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# THE GLOBAL EXISTENCE ISSUE FOR THE COMPRESSIBLE EULER SYSTEM WITH POISSON OR HELMHOLTZ COUPLINGS

X. BLANC, R. DANCHIN, B. DUCOMET AND Š. NEČASOVÁ

**ABSTRACT.** We consider the Cauchy problem for the barotropic Euler system coupled to Helmholtz or Poisson equations, in the whole space. We assume that the initial density is small enough, and that the initial velocity is close to some reference vector field  $u_0$  such that the spectrum of  $Du_0$  is bounded away from zero. We prove the existence of a global unique solution with (fractional) Sobolev regularity, and algebraic time decay estimates. Our work extends the papers by D. Serre and M. Grassin [20, 21, 43] dedicated to the compressible Euler system without coupling and integer regularity exponents.

**Keywords:** compressible Euler system, Helmholtz, Poisson, global solution, decay.

**AMS subject classification:** 35Q30, 76N10

## 1. INTRODUCTION

It is well-known from physics textbooks that Euler-Poisson system is relevant either *in the attractive case* for describing plasma dynamics when the compressible electron fluid interacts with a constant charged ionic background [14], or *in the repulsive case* as a model of evolution of self-gravitating gaseous stars [9].

The Euler-Helmholtz system has been introduced more recently to investigate some vortex flows of compressible media [10].

In the plasma context, the inertia of ions being much larger than that of electrons, a standard approximation consists in assuming that the dynamics reduces to the motion of electrons in a constant ionic background. A simple description is given by the following barotropic Euler system:

$$(1.1) \quad \partial_t n + \operatorname{div}_x(nu) = 0,$$

$$(1.2) \quad \partial_t(nu) + \operatorname{div}_x(nu \otimes u) + \frac{1}{m_e} \nabla_x P(n) = \frac{ne}{m_e} \nabla_x \phi,$$

with initial data

$$(1.3) \quad (n, u)(x, 0) = (n_0, u_0)(x),$$

where the electric potential  $\phi$  satisfies the Helmholtz equation

$$(1.4) \quad \Delta \phi - \mu^2 \phi = 4\pi e(n - \bar{n}) \quad \text{with} \quad |\phi| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty.$$

Here,  $n(t, x)$  is the density of electrons of charge  $e$  and mass  $m_e$  ( $t$  denotes the time variable and  $x \in \mathbb{R}^3$ , the Eulerian spatial variable),  $\mu$  is a non negative parameter,  $\bar{n}$  is the density of the ionic background,  $u(t, x)$  is the velocity of the electronic flow and  $P(n) = An^\gamma$  ( $\gamma \geq 1$ ) is the electronic pressure.

In the astrophysical context, as explained in e.g. [9] [13], a simple evolutionary model for the density  $\rho = \rho(t, x)$ , the velocity field  $u = u(t, x)$  and the gravitational potential  $\phi = \phi(t, x)$  is the following self-gravitating Euler-Helmholtz system:

$$(1.5) \quad \partial_t \rho + \operatorname{div}_x(\rho u) = 0,$$

$$(1.6) \quad \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x \Pi(\rho) = -\varrho \nabla_x \phi,$$

$$(1.7) \quad \Delta \phi - \mu^2 \phi = 4\pi \Gamma \rho,$$

with initial data

$$(1.8) \quad (\rho, u)(x, 0) = (\rho_0, u_0)(x),$$

where  $\Pi(\rho) = A\rho^\gamma$  is the barotropic pressure with  $A > 0$  and the adiabatic exponent  $\gamma > 1$ , and  $\Gamma$  is the Newton gravitational constant.

In the attractive case (that is, (1.5), (1.6), (1.7)), as the gravitational ingredient in the system (Newton equation) must produce a convergent potential, at least in the massless case ( $\mu = 0$ ), the density has to be decreasing at large distance and a reasonable model for a gaseous star is thus a compactly supported density materializing the domain of the star matter. This introduces the classical difficulty of vacuum (as first observed by Kato [25]) when symmetrizing the system. Despite this, the corresponding Cauchy problem for Euler-Poisson with vacuum for strong solutions was solved locally in time in the eighties by various authors, among them: Makino [33, 34], Makino-Ukai [37], Makino-Pertame [36], Gamblin [17], Bézard [5], Braun and Karp [7] (see also [35] for a clear survey).

Owing to the fact that no dissipative process takes place in the system, existence results are expected to be only local in time even for small data [12] (see blow-up results of Chemin [11] (3D case) or Makino and Perthame [36] (1D spherically symmetric case)). However, in a series of papers [43] [20] [21], D. Serre and M. Grassin pointed out that under a suitable “dispersive” spectral condition on the initial velocity that will be specified in the next section, and a smallness hypothesis on the initial density, the compressible Euler system admits a unique global smooth solution. They first considered the isentropic case for a compactly supported density, then relaxed the support condition and finally extended the result to the non-isentropic case. In [28], Lécureux-Mercier obtained similar results for the Van der Waals equation of state.

The main purpose of the present work is to show that the Serre-Grassin global existence result extends to the compressible Euler system coupled with the Poisson or Helmholtz equations, *regardless of the sign of the coupling*.

Compared to the pure Euler system, once the global existence problem has been solved, a new situation arises regarding the study of the static solutions of the Euler-Poisson (resp. Euler-Helmholtz) system and their stability. In the gravitational Poisson case, this is a classical question in astrophysics (see [11] [42]) which is difficult to address in the general case for technical reasons (see [30] for a survey in the 1D case).

The rest of the paper is structured as follows. In the next section, we state our main results and give some insights on our strategy. In Section 3, we establish decay estimates in Sobolev spaces first for the multi-dimensional Burgers equation (that is expected to provide an approximate solution for our system), and next for the compressible Euler equation coupled with the Poisson or the Helmholtz equation. The next section is devoted to the proofs of the main global existence results, then we show the uniqueness of the solution. Some technical results like, in particular, first and second order commutator estimates are proved in the appendix.

Throughout the paper,  $C$  denotes a harmless ‘constant’ that may change from line to line, and we use sometimes the notation  $A \lesssim B$  to mean that  $A \leq CB$ . The notation  $A \approx B$  is used if both  $A \lesssim B$  and  $B \lesssim A$ .

## 2. MAIN RESULTS

In order to reformulate Systems (1.1)-(1.4) and (1.5)-(1.8) in a unified way, let us consider the following general system for the density  $\varrho = \varrho(t, x)$ , velocity field  $u = u(t, x)$  and potential  $\phi = \phi(t, x)$ :

$$(2.9) \quad \partial_t \varrho + \operatorname{div}_x(\varrho u) = 0,$$

$$(2.10) \quad \partial_t(\varrho u) + \operatorname{div}_x(\varrho u \otimes u) + \nabla_x p(\varrho) = \kappa \varrho \nabla_x \phi,$$

$$(2.11) \quad \Delta \phi - \mu^2 \phi = 4\pi G \varrho,$$

supplemented with initial data

$$(2.12) \quad (\varrho, u)(0, x) = (\varrho_0, u_0)(x).$$

That system encompasses the previous ones: take  $\bar{n} = 0$ ,  $\varrho(t, x) = n(t, x)$ ,  $p(\varrho) = \frac{1}{m_e} P(n)$ ,  $G = e$  and  $\kappa = e/m_e$  (resp.  $\varrho(t, x) = \rho(t, x)$ ,  $p(\varrho) = \Pi(\rho)$ ,  $G = \Gamma$  and  $\kappa = -1$ ). By analogy with the original systems, the case  $\kappa < 0$  will be named attractive, the case  $\kappa > 0$ , repulsive and the case  $\kappa = 0$ , pure Euler system. For  $\kappa \neq 0$ , the Poisson and Helmholtz couplings correspond to  $\mu = 0$  and  $\mu > 0$ , respectively.

Since our functional framework will force the density to tend to 0 at infinity, the standard symmetrization for the compressible Euler equations is not appropriate. For that reason, we shall rather use the one that has been introduced by T. Makino in [33], namely, we set

$$(2.13) \quad \rho := \frac{2\sqrt{A\gamma}}{\gamma - 1} \varrho^{\frac{\gamma-1}{2}}.$$

After that change of unknown, System (2.9)-(2.12) rewrites

$$(MP) \quad \begin{cases} (\partial_t + u \cdot \nabla) \rho + \frac{\gamma-1}{2} \rho \operatorname{div} u = 0, \\ (\partial_t + u \cdot \nabla) u + \frac{\gamma-1}{2} \rho \nabla \rho = \kappa \nabla \phi, \\ \Delta \phi - \mu^2 \phi = \tilde{G} \rho^{\frac{2}{\gamma-1}} \quad \text{with} \quad \tilde{G} := 4\pi G \left( \frac{(\gamma-1)^2}{4A\gamma} \right)^{\frac{1}{\gamma-1}}. \end{cases}$$

Consider the auxiliary Cauchy problem for the  $d$ -dimensional Burgers equation:

$$(2.14) \quad \partial_t v + v \cdot \nabla_x v = 0,$$

with initial data

$$(2.15) \quad v(0, x) = v_0(x).$$

In the particular case of the Euler equation (that is  $\kappa = 0$ ) and under suitable spectral conditions on  $Du_0$ , it is known [19],[20] that (2.14) is a good approximation of (2.9)-(2.11) provided the density of the fluid is small enough. The formal heuristics is that if one divides by  $\varrho$ , then equation (2.10) becomes

$$(2.16) \quad \partial_t u + u \cdot \nabla_x u + \frac{A\gamma}{\gamma-1} \nabla_x(\varrho^{\gamma-1}) = \kappa \nabla_x \phi,$$

and neglecting  $\phi$  and  $\nabla_x(\varrho^{\gamma-1})$  in (2.16), we get (2.14).

In order to state our results, we need to introduce the following function space:

$$E^s := \{z \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}^d), Dz \in L^\infty \quad \text{and} \quad D^2 z \in H^{s-2}\}.$$

The following result has been first proved in [19], [20] in the case of integer regularity exponents. Here, we extend it to *real* exponents.

**Proposition 2.1.** *Let  $v_0$  be in  $E^s(\mathbb{R}^d)$  with  $s > 1 + d/2$  and satisfy:*

(H0) *there exists  $\varepsilon > 0$  such that for any  $x \in \mathbb{R}^d$ ,  $\text{dist}(\text{Sp}(Dv_0(x)), \mathbb{R}_-) \geq \varepsilon$ ,*

*where  $\text{Sp } A$  denotes the spectrum of the matrix  $A$ .*

*Then (2.14)-(2.15) has a classical solution  $v$  on  $\mathbb{R}_+ \times \mathbb{R}^d$  such that*

$$D^2v \in \mathcal{C}^j(\mathbb{R}_+; H^{s-2-j}(\mathbb{R}^d)) \quad \text{for } j = 0, 1.$$

*Moreover,  $Dv \in \mathcal{C}_b(\mathbb{R}_+ \times \mathbb{R}^d)$  and we have for any  $t \geq 0$  and  $x \in \mathbb{R}^d$ ,*

$$(2.17) \quad Dv(t, x) = (1+t)^{-1}I + (1+t)^{-2}K(t, x)$$

*for some function  $K \in \mathcal{C}_b(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^d \times \mathbb{R}^d)$  that satisfies also*

$$(2.18) \quad \|K(t)\|_{\dot{H}^\sigma} \leq K_\sigma(1+t)^{d/2-\sigma} \quad \text{for all } 0 < \sigma \leq s-1.$$

*Finally, if  $D^2v_0$  is bounded, then we have*

$$(2.19) \quad \|D^2v(t)\|_{L^\infty} \leq C(1+t)^{-3}.$$

The main goal of the paper is to prove the following global existence and uniqueness result for System (2.9)-(2.12).

**Theorem 2.1.** *Let  $s > 1 + d/2$  and  $\gamma > 1$ . Assume that the initial data  $(\rho_0, u_0)$  satisfy:*

- (H1) *there exists  $v_0$  in  $E^{s+1}$  satisfying (H0) and such that  $u_0 - v_0$  is small in  $H^s$ ;*
- (H2)  *$\varrho_0^{\frac{\gamma-1}{2}}$  is small enough in  $H^s$ .*

*Denote by  $v$  the global solution of (2.14)-(2.15) given by Proposition 2.1.*

*Then, there exists a unique global solution  $(\varrho, u, \phi)$  to (2.9)-(2.12), such that*

$$(\varrho^{\frac{\gamma-1}{2}}, u - v) \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{R}^d)),$$

*provided  $d, \gamma$  and  $s$  satisfy the following additional conditions:*

- *Pure Euler case  $\kappa = \mu = 0$ : no additional condition on  $d, \gamma$  and  $s$ .*
- *Poisson case  $\kappa \neq 0$  and  $\mu = 0$ :  $d \geq 3$ ,  $\gamma < \min(\frac{5}{3}, 1 + \frac{4}{d-1})$  and  $s < \frac{3}{2} + \frac{2}{\gamma-1}$  (no upper bound for  $s$  if  $\gamma = 1 + \frac{2}{k}$  for some integer  $k$ ).*
- *Helmholtz case  $\kappa \neq 0$  and  $\mu \neq 0$ : either  $(d, s, \gamma)$  are as for the Poisson case, or  $d \geq 2$ ,  $s < \frac{1}{2} + \frac{2}{\gamma-1}$  and  $\gamma < 1 + \frac{4}{d+1}$  (no upper bound for  $s$  if  $\gamma = 1 + \frac{2}{k}$  for some integer  $k$ ).*

**Remark 2.1.** *The fact that  $\varrho^{\frac{\gamma-1}{2}} \in \mathcal{C}(\mathbb{R}_+; H^s(\mathbb{R}^d))$  implies that  $\varrho \in \mathcal{C}(\mathbb{R}_+; L^1(\mathbb{R}^d))$  whenever  $\gamma \leq 2$  (a condition which is always satisfied in the Poisson and Helmholtz cases with  $d = 3$ ). Hence, the constructed solutions have finite mass, and one can show that it is conserved through the evolution.*

**Remark 2.2.** *Even for the pure Euler case, our results extend those of Grassin and Serre [21, 20, 43] since we are able to treat any dimension  $d \geq 1$  and real Sobolev exponent  $s > 1 + d/2$ .*

The following decay estimates are satisfied by the global solutions constructed in the previous theorem (to be compared with those for the Burgers equation).

**Theorem 2.2.** *Let all the assumptions of Theorem 2.1 be in force. Then, for all  $\sigma$  in  $[0, s]$ , the solution  $(\varrho, u, \phi)$  constructed therein satisfies*

$$\left\| \varrho^{\frac{\gamma-1}{2}}, u - v \right\|_{\dot{H}^\sigma} \leq C_\sigma (1+t)^{\frac{d}{2}-\sigma-\min(1, d(\frac{\gamma-1}{2}))},$$

where  $C_\sigma$  depends only on the initial data, on  $d$ ,  $\gamma$ ,  $\mu$ , and on  $\sigma$ .

Regarding the Poisson coupling in the repulsive case (that is  $\mu = 0$  and  $\kappa > 0$ ), our results have to be compared with the remarkable work of Y. Guo in [19]. There, having  $\kappa > 0$  and  $\nabla \times u_0 = 0$  is fundamental, and the result holds for small perturbations of the state  $(\rho, u) = (\bar{\rho}, 0)$  with  $\bar{\rho}$  a *positive* constant. He also proved that  $\|\rho(t) - \bar{\rho}\|_{L^\infty}$  and  $\|u(t)\|_{L^\infty}$  decay as  $(1+t)^{-p}$  for any  $p < 3/2$  when  $t \rightarrow \infty$ , which does not correspond to the decay we here establish here.

Guo's approach is somehow orthogonal to ours since it strongly relies on the stability properties of the linearized Euler-Poisson system about  $(\bar{\rho}, 0)$ .

### 3. DECAY ESTIMATES IN SOBOLEV SPACES

The goal of the present section is to prove a priori decay estimates in Sobolev spaces first for the multi-dimensional Burgers equation (2.14) and, next, for the discrepancy between the solution to (MP) and to (2.14). Those estimates will play a fundamental role in the proof of our global existence result.

**3.1. Decay estimates for the Burgers equation.** The purpose of this part is to prove Proposition 2.1 for any real regularity exponent  $s > 1 + d/2$ .

Let  $X$  be the flow of  $v$ . The proof relies on the fact that the matrix valued function  $A : (t, y) \mapsto Dv(t, X(t, y))$  satisfies the Ricatti equation

$$A' + A^2 = 0, \quad A|_{t=0} = Dv_0.$$

From Hypothesis **(H0)**, one can deduce that  $v(t, y)$  is defined for all  $t \geq 0$  and  $y \in \mathbb{R}^d$ , and that

$$Dv(t, X(t, y)) = (\text{Id} + tDv_0(y))^{-1} Dv_0(y) \quad \text{with} \quad X(t, y) = y + tv_0(y).$$

Therefore, denoting  $X_t : y \mapsto X(t, y)$ , we have

$$(3.20) \quad Dv(t, x) = \frac{\text{Id}}{1+t} + \frac{K(t, x)}{(1+t)^2}$$

with  $K(t, x) := (1+t)(\text{Id} + tDv_0(X_t^{-1}(x)))^{-1}(Dv_0(X_t^{-1}(x)) - \text{Id}).$

In particular, we have

$$(3.21) \quad \text{div } v(t, y + tv_0(y)) = \frac{d}{1+t} + \frac{\text{Tr } K(t, y + tv_0(y))}{(1+t)^2}.$$

Furthermore, Hypothesis **(H0)** implies that

$$(3.22) \quad \|(\text{Id} + tDv_0)^{-1}\|_{L^\infty} \lesssim (1 + \varepsilon t)^{-1},$$

and  $K$  is thus bounded on  $\mathbb{R}_+ \times \mathbb{R}^d$ .

For the proof of (2.19) we refer to [21]. We proceed with the proof of (2.18) in the case  $\sigma \in ]0, 1[$  (for the integer case, see [21]). To bound  $\tilde{K}_t := (1+t)^{-1}K(t, \cdot)$  in  $\dot{H}^\sigma$ , we use the following characterization of Sobolev norms by finite differences:

$$\|\tilde{K}_t\|_{\dot{H}^\sigma}^2 \approx \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\tilde{K}_t(y) - \tilde{K}_t(x)|^2}{|y - x|^{d+2\sigma}} dx dy$$

and the fact that  $\tilde{K}_t(y) - \tilde{K}_t(x) = I_t^1(x, y) + I_t^2(x, y)$  with

$$\begin{aligned} I_t^1(x, y) &= (\text{Id} + tDv_0(X_t^{-1}(y)))^{-1} (Dv_0(X_t^{-1}(y)) - Dv_0(X_t^{-1}(x))), \\ I_t^2(x, y) &= t(\text{Id} + tDv_0(X_t^{-1}(y)))^{-1} (Dv_0(X_t^{-1}(x)) - Dv_0(X_t^{-1}(y))) \\ &\quad \times (\text{Id} + tDv_0(X_t^{-1}(x)))^{-1} (Dv_0(X_t^{-1}(x)) - \text{Id}). \end{aligned}$$

We see, thanks to (3.22) and to the change of variable  $x' = X_t^{-1}(x)$  and  $y' = X_t^{-1}(y)$ , that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_t^1(x, y)|^2}{|y - x|^{d+2\sigma}} dx dy \leq \frac{C}{(1 + \varepsilon t)^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|Dv_0(y') - Dv_0(x')|^2}{|X_t(y') - X_t(x')|^{d+2\sigma}} J_{X_t}(x') J_{X_t}(y') dx' dy'.$$

Therefore, using the fact that  $\|J_{X_t}\|_{L^\infty} \leq C(1 + \varepsilon t)^d$  and that

$$\begin{aligned} |y' - x'| &= |X_t^{-1}(X_t(y')) - X_t^{-1}(X_t(x'))| \leq \|DX_t^{-1}\|_{L^\infty} |X_t(y') - X_t(x')| \\ &\leq \frac{C}{1 + \varepsilon t} |X_t(y') - X_t(x')|, \end{aligned}$$

we get that

$$(3.23) \quad \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_t^1(x, y)|^2}{|y - x|^{d+2\sigma}} dx dy \leq C(1 + \varepsilon t)^{d-2-2\sigma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|Dv_0(y') - Dv_0(x')|^2}{|y' - x'|^{d+2\sigma}} dx dy.$$

Similarly, (3.22) and the change of variable  $x' = X_t^{-1}(x)$  and  $y' = X_t^{-1}(y)$  imply that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|I_t^2(x, y)|^2}{|y - x|^{d+2\sigma}} dx dy \leq \frac{Ct^2}{(1 + \varepsilon t)^4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|Dv_0(y') - Dv_0(x')|^2}{|X_t(y') - X_t(x')|^{d+2\sigma}} J_{X_t}(x') J_{X_t}(y') dx' dy',$$

and we thus also have (3.23) for  $I_t^2$ . As a conclusion, using the characterization of  $\|Dv_0\|_{\dot{H}^\sigma}$  by finite difference, we get

$$\|\tilde{K}_t\|_{\dot{H}^\sigma} \leq C(1 + \varepsilon t)^{\frac{d}{2}-1-\sigma} \|Dv_0\|_{\dot{H}^\sigma},$$

which gives the desired estimate for  $\sigma \in ]0, 1[$ .

Proving the result for higher order regularity exponents may be done by taking advantage of the explicit formula for partial derivatives of  $\tilde{K}_t$  that has been derived by M. Grassin in [21, p. 1404]. The same method as in the case  $\sigma \in ]0, 1[$  has to be applied to each term of the formula. The details are left to the reader.  $\square$

**3.2. Sobolev estimates for System (MP).** Let  $v$  be the solution of the Burgers equation given by Proposition 2.1. Let us set  $w := u - v$  where  $(\rho, u, \phi)$  stands for a sufficiently smooth solution of (MP) on  $[0, T] \times \mathbb{R}^d$ . Then  $(\rho, w, \phi)$  satisfies:

$$(BB) \quad \begin{cases} (\partial_t + w \cdot \nabla) \rho + \frac{\gamma-1}{2} \rho \operatorname{div} w + v \cdot \nabla \rho + \frac{\gamma-1}{2} \rho \operatorname{div} v = 0, \\ (\partial_t + w \cdot \nabla) w + \frac{\gamma-1}{2} \rho \nabla \rho + v \cdot \nabla w + w \cdot \nabla v = \kappa \nabla \phi, \\ \Delta \phi - \mu^2 \phi = \tilde{G} \rho^{\frac{2}{\gamma-1}}. \end{cases}$$

Our aim is to prove decay estimates in  $\dot{H}^\sigma$  for (BB), for all  $0 \leq \sigma \leq s$ . Clearly, arguing by interpolation, it suffices to consider the border cases  $\sigma = 0$  and  $\sigma = s$ .

Let us start with  $\sigma = 0$ . Taking the  $L^2$  scalar product of the first two equations of (BB) with  $(\rho, w)$  gives

$$(3.24) \quad \frac{1}{2} \frac{d}{dt} \|(\rho, w)\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^d} (\rho^2 + |w|^2) \operatorname{div} v \, dx - \frac{1}{2} \int_{\mathbb{R}^d} (\rho^2 + |w|^2) \operatorname{div} w \, dx \\ + \frac{\gamma-1}{2} \int_{\mathbb{R}^d} \rho^2 \operatorname{div} v \, dx + \int_{\mathbb{R}^d} (w \cdot \nabla v) \cdot w \, dx + \frac{\gamma-1}{4} \int_{\mathbb{R}^d} \rho^2 \operatorname{div} w \, dx = \kappa \int_{\mathbb{R}^d} \nabla \phi \cdot w \, dx.$$

Let

$$(3.25) \quad c_{d,\gamma} := \min \left( 1, d \frac{\gamma-1}{2} \right) - \frac{d}{2}.$$

From (3.24), Cauchy-Schwarz inequality and (3.21), we deduce that, denoting by  $M$  a bound of  $K$ ,

$$(3.26) \quad \frac{1}{2} \frac{d}{dt} \|(\rho, w)\|_{L^2}^2 + \frac{c_{d,\gamma}}{1+t} \|(\rho, w)\|_{L^2}^2 \leq |\kappa| \|\nabla \phi\|_{L^2} \|w\|_{L^2} \\ + \frac{M \max(1 + \frac{d}{2}, |\gamma-2|\frac{d}{2})}{(1+t)^2} \|(\rho, w)\|_{L^2}^2 + \max \left( \frac{1}{2}, \frac{|\gamma-3|}{4} \right) \|\operatorname{div} w\|_{L^\infty} \|(\rho, w)\|_{L^2}^2.$$

Even if  $\kappa = 0$  (i.e. standard compressible Euler equation), proving existence results for (BB) requires a control on  $\|\operatorname{div} w\|_{L^\infty}$ . Owing to the hyperbolicity of the system, it seems (at least in the multi-dimensional case) difficult to go beyond the energy framework, and it is thus natural to look for a priori estimates in Sobolev spaces  $H^s$ . Now, owing to Sobolev embedding, the minimal requirement to get eventually a bound on  $\|\operatorname{div} w\|_{L^\infty}$  is that  $s > 1 + d/2$ .

In order to prove Sobolev estimates, we introduce the homogeneous fractional derivation operator  $\dot{\Lambda}^s$  defined by  $\mathcal{F}(\dot{\Lambda}^s f)(\xi) := |\xi|^s \mathcal{F}f(\xi)$  and observe that  $\rho_s := \dot{\Lambda}^s \rho$ ,  $w_s := \dot{\Lambda}^s w$  and  $\phi_s := \dot{\Lambda}^s \phi$  satisfy (with the usual summation convention over repeated indices)

$$(BB_s) \quad \begin{cases} (\partial_t + w \cdot \nabla) \rho_s + \frac{\gamma-1}{2} \rho \operatorname{div} w_s + v \cdot \nabla \rho_s - s \partial_j v^k \dot{\Lambda}^{-2} \partial_{jk}^2 \rho_s + \frac{\gamma-1}{2} \dot{\Lambda}^s (\rho \operatorname{div} v) \\ \quad \quad \quad = \dot{R}_s^1 + \dot{R}_s^2 + \dot{R}_s^3, \\ (\partial_t + w \cdot \nabla) w_s + \frac{\gamma-1}{2} \rho \nabla \rho_s + v \cdot \nabla w_s - s \partial_j v^k \dot{\Lambda}^{-2} \partial_{jk}^2 w_s + \dot{\Lambda}^s (w \cdot \nabla v) \\ \quad \quad \quad = \dot{R}_s^4 + \dot{R}_s^5 + \dot{R}_s^6 + \kappa \nabla \phi_s, \\ \Delta \phi_s - m^2 \phi_s = \tilde{G} \dot{\Lambda}^s (\rho^{\frac{2}{\gamma-1}}), \end{cases}$$

with

$$\begin{aligned} \dot{R}_s^1 &:= [w, \dot{\Lambda}^s] \nabla \rho, & \dot{R}_s^4 &:= [w, \dot{\Lambda}^s] \nabla w, \\ \dot{R}_s^2 &:= \frac{\gamma-1}{2} [\rho, \dot{\Lambda}^s] \operatorname{div} w, & \dot{R}_s^5 &:= \frac{\gamma-1}{2} [\rho, \dot{\Lambda}^s] \nabla \rho, \\ \dot{R}_s^3 &:= [v, \dot{\Lambda}^s] \nabla \rho - s \partial_j v^k \dot{\Lambda}^{-2} \partial_{jk}^2 \rho_s, & \dot{R}_s^6 &:= [v, \dot{\Lambda}^s] \nabla w - s \partial_j v^k \dot{\Lambda}^{-2} \partial_{jk}^2 w_s. \end{aligned}$$

The definition of  $\dot{R}_s^3$  and  $\dot{R}_s^6$  is motivated by the fact that, according to the classical theory of pseudo-differential operators, we expect to have

$$[\dot{\Lambda}^s, v] \cdot \nabla z = \frac{1}{i} \{ |\xi|^s, v(x) \} (D) \nabla z + \text{remainder}.$$

Computing the *Poisson bracket* in the right-hand side yields

$$\frac{1}{i} \{ |\xi|^s, v(x) \} (D) = -s \partial_j v \dot{\Lambda}^{s-2} \partial_j.$$



Now, taking advantage of (3.20), we get

$$-\partial_j v^k \dot{\Lambda}^{-2} \partial_{jk}^2 z = \frac{1}{1+t} z - \frac{K_{kj}}{(1+t)^2} \dot{\Lambda}^{-2} \partial_{jk}^2 z,$$

and using (3.21) yields

$$\begin{aligned} \dot{\Lambda}^s(\rho \operatorname{div} v) &= \frac{d}{1+t} \rho_s + \frac{1}{(1+t)^2} \dot{\Lambda}^s(\rho \operatorname{Tr} K) \\ \text{and } \dot{\Lambda}^s(w \cdot \nabla v) &= \frac{1}{1+t} w_s + \frac{1}{(1+t)^2} \dot{\Lambda}^s(K \cdot w). \end{aligned}$$

Hence, taking the  $L^2$  inner product of  $(BB_s)$  with  $(\rho_s, w_s)$ , and denoting

$$(3.27) \quad c_{d,\gamma,s} := c_{d,\gamma} + s,$$

we discover that

$$\begin{aligned} (3.28) \quad & \frac{1}{2} \frac{d}{dt} \|(\rho_s, w_s)\|_{L^2}^2 + \frac{c_{d,\gamma,s}}{1+t} \|(\rho_s, w_s)\|_{L^2}^2 \leq |\kappa| \|\nabla \phi_s\|_{L^2} \|w_s\|_{L^2} \\ & + \frac{\|\operatorname{div} w\|_{L^\infty}}{2} \|(\rho_s, w_s)\|_{L^2}^2 + \frac{\gamma-1}{2} \|\nabla \rho\|_{L^\infty} \|\rho_s\|_{L^2} \|w_s\|_{L^2} + \frac{sM}{(1+t)^2} \|(\rho_s, w_s)\|_{L^2}^2 \\ & + \left( \frac{1}{(1+t)^2} \left( \frac{\gamma-1}{2} \|\dot{\Lambda}^s(\rho \operatorname{Tr} K)\|_{L^2} + \|\dot{\Lambda}^s(K \cdot w)\|_{L^2} \right) + \sum_{j=1}^6 \|\dot{R}_s^j\|_{L^2} \right) \|(\rho_s, w_s)\|_{L^2}. \end{aligned}$$

The terms  $\dot{R}_s^1$ ,  $\dot{R}_s^2$ ,  $\dot{R}_s^4$  and  $\dot{R}_s^5$  may be treated according to the homogeneous version of Kato and Ponce commutator estimates (see Lemma 4.3 in the Appendix). We get

$$\begin{aligned} \|\dot{R}_s^1\|_{L^2} &\lesssim \|\nabla \rho\|_{L^\infty} \|\nabla w\|_{\dot{H}^{s-1}} + \|\nabla w\|_{L^\infty} \|\rho\|_{\dot{H}^s}, \\ \|\dot{R}_s^4\|_{L^2} &\lesssim \|\nabla w\|_{L^\infty} \|w\|_{\dot{H}^s}, \\ \|\dot{R}_s^2\|_{L^2} &\lesssim \|\operatorname{div} w\|_{L^\infty} \|\rho\|_{\dot{H}^s} + \|\nabla \rho\|_{L^\infty} \|\operatorname{div} w\|_{\dot{H}^{s-1}}, \\ \|\dot{R}_s^5\|_{L^2} &\lesssim \|\nabla \rho\|_{L^\infty} \|\rho\|_{\dot{H}^s}. \end{aligned}$$

The (more involved) terms  $\dot{R}_s^3$  and  $\dot{R}_s^6$  may be handled thanks to Lemma 4.4. We get

$$\begin{aligned} \|\dot{R}_s^3\|_{L^2} &\lesssim \|\nabla \rho\|_{L^\infty} \|v\|_{\dot{H}^s} + \|\nabla^2 v\|_{L^\infty} \|\nabla \rho\|_{\dot{H}^{s-2}}, \\ \|\dot{R}_s^6\|_{L^2} &\lesssim \|\nabla w\|_{L^\infty} \|v\|_{\dot{H}^s} + \|\nabla^2 v\|_{L^\infty} \|\nabla w\|_{\dot{H}^{s-2}}. \end{aligned}$$

Finally, the standard Sobolev tame estimate yields

$$\begin{aligned} \|\dot{\Lambda}^s(\rho \operatorname{Tr} K)\|_{L^2} &\lesssim \|\rho\|_{L^\infty} \|\nabla(\operatorname{Tr} K)\|_{\dot{H}^{s-1}} + \|\operatorname{Tr} K\|_{L^\infty} \|\rho\|_{\dot{H}^s} \\ \text{and } \|\dot{\Lambda}^s(K \cdot w)\|_{L^2} &\lesssim \|w\|_{L^\infty} \|\nabla K\|_{\dot{H}^{s-1}} + \|K\|_{L^\infty} \|w\|_{\dot{H}^s}. \end{aligned}$$

Plugging all the above estimates in (3.28) and using Proposition 2.1, we end up with

$$\begin{aligned} (3.29) \quad & \frac{1}{2} \frac{d}{dt} \|(\rho, w)\|_{\dot{H}^s}^2 + \frac{c_{d,\gamma,s}}{1+t} \|(\rho, w)\|_{\dot{H}^s}^2 \leq |\kappa| \|\nabla \phi\|_{\dot{H}^s} \|w\|_{\dot{H}^s} \\ & + \frac{CM}{(1+t)^2} \|(\rho, w)\|_{\dot{H}^s}^2 + C \|D^2 v\|_{\dot{H}^{s-1}} \|(\rho, w)\|_{L^\infty} \|(\rho, w)\|_{\dot{H}^s} \\ & + \|(\nabla \rho, \nabla w)\|_{L^\infty} \|(\rho, w)\|_{\dot{H}^s}^2 + \|\nabla^2 v\|_{L^\infty} \|(\rho, w)\|_{\dot{H}^{s-1}} \|(\rho, w)\|_{\dot{H}^s}. \end{aligned}$$

Let us introduce the notation

$$\dot{X}_\sigma := \|(\rho, w)\|_{\dot{H}^\sigma} \quad \text{and} \quad X_\sigma := \sqrt{\dot{X}_0^2 + \dot{X}_\sigma^2} \approx \|(\rho, w)\|_{H^\sigma} \quad \text{for } \sigma \geq 0.$$

Our aim is to bound the right-hand side of (3.26) and (3.29) in terms of  $\dot{X}_0$  and  $\dot{X}_s$  only.

*The standard compressible Euler equations* ( $\kappa = \mu = 0$ ). Arguing by interpolation, we get

$$(3.30) \quad \|(\rho, w)\|_{L^\infty} \lesssim \dot{X}_0^{1-\frac{d}{2s}} \dot{X}_s^{\frac{d}{2s}},$$

$$(3.31) \quad \|(D\rho, Dw)\|_{L^\infty} \lesssim \dot{X}_0^{1-\frac{1}{s}(\frac{d}{2}+1)} \dot{X}_s^{\frac{1}{s}(\frac{d}{2}+1)},$$

$$(3.32) \quad \|(\rho, w)\|_{\dot{H}^{s-1}} \lesssim \dot{X}_0^{\frac{1}{s}} \dot{X}_s^{1-\frac{1}{s}}.$$

Then, plugging these inequalities and those of Proposition 2.1 in (3.26) and (3.29) yields

$$\begin{aligned} \frac{d}{dt} \dot{X}_0 + \frac{c_{d,\gamma}}{1+t} \dot{X}_0 &\lesssim \frac{\dot{X}_0}{(1+t)^2} + \dot{X}_0^{2-\frac{1}{s}(\frac{d}{2}+1)} \dot{X}_s^{\frac{1}{s}(\frac{d}{2}+1)}, \\ \frac{d}{dt} \dot{X}_s + \frac{c_{d,\gamma,s}}{1+t} \dot{X}_s &\lesssim \frac{\dot{X}_s}{(1+t)^2} + \frac{\dot{X}_0^{1-\frac{d}{2s}} \dot{X}_s^{\frac{d}{2s}}}{(1+t)^{s+2-\frac{d}{2}}} + \dot{X}_0^{1-\frac{1}{s}(\frac{d}{2}+1)} \dot{X}_s^{1+\frac{1}{s}(\frac{d}{2}+1)} + \frac{\dot{X}_0^{\frac{1}{s}} \dot{X}_s^{1-\frac{1}{s}}}{(1+t)^3}. \end{aligned}$$

Since we expect to have  $\dot{X}_\sigma \lesssim (1+t)^{-c_{d,\gamma,\sigma}}$  for all  $\sigma \in [0, s]$ , it is natural to introduce the function  $\dot{Y}_\sigma := (1+t)^{c_{d,\gamma,\sigma}} \dot{X}_\sigma$ . However, for technical reasons, we proceed as in [21] and work with

$$(3.33) \quad \dot{Y}_\sigma := (1+t)^{c_{d,\gamma,\sigma}-a} \dot{X}_\sigma \quad \text{for some } a > 1.$$

Then, observing that

$$\frac{d}{dt} \dot{Y}_\sigma + \frac{a}{1+t} \dot{Y}_\sigma = (1+t)^{c_{d,\gamma,\sigma}-a} \left( \frac{d}{dt} \dot{X}_\sigma + \frac{c_{d,\gamma,\sigma}}{1+t} \dot{X}_\sigma \right),$$

the above inequalities for  $\dot{X}_0$  and  $\dot{X}_s$  lead us to

$$\begin{aligned} \frac{d}{dt} \dot{Y}_0 + \frac{a}{1+t} \dot{Y}_0 &\lesssim \frac{\dot{Y}_0}{(1+t)^2} + \frac{\dot{Y}_0^{2-\frac{1}{s}(\frac{d}{2}+1)} \dot{Y}_s^{\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}}, \\ \frac{d}{dt} \dot{Y}_s + \frac{a}{1+t} \dot{Y}_s &\lesssim \frac{\dot{Y}_s}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{d}{2s}} \dot{Y}_s^{\frac{d}{2s}}}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{1}{s}(\frac{d}{2}+1)} \dot{Y}_s^{1+\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{\dot{Y}_0^{\frac{1}{s}} \dot{Y}_s^{1-\frac{1}{s}}}{(1+t)^2}, \end{aligned}$$

whence, introducing the notation  $Y_\sigma := \sqrt{\dot{Y}_0^2 + \dot{Y}_\sigma^2}$ ,

$$\frac{d}{dt} Y_s + \frac{a}{1+t} Y_s \lesssim \frac{Y_s}{(1+t)^2} + \frac{Y_s^2}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}}.$$

At this stage, one can take  $a = 1 + \frac{d}{2} + c_{d,\gamma}$  so that  $a > 1$  is satisfied as soon as  $\gamma > 1$ . Then, we eventually get the differential inequality

$$\frac{d}{dt} Y_s + \frac{a}{1+t} Y_s \lesssim \frac{Y_s}{(1+t)^2} + Y_s^2,$$

from which one can get a global control of  $Y_s$  *without any restriction on  $\gamma > 1$  and  $s > \frac{d}{2} + 1$* , see Proposition 4 in [21], or Lemma 4.1 below<sup>1</sup>.

<sup>1</sup>Having  $\gamma = 1$  would force us to take  $a = 1$  in Lemma 4.1, in which case the expected inequality unfortunately fails.

*Poisson coupling* ( $\kappa \neq 0$  and  $\mu = 0$ ). Compared to the previous paragraph, one has to bound the additional term  $\nabla\phi$  in  $H^s$  or, equivalently,  $\nabla(-\Delta)^{-1}(\rho^{\frac{2}{\gamma-1}})$  in  $L^2$  and in  $\dot{H}^s$ .

To bound the  $L^2$  norm, we will have to assume that  $d \geq 3$ . Then, by virtue of the Sobolev embedding  $L^p \hookrightarrow \dot{H}^{-1}$  with  $\frac{d}{p} = 1 + \frac{d}{2}$ , we get

$$\|\nabla(-\Delta)^{-1}(\rho^{\frac{2}{\gamma-1}})\|_{L^2} \lesssim \|\rho^{\frac{2}{\gamma-1}}\|_{L^p} \lesssim \|\rho\|_{L^q}^{\frac{2}{\gamma-1}} \quad \text{with} \quad q = \frac{2p}{\gamma-1} = \frac{4d}{(\gamma-1)(d+2)}.$$

Bounding the  $L^q$  norm from the Sobolev norm  $H^s$  requires  $q \geq 2$ , whence the constraint

$$(3.34) \quad 1 < \gamma \leq \frac{3d+2}{d+2}.$$

Then, remembering the definition of  $q$ , one may argue by interpolation as follows:

$$\|\nabla(-\Delta)^{-1}(\rho^{\frac{2}{\gamma-1}})\|_{L^2} \lesssim \left( \|\rho\|_{L^2}^{1-\theta} \|\rho\|_{\dot{H}^s}^\theta \right)^{\frac{2}{\gamma-1}} \quad \text{with} \quad \theta = \frac{1}{s} \left( \frac{d}{2} - \frac{(\gamma-1)(d+2)}{4} \right).$$

Using the definition of  $\theta$ , this eventually leads to

$$\frac{d}{dt}\dot{Y}_0 + \frac{a}{1+t}\dot{Y}_0 \lesssim \frac{\dot{Y}_0}{(1+t)^2} + \frac{\dot{Y}_0^{2-\frac{1}{s}(\frac{d}{2}+1)}\dot{Y}_s^{\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{(\dot{Y}_0^{1-\theta}\dot{Y}_s^\theta)^{\frac{2}{\gamma-1}}}{(1+t)^{(c_{d,\gamma}+\frac{d}{2}-a)(\frac{2}{\gamma-1}-1)-1}}.$$

Bounding the  $\dot{H}^s$  norm of the potential term relies on Lemma 4.2 in the Appendix, from which we get if  $\frac{2}{\gamma-1} \geq 1$ , taking  $z = \rho$ ,  $\alpha = \frac{2}{\gamma-1}$  and  $\sigma = s-1$ ,

$$(3.35) \quad \|\nabla(-\Delta)^{-1}\rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^s} \leq \|\rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^{s-1}} \lesssim \|\rho\|_{L^\infty}^{\frac{2}{\gamma-1}-1} \|\rho\|_{\dot{H}^{s-1}} \quad \text{if} \quad 0 \leq s-1 < \frac{2}{\gamma-1} + \frac{1}{2}.$$

Since we need  $s > 1 + \frac{d}{2}$ , Inequality (3.35) yields the following additional restrictions on  $\gamma$ :

$$(3.36) \quad 1 < \gamma \leq 3 \quad \text{and} \quad \gamma < 1 + \frac{4}{d-1}.$$

From that point, arguing as in the previous paragraph, we obtain that

$$\begin{aligned} \frac{d}{dt}\dot{Y}_s + \frac{a}{1+t}\dot{Y}_s &\lesssim \frac{\dot{Y}_s}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{d}{2s}}\dot{Y}_s^{\frac{d}{2s}}}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{1}{s}(\frac{d}{2}+1)}\dot{Y}_s^{1+\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} \\ &\quad + \frac{\dot{Y}_0^{\frac{1}{s}}\dot{Y}_s^{1-\frac{1}{s}}}{(1+t)^2} + \frac{(\dot{Y}_0^{1-\frac{d}{2s}}\dot{Y}_s^{\frac{d}{2s}})^{\frac{2}{\gamma-1}-1}\dot{Y}_0^{\frac{1}{s}}\dot{Y}_s^{1-\frac{1}{s}}}{(1+t)^{(c_{d,\gamma}+\frac{d}{2}-a)(\frac{2}{\gamma-1}-1)-1}}. \end{aligned}$$

Therefore, still denoting  $Y_s := \sqrt{\dot{Y}_0^2 + \dot{Y}_s^2}$ , we get

$$(3.37) \quad \frac{d}{dt}Y_s + \frac{a}{1+t}Y_s \lesssim \frac{Y_s}{(1+t)^2} + \frac{Y_s^2}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{Y_s^{\frac{2}{\gamma-1}}}{(1+t)^{(c_{d,\gamma}+\frac{d}{2}-a)(\frac{2}{\gamma-1}-1)-1}}.$$

At this stage, one may apply Lemma 4.1 with

$$m = \frac{2}{\gamma-1} - 1 \quad \text{and} \quad m' = 2 + \left( \frac{2}{\gamma-1} - 1 \right) \left( a - c_{d,\gamma} - \frac{d}{2} \right)$$

and eventually get, provided  $\|(\rho_0, w_0)\|_{H^s}$  is small enough:

$$(3.38) \quad \sqrt{(1+t)^{2s}\|(\rho, w)\|_{H^s}^2 + \|(\rho, w)\|_{L^2}^2} \leq 2 \frac{e^{\frac{Ct}{1+t}}}{(1+t)^{c_{d,\gamma}}} \|(\rho_0, w_0)\|_{H^s}.$$

Let us emphasize that in order to apply Lemma 4.1, we need  $a > 1$  and  $m' < ma$ , the second condition being equivalent to

$$2 + \left( \frac{2}{\gamma - 1} - 1 \right) \left( a - c_{d,\gamma} - \frac{d}{2} \right) < a \left( \frac{2}{\gamma - 1} - 1 \right),$$

that is to say

$$\min \left( 1, d \left( \frac{\gamma - 1}{2} \right) \right) \left( \frac{2}{\gamma - 1} - 1 \right) > 2.$$

For  $d \geq 3$ , that latter inequality is equivalent to  $\gamma < 5/3$  (and for  $d \leq 2$ , it is never satisfied). Keeping in mind the constraints (3.34), (3.35) and (3.36), one can conclude that (3.38) holds true whenever

$$(3.39) \quad 1 < \gamma < \min \left( \frac{5}{3}, 1 + \frac{4}{d-1} \right), \quad 1 + \frac{d}{2} < s < \frac{3}{2} + \frac{2}{\gamma - 1} \quad \text{and} \quad d \geq 3.$$

Of course, (3.38) is valid for all  $s > 1 + \frac{d}{2}$  in the case where  $\frac{2}{\gamma-1}$  is an integer, that is to say, if  $\gamma = 1 + \frac{2}{k}$  for some integer  $k > \max(3, \frac{d}{2})$ .

*Helmholtz coupling.* Let us finally consider the case  $\mu > 0$  and  $\kappa \neq 0$ . Since  $\mu > 0$ , we have on the one hand, if  $\frac{2}{\gamma-1} - 1 \geq 0$  (that is  $\gamma \leq 3$ ),

$$\|\nabla(\mu^2 - \Delta)^{-1} \rho^{\frac{2}{\gamma-1}}\|_{L^2} \lesssim \|\nabla(\rho^{\frac{2}{\gamma-1}})\|_{L^2} \lesssim \|\rho\|_{L^\infty}^{\frac{2}{\gamma-1}-1} \|\nabla \rho\|_{L^2}$$

and, on the other hand, as may be seen after decomposing into small and large  $\xi$ 's in the definition of the norm in  $\dot{H}^s$ ,

$$\|\nabla(\mu^2 - \Delta)^{-1} \rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^s} \lesssim \|\rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^{s-1}}.$$

The right-hand side may be bounded according to (3.35). Hence, arguing as in the case  $m = 0$ , we end up with

$$\begin{aligned} \frac{d}{dt} \dot{Y}_0 + \frac{a}{1+t} \dot{Y}_0 &\lesssim \frac{\dot{Y}_0}{(1+t)^2} + \frac{\dot{Y}_0^{2-\frac{1}{s}(\frac{d}{2}+1)} \dot{Y}_s^{\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{(\dot{Y}_0^{1-\frac{d}{2s}} \dot{Y}_s^{\frac{d}{2s}})^{\frac{2}{\gamma-1}-1} \dot{Y}_0^{1-\frac{1}{s}} \dot{Y}_s^{\frac{1}{s}}}{(1+t)^{(\frac{d}{2}+c_{d,\gamma}-a)(\frac{2}{\gamma-1}-1)+1}}, \\ \frac{d}{dt} \dot{Y}_s + \frac{a}{1+t} \dot{Y}_s &\lesssim \frac{\dot{Y}_s}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{d}{2s}} \dot{Y}_s^{\frac{d}{2s}}}{(1+t)^2} + \frac{\dot{Y}_0^{1-\frac{1}{s}(\frac{d}{2}+1)} \dot{Y}_s^{1+\frac{1}{s}(\frac{d}{2}+1)}}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{\dot{Y}_0^{\frac{1}{s}} \dot{Y}_s^{1-\frac{1}{s}}}{(1+t)^2} \\ &\quad + \frac{(\dot{Y}_0^{1-\frac{d}{2s}} \dot{Y}_s^{\frac{d}{2s}})^{\frac{2}{\gamma-1}-1} \dot{Y}_0^{\frac{1}{s}} \dot{Y}_s^{1-\frac{1}{s}}}{(1+t)^{(\frac{d}{2}+c_{d,\gamma}-a)(\frac{2}{\gamma-1}-1)-1}}. \end{aligned}$$

As expected, the estimate for  $\dot{Y}_0$  is “better” than in the Poisson case and, at this stage, one does not have any constraint on the dimension. Unfortunately, this is not of much help since the inequality for  $\dot{Y}_s$  is the same as before, leading us again to

$$(3.40) \quad \frac{d}{dt} Y_s + \frac{a}{1+t} Y_s \lesssim \frac{Y_s}{(1+t)^2} + \frac{Y_s^2}{(1+t)^{1+\frac{d}{2}+c_{d,\gamma}-a}} + \frac{Y_s^{\frac{2}{\gamma-1}}}{(1+t)^{(\frac{d}{2}+c_{d,\gamma}-a)(\frac{2}{\gamma-1}-1)-1}}$$

and thus to (3.38) under condition (3.39).

To handle the case  $d = 2$ , one may use the fact that we also have

$$\|\nabla(\mu^2 - \Delta)^{-1} \rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^s} \lesssim \|\rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^s}.$$

Hence, using Lemma 4.2 yields

$$\|\nabla(\mu^2 - \Delta)^{-1} \rho^{\frac{2}{\gamma-1}}\|_{\dot{H}^s} \lesssim \|\rho\|_{L^\infty}^{\frac{2}{\gamma-1}-1} \|\rho\|_{\dot{H}^s} \quad \text{if } s < \frac{2}{\gamma-1} + \frac{1}{2}.$$

Consequently the denominator of the inequality for  $\dot{Y}_s$  becomes  $(1+t)^{(\frac{d}{2}+c_{d,\gamma}-a)(\frac{2}{\gamma-1}-1)}$  so that, using Lemma 4.1, we now arrive, if  $d = 2$ , at the constraints

$$1 < \gamma < \frac{7}{3} \quad \text{and} \quad s < \frac{2}{\gamma-1} + \frac{1}{2}.$$

Using the same method as for  $d \geq 3$  (one has to take  $m' = 1 + (\frac{2}{\gamma-1})(a - c_{d,\gamma} - \frac{d}{2})$  and  $m = \frac{2}{\gamma-1} - 1$  in Lemma 4.1), one can still complete the proof if

$$1 < \gamma < 1 + \frac{4}{d+1} \quad \text{and} \quad s < \frac{2}{\gamma-1} + \frac{1}{2}.$$

Note that, if  $\gamma = 1 + \frac{2}{k}$  for some  $k > 2$ , then  $\frac{2}{\gamma-1}$  is an integer so that the only remaining constraint on  $s$  is that  $s > 1 + \frac{d}{2}$ .

#### 4. PROVING THEOREM 2.1

A number of works have been devoted to the local existence issue for the Euler-Poisson system, in various functional settings (see e.g. Makino [33], Gamblin [17], Bézard [5] and Brauer-Karp [7]). However, to the best of our knowledge, none of them treats also the case  $\mu \neq 0$  and Sobolev spaces with fractional regularity (furthermore, our data are not exactly in uniformly local Sobolev spaces). For the reader's convenience, we sketch the proof of global existence for (2.9)–(2.12) in the functional setting of Theorem 2.1, then establish uniqueness by means of a classical energy method.

**4.1. Existence.** Here we are given  $(\rho_0, u_0)$  satisfying the assumptions of Theorem 2.1. Our goal is to prove the existence of a global-in-time solution.

*Step 1: Solving an approximate system.* Fix some cut-off function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  supported in, say, the ball  $B(0, 4/3)$  and with value 1 on  $B(0, 3/4)$ . Set  $v^n := \chi(n^{-1} \cdot) v$ . Let  $J_n$  be the Friedrichs' truncation operator defined by  $J_n z := \mathcal{F}^{-1}(1_{B(0,n)} \mathcal{F} z)$ .

For all  $n \geq 1$ , we consider the following regularization of  $(BB)$ :

$$(BB_n) \quad \begin{cases} \partial_t \rho + J_n((v^n + J_n w) \cdot \nabla J_n \rho) + \frac{\gamma-1}{2} J_n(J_n \rho \operatorname{div}(v^n + J_n w)) = 0, \\ \partial_t w + J_n((v^n + J_n w) \cdot \nabla J_n w) + \frac{\gamma-1}{2} J_n(J_n \rho \nabla J_n \rho) + J_n(J_n w \cdot \nabla v^n) \\ \quad = -\kappa \tilde{G} J_n \nabla(\mu^2 \operatorname{Id} - \Delta)^{-1}(J_n \rho)^{\frac{2}{\gamma-1}}, \end{cases}$$

supplemented with initial data  $(J_n \rho_0, J_n w_0)$ .

Note that  $v^n$  is in  $\mathcal{C}(\mathbb{R}_+; H^{s+1})$ . Hence the above system may be seen as an ODE in  $L^2(\mathbb{R}^d; \mathbb{R} \times \mathbb{R}^d)$ . Applying the standard Cauchy-Lipschitz theorem thus ensures that there exists a unique maximal solution  $(\rho^n, w^n) \in \mathcal{C}^1([0, T^n]; L^2)$  to  $(BB_n)$ .

Now, from  $J_n^2 = J_n$ , we deduce that  $(J_n \rho^n, J_n w^n)$  also satisfies  $(BB_n)$ . Hence, uniqueness of the solution entails that  $J_n \rho^n = \rho^n$  and  $J_n w^n = w^n$ . In other words,  $(\rho^n, w^n)$  is spectrally localized in the ball  $B(0, n)$ , and one can thus assert that  $(\rho^n, w^n) \in \mathcal{C}^1([0, T^n]; H^\sigma)$

for all  $\sigma$  in  $\mathbb{R}$ , and actually satisfies:

$$(4.41) \quad \begin{cases} \partial_t \rho^n + J_n((v^n + w^n) \cdot \nabla \rho^n) + \frac{\gamma-1}{2} J_n(\rho^n \operatorname{div}(v^n + w^n)) = 0, \\ \partial_t w^n + J_n((v^n + w^n) \cdot \nabla w^n) + \frac{\gamma-1}{2} J_n(\rho^n \nabla \rho^n) + J_n(w^n \cdot \nabla v^n) \\ \quad \quad \quad = -\kappa \tilde{G} J_n \nabla (\mu^2 \operatorname{Id} - \Delta)^{-1} (\rho^n)^{\frac{2}{\gamma-1}}, \\ (\rho^n, w^n)|_{t=0} = (J_n \rho_0, J_n w_0). \end{cases}$$

*Step 2: Uniform a priori estimates in the solution space.* Since  $J_n$  is an orthogonal projector in any Sobolev space and  $(J_n \rho^n, J_n w^n) = (\rho^n, w^n)$ , one can repeat verbatim (and rigorously) the computations of subsection 3.2. The only change is that since  $Dv^n = Dv + \mathcal{O}(n^{-1})$ , the final estimates therein only hold on the time interval  $[0, \min(cn, T_n))$  for some  $c > 0$ , which eventually implies that  $T_n \geq cn$ .

The conclusion of this step is that for any fixed  $T > 0$ , the couple  $(\rho^n, w^n)$  for  $n$  large enough is defined on  $[0, T]$ , belongs to  $\mathcal{C}^1([0, T]; H^\sigma)$  for all  $\sigma \in \mathbb{R}$  and is bounded in  $L^\infty([0, T]; H^s)$ .

*Step 3: Convergence.* Let us fix some  $T > 0$ . Given the uniform bounds of the previous step, the weak \* compactness theorem ensures that (up to an omitted extraction), there exists some  $(\rho, w) \in L^\infty([0, T]; H^s)$  so that

$$(\rho^n, w^n) \rightharpoonup (\rho, w) \text{ weak } * \text{ in } L^\infty([0, T]; H^s).$$

Furthermore, computing  $\partial_t \rho^n$  and  $\partial_t w^n$  by means of (4.41) and using standard product laws in Sobolev spaces, one can prove that for all  $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ ,  $(\theta \partial_t \rho^n, \theta \partial_t w^n)$  is bounded in  $L^\infty([0, T]; H^{s-1})$ . Hence, from Aubin-Lions lemma, interpolation and Cantor diagonal process, we gather that (still up to an omitted extraction), we have

$$(\theta \rho^n, \theta w^n) \rightarrow (\theta \rho, \theta w) \text{ in } L^\infty([0, T]; H^{s'}) \text{ for all } \theta \in \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ and } s' < s.$$

This allows to pass to the limit in (4.41) and to conclude that  $(\rho, w)$  satisfies (BB) on  $[0, T] \times \mathbb{R}^d$ . Of course, since  $T > 0$  is arbitrary,  $(\rho, w)$  actually satisfies (BB) on  $\mathbb{R}_+ \times \mathbb{R}^d$  and belongs to  $L_{loc}^\infty(\mathbb{R}_+; H^s)$ .

*Step 4: Time continuity.* That  $(\rho, w)$  lies in  $\mathcal{C}(\mathbb{R}_+; H^s)$  may be achieved either by adapting the arguments of Kato in [25] or those of [1, Chap. 4]. Note that ref. [25] allows in addition to prove the continuity of the flow map in the space  $\mathcal{C}(\mathbb{R}_+; H^s)$ .

**4.2. Uniqueness.** This part is devoted to the proof of the following uniqueness result.

**Proposition 4.1.** *Let the assumptions of Theorem 2.1 be in force and assume that  $(\varrho^1, u^1)$  and  $(\varrho^2, u^2)$  are two solutions of (MP) on  $[0, T] \times \mathbb{R}^d$  such that  $(\varrho^i)^{\frac{\gamma-1}{2}} \in \mathcal{C}([0, T]; L^2 \cap L^\infty)$ ,  $\nabla \varrho^i \in L^\infty([0, T] \times \mathbb{R}^d)$ ,  $(u^i - v^i) \in \mathcal{C}([0, T]; L^2)$  and  $\nabla u^i \in L^1([0, T]; L^\infty)$  for  $i = 1, 2$ .*

*If, in addition,  $(\varrho^1, u^1)$  and  $(\varrho^2, u^2)$  coincide at time  $t = 0$ , then the two solutions are the same on  $[0, T] \times \mathbb{R}^d$ .*

**Proof:** We define  $w^i = u^i - v^i$  for  $i = 1, 2$ , and  $(\delta \rho, \delta w, \delta \phi) := (\rho^2 - \rho^1, w^2 - w^1, \phi^2 - \phi^1)$ . Since  $(\varrho^1, u^1)$  and  $(\varrho^2, u^2)$  coincide at time  $t = 0$ , so do  $v^1$  and  $v^2$ . Hence, uniqueness for

Burgers equation implies  $v^1 = v^2 := v$ . We thus have

$$\begin{cases} (\partial_t + w^2 \cdot \nabla) \delta \rho + \frac{\gamma-1}{2} \rho^2 \operatorname{div} \delta w + v \cdot \nabla \delta \rho + \frac{\gamma-1}{2} \delta \rho \operatorname{div} v = -\delta w \cdot \nabla \rho^1 - \frac{\gamma-1}{2} \delta \rho \operatorname{div} w^1, \\ (\partial_t + w^2 \cdot \nabla) \delta w + \frac{\gamma-1}{2} \rho^2 \nabla \delta \rho + v \cdot \nabla \delta w + \delta w \cdot \nabla v \\ \quad \quad \quad = -\delta w \cdot \nabla w^1 - \frac{\gamma-1}{2} \delta \rho \nabla \rho^1 + \kappa \nabla \delta \phi, \\ \Delta \delta \phi - \mu^2 \delta \phi = \tilde{G}((\rho^2)^{\frac{2}{\gamma-1}} - (\rho^1)^{\frac{2}{\gamma-1}}). \end{cases}$$

Taking the  $L^2$  scalar product of the first and second equations with  $\delta \rho$  and  $\delta w$ , respectively, and arguing as for proving (3.24), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(\delta \rho, \delta w)\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{R}^d} (\delta \rho^2 + |\delta w|^2) \operatorname{div} (v + w^2) dx + \frac{\gamma-1}{2} \int_{\mathbb{R}^d} \delta \rho \delta w \cdot \nabla (\rho^2 - \rho^1) dx \\ &\quad - \int_{\mathbb{R}^d} \delta w \cdot \nabla \rho^1 \delta \rho dx - \frac{\gamma-1}{2} \int_{\mathbb{R}^d} (\delta \rho)^2 \operatorname{div} (v + w^1) dx \\ &\quad - \int_{\mathbb{R}^d} (\delta w \cdot \nabla (v + w^1)) \cdot \delta w dx + \kappa \int_{\mathbb{R}^d} \nabla \delta \phi \cdot \delta w dx, \end{aligned}$$

whence

$$\frac{d}{dt} \|(\delta \rho, \delta w)\|_{L^2}^2 \leq C \|\nabla v, \nabla w^1, \nabla w^2, \nabla \rho^1, \nabla \rho^2\|_{L^\infty} \|(\delta \rho, \delta w)\|_{L^2}^2 + |\kappa| \|\nabla \delta \phi\|_{L^2} \|\delta w\|_{L^2}.$$

After time integration, this gives for all  $t \in [0, T]$  (since  $\delta \rho|_{t=0} = 0$  and  $\delta w|_{t=0} = 0$ ):

$$\|(\delta \rho, \delta w)(t)\|_{L^2} \leq C \int_0^t \|\nabla v, \nabla w^1, \nabla w^2, \nabla \rho^1, \nabla \rho^2\|_{L^\infty} \|(\delta \rho, \delta w)\|_{L^2} d\tau + |\kappa| \int_0^t \|\nabla \delta \phi\|_{L^2} d\tau.$$

- In the pure Euler case (namely  $\kappa = 0$ ), applying Gronwall lemma readily gives  $(\delta \rho, \delta w) \equiv (0, 0)$  on  $[0, T]$ .
- In the Poisson case ( $\kappa \neq 0$  and  $\mu = 0$ ) with  $d \geq 3$  one can use the fact that operator  $\nabla(-\Delta)^{-1}$  maps  $L^p$  (with  $\frac{1}{p} = \frac{1}{d} + \frac{1}{2}$ ) to  $L^2$  and that

$$(\rho^2)^{\frac{2}{\gamma-1}} - (\rho^1)^{\frac{2}{\gamma-1}} = \frac{2}{\gamma-1} \delta \rho \int_0^1 (\rho^1 + \eta \delta \rho)^{\frac{2}{\gamma-1}-1} d\eta.$$

Hence

$$\|\nabla \delta \phi\|_{L^2} \leq C \|\delta \rho\|_{L^2} \left\| \int_0^1 (\rho^1 + \eta \delta \rho)^{\frac{2}{\gamma-1}-1} d\eta \right\|_{L^d}.$$

Note that since  $\gamma \leq 1 + 2d/(d+2)$ , and  $\rho^i \in L^\infty(0, T; L^2 \cap L^\infty)$  for  $i = 1, 2$ , we are guaranteed that the integral is bounded in terms of  $\rho^1$  and  $\rho^2$ . Hence, we have

$$\|\nabla \delta \phi\|_{L^2} \leq C_{\rho^1, \rho^2} \|\delta \rho\|_{L^2}$$

and applying Gronwall lemma still ensures uniqueness.

- In the Helmholtz case ( $\kappa \neq 0$  and  $\mu \neq 0$ ) with  $d \geq 2$ , we just use the fact that  $\nabla(\mu - \Delta)^{-1}$  maps  $L^2$  to itself, and thus, arguing as above,

$$\|\nabla \delta \phi\|_{L^2} \leq C \|\delta \rho\|_{L^2} \left\| \int_0^1 (\rho^1 + \eta \delta \rho)^{\frac{2}{\gamma-1}-1} d\eta \right\|_{L^\infty}.$$

Since  $\rho^1$  and  $\rho^2$  are bounded, one can apply Gronwall lemma to prove that the two solutions coincide.

This completes the proof of the proposition.  $\square$

## APPENDIX

For the reader's convenience, we prove here some technical results that have been used in the paper. Let us start with an ODE estimate.

**Lemma 4.1.** *Let  $Y : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfy the differential inequality*

$$\frac{d}{dt}Y + \frac{a}{1+t}Y \leq C \left( \frac{Y}{(1+t)^2} + Y^2 + (1+t)^{m'-1}Y^{m+1} \right) \quad \text{on } \mathbb{R}_+$$

*for some  $C > 0$ ,  $a > 1$ ,  $m > 0$  and  $m' < ma$ . Then, there exists  $c = c(a, m, m', C)$  such that if  $Y(0) \leq c$ , then we have*

$$Y(t) \leq 2e^{\frac{Ct}{1+t}} \frac{Y_0}{(1+t)^a} \quad \text{for all } t \geq 0.$$

*Proof.* We set  $Z(t) := (1+t)^a e^{-\frac{Ct}{1+t}} Y(t)$  and observe that the above differential inequality recasts in

$$\frac{d}{dt}Z \leq C(1+t)^{-a} e^{\frac{Ct}{1+t}} Z^2 + C(1+t)^{m'-ma-1} e^{\frac{Cmt}{1+t}} Z^{m+1},$$

which implies that

$$(4.42) \quad \frac{d}{dt}Z \leq Ce^C (1+t)^{-a} Z^2 + C(1+t)^{m'-ma-1} e^{Cm} Z^{m+1}.$$

The conclusion stems from a bootstrap argument : let  $Z_0 := Z(0)$  and assume that

$$(4.43) \quad Z(t) \leq 2Z_0 \quad \text{on } [0, T].$$

Then (4.42) implies that

$$\frac{d}{dt}Z \leq 4Ce^C (1+t)^{-a} Z_0^2 + C(1+t)^{m'-ma-1} e^{Cm} (2Z_0)^{m+1}.$$

Hence, integrating in time, we discover that on  $[0, T]$ , we have

$$Z(t) \leq Z_0 + \frac{4Ce^C}{a-1} Z_0^2 (1 - (1+t)^{1-a}) + \frac{Ce^{Cm} (2Z_0)^{m+1}}{ma - m'} (1 - (1+t)^{m'-ma}).$$

Let us discard the obvious case  $Z_0 = 0$ . Then, if  $Z_0$  is so small as to satisfy

$$\frac{4Ce^C}{a-1} Z_0 + \frac{2^{m+1} Ce^{Cm} Z_0^m}{ma - m'} \leq 1,$$

the above inequality ensures that we actually have  $Z(t) < 2Z_0$  on  $[0, T]$ . Therefore the supremum of  $T > 0$  satisfying (4.43) has to be infinite.  $\square$

The following result has been used to bound the potential term.

**Lemma 4.2.** *Let  $\alpha \geq 1$  and  $0 \leq \sigma < \alpha + \frac{1}{2}$ . Then we have the inequality*

$$(4.44) \quad \| |z|^\alpha \|_{\dot{H}^\sigma} \lesssim \|z\|_{L^\infty}^{\alpha-1} \|z\|_{\dot{H}^\sigma}.$$

*Proof.* The corresponding inequality for nonhomogeneous Sobolev spaces, namely

$$\| |z|^\alpha \|_{H^\sigma} \lesssim \|z\|_{L^\infty}^{\alpha-1} \|z\|_{H^\sigma}$$

is a particular case of [24, Thm. 1.2].



Applying that inequality to  $z(\lambda \cdot)$  for all  $\lambda > 0$ , and using the fact that

$$\| |z(\lambda \cdot)|^\alpha \|_{\dot{H}^\sigma} = \lambda^{\sigma - \frac{d}{2}} \| |z|^\alpha \|_{\dot{H}^\sigma} \quad \text{and} \quad \| z(\lambda \cdot) \|_{H^\sigma} \approx \lambda^{-\frac{d}{2}} (\| z \|_{L^2} + \lambda^\sigma \| z \|_{\dot{H}^\sigma}),$$

we get the desired inequality after having  $\lambda$  tend to  $+\infty$ .  $\square$

We also used the following first order commutator estimate that corresponds to the end of [29, Rem. 1.5], or may be seen as a straightforward adaptation to the homogeneous framework of the second inequality of [2, Lemma A.2]:

**Lemma 4.3.** *If  $s > 0$ , then we have:*

$$\| [v, \dot{\Lambda}^s] u \|_{L^2} \lesssim \| v \|_{\dot{H}^s} \| u \|_{L^\infty} + \| \nabla v \|_{L^\infty} \| u \|_{\dot{H}^{s-1}}.$$

The following second order commutator inequality played a key role in the proof of Sobolev estimates with noninteger exponent for the solution to  $(BB)$ .

**Lemma 4.4.** *If  $s > 1$ , then we have:*

$$\| [v, \dot{\Lambda}^s] u - s \nabla v \cdot \dot{\Lambda}^{s-2} \nabla u \|_{L^2} \lesssim \| v \|_{\dot{H}^s} \| u \|_{L^\infty} + \| \nabla^2 v \|_{L^\infty} \| u \|_{\dot{H}^{s-2}}.$$

*Proof.* The proof follows the lines of that of [2, Lemma A.3], which deals with the non-homogeneous case. It relies on Bony's decomposition and on continuity results for the paraproduct and remainder operators. For the reader convenience, let us shortly recall how it works. Fix some smooth radial function  $\chi$  supported in (say) the ball  $B(0, 4/3)$  and with value 1 on  $B(0, 3/4)$ , then set  $\varphi := \chi(\cdot/2) - \chi$ . For all  $j \in \mathbb{Z}$ , we define the spectral cut-off operators  $\dot{\Delta}_j$  and  $\dot{S}_j$  acting on tempered distributions  $u$  as follows:

$$\begin{aligned} \dot{\Delta}_j u &:= \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) = 2^{jd} (\mathcal{F}^{-1} \varphi)(2^j \cdot) \star u, \\ \dot{S}_j u &:= \mathcal{F}^{-1}(\chi(2^{-j} \cdot) \mathcal{F} u) = 2^{jd} (\mathcal{F}^{-1} \chi)(2^j \cdot) \star u, \end{aligned}$$

where  $\mathcal{F}$  denotes the Fourier transform on  $\mathbb{R}^d$ .

Whenever the product  $uv$  of two tempered distributions  $u$  and  $v$  is defined, its so-called Bony's decomposition (first introduced in [6]) reads:

$$uv = T_u v + T_v u + R(u, v)$$

where the *paraproduct* operator  $T$  and *remainder* operator  $R$  are defined by

$$T_u v := \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v \quad \text{and} \quad R(u, v) := \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u (\dot{\Delta}_{j-1} v + \dot{\Delta}_j v + \dot{\Delta}_{j+1} v).$$

One can now start the proof. Decomposing the terms  $v \dot{\Lambda}^s u$ ,  $\dot{\Lambda}^s(uv)$  and  $\nabla v \cdot \dot{\Lambda}^{s-2} \nabla u$  according to Bony's decomposition, we get with the usual summation convention,

$$\begin{aligned} [v, \dot{\Lambda}^s] u - s \nabla v \cdot \dot{\Lambda}^{s-2} \nabla u &= \underbrace{[T_v, \dot{\Lambda}^s] u}_{R^1} - s \underbrace{T_{\partial_k v} \partial_k \dot{\Lambda}^{s-2} u}_{R^4} + \underbrace{T_{\dot{\Lambda}^s u} v}_{R^2} - \underbrace{\dot{\Lambda}^s T_u v}_{R^3} \\ &\quad - s \underbrace{T_{\partial_k \dot{\Lambda}^{s-2} u} \partial_k v}_{R^4} + \underbrace{R(v, \dot{\Lambda}^s u)}_{R^5} - \underbrace{\dot{\Lambda}^s R(v, u)}_{R^6} - s \underbrace{R(\partial_k v, \partial_k \dot{\Lambda}^{s-2} u)}_{R^7}. \end{aligned}$$

*Bounding  $R^1$ .* From the definition of the paraproduct, we have  $R^1 = \sum_{j \in \mathbb{Z}} R_j^1$  with  $R_j^1 := \dot{S}_{j-1} v \tilde{\varphi}(2^{-j} D) \dot{\Lambda}^s \dot{\Delta}_j u - \tilde{\varphi}(2^{-j} D) \dot{\Lambda}^s (\dot{S}_{j-1} v \dot{\Delta}_j u) - s \dot{S}_{j-1} \partial_k v \tilde{\varphi}(2^{-j} D) (\partial^k \dot{\Lambda}^{s-2}) \dot{\Delta}_j u$ , for some suitable  $\tilde{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  supported in an annulus. From second order Taylor's formula, we gather

$$R_j^1 = - \int_{\mathbb{R}^d} \int_0^1 h_{s,j}(y) y \cdot D^2 \dot{S}_{j-1} v(x - \tau y) \cdot y \dot{\Delta}_j u(x - y) (1 - \tau) d\tau dy,$$

where  $h_{s,j} := \mathcal{F}^{-1}(| \cdot |^s \tilde{\varphi}(2^{-j} \cdot)) = 2^{js} \mathcal{F}^{-1}(\tilde{\varphi}_s(2^{-j} \cdot))$  and  $\tilde{\varphi}_s(\xi) := |\xi|^s \tilde{\varphi}(\xi)$ .

Therefore,

$$\begin{aligned} \|R_j^1\|_{L^2} &\leq 2^{j(s-2)} \| | \cdot |^2 \mathcal{F}^{-1} \tilde{\varphi}_s \|_{L^1} \|D^2 \dot{S}_{j-1} v\|_{L^\infty} \|\dot{\Delta}_j u\|_{L^2} \\ &\lesssim \|D^2 v\|_{L^\infty} 2^{j(s-2)} \|\dot{\Delta}_j u\|_{L^2}. \end{aligned}$$

Since the spectral localization of the terms  $R_j^1$  implies that

$$\|R^1\|_{L^2}^2 \approx \sum_{j \in \mathbb{Z}} \|R_j^1\|_{L^2}^2,$$

and because, for any  $\sigma \in \mathbb{R}$ , we have

$$\|z\|_{\dot{H}^\sigma}^2 \approx \sum_{j \in \mathbb{Z}} 2^{2j\sigma} \|\dot{\Delta}_j z\|_{L^2}^2,$$

one ends up with

$$\|R^1\|_{L^2} \lesssim \|\nabla^2 v\|_{L^\infty} \|u\|_{\dot{H}^{s-2}}.$$

*Bounding  $R^2$ .* Combining a standard continuity result for the paraproduct (see e.g. [1, Chap. 2]) with the fact that  $\dot{\Lambda}^s : L^\infty \rightarrow \dot{B}_{\infty,\infty}^{-s}$  is continuous implies if  $s > 0$ ,

$$\|R^2\|_{L^2} \lesssim \|\dot{\Lambda}^s u\|_{\dot{B}_{\infty,\infty}^{-s}} \|v\|_{\dot{H}^s} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{H}^s}.$$

*Bounding  $R^3$ .* Since  $\dot{\Lambda}^s$  maps  $\dot{H}^s$  to  $L^2$ , we have for all  $s \in \mathbb{R}$ ,

$$\|R^3\|_{L^2} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{H}^s}.$$

*Bounding  $R^4$ .* Standard continuity results for the paraproduct combined with the fact that  $(\partial^k | \cdot |^s)(D)$  is a homogeneous multiplier of degree  $s - 1$  yield, if  $s > 1$ ,

$$\|R^4\|_{L^2} \lesssim \|\nabla \dot{\Lambda}^{s-2} u\|_{\dot{B}_{\infty,\infty}^{1-s}} \|\nabla v\|_{\dot{H}^{s-1}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{H}^s}.$$

*Bounding  $R^5$ .* Basic results of continuity for the remainder, see e.g. [41], ensure that for  $s \in \mathbb{R}$ , since  $\dot{\Lambda}^s$  maps  $L^\infty$  to  $\dot{F}_{\infty,2}^{-s}$ ,

$$\|R^5\|_{L^2} \lesssim \|v\|_{\dot{H}^s} \|\dot{\Lambda}^s u\|_{\dot{F}_{\infty,2}^{-s}} \lesssim \|v\|_{\dot{H}^s} \|u\|_{L^\infty}.$$

*Bounding  $R^6$ .* If  $s > 0$ , then we have

$$\|R^6\|_{L^2} \lesssim \|v\|_{\dot{H}^s} \|u\|_{L^\infty},$$

*Bounding  $R^7$ .* Since  $\partial^k \dot{\Lambda}^{s-2}$  is a homogeneous multiplier of degree  $s - 1$ , we have

$$\|R^7\|_{L^2} \lesssim \|\nabla v\|_{\dot{H}^{s-1}} \|\partial^k \dot{\Lambda}^{s-2} u\|_{\dot{F}_{\infty,2}^{1-s}} \lesssim \|\nabla v\|_{\dot{H}^{s-1}} \|u\|_{L^\infty}.$$

Putting together all the above estimates gives what we want.  $\square$

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