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## ARTICLE TYPE

# Existence and uniqueness of the entropy solution of a stochastic conservation law with a $Q$ -Brownian motion

Tadahisa FUNAKI<sup>1</sup> | Yueyuan GAO<sup>2</sup> | Danielle HILHORST\*<sup>3</sup>

<sup>1</sup>Department of Mathematics, School of Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

<sup>2</sup>MathAM-OIL, AIST c/o AIMR, Tohoku University, 2-1-1, Katahira, Aoba-ku, Sendai, 980-8577, Japan

<sup>3</sup>CNRS and Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Orsay Cedex, 91405, France

## Correspondence

\*Danielle Hilhorst. Email: danielle.hilhorst@math.u-psud.fr

## Present Address

CNRS and Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay, Orsay Cedex, 91405, France

## Summary

In this paper, we prove the existence and uniqueness of the entropy solution for a first order stochastic conservation law with a multiplicative source term involving a  $Q$ -Brownian motion.

After having defined a measure-valued weak entropy solution of the stochastic conservation law, we present the Kato inequality and as a corollary we deduce the uniqueness of the measure-valued weak entropy solution which coincides with the unique weak entropy solution of the problem. The Kato inequality is proved by a doubling of variables method; to that purpose, we prove the existence and the uniqueness of the strong solution of an associated stochastic nonlinear parabolic problem by means of an implicit time discretization scheme; we also prove its convergence to a measure-valued entropy solution of the stochastic conservation law, which proves the existence of the measure-valued entropy solution.

## KEYWORDS:

Stochastic first order conservation law,  $Q$ -Brownian motion, Kato inequality, Existence and uniqueness of the entropy solution, Associated parabolic problem

## 1 | INTRODUCTION

In this paper we prove the existence and uniqueness of the entropy solution for the first order stochastic conservation law

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t), & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x), & \omega \in \Omega, x \in \mathbb{T}^d. \end{cases} \quad (1)$$

where  $\mathbb{T}^d$  is the  $d$ -dimensional torus,  $W(x, t)$  is a  $Q$ -Brownian motion, satisfying the two conditions (4) and (5) in section 2, with the following hypotheses ( $H$ ):

- $u_0 \in L^2(\mathbb{T}^d)$ .
- $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function with Lipschitz constant  $C_f$  such that  $f(0) = 0$ .
- $g : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function with Lipschitz constant  $C_g$ ; moreover we suppose that  $g$  is bounded such that  $|g| < M_g$  for a positive constant  $M_g$ .
- $\mathbf{v} \in C^1(\mathbb{T}^d \times [0, T])$  with  $\operatorname{div} \mathbf{v} = 0$  for all  $(x, t) \in \mathbb{T}^d \times [0, T]$ , so that there exists  $V < \infty$  such that  $|\mathbf{v}(x, t)| \leq V$  for all  $(x, t) \in \mathbb{T}^d \times [0, T]$ .

Several articles have been devoted to the study of stochastic perturbations of nonlinear first order hyperbolic problems. Let us mention the article of Bauzet-Vallet-Wittbold<sup>2</sup> who prove the existence and uniqueness of a stochastic entropy solution of the Cauchy problem for a stochastic conservation law with a multiplicative one-dimensional noise in time. They apply an artificial viscosity method to prove the existence of a solution, and an adaptation of the Kruzhkov doubling of variables method to prove that any stochastic entropy solution coincides with the solution obtained by the artificial viscosity method. Then as an extension, Bauzet-Vallet-Wittbold<sup>3</sup> prove an existence and uniqueness result for the stochastic entropy solution of a Dirichlet problem for the stochastic conservation law with multiplicative noise in time.

As for the uniqueness result, Debussche-Vovelle<sup>5</sup> consider the  $d$ -dimensional problem with multiplicative noise  $du + f(u)_x dt = h(u)dW$ . The noise term  $dW$  is a function of space and time. The authors prove the existence and uniqueness of the kinetic solution. We believe however that our present results are important since entropy solutions are much easier to approximate numerically than kinetic solutions.

Funaki-Gao-Hilhorst<sup>7</sup> prove the convergence of a finite volume scheme for the stochastic conservation law (1). This convergence result implies the existence of a measure-valued weak entropy solution. In this paper, we extend the proof of the existence and uniqueness of the weak entropy solution of Bauzet-Vallet-Wittbold<sup>2,3</sup> to the case of equation (1) where the noise involves a  $Q$ -Brownian motion.

This paper is organized as follows: in section 2, we recall the definitions of a weak entropy solution and of a measure-valued entropy solution of Problem (1) and we present the associated parabolic problem and an inequality satisfied by its strong solution. We prove that the solution of the associated parabolic problem converges to a measure-valued entropy solution of the stochastic conservation law when the diffusion coefficient tends to 0.

In section 3, we first present the Kato inequality and as a corollary, we first deduce the uniqueness of the measure-valued weak entropy solution and then the uniqueness of the weak entropy solution. As a consequence of the proofs, the measure-valued weak entropy solution is unique and coincides with the unique weak entropy solution. The Kato inequality is proved by a doubling of variables method for which we need entropy type inequalities for measure-valued weak entropy solutions and for the strong solutions of the associated parabolic problem.

In section 4, we show the well-posedness of the associated parabolic problem. We apply an implicit time discretization to obtain a semi-discrete solution of the associated stochastic nonlinear parabolic problem. By Itô's formula and a priori estimates, we prove that when the time step tends to zero, the limit of this discrete solution is a strong solution of the corresponding parabolic problem. We then prove the uniqueness of the strong solution.

## 2 | ENTROPY FORMULATION

We recall the definitions of a  $Q$ -Brownian motion and of an entropy solution of Problem (1).

**Definition 1** ( $Q$ -Brownian motion in  $L^2(\mathbb{T}^d)$ ). Let  $Q \geq 0$  be a symmetric trace class operator on  $L^2(\mathbb{T}^d)$ , let  $\{e_j\}_{j \geq 1}$  be an orthonormal basis in  $L^2(\mathbb{T}^d)$  diagonalizing  $Q$  and  $\{\lambda_j \geq 0\}_{j \geq 1}$  be the corresponding eigenvalues, such that

$$Qe_j = \lambda_j e_j$$

for all  $j \geq 1$ . We recall that  $Q$  is of trace class, namely that

$$\text{Tr } Q = \sum_{j=1}^{\infty} \langle Qe_j, e_j \rangle_{L^2(\mathbb{T}^d)} = \sum_{j=1}^{\infty} \lambda_j \leq \Lambda_0, \quad (2)$$

for some positive constant  $\Lambda_0$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space equipped with a filtration  $(\mathcal{F}_t)$ <sup>8</sup> and  $\{\beta_j\}_{j \geq 1}$  be a sequence of independent  $(\mathcal{F}_t)$ -Brownian motions defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ . We recall that a Brownian motion  $\beta(t)$  is called an  $(\mathcal{F}_t)$ -Brownian motion if it is  $(\mathcal{F}_t)$ -adapted and the increment  $\beta(t) - \beta(s)$  is independent of  $\mathcal{F}_s$  for every  $0 \leq s < t$ . The process  $W$  defined by

$$W(x, t) = \sum_{j=1}^{\infty} \beta_j(t) Q^{\frac{1}{2}} e_j(x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j(x) \quad (3)$$

is a  $Q$ -Brownian motion in  $L^2(\mathbb{T}^d)$ , and the series defined by (3) is convergent in  $L^2(\Omega, C([0, T], L^2(\mathbb{T}^d)))$  [cf. 4].

We suppose furthermore that  $e_j \in C(\mathbb{T}^d)$  for  $j = 1, 2, \dots$  and that there exists a positive constant  $\Lambda_1$  such that

$$\sum_{j=1}^{\infty} \lambda_j \|e_j\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (4)$$

In the proof of the existence of a strong solution of the associated parabolic problem, we need the further assumption that there exists a positive constant  $\Lambda_2$ , such that

$$\sum_{j=1}^{\infty} \lambda_j \|\nabla e_j\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_2. \quad (5)$$

**Definition 2** (Weak entropy solution of Problem (1)). A function  $u \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d))$  is a weak entropy solution of the stochastic scalar conservation law (1) with the initial condition  $u_0 \in L^2(\mathbb{T}^d)$ , if P-a.s. in  $\Omega$ ,

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x) - k) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u - k) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u, k) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} \eta'(u - k) g(u) \varphi(x, t) dW(x, t) dx + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u - k) g^2(u) \varphi(x, t) Q(x, x) dx dt \\ & \geq 0, \end{aligned}$$

where

$$F^\eta(a, b) = \int_b^a \eta'(\sigma - b) f'(\sigma) d\sigma$$

and

$$Q(x, y) = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(y), \quad (6)$$

for all  $\varphi \in \mathcal{C} := \{\varphi \in C_0^\infty(\mathbb{T}^d \times [0, T]), \varphi \geq 0, \varphi(\cdot, T) = 0\}$  and for all  $\eta \in \mathcal{A}$  where  $\mathcal{A}$  is the set of nonnegative  $C^2(\mathbb{R})$  convex functions  $\eta$  such that the support of  $\eta''$  is compact and  $\eta(0) = 0$ .  $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d))$  is the subclass of  $L^2(\Omega \times \mathbb{T}^d \times (0, T))$  consisting of predictable  $L^2(\mathbb{T}^d)$ -valued processes [cf. <sup>4</sup>, page 98].

We remark that for all  $x, y \in \mathbb{T}^d$

$$|Q(x, y)| \leq \sum_{j=1}^{\infty} \lambda_j \|e_j\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (7)$$

**Definition 3** (Measure-valued weak entropy solution of Problem (1)). A function  $\mathbf{u}$  of  $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$  is a measure-valued weak entropy solution of the stochastic scalar conservation law (1) with the initial condition  $u_0 \in L^2(\mathbb{T}^d)$ , if P-a.s. in  $\Omega$ , for all  $\eta \in \mathcal{A}$  and for all  $\varphi \in \mathcal{C}$

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0 - k) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha) - k) \partial_t \varphi(x, t) d\alpha dx dt + \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha), k) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt \\ & + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x, t, \alpha) - k) g(\mathbf{u}(x, t, \alpha)) \varphi(x, t) d\alpha dW(x, t) dx \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x, t, \alpha) - k) g^2(\mathbf{u}(x, t, \alpha)) \varphi(x, t) Q(x, x) d\alpha dx dt \\ & \geq 0. \end{aligned} \quad (8)$$

The main result of this article is the following theorem.

**Theorem 1.** Suppose that the hypotheses (H) hold and that the basis functions  $\{e_j\}_{j \geq 1}$  of the  $Q$ -Brownian motion satisfy the hypotheses (4) and (5); then there exists a unique measure-valued weak entropy solution. Moreover, it coincides with the unique weak entropy solution.

In section 4, we will study the initial value problem

$$(\mathcal{P}^\epsilon) \begin{cases} du^\epsilon - \epsilon \Delta u^\epsilon dt + \operatorname{div}(\mathbf{v}f(u^\epsilon))dt = g(u^\epsilon)dW & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u^\epsilon(x, 0) = u_0^\epsilon(x) & \text{for all } x \in \mathbb{T}^d \end{cases}$$

and prove that Problem  $(\mathcal{P}^\epsilon)$  possesses a unique strong solution.

Next, we let  $\epsilon \rightarrow 0$  in Problem  $(\mathcal{P}^\epsilon)$  to deduce the existence of a measure-valued entropy solution of Problem (1). More precisely, we prove the following result.

**Lemma 1** (Existence of a measure-valued entropy solution). Let  $u^\epsilon$  be the strong solution of the parabolic problem (c.f. Definition 4 in section 4). Then  $u^\epsilon$  tends to a measure-valued entropy solution of Problem (1) as  $\epsilon$  tends to 0 in the sense of Young measures.

*Proof.* We define the functional of  $X = X(x)$

$$\mathcal{G}(X, t) = \int_{\mathbb{T}^d} \eta(X(x) - k)\varphi(x, t)dx,$$

so that

$$\begin{aligned} \mathcal{G}_X(X, t) &= \eta'(X(x) - k)\varphi(x, t) \\ \mathcal{G}_{XX}(X, t) &= \eta''(X(x) - k)\varphi(x, t) \\ \mathcal{G}_t(X, t) &= \int_{\mathbb{T}^d} \eta(X(x) - k)\partial_t \varphi(x, t)dx \end{aligned}$$

and apply Itô's formula [c.f. <sup>4</sup>] for  $\mathcal{G}(u^\epsilon(t), t)$  to obtain

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u^\epsilon(T) - k)\varphi(T)dx \\ &= \int_{\mathbb{T}^d} \eta(u^\epsilon(0) - k)\varphi(0)dx + \int_0^T \int_{\mathbb{T}^d} \eta(u^\epsilon(x, t) - k)\partial_t \varphi(x, t)dxdt \\ & \quad + \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\varphi(x, t)[\epsilon \Delta u^\epsilon(x, t) - \operatorname{div}(\mathbf{v}f(u^\epsilon(x, t)))]dxdt \\ & \quad + \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\varphi(x, t)g(u^\epsilon(x, t))dW(x, t)dx + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u^\epsilon(x, t) - k)\varphi(x, t)g^2(u^\epsilon(x, t))Q(x, x)dxdt. \end{aligned}$$

We first consider the term involving  $\Delta u^\epsilon$ . Note that for all  $\varphi \in \mathcal{C}$ , by integration by parts

$$\begin{aligned} & \epsilon \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t))\varphi(x, t)\Delta u^\epsilon(x, t)dxdt \\ &= -\epsilon \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\nabla_x u^\epsilon(x, t) \cdot \nabla_x \varphi(x, t)dxdt - \epsilon \int_0^T \int_{\mathbb{T}^d} \eta''(u^\epsilon(x, t) - k)|\nabla_x u^\epsilon(x, t)|^2 \varphi(x, t)dxdt. \end{aligned}$$

As for the term involving  $\operatorname{div}(\mathbf{v}f(u^\epsilon(x, t)))$ , we use integration by parts, the chain rule and  $\operatorname{div} v = 0$  to deduce that

$$-\int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\operatorname{div}(\mathbf{v}f(u^\epsilon(x, t)))\varphi(x, t)dxdt = \int_0^T \int_{\mathbb{T}^d} F^\eta(u^\epsilon(x, t), k)\mathbf{v} \cdot \nabla_x \varphi(x, t)dxdt.$$

Remembering that  $\varphi(T) = 0$  and that  $\eta$  is a nonnegative convex function, we deduce the following inequality

$$\begin{aligned}
& \int_{\mathbb{T}^d} \eta(u_0^\epsilon(x) - k)\varphi(x, 0)dx + \int_0^T \int_{\mathbb{T}^d} \eta(u^\epsilon(x, t) - k)\partial_t \varphi(x, t)dxdt \\
& - \epsilon \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\nabla_x u^\epsilon(x, t) \cdot \nabla_x \varphi(x, t)dxdt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u^\epsilon(x, t), k)\mathbf{v} \cdot \nabla_x \varphi(x, t)dxdt \\
& + \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)g(u^\epsilon(x, t))\varphi(x, t)dW(x, t)dx + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u^\epsilon(x, t) - k)g^2(u^\epsilon(x, t))\varphi(x, t)Q(x, x)dxdt \\
& \geq 0.
\end{aligned} \tag{9}$$

We remark that (38) below and the strong convergence of  $\{u^{\Delta t}\}$  to  $u^\epsilon$  in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  [cf. Lemma 12] below imply that  $u^\epsilon$  is uniformly bounded in  $L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ . Thus there exist an entropy process  $\mathbf{u} \in L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$  and a subsequence of  $u^\epsilon$  which we denote again by  $u^\epsilon$  such that  $u^\epsilon$  converges to  $\mathbf{u}$  in the sense of Young measures as  $\epsilon$  tends to 0 [cf. <sup>2</sup>, Page 668]. We multiply (9) by  $\mathbf{1}_A$  where  $A \in \mathcal{F}$  is arbitrary and take the expectation to obtain

$$\begin{aligned}
& \mathbb{E} \left[ \mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0^\epsilon(x) - k)\varphi(x, 0)dx \right] + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u^\epsilon(x, t) - k)\partial_t \varphi(x, t)dxdt \right] \\
& - \mathbb{E} \left[ \epsilon \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\nabla_x u^\epsilon(x, t) \cdot \nabla_x \varphi(x, t)dxdt \right] + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u^\epsilon(x, t), k)\mathbf{v} \cdot \nabla_x \varphi(x, t)dxdt \right] \\
& + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)g(u^\epsilon(x, t))\varphi(x, t)dW(x, t)dx \right] \\
& + \frac{1}{2} \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u^\epsilon(x, t) - k)g^2(u^\epsilon(x, t))\varphi(x, t)Q(x, x)dxdt \right] \\
& \geq 0.
\end{aligned} \tag{10}$$

The estimate (39) together with the lower semi-continuity of the convex functional  $\|\cdot\|_{L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))}$  implies its weak lower-semi continuity by Ekeland-Temam [cf. <sup>6</sup> Corollary 2.2, p.11], so that

$$\sqrt{\epsilon} \|\nabla u^\epsilon\|_{L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))} \leq C_2. \tag{11}$$

Next we pass to the limit  $\epsilon \rightarrow 0$  in (10); the estimate (11) ensures that

$$- \mathbb{E} \left[ \epsilon \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u^\epsilon(x, t) - k)\nabla u^\epsilon(x, t) \cdot \nabla \varphi(x, t)dxdt \right] \rightarrow 0$$

as  $\epsilon \rightarrow 0$ . We suppose that  $u_0^\epsilon \rightarrow u_0$  strongly in  $L^2(\mathbb{T}^d)$  and use similar arguments as in <sup>7</sup> for the other terms to deduce that

$$\begin{aligned}
& \mathbb{E} \left[ \mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0 - k)\varphi(x, 0)dx \right] + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha) - k)\partial_t \varphi(x, t)d\alpha dxdt \right] \\
& + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha), k)\mathbf{v} \cdot \nabla_x \varphi(x, t)d\alpha dxdt \right] \\
& + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x, t, \alpha) - k)g(\mathbf{u}(x, t, \alpha))\varphi(x, t)d\alpha dW(x, t)dx \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x, t, \alpha) - k) g^2(\mathbf{u}(x, t, \alpha)) \varphi(x, t) \underline{Q}(x, x) d\alpha dx dt \right] \\
& \geq 0,
\end{aligned}$$

which shows that  $\mathbf{u}$  is a measure-valued entropy solution of Problem (1).  $\square$

### 3 | UNIQUENESS OF THE MEASURE-VALUED WEAK ENTROPY SOLUTION

The main part of this section is to prove the Kato inequality for measure-valued weak entropy solutions of Problem (1). The Kato inequality will imply the uniqueness of the measure-valued weak entropy solution and as a consequence the uniqueness of the weak entropy solution.

**Proposition 1** (Kato inequality). Let  $\mathbf{u}, \hat{\mathbf{u}}$  be 2 measure-valued weak entropy solutions to (1) with initial data  $u_0, \hat{u}_0 \in L^2(\mathbb{T}^d)$  respectively. Then, for any nonnegative function  $\varphi$  with compact support belonging to  $H^1(\mathbb{T}^d \times [0, T])$ , there holds

$$\begin{aligned}
& \int_{\mathbb{T}^d} |u_0 - \hat{u}_0| \varphi(0) dx + \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 |\mathbf{u}(x, t, \alpha) - \hat{\mathbf{u}}(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 F(\mathbf{u}(x, t, \alpha), \hat{\mathbf{u}}(x, t, \beta)) \mathbf{v} \cdot \nabla \varphi(x, t) d\alpha d\beta dx dt \right] \\
& \geq 0.
\end{aligned} \tag{12}$$

where  $Q_T = \mathbb{T}^d \times (0, T)$  and  $F(a, b) = \text{sgn}(a - b)[f(a) - f(b)]$  with  $\text{sgn}(a) = \frac{a}{|a|}$  for all  $a \neq 0$ . Note that  $F$  is a Lipschitz continuous function with constant  $C_f$ .

**Proof of Theorem 1:** We use below the fact that the Kato inequality holds for all nonnegative test functions  $\varphi \in H^1(Q_T)$ . Let  $K$  be a positive constant, and denote by  $\psi$  any smooth nonincreasing function such that  $\mathbf{1}_{(-\infty, K]}(\cdot) \leq \psi(\cdot) \leq \mathbf{1}_{(-\infty, K+1]}(\cdot)$ . Setting  $\varphi(x, t) = \psi(|x| - VC_f t) \gamma(t)$  with  $\gamma(t) = \frac{T-t}{T}$  and substituting  $\varphi$  in (12) yields

$$\begin{aligned}
& \mathbb{E} \left[ \int_{\mathbb{T}^d} |\hat{u}_0 - u_0| \psi(|x|) \gamma(0) \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 \psi(|x| - VC_f t) |\mathbf{u}(x, t, \beta) - \hat{\mathbf{u}}(x, t, \alpha)| \gamma'(t) d\alpha d\beta dx dt \right] \\
& - \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 VC_f \psi'(|x| - VC_f t) |\mathbf{u}(x, t, \beta) - \hat{\mathbf{u}}(x, t, \alpha)| \gamma(t) d\alpha d\beta dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 \psi'(|x| - VC_f t) F(\mathbf{u}(x, t, \beta), \hat{\mathbf{u}}(x, t, \alpha)) \mathbf{v} \cdot \frac{x}{|x|} \gamma(t) d\alpha d\beta dx dt \right] \\
& \geq 0.
\end{aligned} \tag{13}$$

Next, we study the signs of the last two terms in (13). Since  $\psi$  is a nonincreasing function, we have that the sum of the last two terms is bounded from above by

$$\mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 (VC_f - VC_f) \psi'(|x| - VC_f t) |\mathbf{u}(x, t, \beta) - \hat{\mathbf{u}}(x, t, \alpha)| \gamma(t) d\alpha d\beta dx dt \right] = 0,$$

which combined with (13) yields

$$\mathbb{E} \left[ \int_{\mathbb{T}^d} |\hat{u}_0 - u_0| \psi(|x|) \right] \gamma(0) + \mathbb{E} \left[ \int_{Q_T} \gamma'(t) \int_0^1 \int_0^1 \psi(|x| - VC_f t) |\mathbf{u}(x, t, \beta) - \hat{\mathbf{u}}(x, t, \alpha)| d\alpha d\beta dx dt \right] \geq 0.$$

We suppose that  $\hat{u}_0 = u_0$  and choose  $K = \sup_{x \in \mathbb{T}^d} |x| - VC_f T$  to deduce that  $\psi(|x| - VC_f t) = 1$ . Also in view of the fact that  $\gamma'(t) = -\frac{1}{T}$ , we deduce that

$$\mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 |\mathbf{u}(x, t, \beta) - \hat{\mathbf{u}}(x, t, \alpha)| d\alpha d\beta dx dt \right] = 0 \quad (14)$$

for all  $\alpha, \beta \in (0, 1)$ . This implies that  $\mathbf{u}(x, t, \beta) = \hat{\mathbf{u}}(x, t, \alpha)$  for almost every  $x \in \mathbb{T}^d, t \in (0, T), \omega \in \Omega, \alpha, \beta \in (0, 1)$ . Setting  $\alpha = \beta$ , we deduce that  $\mathbf{u}(x, t, \alpha) = \hat{\mathbf{u}}(x, t, \alpha)$  for almost every  $x \in \mathbb{T}^d, t \in (0, T)$  so that Problem (1) has a unique measure-valued weak entropy solution, say  $\mathbf{u} = \mathbf{u}(x, t, \alpha)$ . Then setting  $\alpha \neq \beta$ , we deduce that  $\mathbf{u}(x, t, \alpha) = \mathbf{u}(x, t, \beta)$  which implies that the measure-valued weak entropy solution does not depend on the third parameter. Thus it is a weak entropy solution and (14) also implies the uniqueness of the weak entropy solution of Problem (1).  $\square$

In the sequel we prove the Kato inequality, Proposition 1.

**Proof of Proposition 1:** We suppose that  $\rho_m$  and  $\rho_n$  are two mollifier sequences in  $\mathbb{T}^d$  and  $\mathbb{R}$  respectively, with  $\text{supp } \rho_m \subset [-\frac{2}{m}, 0]^d$  and  $\text{supp } \rho_n \subset [-\frac{2}{n}, 0]$ . Suppose that  $\rho_l$  is a mollifier sequence with  $\text{supp } \rho_l \subset [-\frac{1}{l}, \frac{1}{l}]$ . Remark that  $|\rho_m| \leq Cm^d, |\rho_n| \leq Cn$  and  $|\rho_l| \leq Cl$  for some constant  $C$  and that  $\rho_l$  is an even function for all  $l$ . Let  $u^\epsilon$  be the strong solution of the associated parabolic problem in the sense of Definition 4. We use the notations  $\hat{\mathbf{u}} = \hat{\mathbf{u}}(x, t, \alpha)$  for a measure-valued solution and  $\hat{u}_0$  for the corresponding initial condition, and set  $\mathcal{A}_k^l := \rho_l(\hat{\mathbf{u}}(x, t, \alpha) - k)$  and  $\mathcal{B}_k^l := \rho_l(u^\epsilon(y, s) - k)$ . Next, we deduce two inequalities based on (8) and (9) respectively. We choose  $\eta = \eta_\delta \in \mathcal{A}$  such that

$$\eta'_\delta(r) = \begin{cases} 1 & \text{if } r > \delta \\ \sin(\frac{\pi r}{2\delta}) & \text{if } |r| \leq \delta \\ -1 & \text{if } r < -\delta \end{cases}$$

in the inequality (8) as well as in the inequality (9). Note that  $\eta'_\delta$  is an even Lipschitz continuous function. In the inequality (8), we replace the test function  $\varphi(x, t)$  by  $\varphi(y, s)\rho_m(x - y)\rho_n(t - s)$  and then multiply it by  $\mathcal{B}_k^l$  and integrate  $k$  on  $\mathbb{R}, y$  on  $\mathbb{T}^d$  and  $s$  on  $(0, T)$  and write last the term coming from the stochastic integral to obtain

$$\begin{aligned} & \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \eta_\delta(\hat{u}_0(x) - k) \varphi(y, s) \rho_n(-s) \rho_m(x - y) dx \mathcal{B}_k^l dk dy ds \right] \\ & + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 \eta_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) \rho_n(t - s) \partial_t \varphi(y, s) \rho_m(x - y) d\alpha dx dt \mathcal{B}_k^l dk dy ds \right] \\ & + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 \eta_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) \varphi(y, s) \partial_t \rho_n(t - s) \rho_m(x - y) d\alpha dx dt \mathcal{B}_k^l dk dy ds \right] \\ & + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 F^{\eta_\delta}(\hat{\mathbf{u}}(x, t, \alpha), k) \mathbf{v} \cdot \rho_m(x - y) \nabla_x \varphi(y, s) \rho_n(t - s) d\alpha dx dt \mathcal{B}_k^l dk dy ds \right] \\ & + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 F^{\eta_\delta}(\hat{\mathbf{u}}(x, t, \alpha), k) \mathbf{v} \cdot \nabla_x \rho_m(x - y) \rho_n(t - s) \varphi(y - s) d\alpha dx dt \mathcal{B}_k^l dk dy ds \right] \\ & + \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) g^2(\hat{\mathbf{u}}(x, t, \alpha)) \rho_m(x - y) \rho_n(t - s) \varphi(y, s) Q(x, x) d\alpha dx dt \mathcal{B}_k^l dk dy ds \right] \end{aligned}$$



$$\begin{aligned}
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \varphi(y, s) \rho_m(x - y) \rho_n(t - s) dx dy ds dW(x, t) \mathcal{B}_k^l \right] \\
& := I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \\
& \geq 0.
\end{aligned}$$

Next in (9) we change the variables  $(x, t)$  to  $(y, s)$ , replace  $\varphi(y, s)$  by  $\varphi(y, s) \rho_m(x - y) \rho_n(t - s)$  and write last the term coming from  $\epsilon \Delta u^\epsilon$ , then multiply by  $\mathcal{A}_k^l$  and integrate on  $\alpha, k, x$  and  $t$ , to obtain

$$\begin{aligned}
& \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \eta_\delta(u_0^\epsilon(y) - k) \varphi(y, 0) \rho_n(t) \rho_m(x - y) dy \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \eta_\delta(u^\epsilon(y, s) - k) \rho_n(t - s) \partial_s \varphi(y, s) \rho_m(x - y) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \eta_\delta(u^\epsilon(y, s) - k) \partial_s \rho_n(t - s) \varphi(y, s) \rho_m(x - y) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} F^{\eta_\delta}(u^\epsilon(y, s), k) \mathbf{v} \cdot \rho_m(x - y) \nabla_y \varphi(y, s) \rho_n(t - s) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} F^{\eta_\delta}(u^\epsilon(y, s), k) \mathbf{v} \cdot \nabla_y \rho_m(x - y) \varphi(y, s) \rho_n(t - s) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \eta''_\delta(u^\epsilon(y, s) - k) g^2(u^\epsilon(y, s)) \rho_m(x - y) \rho_n(t - s) \varphi(y, s) Q(y, y) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& + \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_0^T \int_{\mathbb{T}^d} \eta'_\delta(u^\epsilon(y, s) - k) g(u^\epsilon(y, s)) \varphi(y, s) \rho_m(x - y) \rho_n(t - s) dW(y, s) dy \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& - \epsilon \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \eta'_\delta(u^\epsilon(y, s) - k) \nabla_y u^\epsilon(y, s) \nabla_y [\varphi(y, s) \rho_m(x - y)] \rho_n(t - s) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& := J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 \\
& \geq 0.
\end{aligned}$$

We will show that the sum of  $I_i + J_i$  for  $i = 1, 2, \dots, 7$  and of the term  $J_8$  converges to the left-hand-side of Kato inequality (12). More precisely, we denote the limit of  $u^\epsilon$  by  $\mathbf{u}$ ; we should prove that

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_1 + J_1] = \mathbb{E} \left[ \int_{\mathbb{T}^d} |u_0 - \hat{u}_0| \varphi(0) dx \right], \quad (15)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_2 + J_2] = \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 |\mathbf{u}(x, t, \alpha) - \hat{\mathbf{u}}(x, t, \beta)| \partial_t \varphi(x, t) d\alpha d\beta dx dt \right], \quad (16)$$

$$I_3 + J_3 = 0, \quad (17)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_4 + J_4] = \mathbb{E} \left[ \int_{Q_T} \int_0^1 \int_0^1 F(\mathbf{u}(x, t, \beta), \hat{\mathbf{u}}(x, t, \alpha)) \mathbf{v} \cdot \nabla \varphi(x, t) d\alpha d\beta dx dt \right], \quad (18)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_5 + J_5] = 0, \quad (19)$$

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7 + J_7] = 0, \quad (20)$$

and

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} J_8 = 0. \quad (21)$$

We refer to<sup>2</sup> for the detailed proofs of (15) - (19) and of (21). Here we will only prove (20). Since we are dealing with a  $Q$ -Brownian motion instead of a one-dimensional Brownian motion, the proof of (20) is novel with respect to the proofs given in<sup>2</sup>. However, let us remark some simple facts. The partial derivatives  $\partial_t[\varphi(y, s)]$  and  $\nabla_x[\varphi(y, s)]$  in the terms  $I_2$  and  $I_4$  respectively vanish. Therefore  $I_2 = I_4 = 0$ . Moreover, since  $t \in (0, T)$  and since the support of  $\rho_n$  is included in  $\mathbb{R}^-$ , the term  $J_1$  vanishes. Let us now prove (20). We start by showing the following result.

**Lemma 2.**  $J_7 = 0$ .

*Proof.* We set  $\mathcal{G}_n^m = \rho_m(x - y)\rho_n(t - s)$ , and use the fact that  $\text{supp } \rho_n \subset \mathbb{R}^-$  to obtain

$$\begin{aligned} J_7 &= \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_0^T \int_{\mathbb{T}^d} \eta'_\delta(u^\epsilon - k) g(u^\epsilon) \varphi(y, s) \mathcal{G}_n^m dW(y, s) dy \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\ &= \int_{Q_T} \int_{\mathbb{R}} \mathbb{E} \left[ \int_t^T \int_{\mathbb{T}^d} \eta'_\delta(u^\epsilon - k) g(u^\epsilon) \varphi(y, s) \mathcal{G}_n^m dW(y, s) dy \int_0^1 \mathcal{A}_k^l d\alpha \right] dk dx dt. \end{aligned}$$

The above expectation vanishes, since  $\int_0^1 \mathcal{A}_k^l d\alpha$  is  $\mathcal{F}_t$ -measurable and  $\int_0^t g(u^\epsilon) \eta'_\delta(u^\epsilon - k) \varphi(y, s) \rho_m(x - y) \rho_n(t - s) dW(y, s)$  is  $\mathcal{F}_t$ -martingale for each  $y$ . This yields that  $J_7 = 0$ .  $\square$

Next we study the limits of the sum of the Itô correction terms  $I_6 + J_6$  as  $n$  and  $l$  tend to infinity.

**Lemma 3.** We have the following limit properties:

$$\begin{aligned} I_6 + J_6 &\xrightarrow{n \rightarrow \infty} \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) g^2(\hat{\mathbf{u}}(x, t, \alpha)) \rho_m(x - y) \varphi(y, t) Q(x, x) \rho_l(u^\epsilon(y, t) - k) d\alpha dk dy dx dt \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \eta''_\delta(u^\epsilon(y, t) - k) g^2(u^\epsilon(y, t)) \rho_m(x - y) \varphi(y, t) Q(y, y) \int_0^1 \rho_l(\hat{\mathbf{u}}(x, t, \alpha) - k) d\alpha dx dt dk dy \right] \\ &\xrightarrow{l \rightarrow \infty} \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{Q_T} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) g^2(\hat{\mathbf{u}}(x, t, \alpha)) \rho_m(x - y) \varphi(y, t) Q(x, x) d\alpha dx dt dy \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{Q_T} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) g^2(u^\epsilon(y, t)) \rho_m(x - y) \varphi(y, t) Q(y, y) d\alpha dx dt dy \right]. \end{aligned}$$

*Proof.* We only justify the limit as  $n \rightarrow \infty$ . We define  $A_1$  as the difference of  $I_6 + J_6$  and the limit term and perform a change of variables from  $k$  to  $u^\epsilon(y, s) - k$  when  $y$  and  $s$  are fixed to obtain

$$A_1 = \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \int_0^1 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - k) g^2(\hat{\mathbf{u}}(x, t, \alpha)) \rho_m(x - y) \rho_n(t - s) \varphi(y, s) Q(x, x) d\alpha \rho_l(u^\epsilon(y, s) - k) dk dy ds dx dt \right]$$

$$\begin{aligned}
& -\frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_0^1 \eta''_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - k) g^2(\hat{\mathbf{u}}(x, t, \alpha)) \rho_m(x - y) \varphi(y, t) Q(x, x) \rho_l(u^{\epsilon}(y, t) - k) d\alpha dk dy dx dt \right] \\
& + \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{Q_T} \eta''_{\delta}(u^{\epsilon}(y, s) - k) g^2(u^{\epsilon}(y, s)) \rho_m(x - y) \rho_n(t - s) \varphi(y, s) Q(y, y) dy ds \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right] \\
& - \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \eta''_{\delta}(u^{\epsilon}(y, t) - k) g^2(u^{\epsilon}(y, t)) \rho_m(x - y) \varphi(y, t) Q(y, y) dy \int_0^1 \mathcal{A}_k^l d\alpha dk dx dt \right].
\end{aligned}$$

Therefore, since  $\eta''_{\delta}$  is bounded and Lipschitz-continuous,  $\varphi \in C$ ,  $|Q(x, x)| \leq \Lambda_1$  for all  $x \in \mathbb{T}^d$  and  $g$  is a bounded function, after some computations, we deduce that

$$\begin{aligned}
|A_1| & \leq \frac{c(m, \varphi, \Lambda_1, M_g^2)}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \int_0^T \int_0^1 \min(2\|\eta''_{\delta}\|_{\infty}, |u^{\epsilon}(y, s) - u^{\epsilon}(y, t)|) \rho_n(t - s) d\alpha dx dt dy ds \right] \\
& + \frac{c(\varphi, \Lambda_1, M_g^2)}{2} \mathbb{E} \left[ \int_0^T \int_{Q_T} \min(2\|\eta''_{\delta}\|_{\infty}, |u^{\epsilon}(y, s) - u^{\epsilon}(y, t)|) \rho_n(t - s) dy ds dt \right] \\
& + \frac{c(\varphi, \eta''_{\delta}, \Lambda_1)}{2} \mathbb{E} \left[ \int_0^T \int_{Q_T} [g^2(u^{\epsilon}(y, s)) - g^2(u^{\epsilon}(y, t))] \rho_n(t - s) dy ds dt \right] \\
& + \frac{c(\varphi, \eta''_{\delta}, \Lambda_1)}{2n} M_g^2 |\mathbb{T}^d|.
\end{aligned}$$

One can then prove that the 4 terms above tend to 0 as  $n \rightarrow \infty$ . One can justify the limit as  $l \rightarrow \infty$  in a similar way as in<sup>2</sup>.  $\square$

Next we study the stochastic integral term, namely  $I_7$ .

**Lemma 4.**

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} I_7 = - \int_{Q_T} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_0^1 \eta'_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - u^{\epsilon}(y, t)) g(\hat{\mathbf{u}}(x, t, \alpha)) g(u^{\epsilon}(y, t)) d\alpha \right] \rho_m(x - y) \varphi(y, t) Q(x, y) dy dx dt.$$

*Proof.* Using the same type of arguments as in the proof of Lemma 2, we deduce that

$$\mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s-\frac{2}{n}}^s \int_0^1 \eta'_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) dx dW(x, t) \varphi(y, s) \rho_m(x - y) \rho_l(u^{\epsilon}(y, s - \frac{2}{n}) - k) dk dy ds \right] \quad (22)$$

$$= 0.$$

Therefore subtracting equality (22) from the expression of  $I_7$ , we deduce that

$$\begin{aligned}
I_7 & = \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s-\frac{2}{n}}^s \int_0^1 \eta'_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) dW(x, t) \varphi(y, s) \rho_m(x - y) dx \rho_l(u^{\epsilon}(y, s) - k) dk dy ds \right] \\
& - \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s-\frac{2}{n}}^s \int_0^1 \eta'_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) dx dW(x, t) \varphi(y, s) \rho_m(x - y) \rho_l(u^{\epsilon}(y, s - \frac{2}{n}) - k) dk dy ds \right] \\
& = \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s-\frac{2}{n}}^s \int_0^1 \eta'_{\delta}(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) dW(x, t) \rho_m(x - y) dx \right]
\end{aligned}$$

$$\times \varphi(y, s) \left( \rho_l(u^\epsilon(y, s) - k) - \rho_l(u^\epsilon(y, s - \frac{2}{n}) - k) \right) dk dy ds \Big].$$

We define  $A_\epsilon = \epsilon \Delta u^\epsilon - \operatorname{div}(v f(u^\epsilon))$  and apply Itô's formula for  $\mathcal{G}(u^\epsilon(t), t)$  taking  $\mathcal{G}(X, t) = \rho_l(X)$ . Then, noting

$$\mathcal{G}_X(X, t) = \rho_l'(X), \quad \mathcal{G}_{XX}(X, t) = \rho_l''(X), \quad \mathcal{G}_t(X, t) = 0$$

we deduce that for almost all  $y \in \mathbb{T}^d$  and almost all  $s \in (0, T)$ , and  $n \geq \frac{2}{s}$

$$\begin{aligned} & \rho_l(u^\epsilon(y, s) - k) - \rho_l(u^\epsilon(y, s - \frac{2}{n}) - k) \\ &= \int_{s - \frac{2}{n}}^s \rho_l'(u^\epsilon(y, \sigma) - k) A_\epsilon(y, \sigma) d\sigma + \int_{s - \frac{2}{n}}^s \rho_l'(u^\epsilon(y, \sigma) - k) g(u^\epsilon(y, \sigma)) dW(y, \sigma) + \frac{1}{2} \int_{s - \frac{2}{n}}^s \rho_l''(u^\epsilon(y, \sigma) - k) g^2(u^\epsilon(y, \sigma)) Q(y, y) d\sigma \\ &= -\frac{\partial}{\partial k} \left[ \int_{s - \frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) A_\epsilon(y, \sigma) d\sigma + \int_{s - \frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) g(u^\epsilon(y, \sigma)) dW(y, \sigma) \right. \\ & \quad \left. + \frac{1}{2} \int_{s - \frac{2}{n}}^s \rho_l'(u^\epsilon(y, \sigma) - k) g^2(u^\epsilon(y, \sigma)) Q(y, y) d\sigma \right]. \end{aligned}$$

After some computation, we obtain

$$\begin{aligned} I_7 &= -\mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s - \frac{2}{n}}^s \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) dW(x, t) \rho_m(x - y) \varphi(y, s) dx \right. \\ & \quad \times \left[ \int_{s - \frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) A_\epsilon(y, \sigma) d\sigma + \int_{s - \frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) g(u^\epsilon(y, \sigma)) dW(y, \sigma) \right. \\ & \quad \left. \left. + \frac{1}{2} \int_{s - \frac{2}{n}}^s \rho_l'(u^\epsilon(y, \sigma) - k) g^2(u^\epsilon(y, \sigma)) Q(y, y) d\sigma \right] dk dy ds \right] \\ &=: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 \end{aligned}$$

In the sequel, we prove the limits

$$\lim_{n \rightarrow \infty} \mathbb{I}_1 = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{I}_3 = 0;$$

and

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{I}_2 = - \int_{Q_T} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) g(\hat{\mathbf{u}}(x, t, \alpha)) g(u^\epsilon(y, t)) d\alpha \right] \rho_m(x - y) Q(x, y) \varphi(y, t) dx dy dt.$$

We start by estimating  $\mathbb{I}_1$ . Setting

$$G_{\mathbb{I}_1} := \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t - s) \rho_m(x - y) \varphi(y, t),$$

recalling the assumption that  $\sum_{j=1}^{\infty} \lambda_j \|e_j\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1$ , and using the properties that  $|\rho_m| \leq C m^d$ ,  $|\rho_n| \leq C n$  and  $|\rho_l| \leq C l$  for some constant  $C$ , Cauchy-Schwarz inequality and the Itô isometry, we obtain after some computations that

$$|\mathbb{I}_1| = \left| \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_{s - \frac{2}{n}}^s G_{\mathbb{I}_1} dW(x, t) \right] \left( \int_{s - \frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) A_\epsilon(y, \sigma) d\sigma \right) dx dk dy ds \right|$$

$$\leq \frac{C(\varphi, \eta_\delta) l \sqrt{\Lambda_1}}{\sqrt{n}} \left[ M_g^2 + \|A_\epsilon\|_{L^2((0,T) \times \Omega \times \mathbb{T}^d)}^2 \right],$$

which tends to 0 as  $n$  tends to  $\infty$ . Lemma 14 in section 4 ensures that  $\|A_\epsilon\|_{L^2((0,T) \times \Omega \times \mathbb{T}^d)}^2$  is bounded. Using a similar idea and the fact that  $|Q(x, y)| \leq \Lambda_1$  for all  $x, y \in \mathbb{T}^d$  we prove that  $\lim_{n \rightarrow \infty} \mathbb{I}_3 = 0$ .

We now consider the term  $\mathbb{I}_2$ . Applying the Itô isometry by observing that  $dW(x, t)dW(y, \sigma) = Q(x, y)\delta(t - \sigma)dt$ , we obtain:

$$\begin{aligned} \mathbb{I}_2 &= -\mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \int_{s-\frac{2}{n}}^s \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t-s) dW(x, t) \varphi(y, s) \rho_m(x-y) dx \right. \\ &\quad \left. \times \left( \int_{s-\frac{2}{n}}^s \rho_l(u^\epsilon(y, \sigma) - k) g(u^\epsilon(y, \sigma)) dW(y, \sigma) \right) dk dy ds \right] \\ &= -\int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_{s-\frac{2}{n}}^s \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_n(t-s) \rho_l(u^\epsilon(y, t) - k) g(u^\epsilon(y, t)) dt \right. \\ &\quad \left. \times \varphi(y, s) \rho_m(x-y) Q(x, y) dx dk dy ds \right] \\ &\xrightarrow{n \rightarrow \infty} -\int_{Q_T} \int_{\mathbb{R}} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - k) g(\hat{\mathbf{u}}(x, t, \alpha)) d\alpha \rho_l(u^\epsilon(y, t) - k) g(u^\epsilon(y, t)) \right] \varphi(y, t) \rho_m(x-y) Q(x, y) dx dk dy dt \\ &\xrightarrow{l \rightarrow \infty} -\int_{Q_T} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) g(\hat{\mathbf{u}}(x, t, \alpha)) g(u^\epsilon(y, t)) d\alpha \right] \varphi(y, t) \rho_m(x-y) Q(x, y) dx dy dt. \end{aligned}$$

The limits of  $n \rightarrow \infty$  and  $l \rightarrow \infty$  are proved by using the fact that  $g$  is a bounded function,  $|Q(x, y)| \leq \Lambda_1$ ,  $\eta_\delta''$  is a Lipschitz-continuous function and  $\text{supp } \rho_l \subset [-\frac{1}{l}, \frac{1}{l}]$ .  $\square$

Combining the Lemmas 2, 3 and 4 yields

$$\begin{aligned} &\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7 + J_7] = \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7] \\ &= \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \int_0^1 g^2(\hat{\mathbf{u}}(x, t, \alpha)) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \rho_m(x-y) \varphi(y, t) Q(x, y) d\alpha dy dx dt \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \int_0^1 g^2(u^\epsilon(y, t)) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \rho_m(x-y) \varphi(y, t) Q(y, y) d\alpha dy dx dt \right] \\ &\quad - \int_{Q_T} \int_{\mathbb{T}^d} \mathbb{E} \left[ \int_0^1 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) g(\hat{\mathbf{u}}(x, t, \alpha)) g(u^\epsilon(y, t)) d\alpha \right] \varphi(y, t) \rho_m(x-y) Q(x, y) dy dx dt \\ &= \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \int_0^1 \sum_{j=1}^{\infty} \lambda_j [g(\hat{\mathbf{u}}(x, t, \alpha)) e_j(x) - g(u^\epsilon(y, t)) e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \rho_m(x-y) \varphi(y, t) d\alpha dy dx dt \right]. \end{aligned}$$

Next, we prove the following result

**Lemma 5.** There holds

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7] = 0.$$

*Proof.* We remark that

$$\begin{aligned}
& \sum_{j=1}^{\infty} \lambda_j [g(\hat{\mathbf{u}}(x, t, \alpha))e_j(x) - g(u^\varepsilon(y, t))e_j(y)]^2 \\
&= \sum_{j=1}^{\infty} \lambda_j [(g(\hat{\mathbf{u}}(x, t, \alpha)) - g(u^\varepsilon(y, t)))e_j(x) + g(u^\varepsilon(y, t))(e_j(x) - e_j(y))]^2 \\
&\leq 2Q(x, x) [g(\hat{\mathbf{u}}(x, t, \alpha)) - g(u^\varepsilon(y, t))]^2 + 2M_g^2 \sum_{j=1}^{\infty} \lambda_j [e_j(x) - e_j(y)]^2,
\end{aligned}$$

so that

$$\begin{aligned}
& \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7] \\
&= \frac{1}{2} \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \sum_{j=1}^{\infty} \lambda_j [g(\hat{\mathbf{u}}(x, t, \alpha))e_j(x) - g(u^\varepsilon(y, t))e_j(y)]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \rho_m(x - y) \varphi(y, t) dy dx dt \right] \\
&\leq \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} Q(y, y) [g(\hat{\mathbf{u}}(x, t, \alpha)) - g(u^\varepsilon(y, t))]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \rho_m(x - y) \varphi(y, t) dy dx dt \right] \\
&+ \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} M_g^2 \sum_{j=1}^{\infty} \lambda_j [e_j(x) - e_j(y)]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \rho_m(x - y) \varphi(y, t) dy dx dt \right].
\end{aligned} \tag{23}$$

First we estimate the first term on the right hand-side of (23); from (7) and the hypothesis (H) for  $g$  and noting that  $\text{supp } \eta''_\delta \subset [-\delta, \delta]$  and  $0 \leq \eta''_\delta \leq 2\pi/\delta$ , it is bounded from above by

$$\begin{aligned}
& \Lambda_1 C_g^2 \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} (\hat{\mathbf{u}} - u^\varepsilon)^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \rho_m(x - y) \varphi(y, t) dy dx dt \right] \\
&\leq 2\pi \Lambda_1 C_g^2 \delta \mathbb{E} \left[ \int_{Q_T} \int_{\mathbb{T}^d} \rho_m(x - y) \varphi(y, t) dy dx dt \right] \\
&\leq 2\pi \Lambda_1 C_g^2 \delta \int_{Q_T} \varphi(y, t) dy dt,
\end{aligned}$$

which tends to 0 as  $\delta$  tends to 0. In what follows, we prove that the second term on the right-hand-side of (23) tends to 0 as  $m \rightarrow \infty$ . Recalling (4) for each  $\varepsilon_1 > 0$ , there exists  $K_0$ , such that for all  $K > K_0$ ,

$$\sum_{j=K+1}^{\infty} \lambda_j [e_j(x) - e_j(y)]^2 \leq 2 \sum_{j=K+1}^{\infty} \lambda_j e_j^2(x) + 2 \sum_{j=K+1}^{\infty} \lambda_j e_j^2(y) \leq \varepsilon_1, \tag{24}$$

we choose  $K > K_0$  to divide the infinite sum in the second term of (23) into 2 parts, forgetting the constant  $M_g^2$

$$\begin{aligned}
& \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=1}^{\infty} \lambda_j [e_j(x) - e_j(y)]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \varphi(y, t) dx dy dt \\
&= \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=1}^K \lambda_j [e_j(x) - e_j(y)]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \varphi(y, t) dx dy dt \\
&+ \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=K+1}^{\infty} \lambda_j [e_j(x) - e_j(y)]^2 \eta''_\delta(\hat{\mathbf{u}}(x, t, \alpha) - u^\varepsilon(y, t)) \varphi(y, t) dx dy dt.
\end{aligned} \tag{25}$$

We study the first term on the right-hand side of (25). Using the fact that  $\text{supp } \rho_m \subset [-\frac{1}{2m}, \frac{1}{2m}]^d$ , as well as the fact that  $e_j \in C(\mathbb{T}^d)$ , setting  $x - y := z$  and  $(e_j(x) - e_j(y))^2 := f_j(z)$ , we deduce that

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \int_{\mathbb{T}^d} \int_0^T \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=1}^K \lambda_j [e_j(x) - e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dx dy dt \\
&= \sum_{j=1}^K \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \lambda_j \rho_m(z) f_j(z) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dz dy dt \\
&= \sum_{j=1}^K \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \lambda_j m^d k_\rho e^{-\frac{1}{1-|mz+1|}} \chi_{\{z \in \mathbb{T}^d, |mz+1| < 1\}} f_j(z) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dz dy dt \\
&= \sum_{j=1}^K \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \lambda_j m^d k_\rho e^{-\frac{1}{1-|\tau|}} \chi_{\{\tau \in \mathbb{T}^d, |\tau| < 1\}} f_j(\frac{\tau-1}{m}) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) \frac{d\tau}{m^d} dy dt \\
&= \sum_{j=1}^K \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \lambda_j k_\rho e^{-\frac{1}{1-|\tau|}} \chi_{\{\tau \in \mathbb{T}^d, |\tau| < 1\}} f_j(\frac{\tau-1}{m}) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) d\tau dy dt \\
&= 0.
\end{aligned} \tag{26}$$

As for the second term, we use (24) to obtain,

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=K+1}^\infty \lambda_j [e_j(x) - e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dx dy dt \\
&\leq \epsilon_1 \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dx dy dt \\
&\leq \epsilon_1 T |\mathbb{T}^d| \|\eta_\delta''\|_{L^\infty((0, T) \times \mathbb{T}^d)} \|\varphi\|_{L^\infty((0, T) \times \mathbb{T}^d)}.
\end{aligned}$$

In view of (25) and (26),

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=1}^\infty \lambda_j [e_j(x) - e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dx dy dt \\
&\leq \epsilon_1 T |\mathbb{T}^d| \|\eta_\delta''\|_{L^\infty(\mathbb{T}^d)} \|\varphi\|_{L^\infty(\mathbb{T}^d)}
\end{aligned}$$

for all  $\epsilon_1 > 0$ , i.e.

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho_m(x - y) \sum_{j=1}^\infty \lambda_j [e_j(x) - e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \varphi(y, t) dx dy dt = 0.$$

Recall that

$$\begin{aligned}
& \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7] \\
&= \frac{1}{2} \mathbb{E} \left[ \int_{\mathcal{Q}_T} \int_{\mathbb{T}^d} \int_0^1 \sum_{j=1}^\infty \lambda_j [g(\hat{\mathbf{u}}(x, t, \alpha)) e_j(x) - g(u^\epsilon(y, t)) e_j(y)]^2 \eta_\delta''(\hat{\mathbf{u}}(x, t, \alpha) - u^\epsilon(y, t)) \rho_m(x - y) \varphi(y, t) d\alpha dy dx dt \right];
\end{aligned}$$

then in view of (23), we deduce that

$$\lim_{\epsilon \rightarrow 0} \lim_{\delta \rightarrow 0} \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} [I_6 + J_6 + I_7] = 0.$$

which complete the proof of Lemma 5.  $\square$

Passing to the limits  $n, l \rightarrow \infty, \delta, \epsilon \rightarrow 0$  and  $m \rightarrow \infty$  in  $\sum_{i=1}^7 \{I_i + J_i\} + J_8$  by , we obtain (12), which completes the proof of the Kato inequality. □

## 4 | EXISTENCE AND UNIQUENESS OF THE STRONG SOLUTION OF THE ASSOCIATED PARABOLIC PROBLEM

In this section, we prove the existence and uniqueness of the strong solution of the associated stochastic nonlinear parabolic problem; to that purpose we apply a time discretization method.

**Definition 4.** We say that  $u^\epsilon$  is a strong solution of the problem

$$(\mathcal{P}^\epsilon) \begin{cases} du^\epsilon - \epsilon \Delta u^\epsilon dt + \operatorname{div}(\mathbf{v}f(u^\epsilon))dt = g(u^\epsilon)dW & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u^\epsilon(x, 0) = u_0^\epsilon(x) & \text{for all } x \in \mathbb{T}^d \end{cases} \quad (27)$$

with initial condition  $u_0^\epsilon(x) \in H^1(\mathbb{T}^d)$ , if  $u^\epsilon$  satisfies

1.  $u^\epsilon \in L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d))) \cap L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$  is an  $L^2(\mathbb{T}^d)$ -adapted process to the filtration  $(\mathcal{F}_t)$ ;
2. for almost all  $t \in (0, T)$ , P-a.s. there holds,

$$u^\epsilon(\cdot, t) = u_0^\epsilon + \epsilon \int_0^t \Delta u^\epsilon ds - \int_0^t \operatorname{div}(\mathbf{v}f(u^\epsilon))ds + \int_0^t g(u^\epsilon)dW(s), \quad (28)$$

in  $L^2(\mathbb{T}^d)$ .

We suppose that  $u_0^\epsilon \rightarrow u_0$  strongly in  $L^2(\mathbb{T}^d)$  as  $\epsilon \rightarrow 0$ , where  $u_0$  is the initial condition of Problem (1). The following result is the analog of Proposition A.2 in<sup>2</sup>.

**Theorem 2.** Let  $\epsilon > 0$  be arbitrary. Problem  $(\mathcal{P}^\epsilon)$  admits a unique strong solution.

*Proof.* We perform an implicit time discretization with  $\Delta t = T/N$ . The scheme is as follows: for a given small positive constant  $\Delta t$  and  $u_n^\epsilon$  in  $L^2(\Omega; H^1(\mathbb{T}^d))$ ,  $\mathcal{F}_{n\Delta t}$ -measurable, find  $u_{n+1}^\epsilon$  in  $L^2(\Omega; H^1(\mathbb{T}^d))$ ,  $\mathcal{F}_{(n+1)\Delta t}$ -measurable, such that P-a.s. and for all  $v$  in  $H^1(\mathbb{T}^d)$

$$\int_{\mathbb{T}^d} [(u_{n+1}^\epsilon - u_n^\epsilon)v + \Delta t(\epsilon \nabla u_{n+1}^\epsilon \cdot \nabla v - f(u_{n+1}^\epsilon)\mathbf{v} \cdot \nabla v)] dx = \int_{\mathbb{T}^d} g(u_n^\epsilon)v(W_{n+1} - W_n) dx \quad (29)$$

where  $W_n = W(t_n) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t_n) e_j(x)$ ; we remark that  $u_0^\epsilon = u_0^\epsilon(x)$ . Before pursuing the proof of Theorem 2, we first show the existence of the sequence  $\{u_n^\epsilon\}_{n \geq 0}$ .

**Lemma 6.** If  $\Delta t < \frac{2\epsilon}{(VC_f)^2}$ , Problem (29) possesses a unique solution  $u_{n+1}^\epsilon \in L^2(\Omega; H^1(\mathbb{T}^d))$ .

*Proof.* The proof is similar to that of [c.f.<sup>2</sup>, Lemma A.3.] □

Next, we prove some a priori estimates.

**Lemma 7.** We define

$$u^{\Delta t}(t) = \sum_{n=0}^{N-1} u_{n+1}^\epsilon \cdot \mathbf{1}_{[n\Delta t, (n+1)\Delta t)}(t) \quad (30)$$

where  $N = T/\Delta t$ ; then the following hold

$$\begin{aligned} \|u^{\Delta t}\|_{L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))} &\leq C_1, \\ \|\nabla u^{\Delta t}\|_{L^2(0, T; (L^2(\Omega; L^2(\mathbb{T}^d)))^d)} &\leq C_2, \end{aligned}$$



where the constant  $C_1$  depends on  $u_0, \Lambda_1, M_g$  and  $T$ , and the constant  $C_2$  depends on  $\epsilon, u_0, \Lambda_1, M_g$  and  $T$ . Thus,

$$\|u^{\Delta t}\|_{L^2(0,T;L^2(\Omega;H^1(\mathbb{T}^d)))} \leq C$$

for a constant  $C$  which depends on  $\epsilon, u_0, \Lambda_1, M_g$  and  $T$ .

We define the linear interpolation function

$$\tilde{u}^{\Delta t} = \sum_{n=1}^N \left[ \frac{u_n^\epsilon - u_{n-1}^\epsilon}{\Delta t} [t - (n-1)\Delta t] + u_{n-1}^\epsilon \right] \cdot \mathbf{1}_{[(n-1)\Delta t, n\Delta t)}. \quad (31)$$

The set  $\{\tilde{u}^{\Delta t}\}$  is bounded in  $L^2(0,T;L^2(\Omega;H^1(\mathbb{T}^d))) \cap L^\infty(0,T;L^2(\Omega;L^2(\mathbb{T}^d)))$ , and  $\tilde{u}^{\Delta t} - u^{\Delta t}$  converges to 0 in  $L^2(0,T;L^2(\Omega;H^1(\mathbb{T}^d)))$ .

*Proof.* Setting  $v = u_{n+1}^\epsilon$  in (29) and taking the expectation yields

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} (|u_{n+1}^\epsilon|^2 - |u_n^\epsilon|^2 + |u_{n+1}^\epsilon - u_n^\epsilon|^2) dx \right] + \Delta t \epsilon \mathbb{E} \left[ \int_{\mathbb{T}^d} |\nabla u_{n+1}^\epsilon|^2 dx \right] - \Delta t \mathbb{E} \left[ \int_{\mathbb{T}^d} f(u_{n+1}^\epsilon) \mathbf{v} \cdot \nabla u_{n+1}^\epsilon dx \right] \\ &= \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) g(u_n^\epsilon) (u_{n+1}^\epsilon - u_n^\epsilon) dx \right] + \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) g(u_n^\epsilon) u_n^\epsilon dx \right]. \end{aligned} \quad (32)$$

Define  $F(s) = \int_0^s f(\tau) d\tau$  and use the fact that  $\operatorname{div} \mathbf{v} = 0$ ; then for all functions  $v \in H^1(\mathbb{T}^d)$

$$\int_{\mathbb{T}^d} f(v) \mathbf{v} \cdot \nabla v dx = \int_{\mathbb{T}^d} \mathbf{v} \cdot \nabla F(v) dx = - \int_{\mathbb{T}^d} F(v) \operatorname{div} \mathbf{v} dx = 0.$$

Next we use (6), (7), the equality  $\mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) g(u_n^\epsilon) u_n^\epsilon dx \right] = 0$ , and the monotone convergence theorem to deduce that

$$\begin{aligned} \mathbb{E} [(W_{n+1} - W_n)^2] &= \mathbb{E} \left[ \left( \sum_{j=0}^{\infty} \sqrt{\lambda_j} (\beta_j(t_{n+1}) - \beta_j(t_n)) e_j(x) \right)^2 \right] \\ &= \mathbb{E} \left[ \sum_{j=0}^{\infty} \lambda_j (\beta_j(t_{n+1}) - \beta_j(t_n))^2 e_j^2(x) \right] \\ &= \sum_{j=0}^{\infty} \mathbb{E} \left[ \lambda_j (\beta_j(t_{n+1}) - \beta_j(t_n))^2 e_j^2(x) \right] \\ &= \Delta t Q(x, x) \leq \Lambda_1 \Delta t. \end{aligned} \quad (33)$$

Applying Young's inequality and (33) yields

$$\begin{aligned} \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) g(u_n^\epsilon) (u_{n+1}^\epsilon - u_n^\epsilon) dx \right] &\leq \frac{1}{2\bar{\epsilon}} \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n)^2 g^2(u_n^\epsilon) dx \right] + \frac{\bar{\epsilon}}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_{n+1}^\epsilon - u_n^\epsilon)^2 dx \right] \\ &\leq \frac{\Delta t}{2\bar{\epsilon}} \int_{\mathbb{T}^d} \mathbb{E} [g^2(u_n^\epsilon)] Q(x, x) dx + \frac{\bar{\epsilon}}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_{n+1}^\epsilon - u_n^\epsilon)^2 dx \right], \end{aligned} \quad (34)$$

for all  $\bar{\epsilon} > 0$ . We choose  $\bar{\epsilon} = \frac{1}{2}$  and deduce from (32) that

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left[ \int_{\mathbb{T}^d} (|u_{n+1}^\epsilon|^2 - |u_n^\epsilon|^2 + |u_{n+1}^\epsilon - u_n^\epsilon|^2) dx \right] + \Delta t \epsilon \mathbb{E} \left[ \int_{\mathbb{T}^d} |\nabla u_{n+1}^\epsilon|^2 dx \right] \\ & \leq \Delta t \mathbb{E} \left[ \int_{\mathbb{T}^d} g^2(u_n^\epsilon) Q(x, x) dx \right] + \frac{1}{4} \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_{n+1}^\epsilon - u_n^\epsilon)^2 dx \right] \\ & \leq \Lambda_1 \Delta t \int_{\mathbb{T}^d} \mathbb{E} [g^2(u_n^\epsilon)] dx + \frac{1}{4} \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_{n+1}^\epsilon - u_n^\epsilon)^2 dx \right]. \end{aligned} \quad (35)$$

We denote by  $\|\cdot\|$  the norm in  $L^2(\mathbb{T}^d)$ , change the index in (35) to  $k$  and sum from 0 to  $n-1$ , for all  $n \geq 1$ , to obtain,

$$\begin{aligned} \frac{1}{2} \mathbb{E} [\|u_n^\epsilon\|^2] + \frac{1}{4} \sum_{k=0}^{n-1} \mathbb{E} [\|u_{k+1}^\epsilon - u_k^\epsilon\|^2] + \Delta t \epsilon \sum_{k=0}^{n-1} \mathbb{E} [\|\nabla u_{k+1}^\epsilon\|^2] & \leq \frac{1}{2} \|u_0^\epsilon\|^2 + \Lambda_1 \Delta t \sum_{k=0}^{n-1} \mathbb{E} [\|g(u_k^\epsilon)\|^2] \\ & \leq \frac{1}{2} \|u_0^\epsilon\|^2 + \Lambda_1 T M_g^2, \end{aligned} \quad (36)$$

so that

$$\mathbb{E} [\|u_n^\epsilon\|^2] \leq C_1, \quad (37)$$

where the constant  $C_1$  depends on  $u_0^\epsilon$ ,  $\Lambda_1$ ,  $M_g$  and  $T$ , and does not depend on  $n$ . In view of (37), and the definition of  $u^{\Delta t}$  we obtain

$$\|u^{\Delta t}\|_{L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))} \leq \sqrt{C_1}, \quad (38)$$

and we deduce from (36) that

$$\sqrt{\epsilon} \|\nabla u^{\Delta t}\|_{L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))} \leq C_2, \quad (39)$$

where the constant  $C_2$  depends on  $C_1$ . In turn (38) and (39) imply that

$$\|u^{\Delta t}\|_{L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))} \leq C, \quad (40)$$

for a constant  $C$  which depends on  $C_1$ ,  $\epsilon$  and  $T$ . In view of (36) and (37)

$$\begin{aligned} \|\tilde{u}^{\Delta t} - u^{\Delta t}\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}^2 & = \mathbb{E} \left[ \int_{\mathbb{T}^d} \sum_{n=1}^N \int_{(n-1)\Delta t}^{n\Delta t} (u_n^\epsilon - u_{n-1}^\epsilon)^2 \left( \frac{t - n\Delta t}{\Delta t} \right)^2 \mathbf{1}_{[(n-1)\Delta t, n\Delta t]} dt dx \right] \\ & = \frac{1}{3} \Delta t \sum_{n=1}^N \mathbb{E} [\|u_n^\epsilon - u_{n-1}^\epsilon\|_{L^2(\mathbb{T}^d)}^2] \\ & \leq \frac{4}{3} \Delta t \left( \frac{1}{2} \|u_0^\epsilon\|^2 + \Lambda_1 T M_g^2 \right) \end{aligned} \quad (41)$$

which converges to 0 as  $\Delta t$  tends to 0. Next we prove that

$$\nabla(u^{\Delta t} - \tilde{u}^{\Delta t}) \rightharpoonup 0,$$

weakly in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)^d))$ .

Consider a test function  $\phi(x, t) = \sum_{n=1}^N \varphi(x) \frac{n\Delta t - t}{\Delta t} \mathbf{1}_{[(n-1)\Delta t, n\Delta t]}$  with  $\varphi(x) \in H^1(\mathbb{T}^d)^d$ , so that  $\phi \in L^2(0, T; H^1(\mathbb{T}^d)^d)$ . In view of the definitions of  $u^{\Delta t}$  and of  $\tilde{u}^{\Delta t}$ , we deduce that

$$\begin{aligned} \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \nabla(u^{\Delta t} - \tilde{u}^{\Delta t}) \phi(x, t) dt dx \right] & = \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \nabla \left( u_n^\epsilon - \sum_{n=1}^N \frac{u_n^\epsilon - u_{n-1}^\epsilon}{\Delta t} [t - (n-1)\Delta t] - u_{n-1}^\epsilon \right) \varphi(x) \mathbf{1}_{[(n-1)\Delta t, n\Delta t]} dt dx \right] \\ & = \mathbb{E} \left[ \int_{\mathbb{T}^d} \sum_{n=1}^N \int_{(n-1)\Delta t}^{n\Delta t} \left( 1 - \frac{t - (n-1)\Delta t}{\Delta t} \right) (\nabla u_n^\epsilon - \nabla u_{n-1}^\epsilon) \varphi(x) dt dx \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[ \frac{\Delta t}{2} \int_{\mathbb{T}^d} \sum_{n=1}^N (\nabla u_n^\varepsilon - \nabla u_{n-1}^\varepsilon) \varphi(x) dx \right] \\
&= - \mathbb{E} \left[ \frac{\Delta t}{2} \sum_{n=1}^N \int_{\mathbb{T}^d} (u_n^\varepsilon - u_{n-1}^\varepsilon) \operatorname{div} \varphi(x) dx \right].
\end{aligned}$$

Since by Cauchy-Schwarz inequality

$$\left| \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_n^\varepsilon - u_{n-1}^\varepsilon) \operatorname{div} \varphi(x) dx \right] \right| \leq \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_n^\varepsilon - u_{n-1}^\varepsilon)^2 dx \right] \mathbb{E} \left[ \int_{\mathbb{T}^d} |\operatorname{div} \varphi(x)|^2 dx \right],$$

it follows that

$$\begin{aligned}
\left| \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \nabla(u^{\Delta t} - \tilde{u}^{\Delta t}) \phi(x, t) dt dx \right] \right| &= \left| \mathbb{E} \left[ \frac{\Delta t}{2} \sum_{n=1}^N \int_{\mathbb{T}^d} (u_n^\varepsilon - u_{n-1}^\varepsilon) \operatorname{div} \varphi(x) dx \right] \right| \\
&\leq \frac{\Delta t}{2} \sum_{n=1}^N \mathbb{E} \left[ \int_{\mathbb{T}^d} (u_n^\varepsilon - u_{n-1}^\varepsilon)^2 dx \right] \mathbb{E} \left[ \int_{\mathbb{T}^d} |\operatorname{div} \varphi(x)|^2 dx \right]
\end{aligned}$$

which converges to 0 in view of (36). This implies  $u^{\Delta t} - \tilde{u}^{\Delta t}$  converges weakly to 0 in  $L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$ .  $\square$

We deduce from Lemma 7 that there exist a subsequence of  $\{\Delta t\}$  which we denote by  $\{\Delta t_j\}$  and functions  $u \in L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$ ,  $f_u \in L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  and  $g_u \in L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  such that

$$u^{\Delta t_j} \rightharpoonup u \quad \text{and} \quad \tilde{u}^{\Delta t_j} \rightharpoonup u$$

weakly in  $L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$  as  $\Delta t_j \rightarrow 0$ , and

$$f(u^{\Delta t_j}) \rightharpoonup f_u \quad \text{and} \quad g(u^{\Delta t_j}) \rightharpoonup g_u$$

weakly in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  as  $\Delta t_j \rightarrow 0$ . Since  $g$  is a bounded function,  $g(u^{\Delta t})$  is bounded in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  uniformly in  $\Delta t$ . Moreover

$$\mathbb{E} [\|f(u_n)\|^2] = \mathbb{E} [\|f(u_n) - f(0)\|^2] \leq C_f^2 \mathbb{E} [\|u_n\|^2] \leq C_f^2 C_1, \quad (42)$$

so that  $f(u^{\Delta t})$  is bounded in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  uniformly in  $\Delta t$ . In view of (40), there exist a function  $u \in L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$  and a subsequence  $\{u^{\Delta t_j}\}$  of  $\{u^{\Delta t}\}$  which converges weakly to  $u$  in  $L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$  as  $\Delta t_j$  tends to zero. Moreover (42) and the assumption on  $g$  imply that there exist functions  $f_u$  and  $g_u$  and a subsequence of  $\{u^{\Delta t_j}\}$ , which we denote again by  $\{u^{\Delta t_j}\}$ , such that  $f(u^{\Delta t_j})$  converges weakly to  $f_u$  and  $g(u^{\Delta t_j})$  converges weakly to  $g_u$  in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  as  $\Delta t_j$  tends to zero. For later use, we define

$$\tilde{B}^{\Delta t_j} = \sum_{n=1}^N \left[ \frac{B_n - B_{n-1}}{\Delta t_j} [t - (n-1)\Delta t_j] + B_{n-1} \right] \cdot \mathbf{1}_{[(n-1)\Delta t_j, n\Delta t_j)} \quad (43)$$

where

$$B_n = \sum_{k=0}^{n-1} (W_{k+1} - W_k) g(u_k) = \int_0^{n\Delta t_j} g(u^{\Delta t_j}(\cdot - \Delta t_j)) dW, \quad (44)$$

in which we set  $u^{\Delta t_j}(s) = u^{\Delta t_j}(0) = u_0$ , for all  $s < 0$ . One can check that  $\{\tilde{B}^{\Delta t_j}\}$  is uniformly bounded in  $L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d))$ . Next, we identify the weak limit of  $\int_0^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s)$  when  $\Delta t_j \rightarrow 0$  in  $L^2(\Omega; L^2(\mathbb{T}^d))$  for all  $t \in [0, T]$ .

**Lemma 8.** We have the following limit property

$$\int_0^t g(u^{\Delta t_j}(\cdot - \Delta t_j)) dW(x, t) \rightarrow \int_0^t g_u dW(x, t) \quad (45)$$

as  $\Delta t_j$ , weakly in  $L^2(\Omega; L^2(\mathbb{T}^d))$  for all  $t \in [0, T]$ .

*Proof.* We set  $u^{\Delta t_j}(s) = u_0$ , for all  $s < 0$ .

We define the operator  $\tilde{\mathcal{T}}$  as

$$\tilde{\mathcal{T}}(\mathcal{U}) = \int_0^t \mathcal{U} dW(x, s) \quad (46)$$

where  $W(x, t)$  is the  $Q$ -Brownian motion defined by (3) satisfying the condition (4). Using the same idea as in the proof of (45) in [c.f. <sup>7</sup>, Lemma 5.4], we conclude that along a subsequence

$$\tilde{\mathcal{T}}(g(u^{\Delta t_j}(\cdot - \Delta t_j))) \rightharpoonup \tilde{\mathcal{T}}(g_u)$$

weakly in  $L^2(\Omega \times \mathbb{T}^d)$ , as  $\Delta t_j \rightarrow 0$ , or in other words, in view of the definition (46) of  $\tilde{\mathcal{T}}$ ,

$$\int_0^t g(u^{\Delta t_j}(\cdot - \Delta t_j)) dW(x, t) \rightarrow \int_0^t g_u dW(x, t)$$

weakly in  $L^2(\Omega; L^2(\mathbb{T}^d))$  for all  $t \in [0, T]$ . □

**Lemma 9.** For almost all  $t \in (0, T)$ , P-a.s., there holds,

$$u(t) = u_0^\epsilon + \epsilon \int_0^t \Delta u ds - \int_0^t \operatorname{div}(\mathbf{v} f_u) ds + \int_0^t g_u dW(s),$$

in  $(H^1(\mathbb{T}^d))'$ , where  $u$  is the weak limit of  $u^{\Delta t_j}$  in  $L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))$  and its weak-\* limit in  $L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ .

*Proof.* For all  $t \in [0, T]$ , there exists  $n$  such that  $t \in [n\Delta t_j, (n+1)\Delta t_j]$ . We return to (29) and suppose that the test function  $v \in L^2(\Omega; H^1(\mathbb{T}^d))$  and is  $\mathcal{F}_t$ -adapted, and sum  $k$  from 0 to  $n$  to deduce that P-a.s.

$$\sum_{k=0}^n \int_{\mathbb{T}^d} (u_{k+1}^\epsilon - u_k^\epsilon) v dx + \Delta t_j \sum_{k=0}^n \int_{\mathbb{T}^d} (\epsilon \nabla u_{k+1}^\epsilon \cdot \nabla v - f(u_{k+1}^\epsilon) \mathbf{v} \cdot \nabla v) dx = \sum_{k=0}^n \int_{\mathbb{T}^d} [v g(u_k^\epsilon) (W_{k+1} - W_k)] dx,$$

which yields

$$\int_{\mathbb{T}^d} (u_{n+1}^\epsilon v - u_0^\epsilon v) dx + \Delta t_j \sum_{k=0}^n \int_{\mathbb{T}^d} (\epsilon \nabla u_{k+1}^\epsilon \cdot \nabla v - f(u_{k+1}^\epsilon) \mathbf{v} \cdot \nabla v) dx = \Delta t_j \sum_{k=0}^n \int_{\mathbb{T}^d} \left[ v g(u_k^\epsilon) \frac{W_{k+1} - W_k}{\Delta t_j} \right] dx$$

and by definition (30) it is equivalent to

$$\begin{aligned} & \mathbb{E} \left[ \int_{\mathbb{T}^d} (u^{\Delta t_j}(t) v - u_0^\epsilon v) dx \right] + \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_0^t \epsilon \nabla u^{\Delta t_j}(s) - f(u^{\Delta t_j}(s)) \mathbf{v} ds \right) \cdot \nabla v dx \right] \\ & - \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_{n\Delta t_j}^t \epsilon \nabla u^{\Delta t_j}(s) - f(u^{\Delta t_j}(s)) \mathbf{v} ds \right) \cdot \nabla v dx \right] \\ & = \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_0^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s) \right) v dx \right] - \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_{n\Delta t_j}^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s) \right) v dx \right]. \end{aligned} \quad (47)$$

Next we estimate the terms involving the time integral on  $[n\Delta t_j, t)$ .

$$\begin{aligned} \left| \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{n\Delta t_j}^t \epsilon \nabla u^{\Delta t_j}(s) \cdot \nabla v ds dx \right] \right| &\leq \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{n\Delta t_j}^t |\epsilon \nabla u^{\Delta t_j}(s) \cdot \nabla v| ds dx \right] \\ &\leq \left( \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{n\Delta t_j}^t |\nabla v|^2 ds dx \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{n\Delta t_j}^t |\epsilon \nabla u^{\Delta t_j}(s)|^2 ds dx \right] \right)^{1/2} \\ &\leq \epsilon \sqrt{\Delta t_j} \|v\|_{L^2(\Omega; H^1(\mathbb{T}^d))} \|u^{\Delta t_j}(s)\|_{L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d)))}. \end{aligned}$$

We deduce from a similar computation that

$$\left| \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_{n\Delta t_j}^t f(u^{\Delta t_j}(s)) \mathbf{v} ds \cdot \nabla v dx \right] \right| \leq \epsilon V \sqrt{\Delta t_j} \|v\|_{L^2(\Omega; H^1(\mathbb{T}^d))} \|f(u^{\Delta t_j}(s))\|_{L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))}.$$

Also in view of (33),

$$\begin{aligned} \left| \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_{n\Delta t_j}^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s) \right) v dx \right] \right| &\leq \left( \mathbb{E} \left[ \int_{\mathbb{T}^d} v^2 dx \right] \right)^{1/2} \left( \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_{n\Delta t_j}^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s) \right)^2 dx \right] \right)^{1/2} \\ &= \|v\|_{L^2(\Omega; L^2(\mathbb{T}^d))} M_g \left( \mathbb{E} \left[ \int_{\mathbb{T}^d} (W(t) - W_n)^2 dx \right] \right)^{1/2} \\ &\leq \Delta t_j \Lambda_1 M_g \|v\|_{L^2(\Omega; L^2(\mathbb{T}^d))} \end{aligned}$$

We recall that as  $\Delta t_j \rightarrow 0$ ,

$$\begin{aligned} u^{\Delta t_j} &\rightharpoonup u \quad \text{in } L^2(0, T; L^2(\Omega; H^1(\mathbb{T}^d))); \\ f(u^{\Delta t_j}) &\rightharpoonup f_u \quad \text{in } L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d))); \\ \int_0^t g(u^{\Delta t_j}(s - \Delta t_j)) dW(s) &\rightharpoonup \int_0^t g_u dW(s) \quad \text{in } L^2(\Omega; L^2(\mathbb{T}^d)) \text{ for all } t \in [0, T], \end{aligned}$$

and also using that  $u^{\Delta t_j}$  is bounded in  $L^\infty(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ , we deduce that for a.e.  $t > 0$ , there exists a subsequence of  $u^{\Delta t_j}(t)$  which we denote again by  $u^{\Delta t_j}(t)$  which converges to  $u(t)$  weakly in  $L^2(\Omega; L^2(\mathbb{T}^d))$ . We let  $\Delta t_j \rightarrow 0$  in (47) to deduce that

$$\mathbb{E} \left[ \int_{\mathbb{T}^d} u(t) v dx \right] - \mathbb{E} \left[ \int_{\mathbb{T}^d} u_0^\epsilon v dx \right] + \epsilon \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} \nabla u \cdot \nabla v dx ds \right] - \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} f_u \mathbf{v} \cdot \nabla v dx ds \right] = \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^t g_u dW(s) v dx \right]$$

for all  $v \in L^2(\Omega; H^1(\mathbb{T}^d))$  and all  $t \in [0, T]$ , so that

$$\mathbb{E} \left[ \int_{\mathbb{T}^d} \left( u(t) - \int_0^t g_u dW(s) \right) v dx \right] - \mathbb{E} \left[ \int_{\mathbb{T}^d} u_0^\epsilon v dx \right] + \epsilon \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} \nabla u \cdot \nabla v dx ds \right] - \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} f_u \mathbf{v} \cdot \nabla v dx ds \right] = 0$$

which is equivalent to

$$\mathbb{E} \left[ \int_{\mathbb{T}^d} \left( u(t) - \int_0^t g_u dW(s) \right) v dx \right] - \mathbb{E} \left[ \int_{\mathbb{T}^d} u_0^\epsilon v dx \right] + \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} \langle -\epsilon \Delta u + \operatorname{div}(f_u \mathbf{v}), v \rangle_{(H^1)^Y, H^1} dx ds \right] = 0.$$

Thus for almost all  $t \in (0, T)$ , P-a.s., there holds,

$$u(t) = u_0^\epsilon(\cdot) + \epsilon \int_0^t \Delta u ds - \int_0^t \operatorname{div}(\mathbf{v} f_u) ds + \int_0^t g_u dW(s), \quad (48)$$

in  $(H^1(\mathbb{T}^d))'$ . In the following we will identify  $f_u$  and  $g_u$ .  $\square$

**Lemma 10.** Suppose that the hypothesis (5) holds for some positive constant  $\Lambda_2$ . Then  $\partial_t [\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j}]$  is bounded in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ .

*Proof.* The proof is similar to that of Bauzet-Vallet-Wittbold [c.f.<sup>2</sup>, proof of Lemma A.3, p.697]. We just consider the new terms  $\mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) \nabla(u_{n+1}^\epsilon - u_n^\epsilon) \cdot \nabla g(u_n^\epsilon) dx \right]$  and  $\mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) \nabla(u_{n+1}^\epsilon - u_n^\epsilon) \cdot \nabla g(u_n^\epsilon) dx \right]$  which should be estimated instead of  $E(w_{n+1} - w_n) \int_{\mathbb{R}^d} \nabla[u_{n+1} - u_n] \cdot \nabla h(u_n)$  in c.f.<sup>2</sup>. In view of (33), we deduce that

$$\begin{aligned} \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) \nabla(u_{n+1}^\epsilon - u_n^\epsilon) \cdot \nabla g(u_n^\epsilon) dx \right] &\leq \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n)^2 |\nabla g(u_n^\epsilon)|^2 dx + \frac{1}{4} \int_{\mathbb{T}^d} |\nabla(u_{n+1}^\epsilon - u_n^\epsilon)|^2 dx \right] \\ &\leq \int_{\mathbb{T}^d} \mathbb{E} [(W_{n+1} - W_n)^2] \mathbb{E} [|\nabla g(u_n^\epsilon)|^2] dx + \frac{1}{4} \mathbb{E} \left[ \int_{\mathbb{T}^d} |\nabla(u_{n+1}^\epsilon - u_n^\epsilon)|^2 dx \right] \\ &\leq \Lambda_1 \Delta t_j \int_{\mathbb{T}^d} \mathbb{E} [|\nabla g(u_n^\epsilon)|^2] dx + \frac{1}{4} \mathbb{E} \left[ \int_{\mathbb{T}^d} |\nabla(u_{n+1}^\epsilon - u_n^\epsilon)|^2 dx \right]. \end{aligned}$$

One can obtain an estimate for the term  $\mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{n+1} - W_n) \nabla(u_{n+1}^\epsilon - u_n^\epsilon) \cdot \nabla g(u_n^\epsilon) dx \right]$  in a similar fashion by using (5). We deduce that

$$\begin{aligned} &\mathbb{E} \left[ \|u_{n+1}^\epsilon - u_n^\epsilon - (W_{n+1} - W_n)g(u_n^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 \right] + \Delta t_j \epsilon \mathbb{E} \left[ \|\nabla u_{n+1}^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 - \|\nabla u_n^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 + \frac{1}{2} \|\nabla(u_{n+1}^\epsilon - u_n^\epsilon)\|_{L^2(\mathbb{T}^d)^d}^2 \right] \\ &\leq \Lambda_1 (\Delta t_j)^2 \epsilon \mathbb{E} \left[ \|\nabla g(u_n^\epsilon)\|_{L^2(\mathbb{T}^d)^d}^2 \right] + \Lambda_2 (\Delta t_j)^2 \epsilon \mathbb{E} \left[ \|g(u_n^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 \right] + C(\mathbf{v}, C_f) (\Delta t_j)^2 \mathbb{E} \left[ \|\nabla u_{n+1}^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 \right]. \end{aligned}$$

We sum the above inequality on the variable  $n$  from 0 to  $N-1$  and apply Lemma 7 to deduce that

$$\begin{aligned} &\sum_{n=0}^{N-1} \Delta t_j \mathbb{E} \left[ \left\| \frac{u_{n+1}^\epsilon - u_n^\epsilon - (W_{n+1} - W_n)g(u_n^\epsilon)}{\Delta t_j} \right\|_{L^2(\mathbb{T}^d)}^2 \right] + \epsilon \mathbb{E} \left[ \|\nabla u_N^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 \right] + \frac{\epsilon}{2} \sum_{n=0}^{N-1} \mathbb{E} \left[ \|\nabla(u_{n+1}^\epsilon - u_n^\epsilon)\|_{L^2(\mathbb{T}^d)^d}^2 \right] \\ &\leq C(\mathbf{v}, C_f, C_g, \epsilon, \Lambda_1) \Delta t_j \sum_{n=1}^N \mathbb{E} \left[ \|\nabla u_n^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 \right] + \epsilon \mathbb{E} \left[ \|\nabla u_0^\epsilon\|_{L^2(\mathbb{T}^d)^d}^2 \right] + \Lambda_2 \Delta t_j \epsilon \sum_{n=0}^N \mathbb{E} \left[ \|g(u_n^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 \right] \\ &\leq C, \end{aligned} \quad (49)$$

where  $C$  depends on  $\mathbf{v}, C_f, C_g, \Lambda_1, \Lambda_2, u_0$  and  $\epsilon$ . We take the difference of (31) and (43) to deduce that

$$\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j} = \sum_{n=1}^N \left[ \frac{u_n - u_{n-1} - (B_n - B_{n-1})}{\Delta t_j} (t - (n-1)\Delta t_j) + (u_{n-1} - B_{n-1}) \right] \mathbf{1}_{[(n-1)\Delta t_j, n\Delta t_j)},$$

which we differentiate with respect to  $t$  to obtain

$$\partial_t [\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j}] = \sum_{n=1}^N \left[ \frac{u_n^\epsilon - u_{n-1}^\epsilon - (B_n - B_{n-1})}{\Delta t_j} \right] \mathbf{1}_{[(n-1)\Delta t_j, n\Delta t_j)} = \sum_{n=1}^N \left[ \frac{u_n^\epsilon - u_{n-1}^\epsilon - (W_n - W_{n-1})g(u_{n-1}^\epsilon)}{\Delta t_j} \right] \mathbf{1}_{[(n-1)\Delta t_j, n\Delta t_j)}$$

where we have also used the definition of  $B_n$  in (44). The result of Lemma 10 then follows from (49).  $\square$

**Lemma 11.** P-a.s.,  $\int_0^t g_u dW(s)$  is continuous in time and  $u$  defined as (48) is in  $C([0, T]; (H^1(\mathbb{T}^d))')$ .

*Proof.* We have proved that, for almost every  $\omega \in \Omega$ ,

$$\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j} \in L^\infty(0, T; L^2(\mathbb{T}^d));$$

$$\partial_t[\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j}] \in L^2(0, T; L^2(\mathbb{T}^d)) \subset L^2\left(0, T; (H^1(\mathbb{T}^d))'\right).$$

Using the fact that the embedding from  $L^2(\mathbb{T}^d)$  into  $(H^1(\mathbb{T}^d))'$  is compact, and applying the Theorem of Aubin-Lions-Simon [cf. <sup>1</sup>, Theorem II.5.16], we deduce that the sequence  $\{\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j}\}$  is relatively compact in  $C([0, T]; (H^1(\mathbb{T}^d))')$ . Therefore

$\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j} \rightarrow u - \int_0^t g_u dW(s)$  strongly in  $C([0, T]; (H^1(\mathbb{T}^d))')$  as  $\Delta t_j \rightarrow 0$ . If  $\int_0^t g_u dW(s)$  is continuous in time, then  $u(t)$  is continuous in time. We define

$$\begin{aligned} w(x, t) &= \int_0^t g_u dW(x, s) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j(x) \int_0^t g_u d\beta_j(s), \\ w_N(x, t) &= \sum_{j=1}^N \sqrt{\lambda_j} e_j(x) \int_0^t g_u d\beta_j(s) \end{aligned}$$

for all  $N \in \mathbb{N}$ . Thus for all  $1 \leq N < M$ ,  $M, N \in \mathbb{N}$ , by Doob's inequality and Itô isometry,

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} (w_M(x, t) - w_N(x, t))^2 \right] \leq 4 \mathbb{E} [(w_M(x, T) - w_N(x, T))^2] = 4 \sum_{j=N+1}^M \lambda_j e_j^2(x) \int_0^T \mathbb{E} [g_u^2] ds. \quad (50)$$

Recalling the assumption that  $\sum_{j=1}^{\infty} \lambda_j \|e_j\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1$  and using the fact that  $g_u \in L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ , we deduce that

$$\sum_{j=1}^{\infty} \lambda_j \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} e_j^2 g_u^2 ds dx \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \sum_{j=1}^{\infty} \lambda_j e_j^2 g_u^2 ds dx \right] \leq \Lambda_1 \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} g_u^2 ds dx \right] < \infty,$$

so that in view of (50)

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|w_M(t) - w_N(t)\|_{L^2(\mathbb{T}^d)}^2 \right] \leq \int_{\mathbb{T}^d} dx \mathbb{E} \left[ \sup_{0 \leq t \leq T} (w_M(t) - w_N(t))^2 \right] \leq 4 \sum_{j=N+1}^M \lambda_j \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} e_j^2 g_u^2 ds dx \right] \rightarrow 0,$$

as  $N < M$  and  $N \rightarrow \infty$ . This implies with the help of Borel-Cantelli's lemma that  $\{w_N(t)\}$  has a Cauchy subsequence  $\{w_{N'}(t)\}$  in  $C([0, T], L^2(\mathbb{T}^d))$ , P-a.s.. Therefore since  $\{w_{N'}\}$  converges to  $w$  in  $C([0, T], L^2(\mathbb{T}^d))$ , it follows that  $w \in C([0, T], L^2(\mathbb{T}^d))$ .

As P-a.s.,  $u - \int_0^t g_u dW(s)$  is in  $C([0, T], H^1(\mathbb{T}^d)')$ , we deduce that P-a.s.,  $u \in C([0, T], H^1(\mathbb{T}^d)')$ .  $\square$

Next we identify  $f_u$  with  $f(u)$ ,  $g_u$  with  $g(u)$  and  $u$  with  $u^\epsilon$ .

**Lemma 12.** We have that  $u^{\Delta t_j} \rightarrow u$  strongly in  $L^2(0, T; L^2(\Omega \times \mathbb{T}^d))$  as  $\Delta t_j \rightarrow 0$ . In particular, this implies that that  $f_u = f(u)$  and  $g_u = g(u)$ . And  $u$  is a strong solution of Problem  $(\mathcal{P}^\epsilon)$ , we will denote it by  $u^\epsilon$ .

Before proving the Lemma 12, we show a preliminary estimate.

**Lemma 13.** There holds

$$\begin{aligned} & \limsup_{\Delta t_j \rightarrow 0} \int_0^T e^{-ct} \mathbb{E} [\|u^{\Delta t_j}(t)\|^2] dt \\ & \leq T \|u_0\|^2 + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} f_u \mathbf{v} \cdot \nabla u dx \right] ds dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} g_u g(u) Q(x, x) dx \right] ds dt \end{aligned} \quad (51)$$

$$- \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} g^2(u) Q(x, x) dx \right] ds dt - c \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u\|^2] ds dt - 2\epsilon \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u\|^2] ds dt.$$

One can prove this inequality by using similar ideas as in<sup>2</sup> and the fact that  $|Q(x, x)| \leq \Lambda_1$ . Next, we use this result to prove Lemma 12.

*Proof.* We apply the Itô's formula [cf. <sup>10</sup>, Theorem 4.2.5] and also using the equality (4.2.28) in<sup>10</sup>, to deduce that

$$\begin{aligned} & e^{-ct} \mathbb{E} [\|u(t)\|^2] + 2\epsilon \int_0^t e^{-cs} \mathbb{E} [\|\nabla u\|^2] ds - 2 \int_0^t \mathbb{E} \left[ \int_{\mathbb{T}^d} e^{-cs} \mathbf{v} f_u \cdot \nabla u \right] dx ds \\ &= \mathbb{E} [\|u_0\|^2] - c \int_0^t e^{-cs} \mathbb{E} [\|u(s)\|^2] ds + \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} g_u^2 Q(x, x) dx \right] ds. \end{aligned} \quad (52)$$

We integrate (52) on time interval  $[0, T)$  which yields

$$\begin{aligned} & T \|u_0\|^2 + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} f_u \mathbf{v} \cdot \nabla u dx \right] ds dt - 2\epsilon \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u\|^2] ds dt - c \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u\|^2] ds dt \\ &= \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|^2] dt - \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} g_u^2 Q(x, x) dx \right] ds dt. \end{aligned} \quad (53)$$

We substitute (53) into (51) to obtain

$$\limsup_{\Delta t_j \rightarrow 0} \int_0^T e^{-ct} \mathbb{E} [\|u^{\Delta t_j}(t)\|^2] dt + \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[ \int_{\mathbb{T}^d} (g_u - g(u))^2 Q(x, x) dx \right] ds dt \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|^2] dt,$$

which yields the inequality

$$\limsup_{\Delta t_j \rightarrow 0} \int_0^T e^{-ct} \mathbb{E} [\|u^{\Delta t_j}(t)\|^2] dt \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|^2] dt. \quad (54)$$

In what follows, we prove that  $u^{\Delta t_j}$  converges to  $u$  strongly in  $L^2(0, T; L^2(\Omega \times \mathbb{T}^d))$ . We have that

$$\begin{aligned} \|e^{-ct/2}(u^{\Delta t_j} - u)\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}^2 &= \mathbb{E} \left[ \int_0^T e^{-ct} \int_{\mathbb{T}^d} (u^{\Delta t_j} - u)(u^{\Delta t_j} - u) dx dt \right] \\ &= \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} (e^{-ct/2} u^{\Delta t_j})^2 dx dt \right] - 2 \mathbb{E} \left[ \int_0^T e^{-ct} \int_{\mathbb{T}^d} u u^{\Delta t_j} dx dt \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} e^{-ct} u^2 dx dt \right]. \end{aligned}$$

Passing to the limit and using (54), we deduce that

$$\lim_{\Delta t_j \rightarrow 0} \|e^{-ct/2}(u^{\Delta t_j} - u)\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}^2 \leq 0,$$

so that also

$$\lim_{\Delta t_j \rightarrow 0} \|u^{\Delta t_j} - u\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}^2 \leq e^{cT} \lim_{\Delta t_j \rightarrow 0} \|e^{-ct/2}(u^{\Delta t_j} - u)\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}^2 = 0.$$

Finally we prove that  $f_u = f(u)$  and  $g_u = g(u)$ . Since  $f$  is Lipschitz continuous we have

$$\|f(u) - f(u^{\Delta t_j})\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))} \leq C_f \|u - u^{\Delta t_j}\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))}.$$

Using the fact that  $\|u - u^{\Delta t_j}\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))} \rightarrow 0$ , we deduce that  $\|f(u) - f(u^{\Delta t_j})\|_{L^2(0, T; L^2(\Omega \times \mathbb{T}^d))} \rightarrow 0$ . Hence,  $f(u^{\Delta t_j}) \rightarrow f(u)$  strongly in  $L^2(0, T; L^2(\Omega \times \mathbb{T}^d))$  as  $\Delta t_j \rightarrow 0$ , which in turn implies that  $f_u = f(u)$ . A similar proof implies that  $g_u = g(u)$ .



Thus for almost all  $t \in (0, T)$ , P-a.s., there holds,

$$u(\cdot, t) = u_0^\epsilon + \epsilon \int_0^t \Delta u ds - \int_0^t \operatorname{div}(\mathbf{v}f(u)) ds + \int_0^t g(u) dW(s), \quad (55)$$

in  $(H^1(\mathbb{T}^d))'$ . Next we prove that this equality also holds in  $L^2(\mathbb{T}^d)$ . Since

$$\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j} \rightharpoonup u(\cdot, t) - \int_0^t g(u) dW(s)$$

is in  $L^2(0, T; L^2(\mathbb{T}^d))$ , we have that  $u(\cdot, t) - \int_0^t g(u) dW(s) \in L^2(0, T; L^2(\mathbb{T}^d))$  P-a.s.. In view of Lemma 11 and the identification  $g_u = g(u)$ , we deduce that  $\int_0^t g(u) dW(s) \in C([0, T]; L^2(\mathbb{T}^d))$  and thus  $u \in L^2(0, T; L^2(\mathbb{T}^d))$ . This completes the proof that for almost all  $t \in (0, T)$ , P-a.s., there holds,

$$u(\cdot, t) = u_0^\epsilon + \epsilon \int_0^t \Delta u ds - \int_0^t \operatorname{div}(\mathbf{v}f(u)) ds + \int_0^t g(u) dW(s), \quad (56)$$

in  $L^2(\mathbb{T}^d)$ , so that  $u$  is a strong solution in the sense of Definition 4, which can be denoted by  $u^\epsilon$ .  $\square$

**Lemma 14.**  $\epsilon \Delta u^\epsilon - \operatorname{div}(\mathbf{v}f(u^\epsilon))$  belongs to  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ .

*Proof.* According to Lemma 10 and the facts that  $\tilde{u}^{\Delta t_j} \rightharpoonup u^\epsilon$  weakly in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$  and  $\tilde{B}^{\Delta t_j} \rightarrow \int_0^t g(u^\epsilon) dW(s)$  in  $L^2(\Omega; L^2(\mathbb{T}^d))$  for all  $t$ , we obtain that  $\partial_t [\tilde{u}^{\Delta t_j} - \tilde{B}^{\Delta t_j}]$  converges up to a subsequence to  $\partial_t [u^\epsilon - \int_0^t g(u^\epsilon) dW(s)]$  in  $L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ . Since

$$\partial_t \left[ u^\epsilon - \int_0^t g(u^\epsilon) dW(s) \right] = \epsilon \Delta u^\epsilon - \operatorname{div}(\mathbf{v}f(u^\epsilon)),$$

it follows that  $\epsilon \Delta u^\epsilon - \operatorname{div}(\mathbf{v}f(u^\epsilon)) \in L^2(0, T; L^2(\Omega; L^2(\mathbb{T}^d)))$ . This result is used in the proof of Lemma 4.  $\square$

## The uniqueness of the strong solution $u^\epsilon$

Finally, we prove the uniqueness of the strong solution of Problem  $(\mathcal{P}^\epsilon)$ . Let  $u_1^\epsilon$  and  $u_2^\epsilon$  be two strong solutions of Problem  $(\mathcal{P}^\epsilon)$ . Applying Itô's formula to the difference of the equations for  $u_1^\epsilon$  and  $u_2^\epsilon$ , we deduce that

$$\begin{aligned} \|u_1^\epsilon(t) - u_2^\epsilon(t)\|_{L^2(\mathbb{T}^d)}^2 &= 2\epsilon \int_0^t \langle \Delta(u_1^\epsilon - u_2^\epsilon), u_1^\epsilon - u_2^\epsilon \rangle_{L^2(\mathbb{T}^d)} - 2 \int_0^t \langle \operatorname{div}(\mathbf{v}f(u_1^\epsilon)) - \operatorname{div}(\mathbf{v}f(u_2^\epsilon)), u_1^\epsilon - u_2^\epsilon \rangle_{L^2(\mathbb{T}^d)} \\ &\quad + 2 \int_0^t \langle g(u_1^\epsilon) - g(u_2^\epsilon), u_1^\epsilon - u_2^\epsilon \rangle_{L^2(\mathbb{T}^d)} dW(\cdot, s) + \int_0^t \sum_{j=0}^{\infty} \langle [g(u_1^\epsilon) - g(u_2^\epsilon)]^2 \lambda_j e_j, e_j \rangle_{L^2(\mathbb{T}^d)} ds. \end{aligned}$$

Next we take the expectation to obtain the estimate

$$\begin{aligned} \mathbf{E} \left[ \|u_1^\epsilon(t) - u_2^\epsilon(t)\|_{L^2(\mathbb{T}^d)}^2 \right] &\leq -2\epsilon \mathbf{E} \left[ \int_0^t \|\nabla(u_1^\epsilon - u_2^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 \right] - 2 \mathbf{E} \left[ \int_0^t \langle \operatorname{div}(\mathbf{v}f(u_1^\epsilon)) - \operatorname{div}(\mathbf{v}f(u_2^\epsilon)), u_1^\epsilon - u_2^\epsilon \rangle ds \right] \\ &\quad + 2 \mathbf{E} \left[ \int_0^t \langle g(u_1^\epsilon) - g(u_2^\epsilon), u_1^\epsilon - u_2^\epsilon \rangle dW(x, s) \right] + \sum_{j=0}^{\infty} \lambda_j \|e_j\|_{L^\infty(\mathbb{T}^d)}^2 \mathbf{E} \left[ \int_0^t \|g(u_1^\epsilon) - g(u_2^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 ds \right], \end{aligned}$$

so that

$$\begin{aligned} & \mathbb{E} \left[ \|u_1^\epsilon(t) - u_2^\epsilon(t)\|_{L^2(\mathbb{T}^d)}^2 \right] + 2\epsilon \mathbb{E} \left[ \int_0^t \|\nabla(u_1^\epsilon - u_2^\epsilon)\|_{L^2(\mathbb{T}^d)}^2 \right] \\ & \leq 2V \mathbb{E} \left[ \int_0^t \int_{\mathbb{T}^d} |\nabla(u_1^\epsilon - u_2^\epsilon)| |f(u_1^\epsilon) - f(u_2^\epsilon)| dx dt \right] + \Lambda_1 C_g \mathbb{E} \left[ \int_0^t \|u_1^\epsilon - u_2^\epsilon\|_{L^2(\mathbb{T}^d)}^2 dt \right], \end{aligned}$$

which combined with Gronwall's lemma yields the uniqueness result of  $u^\epsilon$ .  $\square$

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