work will focus on the asymptotic behavior when ε goes to zero of expressions like (3.16). Note that the application $a(\varepsilon t, t, \cdot) \mapsto u(t, \cdot)$ belongs to the general category of Fourier integral operators, see the book [18]. Non-standard features come here from the large domain of integration in time (of size $s \sim \varepsilon^{-1}$) and from the unusual properties of the phase Φ (during large times $s \sim \varepsilon^{-1}$). Typically, the phase $\varphi(s, y)$ is linear in y for fixed time, but with an increasing coefficient s , which enhances new phenomena.

3.2. Main assumptions. Consider (3.13). By incorporating the action of $\zeta(\varepsilon D_x)$ inside the definition of F , we find a scalar equation in one space dimension like

$$(3.18) \quad \partial_t u - \frac{i}{\varepsilon} p(\varepsilon D_x) u + F = 0, \quad u|_{t=0} = 0.$$

As in Paragraph 2.1.2.2, we can decompose the source term F into $F = F_L + F_{NL}$ with F_L as in (2.22). Below, we state our general assumptions regarding the terms which are present in (3.18).

3.2.1. Assumptions on the dispersion relation. We select some j giving rise to (3.6), and we consider the subset of $\text{Char}(L)$ which is associated to the choice of the eigenvalue $\tau_j(S, A, \xi) \equiv \tau_j$. For convenience, we will sometimes omit to mention j when dealing with τ_j or related expressions. The surface

$$\mathcal{V} \equiv \mathcal{V}_j := \{(\tau_j(\xi), \xi); \xi \in \mathbb{R}^d\} \subset \mathbb{R}^{1+d}$$

does not depend on (t, x) , and it is contained in all sections of $\text{Char}(L)$. Our aim is to study the equation (3.18) with a pseudo-differential operator $p(-i\varepsilon \partial_x)$ whose symbol $p = (1 - \underline{\chi}_c)\tau$ is satisfying assumptions which are inspired by (2.12). With this in mind, we impose the following conditions.

Assumption 3.3 (Existence of a resonance). *The symbol $p \in C^\infty(\mathbb{R}; \mathbb{R})$ satisfies:*

- (a) *There exists $\xi_c \in [0, 1/2]$ such that $p|_{[0, \xi_c]} \equiv 0$;*
- (b) *The function p' is positive on the interval $]\xi_c, +\infty[$;*
- (c) *The derivative $p'(\xi)$ converges to zero when ξ tends to $+\infty$;*
- (d) *The function p is such that*

$$(3.19) \quad \exists q \geq 2 \text{ and } \ell < 0; \quad \lim_{\xi \rightarrow +\infty} \xi^{q+2} p''(\xi) = \ell;$$

- (e) *The function p is even.*

In other words, we have (3.19) together with

$$(3.20a) \quad \forall \xi \in [0, \xi_c], \quad p(\xi) = 0,$$

$$(3.20b) \quad \forall \xi \in]\xi_c, +\infty[, \quad 0 < p'(\xi),$$

$$(3.20c) \quad \forall \xi \in [0, +\infty[, \quad 0 \leq p(\xi) = p(-\xi),$$

as well as

$$(3.21) \quad \lim_{\xi \rightarrow +\infty} p'(\xi) = 0.$$

As seen in Section 2, the formula (2.12) furnishes a typical example of symbol p , which is issued from electromagnetism.

Lemma 3.4. *Assumption 3.3 is satisfied by the function p of (2.12).*

Proof. Recall that G_-^{-1} is strictly decreasing from $[0, +\infty[$ to $]0, 1]$. Taking into account (2.11) and (2.12), we have (a) with $\xi_c = 5/8$. Compute

$$p'(\xi) = -\chi'(\xi) G_-^{-1}(\xi^{-2}) - 2(1 - \chi(\xi)) \xi^{-3} G'_- \circ G_-^{-1}(\xi^{-2})^{-1}.$$

For $\xi > \xi_c$, in view of (2.8) and (2.11b), $p'(\xi)$ is the sum of two positive expressions. This furnishes (b). On the other hand, using (2.9), we find

$$\lim_{\xi \rightarrow +\infty} \xi^3 p'(\xi) = -2 \lim_{\xi \rightarrow +\infty} G'_- \circ G_-^{-1}(\xi^{-2})^{-1} = -2 G'_-(1)^{-1} = 2.$$

As a direct consequence, we have (c). For $\xi > 1$, there remains

$$p''(\xi) = 6 \xi^{-4} G'_- \circ G_-^{-1}(\xi^{-2})^{-1} - 4 \xi^{-6} G''_- \circ G_-^{-1}(\xi^{-2}) G'_- \circ G_-^{-1}(\xi^{-2})^{-3}.$$

Passing to the limit $\xi \rightarrow +\infty$, we recover (3.19) with $q = 2$ and $\ell = -6 < 0$. Finally, by construction, the function p is the product of two even functions, and therefore we have (e). \square

Remark 3.5 (About the optimality of Assumption 3.3). *Most of our results remain valid when $q \geq 1$, and some of them hold true when (3.19) is relaxed according to*

$$(3.22) \quad \exists (q, \ell) \in]0, +\infty[\times \mathbb{R}_-; \quad \lim_{\xi \rightarrow +\infty} \xi^{q+2} p''(\xi) = \ell.$$

For instance, with χ as in (2.11) and $p(\xi) = [1 - \chi(\xi)]\tau(\xi)$ as in (3.12), we could also consider the following choices

$$\begin{aligned} \tau(\xi) &= \frac{2}{\pi} \arctan |\xi|, & q &= 1, & \ell &= -\frac{4}{\pi}, \\ \tau(\xi) &= \int_0^\xi \frac{ds}{1 + |s|^{1+q}}, & q &> 0, & \ell &= -1 - q. \end{aligned}$$

However, the precise description of the large time behavior of u (for t of order $1/\varepsilon$, as in Proposition 4.16) does require $\ell < 0$. As a matter of fact, the case $\ell = 0$ is an option which does not allow to quantify the dispersive effects (as in Lemma 4.7).

From (3.19), we obtain that p'' is integrable on \mathbb{R}_+ . Using (3.21), this yields

$$p'(\xi) = - \int_\xi^{+\infty} p''(s) ds.$$

Then, exploiting (3.22), we can obtain

$$(3.23) \quad \lim_{\xi \rightarrow +\infty} \xi^{q+1} p'(\xi) = -\frac{\ell}{q+1} > 0.$$

Note that the condition (3.20b) implies that the limit in the right hand side of (3.23) should be nonnegative. This is compatible with the condition $\ell < 0$ of (3.19) or with the condition $\ell \leq 0$ of (3.22). Now, from (3.23), we know that p' is integrable on \mathbb{R}_+ . Thus, we can find a number $\omega_+^\infty > 0$ such that

$$(3.24) \quad \lim_{\xi \rightarrow +\infty} p(\xi) = \omega_+^\infty := \int_0^{+\infty} p'(\xi) d\xi > 0.$$

The limit ω_+^∞ has clear physical meaning, in the sense of a resonance frequency. In the context of SMP, the number ω_+^∞ coincides with the electron cyclotron resonance frequency ω_c introduced at the level of (2.4) and (2.5). In the text [8], it is called *a resonance of the first type*.

In the scalar framework (3.18) which involves only one dispersion relation, changing the time scale t into $\omega_+^\infty t$, the symbol p and the source term F are respectively replaced by $(\omega_+^\infty)^{-1}p$ and $(\omega_+^\infty)^{-1}F$. By this way, we can ensure that $\omega_+^\infty = 1$.

Assumption 3.6 (Normalization of the resonance). *The resonance frequency, that is the limit ω_+^∞ , is normalized to unity.*

$$(3.25) \quad \lim_{\xi \rightarrow +\infty} p(\xi) = 1.$$

Consequently, for ξ large enough, the dispersion relation $\tau = p(\xi)$ does mimic the choice $p \equiv 1$ of (1.3). However, in Assumption 3.3, due to (b), the function p is definitely not constant, and therefore the variety \mathcal{V} is curved. Again, this is a hint that some kind of *dispersive effects* are present. Let us clarify this point. On the one hand, combining (3.23) and (3.25), we get easily

$$\lim_{\xi \rightarrow +\infty} p'(\xi) \xi p(\xi)^{-1} = 0 \neq 1.$$

In the vocabulary of geometric optics, this means that the group velocity $p'(\xi)$ and the phase velocity $p(\xi)/\xi$ are (asymptotically) different, and hence dispersive effects persist (see e.g. [34]). On the other hand, because the symbol p is bounded, there are no (local in time) Strichartz estimates which could be associated to the propagator $e^{itp(D_x)}$, improving Sobolev embeddings, in the sense that if

$$\|e^{itp(D_x)} f\|_{L^a([0,T];L^b(\mathbb{R}))} \leq C(T) \|f\|_{H^k(\mathbb{R})}, \quad \forall f \in H^k(\mathbb{R}),$$

for some $T > 0$ and (a, b) satisfying $\frac{2}{a} = \frac{1}{2} - \frac{1}{b}$ (admissible pair), then necessarily, $k \geq \frac{1}{2} - \frac{1}{b}$ (see [4]). On the other hand, *frequency localized* Strichartz estimates are available (see [5]): since we consider high frequency phenomena, these localized estimates are not helpful. One thus has to be cautious about the notion of dispersive effects that is involved. In this article, it refers to the first interpretation.

Remark 3.7. *The equation (1.3) of the introduction can be put in the form (3.18) with $p \equiv 1$. It is also dispersive since, for $\xi \neq 0$, the derivative $p'(\xi) = 0$ is different from $\xi^{-1}p(\xi) = \xi^{-1}$. It satisfies Assumption 3.6 but not Assumption 3.3. Indeed, in contradiction with (3.20b), the group velocity $p'(\xi)$ is simply zero.*

Now, let us come back to the content of (a), (b), (c), (d) and (e).

- (a). By multiplying the eigenvalue τ by the cut-off function $1 - \chi_c$ with χ_c as in (3.10), we can always guarantee (a) for $p = (1 - \chi_c)\tau$.

- (b). To better understand the origin of (b) and the conditions under which the property (b) is indeed accessible, we can examine what happens in the one dimensional framework ($d = 1$). Starting from (3.2), this means to fix $\sigma \in \mathbb{S}^{d-1}$, to identify $L(S, 0, \sigma)$ with some symmetric matrix S , and to look at

$$(3.26) \quad \partial_t u + S \partial_x u + \varepsilon^{-1} A u + F = 0, \quad u|_{t=0} = 0, \quad x \in \mathbb{R}.$$

As already explained, to obtain (3.6) for some index j , zero must be an eigenvalue of the symmetric matrix $iL(S, 0, 1)$, which coincides with $-S$. Let P be the orthogonal projector onto the kernel of S . If the matrix S commutes with A , that is if $[S, A] = 0$, the two matrices S and A are simultaneously diagonalizable, and the system (3.26) can be decoupled into distinct transport equations. In particular, from (3.26), we can extract

$$(3.27) \quad \partial_t P u + \varepsilon^{-1} (P A P) P u + P F = 0, \quad P u|_{t=0} = 0.$$

Among the eigenvalues $\tau_j(S, A, \xi)$ of $iL(S, A, \xi)$, we can distinguish those coming from (3.27), which are simply eigenvalues of PAP , and therefore constant in ξ . Then, the condition (b) is not verified. This means that interesting situations may arise only on condition that $[S, A] \neq 0$.

In fact, in order to have the condition (b), the important thing is the presence asymptotically, say for $\xi \geq \xi_p$ with $\xi_p \in \mathbb{R}_+$, of some nontrivial monotone behavior of τ . Then, changing t into $-t$ if necessary, we get the growth criterion $p'(\xi) \geq 0$ for $\xi \geq \xi_p$. Changing x into μx with $\mu \in \mathbb{R}$, we can obtain $0 \leq \xi_p \leq \xi_c \leq 1/2$. Then, multiplying τ_j by $1 - \underline{\chi}_c$ as in (3.12), the situation does fit in with (b). The supplementary restriction $p'(\xi) > 0$ is aimed to guarantee that dispersive effects associated with the variations of p actually occur.

- (c). The condition (c) is useful to infer (3.24) from (3.19). Subject to (3.6), it is an easy consequence of (b). Let us briefly explain why. In view of Remark 3.1, the condition (3.6) gives rise to the existence of a finite limit $\tau_j^\infty \equiv \tau_j^\infty(S, A, \sigma)$. At this stage, there is no sign condition on $\tau_j^\infty \in \mathbb{R}$. Now, given any $\lambda \in \mathbb{R}$, we can change the solution $u(t, \cdot)$ into $\tilde{u}(t, \cdot) := \exp(i\lambda t/\varepsilon) u(t, \cdot)$. This gauge transformation is not without consequence on the source term F (see Paragraph 2.1.2.2), which must be adjusted accordingly. It also modifies the matrix A into $\tilde{A} := A - i\lambda$, and the symbol $\tau_j(S, A, \xi)$ into

$$\tilde{\tau}_j(S, \tilde{A}, \xi) := \tau_j(S, \tilde{A}, \xi) = \tau_j(S, A, \xi) + \lambda.$$

It follows that τ_j^∞ and $\tau_j(S, A, 0)$ are respectively turned into

$$\tilde{\tau}_j^\infty := \tau_j^\infty + \lambda, \quad \tilde{\tau}_j(S, \tilde{A}, 0) = \tau_j(S, A, 0) + \lambda.$$

For the choice of a sufficiently large number λ , the new limit $\tilde{\tau}_j^\infty$ becomes positive, like ω_+^∞ in (3.24). Moreover, provided that $\tau_j(S, A, \cdot)$ is increasing on \mathbb{R}_+ , which means that we can take $\xi_p = 0$ as well as $\xi_c > 0$ arbitrarily small, by selecting $\lambda = -\tau_j(S, A, 0)$, we can ensure that $\tilde{\tau}_j(S, \tilde{A}, 0) = 0$ and $\tilde{\tau}_j^\infty \equiv \omega_+^\infty > 0$. Then, as in the whistler case, the function $\tilde{\tau}_j(S, \tilde{A}, \cdot)$ connects some zero eigenvalue of S (when $\xi \rightarrow +\infty$) to some zero eigenvalue of \tilde{A} (when $\xi = 0$).

- (d). The condition (3.19) does not only guarantee the existence of a resonance frequency. It is much more restrictive. It provides information about the asymptotic behavior inside (3.7). Indeed, from (3.23), we can extract the rate of convergence

$$(3.28) \quad \lim_{\xi \rightarrow +\infty} \xi^q [\omega_+^\infty - p(\xi)] = -\frac{\ell}{q(q+1)} > 0.$$

- (e). The last restriction (e) is not essential. It is inspired by the whistler dispersion relation τ_w which is an even function, and for which a global analysis up to the zero frequency $\xi_p = 0$ is directly available, without involving a cut-off function $\underline{\chi}_c$ with ξ_c large. It is imposed here for the sake of simplicity. It can be avoided, albeit with technicalities to distinguish what happens in the two directions $\pm\sigma$.

To conclude, let us illustrate the above discussion in the context of equation (3.26), when $N = 2$. After adequate rescalings, the framework can be reduced to

$$S = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -ia & -b - ic \\ b - ic & -id \end{pmatrix}, \quad (a, b, c, d) \in \mathbb{R}^4.$$

The pertinent eigenvalue $\tau(\xi)$ of

$$iL(s, A, \xi) = -\xi S + iA = \begin{pmatrix} a & c - ib \\ c + ib & d + \xi \end{pmatrix}$$

is the one issued from the zero eigenvalue of S . Thus, it must be zero when $A = 0$. For $\xi \geq 0$, this means to select

$$\tau(\xi) := \frac{1}{2}(\xi + a + d - \sqrt{(\xi - a + d)^2 + 4b^2 + 4c^2}), \quad \lim_{\xi \rightarrow +\infty} \tau(\xi) = a.$$

For large values of $\xi \geq 0$, the function τ is constant (equal to a) if and only if $b = c = 0$, or equivalently if and only if $[S, A] = 0$. We have to address the opposite case, when $[S, A] \neq 0$ or when $bc \neq 0$. By adjusting the gauge parameter λ , we can work with $a = 1$. Multiply τ by $1 - \chi_c$ to form $p = (1 - \chi_c)\tau$. Then, for all choice of $\xi_c > 0$, we obtain (a), (b) and (c). We also find the substitute (3.22) for (3.19) - or (d) - with $q = 1$. We do not have (e) but, as seen before, this is just a simplifying assumption.

For technical reasons, we need to highlight the following condition.

Assumption 3.8 (Control of derivatives of p).

$$(3.29) \quad \exists D \in \mathbb{N} \setminus \{0, 1\}; \quad \forall n \in \{2, \dots, D\}, \quad \limsup_{\xi \rightarrow +\infty} \frac{|p^{(n)}(\xi)|}{p'(\xi)} < +\infty.$$

Lemma 3.9. *Assumption 3.3 implies Assumption 3.8.*

Proof. When $\ell < 0$ as required in (3.19), from (3.19) and (3.23), we deduce that

$$\lim_{\xi \rightarrow +\infty} \xi \frac{p''(\xi)}{p'(\xi)} = -q - 1,$$

and hence

$$\limsup_{\xi \rightarrow +\infty} \frac{|p''(\xi)|}{p'(\xi)} = 0.$$

By this way, we find (3.29) for $n = D = 2$. □

Come back to (2.12) with G_- as in (2.6). In this case, comparing (2.13) and (3.23), we see that the finiteness condition in Assumption 3.8 is verified for all D . However, in the general case, when $D > 2$, Assumption 3.8 is adding new information, as shown by the example

$$p''(\xi) = \frac{\ell}{(1 + \xi^2)^2} \left(1 + \frac{1}{1 + \xi^2} \cos \xi^6 \right), \quad q = 2, \quad \ell < 0 \text{ and } D = 3.$$

In order to examine the role of D , we will keep track of D in the various estimates.

Remark 3.10 (About other nonlinear effects). *The model (3.18) does not see the interactions that occur between the different modes u_n inside (3.11). Just a quick comment on this. In view of (2.6), the set \mathcal{V} is symmetric with respect to the τ -axis. Consider a phase $\psi(t, 0, 0, x_3)$ satisfying the eikonal equation associated to R-waves*

$$(3.30) \quad \partial_t \psi = p(\partial_{x_3} \psi(t, 0, 0, x_3)), \quad \forall x_3 \in \mathbb{R}.$$

The function $\psi(t, 0, 0, -x_3)$ is also a solution to (3.30). On the other hand, the direction (τ, ξ) is subject to (2.6) if and only if its opposite $(-\tau, -\xi)$, and $(-\tau, \xi)$, is satisfying (2.7). If a phase $\psi(t, 0, 0, x_3)$ is as in (3.30), the function $-\psi(t, 0, 0, x_3)$ satisfies the eikonal equation associated to L-waves.

The same would apply for extended functions $\psi(t, x)$ which could be issued from $\psi(t, 0, 0, x_3)$ by the resolution of the complete eikonal equation, related to some $\xi \in \mathbb{R}^d \setminus \{0\}$. Now, due to possible nonlinear interactions, a non oscillating term can be produced by combining the phases $-\psi$ and $+\psi$. Since the value $\tau = 0$ is still characteristic when $\xi_3 = 0$ (the zero eigenvalue is not completely eliminated, see [9, Lemma 6.1]), this term may propagate and be amplified. There would be at the level of the full system (2.2) a three-wave resonance to study.

The two conditions $N = 1$ and $d = 1$ are of course quite restrictive. They do not allow to take into account a number of multidimensional and nonlinear aspects. But again, the focus here is on resonances, intermittencies and related nonlinear effects, in a framework as accessible as possible.

3.2.2. The source term F_L . After adequate gauge transformations, see Remarks 2.2 and 2.3 as well as the comment about (c) in the preceding Subsection 3.2.1, we look at the equation (3.18) under Assumption 3.3. To simplify matters, we can suppose that $F_L \equiv F_L^0$. On the other hand, to take into account the cut-off by χ_c and the possible pseudo-differential action $\zeta(-\varepsilon D_x)$ introduced in (3.13), we extend below the choice made in (3.14). Given $M \in \mathbb{N}^*$, we consider

$$(3.31) \quad F_L \equiv F_L^0 = \varepsilon^{3/2} \sum_{m \in [-M, M]} A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* e^{im\varphi(t, x)/\varepsilon},$$

where the phase φ is given by (1.26), that is $\varphi(t, x) = t - xt + \gamma(\cos t - 1)$ with $0 < \gamma < 1/4$, and the action of the adjoint A_m^* of the semi-classical pseudo-differential operator $A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)$ corresponds to the right quantization

$$(3.32) \quad A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* u(x) := \iint e^{i(x-y)\xi} A_m(\varepsilon t, t, y, \varepsilon \xi) u(y) dy d\xi.$$

Assumption 3.11 (Choice of the coefficients A_m). *For all integers $m \in [-M, M]$, we impose $A_m(T, t, x, \xi) = \zeta_m(-\xi) a_m(T, t, x)$ where the functions $\zeta_m \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ and $a_m \in \mathcal{C}_b^\infty(\mathbb{R}^3)$ satisfy the following conditions.*

- ζ_m goes to 1 at infinity with

$$\zeta_m(\xi) = 1 + \mathcal{O}\left(\frac{1}{|\xi|}\right), \quad |\xi| \rightarrow \infty;$$

- For $m = 0$, ζ_0 is identically zero near the origin,

$$\exists \xi_0 > 0, \quad \zeta_0|_{[-\xi_0, \xi_0]} \equiv 0;$$

- There exists some $\mathcal{T} > 0$ and some $r \in]0, \gamma/2[$ such that

$$(3.33) \quad \forall m \in [-M, M], \quad \text{supp } a_m \subset]-\infty, \mathcal{T}] \times [1, +\infty[\times [-r, r].$$

Since ζ_m is multiplied by a_m , the asymptotic value $\lim_{|\xi| \rightarrow \infty} \zeta_m(\xi) = 1$ is somehow arbitrary. It could be replaced by any non-zero constant. Also, some coefficients a_m may very well be identically zero. We will see that the most important coefficient is a_1 . All other coefficients will have a negligible contribution in the regime we consider in this paper (this is Fact 1).

As explained in [6, 7], when dealing with strongly magnetized confined plasmas (SMP), the solution to the Vlasov equation is transported by an oscillating flow, see for instance (2.4) and (2.9) in [8]. This induces special structures of the electric current \mathbf{J}_e which does enter in the composition of F at the level of (3.18).

Again, due to the bouncing back and forth (of charged particles) between the mirror points at a frequency which has been normalized in (2.16) to the value 2π , it is expected that the profiles a_m inherit similar periodic structures with respect to $t \in \mathbb{R}$. The same applies in the case of nuclear magnetic resonance (NMR). According to (2.28), the profiles a_m would be constant. But, in practice, they should be periodic due to relaxation phenomena between the repeated action of radio frequency (RF-)excitations.

In fact, the periodicity property is important for our results only as far as the coefficient a_1 is concerned. That is why we just impose the following condition, which is a relaxed version of (1.25).

Assumption 3.12 (a_1 is 2π -periodic in t for large times). *There exists $t_s > 0$ and a function $\underline{a} \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{R})$ such that*

$$(3.34) \quad \forall t \geq t_s, \quad \forall n \in \mathbb{N}, \quad a_1(\cdot, t + 2n\pi, \cdot) \equiv \underline{a}(\cdot, t + 2n\pi, \cdot) \equiv \underline{a}(\cdot, t, \cdot).$$

3.2.3. The nonlinearity F_{NL} . The coupling between “particles” and “waves” could also be described by nonlinear source terms. This induces an additional mixing, and provides a further complication. In the same vein as (1.2) or (1.27), the expression F_{NL} is chosen as a polynomial function in ε^{-1} , ε , u and \bar{u} .

Assumption 3.13 (Choice of the nonlinearity). *Given $J \in \mathbb{N}$ with $2 \leq J \in \mathbb{N}$ and $K \in \mathbb{N}$, as well as complex numbers $\lambda_{j_1 j_2 \nu} \in \mathbb{C}$, we impose*

$$(3.35) \quad F_{NL}(\varepsilon, t, x, u) := \sum_{2 \leq j_1 + j_2 \leq J} \sum_{\nu=2-j_1-j_2}^K \lambda_{j_1 j_2 \nu} F_{j_1 j_2 \nu}(\varepsilon, t, x, u),$$

where $(j_1, j_2, \nu) \in \mathbb{N}^2 \times \mathbb{Z}$, whereas

$$(3.36) \quad F_{j_1 j_2 \nu}(\varepsilon, t, x, u) = \varepsilon^\nu e^{i\omega_{j_1 j_2 \nu} t / \varepsilon} \chi \left(3 - 2 \frac{\varepsilon t}{\mathcal{T}} \right) \chi \left(\frac{x}{r \varepsilon^t} \right) u^{j_1} \bar{u}^{j_2},$$

for a frequency $\omega_{j_1 j_2 \nu} \in \mathbb{R}$ and parameters \mathcal{T} , r and ι satisfying as before the conditions $0 < \mathcal{T}$, $0 < r < \gamma/2 < 1/8$ and $\iota \in [0, 1]$.

We could assume more generally $\nu \in \mathbb{R}$ (taking finitely many values), provided that the size of the nonlinear coupling is at most critical, in the sense that $j_1 + j_2 + \nu \leq 2$. Since the critical case corresponds to the choice $j_1 + j_2 + \nu = 2$, assuming that $\nu \in \mathbb{Z}$ is not a strong restriction.

At the level of (3.36), the nonlinearity undergoes extra time and space localizations. The reasons for doing so will become clear in Section 5. They are related to the nature of the information established in the linear case (Section 4). Indeed, we will get a precise pointwise description of the linear solution only for sufficiently large time ($t \gtrsim 1/\varepsilon$), and only in a small neighborhood of the origin ($|x| \leq r$).

The amplitude $\varepsilon^{3/2}$ of the source term F_L and the nonlinearity (3.35) are inspired by Subsection 1.1. They are adjusted so that nonlinear effects can actually be critical in the limit $\varepsilon \rightarrow 0$ on the long time scale $T \sim 1$ under consideration.

3.3. The notion of quasi-rectification. The term *rectification* has been first introduced in [30] in the context of nonlinear diffractive geometric optics where it means the creation of non-oscillatory waves from highly oscillatory sources. A distinction is made between hyperplanes which are in the characteristic variety (contained in the section \mathcal{V}) and curved sheets.

For wave vectors belonging to flat parts inside \mathcal{V} , the interaction cannot be ignored, while for wave vectors on curved parts, it is negligible at leading order. In the subsequent articles [13, 33], these ideas are extended to dispersive equations and to situations of "almost rectification" (when the resonance comes from the tangent space to the characteristic variety).

In what follows, the expression *quasi-rectification* will be used in reference to the pioneering work [30]. As a matter of fact, as detailed in Paragraph 3.3.1, our approach presents certain similarities with that of [30]. But there are also significant differences that will be emphasized in Paragraph 3.3.2. In order to avoid confusion, it is important to explain clearly what the situation is. In Paragraph 3.3.3, we provide an overview of what quasi-rectification is. The last Paragraph 3.3.4 is aimed to summarize the discussion.

3.3.1. *Analogies.* As in [30], we study a nonlinear hyperbolic equation for long time scales at which nonlinear effects are present. As in [30], the characteristic variety is a mix of curved and flat features. In Figure 2 the red graph is curved while its magenta asymptote is flat. As in [30], amplification phenomena can occur (on small sets). Moreover, in the spirit of [33], we deal with a kind of almost (or quasi) rectification" (at infinity in our case). But here is where the comparisons stop.

3.3.2. *Differences.* The first change involves (2.2). In [30], the authors impose the condition $A = 0$, and they investigate *diffractive* effects. On the contrary, we work here with $A \neq 0$, and we consider *dispersive* effects. When $A = 0$ and $d = 1$, the characteristic variety is a union of lines. By contrast, when $A \neq 0$ and $d = 1$, the section \mathcal{V} does contain curved parts.

Since $N = 1$ and $d = 1$, many aspects of [30] are not present here. For instance, we will not discuss problems related to the interaction of different modes ($N > 1$) of propagation (the interaction between the u_n). Nor are we going to manage the multidimensional ($d > 1$) spreading of waves. However, there will be many new difficulties to deal with.

There is another distinction. In the article [30], the oscillations of the source term come from the oscillations of the initial data after a selection process implying the properties of the differential operator $L(\varepsilon D_x)$. That is not the case here. As a matter of fact, at the level of (3.2), the Cauchy data are simply zero.

Here, the oscillations are imposed from outside, as a part of the source term F . They are issued from the concrete considerations exposed in Section 2. They do not at all involve the differential operator $L(\varepsilon D_x)$. As a matter of fact, the phase φ has no link with $L(\varepsilon D_x)$. In particular, the function φ is not solution to the eikonal equation that is associated with $L(\varepsilon D_x)$.

In [30], there is a clear dichotomy between the flat and curved sheets contained in the characteristic variety \mathcal{V} . In Figure 2, there is no such strict separation. Instead, in view of (3.19), the function p is concave for large values of ξ . Hence, there is no flat part. But there is a progressive transition between a curved dispersion relation and its flat asymptote (for ξ large). While the rectification would refer, among other things, to the presence inside \mathcal{V} of branches without curvature, the *quasi-rectification* exploits the property that the curvature of the section \mathcal{V} asymptotically approaches zero.

3.3.3. *Underlying mechanisms.* The eikonal equation associated to p , the one which could be obtained at the first step of a WKB analysis, would give the value of $\partial_t \psi$ in terms of $\partial_x \psi$ through

$$(3.37) \quad \partial_t \psi(t, x) = p(\partial_x \psi(t, x)) \iff (\partial_t \psi, \partial_x \psi)(t, x) \in \mathcal{V}.$$

Given a generic position (t, x) , the phase φ that is involved in the source term F will not satisfy (3.37). But remark that the spatial derivative $\partial_x \varphi(t, x) = -t$ becomes large when t is growing, while the time derivative $\partial_t \varphi(t, x) = 1 - x - \gamma \sin t$ remains close to 1 (at least for $|x| \ll 1$ and $t \simeq 0$ modulo π). Taking into account (3.25), it follows that $p(\partial_x \varphi(t, x)) = p(-t) = p(t)$ is not far from $\partial_t \varphi(t, x)$ when $t = n\pi$ with n going to infinity. This is the type of resonance which has been illustrated in the introduction through (1.16). This is the reason why the asymptotic direction $\tau = 1$ of \mathcal{V} is physically so important.

Now, let us compare the position of $\nabla_{t,x} \varphi$ relative to \mathcal{V} more precisely. When φ is as in (1.26), the gradient of φ gives rise to a folded Lagrangian manifold $\mathcal{G}(\varphi)$ (see Figure 3 in [8]), which is

$$\mathcal{G}(\varphi) := \{(t, x, \partial_t \varphi, \partial_x \varphi)(t, x) = (t, x, 1 - x - \gamma \sin t, -t); (t, x) \in \mathbb{R} \times \mathbb{R}\}.$$

In general, the direction $(\partial_t \varphi, \partial_x \varphi)(t, x)$ is away from \mathcal{V} . But, near $x = 0$ and for large values of t , it will repeatedly cross the section \mathcal{V} in the course of time. Given some small x , denote by $t_k \equiv t_k(x)$ with $t_k < t_{k+1}$ and $k \in \mathbb{N}^*$ the successive intersection points. The t_k 's form a countably infinite set. By this way, wave packets u_k may be generated. As in (1.13), they look like

$$u_k(t, x) = \sqrt{\varepsilon} a_k(t, x) e^{i\psi_k(t, x)/\varepsilon} + o(\sqrt{\varepsilon}), \quad k \in \mathbb{N},$$

with ψ_k subject to the eikonal equation (3.37).

The phase ψ_k and the profile a_k are determined by the *local* geometrical properties of \mathcal{V} near the position $(\partial_t \varphi, \partial_x \varphi)(t_k(x), x) \in \mathcal{V}$. Since $p'(\xi)$ with $\xi = -t$ is very small for large values of t , especially when $t \sim \varepsilon^{-1}$, the waves u_k become almost stationary for large integers k . Their group velocity is not zero, but it tends to zero. It follows that the emitted waves u_k with k large can strongly interact and produce important local effects during long times $t \sim \varepsilon^{-1}$.

Since the accumulation process of the u_k 's is related to the asymptotic shape of the set \mathcal{V} (for large values of ξ) and of the set $\mathcal{G}(\varphi)$ (for large values of t), what happens ultimately depends on the *global* geometrical properties of \mathcal{V} and $\mathcal{G}(\varphi)$. As will be seen, amplification phenomena can occur, but not everywhere.

The creation, propagation, accumulation and nonlinear interaction at the level of the evolution equation (3.2) of almost standing waves generated near a resonance by highly oscillatory sources like (2.14), with φ as in (1.26), is what is called here *quasi-rectification*. As alluded to above, the study of quasi-rectification requires to combine local and global geometrical features of $\mathcal{G}(\varphi)$ and \mathcal{V} . From a physics viewpoint, the notion of quasi-rectification is well adapted to describe the observed production of quasi-electrostatic waves in SMP [1] or NMR [21], and to measure the relative impacts.

Note that certain mechanisms which are involved show also similarities with what is observed about surface plasmons [29].

The same applies for vortex filaments [2, 3, 15] with Talbot effect. In this case, specific spatial structures appear at special times, while in the present context, we

will obtain specific spatial structures which remain over long time intervals. In Theorems 1.3 and 1.4, the wave function \mathcal{U} is of size $\mathcal{O}(1)$ at integer points only, over large time intervals.

We would also like to cite the recent works of Y. Colin de Verdière and L. Saint-Raymond, see [14] and references therein. The motivations (fluid mechanics/SMP and NMR), the mechanisms (periodic medium/oscillating phase), the structures (PDEs involving variable/constant coefficients, flat/corrugated Lagrangian) and the tools (semiclassical/WKB methods) are distinct. But still, there are similarities and deep connections. The two viewpoints are complementary.

3.3.4. *Back to the physical models, and summary.* In SMP, the motion of charged particles generates an electric current \mathbf{J}_e . In NMR, the precession of the magnetic moment \mathbf{M} creates a magnetization current \mathbf{J}_m . These two sorts of currents oscillate according to the phase φ , and they both appear as a source term inside equations of Maxwell's type. But since the plasmas (in SMP) and the human tissues (in NMR) are inhomogeneous media, dispersion phenomena occur.

The characteristic variety $\text{Char } L := \{(t, x, p(\xi), \xi); (t, x, \xi) \in \mathbb{R}^3\} \subset \mathbb{R}^2 \times \mathbb{R}^2$ is more complicated than hyperplanes such as $\tau = p(\xi) = 1$. In real situations, there are dispersive effects which are encoded in the variations of the symbol p . As we have seen in Subsection 3.1, this happens generically.

In the context of SMP, the dispersive effects can be specified in details. Indeed, the dielectric tensor of magnetized plasmas can be computed, both in the cold case [9] and in the hot case [10]. In SMP, the pertinent function p is available, and it is such that $p'(\xi) \neq 0$. Less information exists concerning NMR, but the situation should be similar.

The reason why the model $p \equiv 1$ is so important is the following. On the one hand, after normalization, the function p will converge to 1 when $|\xi|$ goes to infinity; thus, for large values of $|\xi|$, the *dispersion relation* $\tau = p(\xi)$ mimics the choice $p \equiv 1$ of (1.3). On the other hand, from (1.26), we can deduce that the derivative $\partial_t \varphi$ remains close to 1 and that $\partial_x \varphi = -t$; thus, for large values of t and especially during long times $t \sim \varepsilon^{-1}$, the eikonal equation is almost satisfied. Due to the periodic part inside φ , it is in fact repeatedly verified. By this way, two-dimensional oscillating waves u_k may be emitted.

Vlasov and Bloch equations are completely distinct from Maxwell's equations. The two objects φ and p are issued from different physical laws. As a consequence, the phase φ has nothing to do with the symbol p . But they can intersect incidentally, in the sense of $\mathcal{G} \cap \mathcal{V}$. For the foregoing reasons, they can even cross again and again. Such a configuration is a facet of what can be a *resonance*.

As explained in [8], the cyclotron resonances and the internal repeated emissions of electromagnetic signals are both important phenomena which are involved in all collisionless magnetized plasmas. Applied in the context of SMP, our work helps to better understand the underlying mechanisms which are known to generate in coronas, magnetospheres and fusion devices some heating and some *anomalous transport*. It also sheds a new light on some aspects of NMR.

4. LINEAR ANALYSIS

In this section, we look at the evolution equation (3.18) with a source term F not involving u , that is when $F \equiv F_L$ with F_L as in Paragraph 3.2.2 and φ as in (1.26).

Since the problem is linear, we can study separately each term of the sum present in F_L . Accordingly, we consider here what happens when F reduces to

$$(4.1) \quad F \left(\varepsilon, \varepsilon t, t, x, \frac{\varphi(t, x)}{\varepsilon}, u \right) = \varepsilon^{3/2} A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* e^{im\varphi(t, x)/\varepsilon}.$$

We denote by u^m the solution to (3.18) issued from the choice (4.1). As usual, it is referred as the *mode m*. The purpose is to study the subsequent oscillatory integral $\mathcal{I}(\varepsilon, t, x; m\varphi, a_m)$, which is given by (3.16), with $\zeta = \zeta_m$. In Subsection 4.1, we examine the case of a non-resonant phase. The rest of Section 4 is devoted to the analysis of the resonant situation. This starts in Subsection 4.2 with a presentation of the strategy and results. Then, Subsections 4.3, 4.4 and 4.5 give the details. Given a phase $\psi \in C^\infty(\mathbb{R}; \mathbb{R})$, consider a generalization of (3.16)–(3.17) which is

$$(4.2) \quad \mathcal{I}(\varepsilon, t, x; \psi, \zeta, a) = \frac{\sqrt{\varepsilon}}{2\pi} \int \left(\int_0^t \int e^{-i\Phi(t, x; s, y, \xi)/\varepsilon} \zeta(\xi) a(\varepsilon s, s, y) ds dy \right) d\xi,$$

where

$$(4.3) \quad \Phi(t, x; s, y, \xi) := (s - t)p(\xi) + (x - y)\xi - \psi(s, y).$$

We assume that the phase ψ is *at most quadratic*, in the sense that (for all $n \geq 2$)

$$(4.4) \quad \sup_{2 \leq |\alpha| \leq n} \sup_{(t, x) \in \mathbb{R}^2} |\partial_{t, x}^\alpha \psi(t, x)| =: \mathfrak{B}_n^\psi < \infty.$$

Such a notion is quite standard (see e.g. [23, 31]) in the construction of fundamental solutions which are issued from Schrödinger equations involving a potential that is at most quadratic in space. Then, Hamilton-Jacobi equations with phases which are at most quadratic in space come into play. Here, the right variable is (t, x) . In the case of φ , see (1.26), the phase is of the form *linear+quadratic+bounded*. The quadratic part corresponds to the factor tx , and the function φ is indeed at most quadratic. The asymptotic behavior of \mathcal{I} when ε goes to 0 depends heavily on the existence or not of stationary points when looking at the phase $\Phi(t, x; \cdot)$ as a function of the variables (s, y, ξ) . In view of (4.2), we have

$$(4.5a) \quad \partial_s \Phi(t, x; s, y, \xi) \equiv \partial_s \Phi(s, y, \xi) := p(\xi) - \partial_s \psi(s, y),$$

$$(4.5b) \quad \partial_y \Phi(t, x; s, y, \xi) \equiv \partial_y \Phi(s, y, \xi) := -\xi - \partial_y \psi(s, y).$$

4.1. Non-resonant oscillatory integrals. Unlike the derivative $\partial_\xi \Phi(t, x; \cdot)$, the right hand sides of (4.5) do not involve the parameter t . This makes things easier in the perspective of nonstationary phase arguments (in s and y) applied with $t \sim \varepsilon^{-1}$. For this reason, we introduce the following notion.

Definition 4.1 (Non-resonant phase). *Let $r \in]0, \gamma/2[$ be the number allowing at the level of (3.31) to control uniformly the spatial supports of the profiles A_m inside (3.31). Fix some domain \mathcal{N} satisfying*

$$(4.6) \quad \mathcal{N} \subset \{(s, y, \xi) \in]0, \infty[\times] - r, r[\times \mathbb{R}\}.$$

The phase ψ is said to be non-resonant on the domain \mathcal{N} if there exists a positive constant $\eta > 0$ such that

$$(4.7) \quad \forall (s, y, \xi) \in \mathcal{N}, \quad |p(\xi) - \partial_s \psi(s, y)| + |\xi + \partial_y \psi(s, y)| \geq \eta.$$

As a subset of the characteristic variety $\text{Char}(L)$, one can distinguish

$$\text{Char}(L, \mathcal{N}) := \{(s, y, p(\xi), \xi) ; (s, y, \xi) \in \mathcal{N}\} \subset T^*(\mathbb{R}^2).$$

As a subset of the Lagrangian manifold $\mathcal{G}(\phi) \subset T^*(\mathbb{R}^2)$, one can identify

$$\mathcal{G}_r(\psi) := \{(s, y, \partial_s \psi, \partial_y \psi)(s, y) ; s \in]0, +\infty[, y \in]-r, r[\}.$$

The geometrical interpretation of Definition 4.1 is the following. The phase ψ is non-resonant on the domain \mathcal{N} if and only if the two subsets $\text{Char}(L, \mathcal{N})$ and $\mathcal{G}_r(\psi)$ of the cotangent bundle $T^*(\mathbb{R}^2)$ stay a positive distance η away from each other. We now address the various harmonics $\psi = m\varphi$, where φ is given by (1.26). We will see that two values play a special role, namely $m = 0$ and $m = 1$.

Lemma 4.2 (Non-resonant harmonics $m\varphi$). *Take*

$$\mathcal{N} = \{(s, y, \xi) \in]0, \infty[\times]-r, r[\times \mathbb{R}\}.$$

Let φ as in (1.26). For $m \in \mathbb{Z} \setminus \{0, 1\}$, the phase $m\varphi$ is non-resonant on \mathcal{N} .

Proof. In the case of $\psi = m\varphi$, the condition (4.7) becomes

$$\forall (s, y, \xi) \in \mathcal{N}_m, \quad 0 < \eta \leq |p(\xi) - m + m y + \gamma m \sin s| + |\xi - m s|.$$

For $m < 0$, the properties $p \geq 0$, and $r < \gamma/2$ from Assumption 3.11, yield

$$\begin{aligned} |p(\xi) - m + m y + \gamma m \sin s| &\geq |p(\xi) - m| - |m y + \gamma m \sin s| \\ &\geq |m| (1 - r - \gamma) > 5/8. \end{aligned}$$

For $m \geq 2$, the property $0 \leq p(\xi) \leq 1$ yields

$$|p(\xi) - m + m y + \gamma m \sin s| \geq m (1 - r - \gamma) - 1 > 1/4.$$

It suffices to take $\eta = 1/4$ to get the result. \square

The phase $\psi \equiv 0$, which corresponds to the choice $m\varphi$ with $m = 0$, requires a different treatment, which explains the hypothesis on ζ_0 in Assumption 3.11.

Lemma 4.3 (The non-resonant zero-phase). *For $\delta > 0$, set*

$$\mathcal{N}_0^\delta = \{(s, y, \xi) \in]0, \infty[\times]-r, r[\times \mathbb{R}, \quad |\xi| \geq \delta\}.$$

Then, for all $\delta > 0$, the phase $\psi \equiv 0$ is non-resonant on \mathcal{N}_0^δ .

Proof. It suffices to remark that

$$\forall (s, y, \xi) \in \mathcal{N}_0^\delta, \quad 0 < \delta \leq |\xi| \leq |p(\xi)| + |\xi|.$$

This yields (4.7) in the case $\psi \equiv 0$. \square

Remark 4.4. *The above lemma becomes wrong when $\delta = 0$, even if $\xi_c = 0$. Indeed, in view of Assumption 3.3, the sum $|p(\xi)| + |\xi|$ vanishes at $\xi = 0$.*

Finally, the remaining case $m = 1$ turns out to be the richest. It will be analyzed in details in Subsection 4.2.

The above notion of non-resonant phase is motivated by the following result.

Proposition 4.5 (Vanishing oscillatory integrals). *Let $\psi \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$ be an at most quadratic phase in the sense of (4.4). We also assume that ψ is non-resonant on a domain \mathcal{N} satisfying (4.6). Let ζ and a be two smooth functions which are adjusted as in Assumption 3.11. Then, for all time $t \geq \varepsilon^{-1} \mathcal{T}$, the Fourier integral operator $\mathcal{I}(\varepsilon, t, x; \psi, \zeta, a)$ issued from (4.2) is well-defined as an oscillatory integral. In addition, for all $n \geq 2$, there exists a constant $C_n > 0$ such that, for all $t \leq 2\mathcal{T}/\varepsilon$ and for all $x \in \mathbb{R}$, we have*

$$|\mathcal{I}(\varepsilon, t, x; \psi, \zeta, a)| \leq C_n r \mathcal{T} \varepsilon^{n-\frac{1}{2}} \eta^{1-4n} (1 + \mathfrak{B}_n^\psi)^n \|a\|_{W^{n,\infty}(\mathcal{N})}.$$

As a corollary, we can assert that $\mathcal{I} = \mathcal{O}(\varepsilon^\infty)$.

Proof. The notion of non-resonant phase is designed to apply nonstationary phase arguments with respect to the variables s and y . But the details remain to be worked out. The main problems when dealing with (4.2) are due to the domain of integration which, knowing that $t \sim \varepsilon^{-1}$, is of size ε^{-1} , as well as to the quadratic behavior of ψ .

As a consequence of (4.5) and (4.7), we have $(\partial_s \Phi)^2 + (\partial_y \Phi)^2 \neq 0$ on the domain \mathcal{N} . Thus, we can introduce

$$Q(s, y; \partial_s, \partial_y) := i \left((\partial_s \Phi)^2 + (\partial_y \Phi)^2 \right)^{-1} (\partial_s \Phi \partial_s + \partial_y \Phi \partial_y).$$

By construction, we have

$$(4.8) \quad \varepsilon Q(s, y; \partial_s, \partial_y) e^{-i\Phi/\varepsilon} = e^{-i\Phi/\varepsilon}.$$

Select some $n \geq 1$. After n integrations by parts using the identity (4.8), the expression \mathcal{I} of (4.2) is transformed into

$$(4.9) \quad \mathcal{I} = (-1)^n \frac{\varepsilon^{n+\frac{1}{2}}}{2\pi} \int \left(\iint e^{-i\Phi/\varepsilon} (Q^*)^n (\zeta(\xi) a(\varepsilon s, s, y)) ds dy \right) d\xi,$$

where Q^* is the adjoint operator of Q . In order to compute Q^* , first remark that

$$(4.10) \quad \partial_{ss}^2 \Phi = -\partial_{ss}^2 \psi, \quad \partial_{sy}^2 \Phi = -\partial_{sy}^2 \psi, \quad \partial_{yy}^2 \Phi = -\partial_{yy}^2 \psi.$$

Recall the standard conventions

$$\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \quad |\alpha| := \alpha_1 + \alpha_2, \quad X = (X_1, X_2), \quad X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2}.$$

Introduce the rational functions

$$R_{1,0}^1(X) := \frac{P_{1,0}^1(X)}{X_1^2 + X_2^2}, \quad P_{1,0}^1(X) := X_1, \quad R_{0,1}^1(X) := \frac{P_{0,1}^1(X)}{X_1^2 + X_2^2}, \quad P_{0,1}^1(X) := X_2,$$

as well as

$$R_{0,0}^1(X) := \frac{P_{0,0}^1(X)}{(X_1^2 + X_2^2)^2}, \quad P_{0,0}^1(X) := (-\partial_{ss}^2 \psi + \partial_{yy}^2 \psi) (X_1^2 - X_2^2) + 4 \partial_{sy}^2 \psi X_1 X_2.$$

Using (4.10) and the above definitions, we get

$$-i Q^* = R_{1,0}^1(\partial_s \Phi, \partial_y \Phi) \partial_s + R_{0,1}^1(\partial_s \Phi, \partial_y \Phi) \partial_y + R_{0,0}^1(\partial_s \Phi, \partial_y \Phi).$$

Since ψ is at most quadratic, the functions $P_\star^1(X_1, X_2)$ are in the polynomial ring $\mathcal{C}_b^\infty(\mathbb{R}^2)[X_1, X_2]$, whose elements of degree less than j take the following form

$$P(X) = \sum_{|\alpha| \leq j} C_\alpha(s, y) X^\alpha, \quad C_\alpha \in \mathcal{C}_b^\infty(\mathbb{R}^2).$$

The derivative ∂_\star , with $\partial_\star = \partial_s$ or $\partial_\star = \partial_y$, can act on the coefficients of P . Define

$$\partial_\star P(X) = \sum_{|\alpha| \leq j} (\partial_\star C_\alpha)(s, y) X^\alpha, \quad \partial_\star C_\alpha \in \mathcal{C}_b^\infty(\mathbb{R}^2), \quad \deg(\partial_\star P) \leq j.$$

The two polynomials $P_{1,0}^1$ and $P_{0,1}^1$ are of degree 1, whereas $P_{0,0}^1$ is of degree 2. All the coefficients of these polynomials are bounded by a constant multiplied by \mathfrak{B}_2^ψ , with \mathfrak{B}_2^ψ as in (4.4). Introduce the graded ring

$$\mathcal{R} := \bigoplus_{j=1}^{\infty} \mathcal{R}_j, \quad \mathcal{R}_j := \left\{ R(X) = \frac{P(X)}{(X_1^2 + X_2^2)^j}; P \in \mathcal{C}_b^\infty[X_1, X_2], \deg P \leq j \right\}.$$

We have $R_{1,0}^1 \in \mathcal{R}_1$ and $R_{0,1}^1 \in \mathcal{R}_1$. On the other hand, we find $R_{0,0}^1 \in \mathcal{R}_2$. Now, given $R \in \mathcal{R}_j$ as above, we can compute

$$\begin{aligned} \partial_\star \left[\frac{P(\partial_s \Phi, \partial_y \Phi)}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^j} \right] &= - \frac{(\partial_{\star s}^2 \psi \partial_{X_1} P + \partial_{\star y}^2 \psi \partial_{X_2} P) (\partial_s \Phi^2 + \partial_y \Phi^2)}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j+1}} \\ &\quad + 2j \frac{P (\partial_{\star s}^2 \psi \partial_s \Phi + \partial_{\star y}^2 \psi \partial_y \Phi)}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j+1}} + \frac{\partial_\star P(\partial_s \Phi, \partial_y \Phi)}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^j}. \end{aligned}$$

This implies that $\partial_\star \mathcal{R}_j \subset \mathcal{R}_j \oplus \mathcal{R}_{j+1}$. Now, a simple induction on n shows that

$$(4.11) \quad (-i)^n Q^*(s, y; \partial_s, \partial_y)^n = \sum_{|\alpha| \leq n} R_\alpha^n(\partial_s \Phi, \partial_y \Phi) \partial_{s,y}^\alpha,$$

with

$$(4.12) \quad R_\alpha^n(X) = \sum_{j=n}^{2n-|\alpha|} \frac{P_\alpha^{n,j}(X)}{(X_1^2 + X_2^2)^j}, \quad \deg P_\alpha^{n,j} \leq j, \quad R_\alpha^n \in \bigoplus_{j=n}^{2n-|\alpha|} \mathcal{R}_j.$$

In view of (4.9), (4.11) and (4.12), to get a control on \mathcal{I} , we have to estimate terms which look like

$$\varepsilon^{n+\frac{1}{2}} \int \left(\int_0^{2T/\varepsilon} \int_{-r}^{+r} \frac{|P_\alpha^{n,j}(\partial_s \Phi, \partial_y \Phi)|}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j/2}} \frac{\|a\|_{W^{n,\infty}(\mathcal{N})}}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j/2}} ds dy \right) d\xi.$$

From (4.7), we can deduce that

$$(4.13) \quad 0 < \frac{1}{(\partial_s \Phi)^2 + (\partial_y \Phi)^2} \leq \inf \left(\frac{2}{\eta^2}; \frac{1}{|\xi + \partial_y \psi(s, y)|^2} \right).$$

On the other hand, the coefficients of $P_\alpha^{n,j}$ can be roughly controlled by a constant multiplied by $(1 + \mathfrak{B}_n^\psi)^n$. Note that $\partial_s \Phi$ and $\partial_y \Phi$ are not necessarily bounded in ε for $s \sim \varepsilon^{-1}$. Actually, the expression $\partial_y \Phi$ is not at all bounded in the case of φ . This is why the information $\deg P_\alpha^{n,j} \leq j$ is important. Exploiting (4.13), we can find a constant $C^{n,j}$ such that

$$\forall (s, y, \xi) \in \mathcal{N}, \quad \frac{|P_\alpha^{n,j}(\partial_s \Phi, \partial_y \Phi)|}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j/2}} \leq \frac{C^{n,j}}{\eta^j} (1 + \mathfrak{B}_n^\psi)^n.$$

Now, the idea is to use the right hand side of (4.13) and the condition $2 \leq n \leq j$ inside (4.12) to recover some integrability in ξ . As a matter of fact, we have

$$\begin{aligned} & \int \left(\int_0^{2\mathcal{T}/\varepsilon} \int_{-r}^{+r} \frac{ds dy}{((\partial_s \Phi)^2 + (\partial_y \Phi)^2)^{j/2}} \right) d\xi \\ & \leq \int_0^{2\mathcal{T}/\varepsilon} \int_{-r}^{+r} \left(\int \inf \left(\frac{2^{\frac{j}{2}}}{\eta^j}; \frac{1}{|\xi + \partial_y \psi(s, y)|^j} \right) d\xi \right) ds dy \\ & \leq \frac{4 r \mathcal{T}}{\varepsilon \eta^{j-1}} \int \inf \left(2^{\frac{j}{2}}; \frac{1}{|\xi|^j} \right) d\xi < +\infty. \end{aligned}$$

Combining all the above information, we get the expected result. \square

Preliminary conclusion. In view of Lemma 4.2 together with Proposition 4.5, for all $m \in \mathbb{Z} \setminus \{0, 1\}$, we get $u^m = \mathcal{O}(\varepsilon^\infty)$. In the case $m = 0$, the property $\xi_0 > 0$ in Assumption 3.11, Lemma 4.3 (with $\delta = \xi_0$) as well as Proposition 4.5 also yield $u^0 = \mathcal{O}(\varepsilon^\infty)$. This means that **all modes u^m with $m \neq 1$ are negligible.**

The harmonic $m = 1$ is the only choice which may give rise to interesting phenomena from the viewpoint of quasi-rectification. This is Fact 1 in the PDE context (1.3). With this in mind, in the rest of this section, we focus on the case $m = 1$. We work with $\psi \equiv \varphi$ and $a \equiv a^1$. Unless otherwise specified, for $m = 1$, we will simply use the notations $a \equiv a^1$ and $u \equiv u^1$. Thus, in the next subsection 4.2, the function u represents the solution to

$$(4.14) \quad \partial_t u - \frac{i}{\varepsilon} p(\varepsilon D_x) u + \varepsilon^{3/2} A_1^* e^{i\varphi(t,x)/\varepsilon} = 0, \quad u|_{t=0} = 0.$$

4.2. Resonant oscillatory integrals. The analysis of the expression \mathcal{I} defined by (4.2)-(4.3) with $\psi = \varphi$ relies quite heavily on the explicit formula (1.26). The aim here is to give an overview of next Subsections 4.3, 4.4 and 4.5, where precise results will be established. The reader who is not interested in the details of proofs can read this Subsection 4.2, and then go directly to Section 5. In Paragraph 4.2.1, we clarify what is new in comparison with the phenomena which have been exhibited in the introduction. In Paragraph 4.2.2, we introduce a well-adapted partition of the long time interval $[0, \mathcal{T}/\varepsilon]$ with $\mathcal{T} > 0$. Then, in Paragraph 4.2.3, this leads to a distinction between a dispersive and some almost stationary regime. What happens in these two regimes is described in Paragraphs 4.2.4 and 4.2.5, successively.

4.2.1. Basic mechanisms. The toy model presented in Subsection 1.1 explains how wave packets can be produced over large time. It shows that the creation of wave packets is basically due to the combination of two factors: the first is the presence of a resonance; the second is the introduction as a source term of well-adjusted oscillations. In the toy model of the introduction, the discussion involves basic choices of L ($p \equiv 1$) and φ (which does not depend on x). Now, we want to better understand what happens under the more realistic geometrical conditions, which are Assumptions 3.3 and 3.11, together with formulas (1.26) and (3.31).

When the symbol p is not constant, the emitted signals do propagate spatially. The form of the wave packets is determined by the local geometrical characteristics of \mathcal{V} at the intersection points between $\text{Char}(L)$ and $\mathcal{G}(\varphi)$. But the way in which these wave packets propagate and can accumulate over time depends on the asymptotic properties of \mathcal{V} when $|\xi|$ goes to $+\infty$ and of $\mathcal{G}(\varphi)$ when t goes to $+\infty$.

It follows that the whole picture is a combination of *local* and *global* geometrical features of $\text{Char}(L)$ and $\mathcal{G}(\varphi)$. The resulting effects are constructive and destructive interferences. They appear in the absence of nonlinearity.

4.2.2. *Decomposition into wave packets.* The function Φ of (3.17) with $m = 1$ - or equivalently the function Φ of (4.3) with $\psi = \varphi$ - can be separated into two parts, according to $\Phi = \phi + \gamma - t$. Since the symbol p is even, we find

$$\phi(t, x; s, y, \xi) := (s - t) [p(\xi) - 1] + (x - y) \xi + y s - \gamma \cos s.$$

Accordingly, another way to formulate (4.2) is to write

$$(4.15) \quad \mathcal{I} := \frac{\sqrt{\varepsilon}}{2\pi} e^{i(t-\gamma)/\varepsilon} \int \left(\int_0^t \int e^{-i\phi(t,x;s,y,\xi)/\varepsilon} \zeta(\xi) a(\varepsilon s, s, y) ds dy \right) d\xi.$$

From now on, it is implicitly assumed that t (or $T = \varepsilon t$) runs over a finite period of long times, in coherence with (3.33). More precisely, we wait until the action of the perturbation through a is completed, that is

$$(4.16) \quad \mathcal{T}/\varepsilon \leq t \leq 2\mathcal{T}/\varepsilon, \quad \text{or equivalently} \quad \mathcal{T} \leq T \leq 2\mathcal{T}.$$

Given $t_0 > 0$, with χ as in (2.11), define

$$(4.17) \quad \forall s \in \mathbb{R}, \quad \chi_{t_0}(s) := \chi(s/t_0).$$

In addition, we can tune χ so that it generates a partition of unity with

$$(4.18) \quad \sum_{k \in \mathbb{Z}} \chi_{2\pi/3}(x - k\pi) = 1.$$

We introduce some truncation near the diagonal $s = \xi$, namely

$$(4.19) \quad v(t, x) := \int_0^t \int \int e^{-i\phi(t,x;s,y,\xi)/\varepsilon} \zeta(\xi) a(\varepsilon s, s, y) \chi_{1/4}(s - \xi) ds dy d\xi.$$

The above expression $v(t, x)$ is given by the integral in (s, y, ξ) on the compact domain $[0, t] \times [-r, r] \times [-1/4, 1/4 + t]$ of functions depending smoothly on (ε, t, x) . It is therefore a well-defined smooth function of (ε, t, x) . This choice is motivated by the following lemma.

Lemma 4.6. *For all $\delta > 0$, the phase φ given by (1.26) is non-resonant on the set*

$$\mathcal{N}_1^\delta = \{(s, y, \xi) \in]0, \infty[\times \mathbb{R} \times \mathbb{R}, \quad |\xi - s| \geq \delta\}.$$

Proof. Requiring φ being non-resonant on \mathcal{N}_1^δ amounts to finding $\eta > 0$ such that

$$\forall (s, y, \xi) \in \mathcal{N}_1^\delta, \quad 0 < \eta \leq |p(\xi) - 1 + y + \gamma \sin s| + |\xi - s|.$$

Now, by definition, we have $\delta \leq |\xi - s|$ for all $(s, y, \xi) \in \mathcal{N}_1^\delta$. Thus, the choice $\eta = \delta$ is suitable. \square

In view of Lemma 4.6 with $\delta = 1/4$, Proposition 4.5 yields

$$(4.20) \quad u(t, x) = \frac{\sqrt{\varepsilon}}{2\pi} e^{i(t-\gamma)/\varepsilon} v(t, x) + \mathcal{O}(\varepsilon^\infty), \quad x \in \mathbb{R}, \quad \frac{\mathcal{T}}{\varepsilon} \leq t \leq \frac{2\mathcal{T}}{\varepsilon}.$$

There remains to analyze v . Using again (3.33) with $m = 1$ and $a \equiv a_1$, we can replace (4.19) by an integral on the whole domain \mathbb{R}^3 , which is

$$(4.21) \quad v(t, x) = \iiint e^{-i\phi(t,x;s,y,\xi)/\varepsilon} \zeta(\xi) a(\varepsilon s, s, y) \chi_{1/4}(s - \xi) ds dy d\xi.$$

Using (4.16) and (4.18), we can write

$$(4.22) \quad v(t, x) = \sum_{k \in \mathcal{K}} v_k(t, x), \quad \mathcal{K} := \left\{ k \in \mathbb{N}; k \leq \frac{2}{3} + \frac{\mathcal{T}}{\pi \varepsilon} \right\},$$

where, for $k \in \mathcal{K}$, we have introduced the signal

$$(4.23) \quad v_k(t, x) := \iiint e^{-i \phi(t, x; s, y, \xi) / \varepsilon} \zeta(\xi) a(\varepsilon s, s, y) \\ \times \chi_{1/4}(s - \xi) \chi_{2\pi/3}(s - k\pi) ds dy d\xi.$$

Fix $k \in \mathbb{Z}$, and change s into $s - k\pi$ and ξ into $\xi - k\pi$ in order to obtain

$$(4.24) \quad v_k(t, x) = e^{-i(t - k\pi + k\pi x) / \varepsilon} w_k(t, x),$$

with

$$(4.25) \quad w_k(t, x) := \iiint e^{-i \Phi_k(t, x; s, y, \xi) / \varepsilon} a_k(\varepsilon, s, y, \xi) \\ \times \chi_{1/4}(s - \xi) \chi_{2\pi/3}(s) ds dy d\xi,$$

and where

$$(4.26a) \quad a_k(\varepsilon, s, y, \xi) := \zeta(k\pi + \xi) a(\varepsilon k\pi + \varepsilon s, k\pi + s, y),$$

$$(4.26b) \quad \Phi_k(t, x; s, y, \xi) := (k\pi - t) p(k\pi + \xi) + s [p(k\pi + \xi) - 1] \\ + (x - y) \xi + s y - (-1)^k \gamma \cos s.$$

In view of (4.22), the expression v is the sum of the wave packets v_k . Now, the advantage when working with v_k (or w_k) is that the domain of integration in (s, y, ξ) is compact and independent of ε , k and t . As a counterpart, at the level of line (4.25), the phase Φ_k and the profile a_k involve ε , k and t as parameters.

The expression F_L of (3.31) looks like the one of (1.2). By analogy with (1.8)-(1.16), the expected amplitude of u is $\mathcal{O}(\varepsilon)$. In comparison to this reference threshold, all terms of smaller size $o(\varepsilon)$ will be considered negligible.

4.2.3. *Dispersive vs. almost stationary regime.* The asymptotic behavior of w_k when ε goes to 0 depends heavily on the size of k . In view of (2.11) and (3.33), it suffices to deal with integers k such that $k \in \mathcal{K}$, with \mathcal{K} as in (4.22). Let q be the integer of Assumption 3.3. Given some $c > 0$, we can decompose \mathcal{K} into two separate parts \mathcal{K}_d^c and \mathcal{K}_s^c with

$$(4.27a) \quad \mathcal{K}_d^c \equiv \mathcal{K}_d^c \left(\frac{1}{q+1} \right) := \left\{ k \in \mathbb{N}; k \leq \frac{c}{\varepsilon^{1/(q+1)}} \right\},$$

$$(4.27b) \quad \mathcal{K}_s^c \equiv \mathcal{K}_s^c \left(\frac{1}{q+1} \right) := \left\{ k \in \mathbb{N}; \frac{c}{\varepsilon^{1/(q+1)}} < k \leq \frac{2}{3} + \frac{\mathcal{T}}{\pi \varepsilon} \right\}.$$

In (4.27), the symbol \mathcal{K} has exponent c and subscripts d or s . The exponent c is aimed to specify the choice of the constant c , while the subscripts d or s are used to refer respectively to the words *dispersion* and *almost stationary*.

- In Subsection 4.3, by adjusting $c > 0$ small enough, we can ensure some strong dispersion of the wave packets w_k for values of k in \mathcal{K}_d^c and for t of order $1/\varepsilon$. The contribution of these wave packets is negligible. They mainly emerge from the ball $|x| < r$, with r as in Assumption 3.11.

- In Subsection 4.4, we consider the case $k \in \mathcal{K}_s^c$, for which signals may be emitted at a higher order. Such signals are almost stationary, and therefore they can be detected during long times $t \sim \varepsilon^{-1}$ inside the ball $|x| < r$. In Subsection 4.5, we study the local accumulation (near the position $x = 0$) of these wave packets.

4.2.4. *Dispersive regime* ($k \in \mathcal{K}_d^c$). Using a non-stationary phase argument in ξ , we will establish in Lemma 4.7 that for any given choice of $R > 0$, we can find a constant $c > 0$ such that

$$\sup \left\{ |w_k(t, x)|; 0 \leq k \leq \frac{c}{\varepsilon^{1/(q+1)}}, \frac{\mathcal{T}}{\varepsilon} \leq t \leq \frac{2\mathcal{T}}{\varepsilon}, |x| \leq R \right\} = \mathcal{O}(\varepsilon^{D-1}),$$

where $D \geq 2$ is the integer appearing in Assumption 3.8. Therefore, since $q \geq 2$, the terms w_k with $k \in \mathcal{K}_d^c$ enter into the composition of u through a contribution which can be estimated according to

$$(4.28) \quad \frac{\sqrt{\varepsilon}}{2\pi} \sum_{0 \leq k \leq c\varepsilon^{-1/(q+1)}} |w_k(t, x)| \lesssim \varepsilon^{\frac{1}{2}+D-1-\frac{1}{q+1}} \lesssim \varepsilon^{7/6} \ll \varepsilon.$$

4.2.5. *Stationary regime* ($k \in \mathcal{K}_s^c$). The absence of dispersion may be revealed through the existence of critical points when looking at the phase Φ_k . Accordingly, for $k \in \mathcal{K}_s^c$, the asymptotic behavior of w_k when ε goes to 0 will be analyzed by stationary phase arguments:

- For all $k \in \mathcal{K}_s^c$, the phase Φ_k involved in the definition of w_k in (4.25) has at most one critical point in the domain of integration (Lemma 4.8).
- For all $k \in \mathcal{K}_s^c$, the above mentioned possible critical point is necessarily non-degenerate (Lemma 4.11).
- Possible values of k such that $c \leq \varepsilon^{\frac{1}{q+1}} k < c_1$ with c_1 large enough belong to a transition zone, treated as a black box. For all $k \in \mathcal{K}_s^{c_1}$, the phase Φ_k has indeed a unique critical point (Lemma 4.12).
- When $D \geq 3$, the solution u to (4.14) can be viewed modulo $\mathcal{O}(\varepsilon^{5/3})$ as a sum of wave packets u_k with $k \in \mathcal{K}_s^{c_1}$ (Lemma 4.15).

4.2.6. *Accumulation of the wave packets*. At this stage, Assumption 3.12 (about the property of periodicity for large times) on the amplitude a is needed. Then, a clear asymptotic dichotomy occurs regarding the order of magnitude of u . According to the position in time and space, different orders of magnitude are possible for the wave function u , with a precise expression when u reaches its maximal order of magnitude.

- Constructive interference (Proposition 4.16). For $x = 2j\varepsilon$ with $j \in \mathbb{Z}$ – as forecast by (1.16) with u^1 multiplied by $\varepsilon^{3/2}$ – the solution u to (4.14) is of order ε exactly, see (4.77) and (4.78).
- Destructive interference (Proposition 4.18). On the contrary, for $x = \alpha\varepsilon$ with $\alpha \notin 2\mathbb{Z}$, we find that the amplitude of u is $o(\varepsilon)$, see (4.92).

4.3. **The dispersive regime.** This is when $k \in \mathcal{K}_d^c$ with \mathcal{K}_d^c defined as in (4.27a). In the sum (4.21), the associated waves v_k or w_k are emitted by the source term during relatively small times s . Thus, for t large enough, that is for t as indicated at the level of (4.16), these waves have enough time to (partially) disperse away from any ball $|x| < R$.

It is this idea that is developed and quantified in Lemma 4.7 below. The notations have been set up in Subsection 4.2. In particular, the wave packets v_k and w_k are given by (4.23) and (4.25), respectively.

Lemma 4.7 (Dispersion of waves when $k \in \mathcal{K}_d^c$). *Let w_k be defined as in (4.25). With D as in Assumption 3.8, and for $R > 0$ fixed, there exists a constant $c > 0$ (depending on R) such that*

$$(4.29) \quad \sup \left\{ |w_k(t, x)|; 0 \leq k \leq \frac{c}{\varepsilon^{1/(q+1)}}, \frac{\mathcal{T}}{\varepsilon} \leq t \leq \frac{2\mathcal{T}}{\varepsilon}, |x| \leq R \right\} = \mathcal{O}(\varepsilon^{D-1}).$$

Note that since $|v_k(t, x)| = |w_k(t, x)|$, the same applies to v_k .

Proof. Observe first that the integral defining w_k at the level of (4.25) is restricted to the compact set

$$(4.30) \quad \Upsilon := \left\{ (s, y, \xi) \in \mathbb{R}^3; |s| \leq 2\pi/3, |y| \leq r, |s - \xi| \leq 1/4 \right\},$$

where r comes from Assumption 3.11. We apply the principle of non-stationary phase, but this time through integrations by parts involving ξ . To this end, with Φ_k as in (4.26b), we have to compute

$$(4.31) \quad \partial_\xi \Phi_k(t, x; s, y, \xi) = (k\pi + s - t) p'(k\pi + \xi) + x - y.$$

The situation is the following. Knowing (3.23), for $k \lesssim \varepsilon^{1/(q+1)}$ and $t \gtrsim 1/\varepsilon$, since s, x and y are bounded, we find

$$\partial_\xi \Phi_k(t, x; s, y, \xi) \approx -tp'(k\pi + \xi) + \mathcal{O}(1) \approx \frac{t}{k^{q+1}} + \mathcal{O}(1) \gtrsim \frac{\varepsilon t}{c^{q+1}} - C_0.$$

For c small enough, the right hand side becomes positive. The rest of the proof consists in making this heuristical computation more quantitative.

In view of (3.33), on the domain of integration giving rise to w_k , we have $1 \leq k\pi + s$ as well as (in view of the support of $\chi_{1/4}$) the inequality $-1/4 \leq \xi - s$. It follows that $\xi_c \leq 1/2 < 3/4 \leq k\pi + \xi$. Taking into account (3.20b) and (3.23), knowing that $\ell < 0$, for such values of ξ , we have

$$(4.32) \quad \exists \delta_0 > 0, \quad \delta_0 \leq \sup_{3/4 \leq \eta} \eta^{q+1} p'(\eta) \leq (k\pi + \xi)^{q+1} p'(k\pi + \xi).$$

For t as in (4.16), for (s, y, ξ) as in (4.30), and for $|x| \leq R$, we have

$$\frac{\mathcal{T}}{\varepsilon} - k\pi - \frac{2\pi}{3} \leq t - k\pi - s, \quad k\pi + \xi \leq k\pi + \frac{2\pi}{3} + 1, \quad |x - y| \leq r + R.$$

It follows that

$$(4.33) \quad \delta_0 \left(\frac{\mathcal{T}}{\varepsilon} - k\pi - \frac{2\pi}{3} \right) \left(k\pi + \frac{2\pi}{3} + 1 \right)^{-q-1} - r - R \leq |\partial_\xi \Phi_k(t, x; s, y, \xi)|.$$

Define

$$(4.34) \quad \varepsilon_0 := \frac{3^{q+1}}{(6 + 4\pi)^{q+1}} \frac{\delta_0 \mathcal{T}}{2(\delta_0 + r + R)}, \quad c := \frac{1}{2\pi} \left(\frac{\delta_0 \mathcal{T}}{2(\delta_0 + r + R)} \right)^{\frac{1}{q+1}}.$$

By this way, for $\varepsilon \leq \varepsilon_0$ and $k \leq c\varepsilon^{-1/(q+1)}$, we can assert that

$$\varepsilon^{\frac{1}{q+1}} \left(\frac{2\pi}{3} + 1 \right) \leq \frac{1}{2} \left(\frac{\delta_0 \mathcal{T}}{2(\delta_0 + r + R)} \right)^{\frac{1}{q+1}}, \quad \varepsilon^{\frac{1}{q+1}} k\pi \leq \frac{1}{2} \left(\frac{\delta_0 \mathcal{T}}{2(\delta_0 + r + R)} \right)^{\frac{1}{q+1}}.$$

Sum these two inequalities and rearrange the terms to get

$$(4.35) \quad (\delta_0 + r + R) \left(k\pi + \frac{2\pi}{3} + 1 \right)^{q+1} \leq \frac{\delta_0 \mathcal{T}}{2\varepsilon}.$$

In particular

$$\delta_0 \left(k\pi + \frac{2\pi}{3} + 1 \right) \leq (\delta_0 + r + R) \left(k\pi + \frac{2\pi}{3} + 1 \right)^{q+1} \leq \frac{\delta_0 \mathcal{T}}{2\varepsilon},$$

and hence

$$\frac{\mathcal{T}}{2\varepsilon} \geq k\pi + \frac{2\pi}{3}, \quad \frac{\mathcal{T}}{2\varepsilon} \leq \frac{\mathcal{T}}{\varepsilon} - k\pi - \frac{2\pi}{3}.$$

Exploiting (4.35), it follows that

$$(\delta_0 + r + R) \left(k\pi + \frac{2\pi}{3} + 1 \right)^{q+1} \leq \frac{\delta_0 \mathcal{T}}{2\varepsilon} \leq \delta_0 \left(\frac{\mathcal{T}}{\varepsilon} - k\pi - \frac{2\pi}{3} \right).$$

Coming back to (4.33), this yields

$$(4.36) \quad \forall \varepsilon \in (0, \varepsilon_0], \quad \forall k \in \mathbb{N} \cap [0, c\varepsilon^{-1/(q+1)}], \quad \delta_0 \leq |\partial_\xi \Phi_k(t, x; s, y, \xi)|.$$

As long as $\partial_\xi \Phi_k \neq 0$, the following identity may be used

$$(4.37) \quad e^{-i\Phi_k/\varepsilon} = \mathcal{D}_\xi e^{-i\Phi_k/\varepsilon}, \quad \mathcal{D}_\xi \equiv D(k)_\xi := \frac{\varepsilon i}{\partial_\xi \Phi_k} \partial_\xi.$$

The differential operator \mathcal{D}_ξ is not self-adjoint. We have to deal with

$$(4.38) \quad \mathcal{D}_\xi^* \equiv D(k)_\xi^* = \frac{\varepsilon i}{\partial_\xi \Phi_k(t, x; s, y, \xi)} \partial_\xi - \frac{\varepsilon i \partial_{\xi\xi}^2 \Phi_k(t, x; s, y, \xi)}{(\partial_\xi \Phi_k(t, x; s, y, \xi))^2}.$$

Knowing that (4.36) is verified on the domain of integration giving rise to w_k , an integration by parts (in the variable ξ) using \mathcal{D}_ξ^* yields

$$|w_k(t, x)| := \left| \iiint e^{-i\Phi_k(t, x; s, y, \xi)/\varepsilon} \mathcal{D}_\xi^* [a_k(\varepsilon, s, y) \chi_{\frac{1}{4}}(s - \xi) \chi_{\frac{2\pi}{3}}(s)] ds dy d\xi \right|.$$

Taking (4.36) into account, the application of \mathcal{D}_ξ^* allows to gain a power of ε on condition that

$$(4.39) \quad \frac{|\partial_{\xi\xi}^2 \Phi_k(t, x; s, y, \xi)|}{|\partial_\xi \Phi_k(t, x; s, y, \xi)|^2} = \frac{|(k\pi + s - t) p''(k\pi + \xi)|}{|(k\pi + s - t) p'(k\pi + \xi) + x - y|^2} = \mathcal{O}(1).$$

The difficulty is that, at the level of (4.29), neither k nor t can be bounded uniformly in $\varepsilon \in]0, 1]$. However, with $X := (k\pi + s - t) p'(k\pi + \xi)$, remark that

$$\frac{|x - y|}{|X|} \leq \frac{1}{2} \implies \frac{|\partial_{\xi\xi}^2 \Phi_k|}{|\partial_\xi \Phi_k|} = \left| 1 + \frac{x - y}{X} \right|^{-1} \frac{|p''(k\pi + \xi)|}{p'(k\pi + \xi)} \leq 2 \sup_{s \geq 3/4} \frac{|p''(s)|}{p'(s)}.$$

On the other hand, using (4.36) and then $|x - y| \leq r + R$, we have

$$\frac{|x - y|}{|X|} \geq \frac{1}{2} \implies \frac{|\partial_{\xi\xi}^2 \Phi_k|}{|\partial_\xi \Phi_k|} \leq \frac{|X|}{\delta_0} \frac{|p''(k\pi + \xi)|}{p'(k\pi + \xi)} \leq \frac{2(r + R)}{\delta_0} \sup_{s \geq 3/4} \frac{|p''(s)|}{p'(s)}.$$

From (3.19) and (3.23), we can deduce that

$$C(p) := \sup_{s \geq 3/4} \{ |p''(s)|/p'(s) \} < +\infty.$$

In short, we have (4.39) with

$$\frac{|\partial_{\xi\xi}^2 \Phi_k(t, x; s, y, \xi)|}{\partial_{\xi} \Phi_k(t, x; s, y, \xi)^2} \leq \frac{2}{\delta_0} \max\left(1; \frac{r+R}{\delta_0}\right) C(p) < +\infty.$$

By extension, the action of $(\mathcal{D}_{\xi}^*)^{n-1}$ with $n \leq D$ involves the quotients $\partial_{\xi}^j \Phi_k / \partial_{\xi} \Phi_k$ with $0 \leq j \leq n$. Under Assumption 3.8, the above argument can be repeated $D-1$ times, leading to the estimate (4.29). \square

Note that (4.20) allows to capture any position $x \in \mathbb{R}$. On the contrary, to obtain uniform $\mathcal{O}(\varepsilon^N)$ bounds and to measure more precisely the quantitative aspects of the dispersive effects, we need to restrict the size of the spatial domain, as in (4.29). This is why we will work with $R = r$, where r is fixed as in Assumption 3.11. In agreement with Lemma 4.2, we restrict the discussion to the ball $|x| < r$. From now on, the values of ε_0 and c are adjusted as indicated in (4.34) with $R = r$.

4.4. The regime of standing waves. This is when $k \in \mathcal{K}_s^c$ with \mathcal{K}_s^c as in (4.27b) and c as in (4.34). The novelty when $k \in \mathcal{K}_s^c$ is that the oscillatory integral (4.25) defining w_k may involve stationary points, which are positions (s, y, ξ) satisfying

$$\nabla_{s,y,\xi} \Phi_k(t, x; s, y, \xi) = 0,$$

or equivalently

$$(4.40a) \quad \partial_s \Phi_k(t, x; s, y, \xi) = p(k\pi + \xi) - 1 + y + (-1)^k \gamma \sin s = 0,$$

$$(4.40b) \quad \partial_y \Phi_k(t, x; s, y, \xi) = s - \xi = 0,$$

$$(4.40c) \quad \partial_{\xi} \Phi_k(t, x; s, y, \xi) = (k\pi + s - t) p'(k\pi + \xi) + x - y = 0.$$

In Paragraph 4.4.1, we show that there exists inside Υ , with Υ as in (4.30), at most one critical point which is denoted by $(s_k, y_k, \xi_k)(t, x)$. This means that (s_k, y_k, ξ_k) is as in (4.40), together with

$$(4.41) \quad -2\pi/3 \leq s_k \leq 2\pi/3, \quad -r \leq y_k \leq r, \quad s_k - 1/4 \leq \xi_k \leq s_k + 1/4.$$

In Paragraph 4.4.2, we derive asymptotic formulas describing the behavior of s_k when k goes to infinity. In Paragraph 4.4.3, we remark that the critical points are all non-degenerate. In Paragraph 4.4.4, we take $k \in \mathcal{K}_s^{c_1}$ with $c_1 \geq c$ large enough to find that there exists indeed a critical point. In Paragraph 4.4.5, we consider conditions under which stationary phase arguments can be employed.

4.4.1. Possible existence of critical points. The phase $\Phi_k(t, x; \cdot)$ depends on (k, t, x) and also (implicitly) on $\varepsilon \in]0, \varepsilon_0]$ through the condition $0 \leq t \leq 2\mathcal{T}/\varepsilon$. The same applies to $(s_k, y_k, \xi_k)(t, x)$.

Lemma 4.8 (At most one signal may be emitted from any value $k \in \mathcal{K}_s^c$). *Let us start by fixing ε_0 as in (4.34). Up to decreasing this value of ε_0 , for all $\varepsilon \in]0, \varepsilon_0]$, for all $k \in \mathcal{K}_s^c$ and for all $(t, x) \in [0, 2\mathcal{T}/\varepsilon] \times [-r, r]$, there is inside Υ , in the sense of (4.40), at most one critical point $(s_k, y_k, \xi_k)(t, x)$ of the phase $\Phi_k(t, x; \cdot)$, with positions y_k and ξ_k determined by*

$$(4.42) \quad y_k = 1 - p(k\pi + s_k) - (-1)^k \gamma \sin s_k, \quad \xi_k = s_k.$$

Proof. The constraint (4.40b) reads $s = \xi$. Thus, we can focus on the positions (s, y, s) satisfying (4.41) together with (4.40a) and (4.40c), which become

$$(4.43a) \quad y = 1 - p(k\pi + s) - (-1)^k \gamma \sin s,$$

$$(4.43b) \quad x = h_k(t; s) := 1 - p(k\pi + s) - (-1)^k \gamma \sin s \\ - (k\pi + s - t) p'(k\pi + s).$$

This furnishes already (4.42). Now, we can consider the determination of s . The time s is *a priori* localized as indicated in (4.41). But, using (4.43a), it is possible to get a more precise information on s . We know that $|y| \leq r \leq 1/8$. On the other hand, from (3.28), we can infer that

$$(4.44) \quad \exists C_2 > 0, \quad \forall \xi \geq 1, \quad 0 \leq \xi^q [1 - p(\xi)] \leq C_2.$$

Using the condition $c\varepsilon^{-1/(q+1)} < k$ inside the definition of \mathcal{K}_s^c , as well as (4.44), since $r < \gamma/2$, we have from (4.43a) that

$$(4.45) \quad |\sin s| \leq \frac{r}{\gamma} + \frac{C_2}{\gamma(k\pi + s)^q} \leq \frac{1}{2} + \mathcal{O}(\varepsilon^{q/(q+1)}).$$

In view of (4.45), for $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, we have to deal with the necessary condition $|\sin s| < \sqrt{3}/2$. Since $|s| \leq 2\pi/3$, this means that $1/2 < \cos s$. Given this, we can deduce the preliminary information

$$(4.46) \quad -\pi/3 < s < \pi/3.$$

Compute

$$\partial_s h_k(t; s) := -2 p'(k\pi + s) - (k\pi + s - t) p''(k\pi + s) - (-1)^k \gamma \cos s.$$

Exploiting (3.19) and (3.23), as well as $|t - k\pi - s| \leq 2\mathcal{T}/\varepsilon$, we find

$$|\partial_s h_k(t; s)| = \gamma |\cos s| + \mathcal{O}((k\pi + s)^{-q-1}) + \varepsilon^{-1} \mathcal{O}((k\pi + s)^{-q-2}).$$

Knowing that s must satisfy (4.46) and that $k \in \mathcal{K}_s^c$ is bounded from below as indicated in (4.27b), there remains

$$|\partial_s h_k(t; s)| = \gamma |\cos s| + \mathcal{O}(\varepsilon^{1/(q+1)}) = \mathcal{O}(1).$$

Thus, for $\varepsilon \in]0, \varepsilon_0]$ with ε_0 small enough, we can assert that

$$(4.47) \quad \forall k \in \mathcal{K}_s^c, \quad \forall s \in \left] -\frac{\pi}{3}, \frac{\pi}{3} \right[, \quad 0 < \frac{\gamma}{4} \leq |\partial_s h_k(t; s)|.$$

It follows that the function $h_k(t; \cdot)$ is one-to-one from the interval $] -\pi/3, \pi/3[$ onto its image, which may or may not contain the real number x . At all events, there exists at most one position $s_k(t, x) \in] -\pi/3, \pi/3[$ such that

$$(4.48) \quad h_k(t; s_k(t, x)) = x.$$

In short, any position (s, y, ξ) satisfying (4.40) and (4.41) is subject to $|s| < \pi/3$. Knowing this, as claimed in Lemma 4.8, there exists inside Υ at most one critical point $(s_k, y_k, \xi_k)(t, x)$ and, if any, the value of $s_k(t, x)$ is determined through the implicit relation (4.48), while $(y_k, \xi_k)(t, x)$ is given by (4.42). \square

By construction, the function $s_k(t, \cdot)$ is, for all $t \in [0, 2\mathcal{T}/\varepsilon]$, defined on the interval

$$(4.49) \quad \mathcal{I} s_k(t) := \left\{ h_k(t; s); -\frac{\pi}{3} < s < \frac{\pi}{3} \right\}.$$

Lemma 4.9 (Properties of s_k). *Let ε_0 as in Lemma 4.8, as well as $\varepsilon \in]0, \varepsilon_0]$ and $k \in \mathcal{K}_s^c$. The function s_k determined by (4.48) with h_k as in (4.43b) is smooth on its domain of definition, which is*

$$\mathcal{D}_{s_k} := \{(t, x) \in [0, 2\mathcal{T}/\varepsilon] \times \mathbb{R}; x \in \mathcal{I}_{s_k}(t)\}.$$

For all $\alpha \in \mathbb{N}^2$, we can find $C_\alpha > 0$ giving rise to the uniform estimate

$$(4.50) \quad \sup_{k \in \mathcal{K}_s^c} \sup_{(t,x) \in \mathcal{D}_{s_k}} |\partial_{t,x}^\alpha s_k(t, x)| \leq C_\alpha, \quad \forall \varepsilon \in]0, \varepsilon_0].$$

For $\alpha = (1, 0)$, we get the following more precise estimate

$$(4.51) \quad \sup_{(t,x) \in \mathcal{D}_{s_k}} |\partial_t s_k(t, x)| = \mathcal{O}(k^{-q-1}) = \mathcal{O}(\varepsilon).$$

Proof. The bound (4.50) is, for $\alpha = (0, 0)$, a direct consequence of (4.41). Compute

$$(4.52) \quad \partial_t s_k(t, x) = -\frac{p'(k\pi + s_k(t, x))}{\partial_s h(k, t; s_k(t, x))}, \quad \partial_x s_k(t, x) = \frac{1}{\partial_s h(k, t; s_k(t, x))}.$$

In view of (3.23) and (4.47), this furnishes (4.51) for $\alpha = (1, 0)$, and also (4.50) for $\alpha = (0, 1)$. The general case $|\alpha| > 1$ can be obtained by induction. As a matter of fact, for $|\alpha| > 1$, the expression

$$\partial_s h(k, t; s_k(t, x))^{|\alpha|} \partial_{t,x}^\alpha s_k(t, x)$$

is a finite linear combination of products involving $p^i(k\pi + s_k)$, $\partial_s^j h(k, t; s_k)$ and $\partial_{t,x}^\beta s_k$ with $i \leq |\alpha|$, $j \leq |\alpha|$ and $|\beta| < |\alpha|$. It suffices to remark that all these quantities are uniformly bounded. This comes from Assumption 3.8 concerning p^i and $\partial_s^j h$. This is due to the inductive hypothesis regarding $\partial_{t,x}^\beta s_k$. \square

4.4.2. *Asymptotic formulas related to the critical points.* In the rest of this section and in Section 5, we need to identify the asymptotic behavior of s_k and y_k for large values of k . To this end, introduce

$$(4.53) \quad \tau_k(t; s) := 1 - p(k\pi + s) - (k\pi + s - t)p'(k\pi + s),$$

and remark that (4.48) can also be formulated as

$$(4.54) \quad s_k(t, x) = (-1)^{k+1} \arcsin\left(\frac{x}{\gamma} - \frac{\tau_k(t; s_k(t, x))}{\gamma}\right).$$

Lemma 4.10 (Asymptotic formulas). *Let ε_0 as in Lemma 4.8 as well as $\varepsilon \in]0, \varepsilon_0]$. Uniformly in $k \in \mathcal{K}_s^c$ and $t \in [0, 2\mathcal{T}/\varepsilon]$, the critical point $(s_k, y_k, \xi_k)(t, x)$, if any, is such that*

$$(4.55a) \quad s_k(t, x) = (-1)^{k+1} \arcsin\left(\frac{x}{\gamma} + \frac{\tau_k^0(t)}{\gamma}\right) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right),$$

$$(4.55b) \quad y_k(t, x) = x + (k\pi - t)p'(k\pi) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right) = \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right),$$

where

$$(4.56) \quad \tau_k^0(t) := \tau_k(t; 0) = 1 - p(k\pi) - (k\pi - t)p'(k\pi) = \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right).$$

Recall that

$$\forall k \in \mathcal{K}_s^c, \quad \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right) = \mathcal{O}\left(\frac{1}{k}\right) = \mathcal{O}\left(\varepsilon^{\frac{1}{q+1}}\right),$$

so that (4.55) furnishes indeed an explicit description of $s_k(t, x)$ and $y_k(t, x)$ modulo a small remainder. Remark that $\tau_k^0(t) = \mathcal{O}(1)$ as long as $k \sim \varepsilon^{-1/(q+1)}$ and $t \sim \varepsilon^{-1}$. By contrast, when εk^{q+1} is large, we can exploit the following information

$$(4.57) \quad s_k(t, x) = \xi_k(t, x) = s_k^x + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right), \quad y_k(t, x) = x + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right)$$

with

$$(4.58) \quad s_k^x := (-1)^{k+1} \arcsin\left(\frac{x}{\gamma}\right).$$

Proof. It suffices to write

$$(4.59) \quad \tau_k(t; s) = \tau_k^0(t) + [p(k\pi) - p(k\pi + s)] - (k\pi - t)[p'(k\pi + s) - p'(k\pi)] - sp'(k\pi + s).$$

On the one hand, we have $|s| \leq 2\pi/3$. On the other hand, we can exploit (3.19) and (3.23) to obtain (since εk is bounded when $k \in \mathcal{K}_s^c$)

$$(4.60a) \quad \tau_k(t; s) = \tau_k^0(t) + \mathcal{O}\left(\frac{1}{k^{q+1}}\right) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right) = \tau_k^0(t) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right),$$

$$(4.60b) \quad \tau_k^0(t) = \mathcal{O}\left(\frac{1}{k^q}\right) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right) = \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right).$$

We can deduce (4.55a) from (4.54) and (4.60a). Similarly, (4.55b) is a consequence of the relation (4.40c). \square

4.4.3. *All critical points are non-degenerate.* In view of stationary phase arguments, introduce the Hessian matrix $S_k(t, x)$ of the scalar function $\Phi_k(t, x; \cdot)$, that is

$$\begin{aligned} S_k(t, x) &:= \text{Hess}(\Phi_k)(t, x; s_k(t, x), y_k(t, x), \xi_k(t, x)) \\ &= \begin{pmatrix} \partial_{ss}^2 \Phi_k & \partial_{sy}^2 \Phi_k & \partial_{s\xi}^2 \Phi_k \\ \partial_{ys}^2 \Phi_k & \partial_{yy}^2 \Phi_k & \partial_{y\xi}^2 \Phi_k \\ \partial_{\xi s}^2 \Phi_k & \partial_{\xi y}^2 \Phi_k & \partial_{\xi\xi}^2 \Phi_k \end{pmatrix} (t, x; s_k(t, x), y_k(t, x), \xi_k(t, x)). \end{aligned}$$

It is notable that a control on the invertibility of S_k turns out to be available for all $k \in \mathcal{K}_s^c$. What is even more remarkable is that such a control can be obtained with uniform bounds with respect to $k \in \mathcal{K}_s^c$ and t as in (4.16).

Lemma 4.11 (The critical points are uniformly non-degenerate). *Up to decreasing again the value of $\varepsilon_0 \in]0, 1]$, for all $\varepsilon \in]0, \varepsilon_0]$, for all $k \in \mathcal{K}_s^c$ as well as for all $(t, x) \in [0, 2\mathcal{T}/\varepsilon] \times [-r, r]$, the possible critical point $(s_k, y_k, \xi_k)(t, x)$ of $\Phi_k(t, x; \cdot)$ is non-degenerate, such that $|s_k(t, x)| \leq \pi/3$, and there exists $C \in \mathbb{R}_+^*$ such that*

$$(4.61) \quad \forall (k, t, x) \in \mathcal{K}_s^c \times [0, 2\mathcal{T}/\varepsilon] \times [-r, r], \quad 0 < C \leq |\det S_k(t, x)|.$$

In addition, the signature of the matrix $S_k(t, x)$, that is the number of positive eigenvalues minus the number of negative eigenvalues, is given by

$$(4.62) \quad \text{sign}(S_k(t, x)) = (-1)^k.$$

Proof. We have already proven the condition $|s_k| \leq \pi/3$, see (4.46). Now, taking into account (4.26), we find

$$(4.63) \quad \begin{aligned} S_k &= \begin{pmatrix} (-1)^k \gamma \cos s_k & 1 & p'(k\pi + \xi_k) \\ 1 & 0 & -1 \\ p'(k\pi + \xi_k) & -1 & (k\pi + s_k - t)p''(k\pi + \xi_k) \end{pmatrix} \\ &= S_{(-1)^k}^{\delta_k} + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right) \end{aligned}$$

with the conventions

$$(4.64) \quad S_{\pm}^{\delta_k}(t, x) := \begin{pmatrix} \pm \delta_k & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \frac{\gamma}{2} < \delta_k \equiv \delta_k(t, x) := \gamma \cos s_k < \frac{1}{4}.$$

For $k \in \mathcal{K}_s^c$, we find $\varepsilon^{-1}k^{-q-2} \lesssim \varepsilon^{1/(q+1)} \ll 1$. This furnishes

$$(4.65) \quad \det S_k = \det(S_{(-1)^k}^{\delta_k}) + o(1) = -(-1)^k \delta_k + o(1).$$

In particular, this implies (4.61). On the other hand, we have the algebraic property

$$(4.66) \quad \text{Tr}(S_{\pm}^{\delta_k}) = \pm \delta_k = -\det(S_{\pm}^{\delta_k}) \neq 0.$$

The trace is the sum of the eigenvalues, and the determinant is their product. In view of (4.66), the eigenvalues cannot all have the same sign. Since $S_{\pm}^{\delta_k}$ is a 3×3 matrix, we have only two possibilities:

- The integer k is even. From (4.65), the determinant must be negative. Two eigenvalues are positive and one is negative. The signature is 1.
- The integer k is odd. From (4.65), the determinant must be positive. One eigenvalue is positive and two are negative. The signature is -1 .

Both results are consistent with (4.62). \square

4.4.4. *The existence for sure of critical points when k is large enough.* Let $c_1 \geq c$. Define $\mathcal{K}_s^{c_1}$ as in (4.27b). The inequality $c \leq c_1$ implies that $\mathcal{K}_s^{c_1} \subset \mathcal{K}_s^c$. For c_1 large enough, the content of Lemma 4.8 can be refined.

Lemma 4.12 (Signals from $\mathcal{K}_s^{c_1}$ with c_1 large enough are always detected). *Let ε_0 as in Lemma 4.11 as well as $\varepsilon \in]0, \varepsilon_0]$. There exists $c_1 \geq c$ such that, for all $k \in \mathcal{K}_s^{c_1}$ and for all $(t, x) \in [0, 2\mathcal{T}/\varepsilon] \times [-r, r]$, there is exactly one position $(s_k, y_k, \xi_k)(t, x)$ satisfying the two conditions (4.40) and (4.41).*

Proof. First observe that, given ε_0 as in Lemma 4.11, all preceding estimates remain valid if, like in the case of c_1 , we fix the value of c above the one of (4.34). Consider (4.43b), and remark that

$$(4.67) \quad h_k(t; s) = (-1)^{k+1} \gamma \sin s + \mathcal{O}(k^{-q}) + \varepsilon^{-1} \mathcal{O}(k^{-q-1}).$$

Since $c_1 \varepsilon^{-1/(q+1)} \leq k$, it follows that

$$h_k(t; \pm\pi/3) = \pm(-1)^{k+1} \sqrt{3} \gamma/2 + \mathcal{O}(\varepsilon^{\frac{q}{q+1}}) + \mathcal{O}(c_1^{-q-1}).$$

In particular, for c_1 large enough and ε sufficiently small, we find

$$\pm(-1)^{k+1} h_k(t; \pm\pi/3) > \gamma/2.$$

Taking into account (4.49), we have

$$x \in [-r, r] \subset [-\gamma/2, \gamma/2] \subset \mathcal{I}s_k(t).$$

Since $h_k(t; \cdot)$ is continuous, we can apply the intermediate value theorem. It says that we can find $s_k(t, x) \in] - \pi/3, \pi/3[$ satisfying (4.48). Lemma 4.8 guarantees that such $s_k(t, x)$ is unique in the interval $] - 2\pi/3, 2\pi/3[$. \square

4.4.5. *Towards stationary phase results.* Below, we recall a standard statement, which can be found e.g. in [16, Proposition 5.2] or [38, Theorem 3.16]. It will be used in this section and in the nonlinear analysis of Section 5.3.

Theorem 4.13 (From [16, 38]). *Select $\phi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and $a \in C_c^\infty(\mathbb{R}^n)$ satisfying $\text{supp } a = \Upsilon$. Let $h > 0$. Denote*

$$I_h = I_h(a, \phi) := \int_{\mathbb{R}^n} e^{-i\phi(x)/h} a(x) dx.$$

Suppose that

$$x_0 \in \Upsilon, \quad \nabla_x \phi(x_0) = 0, \quad \det \partial^2 \phi(x_0) \neq 0.$$

Assume further that $\nabla_x \phi(x) \neq 0$ on $\Upsilon \setminus \{x_0\}$. Then, for all $N \in \mathbb{N}^*$, there exist differential operators $M_{2j}(x; D)$ of order less than or equal to $2j$ such that

$$\begin{aligned} \left| I_h - h^{n/2} \sum_{j=0}^{N-1} h^j [M_{2j}(x; D)a(\cdot)]_{x=x_0} e^{-i\phi(x_0)/h} \right| \\ \leq C_N h^{n/2+N} \sum_{|\alpha| \leq 2N+n+1} \|\partial^\alpha a\|_{L^\infty}. \end{aligned}$$

The constant C_N depends on the compact Υ and also on the L^∞ norm of ϕ and its derivatives on Υ . In particular, denoting by $\text{sign } S$ the signature of S , we find

$$(4.68) \quad M_0 = \frac{(2\pi)^{n/2}}{|\det \partial^2 \phi(x_0)|^{1/2}} e^{-i\frac{\pi}{4} \text{sign } \partial^2 \phi(x_0)}.$$

Theorem 4.13 is aimed to be applied to the oscillatory integral defining w_k . When doing this, it is important to get uniform estimates with respect to all parameters k, t and x . Lemma 4.11 is a first indication that this works well. Another aspect is related to the uniform control of the constants C_N . As mentioned above, this can be achieved by looking at the derivatives of Φ_k on Υ .

Lemma 4.14 (Estimates on the derivatives of Φ_k). *Let ε_0 as in Lemma 4.11 as well as $\varepsilon \in]0, \varepsilon_0]$. With the compact set Υ given by (4.30), for all $N \geq 2$, there exists a constant C_N such that uniformly in t as in (4.16) and in x with $|x| \leq r$, we have*

$$(4.69) \quad \sup_{k \in \mathcal{K}_s^c} \sup_{(s, y, \xi) \in \Upsilon} \sum_{1 \leq |\alpha| \leq N} |\partial_{s, y, \xi}^\alpha \Phi_k(t, x; s, y, \xi)| \leq C_N.$$

Proof. Looking at (4.30) and (4.40b), we have indeed (4.69) for the terms which are involving multi-indices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $1 \leq \alpha_2$. For $\alpha_2 = 0$ and $1 \leq \alpha_1$, consider the line (4.40a). When $\alpha_1 = 1$, just apply (3.25). For $1 < \alpha_1$, combine the property (3.23) together with (3.29). Now, assume that $\alpha_1 = \alpha_2 = 0$ and $1 \leq \alpha_3 \leq N$. Then, exploiting (3.29), we find

$$|k\pi + s - t| |p^{(\alpha_3)}(k\pi + \xi)| \lesssim \varepsilon^{-1} p'(k\pi + \xi) = \varepsilon^{-1} \mathcal{O}(k^{-q-1}).$$

Since $k \in \mathcal{K}_s^c$, the right hand side is bounded. Summing the preceding upper bounds over multi-indices α yields (4.69). \square

4.5. The accumulation of wave packets. In Paragraph 4.5.1, the solution u to the equation (4.14) is represented modulo some $o(\varepsilon)$ as a sum of wave packets u_k with $k \in \mathcal{K}_s^{c_1}$. Then, the purpose is to distinguish between situations where constructive **interferences** occur (Paragraph 4.5.2) from those where, on the contrary, destructive **interferences** take place (Paragraph 4.5.3).

4.5.1. *The solution as a sum of wave packets.* Combining (4.20), (4.21) and (4.24), the solution u to (4.14) can be put in the form

$$(4.70) \quad u(t, x) = \sum_{k \in \mathbb{N}} u_k(t, x) + \mathcal{O}(\varepsilon^\infty),$$

with

$$(4.71) \quad u_k(t, x) := \frac{\sqrt{\varepsilon}}{2\pi} e^{i(-\gamma + k\pi - k\pi x)/\varepsilon} w_k(t, x).$$

Lemma 4.15. *Fix c_1 as in Lemma 4.12. Under Assumption 3.8 with $D \geq 3$, we can expand u according to*

$$(4.72) \quad u(t, x) = \sum_{k \in \mathcal{K}_s^{c_1}} u_k(t, x) + \mathcal{O}(\varepsilon^{2 - \frac{1}{q+1}}).$$

The wave packets u_k are of size $\mathcal{O}(\varepsilon^2)$. Assuming that $D \geq 4$, they have the form

$$(4.73) \quad u_k(t, x) = \varepsilon^2 b_k(\varepsilon, t, x) e^{i\Psi_k(t, x, s_k)/\varepsilon} + \mathcal{O}(\varepsilon^{2 + \frac{1}{q+1}}) = \mathcal{O}(\varepsilon^2),$$

with phases Ψ_k and profiles b_k given by

$$(4.74a) \quad \Psi_k(t, x, s_k) := -\gamma + t + (-1)^k \gamma \cos s_k \\ + [1 - p(k\pi + s_k)](k\pi + s_k - t) - (k\pi + s_k)x,$$

$$(4.74b) \quad b_k(\varepsilon, t, x) := (2\pi)^{1/2} \frac{e^{-i(-1)^k \frac{\pi}{4}}}{|\det S_k|^{1/2}} a(\varepsilon k\pi + \varepsilon s_k, k\pi + s_k, y_k).$$

Proof. The sum inside (4.70) can be split into

$$u(t, x) = \underbrace{\sum_{k \in \mathcal{K}_d^c} u_k(t, x)}_{\mathbf{1}} + \underbrace{\sum_{\frac{c}{\varepsilon^{1/(q+1)}} < k \leq \frac{c_1}{\varepsilon^{1/(q+1)}}} u_k(t, x)}_{\mathbf{2}} + \underbrace{\sum_{k \in \mathcal{K}_s^{c_1}} u_k(t, x)}_{\mathbf{3}} + \mathcal{O}(\varepsilon^\infty).$$

We may recognize here the dispersive part **1**, the transitional part **2** which is possibly absent (when $c = c_1$), and the cumulative part **3**. Not all $k \in \mathbb{N}$ have a leading order contribution, and not all with the same size. We will explain separately how to estimate each part.

1 For $k \in \mathcal{K}_d^c$, it suffices to apply Lemma 4.7 to get

$$\left| \sum_{k \in \mathcal{K}_d^c} u_k(t, x) \right| \leq \frac{\sqrt{\varepsilon}}{2\pi} \sum_{k \in \mathcal{K}_d^c} |w_k(t, x)| \leq C \sqrt{\varepsilon} \sum_{k \in \mathcal{K}_d^c} \varepsilon^{D-1} = \mathcal{O}(\varepsilon^{D - \frac{1}{2} - \frac{1}{q+1}}).$$

2 For $k \in \mathcal{K}_s^c$ with c as in (4.34), the idea is to exploit Lemma 4.11 to implement Theorem 4.13 at the level of the oscillatory integral (4.25). To do this, all assumptions must be checked:

- The first, and most important, is (4.30) which guarantees that the integration is on a compact set (independent of k , t or ε).

- The second is (4.69) which enables, away from (s_k, y_k, ξ_k) , to perform $D - 1$ integrations by parts, and still to obtain some $\mathcal{O}(\varepsilon^{D-1})$ error term. When doing this, a major difficulty is that the phase Φ_k still depends on (t, x) . And therefore, according to (4.16), since t may be of size ε^{-1} , it does depend on ε . The aim of the control (4.69) is precisely to overcome this difficulty.

- The main contribution is provided by a small neighborhood of (s_k, y_k, ξ_k) . The implementation of Morse Lemma (usually used when proving Theorem 4.13) is made possible by Lemma 4.11. It requires three derivatives of Φ_k to obtain a \mathcal{C}^1 -diffeomorphism. This implies that D must be at least equal to 3.

- The phase $\Phi_k(t, x; \cdot)$ depends on the three variables $(s, y, \xi) \in \mathbb{R}^3$, and therefore the leading-order term is of amplitude $\varepsilon^{3/2}$ modulo some small $o(\varepsilon^{3/2})$. Then, any extra derivative on Φ_k allows to gain a power of ε in the asymptotic expansion. We must take $D \geq 4$ to be sure of some $\mathcal{O}(\varepsilon^{5/2})$ precision.

Now, the expression w_k of (4.25) can be expanded in powers of ε through Theorem 4.13. To this end, taking into account the definitions (2.11), (4.17) and (4.42) together with Lemma 4.11 which implies $|s_k| \leq \pi/3$, first remark that

$$(4.75) \quad \chi_{1/4}(s_k - \xi_k) \chi_{2\pi/3}(s_k) = \chi_{1/4}(0) \chi_{2\pi/3}(s_k) = 1.$$

By Assumption 3.11, $\zeta(\xi) = 1 + \mathcal{O}(1/|\xi|)$ as $|\xi| \rightarrow \infty$, so

$$\forall k \in \mathcal{K}_s^c, \quad \zeta(k\pi + s_k) = 1 + \mathcal{O}(1/k), \quad 1 - \chi(k\pi + s_k) = 1.$$

Therefore, for all $k \in \mathcal{K}_s^c$, we find

$$A(\varepsilon k\pi + \varepsilon s_k, k\pi + s_k, y_k, k\pi + s_k) = a(\varepsilon k\pi + \varepsilon s_k, k\pi + s_k, y_k) + \mathcal{O}(\varepsilon^{1/(q+1)}).$$

On the other hand, the signature is given by (4.62). Combining all the above information, Theorem 4.13 yields (with $N = 1$)

$$(4.76) \quad w_k(t, x) = (2\pi\varepsilon)^{3/2} \frac{e^{-i(-1)^k \frac{\pi}{4}}}{|\det S_k|^{1/2}} e^{-i\Phi_k(t, x; s_k, y_k, \xi_k)/\varepsilon} \\ \times a(\varepsilon k\pi + \varepsilon s_k, k\pi + s_k, y_k) + \mathcal{O}(\varepsilon^{\frac{3}{2} + \frac{1}{q+1}}),$$

where the remainder term, larger than the one provided by Theorem 4.13, stems from the above approximation of A . In (4.76), the \mathcal{O} is uniform with respect to $k \in \mathcal{K}_s^c$ or t as in (4.16). Whether there exists a stationary point or not, we have $w_k = \mathcal{O}(\varepsilon^{3/2})$, hence $u_k = \mathcal{O}(\varepsilon^2)$. This rough estimation gives rise to

$$\left| \sum_{c\varepsilon^{-1/(q+1)} < k < c_1 \varepsilon^{-1/(q+1)}} u_k(t, x) \right| = \mathcal{O}(\varepsilon^{2 - \frac{1}{q+1}}).$$

③ For $k \in \mathcal{K}_s^{c_1}$ with c_1 as in Lemma 4.12, the content of u_k can be specified. Using the definition of u_k at the level of (4.71) together with (4.76), we find (4.73), with (4.74). Integers $k \in \mathcal{K}_s^{c_1}$ are the most numerous; they may provide the main contribution; and therefore they are set aside at the level of (4.72). Since $D - 1/2 \geq 2$ when $D = 3$, we can retain (4.72). \square

In a similar way to the elementary model of Section 1.1, the superposition of the wave packets u_k can induce a time growth of the solution u to (4.14). The source term of (3.18) is of size $\varepsilon^{3/2}$; in view of (4.73), it can trigger signals u_k of amplitude ε^2 ; at first sight, it can produce during long times $t \sim \varepsilon^{-1} T$ a contribution which may be of size $\varepsilon^2 t \sim \varepsilon T$.

That being said, this cumulative effect is only likely but not certain to occur, due to possible cancellations. The aim of the next Paragraphs 4.5.2 and 4.5.3 is to check what is actually happening.

4.5.2. *Constructive interferences.* In this paragraph, we show that the amplification phenomenon of the preamble does apply at special positions.

Proposition 4.16 (Asymptotic behavior of the solution on some moving lattice).

Under Assumptions 3.3, 3.6 and 3.8 (with $D \geq 4$) on the symbol p , as well as Assumption 3.12 on the profile $a \equiv a^1$, for all $T \in [T, 2T]$ and all $j \in \mathbb{Z}$, the solution u to (4.14) is such that

- If $q = 2$,

$$(4.77) \quad u\left(\frac{T}{\varepsilon}, 2j\varepsilon\right) = o(\varepsilon) + \frac{\varepsilon}{\sqrt{2\pi\gamma}} e^{i\frac{T}{\varepsilon^2}} \left(e^{-i\frac{\pi}{4}} \int_0^{+\infty} e^{-i\frac{\ell}{6}\left(\frac{1}{s} - \frac{T}{s^2}\right)} \underline{a}(s, 0, 0) ds \right. \\ \left. + e^{-i\left(\frac{2\gamma}{\varepsilon} - \frac{\pi}{4}\right)} \int_0^{+\infty} e^{-i\frac{\ell}{6}\left(\frac{1}{s} - \frac{T}{s^2}\right)} \underline{a}(s, \pi, 0) ds \right).$$

- If $q > 2$,

$$(4.78) \quad u\left(\frac{T}{\varepsilon}, 2j\varepsilon\right) = \mathcal{O}\left(\varepsilon^{1+\frac{q-2}{q+1}}\right) + \frac{\varepsilon}{\sqrt{2\pi\gamma}} e^{i\frac{T}{\varepsilon^2}} \left(e^{-i\frac{\pi}{4}} \int_0^{+\infty} \underline{a}(s, 0, 0) ds \right. \\ \left. + e^{-i\left(\frac{2\gamma}{\varepsilon} - \frac{\pi}{4}\right)} \int_0^{+\infty} \underline{a}(s, \pi, 0) ds \right).$$

The leading term in the right hand side of (4.77) and (4.78) does not depend on j . On the other hand, for $j = 0$, the formula (4.77) provides the asymptotic behavior of $u(T/\varepsilon, \cdot)$ at a fixed position, which is the origin $x = 0$.

Now, compare (1.16) multiplied by $\varepsilon^{3/2}$ with (4.78). When $q > 2$ and in the (extended) situation where $a(\cdot, 0, 0) \equiv \mathbf{1}_{[0, T]}$, the two formulas coincide. However, in the critical case $q = 2$, there are some differences. The wave packets u_k have larger group velocities; their wave front sets can mix; they can interact meaningfully. As a matter of fact, the identity (4.77) is more complicated, and the amplification effect can be altered by the oscillatory factor in front of \underline{a} .

We also note that if \underline{a} is not only 2π -periodic in its second argument, but π -periodic, then the above formula boils down to the one stated in Theorem 1.3.

Proof. The starting point is (4.72) together with (4.73). Select some $\alpha \in \mathbb{R}$. Since the $\mathcal{O}(\varepsilon^2)$ inside (4.73) is uniform with respect to k and t , a rough estimate yields

$$(4.79) \quad \varepsilon^{-1} u(t, \varepsilon \alpha) = \sum_{k \in \mathcal{K}_s^{c_1}} \varepsilon b_k(\varepsilon, t, \varepsilon \alpha) e^{i\Psi_k(t, \varepsilon \alpha, s_k)/\varepsilon} + \mathcal{O}\left(\varepsilon^{\frac{q}{q+1}}\right).$$

Recall the definitions inside (4.74). The ingredients Ψ_k and b_k of (4.79) are not free from a dependence on ε which may arise when specifying the choice of k , when replacing t by T/ε , or when substituting x with $\varepsilon \alpha$. A first step in the analysis is to simplify modulo small error terms the content of Ψ_k and b_k . Let us start by reducing Ψ_k . With s_k^x as in (4.58), coming back to (4.74a), compute

$$\Psi_k(t, x, s_k) - \Psi_k(t, x, s_k^x) = (-1)^k \gamma (\cos s_k - \cos s_k^x) - (s_k - s_k^x) x \\ + [1 - p(k\pi + s_k^x)] (s_k - s_k^x) + (k\pi + s_k - t) [p(k\pi + s_k^x) - p(k\pi + s_k)].$$

Combine the mean value theorem with (3.28) and (3.23). For large values of k , this gives rise to

$$\Psi_k(t, x, s_k) - \Psi_k(t, x, s_k^x) = \left[1 + \mathcal{O}\left(\frac{1}{k^q}\right) + \frac{1}{\varepsilon} \mathcal{O}\left(\frac{1}{k^{q+1}}\right) \right] \mathcal{O}(|s_k - s_k^x|).$$

Then, knowing that $k \in \mathcal{K}_s^{c_1}$, we can apply (4.57) to just retain

$$(4.80) \quad \Psi_k(t, x, s_k) = \Psi_k(t, x, s_k^x) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right).$$

For $k \in \mathcal{K}_s^{c_1}$ and ε small enough to be sure that $t_s \leq k\pi + s_k$ with t_s as in (3.34), using (3.34) and (4.57), we find

$$(4.81) \quad \underline{a}(\varepsilon k\pi + \varepsilon s_k, k\pi + s_k, y_k) = \underline{a}(\varepsilon k\pi, k\pi + s_k^x, x) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right).$$

Examine (4.74b). To interpret the quantity $|\det S_k|$, exploit Lemma 4.11 (and its proof). There remains

$$(4.82) \quad \begin{aligned} b_k(\varepsilon, t, x) &= (2\pi)^{1/2} e^{-i(-1)^k \frac{\pi}{4}} (\gamma \cos s_k^x)^{-1/2} \\ &\times \underline{a}(\varepsilon k\pi, k\pi + s_k^x, x) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+1}}\right). \end{aligned}$$

Replace x by $\varepsilon\alpha$. Coming back to (4.58), this yields

$$s_k^{\varepsilon\alpha} = (-1)^{k+1} \arcsin\left(\frac{\varepsilon\alpha}{\gamma}\right) = (-1)^{k+1} \frac{\varepsilon\alpha}{\gamma} + \mathcal{O}(\varepsilon^2).$$

It follows that

$$(4.83a) \quad \Psi_k(t, \varepsilon\alpha, s_k) = \Psi_k^0(t) - k\pi\alpha\varepsilon + \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{-1}k^{-q-1}),$$

$$(4.83b) \quad b_k(\varepsilon, t, \varepsilon\alpha) = b_k^0 + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{-1}k^{-q-1}),$$

with:

$$(4.84a) \quad \Psi_k^0(t) := -\gamma + t + (-1)^k \gamma + [1 - p(k\pi)](k\pi - t),$$

$$(4.84b) \quad b_k^0 := (2\pi)^{1/2} e^{-i(-1)^k \frac{\pi}{4}} \gamma^{-1/2} \underline{a}(\varepsilon k\pi, k\pi, 0).$$

For $k \in \mathcal{K}_s^{c_1}$, a precision like $\mathcal{O}(1/\varepsilon k^{q+1})$ is not enough, since for $k \sim \varepsilon^{-1/(q+1)}$, it is not necessarily small. By contrast, for larger k 's, assuming that $\varepsilon^{-1}\eta \leq k$ for some $\eta \in (0, 1]$, since $q > 1$, we have $\mathcal{O}(1/\varepsilon k^{q+1}) = \mathcal{O}(\varepsilon^q/\eta^{q+1})$, and therefore

$$(4.85) \quad e^{i\Psi_k(t, \varepsilon\alpha, s_k)/\varepsilon} = e^{i\Psi_k^0(t)/\varepsilon} e^{-ik\pi\alpha} + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right).$$

Note the loss of precision by the power ε^{-1} when dividing Ψ_k by ε , as well as a bad dependence upon η near $\eta = 0$ inside the last term above. For the moment, we fix some $\eta \in]0, 1]$. Back to (4.79), for $k \in \mathcal{K}_s^{c_1}$ with $k \leq \varepsilon^{-1}\eta$, just apply (4.73) in the rough form $u_k = \mathcal{O}(\varepsilon^2)$ to get

$$(4.86) \quad \sum_{\mathcal{K}_s^{c_1} \ni k \leq \varepsilon^{-1}\eta} \varepsilon b_k(\varepsilon, t, \varepsilon\alpha) e^{i\Psi_k(t, \varepsilon\alpha, s_k)/\varepsilon} = \mathcal{O}(\eta).$$

For $k \in \mathcal{K}_s^{c_1}$ with $\varepsilon^{-1}\eta \leq k$, we can separate even numbers k from odd numbers k . In other words, we can split \mathcal{K}_s^c into $\mathcal{K}_s^c(e) \cup \mathcal{K}_s^c(o)$ with

$$\mathcal{K}_s^c(e) := \{k \in \mathcal{K}_s^c; k \text{ is even}\}, \quad \mathcal{K}_s^c(o) := \{k \in \mathcal{K}_s^c; k \text{ is odd}\}.$$

By this way, using (4.79), (4.83b), (4.85) and (4.86), we get

$$(4.87) \quad \begin{aligned} \varepsilon^{-1} u(t, \varepsilon \alpha) &= \sum_{\text{par} \in \{e, o\}} \sum_{\varepsilon^{-1} \eta \leq k \in \mathcal{K}_s^{c_1}(\text{par})} \varepsilon b_k^0 e^{i \Psi_k^0(t)/\varepsilon} e^{-i k \pi \alpha} \\ &+ \mathcal{O}(\eta) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right) + \mathcal{O}\left(\frac{\varepsilon^q}{\eta^{q+1}}\right). \end{aligned}$$

Now, we consider the dependence on k when computing the two sums inside (4.87):

- i) For $\alpha = 2j$ as required in (4.77), we have to deal with $e^{-i k \pi \alpha} = e^{-i 2(kj)\pi} = 1$. The phase shift induced by the spatial inhomogeneities of φ is not detected;
- ii) For $\text{par} = e$ or $\text{par} = o$, the power $(-1)^k$ inside (4.84) is simply 1 or -1 ;
- iii) According to $k \in \mathcal{K}_s^{c_1}(e)$ or $k \in \mathcal{K}_s^{c_1}(o)$, we can replace $\underline{a}(\varepsilon k \pi, k \pi, 0)$ by $\underline{a}(\varepsilon k \pi, 0, 0)$, or by $\underline{a}(\varepsilon k \pi, \pi, 0)$, respectively.

After that, a dependence upon k remains inside (4.87). It is examined in detail below. In view of (3.28), remark that

$$(4.88) \quad \begin{aligned} [1 - p(k\pi)] (k\pi - t) &= -\frac{\ell}{q(q+1)} \left(\frac{1}{(k\pi)^{q-1}} - \frac{t}{(k\pi)^q} \right) \\ &+ \left(\frac{t}{k^q} - \frac{\pi}{k^{q-1}} \right) o(1). \end{aligned}$$

For k with $\varepsilon^{-1} \eta \leq k \in \mathcal{K}_s^{c_1}(e)$, since $2 \leq q$, exploiting (4.84) and (4.88) together with ii) and iii), we can deduce that

$$(4.89) \quad \begin{aligned} b_k^0 e^{\frac{i}{\varepsilon} \Psi_k^0(\frac{T}{\varepsilon})} &= \sqrt{\frac{2\pi}{\gamma}} \underline{a}(\varepsilon k \pi, 0, 0) e^{-i \frac{\pi}{4} + \frac{iT}{\varepsilon^2}} e^{-i \frac{\ell}{q(q+1)} \left(\frac{1}{(\varepsilon k \pi)^{q-1}} - \frac{T}{\varepsilon^2 (\varepsilon k \pi)^q} \right)} \\ &+ \frac{T}{\eta^q} o(\varepsilon^{q-2}) + \frac{1}{\eta^{q-1}} o(\varepsilon^{q-2}). \end{aligned}$$

Introduce the symbols $\mathcal{O}_\eta(\varepsilon^k)$ and $o_\eta(\varepsilon^k)$ to mean respectively $C(\eta) \mathcal{O}(\varepsilon^k)$ and $C(\eta) o(\varepsilon^k)$ for some constant $C(\eta)$ which may go to $+\infty$ when η goes to zero. When summing even k at the level of the first sum inside (4.87), with $\text{par} = e$, we recognize a Riemann sum with small width $\varepsilon 2\pi$. Since the regularity of the integrand degenerates at $s = 0$, the rate of convergence is simply $\mathcal{O}_\eta(\varepsilon)$. By this way, in view of (4.89), when $q = 2$, we obtain

$$\begin{aligned} \sum_{\varepsilon^{-1} \eta \leq k \in \mathcal{K}_s^{c_1}(e)} \varepsilon b_k^0 e^{\frac{i}{\varepsilon} \Psi_k^0(\frac{T}{\varepsilon})} &= \frac{e^{i(\frac{T}{\varepsilon^2} - \frac{\pi}{4})}}{\sqrt{2\pi\gamma}} \int_{\pi\eta}^{+\infty} e^{-i \frac{\ell}{6} (\frac{1}{s} - \frac{T}{s^2})} \underline{a}(s, 0, 0) ds \\ &+ o_\eta(\varepsilon^0) + \mathcal{O}_\eta(\varepsilon), \end{aligned}$$

where we have used the property that $k\pi\varepsilon$ goes up to \mathcal{T} , so the integral carries over the whole support of $\underline{a}(\cdot, 0, 0)$. When $q > 2$, since $\eta \leq \varepsilon k \leq \mathcal{T}$, observe that (4.89) involves the factor

$$e^{-i \frac{\ell}{q(q+1)} \left(\frac{1}{(\varepsilon k \pi)^{q-1}} - \frac{T}{\varepsilon^2 (\varepsilon k \pi)^q} \right)} = e^{-i \frac{\ell}{q(q+1)} \left(\frac{1}{(\pi \varepsilon k)^{q-1}} - \frac{T}{(\pi \varepsilon k)^q} \right)} \varepsilon^{q-2} = 1 + \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^q}\right).$$

Hence, when $q > 2$, the Riemann sum argument together with (4.89) yields simply

$$\sum_{\varepsilon^{-1} \eta \leq k \in \mathcal{K}_s^{c_1}(e)} \varepsilon b_k^0 e^{\frac{i}{\varepsilon} \Psi_k^0(\frac{T}{\varepsilon})} = \frac{e^{i(\frac{T}{\varepsilon^2} - \frac{\pi}{4})}}{(2\pi)^{1/2} \gamma^{1/2}} \int_{\pi\eta}^{+\infty} \underline{a}(s, 0, 0) ds + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^q}\right),$$

where we have used the fact that for a smooth integrand (as it is the case with \underline{a} only, that is, without the above singular phase term), a convergence rate (of the order of the discretization parameter) is available in Riemann sums. In the two preceding integrals, the integrand is bounded near $s = 0$. Modulo some $\mathcal{O}(\eta)$, we can integrate from 0 to $+\infty$. Similar considerations apply when dealing with odd values of k . Summing up, we find the leading-order term of (4.77).

Now, come back to (4.87). From the preceding estimates, when $q = 2$, the error term is of the type

$$\varepsilon \left[\mathcal{O}(\eta) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right) + \mathcal{O}\left(\varepsilon^{\frac{q}{q+1}}\right) + o_\eta(\varepsilon^0) + \mathcal{O}_\eta(\varepsilon) \right] = \varepsilon \left[\mathcal{O}(\eta) + o_\eta(\varepsilon^0) \right].$$

This is valid for all $\eta \in]0, 1]$. By fixing η increasingly smaller and then letting ε go to zero, this implies the bound $o(\varepsilon)$, as expected in (4.77). On the contrary, when $q > 2$, we have to deal with an error term like

$$\varepsilon \left(\mathcal{O}(\eta) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right) + \mathcal{O}\left(\varepsilon^{\frac{q}{q+1}}\right) + \mathcal{O}(\varepsilon) + \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^q}\right) \right).$$

Setting $\eta = \varepsilon^{\frac{q-2}{q+1}}$ then yields (4.78). \square

The set $\mathcal{C}_c^0(\mathbb{R}_+)$ of all continuous functions having a compact support is a Banach space when it is equipped with the sup-norm. Define $\lambda(q) = \ell$ if $q = 2$ (with ℓ as in Assumption 3.3) and $\lambda(q) = 0$ if $q > 2$. Given $T \geq 0$ and $q \geq 2$, consider the nontrivial continuous linear form

$$\begin{aligned} \mathcal{L}(T) : \mathcal{C}_c^0(\mathbb{R}_+) &\longrightarrow \mathbb{R} \\ \underline{a} &\longmapsto \mathcal{L}(T)(\underline{a}) := \int_0^{+\infty} e^{-i \frac{\lambda(q)}{6} \left(\frac{1}{s} - \frac{T}{s^2}\right)} \underline{a}(s) ds. \end{aligned}$$

Its kernel $\ker \mathcal{L}(T)$ is a closed vector space of codimension one. Obviously, the complement $(\ker \mathcal{L}(T))^c$ of $\ker \mathcal{L}(T)$ is dense so that, generically, $\underline{a} \in (\ker \mathcal{L}(T))^c$.

Corollary 4.17 (Constructive interference). *Fix any $T \in [\mathcal{T}, 2\mathcal{T}]$. Select \underline{a} as in Assumption 3.12, with moreover $\underline{a}(\cdot, 0, 0)$ or $\underline{a}(\cdot, \pi, 0)$ in $(\ker \mathcal{L}(T))^c$. Looking at the solution u to (4.2) at the time T/ε and at well chosen positions (which may depend on the parameter ε), one can observe some amplification of the sup norm. As a matter of fact, for all $j \in \mathbb{Z}$, we have:*

$$(4.90) \quad \limsup_{\varepsilon \rightarrow 0^+} \left| \frac{1}{\varepsilon} u\left(\frac{T}{\varepsilon}, 2\varepsilon j\right) \right| = \ell_s^{\underline{a}} \neq 0$$

with

$$\ell_s^{\underline{a}} := \frac{1}{\sqrt{2\pi\gamma}} [|\mathcal{L}(T)(\underline{a}(\cdot, 0, 0))| + |\mathcal{L}(T)(\underline{a}(\cdot, \pi, 0))|].$$

Assume moreover that $\underline{a}(\cdot, 0, 0)$ is π -periodic. Then, (4.77) gives rise to

$$(4.91) \quad \left| \frac{1}{\varepsilon} u\left(\frac{T}{\varepsilon}, 2\varepsilon j\right) \right| = \frac{2}{\sqrt{2\pi\gamma}} |\mathcal{L}(T)(\underline{a}(\cdot, 0, 0))| \left| \cos\left(\gamma - \frac{\pi}{4}\right) \frac{1}{\varepsilon} \right| + o(1).$$

Therefore, any number contained in the interval $[0, \ell_s^{\underline{a}}]$ is an adherent point of the family $\{\varepsilon^{-1} |u(\varepsilon^{-1} T, 2\varepsilon j)|\}_{\varepsilon}$. This is typical of a highly oscillating behavior. As mentioned before, the formula (1.16) looks like (4.77) and (4.78). But, as will be seen in the next paragraph, outside the moving lattice $\{2\varepsilon j; j \in \mathbb{Z}\}$, the situation is completely different.

4.5.3. *Destructive interferences.* In this paragraph, we consider the situation where $x = \alpha \varepsilon$ with $\alpha \in \mathbb{R} \setminus \{2\mathbb{Z}\}$. Then, the property i) in the proof of Proposition 4.16 no longer applies. The definition (4.71) of u_k does contain the factor $e^{-ik\pi\alpha}$ which comes from the spatial inhomogeneities of the phase φ and which, after summation, can induce additional cancellations.

Proposition 4.18 (Destructive interference). *Select any $\alpha \in \mathbb{R} \setminus \{2\mathbb{Z}\}$. Suppose that Assumptions 3.3, 3.6, 3.8 (with $D \geq 4$) and 3.12 are satisfied. Then for all $T \in [\mathcal{T}, 2\mathcal{T}]$, the solution u to (4.14) is such that*

$$(4.92) \quad u\left(\frac{T}{\varepsilon}, \alpha\varepsilon\right) = \begin{cases} o(\varepsilon) & \text{if } q = 2, \\ \mathcal{O}\left(\varepsilon^{\frac{6q-2}{5q}}\right) & \text{if } q > 2. \end{cases}$$

Proof. We resume (4.87), which holds for all $\eta > 0$. Exploiting (4.88) and (4.89), this becomes

$$\begin{aligned} \frac{1}{\varepsilon} u\left(\frac{T}{\varepsilon}, \alpha\varepsilon\right) &= (2\pi)^{1/2} \gamma^{-1/2} e^{i\left(\frac{T}{\varepsilon^2} - \frac{\pi}{4}\right)} \sum_{\substack{\eta/\varepsilon \leq k \text{ even} \leq \frac{T}{\pi\varepsilon}} \varepsilon e^{-ik\pi\alpha} G_q^e(\varepsilon, \varepsilon k\pi) \\ &\quad + (2\pi)^{1/2} \gamma^{-1/2} e^{i\left(\frac{T}{\varepsilon^2} - \frac{\pi}{4} - \frac{2\gamma}{\varepsilon}\right)} \sum_{\substack{\eta/\varepsilon \leq k \text{ odd} \leq \frac{T}{\pi\varepsilon}} \varepsilon e^{-ik\pi\alpha} G_q^o(\varepsilon, \varepsilon k\pi) \\ &\quad + \mathcal{O}(\eta) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right) + \mathcal{O}\left(\varepsilon^{\frac{q}{q+1}}\right) + o\left(\frac{\varepsilon^{q-2}}{\eta^q}\right), \end{aligned}$$

where by definition

$$G_q^e(\varepsilon, s) := e^{-i\frac{\ell}{6}\left(\frac{1}{s} - \frac{T}{s^2}\right)\varepsilon^{q-2}} \underline{a}(s, 0, 0), \quad G_q^o(\varepsilon, s) := e^{-i\frac{\ell}{6}\left(\frac{1}{s} - \frac{T}{s^2}\right)\varepsilon^{q-2}} \underline{a}(s, \pi, 0).$$

We consider separately the two above sums. We discuss the case k even, the case k odd being similar. The idea is to use Abel's summation formula. To this end, given some $\delta \in]0, 1]$, we interpret the sum as follows

$$\sum_{\substack{\eta/\varepsilon \leq k \text{ even} \leq \frac{T}{\pi\varepsilon}} \varepsilon e^{-ik\pi\alpha} G_q^e(\varepsilon, \varepsilon k\pi) = \sum_{\substack{\eta/\delta \leq j \leq \frac{T}{2\pi\delta}} \sum_{\substack{j\delta \leq k \text{ even} \leq \frac{(j+1)\delta}{\varepsilon}}} \varepsilon e^{-ik\pi\alpha} G_q^e(\varepsilon, \varepsilon k\pi).$$

For all j , fix some k_j even inside $\left[\frac{j\delta}{\varepsilon}, \frac{(j+1)\delta}{\varepsilon}\right]$. For all k in this interval, Taylor's formula gives rise to

$$|G_q^e(\varepsilon, \varepsilon k\pi) - G_q^e(\varepsilon, \varepsilon k_j\pi)| \leq \pi\delta \sup_{s \geq \pi\eta} |\partial_s G_q^e(\varepsilon, s)| = \delta \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^3}\right) + \delta \mathcal{O}(1).$$

It follows that

$$\sum_{\substack{\eta/\varepsilon \leq k \text{ even} \leq \frac{T}{\pi\varepsilon}} \varepsilon e^{-ik\pi\alpha} G_q^e(\varepsilon, \varepsilon k_j\pi) = \mathcal{E}_r^e + \sum_{\eta/\delta \leq j \leq \frac{T}{2\pi\delta}} \varepsilon G_q^e(\varepsilon, \varepsilon k_j\pi) \times \sum_{\substack{j\delta \leq k \text{ even} \leq \frac{(j+1)\delta}{\varepsilon}}} e^{-ik\pi\alpha}.$$

The error term \mathcal{E}_r^e can be estimated according to

$$\mathcal{E}_r^e = \frac{\mathcal{T}}{2\pi\delta} \frac{\delta}{\varepsilon} \varepsilon \left[\delta \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^3}\right) + \delta \mathcal{O}(1) \right] = \delta \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^3}\right) + \delta \mathcal{O}(1).$$

Since $e^{-i\pi\alpha} \neq 1$, we have

$$\left| \sum_{\eta/\delta \leq j \leq \frac{T}{2\pi\delta}} \varepsilon G_q^e(\varepsilon, \varepsilon k_j\pi) \times \sum_{\substack{j\delta \leq k \text{ even} \leq \frac{(j+1)\delta}{\varepsilon}}} e^{-ik\pi\alpha} \right| = \frac{1}{\delta} \mathcal{O}(\varepsilon).$$

In short, we have

$$\begin{aligned} \frac{1}{\varepsilon} \left| u\left(\frac{T}{\varepsilon}, \alpha \varepsilon\right) \right| &= \mathcal{O}(\eta) + \mathcal{O}\left(\frac{\varepsilon^{q-1}}{\eta^{q+1}}\right) + \mathcal{O}\left(\varepsilon^{\frac{q}{q+1}}\right) + o\left(\frac{\varepsilon^{q-2}}{\eta^q}\right) \\ &\quad + \delta \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^3}\right) + \delta \mathcal{O}(1) + \frac{1}{\delta} \mathcal{O}(\varepsilon). \end{aligned}$$

This is valid for all $(\eta, \delta) \in]0, 1]^2$. We fix $\delta = \eta^4$ so that

$$\mathcal{O}(\eta) + \delta \mathcal{O}\left(\frac{\varepsilon^{q-2}}{\eta^3}\right) + \delta \mathcal{O}(1) + \frac{1}{\delta} \mathcal{O}(\varepsilon) = \mathcal{O}(\eta) + \frac{1}{\eta^4} \mathcal{O}(\varepsilon).$$

By fixing η increasingly smaller and then letting ε goes to zero, we can recover some $o(\varepsilon^0)$ or, after multiplication by ε , some $o(\varepsilon)$ as announced in (4.92). When $q > 2$, a better estimate is available by optimizing the choice of η . Just take $\eta = \varepsilon^{(q-2)/5q}$ to obtain (4.92). \square

Remark 4.19 (Contrast between constructive and destructive interferences). *The controls of the error terms inside (4.77), (4.78) and (4.92) are not claimed to be sharp. For instance, by specifying a rate of convergence at the level of (3.22), the precision $o(\varepsilon)$ in (4.77) and (4.92) could be improved into $\mathcal{O}(\varepsilon^{1+\kappa})$ for some $\kappa > 0$. At all events, the amplitude of the solution u to (4.2) is asymptotically maximal on a set of Lebesgue measure zero, which is the lattice $\varepsilon\mathbb{Z}$ moving with $\varepsilon \in]0, 1]$. Everywhere else, it is smaller.*

5. NONLINEAR ANALYSIS

In this chapter, we prove the nonlinear information (2) of Theorem 1.3, as well as Theorem 1.4. In Section 5.1, we precise the framework, and we collect various estimates about the solution $u^{(0)}$ of (1.28a). In Section 5.2, we measure the influence of different types of nonlinearity according to gauge parameters \mathfrak{g} that characterize them. We prove that nonlinear effects are not detected at leading order as long as $\mathfrak{g} \neq 1$. This is Fact 2 in the PDE context. As a consequence, when $\mathfrak{g} \neq 1$, the distinction between constructive and destructive interferences remains in the same state as in the linear case. This dichotomy does persist when $\mathfrak{g} = 1$. But, as will be seen in Section 5.3, the profiles exhibited in (4.77) must be modified accordingly, in order to take into account the nontrivial effects of nonlinearity.

5.1. General setting. In Paragraph 5.1.1, we recall the main assumptions, and we start the discussion about nonlinear effects. In Paragraph 5.1.2, we study the kernel of a singular operator, which appears when seeking sup norm estimates. In Paragraph 5.1.3, we classify the different sorts of gauge parameters, and we illustrate them by examples. In Paragraph 5.1.4, we establish various estimates concerning the solution $u^{(0)}$ of (1.28a).

5.1.1. Main assumptions. We work under the hypotheses of Theorems 1.3 and 1.4 concerning p and φ . In particular, we suppose that $q = 2$ and $D \geq 4$. The phase φ is subject to Assumption 1.2. The expression $u^{(0)}$ is obtained by solving the linear equation (1.28a), with F_L as in (3.31). In (3.31), the sum is assumed to be finite, to avoid extra discussions about the convergence of infinite sums which can appear in the approximating process.

In view of Propositions 4.16 and 4.18, the function $u^{(0)}$ is of size ε in L^∞ . It follows that the quadratic nonlinearity of (1.36) can be expected to play a role at leading order for long times $t \sim \varepsilon^{-1}$. The right hand side of (1.36) may seem quite specific. It is adjusted in order to generate through (1.36) a solution $u^{(1)}$ of size comparable to $u^{(0)}$. To understand why, and also to discern the possible effects of other nonlinearities, it is interesting to generalize (1.36) up to some extent. With this in mind, we replace (1.36) by (1.28b) with F_{NL} satisfying Assumption 3.13. Following (1.29), we introduce

$$(5.1) \quad u^{(j)}(t, x) = \varepsilon e^{it/\varepsilon} \mathcal{U}^{(j)}\left(\varepsilon t, \frac{x}{\varepsilon}\right), \quad \mathcal{U}^{(j)}(T, z) = \frac{1}{\varepsilon} e^{-iT/\varepsilon^2} u^{(j)}\left(\frac{T}{\varepsilon}, \varepsilon z\right).$$

The solution $u^{(1)}$ to (1.28b) is a superposition of the contributions brought by the different terms $F_{j_1 j_2 \nu}$ composing F_{NL} , see (3.36). Thus, we can study separately what happens for a fixed choice of $(j_1, j_2, \nu) \in \mathbb{N}^2 \times \mathbb{Z}$. With this in mind, we focus our attention on a single monomial having the form

$$(5.2) \quad F_{NL} \equiv F_{j_1 j_2 \nu} = \varepsilon^\nu e^{i\omega t/\varepsilon} \chi\left(3 - 2\frac{\varepsilon t}{T}\right) \chi\left(\frac{x}{r\varepsilon^\iota}\right) u^{j_1} \bar{u}^{j_2}.$$

We have seen in Subsection 1.1 that the gauge parameter \mathfrak{g} is a good indicator of the time oscillations which remain in the source term of equation (1.5) after filtering out of the equation (1.3) through the change (1.4). When dealing with (5.2), a similar definition applies.

Definition 5.1 (Gauge parameter). *The gauge parameter associated with $F_{j_1 j_2 \nu}$ is the real number $\mathfrak{g}_{j_1 j_2 \nu}$ defined by $\mathfrak{g}_{j_1 j_2 \nu} := \omega + j_1 - j_2$.*

From now on, we fix F_{NL} as in (5.2) with indices ν , j_1 and j_2 adjusted in such a way that $\nu + j_1 + j_2 \geq 2$. We will sometimes simply note $\mathfrak{g} \equiv \mathfrak{g}_{j_1 j_2 \nu} \in \mathbb{R}$. In (5.2), the coefficient which appears in front of $u^{j_1} \bar{u}^{j_2}$ is the product of three factors.

In the light of the first factor $e^{i\omega t/\varepsilon}$, in the case of a non-zero frequency $\omega \neq 0$, the source term F_{NL} does involve time oscillations. Reasons for introducing $e^{i\omega t/\varepsilon}$ have been explained in Remark 2.2, and also in Paragraph 3.2.1 when adjusting p in order to recover (3.24).

Looking at the second factor, the source term F_{NL} is switched on after all signals have been emitted, that is during the long time interval $[T/\varepsilon, 2T/\varepsilon]$, which could be replaced by $[\eta/\varepsilon, 1/(\eta\varepsilon)]$ for any $\eta \in]0, 1]$. But a positive gap ($\eta > 0$) seems to be needed. Indeed, Lemma 4.7 makes a first group of wave packets which, due to a dispersive phenomenon, is negligible in the limit $\varepsilon \rightarrow 0$. It requires to consider sufficiently large times, so the phase Φ_k could be uniformly non-stationary in ξ .

In the light of the third factor, the source term F_{NL} is spatially localized in a ball of size $r\varepsilon^\iota$. The impact of F_{NL} is potentially all the more stronger that ι is small. The choice of a large negative parameter ι , with $\iota \ll -1$, involves almost no spatial localization. The case $\iota = 0$ corresponds to a diluted source which acts on the domain where Propositions 4.16 and 4.18 furnish some refined information. Finally, the selection of the limiting value $\iota = 1$ implies a concentrated source which, for convenience, is placed here at the origin. Larger values of ι , with $\iota \geq 1$, will not be investigated because they have little interest.

The impact of the nonlinearity (5.2) can be measured by looking at the difference $\mathcal{W} := \mathcal{U}^{(1)} - \mathcal{U}^{(0)}$. From (1.28a) and (1.28b), it is easy to deduce that

$$(5.3) \quad \partial_T \mathcal{W} - \frac{i}{\varepsilon^2} (p(-i \partial_z) - 1) \mathcal{W} = \varepsilon^{\nu+j_1+j_2-2} e^{i(\mathfrak{g}-1)T/\varepsilon^2} \mathcal{G}^\varepsilon, \quad \mathcal{W}|_{t=0} \equiv 0,$$

where $\mathfrak{g} \equiv \mathfrak{g}_{j_1 j_2 \nu}$ is as in Definition 5.1, and the source term \mathcal{G}^ε is determined by

$$(5.4) \quad \mathcal{G}^\varepsilon(T, z) = \chi\left(3 - 2\frac{T}{\mathcal{T}}\right) \chi\left(\frac{z}{r\varepsilon^{\iota-1}}\right) \mathcal{U}^{(0)}(T, z)^{j_1} \bar{\mathcal{U}}^{(0)}(T, z)^{j_2}.$$

By construction, the function \mathcal{G}^ε is smooth and compactly supported in (T, z) . We have seen in Section 4 that $\mathcal{U}^{(0)}(T, z)$, and therefore $\mathcal{G}^\varepsilon(T, z)$, is some $\mathcal{O}(1)$ as long as (T, z) is such that $\mathcal{T} \leq T \leq 2\mathcal{T}$ and $|z| \leq r/\varepsilon$. Moreover, this control is sharp when $z \in \mathbb{Z}$. Coming back to (5.3), Duhamel's formula reads

$$(5.5) \quad \begin{aligned} \mathcal{W}(T, z) &= \varepsilon^{\nu+j_1+j_2-2} (2\pi)^{-1} \\ &\times \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1) + i(\mathfrak{g}-1)\frac{s}{\varepsilon^2}} \mathcal{G}^\varepsilon(s, y) ds dy d\xi. \end{aligned}$$

Our aim is to study $\mathcal{W}(T, z)$ through (5.5). As a first step, we would like to establish that, for \mathcal{G}^ε as in (5.4), we have

$$(5.6) \quad \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1) + i(\mathfrak{g}-1)\frac{s}{\varepsilon^2}} \mathcal{G}^\varepsilon(s, y) ds dy d\xi = \mathcal{O}(1) \text{ in } L^\infty.$$

We know already that the function \mathcal{G}^ε is of size 1 at integer points, and that it is of smaller amplitude at all other spatial positions. Thus, the matter is to understand how the integral operator inside (5.6) acts on L^∞ . The main problem when dealing with (5.6) is the global domain of integration in ξ and (for $\iota < 1$) the large domain (of size $\varepsilon^{\iota-1}$) of integration in y . This difficulty is examined, and partly solved, in the next paragraph.

5.1.2. *A singular integral operator.* Following the convention (3.15), denote by \mathcal{F}_\star the partial Fourier transform with respect to the variable $\star \in \{y, \xi\}$. Given $\tau \in \mathbb{R}$ and $\Lambda \in L^\infty(\mathbb{R})$, define the operator B_τ^Λ by

$$(5.7) \quad \begin{aligned} B_\tau^\Lambda \mathcal{G}^\varepsilon(z) &:= \mathcal{F}_\xi(2\pi(e^{i\tau(p(\xi)-1)} - 1)\Lambda(\xi)(\mathcal{F}_y^{-1}\mathcal{G}^\varepsilon)(\xi))(z) \\ &= \iint e^{-i(z-y)\xi} (e^{i\tau(p(\xi)-1)} - 1)\Lambda(\xi)\mathcal{G}^\varepsilon(y) dy d\xi. \end{aligned}$$

When $\Lambda \equiv \mathbf{1}_\mathbb{R}$, the operator B_τ^Λ is simply denoted by $B_\tau := B_\tau^{\mathbf{1}_\mathbb{R}}$. Looking at (5.7), it is clear that $B_\tau^\Lambda : H^\sigma(\mathbb{R}) \rightarrow H^\sigma(\mathbb{R})$ is a bounded operator for all $\sigma \in \mathbb{R}$, with

$$(5.8) \quad \|B_\tau^\Lambda \mathcal{G}^\varepsilon\|_{H^\sigma(\mathbb{R})} \leq 2\|\Lambda\|_{L^\infty(\mathbb{R})} \|\mathcal{G}^\varepsilon\|_{H^\sigma(\mathbb{R})}.$$

In (5.7), we first integrate in y and then in ξ . Another viewpoint, which is more adapted to get L^∞ -estimates, is to first integrate in ξ and then in y . By this way, we find $B_\tau^\Lambda \mathcal{G}^\varepsilon = K_\tau^\Lambda * \mathcal{G}^\varepsilon$ with a kernel K_τ^Λ given by

$$(5.9) \quad K_\tau^\Lambda(y) := \int e^{-iy\xi} (e^{i\tau(p(\xi)-1)} - 1)\Lambda(\xi) d\xi, \quad K_\tau := K_\tau^{\mathbf{1}_\mathbb{R}}.$$

In view of (3.28), where $\omega_+^\infty = 1$ and $q = 2$, the integrand inside (5.9) is integrable, and it depends smoothly on the parameters τ and y . The expression $K_\tau^\Lambda(y)$ is therefore well defined. It is continuous with respect to (τ, y) , and in view of (3.28),

$$(5.10) \quad |K_\tau^\Lambda(y)| \leq C|\tau| \left(\int \frac{d\xi}{1+\xi^2} \right) \|\Lambda\|_{L^\infty(\mathbb{R})}.$$

Now, the solution \mathcal{W} to (5.3) can be decomposed into $\mathcal{W}_l + \mathcal{W}_{nl}$ with

$$(5.11) \quad \mathcal{W}_l(T, z) := \varepsilon^{\nu+j_1+j_2-2} \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} \mathcal{G}^\varepsilon(s, z) ds,$$

$$(5.12) \quad \mathcal{W}_{nl}(T, z) := \frac{\varepsilon^{\nu+j_1+j_2-2}}{2\pi} \int_0^T e^{i(\mathfrak{g}-1)s/\varepsilon^2} B_{(T-s)/\varepsilon^2} \mathcal{G}^\varepsilon(s, z) ds.$$

By this way, the improper integral inside (5.5) is defined without ambiguity. Indeed, both (5.11) and (5.12) involve local integrals with respect to s of bounded functions. For (5.11), this is obvious since $\mathcal{G}^\varepsilon(s, z) = \mathcal{O}(1)$. Concerning (5.12), this results from the pointwise estimate

$$|B_{(T-s)/\varepsilon^2} \mathcal{G}^\varepsilon(s, z)| \leq \|K_{(T-s)/\varepsilon^2}\|_{L^\infty(\mathbb{R})} \|\mathcal{G}^\varepsilon(s, \cdot)\|_{L^1(\mathbb{R})} \lesssim \varepsilon^{\iota-3}.$$

Let us look more closely at $\mathcal{W}_l(T, z)$. To get $\mathcal{W}_l(T, z)$, it suffices to know $\mathcal{G}^\varepsilon(\cdot, z)$, that is $\mathcal{U}^{(0)}(\cdot, z)$. In this sense, the action on \mathcal{G}^ε leading to \mathcal{W}_l is local in space, and therefore it is consistent with the dichotomy between constructive and destructive interferences exhibited in Propositions 4.16 and 4.18. In fact, a precise asymptotic description of \mathcal{W}_l is available.

Lemma 5.2 (Description of the part \mathcal{W}_l). *We work under Assumption 3.12, with moreover $\underline{a}(T, \cdot, x)$ periodic of period π . Then, for all $z \in \mathbb{R} \setminus \{2\mathbb{Z}\}$, we find that $\mathcal{W}_l(T, z) = o(1)$. When $\nu + j_1 + j_2 > 2$ or when $\mathfrak{g} \neq 1$, for all $z = 2j$ with $j \in \mathbb{Z}$, we have again $\mathcal{W}_l(T, 2j) = o(1)$. On the contrary, when $\iota \in [0, 1[$, $\nu + j_1 + j_2 = 2$ and $\mathfrak{g} = 1$, we obtain that*

$$(5.13) \quad \mathcal{W}_l(T, 2j) = o(1) + \left[\sqrt{\frac{2}{\pi\gamma}} \cos\left(\frac{\gamma}{\varepsilon} - \frac{\pi}{4}\right) \right]^{j_1+j_2} e^{i(j_2-j_1)\gamma/\varepsilon} \int_0^T \chi\left(3 - 2\frac{s}{T}\right) \times \\ \left(\int_0^{+\infty} e^{-i\frac{\xi}{6}\left(\frac{1}{\sigma_1} - \frac{s}{\sigma_1^2}\right)} \underline{a}(\sigma_1, 0, 0) d\sigma_1 \right)^{j_1} \left(\int_0^{+\infty} e^{i\frac{\xi}{6}\left(\frac{1}{\sigma_2} - \frac{s}{\sigma_2^2}\right)} \underline{a}(\sigma_2, 0, 0) d\sigma_2 \right)^{j_2} ds.$$

Lemma 5.2 is instructive. It indicates, among other things, that the constructive interferences do not impact \mathcal{W}_l when $\mathfrak{g} \neq 1$. As will be seen, this principle also applies to \mathcal{W}_{nl} .

Proof. First, recall that $\mathcal{W}_l(T, z) = 0$ when $0 \leq T \leq \mathcal{T}$. For $\mathcal{T} \leq T$, observe that

$$(5.14) \quad |\mathcal{W}_l(T, z)| \leq \varepsilon^{\nu+j_1+j_2-2} \int_{\mathcal{T}}^T |\mathcal{U}^{(0)}(s, z)|^{j_1+j_2} ds = \mathcal{O}(\varepsilon^{\nu+j_1+j_2-2}).$$

When $\nu + j_1 + j_2 > 2$, the smallness of $\mathcal{W}_l(T, z)$ follows directly from (5.14). Now, assume that $\nu + j_1 + j_2 = 2$. The first assertion of Lemma 5.2, the one implying positions $z \in \mathbb{R} \setminus \{2\mathbb{Z}\}$, is a direct consequence of (5.14), Proposition 4.18 and Lebesgue's dominated convergence theorem. Finally, consider the case $z = 2j$ with $j \in \mathbb{Z}$. From the forthcoming bound (5.29), that will be derived independently in Paragraph 5.1.4, we know that

$$|\partial_s \mathcal{G}^\varepsilon(s, z)| = \mathcal{O}(|\partial_s \mathcal{U}^{(0)}(s, z)|) = \mathcal{O}(\varepsilon^{-2/3}).$$

When $\mathfrak{g} \neq 1$, an integration by parts in s performed at the level of (5.11) indicates that $|\mathcal{W}_l(s, z)| = \mathcal{O}(\varepsilon^{4/3})$. When $\mathfrak{g} = 1$, the time oscillating factor disappears from (5.11). Plug (4.77) into (5.1) to recover an asymptotic description of $\mathcal{U}^{(0)}$. When $\iota \in [0, 1[$, for small values of ε , we find $\chi(2j/r\varepsilon^{\iota-1}) = 1$, yielding (5.13). \square

It should be noticed that the formula (5.13) with $(j_1, j_2) = (2, 0)$ differs from (1.37). In (5.13), the two integrals in $d\sigma_1$ and $d\sigma_2$ are separated while, at the level of (1.37), they are correlated through a nontrivial factor. The reason of this difference is that the contribution \mathcal{W}_{nl} is not at all a small perturbation of \mathcal{W}_l . As will be seen, the decomposition of \mathcal{W} into \mathcal{W}_l and \mathcal{W}_{nl} is suitable to show (at least when $\mathfrak{g} \neq 1$) the sup norm decreasing of \mathcal{W} . But it is not sufficiently precise to obtain (1.37). When $\mathfrak{g} = 1$, the two terms \mathcal{W}_l and \mathcal{W}_{nl} combine asymptotically to form (1.37), which provides with the correct prediction.

There remains to study \mathcal{W}_{nl} . The access to \mathcal{W}_{nl} is more complicated than for \mathcal{W}_l . Indeed, in (5.12), the action of B_τ is nl on local in space, and it is also singular in terms of ε when ε goes to zero. Let us examine this in more detail. From Young's convolution inequality, we know that

$$(5.15) \quad \|B_\tau^\Lambda \mathcal{G}^\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|K_\tau^\Lambda\|_{L^p(\mathbb{R})} \|\mathcal{G}^\varepsilon\|_{L^{p/(p-1)}(\mathbb{R})}, \quad \forall p \in [1, +\infty].$$

Come back to (5.12). Since τ is aimed to be replaced by $(T-s)/\varepsilon^2$, the access to \mathcal{W}_{nl} needs to consider large values of τ , say $\tau \in [1, +\infty[$.

Lemma 5.3 (Estimates on the L^2 and L^∞ norms of the kernel K_τ^Λ). *Fix $\rho \geq 0$, and assume that $\Lambda \in L^\infty(\mathbb{R})$ is such that $\Lambda(\xi) = \mathcal{O}(|\xi|^{-\rho})$ when $|\xi|$ goes to $+\infty$. Denote by $\tilde{q} := \tilde{p}/(\tilde{p}-1)$ the Hölder conjugate of \tilde{p} . By convention, we have $\tilde{q} = 1$ when $\tilde{p} = +\infty$, and $\tilde{q} = 2$ when $\tilde{p} = 2$. Then, for large values of τ , we find*

$$(5.16) \quad \forall \tilde{p} \in \{+\infty, 2\}, \quad \|K_\tau^\Lambda\|_{L^{\tilde{p}}(\mathbb{R})} \lesssim 1 + \tau^{(1/\tilde{q}\tilde{q}) - (\rho/q)},$$

where $q \geq 2$ is the number stemming from (3.19).

When Λ is just bounded ($\rho = 0$), the estimate (5.16) helps control the explosion when $\tau \rightarrow +\infty$ of the $L^{\tilde{p}}(\mathbb{R})$ -norm of K_τ^Λ . The situation is improving when $\rho > 0$. In particular, when $\tilde{q} = 1$ and $\rho = 1$, the family $(K_\tau^\Lambda)_\tau$ is bounded in $L^\infty(\mathbb{R})$.

Proof. We can assert that

$$\|K_\tau^\Lambda\|_{L^{\tilde{p}}(\mathbb{R})} \lesssim \left(1 + \int_1^{+\infty} |1 - \cos(\tau(1-p(\xi)))|^{\tilde{q}/2} |\Lambda(\xi)|^{\tilde{q}} d\xi\right)^{1/\tilde{q}}.$$

This is obvious when $\tilde{p} = +\infty$. This is a consequence of (5.9) and Plancherel theorem when $\tilde{p} = 2$. The change of variables $\eta = 1 - p(\xi)$ sends $\xi = 1$ to the positive value $\eta_1 := 1 - p(1)$, and $\xi = +\infty$ to $\eta_\infty = 0$. It gives rise to

$$\|K_\tau^\Lambda\|_{L^{\tilde{p}}(\mathbb{R})} \lesssim \left(1 + \int_0^{\eta_1} \frac{|1 - \cos(\tau\eta)|^{\tilde{q}/2}}{p' \circ (1-p)^{-1}(\eta)} |\Lambda \circ (1-p)^{-1}(\eta)|^{\tilde{q}} d\eta\right)^{1/\tilde{q}},$$

where $(1-p)^{-1} :]0, \eta_1] \rightarrow [1, +\infty[$ is the inverse function of $1-p$. From (3.23) and (3.28), it is easy to infer that

$$\begin{aligned} \exists C > 0, \quad p' \circ (1-p)^{-1}(\eta) &\geq C\eta^{(q+1)/q}, \quad \forall \eta \in]0, \eta_1], \\ \exists C > 0, \quad |\Lambda \circ (1-p)^{-1}(\eta)|^{\tilde{q}} &\leq C\eta^{\rho\tilde{q}/q}, \quad \forall \eta \in]0, \eta_1]. \end{aligned}$$

It follows that

$$(5.17) \quad \|K_\tau^\Lambda\|_{L^{\tilde{p}}(\mathbb{R})} \lesssim \left(1 + \tau^{(1-\rho\tilde{q})/q} \int_0^{\tau\eta_1} |1 - \cos \eta|^{\tilde{q}/2} \eta^{(-q-1+\rho\tilde{q})/q} d\eta\right)^{1/\tilde{q}}.$$

The integral on the right hand side of (5.17) is convergent near $\eta_\infty = 0$ because

$$\forall \tilde{q} \in \{1, 2\}, \quad \tilde{q} - 1 - (1/q) + (\rho\tilde{q}/q) > -1.$$

When $\rho\tilde{q} = 1$, (5.16) is a direct consequence of (5.17). Otherwise, remark that

$$(5.18) \quad \int_{\eta_1}^{\tau\eta_1} |1 - \cos \eta|^{\tilde{q}/2} \eta^{(-q-1+\rho\tilde{q})/q} d\eta \leq \frac{2q}{1-\rho\tilde{q}} \eta_1^{(\rho\tilde{q}-1)/q} (1 - \tau^{-(1-\rho\tilde{q})/q}).$$

From (5.17) and (5.18), we can deduce (5.16). \square

Lemma 5.4 (Pointwise estimates on the kernel K_τ). *In the case $\Lambda \equiv \mathbf{1}_{\mathbb{R}}$ and $q = 2$, we find (for some $c \in \mathbb{C}$) that*

$$(5.19) \quad K_\tau(0) \sim c\tau^{1/2}, \quad \text{whereas: } \forall y \neq 0, \quad K_\tau(y) = \mathcal{O}(\tau^{1/6}).$$

Proof. For large values of $|\xi|$, we know that $1 - p(\xi) \sim c\xi^{-2}$ for some positive constant c , say $c = 1$. In what follows, to simplify the discussion, we directly replace $1 - p(\xi)$ by ξ^{-2} . Then, the change of variables $\tau\xi^{-2} = \eta$ gives rise to

$$K_\tau(0) = \tau^{1/2} \int_0^{+\infty} (e^{-i\eta} - 1)\eta^{-3/2} d\eta \sim c\tau^{1/2}.$$

This furnishes the left part of (5.19), and this indicates that (5.16) is sharp (at least when $\tilde{p} = +\infty$ and $\rho = 0$). Now, fix some $y \neq 0$, and decompose $K_\tau(y)$ into

$$\begin{aligned} K_\tau(y) &= \int_{|\xi| \leq 1} e^{-iy\xi} (e^{-i\tau\xi^{-2}} - 1) d\xi + \frac{1}{iy} \int_{1 < |\xi| \leq c\tau^{1/3}} \partial_\xi (e^{-iy\xi}) d\xi \\ &\quad + \int_{1 < |\xi| \leq c\tau^{1/3}} e^{-i(y\xi + \tau\xi^{-2})} d\xi + \int_{c\tau^{1/3} \leq |\xi|} e^{-iy\xi} (e^{-i\tau\xi^{-2}} - 1) d\xi. \end{aligned}$$

The first line is clearly some $\mathcal{O}(1)$. An integration by parts in the last term yields, modulo $\mathcal{O}(1)$, a better decreasing in ξ , namely

$$\int_{c\tau^{1/3} \leq |\xi|} e^{-iy\xi} (e^{-i\tau\xi^{-2}} - 1) d\xi = \mathcal{O}(1) + \frac{2\tau}{y} \int_{c\tau^{1/3} \leq |\xi|} \xi^{-3} e^{-i(y\xi + \tau\xi^{-2})} d\xi.$$

Then, apply the change of variables $\xi = \tau^{1/3}\eta$ to obtain

$$K_\tau(y) = \mathcal{O}(1) + \tau^{1/3} \int_{1 < |\eta| \leq c} e^{-i\tau^{1/3}(y\eta + \eta^{-2})} d\eta + \frac{2\tau^{1/3}}{y} \int_{c \leq |\eta|} \eta^{-3} e^{-i\tau^{1/3}(y\eta + \eta^{-2})} d\eta.$$

Use the principle of non-stationary phase to restrict the domain of integration near the (unique) critical point $\eta = (2/y)^{1/3}$. When doing this, note that the boundary terms can be avoided by smoothing the above localizations. After stationary phase approximation, there remains some $\mathcal{O}(\tau^{1/6})$ as expected. \square

Lemma 5.4 indicates that, when $|\tau|$ goes to $+\infty$, the function K_τ may explode more rapidly near the origin than elsewhere. In view of (5.19), the L^2 -information contained in (5.16) appears as an intermediate information between the two extreme behaviors at $y = 0$ and $y \neq 0$. It is more precise than the local L^2 -estimate that could be deduced from (5.16) when $\tilde{p} = +\infty$.

Corollary 5.5. *Fix $\rho \geq 0$, and assume that $\Lambda \in L^\infty(\mathbb{R})$ is such that $\Lambda(\xi) = \mathcal{O}(|\xi|^{-\rho})$ when $|\xi|$ goes to $+\infty$. Let $h \in L^\infty(\mathbb{R})$. Define*

$$\mathcal{G}^\varepsilon(z) := \chi\left(\frac{\varepsilon z}{r\varepsilon^\iota}\right) h(z).$$

Then, for all $\iota \in [0, 1]$, we have

$$(5.20) \quad \|B_{(T-s)/\varepsilon^2}^\Lambda \mathcal{G}^\varepsilon\|_{L^\infty(\mathbb{R})} \lesssim \|h\|_{L^\infty(\mathbb{R})} \left(\varepsilon^{(\iota-1)/2} + \varepsilon^{(\iota/2)+\rho-1} \right).$$

The loss in the right hand side of (5.20) is decreasing when ρ is growing to $\rho = 1/2$, and then it is saturated for $\rho = 1/2$ at the value $\varepsilon^{(\iota-1)/2}$. As will be seen in the proof below, this residual loss (when $\iota \in [0, 1]$) is coming from the L^2 -impact of the spatial localization in a domain of size $\varepsilon^{\iota-1}$.

Proof. Exploit (5.15) with $p = 2$, and then (5.16) with $\tilde{p} = 2$ (and $q = 2$) to get

$$\begin{aligned} \|B_{(T-s)/\varepsilon^2}^\Lambda \mathcal{G}^\varepsilon\|_{L^\infty(\mathbb{R})} &\lesssim \left(1 + \frac{T-s}{\varepsilon^2}\right)^{(1/4)-(\rho/2)} \left[\int \chi\left(\frac{z}{r\varepsilon^{\iota-1}}\right)^2 h(z)^2 dz\right]^{1/2} \\ &\lesssim \|h\|_{L^\infty(\mathbb{R})} \left(1 + \varepsilon^{-(1/2)+\rho}\right) \varepsilon^{(\iota-1)/2}, \end{aligned}$$

which yields (5.20) since $\iota \leq 1$. \square

Applied in the context of \mathcal{W}_{nl} , this furnishes

$$(5.21) \quad \|\mathcal{W}_{nl}(T, \cdot)\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{\nu+j_1+j_2-2} \|\mathcal{U}^{(0)}\|_{L^\infty([0,T] \times \mathbb{R})}^{j_1+j_2} (1 + \varepsilon^{(\iota/2)-1}).$$

The influence of nonlinearities is clearly stronger when $\nu + j_1 + j_2$ is small. In view of (5.21), for $\nu + j_1 + j_2 + \frac{\iota}{2} - 3 > 0$, it may be neglected. On the contrary, when $\nu + j_1 + j_2 + \frac{\iota}{2} - 3 < 0$, the control (5.21) does not help to extract some uniform bound in L^∞ . For instance, it is clearly insufficient when $\nu + j_1 + j_2 = 2$, even in the most favorable case $\iota = 1$.

The preceding analysis does not take into account the time oscillations (with respect to the variable s) which can lead to further cancellations when computing the integral term inside (5.12). Observe that the time derivative (in s) of the phase involved in (5.5) is $\mathfrak{g} - p(\xi)$. This is why the discussion in the next paragraph is organized around the zeroes of this function.

5.1.3. Classification of gauge parameters. Recall that the gauge parameter \mathfrak{g} has been introduced at the level of Definition 1.1 (or 5.1). Compare the oscillating integral (3.16) with (5.5). For the choice $\varphi(s, y) = \mathfrak{g}s$ in (3.17), the phase Φ of (3.16) coincides with the one which is coming from (5.5). But, in comparison with Section 3, the novelty is that the expression $\mathcal{G}^\varepsilon(T, z)$, in contrast to $a(T, s, x)$, is not strictly speaking a “profile”. The spatial variables z and x are not the same. The support of $\mathcal{G}^\varepsilon(T, \cdot)$ is (at least when $\iota < 1$) of size $\varepsilon^{\iota-1} \gg 1$, while the support of $a(T, s, \cdot)$ is of size one. Dealing with \mathcal{G}^ε in the original variable x (instead of z) would mean to involve an expression that is expected to be rapidly oscillating in x , and therefore that is not compatible with integrations by parts in x .

Much less information is available on \mathcal{G}^ε than on a . However, due to the filtering procedure (5.1), it could be expected that $\partial_s \mathcal{G}^\varepsilon$ is (to some extent) under control. This forecast, that will be confirmed in what follows, explains why integrations by parts in s should remain effective. But, to this end, the criterion (4.7) must be restricted. We must now focus on the role of $\partial_s \Phi \equiv p(\xi) - \mathfrak{g}$. In place of η in (4.7), define the threshold

$$(5.22) \quad c_{\mathfrak{g}} := \inf \{ |p(\xi) - \mathfrak{g}|; \xi \in \mathbb{R} \}.$$

Different situations can occur.

Definition 5.6. *The gauge parameter \mathfrak{g} is said:*

- non resonant when $\mathfrak{g} \notin [0, 1]$ so that $c_{\mathfrak{g}} > 0$;
- transitionally resonant when $\mathfrak{g} = 0$ so that $c_{\mathfrak{g}} = 0$. Then, $p \equiv \mathfrak{g} = 0$ on the whole interval $[-\xi_c, \xi_c]$, and it becomes non zero near $\pm \xi_c$;

- pointwise resonant when $\mathfrak{g} \in]0, 1[$ so that $c_{\mathfrak{g}} = 0$, and we can find a unique position $\xi_{\mathfrak{g}} \in]\xi_c, +\infty[$ such that $p(\xi_{\mathfrak{g}}) = p(-\xi_{\mathfrak{g}}) = \mathfrak{g}$;
- completely resonant when $\mathfrak{g} = 1$ so that $c_{\mathfrak{g}} = 0$. Then, the function $p(\xi)$ can become arbitrarily closed to $\mathfrak{g} = 1$ when $|\xi|$ goes to $+\infty$.

Below, we list some examples of F_{NL} , where for the simplicity of the presentation, we leave out the localizations involving the function χ .

Example 5.7 (A standard choice). *Just take $F_{NL} = u^2$ so that $(j_1, j_2, \nu) = (2, 0, 0)$ and $\omega = 0$. We find $\nu + j_1 + j_2 = 2$ (critical size). The gauge parameter is $\mathfrak{g} = 2$. It is non resonant.*

Example 5.8 (Quadratic nonlinearity in $|u|$). *For the selection of $F_{NL} = |u|^2 = u\bar{u}$, we find $(j_1, j_2, \nu) = (1, 1, 0)$ and $\omega = 0$. We still have $\nu + j_1 + j_2 = 2$ (critical size), but this time, the gauge parameter is $\mathfrak{g} = 0$. It is transitionally resonant.*

Example 5.9 (Presence of time oscillations). *For $F_{NL} = e^{i\omega t/\varepsilon}|u|^2$ with $\omega \in]0, 1[$, we find $\mathfrak{g} \in]0, 1[$. The gauge parameter is pointwise resonant.*

Example 5.10 (The nonlinearity investigated in Theorem 1.3 and 1.4). *The choice made in equation (1.36), that is $F_{NL} = e^{-it/\varepsilon}u^2$, is built with $(j_1, j_2, \nu) = (2, 0, 0)$ and $\omega = -1$, so that $\nu + j_1 + j_2 = 2$ and $\mathfrak{g} = 1$. The size is critical and the gauge parameter is completely resonant.*

Example 5.11 (Critical cubic nonlinearity). *The critical size can be achieved for a cubic nonlinearity like $F_{NL} = \varepsilon^{-1}e^{i\omega t/\varepsilon}|u|^2u$, in which case $(j_1, j_2, \nu) = (2, 1, -1)$. When $\omega = 0$, the gauge parameter is completely non resonant. This situation is expected to involve leading order nonlinear effects, like in Example 5.10.*

5.1.4. *Various estimates involving $\mathcal{U}^{(0)}$.* The purpose of this paragraph is to list estimates that are accessible concerning $\mathcal{U}^{(0)}$, and therefore that could be used when dealing with \mathcal{G}^ε at the level of (5.5). Propositions 4.16 and 4.18 already furnish the following optimal local (for $|x| \leq r$ or $|z| \leq r/\varepsilon$) sup norm estimate

$$(5.23) \quad \exists C > 0, \quad \forall T \in [\mathcal{T}, 2\mathcal{T}], \quad \forall |z| \leq r/\varepsilon, \quad |\mathcal{U}^{(0)}(T, z)| \leq C.$$

Global L^2 and L^∞ controls are also available. Unlike (5.23), they may not be sharp.

Lemma 5.12 (Global control in L^2 and L^∞ -norm). *For all $(j, n) \in \mathbb{N}^2$, we have*

$$(5.24) \quad \forall T \in [0, 2\mathcal{T}], \quad \|\partial_T^j \partial_z^n \mathcal{U}^{(0)}(T, \cdot)\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^{-2j-n-1}),$$

$$(5.25) \quad \forall T \in [0, 2\mathcal{T}], \quad \|\partial_T^j \partial_z^n \mathcal{U}^{(0)}(T, \cdot)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\varepsilon^{-2j-n-\frac{3}{2}}).$$

We emphasize the discrepancy between the optimal uniform L^∞ -bound inside (5.23) (for data localized in space), and the bound (5.25) with $j = n = 0$, which holds globally in space. This loss of an $\varepsilon^{-3/2}$ factor is most likely “only” technical. It explains why in the forthcoming analysis, the presence at the level of (5.4) of some spatial cut-off (driven by ι with $0 \leq \iota \leq 1$) is needed.

The derivatives ∂_t and ∂_x applied to oscillations of the form (3.31) with φ as in (1.26) produce respectively the factors $\varepsilon^{-1}\partial_t\varphi \sim \varepsilon^{-1}$ and $\varepsilon^{-1}\partial_x\varphi \sim \varepsilon^{-1}t$. Thus, for long times $t \sim \varepsilon^{-1}$, it could be expected that the action of ∂_{Tz}^α takes the form of a loss similar to

$$(5.26) \quad \forall \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2, \quad \partial_{Tz}^\alpha \sim \varepsilon^{-\alpha_1+\alpha_2} \partial_{tx}^\alpha \sim \varepsilon^{-2\alpha_1} t^{\alpha_2} \sim \varepsilon^{-2\alpha_1-\alpha_2}.$$

The bounds (5.24) and (5.25) are both in agreement with this prediction since the application of $\partial_T = \varepsilon^{-1}\partial_t$ and $\partial_z = \varepsilon\partial_x$ cost respectively ε^{-2} and ε^{-1} .

Proof. Consider the equation (1.28a) with $F_L \equiv F$ as in (4.1). Since $p(\varepsilon D_x)$ is a pseudo-differential operator with constant coefficients, it does commute with the derivative $\partial_t^j \partial_x^n$. Thus, through usual L^2 -energy estimates, we can infer that

$$\partial_t \|\partial_t^j \partial_x^n u^{(0)}\|_{L^2(\mathbb{R})}^2 \lesssim \varepsilon^{3/2} \sum_{|m| \leq M} \|\partial_t^j \partial_x^n u^{(0)}\|_{L^2(\mathbb{R})} \|\partial_t^j \partial_x^n A_m^* e^{im\varphi/\varepsilon}\|_{L^2(\mathbb{R})}.$$

Remark that

$$\partial_t^j \partial_x^n A_m^* = \sum_{i_n=0}^n \sum_{i_j=0}^j \binom{n}{i_n} \binom{j}{i_j} (\partial_t^{i_j} \partial_x^{i_n} A_m)^* \partial_t^{j-i_j} \partial_x^{n-i_n}.$$

By assumption, the symbol $\partial_t^{i_j} \partial_x^{i_n} A_m(\varepsilon t, t, \cdot)$ is smooth, and its derivatives with respect to x and ξ are uniformly bounded in ε . From the Calderón–Vaillancourt Theorem, we know that $(\partial_t^{i_j} \partial_x^{i_n} A_m)^*$ acts continuously on $L^2(\mathbb{R})$. On the other hand, the function a_m , and therefore $\partial_t^{i_j} \partial_x^{i_n} A_m$, is spatially supported in the ball $|x| < r$. Thus, we can replace $e^{im\varphi/\varepsilon}$ by the L^2 -function $\chi(|x|/\tilde{r})e^{im\varphi/\varepsilon}$ where $\tilde{r} := 8r/5$. And thereby, we have to estimate a sum of terms similar to

$$\|\partial_t^{j-i_j} \partial_x^{n-i_n} (\chi(x/\tilde{r})e^{im\varphi/\varepsilon})\|_{L^2(\mathbb{R})}.$$

The derivatives ∂_t and ∂_x , when they are applied to the oscillation $e^{im\varphi/\varepsilon}$, produce respectively the singular factors $\varepsilon^{-1} \partial_t \varphi \sim \varepsilon^{-1}$ and $\varepsilon^{-1} \partial_x \varphi \sim t\varepsilon^{-1}$. The worst term arises when $(i_j, i_n) = (0, 0)$. As a consequence, we find that

$$\partial_t \|\partial_t^j \partial_x^n u^{(0)}\|_{L^2(\mathbb{R})}^2 \lesssim \varepsilon^{3/2} \varepsilon^{-j} t^n \varepsilon^{-n} \|\partial_t^j \partial_x^n u^{(0)}\|_{L^2(\mathbb{R})},$$

and therefore, by Grönwall's lemma, that

$$\|\partial_t^j \partial_x^n u^{(0)}(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim t^{n+1} \varepsilon^{(3/2)-j-n}.$$

Then, in line with (5.24), we get that

$$\begin{aligned} \|\partial_T^j \partial_z^n \mathcal{U}^{(0)}(T, \cdot)\|_{L^2(\mathbb{R})} &= \frac{1}{\varepsilon} \left\| \sum_{i_j=0}^j \binom{j}{i_j} \partial_T^{i_j} [e^{-iT/\varepsilon^2}] \times \partial_T^{j-i_j} \partial_z^n [u^{(0)}\left(\frac{T}{\varepsilon}, \varepsilon \cdot\right)] \right\|_{L^2(\mathbb{R})} \\ &\lesssim \varepsilon^{-1+n-j} \sum_{i_j=0}^j \varepsilon^{-i_j} \left\| (\partial_t^{j-i_j} \partial_x^n u^{(0)})\left(\frac{T}{\varepsilon}, \varepsilon \cdot\right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \varepsilon^{-1+n-j} \sum_{i_j=0}^j \varepsilon^{-i_j} \frac{1}{\sqrt{\varepsilon}} \left\| (\partial_t^{j-i_j} \partial_x^n u^{(0)})\left(\frac{T}{\varepsilon}, \cdot\right) \right\|_{L^2(\mathbb{R})} \\ &\lesssim \varepsilon^{-1+n-j} \sum_{i_j=0}^j \varepsilon^{-i_j} \frac{1}{\sqrt{\varepsilon}} \frac{T}{\varepsilon} \varepsilon^{3/2} \varepsilon^{-(j-i_j)} \left(\frac{T}{\varepsilon}\right)^n \varepsilon^{-n} = \mathcal{O}(\varepsilon^{-2j-n-1}). \end{aligned}$$

This furnishes (5.24). The sup norm control (5.25) is then a consequence of the standard one-dimensional Gagliardo–Nirenberg inequality

$$(5.27) \quad \|\mathcal{V}\|_{L^\infty(\mathbb{R})} \leq \sqrt{2} \|\mathcal{V}\|_{L^2(\mathbb{R})}^{1/2} \|\partial_z \mathcal{V}\|_{L^2(\mathbb{R})}^{1/2}. \quad \square$$

The interest of using $\mathcal{U}^{(0)}$ instead of $u^{(0)}$ is twofold. First, as noted in (5.23), the amplitude of the wave becomes of size one. Secondly, when passing from $u^{(0)}$ to $\mathcal{U}^{(0)}$, the main temporal oscillations are locally filtered out in the following sense.

Lemma 5.13 (Local sup norm controls involving derivatives of $\mathcal{U}^{(0)}$).

$$(5.28) \quad \exists C > 0, \quad \forall T \in [\mathcal{T}, 2\mathcal{T}], \quad \forall |z| \leq r/\varepsilon, \quad |\partial_z \mathcal{U}^{(0)}(T, z)| \leq C\varepsilon^{-1},$$

$$(5.29) \quad \exists C > 0, \quad \forall T \in [\mathcal{T}, 2\mathcal{T}], \quad \forall |z| \leq r/\varepsilon, \quad |\partial_T \mathcal{U}^{(0)}(T, z)| \leq C\varepsilon^{-2/3},$$

$$(5.30) \quad \exists C > 0, \quad \forall T \in [\mathcal{T}, 2\mathcal{T}], \quad \forall |z| \leq r/\varepsilon, \quad |\partial_{Tz}^2 \mathcal{U}^{(0)}(T, z)| \leq C\varepsilon^{-3/2}.$$

Proposition 4.16 implies that the information (5.23) is sharp. Starting from (5.23), the two controls (5.29) and (5.30), which both involve time derivatives, represent improvements in comparison to what would be provided by (5.26). This means that the time oscillations contained in $u^{(0)}$ have indeed been somewhat filtered out when passing from $u^{(0)}$ to $\mathcal{U}^{(0)}$.

Proof. Denote by $u_m^{(j)}$ with $j \in \{0, 1\}$ the m^{th} harmonic of $u^{(j)}$. In particular, $u_m^{(0)}$ can be obtained by solving

$$(5.31) \quad \partial_t u_m^{(0)} - \frac{i}{\varepsilon} p(-i\varepsilon \partial_x) u_m^{(0)} = \varepsilon^{3/2} A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* e^{im\varphi(t,x)/\varepsilon},$$

with initial data $u_m^{(0)}|_{t=0} \equiv 0$. The situation is as in Paragraph 3.1.3, see (3.14) and (3.17), with

$$\Phi(t, x; s, y, \xi) := (s - t)p(\xi) + (x - y)\xi - m\varphi(s, y).$$

In order to lighten the notations, we drop the dependence of the phase upon m . As in Paragraph 4.2.2, we can separate Φ according to $\Phi = \phi + m(\gamma - t)$ to deal with

$$\phi(t, x; s, y, \xi) := (s - t)[p(\xi) - m] + (x - y)\xi + mys - m\gamma \cos s.$$

The function $u_m^{(0)}$ looks like u in (4.15), that is

$$u_m^{(0)}(t, x) = \frac{\sqrt{\varepsilon}}{2\pi} e^{im(t-\gamma)/\varepsilon} \int_0^t \iint e^{-i\phi(t,x;s,y,\xi)/\varepsilon} \zeta_m(\xi) a_m(\varepsilon s, s, y) ds dy d\xi.$$

Apply the derivative ∂_x to the above relation. This introduces a factor ξ/ε in the integral. Since ξ is like $\lesssim \varepsilon^{-1}$ at the critical points, a rough estimate furnishes

$$(5.32) \quad \left| \partial_x u_m^{(0)}(t, x) \right| \lesssim \sqrt{\varepsilon} \varepsilon^{3/2} \varepsilon^{-2} \varepsilon^{-1} = \varepsilon^{-1},$$

where the products of powers of ε follows, one after another, from the amplitude, the stationary phase approximation (in dimension 3), the term $\xi/\varepsilon \sim \varepsilon^{-2}$, and the number ε^{-1} of critical points. The bound (5.32) is equivalent to (5.28).

From (5.31), we can deduce that

$$\partial_t (e^{-it/\varepsilon} u_m^{(0)}) = \varepsilon^{3/2} A_m^* e^{i(m\varphi-t)/\varepsilon} + \frac{i}{\varepsilon} e^{-it/\varepsilon} (p(-i\varepsilon \partial_x) - 1) u_m^{(0)}.$$

Then, from the above integral representation of $u_m^{(0)}$, we get

$$(5.33) \quad \begin{aligned} \partial_t (e^{-it/\varepsilon} u_m^{(0)}(t, x)) &= \varepsilon^{3/2} A_m^* e^{i(m\varphi-t)/\varepsilon} + \frac{i}{2\pi\sqrt{\varepsilon}} e^{i(mt-t-m\gamma)/\varepsilon} \\ &\quad \times \underbrace{\int_0^t \iint e^{-i\phi(t,x;s,y,\xi)/\varepsilon} (p(\xi) - 1) \zeta_m(\xi) a_m(\varepsilon s, s, y) ds dy d\xi}_{=:\mathcal{P}_m(t,x)}. \end{aligned}$$

Coming back to (3.32) where u is replaced adequately, the first term in (5.33) can be expressed as

$$|A_m^* e^{i(m\varphi-t)/\varepsilon}| = \frac{1}{\varepsilon} \left| \iint e^{i(x\xi - y\xi - myt)/\varepsilon} \zeta_m(\xi) a_m(\varepsilon t, t, y) dy d\xi \right|.$$

The phase involved has only one critical point $(y, \xi) = (x, -mt)$ which is non degenerate. Since the dimension is two, this allows to gain the factor ε so that

$$(5.34) \quad \varepsilon^{3/2} A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* e^{i(m\varphi(t,x)-t)/\varepsilon} = \mathcal{O}(\varepsilon^{3/2}).$$

Let us now consider the expression \mathcal{P}_m emphasized in (5.33). In view of Lemma 4.2 and Lemma 4.3, and since $\zeta_0 = 0$ near the origin (Assumption 3.11), Proposition 4.5 implies that $\mathcal{P}_m = \mathcal{O}(\varepsilon^\infty)$ when $m \neq 1$, so that

$$(5.35) \quad \forall m \neq 1, \quad \partial_t(e^{-it/\varepsilon} u_m^{(0)}(t, x)) = \mathcal{O}(\varepsilon^{3/2}), \quad |x| \leq r, \quad \frac{\mathcal{T}}{\varepsilon} \leq t \leq \frac{2\mathcal{T}}{\varepsilon}.$$

We thus focus on the resonant case $m = 1$. We decompose \mathcal{P}_1 as we did concerning v at the level of (4.22) and (4.23), to get

$$(5.36) \quad \mathcal{P}_1(t, x) = \sum_{k \in \mathcal{K}} \mathcal{P}_{1,k}(t, x), \quad \mathcal{K} = \left\{ k \in \mathbb{N}; k \leq \frac{2}{3} + \frac{\mathcal{T}}{\pi \varepsilon} \right\}.$$

The analysis of Section 4.3 readily shows that we can find a constant $c > 0$ such that (recall that here, $q = 2$)

$$(5.37) \quad \sum_{0 \leq k \leq c\varepsilon^{-1/3}} |\mathcal{P}_{1,k}(t, x)| \lesssim \sum_{0 \leq k \leq c\varepsilon^{-1/3}} \varepsilon^{D-1} \lesssim \varepsilon^{D-4/3}.$$

For $k \in \mathcal{K}$ with $c\varepsilon^{-1/3} \leq k$, as in Section 4.4, we can rely on a stationary phase argument. The only difference in the treatment of \mathcal{P}_1 compared to the preceding analysis of $u^{(0)}$ is the presence of the factor $p(\xi) - 1$ in the integral. Remark that

$$|p(k\pi + \xi) - 1| = \mathcal{O}(k^{-2}), \quad \text{uniformly in } |\xi| \leq 1.$$

This property allows to improve the convergence of the sum of wave packets. As a matter of fact, resuming the stationary phase argument in (s, y, ξ) and relying on the decay in k which is provided by the factor to gain the convergence of the series in k , we come up with

$$(5.38) \quad \sum_{c\varepsilon^{-1/3} \leq k \in \mathcal{K}} |\mathcal{P}_{1,k}(t, x)| \lesssim \varepsilon^{3/2} \sum_{c\varepsilon^{-1/3} \leq k \in \mathcal{K}} \frac{1}{k^2} \lesssim \varepsilon^{3/2} \varepsilon^{1/3} = \varepsilon^{11/6}.$$

As prescribed at the level of (5.33), divide (5.37) and (5.38) by $\sqrt{\varepsilon}$. Since $D \geq 4$, we can retain that

$$(5.39) \quad \partial_t(e^{-it/\varepsilon} u_1^{(0)}(t, x)) = \mathcal{O}(\varepsilon^{4/3}), \quad |x| \leq r, \quad 0 \leq t \leq \frac{2\mathcal{T}}{\varepsilon}.$$

Compute

$$\partial_T \mathcal{U}^{(0)}(T, z) = \frac{1}{\varepsilon^2} \partial_t \left(e^{-it/\varepsilon} u^{(0)}(t, \varepsilon z) \right) \Big|_{t=T/\varepsilon}.$$

To estimate $\partial_T \mathcal{U}^{(0)}$, it suffices to multiply (5.35) and (5.39) by ε^{-2} , and to sum on the finite number of integers m satisfying $|m| \leq M$. This yields (5.29).

Now, in order to get (5.30), just take the derivative of (5.33) with respect to x . Since $\partial_x \varphi / \varepsilon \sim t / \varepsilon \sim 1 / \varepsilon^2$, we have

$$\varepsilon^{3/2} \partial_x \left(A_m(\varepsilon t, t, x, -i\varepsilon \partial_x)^* e^{i(m\varphi(t,x)-t)/\varepsilon} \right) = \mathcal{O}(\varepsilon^{-1/2}).$$

This term turns out to bring the largest contribution. As a matter of fact, at the level of the oscillating integral in the second line of (5.33), the x -derivative produces the factor $\partial_x \phi / \varepsilon = \xi / \varepsilon$. The multiplication by ξ is compensated by the decreasing of $p - 1$. Due to the assumptions on p , the symbol $\xi(p(\xi) - 1)\zeta_m(\xi)$ remains in a convenient class. We can still apply Lemma 4.7, except that the control inside (5.37) must be replaced by some $\mathcal{O}(\varepsilon^{D-7/3})$. On the other hand, at the critical points, ξ/ε behaves like k/ε , so the estimate (5.38) becomes

$$\varepsilon^{3/2} \sum_{c\varepsilon^{-1/3} \leq k \in \mathcal{K}} \frac{1}{k^2} \frac{k}{\varepsilon} = \varepsilon^{1/2} \sum_{c\varepsilon^{-1/3} \leq k \in \mathcal{K}} \frac{1}{k} \lesssim \sqrt{\varepsilon} \ln \frac{1}{\varepsilon}.$$

Gathering the above three estimates and recalling that $\partial_z = \varepsilon \partial_x$, we can easily infer the content of (5.30). \square

So far, we have not exploited the fact that (5.23) is achieved on a set of zero Lebesgue measure. This property is useful for what follows.

Lemma 5.14 (Local vanishing properties). *Let $(m, n) \in \mathbb{N}^* \times \mathbb{N}$. Given a function w in the Schwartz space $\mathcal{S}(\mathbb{R})$, define*

$$\mathcal{J}_\varepsilon \equiv \mathcal{J}_\varepsilon(T) \equiv \mathcal{J}_\varepsilon(m, n, w; T) := \int |w(y)| |\mathcal{U}^{(0)}(T, y)|^m |\varepsilon^{2/3} \partial_T \mathcal{U}^{(0)}(T, y)|^n dy.$$

Then, for all $T \in [\mathcal{T}, 2\mathcal{T}]$, we have $\mathcal{J}_\varepsilon = o(1)$.

Proof. Decompose \mathcal{J}_ε into $\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon^+ + \mathcal{J}_\varepsilon^-$ with

$$\mathcal{J}_\varepsilon^\pm := \int_{\pm\varepsilon|y| \leq \pm r} |w(y)| |\mathcal{U}^{(0)}(T, y)|^m |\varepsilon^{2/3} \partial_T \mathcal{U}^{(0)}(T, y)|^n dy.$$

Exploit the global estimate (5.25) and the decreasing of $w \in \mathcal{S}(\mathbb{R})$ to obtain

$$|\mathcal{J}_\varepsilon^-| \leq C \varepsilon^{-(3m/2) - (17n/6)} \|w\|_{L^1(r \leq \varepsilon|y|)} = \mathcal{O}(\varepsilon^\infty).$$

On the other hand, due to (5.23) and (5.29), the family

$$\mathbb{1}_{[-r/\varepsilon, r/\varepsilon]}(y) |w(y)| |\mathcal{U}^{(0)}(T, y)|^m |\varepsilon^{2/3} \partial_T \mathcal{U}^{(0)}(T, y)|^n, \quad \varepsilon \in]0, 1]$$

is uniformly bounded by $C|w| \in L^1(\mathbb{R})$. Applying Proposition 4.18, it converges to zero out of the set $2\mathbb{Z}$, which is of Lebesgue measure 0 in \mathbb{R} . Under such hypotheses, the Lebesgue dominated convergence theorem guarantees that $\mathcal{J}_\varepsilon^+ = o(1)$. \square

5.1.5. *General estimates involving $\mathcal{W} = \mathcal{U}^{(1)} - \mathcal{U}^{(0)}$.* Depending on the choice of the parameters ν, j_1, j_2, ω and ι , the source term F_{NL} can bring a contribution which is of the same size of $\mathcal{U}^{(0)}$, or not. To understand what happens, it is interesting to first investigate a situation implying no condition on ν, j_1, j_2, ω , and no particular assumptions (through ι) on the spatial localization. To this end, we could directly exploit (5.25) with $j = n = 0$ at the level of (5.21) to obtain a preliminary sup norm control on \mathcal{W} . But, knowing (5.28), it is possible to improve this first bound.

Lemma 5.15 (Global sup norm control on $\mathcal{U}^{(1)} - \mathcal{U}^{(0)}$). *Fix any $\iota \in]-\infty, 1]$. Then*

$$(5.40) \quad \|(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, \cdot)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\varepsilon^{\nu - \frac{j_1}{2} - \frac{j_2}{2} - 2}).$$

Proof. Recall (5.27) and (5.8) which allow to control $B_\tau^\Lambda \mathcal{V}$ in sup norm uniformly in τ through the L^2 -norms of \mathcal{V} and $\partial_z \mathcal{V}$ as indicated below

$$(5.41) \quad \begin{aligned} \|B_\tau^\Lambda \mathcal{V}\|_{L^\infty(\mathbb{R})} &\lesssim \|B_\tau^\Lambda \mathcal{V}\|_{L^2(\mathbb{R})}^{1/2} \|\partial_z(B_\tau^\Lambda \mathcal{V})\|_{L^2(\mathbb{R})}^{1/2} \\ &\lesssim \|\Lambda\|_{L^\infty(\mathbb{R})} \|\mathcal{V}\|_{L^2(\mathbb{R})}^{1/2} \|\partial_z \mathcal{V}\|_{L^2(\mathbb{R})}^{1/2}. \end{aligned}$$

On the other hand, using (5.24) and (5.25) with $j = 0$ and $n \in \{0, 1\}$, we can infer that, for all $k \in \{0, 1\}$, we have

$$(5.42) \quad \begin{aligned} \|\partial_y^k [\chi(\varepsilon^{1-\iota} r^{-1} | \cdot |) \mathcal{U}^{(0)}(s, \cdot)^{j_1} \bar{\mathcal{U}}^{(0)}(s, \cdot)^{j_2}]\|_{L^2(\mathbb{R})} \\ \lesssim \|\mathcal{U}^{(0)}(s, \cdot)\|_{L^\infty(\mathbb{R})}^{j_1+j_2-1} (\|\mathcal{U}^{(0)}(s, \cdot)\|_{L^2(\mathbb{R})} + \|\partial_y^k \mathcal{U}^{(0)}(s, \cdot)\|_{L^2(\mathbb{R})}) \\ \lesssim (\varepsilon^{-\frac{3}{2}})^{j_1+j_2-1} (\varepsilon^{-1} + \varepsilon^{-k-1}). \end{aligned}$$

Recall that $\mathcal{W} = \mathcal{U}^{(1)} - \mathcal{U}^{(0)} = \mathcal{W}_l + \mathcal{W}_{nl}$ with \mathcal{W}_l and \mathcal{W}_{nl} as in (5.11) and (5.12). The part \mathcal{W}_l can be controlled according to

$$\|\mathcal{W}_l\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{\nu+j_1+j_2-2} \|\mathcal{G}^\varepsilon\|_{L^\infty(\mathbb{R})}.$$

From (5.4) and (5.25) with $j = n = 0$, we can easily deduce (5.40). On the other hand, combine (5.41) and (5.42) at the level of (5.12) to get

$$\begin{aligned} \|\mathcal{W}_{nl}\|_{L^\infty(\mathbb{R})} &\lesssim \varepsilon^{\nu+j_1+j_2-2} \|\mathcal{G}^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \|\partial_z \mathcal{G}^\varepsilon\|_{L^2(\mathbb{R})}^{1/2} \\ &\lesssim \varepsilon^{\nu+j_1+j_2-2} (\varepsilon^{-\frac{3}{4}})^{j_1+j_2-1} \varepsilon^{-\frac{1}{2}} (\varepsilon^{-\frac{3}{4}})^{j_1+j_2-1} \varepsilon^{-1}, \end{aligned}$$

which leads directly to (5.40). \square

The preliminary estimate (5.40) is far from enough to reach some $\mathcal{O}(1)$ or less, under the sole condition $\nu + j_1 + j_2 \geq 2$. More specific arguments (involving especially the spatial localization) are needed to improve the above analysis.

5.2. Sorting of gauge parameters. The constructive interferences of Theorem 1.3 occur on a set of Lebesgue measure zero. From this viewpoint, comparisons in L^p -norms with $p < +\infty$ cannot be relevant. We must stick to the use of the sup norm. This motivates the following definition, which is inspired by a notion of linearizability introduced in [25].

Definition 5.16 (Linearizability during long times). *We say that the nonlinearity plays no role at leading order during long times when*

$$(5.43) \quad \sup_{0 \leq T \leq 2T} \|(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, \cdot)\|_{L^\infty(\mathbb{R})} = o(1) \quad \text{as } \varepsilon \rightarrow 0.$$

In this subsection, we show that when $\mathfrak{g} \neq 1$, nonlinear effects are absent at leading order during long times (in the sense of Definition 5.16), provided that $\iota = 1$. When $\mathfrak{g} \notin \{0, 1\}$, the assumption $\iota = 1$ may be relaxed to $\iota \in [0, 1]$. In the next paragraphs 5.2.1, 5.2.2 and 5.2.3, following the distinctions which have been made in Definition 5.6, we examine successively the cases $\mathfrak{g} \notin [0, 1]$, $\mathfrak{g} \in]0, 1[$, and $\mathfrak{g} = 0$.

Remark 5.17. *The results of this subsection, Propositions 5.18, 5.19 and 5.20, rely on the estimates (5.23), (5.28), (5.29) and (5.30), which have been established only for $T \geq \mathcal{T}$, hence the time localizing factor $\chi(3 - 2\varepsilon t/\mathcal{T})$ in \mathcal{G}^ε . We will see later that these estimates could be adapted for $T \geq \eta$ with $\eta > 0$. But the case of smaller times $t \ll \eta/\varepsilon$ is not straightforward.*

5.2.1. *The case of non resonant gauge parameters.* This is when $\mathfrak{g} \notin [0, 1]$. Then, the distance from $p(\xi)$ to \mathfrak{g} remains bounded below by a positive constant. In other words, the function

$$\begin{aligned} \Gamma : \mathbb{R} &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \Gamma(\xi) := (p(\xi) - \mathfrak{g})^{-1} \end{aligned}$$

is bounded, that is $\Gamma \in L^\infty(\mathbb{R})$.

Proposition 5.18. *Assume that $\mathfrak{g} \notin [0, 1]$ and $\iota \in [0, 1]$. Then, the nonlinearity plays no role at leading order during long times. More precisely*

$$(5.44) \quad \forall T \in [0, 2\mathcal{T}], \quad \|(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, \cdot)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\varepsilon^{\iota/2+1/3}).$$

Proof. It suffices to examine the critical size case, where $\nu + j_1 + j_2 = 2$. In the case $\nu + j_1 + j_2 > 2$, the above $\mathcal{O}(\varepsilon^{\iota/2+1/3})$ is readily improved to $\mathcal{O}(\varepsilon^{\iota/2+1/3+\nu+j_1+j_2-2})$. The idea is to come back to Duhamel's formula (5.5), and to exploit the oscillations occurring with respect to the time variable s . Integrating by parts in s , we find

$$\begin{aligned} \mathcal{W}(T, z) &= \frac{1}{2\pi} \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1) + i(\mathfrak{g}-1)\frac{s}{\varepsilon^2}} \mathcal{G}^\varepsilon(s, y) ds dy d\xi \\ &= \frac{i\varepsilon^2}{2\pi} \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1) + i(\mathfrak{g}-1)\frac{s}{\varepsilon^2}} \Gamma(\xi) \mathcal{G}^\varepsilon(s, y) dy d\xi \Big|_{s=0}^T \\ &\quad - \frac{i\varepsilon^2}{2\pi} \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1) + i(\mathfrak{g}-1)\frac{s}{\varepsilon^2}} \Gamma(\xi) \partial_s \mathcal{G}^\varepsilon(s, y) ds dy d\xi. \end{aligned}$$

Given $s \in [0, T]$, for $j \in \{0, 1\}$, write the integral in (y, ξ) in the more concise form

$$\frac{1}{2\pi} \iint e^{-i(z-y)\xi + i\frac{\sigma}{\varepsilon^2}(p(\xi)-1)} \Gamma(\xi) \partial_s^j \mathcal{G}^\varepsilon(s, y) dy d\xi = \mathcal{F}_\xi \left(e^{i\frac{\sigma}{\varepsilon^2}(p(\xi)-1)} \Gamma(\xi) \mathcal{F}_y^{-1}(\partial_s^j \mathcal{G}^\varepsilon) \right).$$

Set apart the weight

$$\tilde{\Gamma}(\sigma, \xi) \equiv \tilde{\Gamma}_\varepsilon(\sigma, \xi) := e^{i\frac{\sigma}{\varepsilon^2}(p(\xi)-1)} \Gamma(\xi), \quad \sup_{\varepsilon \in]0, 1]} \|\tilde{\Gamma}_\varepsilon\|_{L^\infty(\mathbb{R}^2)} < +\infty.$$

To estimate such a term in L^∞ , we use the L^2 -norms as intermediary norms, so we lose as little information as possible at the level of Fourier transforms, thanks to Plancherel identity. To do so, we invoke Gagliardo–Nirenberg inequality,

$$\begin{aligned} \left\| \mathcal{F}_\xi \left(\tilde{\Gamma} \mathcal{F}_y^{-1}(\partial_s^j \mathcal{G}^\varepsilon) \right) \right\|_{L^\infty(\mathbb{R})} &\lesssim \left\| \mathcal{F}_\xi \left(\tilde{\Gamma} \mathcal{F}_y^{-1}(\partial_s^j \mathcal{G}^\varepsilon) \right) \right\|_{L^2}^{1/2} \left\| \partial_z \mathcal{F}_\xi \left(\tilde{\Gamma} \mathcal{F}_y^{-1}(\partial_s^j \mathcal{G}^\varepsilon) \right) \right\|_{L^2}^{1/2} \\ &\lesssim \left\| \partial_s^j \mathcal{G}^\varepsilon \right\|_{L^2}^{1/2} \left\| \partial_y \partial_s^j \mathcal{G}^\varepsilon \right\|_{L^2}^{1/2}. \end{aligned}$$

The assumption $\iota \geq 0$ is needed to later invoke the L^∞ estimates of Lemma 5.13, concerning $\mathcal{U}^{(0)}$. Below, to simplify notations, we can drop the exponent in $\mathcal{U}^{(0)}$. For $j \in \{0, 1\}$, since $\iota \in [0, 1]$, we can assert that

$$\begin{aligned} \left\| \partial_y^j \left[\chi \left(\frac{\cdot}{\varepsilon^{\iota-1} r} \right) \partial_s (\mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}) \right] \right\|_{L^2(\mathbb{R})} &\lesssim \\ &\varepsilon^{(1-\iota)/2} \|\mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)}^{j_1+j_2-1} \|\partial_s \mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)} \\ &+ \varepsilon^{(\iota-1)/2} \|\mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)}^{j_1+j_2-2} \|\partial_y^j \mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)} \|\partial_s \mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)} \\ &+ \varepsilon^{(\iota-1)/2} \|\mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)}^{j_1+j_2-1} \|\partial_{sy}^{(1,j)} \mathcal{U}(s)\|_{L^\infty(|z| \leq r/\varepsilon)}. \end{aligned}$$

Since $\iota \in [0, 1]$, we can exploit the local sup norm estimates (5.23), (5.28), (5.29) and (5.30), so the above estimate yields

$$\begin{aligned} & \left\| \partial_y^j \left[\chi \left(\frac{\cdot}{\varepsilon^{\iota-1} r} \right) \partial_s (\mathcal{U}^{j_1} \bar{\mathcal{U}}^{j_2}) \right] \right\|_{L^2(\mathbb{R})} \\ & \lesssim \varepsilon^{(1-\iota)/2} \varepsilon^{-2/3} + \varepsilon^{(\iota-1)/2} \varepsilon^{-j} \varepsilon^{-2/3} + \varepsilon^{(\iota-1)/2} \varepsilon^{-2/3-5j/6} \lesssim \varepsilon^{\iota/2} \varepsilon^{-j} \varepsilon^{-7/6}. \end{aligned}$$

We conclude that

$$(5.45) \quad |(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, z)| \lesssim \varepsilon^2 \times \varepsilon^{\iota/4} \varepsilon^{-7/12} \times \varepsilon^{\iota/4} \varepsilon^{-1/2} \varepsilon^{-7/12},$$

which is exactly (5.44). \square

5.2.2. *The case of pointwise resonant gauge parameters.* This is when $\mathfrak{g} \in]0, 1[$. In view of (3.20), we can assert that

$$(5.46) \quad \exists ! \xi_{\mathfrak{g}} \in]\xi_c, +\infty[; \quad p(\xi_{\mathfrak{g}}) = \mathfrak{g}, \quad 0 < p'(\xi_{\mathfrak{g}}).$$

Proposition 5.19. *Assume that $\mathfrak{g} \in]0, 1[$ and that $\iota \in [0, 1]$. Then, the nonlinearity plays no role at leading order during long times. More precisely, for all $\mu < 1/6$, we have*

$$(5.47) \quad \forall T \in [0, 2\mathcal{T}], \quad \|(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, \cdot)\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\varepsilon^{3\iota/4+\mu}).$$

This furnishes again some $o(1)$ in line with Definition 5.16. When $\iota = 0$, this bound is weaker than (5.44).

Proof. We can still work with $\nu + j_1 + j_2 = 2$. Fix $\eta \in [0, 1[$. We perform a frequency localization of size ε^η near the two problematic positions $\pm \xi_{\mathfrak{g}}$. In practice, we insert in the integral (5.5) defining the error $\mathcal{U}^{(1)} - \mathcal{U}^{(0)}$ the decomposition

$$(5.48) \quad 1 = (1 - \chi) \left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta} \right) + \chi \left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta} \right).$$

Concerning the left part of (5.48), that is away from the values $\xi = \pm \xi_{\mathfrak{g}}$, the proof of Proposition 5.18 can be repeated with Γ replaced by

$$\Gamma_\eta(\xi) := \frac{1}{p(\xi) - \mathfrak{g}} (1 - \chi) \left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta} \right).$$

By construction, the function Γ_η is zero on some set of size ε^η containing $\pm \xi_{\mathfrak{g}}$. It follows from (3.20) and (5.46) that Γ_η is globally bounded by $C\varepsilon^{-\eta}$. The integration by parts with respect to the time variable s can still be performed, but now we have to take into account this singular estimate for $\|\Gamma_\eta\|_{L^\infty}$. As a consequence, the gain is $\varepsilon^{2-\eta}$ instead of ε^2 . The corresponding contribution is therefore of size $\varepsilon^{\frac{1}{2}-\eta+\frac{1}{3}}$ instead of being of size $\varepsilon^{\frac{1}{2}+\frac{1}{3}}$ as in (5.44).

From now on, we fix $\eta \in [0, \frac{1}{2} + \frac{1}{3}[$ (so the above estimate yields a small contribution), and we study the contribution coming from the right part of (5.48). The idea is to exploit at the level of (5.5) the oscillations with respect to ξ . To this end, the identity (5.5) may be reformulated as

$$(5.49) \quad \mathcal{W}(T, z) = \frac{1}{2\pi} \int_0^T \int e^{i(\mathfrak{g}s-T)/\varepsilon^2} \mathcal{J}(\varepsilon, T-s, y, z) \mathcal{G}^\varepsilon(s, y) ds dy,$$

where we have put aside the oscillatory integral

$$(5.50) \quad \mathcal{J}(\varepsilon, s, y, z) := \int e^{i\psi(\xi)/\varepsilon^2} \chi\left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta}\right) d\xi,$$

built with the phase $\psi(\xi) \equiv \psi(\varepsilon, s, y, z; \xi)$ given by

$$(5.51) \quad \psi(\xi) := sp(\xi) - \varepsilon^2(z - y)\xi, \quad \psi'(\xi) = sp'(\xi) - \varepsilon^2(z - y).$$

Due to the presence of χ , we have the obvious estimate $|\mathcal{J}| \lesssim \varepsilon^\eta$. Fix $\delta \in [\eta, 1]$. Since $p'(\xi_{\mathfrak{g}}) > 0$, for all time $s \gtrsim \varepsilon^\delta$, we have $\psi'(\xi) \gtrsim \varepsilon^\delta$ for all ξ at a distance $\sim \varepsilon^\eta$ from $\xi_{\mathfrak{g}}$, and for all y and z located at a distance less than $\sim \varepsilon^{-1}$ from the origin. For $s \gtrsim \varepsilon^\delta$, an integration by parts with respect to ξ yields

$$\mathcal{J} = i\varepsilon^2 \int e^{i\psi(\xi)/\varepsilon^2} \left[-\frac{\psi''(\xi)}{\psi'(\xi)^2} \chi\left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta}\right) + \frac{2\xi}{\varepsilon^\eta \psi'(\xi)} \chi'\left(\frac{\xi^2 - \xi_{\mathfrak{g}}^2}{\varepsilon^\eta}\right) \right] d\xi.$$

This indicates a gain of $\varepsilon^{2-2\delta}$ when computing \mathcal{J} . Since this operation may be repeated indefinitely, we deduce that $\mathcal{J} = \mathcal{O}(\varepsilon^\infty)$ for $s \gtrsim \varepsilon^\delta$.

There remains to control the contribution which, in (5.49), is brought by the times s satisfying $T - C\varepsilon^\delta \leq s \leq T$. For such s , a rough estimate based on (5.23) yields, since $|\mathcal{J}| \lesssim \varepsilon^\eta$, some $\mathcal{O}(\varepsilon^{\delta+\eta+\iota-1})$ error term. By optimizing the smallness of $\varepsilon^{\frac{1}{2}-\eta+\frac{1}{3}}$ and $\varepsilon^{\delta+\eta+\iota-1}$ through the selection of $\eta = \frac{2}{3} - \frac{\delta}{2} - \frac{\iota}{4}$, we get some $\varepsilon^{\frac{3\iota}{4} - \frac{1}{3} + \frac{\delta}{2}}$ estimate. Since $\delta < 1$ can be chosen arbitrarily closed to 1, we obtain (5.47). \square

5.2.3. The case of the transitional gauge parameter. This is when $\mathfrak{g} = 0$. In this case, for all ξ in the interval $[-\xi_c, \xi_c]$, we have $p(\xi) = \mathfrak{g} = 0$. On the other hand, in the case $\xi_c > 0$ (which we shall assume in this paragraph), the transitional region near the extreme positions $\pm\xi_c$ is much more degenerate than in Paragraph 5.2.2. The function p is flat near $\pm\xi_c$. Consequently, there is no way to exploit as before, in the vicinity of $\pm\xi_c$, the oscillations with respect to ξ . Still, we can show the following result, by restricting the order of the spatial localization.

Proposition 5.20. *Assume that $\mathfrak{g} = 0$ and that $\iota = 1$. Then, the nonlinearity plays no role at leading order during long times, in the sense of Definition 5.16.*

Proof. Again, we can suppose that the order of magnitude of the nonlinearity is critical, $\nu + j_1 + j_2 = 2$. We recall that we assume here $\xi_c > 0$. If $\xi_c = 0$, the proof of Proposition 5.19 can be repeated. The argument of Paragraph 5.2.1, that is an integration by parts with respect to the time variable, does apply for frequencies ξ located away from $[-2\xi_c, 2\xi_c]$. Then, it suffices to consider

$$(5.52) \quad \mathcal{W}(T, z) = \frac{1}{2\pi} \int_0^T \int e^{-iT/\varepsilon^2} \tilde{\mathcal{J}}(\varepsilon, T - s, y, z) \mathcal{G}^\varepsilon(s, y) ds dy,$$

where, with ψ as in (5.51), we have put aside

$$(5.53) \quad \tilde{\mathcal{J}}(\varepsilon, s, y, z) := \int e^{i\psi(\xi)/\varepsilon^2} \chi\left(\frac{\xi}{4\xi_c}\right) d\xi = \mathcal{O}(1).$$

By this way, using the uniform boundedness of $\mathcal{U}^{(0)}$ and (5.23), we find

$$|(\mathcal{U}^{(1)} - \mathcal{U}^{(0)})(T, z)| \lesssim \int \chi\left(\frac{y}{r}\right) |\mathcal{U}^{(0)}(s, y)| dy.$$

From Lemma 5.14 with $m = 1$ and $n = 0$, the integral on the right hand side goes to 0 with ε . This argument based on the Dominated Convergence Theorem breaks down when $0 \leq \iota < 1$ because the localizing function $\chi(y/(\varepsilon^{\iota-1}r))$ is no longer integrable *uniformly in ε* . \square

The condition $\iota = 1$ is quite restrictive because it requires a concentrated source. In fact, the difficulties raised by the value $\mathfrak{g} = 0$ are somewhat artificial. They are induced by the localization procedure of Paragraph 3.1.2. At the level of (3.12), the symbol is multiplied by $1 - \chi(\xi)$. This operation does not correspond to a physical phenomenon but rather to a technical simplification.

In the next subsection, we examine the remaining situation $\mathfrak{g} = 1$, especially in the interesting and representative framework of equation (1.36).

5.3. The completely resonant situation. We now prove Theorem 1.4, along with some generalizations. When $\mathfrak{g} = 1$, there exists no $\xi \in \mathbb{R}$ such that $p(\xi) = \mathfrak{g}$. But, due to (3.25), the quantity $p(\xi)$ becomes arbitrarily close to the limiting value $\mathfrak{g} = 1$ when $|\xi|$ goes to infinity. Since large values of ξ are addressed when dealing with (5.5), the effects of this approximated resonance are enhanced in the actual context. The study of (1.36) corresponds to the choice $(j_1, j_2, \nu) = (2, 0, 0)$ with $\omega = -1$, so that indeed $\mathfrak{g} = 1$. The expression \mathcal{G}^ε of (5.4) reduces to

$$(5.54) \quad \mathcal{G}^\varepsilon(T, z) := \chi\left(3 - 2\frac{T}{\mathcal{T}}\right) \chi\left(\frac{\varepsilon z}{r\varepsilon^\iota}\right) \mathcal{U}^{(0)}(T, z)^2.$$

And, in this context, Duhamel's formula (5.5) simply reads

$$(5.55) \quad \mathcal{W}(T, z) = \text{Op}(\mathcal{G})(T, z),$$

where we have introduced the integral operator

$$(5.56) \quad \text{Op}(\mathcal{G})(T, z) := \frac{1}{2\pi} \int_0^T \iint e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1)} \mathcal{G}(s, y) ds dy d\xi.$$

To lighten the notations, we will often write u for $u^{(0)}$ and \mathcal{U} for $\mathcal{U}^{(0)}$. The above integral involves the extended spatial cut-off $|y| \leq r\varepsilon^{\iota-1}$, through the introduction of $\chi(\varepsilon \cdot / r\varepsilon^\iota)$ inside \mathcal{G}^ε . Like in the previous subsection, we impose $\iota \in [0, 1]$. This condition is needed because precise information regarding $\mathcal{U}(s, y)$ is available on condition that $|y| \leq r\varepsilon^{-1}$. Indeed, Section 4 provides a description of the solution $u(t, x)$ to (1.28a) only for $|x| \leq r$, that is only for $|y| \leq r\varepsilon^{-1}$.

The formula (5.54) also involves, through the implementation of $\chi(3 - 2s/\mathcal{T})$, the time cut-off $\mathcal{T} \leq s \leq 2\mathcal{T}$. The choice of $[\mathcal{T}, 2\mathcal{T}]$ is inherited from (5.23) and Lemma 5.13. It is introduced for convenience. It could be relaxed by expanding the time domain of integration to any compact set inside $]0, \infty[$. For instance, it could be adapted to any interval of the form $[\eta\mathcal{T}, \eta^{-1}\mathcal{T}]$ with $\eta \in]0, 1]$.

When $\mathfrak{g} = 1$, the estimates of Lemma 5.13 do not suffice to show the smallness of \mathcal{W} . And for good reason: the nonlinearity plays a role at leading order, and modifies the asymptotic behavior by a nontrivial $\mathcal{O}(1)$ -effect that is revealed by (1.37). To see why, the idea is to use the fine description of the function $\mathcal{U} \equiv \mathcal{U}^{(0)}$ which is provided by Section 4, and to inject it into (5.55). Thus, like in Section 4, we impose $q \geq 2$ and $D \geq 4$.

In Paragraph 5.3.1, we explain how to exchange \mathcal{U}^2 inside (5.54)-(5.55)-(5.55) with a more tractable expression made of a sum of wave packets, without changing the content of \mathcal{W} modulo $o(1)$. In Paragraph 5.3.2, we simplify the content of these wave packets, and we exploit their specific structure in order to replace the triple integral (5.56) by a simple integral in time. In the last Paragraph 5.3.3, we perform the asymptotic analysis, showing Theorem 1.4.

5.3.1. *Reduction to a sum of oscillating waves.* The strategy to analyze \mathcal{W} when $g = 1$ is to approximate \mathcal{U} by a sum of oscillating waves indexed by $k \in \mathcal{K}_s^c$. Looking at \mathcal{U}^2 , this yields a bilinear form indexed by $(k_1, k_2) \in \mathcal{K}_s^c \times \mathcal{K}_s^c$. Given $\beta \in [0, 1]$, define

$$(5.57) \quad \mathcal{DK}_s^c(\beta) := \{(k_1, k_2) \in \mathcal{K}_s^c \times \mathcal{K}_s^c; c\varepsilon^{-\beta} < k_1 + k_2\}.$$

In Section 4.4, we have seen that, for all $k \in \mathcal{K}_s^c$, the function $\Phi_k(t, x; \cdot)$ has at most one critical point (s_k, y_k, ξ_k) which satisfies (4.41), which is non-degenerate, and which is such that $\xi_k = s_k$. Recall that s_k and y_k depend smoothly on (t, x) . Using the function $s_k(t, x)$ issued from Lemma 4.8, we define the auxiliary function

$$(5.58) \quad \begin{aligned} \Psi_k(t, x) := & -xk\pi - xs_k(t, x) + (-1)^k \gamma \cos s_k(t, x) \\ & + [1 - p(k\pi + s_k(t, x))] (k\pi + s_k(t, x) - t). \end{aligned}$$

In the statement below, we eliminate from (5.55) a number of terms which seem difficult to identify precisely, but which are small enough.

Proposition 5.21 (The difference $\mathcal{W} = \mathcal{U}^{(1)} - \mathcal{U}^{(0)}$ as a sum of interacting terms). *Fix $\iota \in [0, 1]$, and β such that*

$$(5.59) \quad \frac{1}{q+1} \leq \beta < \frac{3+\iota}{5} \leq 1.$$

Then, the difference $\mathcal{W} = \mathcal{U}^{(1)} - \mathcal{U}^{(0)}$ is such that

$$(5.60) \quad \mathcal{W}(T, z) = o(1) + \sum_{(k_1, k_2) \in \mathcal{DK}_s^c(\beta)} \mathcal{W}_{k_1, k_2}^\varepsilon(T, z),$$

where $\mathcal{DK}_s^c(\beta)$ is as in (5.57), whereas

$$(5.61) \quad \mathcal{W}_{k_1, k_2}^\varepsilon(T, z) := \varepsilon^2 e^{-2i\gamma/\varepsilon} \text{Op}(\mathcal{G}_{k_1, k_2}^\varepsilon)(T, z).$$

With Ψ_k as in (5.58), the bilinear interaction term $\mathcal{G}_{k_1, k_2}^\varepsilon$ of (5.61) is given by

$$(5.62) \quad \mathcal{G}_{k_1, k_2}^\varepsilon(s, y) := \chi\left(3 - 2\frac{s}{\mathcal{T}}\right) \chi\left(\frac{\varepsilon y}{r\varepsilon^\iota}\right) e^{i(\Psi_{k_1} + \Psi_{k_2})(s/\varepsilon, \varepsilon y)/\varepsilon} \mathcal{B}_{k_1, k_2}^\varepsilon(s/\varepsilon, \varepsilon y).$$

The functions $\mathcal{B}_{k_1, k_2}^\varepsilon(t, x)$ are defined on $[0, 2\mathcal{T}] \times [-r, r]$. They take the form

$$(5.63) \quad \mathcal{B}_{k_1, k_2}^\varepsilon := (2\pi)^{-2} \mathcal{A}_{k_1}^\varepsilon \mathcal{A}_{k_2}^\varepsilon, \quad \mathcal{A}_k^\varepsilon = \mathcal{A}_{k,0}^\varepsilon + \varepsilon \mathcal{A}_{k,1}^\varepsilon,$$

with

$$(5.64) \quad \begin{aligned} \mathcal{A}_{k,0}^\varepsilon(t, x) := & (2\pi)^{3/2} e^{-i(-1)^k \frac{\pi}{4}} |\det S_k(t, x)|^{-1/2} \\ & \times \tilde{a}_k(\varepsilon, s_k(t, x), y_k(t, x), s_k(t, x)), \end{aligned}$$

where the matrix S_k is defined at the beginning of Paragraph 4.4.3, with $\det S_k(t, x)$ as in (4.61), whereas \tilde{a}_k is as in (5.67). For all $i \in \{0, 1\}$, we can find a positive constant C_i such that

$$(5.65) \quad \sup_{\varepsilon \in [0, 1]} \sup_{k \in \mathcal{K}_s^c} \sup_{t \in [0, 2\mathcal{T}]} \sup_{x \in [-r, r]} |\mathcal{A}_{k,i}^\varepsilon(t, x)| \leq C_i.$$

Moreover, for all $\alpha \in \mathbb{N}^2$, we can find a constant $C_\alpha > 0$ such that

$$(5.66) \quad \sup_{\varepsilon \in]0,1]} \sup_{(k_1, k_2) \in \mathcal{DK}_s^c(\beta)} \sup_{t \in [0, 2T]} \sup_{x \in [-r, r]} |\partial_{t,x}^\alpha \mathcal{B}_{k_1, k_2}^\varepsilon(t, x)| \leq C_\alpha.$$

The double sum inside (5.60) is actually finite since $k = \mathcal{O}(\varepsilon^{-1})$ when $k \in \mathcal{K}_s^c$: it runs over at most $\mathcal{O}(\varepsilon^{-2})$ terms. It involves fewer terms when β becomes close to the upper bound $(3 + \iota)/5$ which is important because, as will be seen later, other conditions will be needed on β . In comparison to \mathcal{U}^2 , the advantage of working with an expression like \mathcal{G} in (5.62) is a clear separation between an ‘‘explicit phase’’ Ψ and, in view of (5.66), a ‘‘generalized profile’’ \mathcal{B} . Knowing Ψ , this will allow us to compute more precisely the content of \mathcal{W} .

Proof. To prepare the analysis of \mathcal{W} , we have first to resume the stationary phase arguments playing a central role in Section 4. As already explained, see Lemmas 4.2 and 4.3 together with Proposition 4.5, the harmonic $m = 1$ inside (3.31) is the only one which may contribute to $u \equiv u^{(0)}$ or $\mathcal{U} \equiv \mathcal{U}^{(0)}$ modulo $\mathcal{O}(\varepsilon^\infty)$. Thus, it suffices to deal with $a \equiv a_1$. In coherence with (4.26), we work with

$$(5.67) \quad \begin{aligned} a_k(\varepsilon, s, y, \xi) &= \zeta(k\pi + \xi) a(\varepsilon k\pi + \varepsilon s, k\pi + s, y), \\ \tilde{a}_k(\varepsilon, s, y, \xi) &:= a_k(\varepsilon, s, y, \xi) \chi_{1/4}(s - \xi) \chi_{2\pi/3}(s), \\ \Phi_k(t, x; s, y, \xi) &= (k\pi - t) p(k\pi + \xi) + s [p(k\pi + \xi) - 1] + (x - y)\xi \\ &\quad + sy - (-1)^k \gamma \cos s. \end{aligned}$$

Taking into account (5.67), the wave packet w_k of (4.25) becomes

$$(5.68) \quad w_k(t, x) := \iiint e^{-i\Phi_k(t, x; s, y, \xi)/\varepsilon} \tilde{a}_k(\varepsilon, s, y, \xi) ds dy d\xi.$$

Back to u through (4.20), (4.22) and (4.24), we find

$$(5.69) \quad u(t, x) = \mathcal{O}(\varepsilon^\infty) + \frac{\sqrt{\varepsilon}}{2\pi} \sum_{0 \leq k \leq 2/3 + T/(\pi\varepsilon)} e^{i(-\gamma + k\pi - k\pi x)/\varepsilon} w_k(t, x).$$

Recall the distinction (4.27) between \mathcal{K}_d^c and \mathcal{K}_s^c . Then, use (5.1) to translate (5.69) in terms of \mathcal{U} according to

$$(5.70) \quad \begin{aligned} \mathcal{U}(T, z) &= \frac{1}{\varepsilon} e^{-iT/\varepsilon^2} u\left(\frac{T}{\varepsilon}, \varepsilon z\right) = \mathcal{O}(\varepsilon^\infty) + \frac{1}{2\pi\sqrt{\varepsilon}} e^{-iT/\varepsilon^2} \\ &\quad \times \left(\sum_{k \in \mathcal{K}_d^c} + \sum_{k \in \mathcal{K}_s^c} \right) e^{i(-\gamma + k\pi - k\pi\varepsilon z)/\varepsilon} w_k\left(\frac{T}{\varepsilon}, \varepsilon z\right). \end{aligned}$$

We have proved in Section 4.3, see Lemma 4.7, that for $c > 0$ sufficiently small and for $T/\varepsilon \leq t \leq 2T/\varepsilon$, we have $w_k = \mathcal{O}(\varepsilon^{D-1})$ uniformly on \mathcal{K}_d^c . Since $q \geq 2$ and $D \geq 4$, in (5.70), the sum on \mathcal{K}_d^c accounts for some

$$\mathcal{O}\left(\varepsilon^{-\frac{1}{2} - \frac{1}{q+1} + D-1}\right) = \mathcal{O}(\varepsilon^2).$$

Rearranging the terms, we can retain that

$$(5.71) \quad \mathcal{U}(T, z) = \mathcal{O}(\varepsilon^2) + \frac{e^{-i\gamma/\varepsilon}}{2\pi\sqrt{\varepsilon}} \sum_{k \in \mathcal{K}_s^c} e^{-iT/\varepsilon^2} e^{i(k\pi - k\pi\varepsilon z)/\varepsilon} w_k\left(\frac{T}{\varepsilon}, \varepsilon z\right).$$

When $k \in \mathcal{K}_s^c$, the content of the w_k 's can be more detailed through asymptotic expansions in powers of ε . In the absence of a critical point, by nonstationary phase arguments, we just find $w_k = \mathcal{O}(\varepsilon^\infty)$. Otherwise, we can apply Theorem 4.13 to (5.68) in space dimension $n = 3$, with variables $(s, y, \xi) \in \mathbb{R}^3$. This time, by stationary phase arguments, there exist differential operators denoted by $M_{2j}^k(s, y, \xi; D_{s,y,\xi})$, giving rise to functions

$$(5.72) \quad \mathcal{A}_{k,j}^\varepsilon(t, x) := \left[M_{2j}^k(s, y, \xi, D_{s,y,\xi}) \tilde{a}_k(\varepsilon, \cdot) \right]_{|(s,y,\xi)=(s_k, y_k, \xi_k)(t,x)}$$

such that

$$\left| w_k(t, x) - \varepsilon^{3/2} \sum_{j=0}^{N-1} \varepsilon^j \mathcal{A}_{k,j}^\varepsilon(t, x) e^{-i\Phi_k(t,x;s_k(t,x), y_k(t,x), \xi_k(t,x))/\varepsilon} \right| = \mathcal{O}(\varepsilon^{3/2+N}).$$

The expressions $\mathcal{A}_{k,j}^\varepsilon$ depend smoothly on $\varepsilon \in [0, 1]$ through $\tilde{a}_k(\varepsilon, \cdot)$. In fact, taking into account (4.42), they can be viewed as smooth functions of $\varepsilon \in [0, 1]$ and s_k . Then, the smoothness of \tilde{a}_k and s_k is communicated to $\mathcal{A}_{k,j}^\varepsilon$. The expressions $\mathcal{A}_{k,0}^\varepsilon$ and $\mathcal{A}_{k,1}^\varepsilon$ of (5.63) are defined by (5.72). In particular, combining (4.62) and (4.68), we find (5.64).

As a result of Lemma 4.14, the coefficients of the differential operator M_{2j}^k are uniformly bounded with respect to $k \in \mathcal{K}_s^c$. Due to Assumption 3.11, the same applies to all derivatives of \tilde{a}_k and to the preceding $\mathcal{O}(\varepsilon^{3/2+N})$. Applying Lemma 4.9, we can assert that the quantities $\partial_{t,x}^\alpha \mathcal{A}_{k,j}^\varepsilon(t, x)$ are, for all $\alpha \in \mathbb{N}^2$, uniformly bounded with respect to $\varepsilon \in [0, 1]$, $k \in \mathcal{K}_s^c$, $t \in [0, 2\mathcal{T}]$ and $x \in [-r, r]$. And therefore the functions $\mathcal{A}_{k,j}^\varepsilon$ can be viewed as ‘‘generalized profiles’’ satisfying the condition (5.66), which defines an algebra. As a consequence, coming back to the definition (5.63) of $\mathcal{B}_{k_1, k_2}^\varepsilon$, we have (5.66).

Plug the above expansion of w_k with $N = 2$ inside (5.71). The different phases combine to produce in coherence with the definition (5.58) the new phase

$$\Psi_k(t, x) = -t + k\pi - k\pi x - \Phi_k(t, x; s_k(t, x), y_k(t, x), \xi_k(t, x)).$$

When $N = 2$, the remainder is of size

$$\frac{1}{\sqrt{\varepsilon}} \sum_{k \in \mathcal{K}_s^c} \mathcal{O}(\varepsilon^{3/2+2}) = \varepsilon^{-1/2} \varepsilon^{-1} \mathcal{O}(\varepsilon^{3/2+2}) = \mathcal{O}(\varepsilon^2).$$

Recalling the definition $\mathcal{A}_k^\varepsilon = \mathcal{A}_{k,0} + \varepsilon \mathcal{A}_{k,1}$, there remains

$$(5.73) \quad \mathcal{U}(T, z) = \mathcal{O}(\varepsilon^2) + \varepsilon \frac{e^{-i\gamma/\varepsilon}}{2\pi} \sum_{k \in \mathcal{K}_s^c} e^{i\Psi_k(T/\varepsilon, \varepsilon z)/\varepsilon} \mathcal{A}_k^\varepsilon(T/\varepsilon, \varepsilon z),$$

where, by convention, we set $\mathcal{A}_k^\varepsilon \equiv 0$ when there is no critical point. In (5.73), the sum on \mathcal{K}_s^c runs over $\mathcal{O}(1/\varepsilon)$ terms which are all uniformly of size $\mathcal{O}(1)$. Since ε is in factor of the sum, this may furnish some $\mathcal{O}(1)$. Now, we can compute

$$\mathcal{U}(T, z)^2 = \varepsilon^2 \mathcal{R}^\varepsilon(T, z) + \varepsilon^2 e^{-2i\gamma/\varepsilon} \sum_{k_1 \in \mathcal{K}_s^c} \sum_{k_2 \in \mathcal{K}_s^c} \mathcal{B}_{k_1, k_2}^\varepsilon(T/\varepsilon, \varepsilon z) e^{i(\Psi_{k_1} + \Psi_{k_2})(T/\varepsilon, \varepsilon z)/\varepsilon},$$

where $\mathcal{R}^\varepsilon = \mathcal{O}(1)$ and $\mathcal{B}_{k_1, k_2} = (2\pi)^{-2} \mathcal{A}_{k_1} \mathcal{A}_{k_2}$. Define

$$\mathcal{R}\mathcal{G}^\varepsilon(T, z) := \chi \left(3 - 2 \frac{T}{\mathcal{T}} \right) \chi \left(\frac{\varepsilon z}{r\varepsilon^t} \right) \mathcal{R}^\varepsilon(T, z).$$

Replace \mathcal{U}^2 as given by the above representation inside (5.54) and (5.55). This explains the origin of the bilinear interaction term $\mathcal{G}_{k_1, k_2}^\varepsilon = \mathcal{O}(1)$ of (5.62). Also, with $\mathcal{W}_{k_1, k_2}^\varepsilon$ as in (5.61), this gives rise to

$$(5.74) \quad \mathcal{W} = \varepsilon^2 \text{Op}(\mathcal{R}\mathcal{G}^\varepsilon) + \sum_{k_1 \in \mathcal{K}_s^\varepsilon} \sum_{k_2 \in \mathcal{K}_s^\varepsilon} \mathcal{W}_{k_1, k_2}^\varepsilon.$$

To go further, we have now to evaluate the amplitude of $\mathcal{W}_{k_1, k_2}^\varepsilon$. To this end, we interpret (5.61) as we did with \mathcal{W} at the level of (5.11) and (5.12). In other words, we write $\mathcal{W}_{k_1, k_2}^\varepsilon = \mathcal{W}_{k_1, k_2, l}^\varepsilon + \mathcal{W}_{k_1, k_2, nl}^\varepsilon$ with

$$(5.75) \quad \mathcal{W}_{k_1, k_2, l}^\varepsilon(T, z) := \varepsilon^2 (2\pi) e^{-2i\gamma/\varepsilon} \int_{\mathcal{T}} \mathcal{G}_{k_1, k_2}^\varepsilon(s, z) ds = \mathcal{O}(\varepsilon^2),$$

$$(5.76) \quad \mathcal{W}_{k_1, k_2, nl}^\varepsilon(T, z) := \varepsilon^2 e^{-2i\gamma/\varepsilon} \int_{\mathcal{T}} B_{(T-s)/\varepsilon^2} \mathcal{G}_{k_1, k_2}^\varepsilon(s, z) ds.$$

At the level of (5.76), we perform integrations by parts in y , based on the identity

$$\Lambda(\xi)(1 - \partial_y^2)^{1/4} e^{iy\xi} = e^{iy\xi}, \quad \Lambda(\xi) := (1 + |\xi|^2)^{-1/4} = \mathcal{O}(|\xi|^{-1/2}),$$

to get

$$\mathcal{W}_{k_1, k_2, nl}^\varepsilon(T, z) := \varepsilon^2 e^{-2i\gamma/\varepsilon} \int_{\mathcal{T}} B_{(T-s)/\varepsilon^2}^\Lambda ((1 - \partial_y^2)^{1/4} \mathcal{G}_{k_1, k_2}^\varepsilon)(s, z) ds,$$

where we recall that the operator B_τ^Λ is defined in (5.7). Coming back to (5.58), (5.62) and (5.66), we see that a spatial derivative ∂_y applied on $\mathcal{G}_{k_1, k_2}^\varepsilon(s, y)$ induces some loss of size $\mathcal{O}(k_1 + k_2)$, where the \mathcal{O} is uniform in ε whereas its argument $k_1 + k_2$, which comes from the plane wave part $xk\pi$ inside $\Psi_k(\cdot)$, is not, since $k_1, k_2 \in \mathcal{K}_s^c$. By interpolation,

$$((1 - \partial_y^2)^{1/4} \mathcal{G}_{k_1, k_2}^\varepsilon)(s, y) = \mathcal{O}((k_1 + k_2)^{1/2}).$$

Then, from Corollary 5.5 applied with the optimal choice $\rho = 1/2$, we obtain that

$$(5.77) \quad k_1 + k_2 \leq \varepsilon^{-\beta} \implies \mathcal{W}_{k_1, k_2, nl}^\varepsilon(T, z) = \mathcal{O}(\varepsilon^{2+(\iota-1-\beta)/2}).$$

Concerning $\mathcal{R}\mathcal{G}^\varepsilon$, the same type of arguments (this time involving Corollary 5.5 with simply $\rho = 0$) yields a single contribution of the type

$$(5.78) \quad \varepsilon^2 \text{Op}(\mathcal{R}\mathcal{G}^\varepsilon) = \mathcal{O}(\varepsilon^2) + \mathcal{O}(\varepsilon^{1+(\iota/2)}) = \mathcal{O}(\varepsilon^{1+(\iota/2)}).$$

In (5.74), given β as in (5.59), we split the double sum into

$$\sum_{k_1 \in \mathcal{K}_s^\varepsilon} \sum_{k_2 \in \mathcal{K}_s^\varepsilon} = \sum_{k_1 + k_2 \leq \varepsilon^{-\beta}} + \sum_{(k_1, k_2) \in \mathcal{DK}_s^\varepsilon(\beta)}.$$

In the above right hand side, the first sum involves at most $\mathcal{O}(\varepsilon^{-2\beta})$ terms. Thus, using successively (5.78), (5.75) and (5.77), we obtain

$$(5.79) \quad \begin{aligned} \mathcal{W}(T, z) &= \mathcal{O}(\varepsilon^{1+(\iota/2)}) + \varepsilon^{-2\beta} \mathcal{O}(\varepsilon^2) + \varepsilon^{-2\beta} \mathcal{O}(\varepsilon^{2+(\iota-1-\beta)/2}) \\ &+ \sum_{(k_1, k_2) \in \mathcal{DK}_s^\varepsilon(\beta)} \mathcal{W}_{k_1, k_2}^\varepsilon(T, z). \end{aligned}$$

Computing

$$2 + (\iota - 1 - \beta)/2 - 2\beta = (3 + \iota - 5\beta)/2,$$

we obtain a positive number provided (5.59) is satisfied, hence (5.60). \square

At this stage, it is instructive to revisit the proof of Lemma 5.13 on the basis of the representation (5.73) of \mathcal{U} . When computing $\partial_T \mathcal{U}$ through (5.73), we can focus on the sum involving $k \in \mathcal{K}_s^c$, and neglect the $\mathcal{O}(\varepsilon^2)$ term which is presumed to be negligible. Coming back to the definition (5.58) of Ψ_k and because $\mathcal{A}_k^\varepsilon$ can be viewed as a smooth function of $\varepsilon \in [0, 1]$ and s_k , the most significant contributions are brought by

$$\begin{aligned} \partial_T [\Psi_k(T/\varepsilon, \varepsilon z)/\varepsilon] &= \varepsilon^{-2} (\partial_t \Psi_k)(T/\varepsilon, \varepsilon z) \\ &\lesssim \varepsilon^{-2} [(1 + \mathcal{O}(\varepsilon^{-1} k^{-q-1} + k^{-q})) |\partial_t s_k| + |1 - p(k\pi + s_k)|] \\ &\lesssim \mathcal{O}(\varepsilon^{-2}) |\partial_t s_k| + \mathcal{O}(\varepsilon^{-2} k^{-q}), \\ \partial_T [\mathcal{A}_k^\varepsilon(T/\varepsilon, \varepsilon z)/\varepsilon] &\lesssim \varepsilon^{-1} |\partial_t s_k|. \end{aligned}$$

Using (4.51), we can see that the right hand sides are $\mathcal{O}(\varepsilon^{-2} k^{-q})$. This means that the wave packets composing \mathcal{U} inside (5.73) contain less and less time oscillations as k becomes large. This also implies that the derivative $\partial_T \mathcal{U}$ appears as a sum of terms which may be controlled according to (with $q = 2$)

$$|\partial_T \mathcal{U}(T, z)| \lesssim \varepsilon \sum_{k \in \mathcal{K}_s^c} \varepsilon^{-2} k^{-q} \lesssim \varepsilon^{-1} (\varepsilon^{-1/(q+1)})^{-q+1} \lesssim \varepsilon^{-1} \varepsilon^{1/3} \lesssim \varepsilon^{-2/3}.$$

This corresponds exactly to (5.29), which should be therefore optimal. In the same way, starting from (5.58) and exploiting (4.50) with $\alpha = (0, 1)$, we find that

$$\partial_z [\Psi_k(T/\varepsilon, \varepsilon z)/\varepsilon] = \mathcal{O}(k).$$

This yields

$$|\partial_z \mathcal{U}(T, z)| \lesssim \varepsilon \sum_{k \in \mathcal{K}_s^c} k \lesssim \varepsilon^{-1},$$

which indicates that (5.28) could not be improved any further.

5.3.2. Analysis of the bilinear interaction term. We now come back to the study of the operator Op given by (5.56). Since $p(\xi) - 1 \sim 0$ when $|\xi|$ goes to $+\infty$, for large values of ξ , there is no way to gain some smallness in ε by performing integrations by parts with respect to s . Thus, the strategy is to fix s , to integrate in (y, ξ) , and to exploit the special form of $\mathcal{G}_{k_1, k_2}^\varepsilon$ in order to get a more tractable expression. The formula (5.62) reveals the role of the phase

$$\mathfrak{p}_{k_1, k_2}(t, x) := \Psi_{k_1}(t, x) + \Psi_{k_2}(t, x).$$

On the other hand, since $|x| = \varepsilon |y| \lesssim \varepsilon^\iota$ on the domain of integration, for $\iota \in]0, 1]$, only small values of x are involved. This remark indicates that the impact of \mathfrak{p}_{k_1, k_2} should be mainly driven by its Taylor expansion near $x = 0$, which may be written

$$(5.80) \quad \mathfrak{p}_{k_1, k_2}(t, x) = \mathfrak{p}_{k_1, k_2}^0(t) + x \mathfrak{p}_{k_1, k_2}^1(t) + x^2 \mathfrak{r}_{k_1, k_2}(t, x),$$

where \mathfrak{r}_{k_1, k_2} is the smooth function which is issued from the Lagrange remainder at second order.

On the other hand, we find

$$(5.81a) \quad \mathbf{p}_{k_1, k_2}^0(t) := \boldsymbol{\Psi}_{k_1}(t, 0) + \boldsymbol{\Psi}_{k_2}(t, 0) = \sum_{i=1}^2 \left\{ (-1)^{k_i} \gamma \cos s_{k_i}(t, 0) \right. \\ \left. + [1 - p(k_i \pi + s_{k_i}(t, 0))] (k_i \pi + s_{k_i}(t, 0) - t) \right\},$$

$$(5.81b) \quad \mathbf{p}_{k_1, k_2}^1(t) := \partial_x \boldsymbol{\Psi}_{k_1}(t, 0) + \partial_x \boldsymbol{\Psi}_{k_2}(t, 0) = -(k_1 + k_2) \pi \\ + \sum_{i=1}^2 \left\{ -s_{k_i}(t, 0) - (-1)^{k_i} \gamma \partial_x s_{k_i}(t, 0) \sin s_{k_i}(t, 0) \right. \\ \left. + [1 - p(k_i \pi + s_{k_i}(t, 0))] \partial_x s_{k_i}(t, 0) \right. \\ \left. - p'(k_i \pi + s_{k_i}(t, 0)) \partial_x s_{k_i}(t, 0) (k_i \pi + s_{k_i}(t, 0) - t) \right\}.$$

As stated below, both $\mathbf{p}_{k_1, k_2}^0(t)$ and $\mathbf{p}_{k_1, k_2}^1(t)$ are involved in the asymptotic behavior of $\mathcal{W}_{k_1, k_2}^\varepsilon$ at leading order.

Proposition 5.22 (Simplification of the bilinear interaction terms). *Fix $\iota \in]\iota_-, 1]$ with $\iota_- := (13 - \sqrt{89})/8 < 1/2$, and select β such that*

$$(5.82) \quad \begin{cases} \frac{3}{2(q+1)} \leq \frac{3\iota + (3/\iota)(1-2\iota)}{q+1} < \beta < \frac{3+\iota}{5} < 1 & \text{if } \iota \in]\iota_-, 1/2], \\ \frac{3}{2(q+1)} \leq \frac{1+\iota}{q+1} < \beta < \frac{3+\iota}{5} < 1 & \text{if } \iota \in [1/2, 1]. \end{cases}$$

Then, uniformly in $(k_1, k_2) \in \mathcal{DK}_s^c(\beta)$, with $\mathcal{A}_{k,0}^\varepsilon$ as in (5.64), we have

$$(5.83) \quad \mathcal{W}_{k_1, k_2}^\varepsilon(T, z) = o(\varepsilon^2) \\ + \varepsilon^2 (2\pi)^{-2} e^{-2i\gamma/\varepsilon} \int_0^T \chi \left(3 - 2\frac{s}{T} \right) e^{i\mathbf{p}_{k_1, k_2}^0(s/\varepsilon)/\varepsilon} \\ \times e^{iz\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)} e^{i(T-s)[p \circ \mathbf{p}_{k_1, k_2}^1(s/\varepsilon) - 1]/\varepsilon^2} (\mathcal{A}_{k_1, 0}^\varepsilon \mathcal{A}_{k_2, 0}^\varepsilon)(s/\varepsilon, 0) ds.$$

Note that the reduction of (5.61)-(5.62) to (5.83) is quite striking. As a matter of fact, we got rid of the nonlocal aspect in $dyd\xi$ that is involved by the operator Op of (5.56). Indeed, knowing that \mathcal{G} is as in (5.62), the integral in $dyd\xi$ reduces to a multiplication by the factor exhibited in (5.83). This new presentation has many advantages. The action of Op on L^∞ is not uniformly controlled in $\varepsilon \in]0, 1]$ with apparently, in view of Corollary 5.5, an optimal loss of the type $\varepsilon^{(\iota-1)/2}$ when $\rho = 1/2$. Looking at (5.61), we can only say that $\mathcal{W}_{k_1, k_2}^\varepsilon = \mathcal{O}(\varepsilon^{(3+\iota)/2})$. But, looking at (5.83), as a direct consequence of (5.65), we can assert that

$$(5.84) \quad \forall (k_1, k_2) \in \mathcal{DK}_s^c(\beta), \quad \mathcal{W}_{k_1, k_2}^\varepsilon(T, z) = \mathcal{O}(\varepsilon^2).$$

Proof. The oscillating part inside (5.62) can be decomposed according to

$$(5.85) \quad \begin{aligned} e^{i(\boldsymbol{\Psi}_{k_1} + \boldsymbol{\Psi}_{k_2})(s/\varepsilon, \varepsilon y)/\varepsilon} &= e^{i\mathbf{p}_{k_1, k_2}^0(s/\varepsilon)/\varepsilon} && \textcircled{1} \\ &\times e^{i\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)y} && \textcircled{2} \\ &\times e^{ir^2 \varepsilon^{2\iota-1} \mathbf{r}_{k_1, k_2}(s/\varepsilon, \varepsilon y)(r^{-1} \varepsilon^{1-\iota} y)^2} && \textcircled{3} \end{aligned}$$

The general idea of the proof is the following. At s fixed, the contribution $\textcircled{1}$ does not participate to the integration in (y, ξ) , and hence appears as a simple factor at the level of (5.83). The oscillation $\textcircled{2}$ may be combined with $e^{iy\xi}$ to produce after integration with respect to y a phase shift of size $\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)$ on the Fourier side, that is in ξ . Knowing that $\varepsilon^{1-\iota}|y| \lesssim 1$, the contribution $\textcircled{3}$ may be associated with the cut-off $\chi(r^{-1}\varepsilon^{1-\iota}y)$ to produce, for $\iota \geq 1/2$ in the spatial variable $\tilde{y} := r^{-1}\varepsilon^{1-\iota}y$, a localized non oscillating term that may be viewed as a profile ($\iota = 1/2$), or a some modulation ($\iota > 1/2$). The other case $\iota < 1/2$ is slightly different, more difficult, and will be considered separately. Finally, the integration in ξ operates as an inverse Fourier transform, which is reminiscent of a Dirac mass at $\xi = \mathbf{p}_{k_1, k_2}^1(s/\varepsilon)$.

Now, let us get into the specifics. Using Fubini's theorem together with (5.85), we can interpret (5.56), (5.61) and (5.62) according to

$$(5.86) \quad \begin{aligned} \mathcal{W}_{k_1, k_2}^\varepsilon(T, z) &= \frac{\varepsilon^2 e^{-2i\gamma/\varepsilon}}{2\pi} \int_0^T \chi\left(3 - 2\frac{s}{T}\right) e^{i\mathbf{p}_{k_1, k_2}^0(s/\varepsilon)/\varepsilon} \\ &\times \left\{ \int e^{-iz\xi + i(T-s)(p(\xi)-1)/\varepsilon^2} \right. \\ &\quad \left. \times \left\{ \int e^{iy(\xi + \mathbf{p}_{k_1, k_2}^1(s/\varepsilon))} \mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, r^{-1}\varepsilon^{1-\iota}y) dy \right\} d\xi \right\} ds, \end{aligned}$$

where, taking $\tilde{y} = r^{-1}\varepsilon^{1-\iota}y$ as a new variable to work with a spatial localization of size one, we have introduced

$$(5.87) \quad \mathcal{E}_{k_1, k_2}^\varepsilon(t, \tilde{y}) := \chi(\tilde{y}) e^{ir^2\varepsilon^{2\iota-1}\mathbf{r}_{k_1, k_2}(t, r\varepsilon^\iota\tilde{y})\tilde{y}^2} \mathcal{B}_{k_1, k_2}^\varepsilon(t, r\varepsilon^\iota\tilde{y}).$$

Obviously, we have

$$\forall t \geq 0, \quad \text{supp } \mathcal{E}_{k_1, k_2}^\varepsilon(t, \cdot) \subset \text{supp } \chi \subset [-1, 1].$$

As long as $\iota \in [1/2, 1]$, the expression $\mathcal{E}_{k_1, k_2}^\varepsilon(t, \cdot)$ does not oscillate in \tilde{y} . Otherwise, when $\iota \in [0, 1/2[$, each derivative in \tilde{y} causes a loss of $\varepsilon^{2\iota-1}$. On the other hand, the Lagrange remainder \mathbf{r}_{k_1, k_2} is built with second order derivatives of $\Psi_k(t, \cdot)$ which, in view of (4.50), may be uniformly bounded in $(k_1, k_2) \in \mathcal{DK}_s^c(\beta)$. We can combine this information with (5.66) to see that, for all $j \in \mathbb{N}$, we can find a constant $C_j > 0$ such that

$$\sup_{(k_1, k_2) \in \mathcal{DK}_s^c(\beta)} \sup_{t \geq 0} \sup_{\varepsilon \in]0, 1]} \sup_{\tilde{y} \in \mathbb{R}} |\partial_{\tilde{y}}^j \mathcal{E}_{k_1, k_2}^\varepsilon(t, \tilde{y})| \leq C_j (1 + \varepsilon^{(2\iota-1)j}).$$

By integration by parts in y and then interpolation, it follows that for all $\rho \geq 0$ we can find a constant $C_\rho > 0$ such that, uniformly in $(k_1, k_2) \in \mathcal{DK}_s^c(\beta)$, we have

$$(5.88) \quad \sup_{t \geq 0} \sup_{\varepsilon \in]0, 1]} \sup_{\tilde{\xi} \in \mathbb{R}} |(1 + |\tilde{\xi}|^2)^{\rho/2} \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(t, \cdot))(\tilde{\xi})| \leq C_\rho (1 + \varepsilon^{(2\iota-1)\rho}).$$

In (5.86), the integral with respect to y is exactly

$$r\varepsilon^{\iota-1} \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi}), \quad \tilde{\xi} := -r\varepsilon^{\iota-1}(\xi + \mathbf{p}_{k_1, k_2}^1(s/\varepsilon)).$$

In the interest of simplifying notation, we sometimes note \mathbf{p}^1 in place of $\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)$. Then, the last two lines of (5.86) become

$$\int e^{iz(\mathbf{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi})} e^{i(T-s)[p(-\mathbf{p}^1 - r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - 1]/\varepsilon^2} \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi}) d\tilde{\xi} = \textcircled{4} + \textcircled{5},$$

with, since the function p is even

$$\begin{aligned} \textcircled{4} &:= e^{iz\mathfrak{p}^1} e^{i(T-s)(p(-\mathfrak{p}^1)-1)/\varepsilon^2} \int e^{izr^{-1}\varepsilon^{1-\iota}\tilde{\xi}} \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi}) d\tilde{\xi}, \\ &= e^{iz\mathfrak{p}^1} e^{i(T-s)(p(-\mathfrak{p}^1)-1)/\varepsilon^2} (2\pi)^{-1} \mathcal{F}^{-1} \circ \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\varepsilon^{1-\iota}r^{-1}z) \\ &= e^{iz\mathfrak{p}^1} e^{i(T-s)(p(\mathfrak{p}^1)-1)/\varepsilon^2} (2\pi) \mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \varepsilon^{1-\iota}r^{-1}z). \end{aligned}$$

In view of the definition (5.87) together with (5.66), we have

$$\begin{aligned} \mathcal{E}_{k_1, k_2}^\varepsilon(t, \varepsilon^{1-\iota}r^{-1}z) &= \chi(\varepsilon^{1-\iota}r^{-1}z) e^{i\varepsilon z^2 \mathfrak{r}_{k_1, k_2}(t, \varepsilon z)} \mathcal{B}_{k_1, k_2}^\varepsilon(t, \varepsilon z) \\ &= \chi(\varepsilon^{1-\iota}r^{-1}z) (1 + \mathcal{O}(\varepsilon)) (\mathcal{B}_{k_1, k_2}^\varepsilon(t, 0) + \mathcal{O}(\varepsilon)) \\ &= \chi(\varepsilon^{1-\iota}r^{-1}z) \mathcal{B}_{k_1, k_2}^\varepsilon(t, 0) + \mathcal{O}(\varepsilon). \end{aligned}$$

With (5.63) and (5.65), this becomes

$$\mathcal{E}_{k_1, k_2}^\varepsilon(t, \varepsilon^{1-\iota}r^{-1}z) = \chi(\varepsilon^{1-\iota}r^{-1}z) (2\pi)^{-2} (\mathcal{A}_{k_1, 0}^\varepsilon \mathcal{A}_{k_2, 0}^\varepsilon)(t, 0) + \mathcal{O}(\varepsilon).$$

Plug $\textcircled{4}$ with $\mathcal{E}_{k_1, k_2}^\varepsilon$ as above in place of the integral in $dyd\xi$ (the two last lines) of (5.86). This furnishes the leading-order term of (5.83). Thus, it remains to control the part $\textcircled{5}$, which is

$$\begin{aligned} \textcircled{5} &:= \int e^{iz(\mathfrak{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi})} \mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi}) \\ &\quad \times [e^{i(T-s)(p(-\mathfrak{p}^1 - r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - 1)/\varepsilon^2} - e^{i(T-s)(p(-\mathfrak{p}^1) - 1)/\varepsilon^2}] d\tilde{\xi}. \end{aligned}$$

Since p is even, we have

$$\begin{aligned} &|e^{i(T-s)(p(-\mathfrak{p}^1 - r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - 1)/\varepsilon^2} - e^{i(T-s)(p(-\mathfrak{p}^1) - 1)/\varepsilon^2}| \\ (5.89) \quad &\leq 2 \left| \sin\left(\frac{T-s}{2\varepsilon^2} (p(\mathfrak{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - p(\mathfrak{p}^1))\right) \right| \\ &\leq \frac{|T-s|}{\varepsilon^2} |p(\mathfrak{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - p(\mathfrak{p}^1)|. \end{aligned}$$

For the moment, we assume that $0 < \iota < 1$. In the integral defining $\textcircled{5}$, we can distinguish a part where $\varepsilon^{1-\iota}|\tilde{\xi}| \leq 1$ to take advantage of the smallness of the difference $|p(\mathfrak{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - p(\mathfrak{p}^1)|$, and a part where $1 \leq \varepsilon^{1-\iota}|\tilde{\xi}|$ to benefit from the rapid decreasing of $\mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))$. In other words, we use the fact that

$$(5.90) \quad |\textcircled{5}| \leq \textcircled{5}_-^\iota + \textcircled{5}_+^\iota,$$

with (for $0 < \iota < 1$):

$$\textcircled{5}_\pm^\iota := \frac{|T-s|}{\varepsilon^2} \int_{\pm\varepsilon^{1-\iota}|\tilde{\xi}| \leq \pm 1} |p(\mathfrak{p}^1 + r^{-1}\varepsilon^{1-\iota}\tilde{\xi}) - p(\mathfrak{p}^1)| |\mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi})| d\tilde{\xi}.$$

Since $|p|$ is bounded by 1, exploiting (5.88), we find

$$\begin{aligned} \textcircled{5}_-^\iota &\leq \frac{|T-s|}{\varepsilon^2} \int_{1 \leq \varepsilon^{1-\iota}|\tilde{\xi}|} 2C_\rho (1 + \varepsilon^{(2\iota-1)\rho}) (1 + |\tilde{\xi}|^2)^{-\rho/2} d\tilde{\xi} \\ &\leq 2C_\rho \frac{|T-s|}{\varepsilon^2} \left(\int_{1 \leq \varepsilon^{1-\iota}|\tilde{\xi}|} |\varepsilon^{1-\iota}\tilde{\xi}|^{-\rho} d(\varepsilon^{1-\iota}\tilde{\xi}) \right) \varepsilon^{(1-\iota)(\rho-1)} (1 + \varepsilon^{(2\iota-1)\rho}) \\ &\leq 2C_\rho |T-s| \left(\int_{1 \leq |\xi|} |\xi|^{-\rho} d\xi \right) \varepsilon^{(1-\iota)(\rho-1)-2} (1 + \varepsilon^{(2\iota-1)\rho}). \end{aligned}$$

From now on, we fix some ρ satisfying

$$(5.91) \quad \begin{cases} \rho > -1 + 3/\iota & \geq 5 \quad \text{if } \iota \in]0, 1/2], \\ \rho > 1 + 2/(1 - \iota) & \geq 5 \quad \text{if } \iota \in [1/2, 1[. \end{cases}$$

By this way, we can recover $\textcircled{5}_-^\iota = o(1)$.

To control $\textcircled{5}_+^\iota$, we remark that

$$\begin{aligned} \textcircled{5}_+^\iota &\leq \frac{|T - s|}{\varepsilon^2} \left(\sup_{|\xi| \leq r^{-1}} |p'(\mathbf{p}^1 + \xi)| \right) \int_{|\tilde{\xi}| \leq \varepsilon^{\iota-1}} r^{-1} \varepsilon^{1-\iota} |\tilde{\xi}| C_\rho \frac{(1 + \varepsilon^{(2\iota-1)\rho})}{(1 + |\tilde{\xi}|^2)^{\rho/2}} d\tilde{\xi} \\ &\leq \frac{C_\rho}{r} |T - s| \left(\int (1 + |\tilde{\xi}|^2)^{(1-\rho)/2} d\tilde{\xi} \right) \left(\sup_{|\xi| \leq r^{-1}} |p'(\mathbf{p}^1 + \xi)| \right) \frac{1 + \varepsilon^{(2\iota-1)\rho}}{\varepsilon^{1+\iota}}. \end{aligned}$$

Since $(1 - \rho)/2 \leq -2$, the above integral in $\tilde{\xi}$ is finite. Now, we want to extract some additional smallness from the sup term. To this end, we come back to the definition (5.81b) of \mathbf{p}^1 . Knowing that $(k_1, k_2) \in \mathcal{DK}_s^c(\beta)$, we get

$$\begin{aligned} |\mathbf{p}_{k_1, k_2}^1(t)| &\geq c\pi\varepsilon^{-\beta} - \sum_{i=1}^2 \left\{ |s_{k_i}(t, 0)| + \gamma |\partial_x s_{k_i}(t, 0)| \right. \\ &\quad \left. + [1 - p(k_i\pi + s_{k_i}(t, 0))] |\partial_x s_{k_i}(t, 0)| \right. \\ &\quad \left. + 2\mathcal{T}\varepsilon^{-1} p'(k_i\pi + s_{k_i}(t, 0)) |\partial_x s_{k_i}(t, 0)| \right\}. \end{aligned}$$

Since $k_i \in \mathcal{K}_s^c$, we can exploit (4.50) and (3.23) to obtain

$$(5.92) \quad \begin{aligned} |\mathbf{p}_{k_1, k_2}^1(t)| &\geq c\pi\varepsilon^{-\beta} - 2C_{(0,0)} - 2(\gamma + 1)C_{(0,1)} - \sum_{i=1}^2 2\mathcal{T}\varepsilon^{-1} \mathcal{O}(k_i^{-q-1})C_{(0,1)} \\ &\geq c\pi\varepsilon^{-\beta} - \mathcal{O}(1) \gtrsim \varepsilon^{-\beta}. \end{aligned}$$

Then, from (3.23), we get

$$(5.93) \quad \sup_{|\xi| \leq r^{-1}} |p'(\mathbf{p}^1 + \xi)| \lesssim \varepsilon^{\beta(q+1)}.$$

It follows that we can find ρ satisfying (5.91) and so that $\textcircled{5}_+^\iota = o(1)$ if

$$\begin{cases} -3\iota + (3/\iota)(2\iota - 1) + \beta(q + 1) > 0 & \text{if } \iota \in]0, 1/2], \\ -1 - \iota + \beta(q + 1) > 0 & \text{if } \iota \in [1/2, 1[. \end{cases}$$

This provides with a lower bound for β which must be compatible with (5.59). When $\iota \in [1/2, 1[$, we demand

$$-1 - \iota + (q + 1) \frac{3 + \iota}{5} > 0,$$

a condition which is always satisfied for $q \geq 2$ and $\iota \leq 1$. When $\iota \in]0, 1/2]$, we require

$$-3\iota + (3/\iota)(2\iota - 1) + (q + 1) \frac{3 + \iota}{5} > 0,$$

a condition which boils down, in the less favorable case $q = 2$, to

$$12\iota^2 - 39\iota + 15 < 0.$$

Recalling that $\iota < 1$, this condition implies that $\iota_- < \iota$. This is where the specific value ι_- appears.

There remains to discuss the limiting case $\iota = 1$. The definition of $\textcircled{5}_-^\iota$ for $0 < \iota < 1$ could be extended in the case $\iota = 1$. But the preceding argument does not work when $\iota = 1$, because there is no finite choice of ρ satisfying (5.91). For this reason, we adopt the following alternative definition

$$\textcircled{5}_\pm^1 := \frac{|T-s|}{\varepsilon^2} \int_{\pm|\tilde{\xi}| \leq \pm\varepsilon^{-\mu}} |p(\mathbf{p}^1 + r^{-1}\tilde{\xi}) - p(\mathbf{p}^1)| |\mathcal{F}(\mathcal{E}_{k_1, k_2}^\varepsilon(s/\varepsilon, \cdot))(\tilde{\xi})| d\tilde{\xi}.$$

Use (5.88) with $\rho > 1 + (2/\mu) > 3$. When dealing with the sign $-$, the above shift toward high frequencies $\varepsilon^{-\mu} \leq |\tilde{\xi}|$ allows to recover some smallness. Indeed:

$$\begin{aligned} \textcircled{5}_-^1 &\leq \frac{|T-s|}{\varepsilon^2} \int_{\varepsilon^{-\mu} \leq |\tilde{\xi}|} 2C_\rho (1 + \varepsilon^{(2\iota-1)\rho}) (1 + |\tilde{\xi}|^2)^{-\rho/2} d\tilde{\xi} \\ &\lesssim \frac{|T-s|}{\varepsilon^2} \left(\int_{1 \leq |\varepsilon^\mu \tilde{\xi}|} |\varepsilon^\mu \tilde{\xi}|^{-\rho} d(\varepsilon^\mu \tilde{\xi}) \right) \varepsilon^{\mu(\rho-1)} \\ &\lesssim |T-s| \left(\int_{1 \leq |\xi|} |\xi|^{-\rho} d\xi \right) \varepsilon^{\mu(\rho-1)-2} = o(1). \end{aligned}$$

On the other hand, this does not affect the control of the part with the sign $+$. Taking into account (5.92), we find that

$$(5.94) \quad \forall |\tilde{\xi}| \leq r^{-1}\varepsilon^{-\mu}, \quad |\mathbf{p}^1 + r^{-1}\tilde{\xi}| \gtrsim \varepsilon^{-\beta},$$

and therefore, as before, we have

$$\begin{aligned} \textcircled{5}_+^1 &\leq \frac{|T-s|}{\varepsilon^2} \left(\sup_{|\xi| \leq r^{-1}\varepsilon^{-\mu}} |p'(\mathbf{p}^1 + \xi)| \right) \int_{|\tilde{\xi}| \leq \varepsilon^{-\mu}} r^{-1} |\tilde{\xi}| C_\rho \frac{2}{(1 + |\tilde{\xi}|^2)^{\rho/2}} d\tilde{\xi} \\ &\leq 2C_\rho r^{-1} |T-s| \left(\int (1 + |\tilde{\xi}|^2)^{(1-\rho)/2} d\tilde{\xi} \right) \varepsilon^{\beta(q+1)-2}, \end{aligned}$$

which is some $o(1)$ for β as in (5.82). Note that the above argument applied in the case $0 < \iota < 1$ would not improve (5.82). As a matter of fact, the condition (5.82) is issued from the analysis of $\textcircled{5}_+^1$ which is not modified by using (5.94).

At this stage, we have proved

$$\begin{aligned} \mathcal{W}_{k_1, k_2}^\varepsilon(T, z) &= o(\varepsilon^2) \\ &+ \varepsilon^2 (2\pi)^{-2} e^{-2i\gamma/\varepsilon} \chi(\varepsilon^{1-\iota} r^{-1} z) \int_0^T \chi\left(3 - 2\frac{s}{T}\right) e^{ip_{k_1, k_2}^0(s/\varepsilon)/\varepsilon} \\ &\quad \times e^{iz\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)} e^{i(T-s)[p_{k_1, k_2}^1(s/\varepsilon)-1]/\varepsilon^2} (\mathcal{A}_{k_1, 0}^\varepsilon \mathcal{A}_{k_2, 0}^\varepsilon)(s/\varepsilon, 0) ds. \end{aligned}$$

Since $\iota > 0$, we can substitute inside the term $\chi(\varepsilon^{1-\iota} r^{-1} z)$ with $\chi(0) = 1$, hence (5.83). \square

5.3.3. *The asymptotic analysis.* From now on, we fix β as indicated in (5.82). As a consequence of (5.84), for $j = 1$ or $j = 2$, we have

$$\sum_{\substack{k_j \leq \varepsilon^{-\beta} \\ (k_1, k_2) \in \mathcal{DK}_s^c(\beta)}} |\mathcal{W}_{k_1, k_2}^\varepsilon(T, z)| \lesssim \varepsilon^{-\beta} \varepsilon^{-1} \varepsilon^2 = \mathcal{O}(\varepsilon^{1-\beta}) = o(1).$$

Knowing this, we can replace (5.60) by

$$(5.95) \quad \mathcal{W}(T, z) = o(1) + \sum_{\varepsilon^{-\beta} \leq k_1 \in \mathcal{K}_s^c} \sum_{\varepsilon^{-\beta} \leq k_2 \in \mathcal{K}_s^c} \mathcal{W}_{k_1, k_2}^\varepsilon(T, z).$$

The final stage is to exploit the tools and the arguments of Section 4 in order to pass to the limit at the level of (5.95) when ε goes to zero.

Proposition 5.23 (Proof of Theorem 1.4). *Fix $\iota \in]\iota_-, 1]$ and β as in (5.82). Then, the limit of $\mathcal{W}(T, z)$ when ε goes to zero is given by (1.37) when $z = 2j$ with $j \in \mathbb{Z}$, and by (1.38) otherwise.*

Proof. The starting point is (5.83) together with (5.95). The information $\varepsilon^{-\beta} \leq k_j$ inside (5.95) is crucial because it allows to simplify the content of \mathbf{p}^0 , \mathbf{p}^1 and $\mathcal{A}_{k_j, 0}^\varepsilon$ at the level of (5.83). All these expressions depend on s_k and y_k . But, knowing that $\varepsilon^{-\beta} \leq k$, from (4.55a) (4.55b) and (4.56), we have

$$(5.96) \quad s_k(t, 0) = \mathcal{O}(\varepsilon^{(q+1)\beta-1}), \quad y_k = \mathcal{O}(\varepsilon^{(q+1)\beta-1}),$$

where, taking into account (5.82), we are sure that

$$(5.97) \quad 2(q+1)\beta - 2 > 1, \quad q\beta > (q+1)\beta - 1.$$

Now, look at (5.81a) to extract

$$\begin{aligned} \mathbf{p}_{k_1, k_2}^0(t) &= \sum_{i=1}^2 \left\{ (-1)^{k_i \gamma} + \mathcal{O}(s_{k_i}(t, 0)^2) \right. \\ &\quad \left. + [1 - p(k_i \pi + s_{k_i}(t, 0))] (k_i \pi - t) + \mathcal{O}(k_i^{-q}) |s_{k_i}(t, 0)| \right\}. \end{aligned}$$

From (3.28) and (5.96), we can deduce that

$$\begin{aligned} \mathbf{p}_{k_1, k_2}^0(t) &= \sum_{i=1}^2 \left\{ (-1)^{k_i \gamma} + \mathcal{O}(\varepsilon^{2(q+1)\beta-2}) \right. \\ &\quad \left. - \frac{\ell}{q(q+1)} \left(\frac{1}{(k_i \pi)^{q-1}} - \frac{t}{(k_i \pi)^q} \right) (1 + o(1)) + \mathcal{O}(\varepsilon^{q\beta + (q+1)\beta-1}) \right\}. \end{aligned}$$

Using (5.97), there remains

$$(5.98) \quad \begin{aligned} \mathbf{p}_{k_1, k_2}^0(s/\varepsilon)/\varepsilon &= o(1) + ((-1)^{k_1} + (-1)^{k_2}) (\gamma/\varepsilon) \\ &\quad - \sum_{i=1}^2 \left\{ \frac{\ell \varepsilon^{q-2}}{q(q+1)} \left(\frac{1}{(\varepsilon k_i \pi)^{q-1}} - \frac{s}{(\varepsilon k_i \pi)^q} \right) (1 + o(1)) \right\}. \end{aligned}$$

Next, consider (5.81b). Taking into account (4.50) with again (5.96) and (5.97), this gives rise to

$$\begin{aligned} \mathbf{p}_{k_1, k_2}^1(t) &= -(k_1 + k_2)\pi + \sum_{i=1}^2 \left\{ \mathcal{O}(|s_{k_i}(t, 0)|) + \mathcal{O}(k_i^{-q}) + \mathcal{O}(\varepsilon^{-1} k_i^{-q-1}) \right\} \\ &= -(k_1 + k_2)\pi + \mathcal{O}(\varepsilon^{(q+1)\beta-1}) = -(k_1 + k_2)\pi + o(1). \end{aligned}$$

With (3.23), this implies that

$$p \circ \mathbf{p}_{k_1, k_2}^1(t) = p(-(k_1 + k_2)\pi) + \mathcal{O}((k_1 + k_2)^{-q-1}) \times \mathcal{O}(\varepsilon^{(q+1)\beta-1}),$$

and therefore

$$\begin{aligned}
(5.99) \quad \frac{(p \circ \mathbf{p}_{k_1, k_2}^1(s/\varepsilon) - 1)}{\varepsilon^2} &= \frac{(p(k_1\pi + k_2\pi) - 1)}{\varepsilon^2} + \mathcal{O}(\varepsilon^{2(q+1)\beta-3}) \\
&= \frac{(p(k_1\pi + k_2\pi) - 1)}{\varepsilon^2} + o(1) \\
&= -\frac{\ell\varepsilon^{q-2}}{q(q+1)} \frac{1}{(\varepsilon k_1\pi + \varepsilon k_2\pi)^q} (1 + o(1)) + o(1).
\end{aligned}$$

Finally, we examine (5.64). From (4.63), (5.96) and (5.97), we get easily

$$\begin{aligned}
\det S_k &= (-1)^{k+1} \gamma \cos s_k + \mathcal{O}\left(\frac{1}{\varepsilon k^{q+2}}\right) \\
&= (-1)^{k+1} \gamma + \mathcal{O}(\varepsilon^{2(q+1)\beta-2}) + \mathcal{O}(\varepsilon^{-1+\beta(q+2)}) \\
&= (-1)^{k+1} \gamma + \mathcal{O}(\sqrt{\varepsilon}).
\end{aligned}$$

On the other hand, combining (5.67) and (5.96), we find that

$$\tilde{a}_k(\varepsilon, s_k(t, 0), y_k(t, 0), s_k(t, 0)) = \zeta(k\pi) a(\varepsilon k\pi, k\pi, 0) + \mathcal{O}(\sqrt{\varepsilon}).$$

From Assumption 3.11, we know that $\zeta(k\pi) = 1 + \mathcal{O}(k^{-1}) = 1 + \mathcal{O}(\varepsilon^\beta)$. To make things easier, we replace Assumption 3.12 by the more restrictive condition (1.25). Then, we can infer that

$$\begin{aligned}
(5.100) \quad \tilde{a}_k(\varepsilon, s_k(t, 0), y_k(t, 0), s_k(t, 0)) &= a(\varepsilon k\pi, k\pi, 0) + o(1) \\
&= \underline{a}(\varepsilon k\pi, 0, 0) + o(1).
\end{aligned}$$

From now on, we work with the case $q = 2$, which is the most interesting and also the most difficult situation, because (5.98) and (5.99) contain supplementary contributions modulo $o(1)$. We can decompose the sum inside (5.95) into

$$\sum_{\varepsilon^{-\beta} \leq k_1 \in \mathcal{K}_s^c} \sum_{\varepsilon^{-\beta} \leq k_2 \in \mathcal{K}_s^c} = \sum_{\substack{\{\triangleright, \triangleleft\}; \triangleright \text{ is even or odd} \\ \triangleleft \text{ is even or odd}}} \sum_{\substack{\varepsilon^{-\beta} \leq k_1 \in \mathcal{K}_s^c \\ k_1 \text{ is of type } \triangleright}} \sum_{\substack{\varepsilon^{-\beta} \leq k_2 \in \mathcal{K}_s^c \\ k_2 \text{ is of type } \triangleleft}}.$$

We deal below with the sum corresponding to the choice $(\triangleright, \triangleleft) = (\text{even}, \text{even})$, the other cases being completely similar. When k_1 and k_2 are even, by combining (5.98), (5.99) and (5.100), the product inside (5.83) can be reworded into

$$\begin{aligned}
&e^{i\mathbf{p}_{k_1, k_2}^0(s/\varepsilon)/\varepsilon} e^{iz\mathbf{p}_{k_1, k_2}^1(s/\varepsilon)} e^{i(T-s)(p \circ \mathbf{p}_{k_1, k_2}^1(s/\varepsilon) - 1)/\varepsilon^2} (\mathcal{A}_{k_1, 0}^\varepsilon \mathcal{A}_{k_2, 0}^\varepsilon)(s/\varepsilon, 0) \\
&= o(1) - i(2\pi)^3 \gamma^{-1} e^{2i\gamma/\varepsilon} e^{-iz(k_1+k_2)\pi} d_\varepsilon(\varepsilon\pi k_1, \varepsilon\pi k_2, s),
\end{aligned}$$

with

$$\begin{aligned}
(5.101) \quad d_\varepsilon(\sigma_1, \sigma_2, s) &:= e^{-i\frac{\ell}{6}(\frac{1}{\sigma_1} - \frac{s}{\sigma_1^2})(1+o(1))} e^{-i\frac{\ell}{6}(\frac{1}{\sigma_2} - \frac{s}{\sigma_2^2})(1+o(1))} \\
&\times e^{-i\frac{\ell}{6} \frac{T-s}{(\sigma_1+\sigma_2)^2} (1+o(1))} \underline{a}(\sigma_1, 0, 0) \underline{a}(\sigma_2, 0, 0),
\end{aligned}$$

where the dependence on ε is hidden in the $o(1)$. Passing to the limit when ε goes to 0, the $o(1)$ disappears from (5.101). There remains

$$(5.102) \quad d_0(\sigma_1, \sigma_2, s) := e^{-i\frac{\ell}{6} \frac{T-s}{(\sigma_1+\sigma_2)^2}} b(\sigma_1, s) b(\sigma_2, s),$$

where, in coherence with the introduction, we have introduced

$$b(\sigma, s) := e^{-i\frac{\ell}{6}(\frac{1}{\sigma} - \frac{s}{\sigma^2})} \underline{a}(\sigma, 0, 0).$$

When $z \notin 2\mathbb{Z}$, the Abel sum argument can be readily repeated. As before, the terms $e^{-izk_1\pi}$ or $e^{-izk_2\pi}$ compensate (locally in k_1 or in k_2) after summation. By this way, we can recover (1.38). Otherwise, we can recognize a double Riemann sum with a width of $\varepsilon\pi$, which is

$$-\frac{2i}{\gamma\pi} \int_0^T \chi\left(3 - 2\frac{s}{T}\right) \sum_{\substack{\varepsilon^{-\beta} \leq k_1 \in \mathcal{K}_s^c \\ k_1 \text{ is even}}} \sum_{\substack{\varepsilon^{-\beta} \leq k_2 \in \mathcal{K}_s^c \\ k_2 \text{ is even}}} (\varepsilon\pi)^2 d_\varepsilon(\varepsilon\pi k_1, \varepsilon\pi k_2, s) ds.$$

For all $s \in [0, T]$, the function $d_\varepsilon(\cdot, s)$ is defined on the quadrant $\mathcal{Q} := \mathbb{R}_+^* \times \mathbb{R}_+^*$. It is smooth and bounded on the open domain \mathcal{Q} . The singularities of the exponents near $\sigma_1 = 0$ or $\sigma_2 = 0$ translate only into fast oscillations. Moreover, due to (3.33), the support of the function $d_\varepsilon(\cdot, s)$ is uniformly bounded. In particular, the function $d_0(\cdot, s)$ is integrable on \mathcal{Q} . As a consequence, the double Riemann sum does converge (when $\iota < 1$) towards

$$-\frac{2i}{\gamma\pi} \int_0^T \chi\left(3 - 2\frac{s}{T}\right) \left(\int_0^{+\infty} \int_0^{+\infty} d_0(\sigma_1, \sigma_2, s) d\sigma_1 d\sigma_2 \right) ds.$$

The other choices of (\triangleright , \triangleleft) combine to form (1.37). \square

We conclude with a series of comments.

Remark 5.24 (The origin of the correlation coefficient). *As is well known, the weak limit of a product is in general different from the product of the weak limits. Comparing (1.33) and (1.37), this principle applies in the present context. Indeed, the function d_0 of (5.102) is not the product of $b(\sigma_1, s)$ and $b(\sigma_2, s)$. There is a correlation coefficient which is issued from the multiplication inside (5.83) by*

$$e^{i(T-s)(p \circ \mathfrak{p}_{k_1, k_2}^1(s/\varepsilon) - 1)/\varepsilon^2}.$$

Looking at (5.99), we see that this nonlinear effect depends on dispersive properties through the asymptotic behavior of the symbol p when $|\xi|$ goes to $+\infty$. It measures how the various frequencies k_1 and k_2 interact asymptotically (through their sum) in order to affect the profile.

Remark 5.25 (About the spatial localization). *The limiting case $\iota = 1$ could be incorporated just by multiplying (1.37) by $\chi(r^{-1}z)$. By contrast, the case $\iota < \iota_-$ seems more difficult to assess. By pushing the Taylor expansion (5.80) up to the next order 3, for $\iota \in [1/3, 1/2[$, it would be still possible to separate some explicit “oscillating part” from some “generalized profile”. But then, explicit formulas are no more available, and the presence of some extra oscillations can really change the asymptotic behavior (1.37).*

Remark 5.26 (About the critical cubic nonlinearity). *Come back to Example 5.11. This corresponds to the study of*

$$\begin{aligned} \mathcal{U}^{(1)}(T, z) &= \mathcal{U}^{(0)}(T, z) \\ &+ \frac{1}{2\pi} \int_0^T \int \int e^{-i(z-y)\xi + i\frac{T-s}{\varepsilon^2}(p(\xi)-1)} \chi\left(\frac{y}{r\varepsilon^{\iota-1}}\right) |\mathcal{U}^{(0)}(s, y)|^2 \mathcal{U}^{(0)}(s, y) ds dy d\xi. \end{aligned}$$

The above trilinear interaction involves the phase

$$\begin{aligned} \mathfrak{p}_{k_1, k_2, k_3}(t, x) &:= \Psi_{k_1}(t, x) - \Psi_{k_2}(t, x) + \Psi_{k_3}(t, x) \\ &= \mathfrak{p}_{k_1, k_2, k_3}^0(t, x) - (k_1 - k_2 + k_3)\pi + \dots \end{aligned}$$

and thereby, the Dirac mass argument should select the position $(k_1 - k_2 + k_3)\pi$. It could be expected to obtain triple integrals of the form

$$\int_0^T \chi \left(3 - 2 \frac{s}{\mathcal{T}} \right) \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-i \frac{\xi}{6} \frac{T-s}{(\sigma_1 \pm \sigma_2 \pm \sigma_3)^2}} b(\sigma_1, s) \bar{b}(\sigma_2, s) b(\sigma_3, s) ds d\sigma_1 d\sigma_2 d\sigma_3.$$

But the presence of two wave-numbers (k_1 and k_2) with opposite signs may change the situation. Indeed, for large values of k_1 and k_2 with $k_1 - k_2 = \mathcal{O}(1)$, the asymptotic behavior of p is no more involved when computing $p \circ p$. The analysis is apparently different. It should require further development.

Remark 5.27 (About the full nonlinear case). *The description of the solution u to (1.20) with F_{NL} as in (1.27) is a far more complicated task for a number of reasons. Consider for instance the non completely resonant case, when $\mathfrak{g} \neq 1$. The linearizability has been established by nonstationary phase arguments relying on Lemma 5.13. From this perspective, the global controls provided by Lemma 5.12 do not suffice. As a consequence, to prove Theorem 1.4 in the case of the complete nonlinear equation (as opposed to the first two Picard iterates), we would need estimates similar to those from Lemma 5.13. However, those do not seem to be propagated in an easy way by an iterative scheme. Since the proof of Lemma 5.13 actually relies on the wave packets decomposition of Section 4, extending this wave packets decomposition to a nonlinear framework (like in e.g. [12, 24], or [26, 27, 28], which may be understood as generalizations of WKB methods) might be a way to treat the full nonlinear equation. As evoked in the introduction, we will not pursue this question here.*

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