Dispersion equation of a fluid with a surface film

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DISPERSION EQUATION OF A FLUID WITH SURFACE FILM

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Abstract :

The dispersion equation of gravity waves propagating in a liquid covered by a film depends on the nature of the liquid-film interaction as well as on the comportment law of the film.

The study of such effects is important to prevent a gas to be carried to a liquid by a free surface, to approach the problem of moving superficial oceanic layers as well as to explain the exchanges through cellular membranes subject to deformations.

Initially the film is a plane horizontal surface and the liquid is at rest ; small amplitude perturbations propagates through this system and do not cause viscosity effects in the liquid. The possibility of a slip of the film on the liquid is admitted. The interaction between the surface wave and the gas overlaying on the film is neglected and a constant and uniform pressure for this gas is considered.

Using the linearized theory, we may write the Bernoulli equation close to the interface ; this development takes in account the flux of impulsion from the fluid to the film.

A second linearized relation is obtained between the surface density, the velocity potential of the fluid and the surface tension.

The knowledge of the comportment law of the film (elastic, viscoelastic or at constant tension) allows to exprime the superficial tension as a function of the surface density and of its derivatives with respect to space and time variables.

The last relation is obtained between the surface density of the film and the velocity potential of the fluid, using the various possible exchange conditions between the film and the fluid (impermeable film, permeable one with a constant surface density or porous one).

Using solutions of the form $f = \bar{f}\, e^{i(\underline{K}.\underline{r}-\omega t)}$, we deduce the dispersion equation from the three above mentioned coupled relations. The celerity and the decay coefficient of the wave are then written as functions of either the pulsation or the wave length.

Caracteristic curves are plotted and interpreted in the following cases :

- elastic impermeable film
- viscoelastic impermeable film
- permeable film with constant tension and density
- porous film

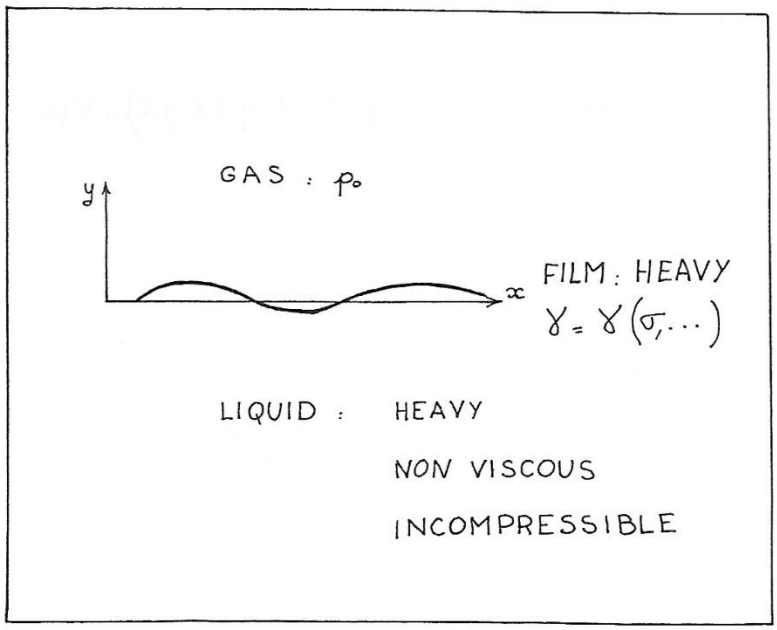
These curves present two branches in relation with two possible celerities for each frequencie ; a singularity take plane for a great viscoelasticity. The existence of these two branches shows the influence of two coupled phenomena : gravity effect in the liquid with free surface, effect of the comportment law of the film.

These results are compared with those of different authors who have studied the general equations or analogous particular configurations although the exchanges between film and liquid are generally of different natures.

The small perturbations method used in this study will certainly be very useful in more complex situations involving irreversible phenomena into the fluid, the film and also during the fluid-film exchanges.

Finally, this study is bounded by the use of the linearized theory. Consequently, the multiple scale method is planed as an extension of this work.

The following text and figures correspond to the oral communication presented at the 14th IUTAM Congress (sept. 1976).



1 - The medium that we are going to considere is made of a film on the surface of a liquid and of a gas on the top of that film. There is no exchange between the gas and the other medium.

The film is a material medium. It is supposed to have no thickness but it will have a surface density and a surface tension. This film is able to slip on the surface of the liquid staying always in contact with it. That is to say no cavitation.

The studied evolution is supposed to be isotherm.

The caracteristic functions of the medium will be :

- first of all, the ones associated to the liquid

$$p \ , \ \rho \ , \ \phi \quad \{ u = \phi,x \ , \ v = \phi,y \ \}$$

- and then, the ones associated to the film

$$\gamma \ , \sigma \ , \ \hat{u} \ , \ \hat{v}$$

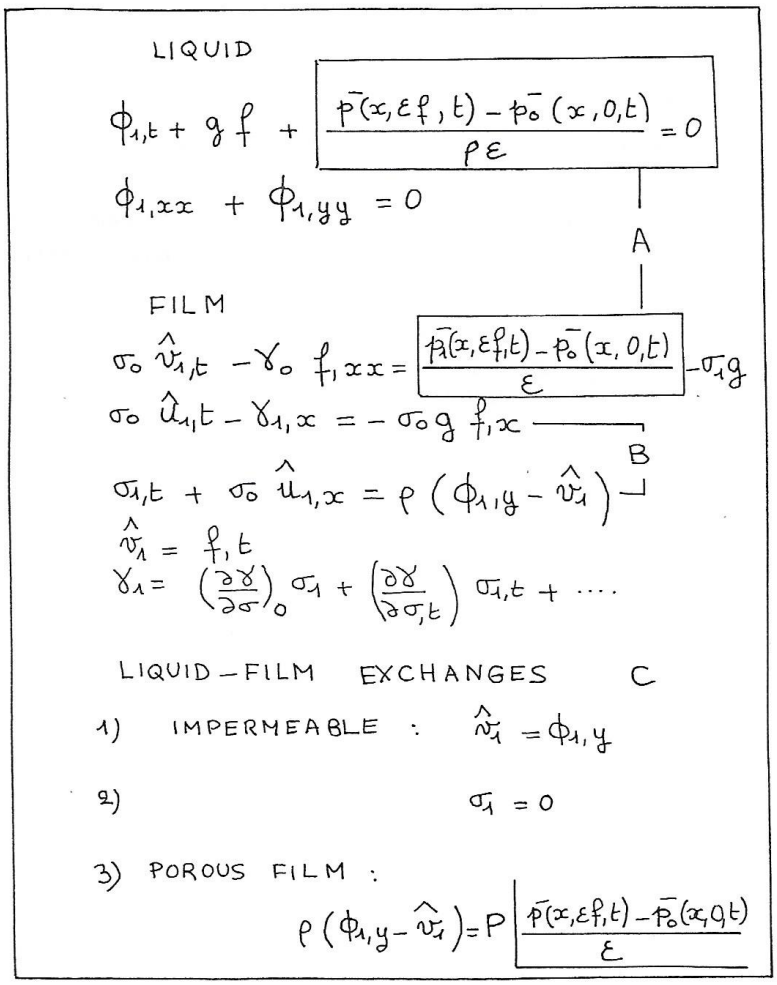
$$\varphi(x,y,t) = \varphi_0(x,y,t) + \varepsilon\,\varphi_1(x,y,t) + 0(\varepsilon^2)$$

$$\begin{cases} y = \varepsilon f(x,t) \\ \varphi(x,y,t) = \varphi_0(x,0,t) \\ \qquad + \varepsilon\left[\varphi_1(x,0,t) + f\left(\dfrac{\partial \varphi_0}{\partial y}\right)_{y=0}\right] + 0(\varepsilon^2) \end{cases}$$

2 - Method of small perturbations

We only retain the first order terms.

The perturbations are of small amplitude and bidimensionnal



3 - Equations of the problem

The first equation used to describe the liquid is the usual Bernoulli's equation. This equation is linearized. As opposed to a problem of free surface, we find here a new term. This term $(\bar{p}-\bar{p_o})/\varepsilon$ related to pressure jump, is due to the presence of a movement of the film.

The second equation of the liquid is the Laplace's equation ; we assumed that the movement is irrotationnal. The film is defined : - by the two components of the balance equation of impulsion - the mass conservation - that equation which describe the fact that the film and the liquid stay in contact - and then the comportement law.

We define now three possibilities for liquid-film exchanges :

1) Impermeable

2) The second exchange condition between liquid and film represent a simple case of permeability. Probably, this case is realistic only if the surface density is negligeable ; for example that will be the case of capillarity films.

3) The third condition used represents the flux of mass proportionnal to the jump of pressure.

$$f = \exp.\left[i\,(K_x x - \omega t)\right]$$
$$\phi_1 = \overline{\Phi}\ \exp.\left[i\,(K_x x + K_y y - \omega t)\right]$$
$$\sigma_1 = \overline{\sigma}\ \exp.\left[i\,(K_x x + K_y y - \omega t)\right]$$

$$\text{A} \quad \begin{cases} \dfrac{\overline{\sigma}}{\rho} - i\Omega\,\dfrac{\overline{\Phi}}{c_\sigma} + 1 - \Omega^2 + \eta_\gamma^2\,\Omega^2\,\Delta_x^2 = 0 \\ \Delta_x^2 + \Delta_y^2 = 0 \end{cases}$$

$$\text{B} \quad \begin{cases} \left[1 + \Gamma(\Omega)\,\Delta_x^2\right]\dfrac{\overline{\sigma}}{\rho} + \Delta_y\,\dfrac{\overline{\Phi}}{c_\sigma} + 1 - \Delta_x^2 = 0 \\ \Gamma(\Omega) = \dfrac{1}{c_\sigma^2}\left[\left(\dfrac{\partial\gamma}{\partial\sigma}\right)_0 - i\Omega\,\omega_\sigma\left(\dfrac{\partial\gamma}{\partial\sigma_{,t}}\right)_0\right] \end{cases}$$

$$\text{C} \quad \begin{cases} 1) \quad \Delta_y\,\dfrac{\overline{\Phi}}{c_\sigma} = -1 \\ 2) \quad \overline{\sigma} = 0 \\ 3) \quad (\Delta_y - \eta_D)\,\dfrac{\overline{\Phi}}{c_\sigma} = i\,\dfrac{\eta_D}{\Omega} - 1 \end{cases}$$

$$\omega_\sigma^2 = \frac{\rho g}{\sigma_0}\ ,\quad c_\sigma^2 = \frac{\sigma_0 g}{\rho}\ ,\quad \eta_\gamma^2 = \frac{\gamma_0\,\rho}{\sigma_0^2\,g}$$

$$\eta_P = c_\sigma\,P$$

$$\Delta_x = c_\sigma\,\frac{K_x}{\omega}\ ,\quad \Delta_y = c_\sigma\,\frac{K_y}{\omega}\ ,\quad \Omega = \frac{\omega}{\omega_\sigma}$$

CHARACTERISTIC THICKNESS OF THE FILM :

σ_0 / ρ

4 - Wave equations

The studied perturbations are of following form : $f = e^{i(K_x x - \omega t)}$. In particular, they are compatible with the presence of an horizontal plane bottom. This bottom is supposed to be located at an infinite distance from the surface.

The dimensionless functions are defined at the bottom of this slide. The function $\Gamma(\Omega)$ caracterise the comportment law of the film. Wa can notice that the caracteristic thickness is included in the definition of the reference functions. In fact, it is possible to show that this thickness set the analytic scale of the linearised problem. More exactly, ours equations are valid only if the wavelength and this thickness have the same order of magnitude ; likewise the pulsations are of the order of ω_σ , etc.

ELASTIC FILM

$$\Delta_y = -i\,\Delta_x$$

$$\Gamma = -\eta^2$$

$$\left(1-\eta^2\Delta_x^2\right)\left(-\Omega\,\Delta_x^{-1}+1-\Omega^2+\eta_\gamma^2\Omega^2\Delta_x^2\right)+\Delta_x^2=0$$

$$\Delta_x = C^{-1}$$

$$\left(C^2-\eta^2\right)\left(-\Omega C+1-\Omega^2+\eta_\gamma^2\Omega^2C^{-2}\right)+1=0$$

CASE $\eta_\gamma = 0$

$$\left(C^2-\eta^2\right)\left(1-\Omega C-\Omega^2\right)+1=0$$

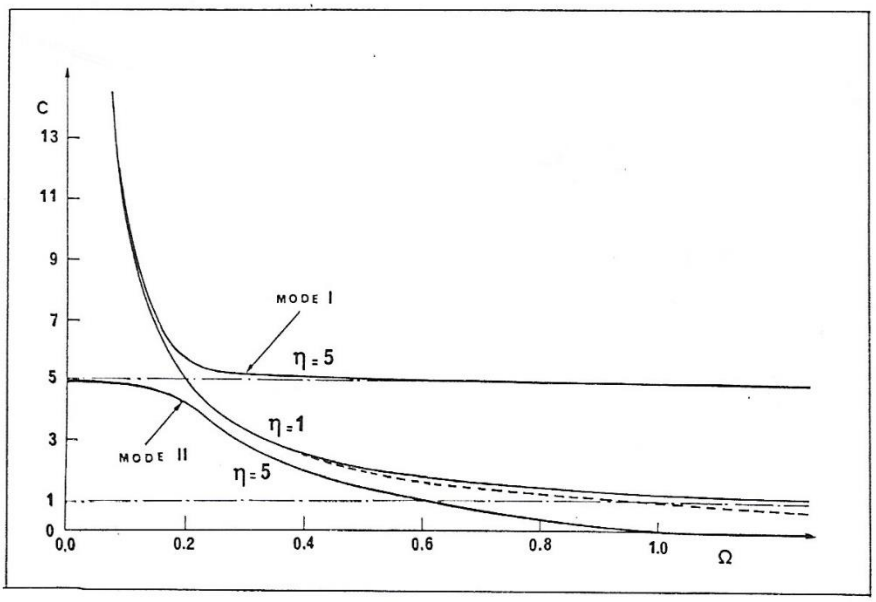
5 - Elastic film

We can find solutions with a real expression for Δ_x (Ω is supposed to be real). We just have to write :

$$\Delta_x = 1/C$$

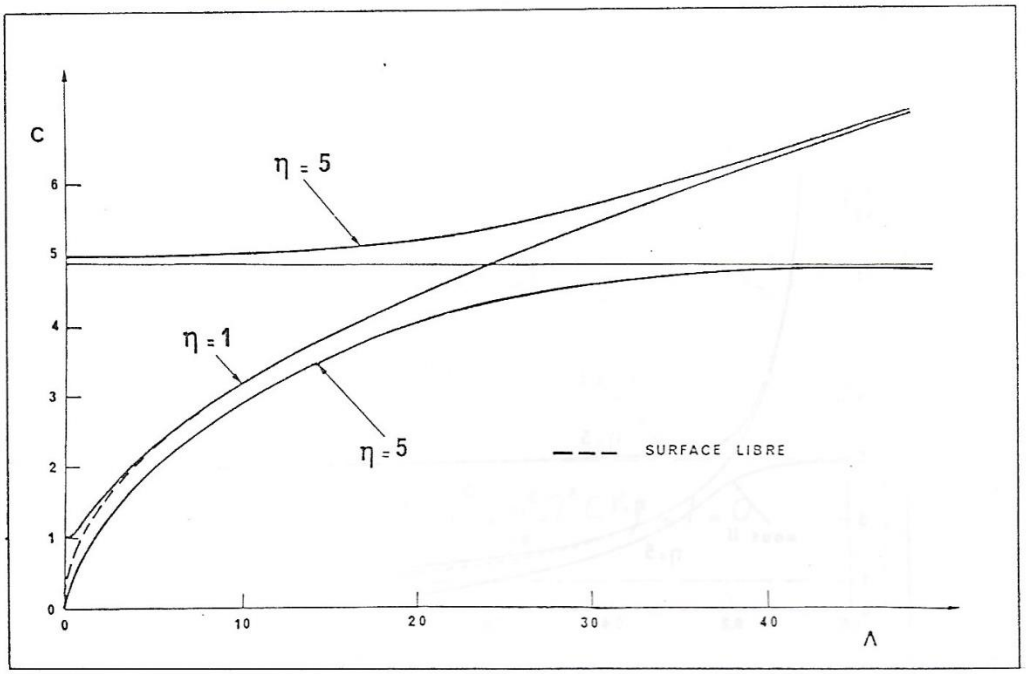
where C is the dimensionless celerity of the wave. (This celerity is defined by the ratio c/c_0).

If the parameter η_γ is equal to zero, the state of rest is associated to the natural state of the elastic film ($\gamma_0=0$). In the opposite case, at rest, the tension of the film is different of zero.



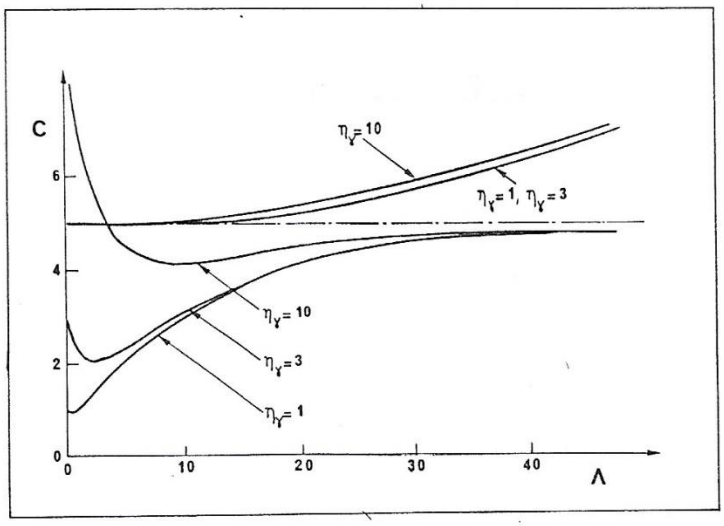
6 - Wave velocity as a function of pulsation

If the parameter η is greater than one, they are two modes of propagation and if η is lower than one, there is only one.



7 - Wave velocity as a function of wave length

For the large wavelength the upper branch is approximate to the solution associated to the free surface ($C^2 = \Lambda$). The lower branch of the curve is caracterized by a limited speed of propagation. This limited celerity is equal to $\sqrt{\eta^2 - 1}$.



8 - Influence of γ_o

The curves $C(\Lambda)$ are plotted with different values of the parameter η_γ. This parameter is associated to the tension of the film at rest.

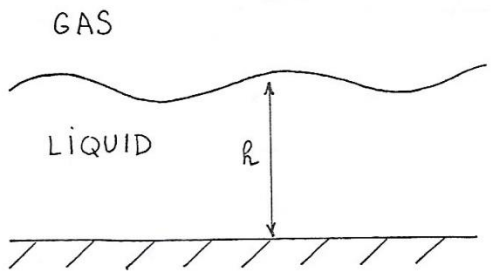
Only the second mode is modified for the small wavelength (η_γ is the limit of C for zero limit of Λ)

The perturbation increase with η_γ and consequently the celerity of the wave increase too.

The first mode is slightly modified for η_γ varying from 0 to 10.

$$\Phi_1 = \left(\bar{\Phi}\, e^{K_x y} + \bar{\Phi}'\, e^{-K_x y} \right) e^{i(K_x x - \omega t)}$$

$$y = -h \longrightarrow \Phi_{1,y} = 0$$



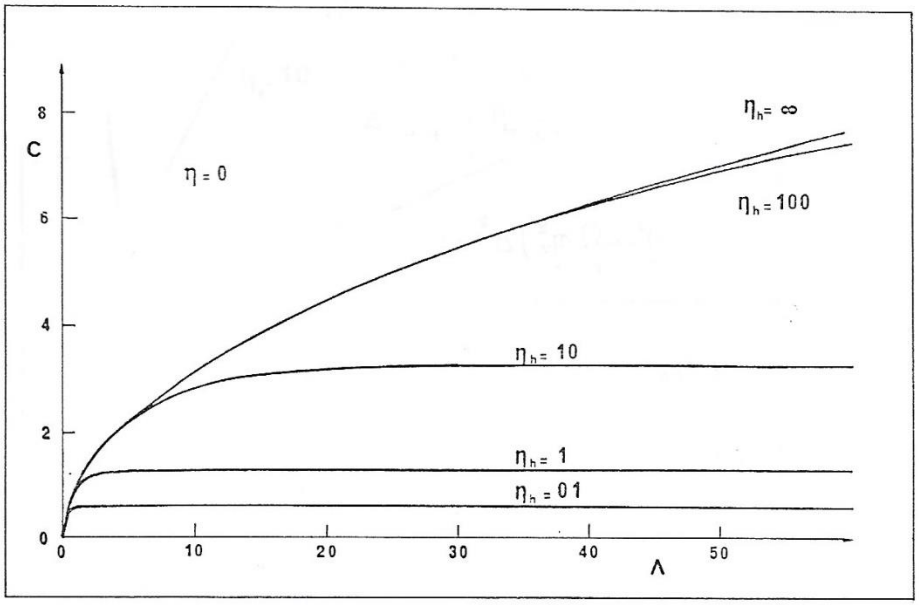
$$\left(1 - \eta^2 \Delta_x^2\right)\left[-\Omega \Delta_x^{-1} \operatorname{cth}\left(\eta_h \Omega \Delta_x\right) + 1 - \Omega^2 + \eta_\gamma^2 \Omega^2 \Delta_x^2 \right] + \Delta_x^2 = 0$$

$$\eta_h = \frac{\rho h}{\sigma_0}$$

9 - In the case of a bottom located at a finite distance from the surface, the solution must be written in the following form :

$$\phi_1 = (\bar{\phi}\, e^{K_x.y} - \bar{\phi}'\, e^{-K_x.y})\, e^{i(K_x.x - \omega t)}$$

We obtain a wave equation in which appears to depth parameter η_h .



10 - For $\eta = 1$, the wave celerity is varying between 1 and a limit whitch depends on η_h. This asymptotic value increases with the depth parameter η_h. For $\eta > 1$, the same effect can be observed for each of the two modes.

VISCOELASTIC FILM.

$$\Gamma = -\eta^2 + i\,\Omega\,\eta_v^2$$

$$\Delta_y = -i\,\Delta_x$$

$$\Delta_x = \frac{1}{C} + i\,A$$

$$\left[1-\left(\eta^2 - i\Omega\eta_v^2\right)\Delta_x^2\right]\left(-\Omega\Delta_x^{-1} + 1 - \Omega^2\right) + \Delta_x^2 = 0$$

$$\left|\begin{array}{l} 2\Omega^3\eta_v^2 CA\left(1+A^2C^2\right) + \Omega^2\left(1+A^2C^2\right)\left(\eta^2 - C^2 - \eta^2A^2C^2 + \eta_v^2AC^2\right) \\ +\Omega\left[-2\eta_v^2CA\left(1+A^2C^2\right) - C\left(C^2-\eta^2\right) + \eta^2C^3A^2\right] \\ +\left(1+A^2C^2\right)\left(C^2-\eta^2+\eta^2A^2C^2+1-A^2C^2\right) = 0 \end{array}\right.$$

$$\left|\begin{array}{l} -\Omega^3\eta_v^2\left(1-A^4C^4\right) + \Omega^2C\left(1+A^2C^2\right)\left(2\eta^2A - \eta_v^2\right) \\ +\Omega\left[\eta_v^2\left(1-A^4C^4\right) + AC^2\left(C^2+\eta^2+\eta^2A^2C^2\right)\right] \\ +2AC\left(1+A^2C^2\right)\left(1-\eta^2\right) = 0 \end{array}\right.$$

11 - The comportment law of the viscoelastic film that we are going to considere gives the surface tension γ as a function of σ and of his time derivatives. Then, the quantity Γ becomes a complex number.

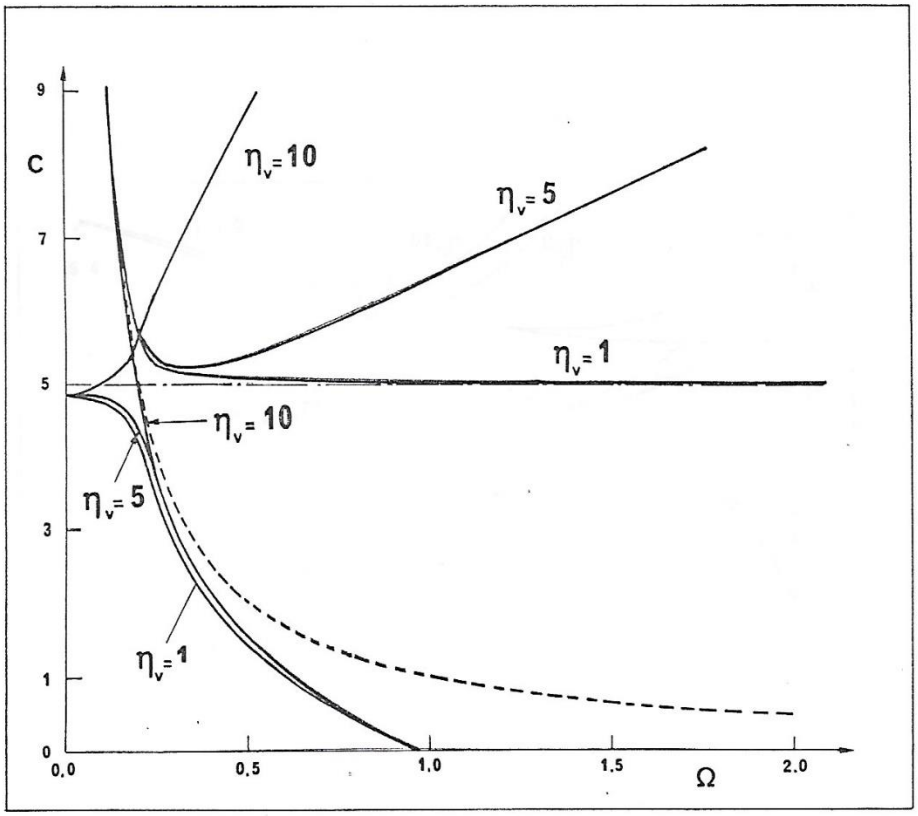
We must introduce a complex wave number and we have :

$$\Delta x = \frac{1}{C} + i\ A.$$

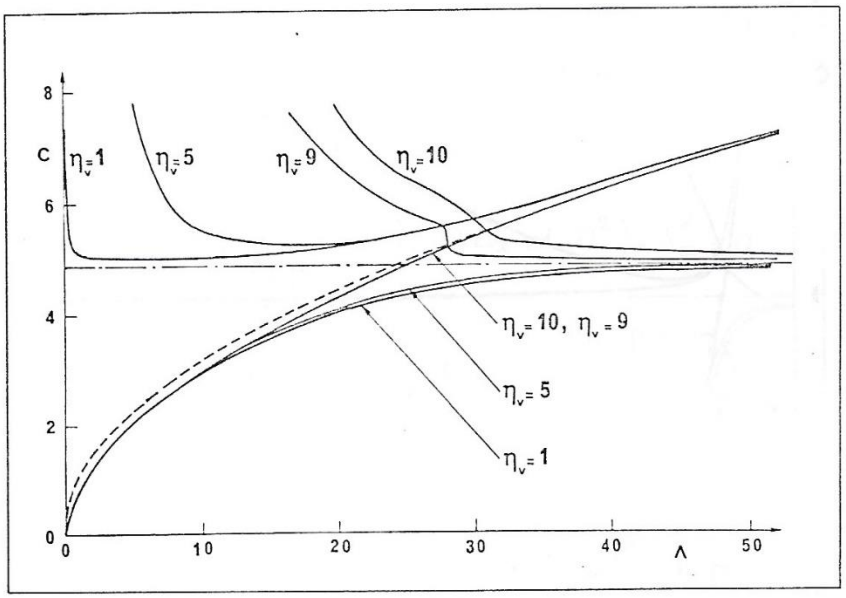
where A is the dimensionless decay coefficient.

We obtain two equations of degree three in Ω from the separation of the real and imaginary parts of the wave equation.

We used a numerical and graphical proceeding to solve this system.

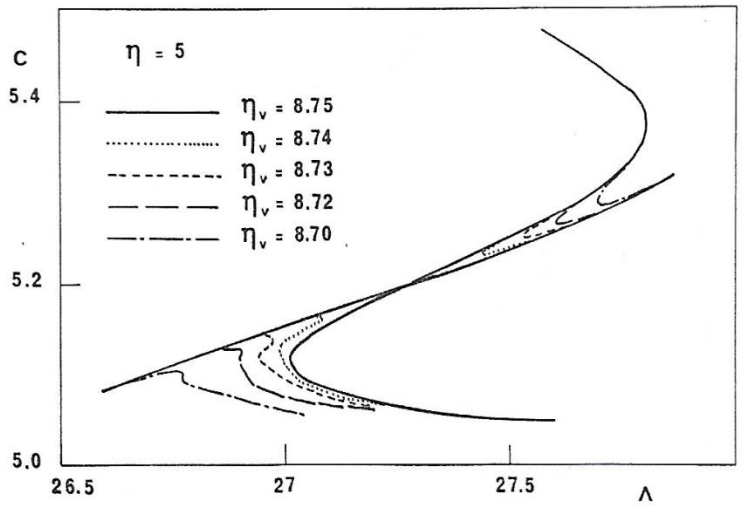


12 - If the parameter η is greater than unity, they are two modes of propagation. This figure shows the influence of the viscoelastic coefficient η_v on these two modes and we note the presence of a singularity for $5 < \eta_v < 10$.

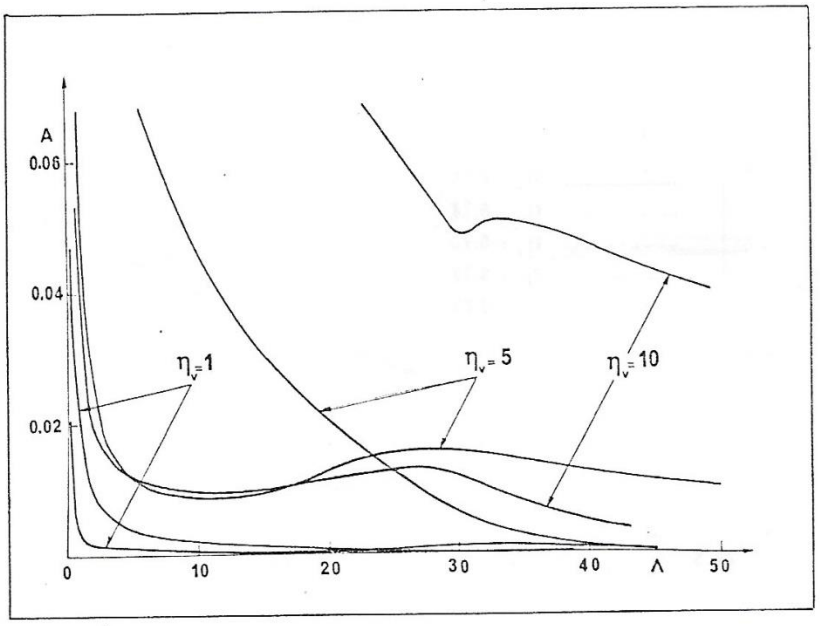


13 - This singularity is obvious in the plane (C,Λ) it is caracterized by an inversion of the two infinite branches of the curves between $\eta_v = 5$ and $\eta_v = 9$.

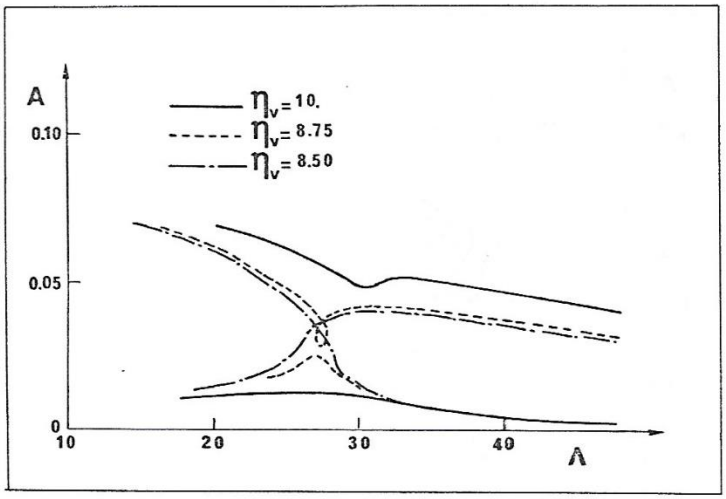
The other effect of η_v is the presence of an infinite value of the wave celerity for $\Lambda = 0$.



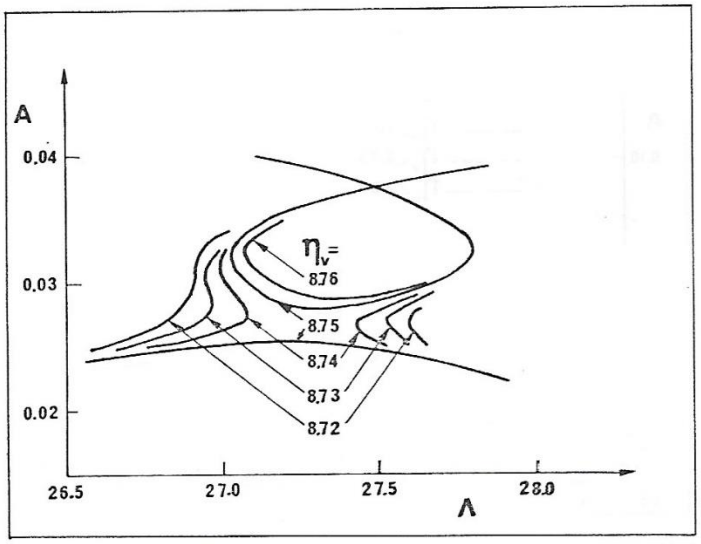
14 - The behaviour of the curves has been studied in the vicinity of the singular point. The singular value of η_v can be located between 8,74 and 8,75.



15 - The reduced decay coefficient A is represented here as a function of reduced wave length Λ . The curves shows the two modes of propagation which are obtained for $\eta = 5$ and the influence of the viscoelastic parameter η_v.

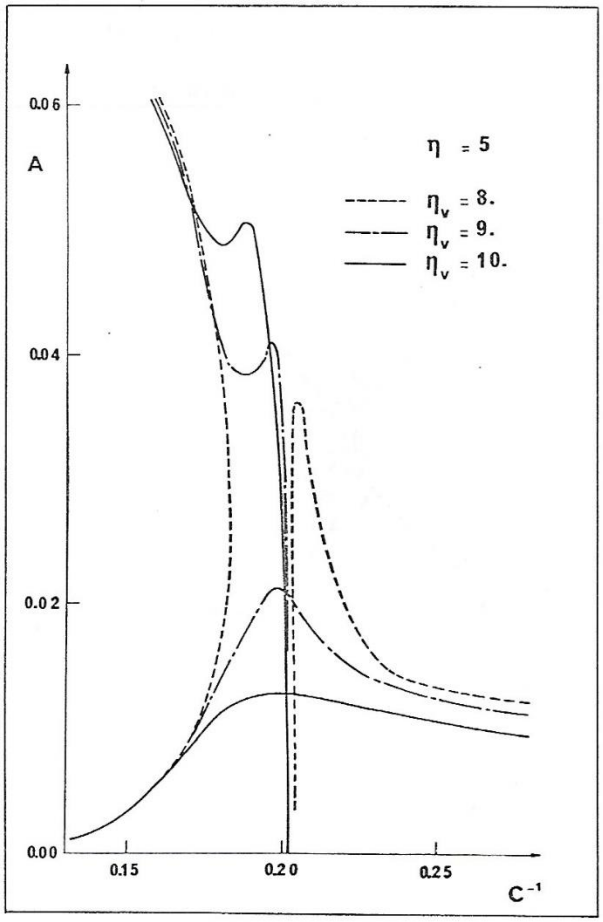


16 - The shape of the three curves obtained for these three values of η_v shows the presence of the singular point. The formation of a is a stage during the evolution of the shape and explains the existence of the intersection.



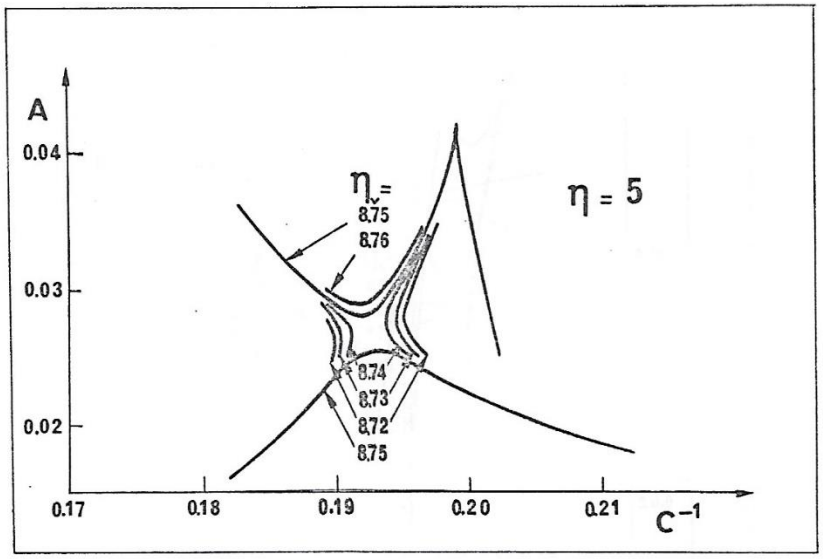
17 - In fact, the behaviour in the immediate neighborough of the singular point is caracterised by the exchange of branches. It is the same way than a family of hyperbolic curves.

We can also say that this singularity is of saddle type.



18 - This figure shows the influence of η_v on the complexe reduced wave number :

$$\Delta x = \frac{1}{C} + i\ A.$$



19 - Expanding the scale, we observe the singular evolution.

FILM WITH A CONSTANT SURFACE DENSITY.

$$\Delta_y = -i\,\Delta_x$$

$$\Omega\left(1-\Delta_x^2\right) - \Delta_x\left(1-\Omega^2+\eta_\gamma^2\Omega^2\Delta_x^2\right) = 0$$

$$\Delta_x = \frac{1}{C} \qquad \Lambda = \frac{C}{\Omega}$$

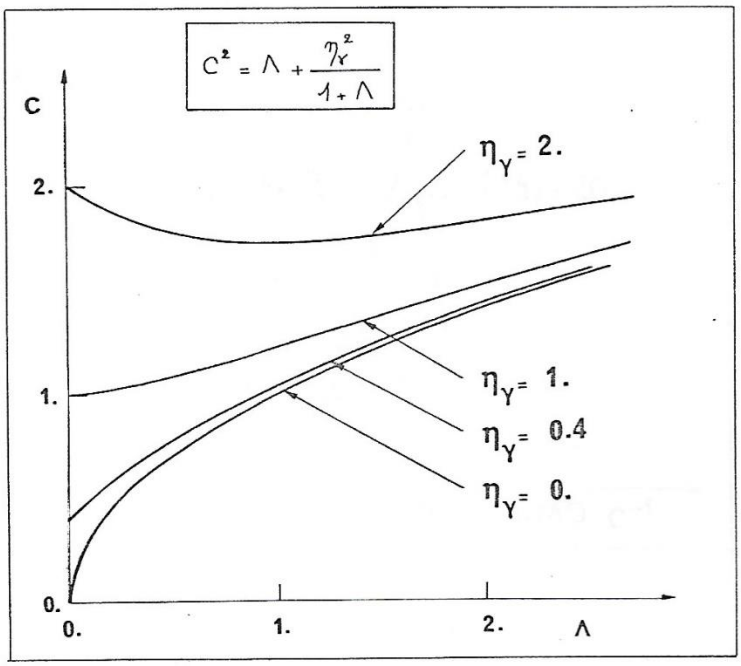
$$\boxed{C^2 = \Lambda + \frac{\eta_\gamma^2}{1+\Lambda}}$$

CASE $\sigma_0 \longrightarrow 0$

$$\boxed{C'^2 = \Lambda' + \frac{\eta'^2_\gamma}{\Lambda'}}$$

$$\eta'^2_\gamma = \frac{\rho\gamma_0}{g\sigma_r^2} \qquad \left.\begin{array}{l} C' = \dfrac{c}{c_r} \\ \Omega' = \dfrac{\omega}{\omega_r} \end{array}\right\} \; g = c_r\omega_r$$

20 - Our study of permeable films beging with a very simple theoretical case which is defined by the existence of a constant surface density. The square of the wave celerity is then an homographic function of the wave length.



21 - We obtain a single mode of propagation which is influenced by the value of the parameter η_γ ; therefore by the value of the surface tension γ_o . When γ_o increases, the wave celerity increases for a given value of Λ.

POROUS FILM.

CASE $\Gamma = 0$, $\eta_\gamma = 0$

1) $\Omega C = 1$ OR $\Delta_x - \Omega = 0$

2) $1 - i\frac{\eta_P}{\Omega} + \left(\Delta_x - i\eta_P\right)\left(\Delta_x + \Omega\right) = 0$

$$\Delta_x = \frac{1}{C} + iA$$

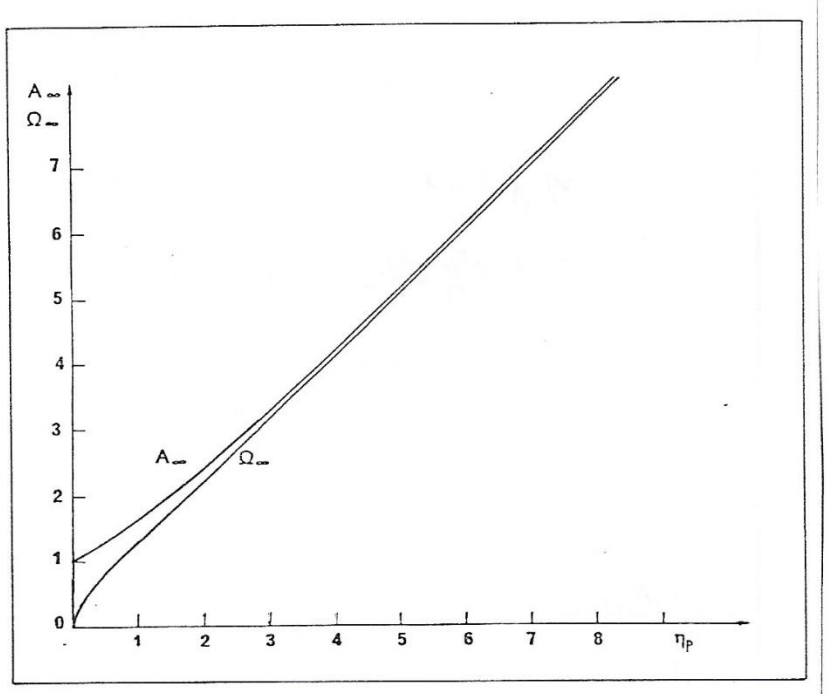
$$A\left(\eta_P - A\right)\Omega^4 + \left[\left(1 - A^2 + \eta_P A\right)\left(2A - \eta_P\right)^2 + \eta_P^2\right]\Omega^2 + \eta_P^2 = 0$$

$$C = \frac{\Omega\left(2A - \eta_P\right)}{\eta_P\left(1 + \Omega^2\right) - A\Omega^2}$$

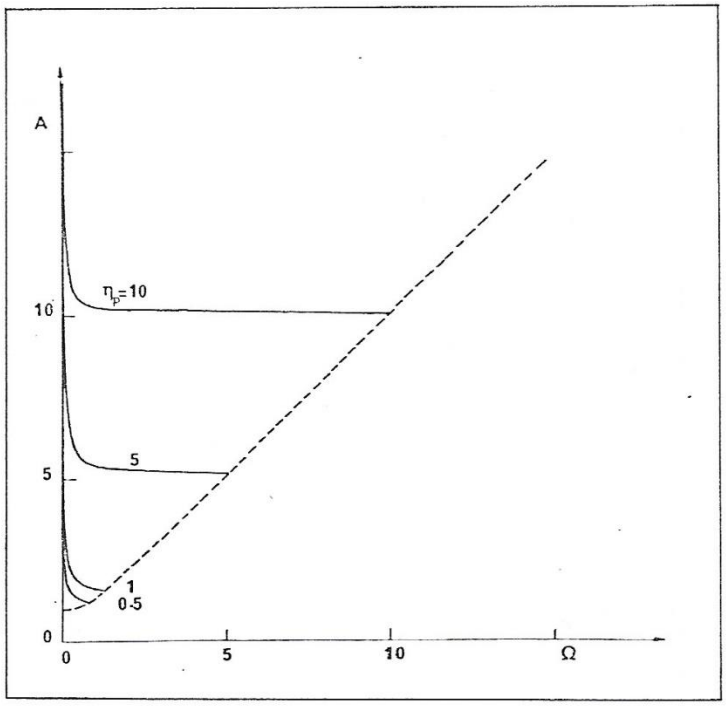
22 - A more complex situation is obtained with a porous film, but the configuration is certainly more realistic than the precedent one.

We have studied in a detailed way the following case caracterized by a zero surface tension.

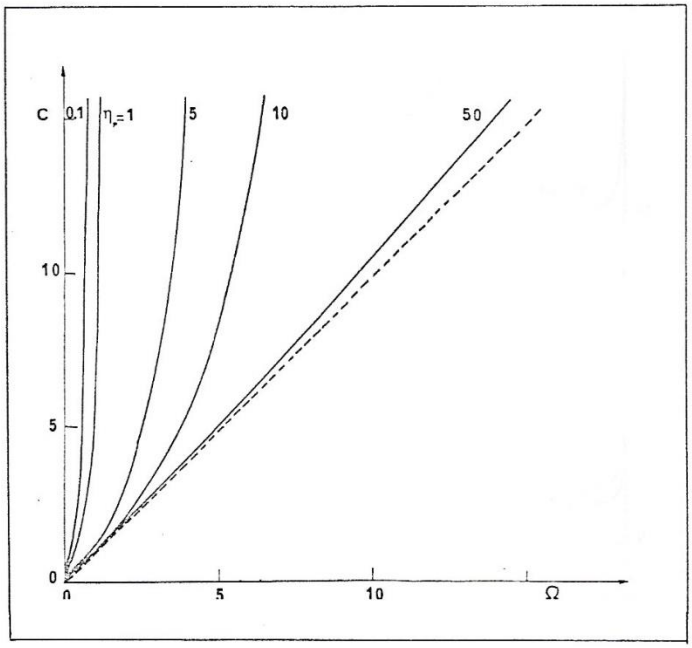
This exchange of matter between film and liquid gives a complex wave number and consequently a decay coefficient.



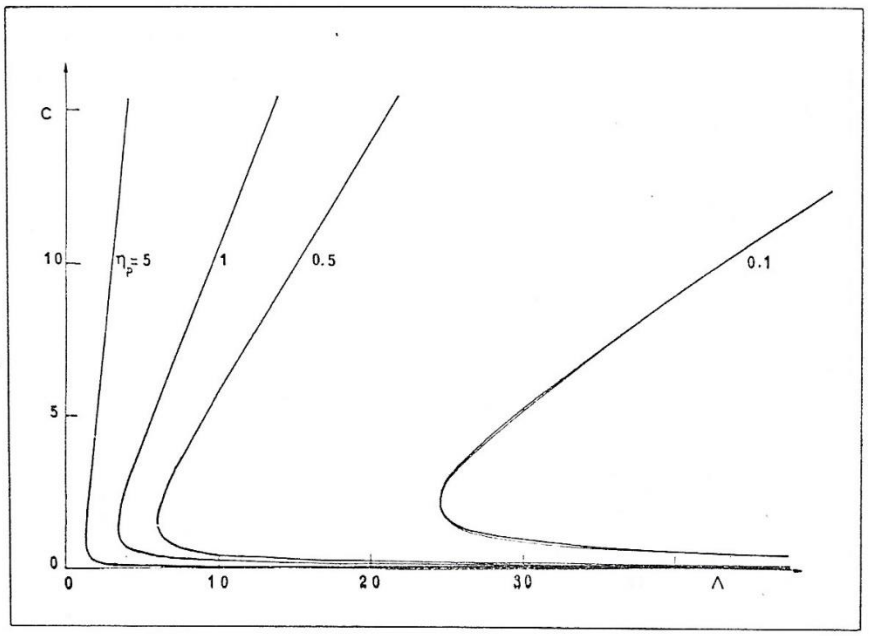
23 - This value Ω_∞ and the corresponding value of the decay coefficient A_∞ are plotted on the figure.



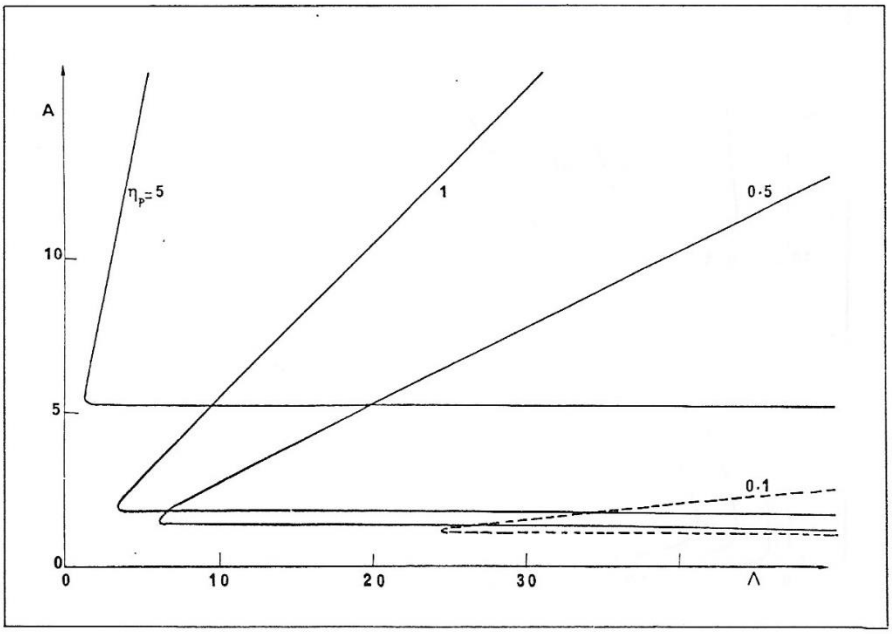
24 - We see here the shapes of the curves $A(\Omega)$. We observe a growth of A with η_p; for a given Ω.



25 - The wave celerity varies from 0 to ∞ when the pulsation grows from 0 to a limit $\Omega\infty$.



26 - The evolution of C is represented as a function of reduced wavelength.



27 - We also obtain the curves A(Λ). None of these curves presents any singularity.

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