Generic transversality of heteroclinic and homoclinic orbits for scalar parabolic equations

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Abstract

In this paper, we consider the scalar reaction-diffusion equations

$$\partial_t u = \Delta u + f(x, u, \nabla u)$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ of class $\mathcal{C}^{2,\gamma}$. We show that the heteroclinic and homoclinic orbits connecting hyperbolic equilibria and hyperbolic periodic orbits are transverse, generically with respect to f. One of the main ingredients of the proof is an accurate study of the singular nodal set of solutions of linear parabolic equations. Our main result is a first step for proving the genericity of Kupka-Smale property, the generic hyperbolicity of periodic orbits remaining unproved.

Key words: transversality, parabolic PDE, Kupka-Smale property, singular nodal set, unique continuation.

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1 Introduction

Let $d \geqslant 2$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class $\mathcal{C}^{2,\gamma}$, where $0 < \gamma \leqslant 1$. Let $p > d$ be fixed, let $X = L^p(\Omega)$ and let

$$\Delta_D \ : \ D(-\Delta_D) = W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) \ \longrightarrow \ X = L^p(\Omega)$$

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be the Laplacian operator with homogeneous Dirichlet boundary conditions. Let $\alpha \in (1/2 + d/2p, 1)$, so that $X^\alpha = D((-\Delta_D)^\alpha) \hookrightarrow W^{2\alpha,p}(\Omega)$ is compactly embedded in $\mathcal{C}^1(\overline{\Omega})$.

We consider the scalar parabolic equation

$$\begin{cases} \partial_t u(x,t) = \Delta_D u(x,t) + f(x, u(x,t), \nabla u(x,t)), & (x,t) \in \Omega \times (0,+\infty) \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ u(x,0) = u_0(x) \in X^\alpha, \end{cases} \tag{1.1}$$

where $f \in \mathcal{C}^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and $u(x,t) \in \mathbb{R}$.

The local existence and uniqueness of classical solutions $u(t) \in \mathcal{C}^0([0,\tau), X^\alpha)$ of Equation (1.1), as well as the continuous dependence of the solutions with respect to the initial data u_0 in X^α, are well known (see [31] for example and Section 2 for more details). Thus, Eq. (1.1) generates a local dynamical system $S(t) \equiv S_f(t)$ on X^α. This dynamical system contains all the features of a classical finite-dimensional system: equilibrium points and periodic orbits, stable and unstable manifolds... We recall the definition of these objects, the definition of hyperbolicity and of transversality in Section 3. There, we also present their construction in our framework. Notice that the realizations results of [14] and [53] show the possible existence of very complicated dynamics for (1.1), such as chaotic dynamics, as soon as $d \geqslant 2$.

In what follows, for any $r \geqslant 2$, we denote by $\mathfrak{C}^r$ the space $\mathcal{C}^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ endowed with the Whitney topology, which is a Baire space (see Appendix A for definitions, including the one of generic subset). In fact, our result still holds if we embed $\mathcal{C}^r$ with another reasonable topology, but the Whitney one is the most classical. See [19] and Appendix A below for more details.

Our main result is as follows.

Theorem 1.1. Generic transversality of connecting orbits
Let $r \geqslant 2$ and let $f_0 \in \mathfrak{C}^r$. Let $\mathcal{C}_0^-$ and $\mathcal{C}_0^+$ be two critical elements of the flow of (1.1), i.e. $\mathcal{C}_0^\pm$ are equilibrium points or periodic orbits, $\mathcal{C}_0^- = \mathcal{C}_0^+$ being possible.

Assume that both $\mathcal{C}_0^-$ and $\mathcal{C}_0^+$ are hyperbolic. Then, there exists a neighborhood $\mathcal{O}$ of f_0 in $\mathfrak{C}^r$ and a generic set $\mathfrak{G} \subset \mathcal{O}$ such that:

i) there exist two families $\mathcal{C}^-(f)$ and $\mathcal{C}^+(f)$ of critical elements (either equilibrium points or periodic orbits) of the flow of (1.1), depending smoothly of $f \in \mathcal{O}$, such that $\mathcal{C}^\pm(f_0) = \mathcal{C}_0^\pm$ and $\mathcal{C}^\pm(f)$ is hyperbolic for any $f \in \mathcal{O}$.

ii) for any f in the generic set $\mathfrak{G} \subset \mathcal{O}$, the unstable manifold $W^u(\mathcal{C}^-(f))$ and the stable manifold $W^s(\mathcal{C}^+(f))$ intersect transversally, i.e. $W^u(\mathcal{C}^-(f)) \pitchfork W^s(\mathcal{C}^+(f))$.

Theorem 1.1 states the generic transversality of connecting orbits, i.e. heteroclinic and homoclinic orbits, between hpyerbolic critical elements (either equilibrium points or periodic orbits). See Figure 1 for an illustration of a typical transversal connecting orbit. This is a first step to obtain the genericity of Kupka-Smale property. Below in this introduction, we recall the historical background and previous results. We discuss about the missing ingredients to obtain the genericity of the whole Kupka-Smale property in Appendix C.

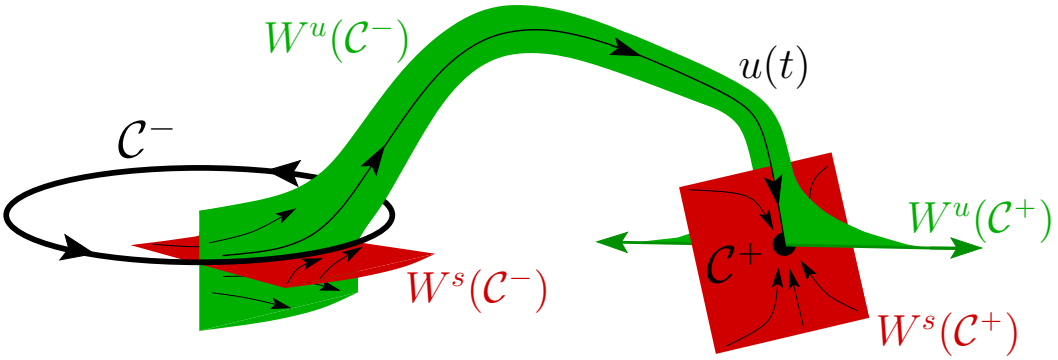

Figure 1: *A typical transversal heteroclinic orbit connecting a periodic orbit $\mathcal{C}^-$ and an equilibrium point $\mathcal{C}^+$. If $\mathcal{C}^\pm$ are hyperbolic, they admit stable and unstable manifolds. Theorem 1.1 states that, the transversality of $u(t)$ in this picture is a generic situation in the parabolic equation* (1.1)*. Here $\mathcal{C}^-$ is a periodic orbit and $\mathcal{C}^+$ is an equilibrium point. This situation is robust to perturbation and yields several important qualitative properties of the dynamics. See the third part of this introduction for the historical background and Section 3 for precise definitions.*

Notice that we do not need to assume global existence of solutions in Theorem 1.1. Indeed, we consider closed and connecting orbits, which are by definition solutions $u(t) \in X^\alpha$ of (1.1), which are defined for any time $t \in \mathbb{R}$ and are also uniformly bounded for $t \in \mathbb{R}$. So, we do not really care about solutions of Eq. (1.1), which do not exist globally. If one wants that all solutions of (1.1) exist for $0 \leqslant t \leqslant \infty$, one has to introduce additional hypotheses on f (see [55] for instance).

We also enhance that our result may apply to settings different from (1.1). Typically, we can choose different boundary conditions or consider systems of parabolic equations. We discuss this kind of straightforward generalizations in Section 7.

Observability of trajectories, unique continuation and singular nodal sets. As in the classical case of generic transversality in ODEs, the proof of Theorem 1.1 consists in finding suitable perturbation of the non-linearity f for breaking the non-transversal orbits. Of course, even if the general patterns and the spirit of the proofs stay the same, working with PDE's instead of ODE's gives rise to several more or less delicate technical problems. For example, for proving generic properties, instead of using Thom's transversality theorem (as in [51]), we will apply a Sard-Smale theorem stated in Appendix B. Here, we want to emphasize that, in the case of PDE's, the main new difficulty arises in the construction of appropriate perturbations. When one wants to prove that a property is dense in the set of ODE's of the form $\dot{y}(t) = g(y(t))$, for each g, one has to construct a particular perturbation εh with small ε such that the flow of $\dot{y}(t) = (g + \varepsilon h)(y(t))$ satisfies the desired property. The vector field h of the perturbation can be chosen freely and localized, so that his support intersects the trajectory of $y(t)$ only in the neighborhood of $y(t_0)$. In the case of PDE's, we have to construct a perturbation h of the non-linearity such that the flow of $\partial_t u(x,t) = \Delta u(x,t) + (f + \varepsilon h)(x, u(x,t), \nabla u(x,t))$ satisfies the desired property. Therefore, the perturbation h of the PDE's is of the

form

$$u(\cdot) \in X^\alpha \longmapsto h(\cdot, u(\cdot), \nabla u(\cdot)) \tag{1.2}$$

Since two distinct functions u_1 and u_2 can take the same value $(u_1(x_0), \nabla u_1(x_0)) = (u_2(x_0), \nabla u_2(x_0))$ at a given $x_0 \in \Omega$, the perturbations of the form (1.2) are in general "non local" in X^α. Given a particular trajectory $u(t)$ and a time t_0, our strategy consists in constructing a perturbation (1.2), whose support, even if it is large, intersects $u(x,t)$ only around (x_0, t_0), which allows to consider (1.2) as a local perturbation. However, this construction is not straightforward and requires deep properties of the PDE. This problem is close to observability questions: how much information on a solution $u(t)$ can we get from the observation at one point x_0 of $u(x_0,t)$ and $\nabla u(x_0,t)$?

To be able to prove Theorem 1.1, we will prove in Section 5 results of the following type.

Theorem 1.2. Injectivity properties of connecting orbits
Let $f \in \mathcal{C}^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $u(t)$ be a heteroclinic or homoclinic orbit connecting two critical elements. Then there exists a dense open set of points $(x_0, t_0) \in \Omega \times \mathbb{R}$ such that the curve $t \mapsto (u(x_0,t), \nabla u(x_0,t))$ is one to one at t_0 in the sense that:

i) $(\partial_t u(x_0,t_0), \nabla \partial_t u(x_0,t_0)) \neq 0$,

ii) for all $t \in \mathbb{R}$, $(u(x_0,t), \nabla(x_0,t)) = (u(x_0,t_0), \nabla(x_0,t_0)) \Longrightarrow t = t_0$.

The above result is a key property to be able to construct a suitable perturbation of the non-linearity f in the proof of Theorem 1.1. The following result is similar: it shows that the period of a periodic orbit of the parabolic equation may be observed very locally. This result is not required in the proof of our main theorem, but it may be interesting by itself and could be a key step to prove the generic hyperbolicity of periodic orbits (see the discussion of Appendix C).

Theorem 1.3. Pointwise observability of the period of periodic orbits
Let $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $p(t)$ be a periodic solution of (1.1) with minimal period $\omega > 0$. Then there exists a dense open set of points $(x_0, t_0) \in \Omega \times \mathbb{R}$ such that

$$(p(x_0,t), \nabla p(x_0,t)) = (p(x_0,t_0), \nabla p(x_0,t_0)) \quad \Longrightarrow \quad t \in t_0 + \mathbb{Z}\omega \ .$$

Notice that in dimension $d = 1$, the above results are true for all (x_0, t_0) and not only for a dense subset (see [37]).

To obtain these injectivity properties of $(x,t) \longmapsto (x, u(x,t), \nabla u(x,t))$, where $u(t) = S_f(t)u_0$ is a bounded complete trajectory of (1.1), we set

$$v(x,t,\tau) = u(x,t) - u(x,t+\tau) \ ,$$

and remark by using the equation (1.1) that $v(x,t)$ is the solution of a linear parabolic equation with parameter of the form

$$\partial_t v(x,t,\tau) = \Delta v(x,t,\tau) + a(x,t,\tau)v(x,t,\tau) + b(x,t,\tau).\nabla_x v(x,t,\tau) \ , \tag{1.3}$$

in the domain Ω of $\mathbb{R}^d$. The non-injectivity points of the image of $(x, u(x,t), \nabla u(x,t))$, $(x,t) \in \Omega \times \mathbb{R}$, are described by the nodal singular set of (1.3), that is, the set of

points (x,t,τ) where $v(x,t,\tau)$ and $\nabla_x v(x,t,\tau)$ both vanish. The singular nodal set of solutions of the parabolic equations, with coefficients independent of the parameter τ, have already been studied in [28] and in [10] for example. Here, generalizing an argument of [29] and applying unique continuations results (recalled in Section 2), we prove the following theorem, see Section 4.

Theorem 1.4. Singular nodal sets for parabolic PDEs with parameter *Let I and J be open intervals of $\mathbb{R}$. Let $a \in \mathcal{C}^\infty(\Omega \times I \times J, \mathbb{R})$ and $b \in \mathcal{C}^\infty(\Omega \times I \times J, \mathbb{R}^d)$ be bounded coefficients. Let v be a strong solution of (1.3) with Dirichlet boundary conditions. Let $r \geqslant 1$ and assume that v is of class $\mathcal{C}^r$ with respect to τ and of class $\mathcal{C}^\infty$ with respect to x and t. Assume moreover that the null solution is not part of the family, that is that, there are no time $t \in I$ and parameter $\tau \in J$ such that $v(.,t,\tau) \equiv 0$.*

Then, the set

$$\{(x_0,t_0) \in \Omega \times I \mid \nexists\, \tau \in J \text{ such that } (v(x_0,t_0,\tau), \nabla v(x_0,t_0,\tau)) = (0,0)\}$$

is generic in $\Omega \times I$. In other words, the projection of all the singular nodal sets of the family of solutions $v(\cdot,\cdot,\tau)$ is negligible in $\Omega \times I$.

Historical background: the Morse-Smale and Kupka-Smale properties.
The transversality of unstable and stable manifolds stated in Theorem 1.1 is related to the local stability of the qualitative dynamics. In the modeling of phenomena in physics or biology, we often work on approximate systems: some phenomena are neglected, only approximate values of the parameters are known, or we work with a discretized version of the system for simulation by computer... Therefore, it is important to know if such small approximations may qualitatively change the dynamics or not. Unfortunately, when perturbing general dynamical systems, drastic changes in the local or global dynamics can occur due for example to bifurcation phenomena. Thus, the common hope is that these bifurcations are rare, that is, that the systems, whose dynamics are robust under perturbations, are dense or generic. Here, we obtain the generic transversality of heteroclinic and homoclinic orbits between critical elements. Roughly, Theorem 1.1 says that if we consider two hyperbolic closed orbits of the flow of the parabolic equation (1.1) and if we observe a connecting orbit between them, then, "almost surely" this connection still remains after small perturbations of the system (numerical computation, changes of the parameters...).

Such stability questions have been extensively studied in the case of vector fields or iterations of maps. In 1937, Andronov and Pontrjagin introduced the fundamental notion of structurally stable vectors fields ("systèmes grossiers" or "coarse systems"), that is, vector fields X_0 which have a neighborhood V_0 in the $\mathcal{C}^1$-topology such that any vector field X in V_0 is topologically equivalent to X_0. In 1959 ([63]), Smale defined the class of nowadays called Morse-Smale dynamical systems on compact $n-$dimensional manifolds, that is, systems for which the non-wandering set consists only in a finite number of hyperbolic equilibria and hyperbolic periodic orbits and for which the intersections of the stable and unstable manifolds of equilibria and periodic orbits are all transversal. Peixoto ([50]) proved that Morse-Smale vector fields are

dense and have structurally stable qualitative dynamics in compact orientable two-dimensional manifolds. In 1968, Palis and Smale ([46], [48]) proved the structural stability of the Morse-Smale dynamical systems in any dimension. However, the density of Morse-Smale systems fails in dimension higher than two, due to "Smale horseshoe". In 1963, Smale ([65]) and also Kupka ([41]) introduced the Kupka-Smale vector fields, that is, the vector fields for which all the equilibria and periodic orbits are hyperbolic and the intersections of the stable and unstable manifolds of equilibria and periodic orbits are all transversal. They both show the density of such systems in any dimension (see also [51]). The qualitative dynamics of Kupka-Smale systems are locally stable: periodic orbits, the local dynamics around them and their connections move smoothly when a parameter of the equation is changing.

For the partial differential equations (PDE's in short), the history of structural stability and of local stability is more recent. Notice that a trajectory of the dynamical system $S(t)$ generated by such a PDE is of the form $t \mapsto S(t)u_0 = u(\cdot, t)$, where $u(x, t)$ is the solution of the PDE with initial data $u_0(x)$. In particular, the trajectory moves in a functions space (often a Sobolev space), which is infinite-dimensional. As a generalization of [46] and [48], [26] and [45] proved that Morse-Smale and Kupka-Smale properties are still meaningful in infinite-dimensional systems for the problem of stability of the qualitative dynamics. Therefore, there is a great interest in obtaining generalizations of the above mentioned finite-dimensional generic results. Notice that, if we want to get a meaningful genericity result, we have to allow perturbations only in the same class of PDE's. Typically, the parameter with respect to which the genericity is obtained is the non-linearity f.

The first example of transversality of unstable and stable manifolds for PDE's is due to Henry ([30]) in 1985 for the reaction-diffusion equation in the segment

$$\partial_t u = u_{xx} + f(x, u, u_x), \quad (x, t) \in (0, 1) \times (0, +\infty) \tag{1.4}$$

with Dirichlet, Neumann or Robin boundary conditions. More strikingly, he obtained the noteworthy property that the stable and unstable manifolds of two hyperbolic equilibria of (1.4) always intersect transversally. A key ingredient for proving this automatic transversality is the use of the non-increase of the "Sturm number" or "zero number" [69] of the solutions of the corresponding linearized parabolic equations. In addition to this automatic transversality, the gradient structure proved in [72] shows the genericity of Morse-Smale property for the flow of (1.4) with separated boundary conditions.

If we consider (1.4) with periodic boundary conditions, that is the parabolic equation on the circle S^1

$$\partial_t u = u_{xx} + f(x, u, u_x), \quad (x, t) \in S^1 \times (0, +\infty) \tag{1.5}$$

then the gradient structure fails but the flow of (1.5) still has particular properties equivalent to the ones of two-dimensional ODEs, such as the Poincaré-Bendixson property proved in [18] (the reader interested in the correspondence between the dynamics of (1.4) and the ones of low-dimensional ODEs may consider the review paper [39]). In 2008, still using the powerful tool of the "zero number", Czaja and Rocha ([13]) proved that, for the parabolic equations on the circle (1.5), the stable and unstable manifolds of hyperbolic periodic orbits always intersect transversally.

In 2010, the second and third authors completed the results of Czaja and Rocha. More precisely, they proved in [37] that the equilibria and periodic orbits are hyperbolic, generically with respect to the nonlinearity f. They also proved that the stable and unstable manifolds of hyperbolic critical elements $\mathcal{C}^-$ and $\mathcal{C}^+$ intersect transversally, unless both critical elements $\mathcal{C}^-$ and $\mathcal{C}^+$ are equilibria of same Morse index and moreover that, generically with respect to f, such connecting orbits between equilibria with the same Morse index ([38]) do not exist. Finally, the Poincaré-Bendixson theorem of [18] yields that, generically with respect to f, the equation (1.5) is Morse-Smale (see [38]).

Concerning spatial dimension higher than $d = 1$, the generic transversality of stable and unstable manifolds has been shown in 1997 by the first author and P. Poláčik ([7]) in the case $f \equiv f(x, u)$, that is, for the equation

$$\partial_t u = \Delta u + f(x, u), \qquad (x, t) \in \Omega \times (0, +\infty) \tag{1.6}$$

with $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$. As a consequence, since (1.6) is a gradient system, they deduce that, under additional dissipative conditions on the non-linearity, the Morse-Smale property holds for the flow (1.6) generically with respect to $f \in \mathcal{C}^2$. It is noteworthy, as shown by Poláčik ([54]), that this generic transversality property is not true if one considers homogeneous functions $f(x, u) \equiv f(u)$ only.

We also mention that generic transversality properties have been shown by the authors for various gradient damped wave equations, see [8] and [36].

Due to the realization results of Dancer and Poláčik, [14] and [53], we know that the dynamics of the flow of the general parabolic equation (1.1) in dimension $d \geqslant 2$ may be as complicated as chaotic flows. We may only hope to prove the genericity of the Kupka-Smale property and not of the Morse-Smale one. Notice that the flow of (1.1) is not gradient (periodic orbits may exist) and the very particular and helpful "zero number property" of spatial dimension $d = 1$ fails. In the present paper, we prove the generic transversality property. The generic hyperbolicity of equilibrium points is already proved in [37] in any space dimension. Thus, the generic hyperbolicity of periodic orbits is the only remaining step to obtain the genericity of the Kupka-Smale property.

Some years ago, in a preliminary draft of this paper, we were convinced to have proved the genericity of the Kupka-Smale property. However, Maxime Percy du Sert pointed to us a gap in the proof of generic hyperbolicity of periodic orbits. We did not manage to fill it. Recently, two of the three authors passed away and we decided to publish the results as obtained together. In particular, we prove the generic transversality only (unlike claimed in [39]). In Appendix C, we quickly discuss our ideas to obtain the generic hyperbolicity of periodic orbits and indicate where the gap remains.

Plan of the article.

In Section 2, we recall the classical existence and uniqueness properties of the solutions of the scalar parabolic equation and the corresponding linear and linear adjoint equations. We also review unique continuation properties, which are fundamental in this paper. In Section 3, we remind some basic definitions such as hyperbolicity of critical elements and we state the main properties of the dynamical system $S_f(t)$,

namely the existence of $\mathcal{C}^1$ immersed finite-codimensional (resp. finite-dimensional) stable (resp. unstable) manifolds of hyperbolic critical elements. Section 4 is devoted to the study of the singular nodal sets and to the proof of Theorem 1.4. In Section 5, we show that Theorem 1.4 leads to one-to-one properties such as Theorems 1.2 and 1.3. Using these tools, in Section 6, we prove Theorem 1.1, i.e. we show the generic transversality of heteroclinic and homoclinic orbits of the parabolic equation (1.1). Section 7 contains discussions about some generalizations of Theorem 1.1. We conclude by two appendices recalling the basic facts about the Whitney topology and Sard-Smale theorems, which will be used in this paper, and one appendix discussing the still open problem of generic hyperbolicity of periodic orbits of (1.1).

Dedication: Very sadly, both Pavol Brunovský and Geneviève Raugel passed away before the publication of this article, respectively in december 2018 and in may 2019. They were still working actively on the manuscript and the present version is exactly the one which have been completed by them. This article is dedicated to their memories.

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2 Some basic results on parabolic PDEs

2.1 Local existence and regularity results of the parabolic equation (1.1)

The solutions of the scalar parabolic equation (1.1) exist locally and are unique, see for example [49] or [31]. In the whole paper, α belongs to the open interval $(\frac{1}{2}+\frac{d}{2p}, 1)$. We recall that we use the notation $f \in \mathcal{C}^r(E,\mathbb{R})$ to indicate the regularity of f, i.e. to say that the function $f : E \to \mathbb{R}$ is of class $\mathcal{C}^r$. Where a topology is required (smooth dependences on f etc.), the notation $\mathfrak{C}^r(E,\mathbb{R})$ refers to the space $\mathcal{C}^r(E,\mathbb{R})$ endowed with the Whitney topology (see Appendix A).

Proposition 2.1. *Let $r \geqslant 1$ and $f \in \mathcal{C}^r(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$.*

i) For any $u_0 \in X^\alpha$, there exists a maximal time $T(u_0) > 0$ such that (1.1) has a unique classical solution $S_f(t)u_0 = u(t) \in \mathcal{C}^0([0,T],X^\alpha) \cap \mathcal{C}^1((0,T],X^\beta) \cap \mathcal{C}^0((0,T],D(-\Delta_D))$, for any $0 \leqslant \beta < 1$ and for any $0 < T \leqslant T(u_0)$. If $T(u_0)$ is finite, then $\|u(t)\|_{X^\alpha}$ goes to $+\infty$ when $t < T(u_0)$ tends to $T(u_0)$.

Moreover, $t \mapsto \partial_t u(t)$ is locally Hölder continuous from $(0,T]$ into X^β, for $0 \leqslant \beta < 1$. In particular, $u(\cdot) \equiv S_f(\cdot)u_0$ belongs to the space $\mathcal{C}^0((0,T],W^{3,p}(\Omega)) \cap \mathcal{C}^1((0,T],W^{s,p}(\Omega))$, for any $s < 2$, and thus belongs to the spaces $\mathcal{C}^0((0,T],\mathcal{C}^2(\overline{\Omega})) \cap \mathcal{C}^1((0,T],\mathcal{C}^1(\overline{\Omega}))$ and $\mathcal{C}^1(\overline{\Omega}\times[\tau,T],\mathbb{R})$, for any $0 < \tau < T$. If, in addition, the first derivatives $D_u f(x,\cdot,\cdot)$ and $D_{\nabla u} f(x,\cdot,\cdot)$ are Lipschitz-continuous on the bounded sets of $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d$, then $u(\cdot)$ belongs to $\mathcal{C}^1((0,T],W^{2,p}(\Omega)) \cap \mathcal{C}^2((0,T],W^{s,p}(\Omega))$, for any $s < 2$ and hence $u(\cdot)$ also belongs to $\mathcal{C}^2(\overline{\Omega}\times[\tau,T],\mathbb{R})$, for any $0 < \tau < T$.

ii) For any $u_0 \in X^\alpha$, for any $T < T(u_0)$, there exist a neighborhood $\mathcal{U} \equiv \mathcal{U}(T)$ of u_0 in X^α and a neighborhood $\mathcal{V} \equiv \mathcal{V}(T)$ of f in $\mathfrak{C}^1$ such that, for any $v_0 \in \mathcal{U}$

and any $g \in \mathcal{V}$, $v(t) \equiv S_g(t)v_0$ is well defined on $[0,T]$, depends continuously on $v_0 \in X^\alpha$ and $g \in \mathfrak{C}^1$, and there exists a positive number $R \equiv R(T,\mathcal{U},\mathcal{V})$ such that $((S_g(t)v_0)(x), (\nabla S_g(t)v_0)(x))$ belongs to the ball $B_{\mathbb{R}^{d+1}}(0,R)$, for all $(t,v_0,g,x) \in [0,T] \times \mathcal{U} \times \mathcal{V} \times \overline{\Omega}$.

iii) Moreover, for any $u_0 \in X^\alpha$, for any $T < T(u_0)$, the map $(t,u_0) \in (0,T] \times \mathcal{U} \mapsto S_f(t)u_0 \in X^\alpha$ is of class $\mathcal{C}^r$ and, in particular, $S_f(t)$ is a local semigroup of class $\mathcal{C}^r$. In addition, there exists a neighborhood $\mathcal{W}$ of f in the space $\mathfrak{C}^r(\overline{\Omega} \times [-2R,2R] \times [-2R,2R]^d, \mathbb{R})$ such that the map $(t,u_0,g) \in (0,T] \times \mathcal{U} \times \mathcal{W} \mapsto S_g(t)u_0 \in X^\alpha$ is of class $\mathcal{C}^r$.

Remarks:

1) The statement (i) is a direct consequence of the existence and regularity results given in [31, Chapter 3] and of elliptic regularity properties. We only want to emphasize that, since the solution $u(\cdot) \equiv S_f(\cdot)u_0$ belongs to $\mathcal{C}^0([0,T], X^\alpha)$ and that X^α is continuously embedded in $\mathcal{C}^1(\overline{\Omega})$, $u(\cdot)$ automatically belongs to the space $\mathcal{C}^0([0,T), \mathcal{C}^1(\overline{\Omega}))$. Since $u(\cdot)$ is a classical solution and belongs to $\mathcal{C}^0((0,T], W^{2,p}(\Omega)) \cap \mathcal{C}^1((0,T], W^{1,p}(\Omega))$, $f(x,u,\nabla u) - \partial_t u$ is in the space $\mathcal{C}^0((0,T], W^{1,p}(\Omega))$ and the regularity properties of the elliptic equation

$$\Delta_D u = \partial_t u - f(x,u,\nabla u) \ ,$$

imply that $u(\cdot)$ belongs to the space $\mathcal{C}^0((0,T], W^{3,p}(\Omega)) \subset \mathcal{C}^0((0,T], \mathcal{C}^2(\overline{\Omega}))$.

2) Statements (ii) and (iii) are also easy consequences of [31, Theorem 3.4.4 and Corollary 3.4.5]. We want to point out that, for any $u_0 \in X^\alpha$ and any $0 < T < T(u_0)$, there exists $R_0 > 0$ such that $(u(x,t), \nabla u(x,t))$, for all $(x,t) \in \overline{\Omega} \times [0,T]$ is bounded in $\mathbb{R}^{d+1}$ by a positive number $R_0 \equiv R_0(u_0,T)$. Since $g(x,u(x,t),\nabla u(x,t))$ depends only on the values of x, $u(x,t)$ and $\nabla u(x,t)$, we can show, by applying the continuity results of [31, Section 3.4], that, for any $R > R_0$, for any $0 < \varepsilon < (R - R_0)/2$, there exists a positive number η such that, for any $g(\cdot,\cdot,\cdot) \in \mathcal{C}^r(\overline{\Omega} \times [-R,R] \times [-R,R]^d, \mathbb{R})$, η-close to f in the classical norm of $\mathfrak{C}^r(\overline{\Omega} \times [-R,R] \times [-R,R]^d, \mathbb{R})$, $((S_g(t)u_0)(x), (\nabla S_g(t)u_0)(x))$ belongs to the ball $B_{\mathbb{R}^{d+1}}(0, R_0 + \varepsilon)$, for all $(x,t) \in \overline{\Omega} \times [0,T]$.

3) Notice that the statement (ii) of Proposition 2.1 implies that the maximal time $T(u_0)$ is a lower-semi-continuous function of the initial data u_0

As we have already seen, the parabolic equation has a smoothing effect at any finite positive time. If the boundary of the domain Ω was of class $\mathcal{C}^\infty$ and f belonged to $\mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, the solutions of Eq.(1.1) would be in $\mathcal{C}^\infty(\overline{\Omega} \times [\tau,T], \mathbb{R})$ for any $0 < \tau < T < T(u_0)$. However, if $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, we can still show that the solutions are regular in the interior of Ω, even if Ω is of class $\mathcal{C}^{2,\alpha}$ only.

In the whole paper, we say that $u(t) : t \in \mathbb{R} \mapsto u(t)$ is a *bounded complete solution (or trajectory)* of (1.1) if it is a solution of (1.1), defined for any $t \in \mathbb{R}$ and bounded in X^α, uniformly with respect to $t \in \mathbb{R}$.

Since we are only interested in the regularity of the bounded complete solutions of (1.1), we will state a $\mathcal{C}^\infty$-regularity result for such solutions.

Proposition 2.2. *Assume that f belongs to $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$. Then, any bounded complete solution $u(t)$ of (1.1) belongs to $\mathcal{C}^\infty(\Omega\times\mathbb{R},\mathbb{R})$. More precisely, for any open set $\mathcal{O}$, such that $\overline{\mathcal{O}}\subset\Omega$, for any $R>0$, any $m\in\mathbb{N}$, any $k\in\mathbb{N}$, and any $q\in[1,\infty]$, there exists a positive constant $K(\mathcal{O},R,m,k,q)$, such that any bounded complete solution $u(t)$, with $\sup_{t\in\mathbb{R}}\ \|u(t)\|_{X^\alpha}\leqslant R$, satisfies*

$$\sup_{t\in\mathbb{R}}\left\|\frac{d^k u}{dt^k}(t)\right\|_{W^{m,q}(\mathcal{O})}\leqslant K(\mathcal{O},R,m,k,q)\ . \tag{2.1}$$

Proof: We will not give all the details of the proof, but will indicate only the main arguments. The proof consists in a recursion argument with respect to k and m. Let $u(t)$ be a bounded complete solution of (1.1) satisfying $\sup_{t\in\mathbb{R}}\|u(t)\|_{X^\alpha}\leqslant R$.

First step: Since f belongs to $\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$, by [31, Corollary 3.4.6], the function $t\in\mathbb{R}\mapsto u(t)\in X^\alpha$ is of class $\mathcal{C}^k$, for any $k\in\mathbb{N}$ and $\frac{d^k u}{dt^k}(t)\in\mathcal{C}^0(\mathbb{R},X^\alpha\cap W^{2,p}(\Omega))\cap\mathcal{C}^1(\mathbb{R},X^\beta)$, for any $\beta<1$, is a classical solution of the equation

$$\frac{d}{dt}(\frac{d^k u}{dt^k})=\Delta\frac{d^k u}{dt^k}+\frac{d^k}{dt^k}(f(x,u,\nabla u))\ . \tag{2.2}$$

We notice that the term $\frac{d^k}{dt^k}(f(x,u,\nabla u))$ can be computed by using the Faa Di Bruno formula [16] and its generalization [9] as follows. We introduce the $(d+1)$-dimensional vector $w(x,t)=(u,\nabla u)(x,t)$, that is $w_1=u$ and $w_{i+1}=\partial_{x_i}u$. Using the generalized Faa Di Bruno formula ([9]), we can write,

$$\begin{aligned}\frac{d^k}{dt^k}(f(x,u(x,t),&\nabla u(x,t)))=\sum_{m_j=1,|m|=1}D_w^m f(x,w(x,t))\frac{d^k}{dt^k}(w_j)(x,t)\\&+\sum_{2\leqslant|m|\leqslant k}D_w^m f(x,w(x,t))\sum_{p(k,m)}k!\Pi_{j=1}^k\frac{\left[\frac{d^{\ell_j}}{dt^{\ell_j}}w\right]^{n_j}}{(n_j!)[\ell_j!]^{|n_j|}}\\&\equiv\sum_{m_j=1,|m|=1}D_w^m f(x,w(x,t))\frac{d^k}{dt^k}(w_j)(x,t)+g_k(x,t)\end{aligned} \tag{2.3}$$

where $p(k,m)=\{(n_1,\dots,n_k;\ell_1,\dots,\ell_k)\,|\,\exists s\in[\![1,k]\!],\ n_i=\ell_i=0\text{ for }1\leqslant i\leqslant n-s\}$ and g_k contains only derivatives with respect to t of order less or equal to $k-1$.

We notice that the estimate (2.1) for $k=0$, $m=2$ and $q=p$ is a direct consequence of the hypothesis and of Proposition 2.1. Using (2.3), the fact that $W^{1,p}(\Omega)$ is an algebra and the bound $\sup_{t\in\mathbb{R}}\|u(t)\|_{X^\alpha}\leqslant R$, one shows by recursion on k that

$$\sup_{t\in\mathbb{R}}\|\frac{d^k u}{dt^k}(t)\|_{W^{2,p}(\Omega)}\leqslant C_2(R,k)\ , \tag{2.4}$$

where $C_2(R,k)$ is a positive constant depending only on R, k (and of f). Like in the remarks following Proposition 2.1, the elliptic regularity properties allow also to deduce from Eq.(2.2) and from the estimate (2.4) that,

$$\sup_{t\in\mathbb{R}}\|\frac{d^k u}{dt^k}(t)\|_{W^{3,p}(\Omega)}\leqslant C_3(R,k)\ , \tag{2.5}$$

where $C_3(R,k)$ is a positive constant depending only on R, k (and of f).

Second step: One easily shows, by recursion on $n \in \mathbb{N}$ (and also k) that,

$$\sup_{t\in\mathbb{R}} \|\frac{d^k u}{dt^k}(t)\|_{W^{3+n,p}(\mathcal{O})} \leqslant C_{3+n}(\mathcal{O}, R, k) \ . \tag{2.6}$$

Indeed, let O_j, $j = 1, 2, \ldots, n+1$, be a sequence of regular open sets such that $\overline{\mathcal{O}} \subset O_{n+1} \subset \overline{O_{n+1}} \subset O_n \subset \ldots \subset O_{j+1} \subset \overline{O}_{j+1} \subset O_j \ldots \subset O_1 \subset \overline{O_1} \subset \Omega$ and φ_j, $j = 1, 2, \ldots$, be a corresponding sequence of regular functions such that $\varphi_j(x) \in [0,1]$, $x \in \overline{\Omega}$, and $\varphi_j(x) \equiv 0$, for $x \in \overline{\Omega}\backslash O_j$ and $\varphi_j(x) \equiv 1$, for $x \in O_{j+1}$. We recall that, by the remarks following Proposition 2.1, one already knows that the estimates (2.5) hold for any $k \in \mathbb{N}$. We remark that $\varphi_1 u$ is a solution of the elliptic equation

$$\Delta(\varphi_1 u) = \varphi_1 \frac{du}{dt} + u\Delta\varphi_1 + 2\nabla u \cdot \nabla\varphi_1 - \varphi_1 f(x, u, \nabla u) \tag{2.7}$$

where $\varphi_1 \frac{du}{dt} + u\Delta\varphi_1 + 2\nabla u \cdot \nabla\varphi_1 - \varphi_1 f(x, u, \nabla u)$ belongs to $W^{3-1,p}(O_1) \cap W_0^{1,p}(O_1)$. By the elliptic regularity results, $\varphi_1 u$ belongs to $W^{3+1,p}(O_1)$ and

$$\sup_{t\in\mathbb{R}} \|\varphi_1 u(t)\|_{W^{3+1,p}(O_1)} \leqslant C_{3+1}(O_1, R, 0, \varphi_1) \ , \tag{2.8}$$

where $C_{3+1}(O_1, R, 0, \varphi_1)$ is a positive constant depending only on O_1, R, φ_1. Likewise, writing the elliptic equality satisfied by $\Delta(\varphi_1(\frac{d^k}{dt^k}u))$ and using the equalities (2.2) and (2.3), one shows, by recursion on k, that $\frac{d^k}{dt^k}(\varphi_1 u)$ belongs to $W^{3+1,p}(O_1)$ and

$$\sup_{t\in\mathbb{R}} \|\frac{d^k}{dt^k}(\varphi_1 u)(t)\|_{W^{3+1,p}(O_1)} \leqslant C_{3+1}(O_1, R, k, \varphi_1) \ , \tag{2.9}$$

where $C_{3+1}(O_1, R, k, \varphi_1)$ is a positive constant depending only on O_1, R, k and φ_1. We notice that $\frac{d^k}{dt^k}(\varphi_1 u)(x) = \frac{d^k}{dt^k}u(x)$, for any $x \in O_2$.

We next assume that $\frac{d^k}{dt^k}(\varphi_j u)$ belongs to $W^{3+j,p}(O_j)$ and that the estimates (2.8) and (2.9) hold with 1 replaced by j. Remarking that $\varphi_{j+1} u$ is a solution of the elliptic equation

$$\Delta(\varphi_{j+1} u) = \varphi_{j+1} \frac{du}{dt} + u\Delta\varphi_{j+1} + 2\nabla u \cdot \nabla\varphi_{j+1} - \varphi_{j+1} f(x, u, \nabla u) \tag{2.10}$$

where $\varphi_{j+1} \frac{du}{dt} + u\Delta\varphi_{j+1} + 2\nabla u \cdot \nabla\varphi_{j+1} - \varphi_{j+1} f(x, u, \nabla u)$ belongs to $W^{3+j-1,p}(O_{j+1}) \cap W_0^{1,p}(O_{j+1})$, we at once show that $\varphi_{j+1} u$ belongs to $W^{3+j+1,p(O_{j+1})\cap W_0^{1,p}(O_{j+1})}$ and that the estimate (2.8) holds with 1 replaced by $j+1$. Likewise, one shows by recursion on k that $\frac{d^k}{dt^k}(\varphi_{j+1} u)$ belongs to $W^{3+j+1,p}(O_{j+1})$ and that the estimate (2.9) holds with 1 replaced by $j+1$. Thus, we have proved by recursion on n and k that $\frac{d^k}{dt^k}(u)$ belongs to $W^{3+n,p}(\mathcal{O})$ and that the estimates (2.6) are satisfied.

The general estimate (2.1) is a direct consequence of the estimates (2.6) and the classical Sobolev embedding theorem. □

2.2 The linear and linear adjoint equations

Let $0 \leqslant s < T$ and let $a(\cdot) \in \mathcal{C}^1([0,T], L^\infty(\Omega))$ and $b(\cdot) \in \mathcal{C}^1([0,T], W^{1,\infty}(\Omega)^d)$. We consider solutions v of the linear parabolic equation

$$\begin{array}{l} v_t(x,t) = \Delta_D v(x,t) + a(x,t)v(x,t) + b(x,t).\nabla v(x,t) \ , \quad t > s, x \in \Omega, \\ v(x,s) = v_s \ . \end{array} \tag{2.11}$$

In what follows, we denote $A(t)$ the operator

$$A(t) = \Delta_D + a(x,t). + b(x,t).\nabla \ .$$

Equation (2.11) arises either when one linearizes the parabolic equation (1.1) along a solution u, in which case we have

$$\left\{ \begin{array}{l} a(x,t) = f'_u(x, u(x,t), \nabla u(x,t)) \\ b(x,t) = f'_{\nabla u}(x, u(x,t), \nabla u(x,t)) \end{array} \right. \tag{2.12}$$

or when one considers the difference $v(t) = u_2(t) - u_1(t)$ between two solutions u_1 and u_2 of (1.1), in which case we have

$$\left\{ \begin{array}{l} a(x,t) = \int_0^1 f'_u(x, (\theta u_2 + (1-\theta)u_1)(x,t), \nabla(\theta u_2 + (1-\theta)u_1)(x,t))d\theta \\ b(x,t) = \int_0^1 f'_{\nabla u}(x, (\theta u_2 + (1-\theta)u_1)(x,t), \nabla(\theta u_2 + (1-\theta)u_1)(x,t))d\theta \end{array} \right. \tag{2.13}$$

Notice that, since f belongs to $\mathcal{C}^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, due to Proposition 2.1, in both cases the coefficients of (2.11) belong to $\mathcal{C}^1((0,T], W^{1,\infty}(\Omega))$. Since in what follows, we are mainly applying the results of this section to bounded complete trajectories, we can consider, without loss of generality, that the coefficients of (2.11) belong to $\mathcal{C}^1([0,T], W^{1,\infty}(\Omega))$.

Proposition 2.3. *Let $r \in [1,\infty)$ and let $v_s \in L^r(\Omega)$. Equation (2.11) has a unique solution $v(t) \equiv U(t,s)v_s \in \mathcal{C}^0([s,T], L^r(\Omega)) \cap \mathcal{C}^1((s,T], L^r(\Omega)) \cap \mathcal{C}^0((s,T], W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega))$ satisfying $v(s) = v_s$. Moreover, $v : t \in (s,T] \mapsto v(t) \in X^\alpha$ is Hölder continuous and belongs to $\mathcal{C}^1((s,T], L^q(\Omega)) \cap \mathcal{C}^0((s,T], W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$ for any $q \in [1,+\infty]$. In particular $v \in \mathcal{C}^0((s,T], \mathcal{C}^1(\overline{\Omega}))$.*

Proof: For the existence, uniqueness and regularity of the solution of $v(t) \equiv U(t,s)v_s$ in $\mathcal{C}^0([s,T], L^r(\Omega)) \cap \mathcal{C}^1((s,T], L^r(\Omega)) \cap \mathcal{C}^0((s,T], W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega))$, we refer to [31, Theorem 7.1.3]. To prove that $v(t)$ belongs to any space $L^q(\Omega)$ (and thus to X^α), we will use a bootstrap argument. Assume that v_s belongs to $L^r(\Omega)$ and set $r = r_0$. By [31, Theorem 7.1.3], $v(s+\delta) \in W^{2,r_0}(\Omega)$ for any $\delta > 0$. If $d - 2r_0 \leqslant 0$, then, $v(s+\delta) \in W^{2,r_0}(\Omega) \subset L^q(\Omega)$, for any positive number $q \geqslant 1$, by the classical Sobolev embedding. If, $d - 2r_0 > 0$, again by the Sobolev embedding theorem, $v(s+\delta) \in W^{2,r_0}(\Omega) \subset L^{r_1}(\Omega)$, for $r_1 = dr_0/(d-2r_0) = r_0 + 2r_0^2/(d-2r_0)$. We again apply [31, Theorem 7.1.3] to deduce that $v(s+2\delta) \in W^{2,r_1}(\Omega)$, for any $\delta > 0$. Again, if $d - 2r_1 > 0$, we obtain that $v(t+2\delta) \in W^{2,r_2}(\Omega) \subset L^{r_2}(\Omega)$, for $r_2 = dr_1/(d-2r_1) \geqslant r_1 + 2r_1^2/(d-2r_1) \geqslant r_0 + 2r_0^2/(d-2r_0) + 2r_1^2/(d-2r_1)$. Clearly, since the increment $r \mapsto 2r/(d-2r)$ is increasing until $d - 2r \leqslant 0$, after a finite

number of steps, we obtain that $v(t) \in L^q(\Omega)$. □

Proposition 2.3 tells that Equation (2.11) generates a family of evolution operators $U(t,s)$ on $L^p(\Omega)$, which is extended to $L^r(\Omega)$ for any $r \geqslant 1$.

Let now $1 < p < +\infty$, which implies that $X = L^p(\Omega)$ is reflexive. Denote by p^* the conjugate exponent of p, that is, $p^* = p/(p-1)$; consider the adjoint space $X^* = (L^p(\Omega))^* = L^{p^*}(\Omega)$ of X and the *adjoint evolution operator* $U(t,s)^* : X^* \to X^*$. Let $T > 0$; for $\psi_T \in L^{p^*}(\Omega)$, we define the function $\psi : s \in [0,T] \mapsto \psi(s) = U(T,s)^*\psi_T$. In general, $\psi(s)$ is only a *weak* solution* of the equation

$$\partial_s \psi(x,s) = -\Delta_D \psi(x,s) - a(x,s)\psi(x,s) + \mathrm{div}(b(x,s)\psi(x,s)) \tag{2.14}$$

with $(x,s) \in \Omega \times (0,T)$ and with final data $\psi(T) = \psi_T$ in the weak-$*$ sense. More precisely, $s \in [0,T) \mapsto \psi(s) \in X^*$ is locally Hölder continuous, for each $\phi \in X$, $\langle \phi, \psi(s) \rangle \to \langle \phi, \psi_T \rangle$ when $s \to T^-$ and, for each $\phi \in D(A^*)$, $(\phi, \psi(s))$ is differentiable on $[0,T)$ with $\partial_t(\phi, \psi(s)) = (A(s)\phi, \psi(s))$.

Usually, $\psi(s) = U(T,s)^*\psi_T$ is only a solution of (2.14) in a weak sense. But here, since $a(\cdot) \in \mathcal{C}^1([0,T], L^\infty(\Omega))$ and $b(\cdot) \in \mathcal{C}^1([0,T], W^{1,\infty}(\Omega)^d)$, $\psi(s)$ is a strong solution of (2.14), as we shall see in the proposition below. Notice that (2.14) is a parabolic equation solved backwards in time.

Proposition 2.4.

1) With the above notations, $\psi(s) = U(T,s)^*\psi_T$ belongs to $\mathcal{C}^1([0,T), X^*) \cap \mathcal{C}^0([0,T), W^{2,p^*}(\Omega) \cap W_0^{1,p^*}(\Omega))$. Moreover, it satisfies (2.14) in the strong sense and $\psi(s)$ belongs to $\mathcal{C}^1([0,T), L^q(\Omega)) \cap \mathcal{C}^0([0,T), W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$ for any $q \geqslant 1$.

2) Let $\tilde{\psi}_T \in X^*$. For any $0 < \eta < T$, $\tilde{\psi}_{T-\eta} = U(T,T-\eta)^*((-\Delta_D)^\alpha)^*\tilde{\psi}_T$ is well defined in X^*. Hence, for $s < T - \eta$, $\tilde{\psi}(s) = U(T-\eta,s)^*\tilde{\psi}_{T-\eta} = U(T,s)^*((-\Delta_D)^\alpha)^*\tilde{\psi}_T$ belongs to $\mathcal{C}^1([0,T-\eta), X^*) \cap \mathcal{C}^0([0,T-\eta), W^{2,p^*}(\Omega) \cap W_0^{1,p^*}(\Omega))$ and a strong solution of (2.14).

Proof: The first part of the proposition is a direct consequence of [31, Theorem 7.3.1] on the existence and regularity of solutions for the adjoint equation and on the fact that the coefficients have the regularity $a(\cdot) \in \mathcal{C}^1([0,T], L^\infty(\Omega))$ and $b(\cdot) \in \mathcal{C}^1([0,T], W^{1,\infty}(\Omega)^d)$. The fact that $\psi(s)$ belongs to any $L^q(\Omega)$ is proved by recursion as in Proposition 2.3.

To show the second part of the proposition, let $\tilde{\psi}_T \in X^*$ and let $\varphi \in X = L^p(\Omega)$. By Proposition 2.3, $U(T,T-\eta)\varphi$ belongs to $X^\alpha = D((-\Delta_D)^\alpha)$ and thus $\langle \tilde{\psi}_T | (-\Delta_D)^\alpha U(T,T-\eta)\varphi \rangle_{L^{p^*},L^p}$ is well defined. Therefore, $U(T,T-\eta)^*((-\Delta_D)^\alpha)^*\tilde{\psi}_T$ is well defined and belongs to $L^{p^*}(\Omega)$. To finish, we apply [31, Theorem 7.3.1] (or the first part of the proposition) to the initial data $\psi_T = U(T,T-\eta)^*((-\Delta_D)^\alpha)^*\tilde{\psi}_T$. □

2.3 Unique continuation properties

In this section, we recall some important unique continuation properties satisfied by the linear parabolic equation (2.11). We enhance that these properties will apply to

solutions $v(t) \in X^\alpha$ of (2.11) with coefficients given by (2.12) or (2.13). Hence, we may apply it to the difference of two solutions of the nonlinear parabolic equation (1.1). In particular, the unique continuation properties below will have fundamental consequences on the properties of the dynamics of (1.1), such as the injectivity of the flow.

The following result is a direct consequence of the backward uniqueness property stated in [4, Theorem II.1].

Proposition 2.5.

1) Let $T > 0$. Let $a(x,t) \in L^\infty(\Omega \times (0,T))$ and let $b(x,t) \in L^\infty(\Omega \times (0,T))^d$. Let $v(t) \in L^2((0,T), H_0^1(\Omega))$ be a solution of the linear parabolic equation (2.11). Then, $v(T) \equiv 0$ in Ω if and only if v vanishes identically in $(0,T) \times \Omega$.

2) Likewise, assume that $a(x,t) \in L^\infty(\Omega \times (0,T))$, that $b(x,t) \in L^\infty(\Omega \times (0,T))^d$ and that $D_{x_i} b(x,t) \in L^\infty(\Omega \times (0,T))^d$, $0 \leqslant i \leqslant d$. Let $\psi(t) \in L^2((0,T), H_0^1(\Omega))$ be a solution of the adjoint linear equation (2.14). Then, $\psi(0) \equiv 0$ in Ω if and only if ψ vanishes identically in $(0,T) \times \Omega$.

Let now u_1 and u_2 be two solutions on the time interval $[0,T]$ of the equation (1.1). We already remarked that $v(t) = u_2(t) - u_1(t)$ satisfies the linear equation (2.11) with the coefficients a and b given by (2.13). By Proposition 2.1, the coefficients a, b and the function $v(t)$ satisfy the regularity assumptions of the above proposition 2.5. Thus, if $u_1(T) = u_2(T)$, then $u_1 \equiv u_2$ on $[0,T]$. This leads to state the following corollary.

Corollary 2.6. *Let $T > 0$. Let $u_1(t)$ and $u_2(t)$ be two solutions on the time interval $[0,T]$ of the equation (1.1). If $u_1(T) = u_2(T)$, then $u_1(t) = u_2(t)$, for any $t \in [0,T]$. In other terms, the local dynamical system $S_f(t)$ generated by (1.1) has the backward uniqueness property.*

The following result is proved in [62] and shows that the set of the zeros of the solutions of the linear parabolic equation is a closed set with empty interior.

Proposition 2.7. *Let $T > 0$, a and b be as in Proposition 2.5. We assume that $v(x,t) \in L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega))$ is a solution of the linear parabolic equation (2.11). If $v(x,t)$ vanishes on an open non-empty subset of $\Omega \times (0,T)$, then $v(x,t)$ identically vanishes on $\Omega \times (0,T)$.*

A similar result has been obtained for the strong solutions of the adjoint equation in [17, Corollary 2.12].

Proposition 2.8. *Let $T > 0$. Let $a(x,s) \in L^\infty(\Omega \times (0,T))$ and let $b(x,s) \in L^\infty(\Omega \times (0,T))^d$. Let $\psi(s) \in L^2((0,T), H^2(\Omega) \cap H_0^1(\Omega))$ be a solution of the adjoint equation (2.14). If $\psi(x,t)$ vanishes on an open non-empty subset of $\Omega \times (0,T)$, then $\psi(x,t)$ identically vanishes on $\Omega \times (0,T)$.*

In the particular case of smooth solutions of (2.11) (typically if one considers global bounded solutions and a smooth non-linearity f), we will need stronger properties on the zeros of the solutions in Section 4.

We say that v vanishes to infinite order in both the space and time variables at (x_0, t_0) if, for any $k \geqslant 1$, there is a constant $C_k > 0$, such that, for any $(x,t) \in \Omega \times [-T, 0]$,

$$|v(x,t)| \leqslant C_k(|x - x_0|^2 + |t - t_0|)^{k/2} \ . \tag{2.15}$$

We shall often apply the following unique continuation result of Escauriaza and Fernández [15].

Proposition 2.9. *Assume that $v \in \mathcal{C}^0((-T,0], \mathcal{C}^2(\overline{\Omega})) \cap \mathcal{C}^1((-T,0], \mathcal{C}^1(\overline{\Omega}))$ is a solution of (2.11) and satisfies either homogeneous Dirichlet or homogeneous Neumann boundary conditions. Suppose that v vanishes to infinite order at $(x_0, 0)$ in both the space and time variables in the sense of (2.15). Assume moreover that there exists a positive constant K such that for any $(x,t) \in \Omega \times (-T, 0]$,*

$$|v_t(x,t) - \Delta v(x,t)| \leqslant K(|\nabla v(x,t)| + |v(x,t)|) \ . \tag{2.16}$$

Then, $v(x,0)$ vanishes for any $x \in \Omega$ and therefore $v(x,t)$ identically vanishes in $\Omega \times [-T, 0]$.

We say that v vanishes to infinite order in space at (x_0, t_0) if, for any $k \geqslant 1$, there is a constant $C_k > 0$, such that

$$|v(x,t_0)| \leqslant C_k|x - x_0|^k \ . \tag{2.17}$$

From Proposition 2.9 and [2, Theorem 1], we deduce the following unique continuation result for solutions $v \in \mathcal{C}^0((-T,0], \mathcal{C}^2(\overline{\Omega})) \cap \mathcal{C}^1((-T,0], \mathcal{C}^1(\overline{\Omega}))$ of (2.16), which vanish to infinite order in space. The following result can also be deduced from Proposition 2.9, a simple computation and, a recursion argument when $v(x,t)$ is a $\mathcal{C}^\infty$-function in the variables (x,t). Indeed, if for example $v(x,t_0)$ vanishes to order 2 (resp. 4) in space at (x_0,t_0), then, due to the equation (2.11), $v_t(x,t_0)$ vanishes to order 0 (resp. 2) in space at (x_0,t_0). Moreover, if $v(x,t_0)$ vanishes to order 4 in space at (x_0,t_0), deriving the equation (2.11) with respect to t, one shows that $v_{tt}(x,t)$ vanishes at order 0 in space. Finally, continuing the recursion argument on k and on the derivatives with respect to t, one shows that v vanishes to infinite order at (x_0,t_0) in both the space and time variables in the sense of (2.15)

Proposition 2.10. *Assume that $v \in \mathcal{C}^0((-T,0], \mathcal{C}^2(\overline{\Omega})) \cap \mathcal{C}^1((-T,0], \mathcal{C}^1(\overline{\Omega}))$ satisfies the inequality (2.16) and either homogeneous Dirichlet or homogeneous Neumann boundary conditions. Suppose also that v vanishes to infinite order in space at $(x_0, 0)$, for some $x_0 \in \Omega$. Then, $v(x,0)$ vanishes for any $x \in \Omega$ and therefore $v(x,t)$ identically vanishes in $\Omega \times [-T, 0]$.*

3 The local infinite-dimensional dynamical system $S_f(t)$

In this section, we recall some basic properties of the local dynamical system $S_f(t)$ generated by the parabolic equation (1.1) on X^α (if the dependence on f is clear, we simply write $S(t)$). As we have seen in the introduction, the hyperbolicity of the critical elements (that is, the equilibrium points and periodic orbits) and the transversality of the stable and unstable manifolds play a primordial role. Thus, we will focus on recalling the definitions and main properties of these objects.

3.1 Critical elements and hyperbolicity

Let $e \in X^\alpha$ be an equilibrium point of (1.1). The linearization $(D_u S(t)e)$ of the dynamical system $S(t)$ at e is given by the linear semigroup $e^{L_e t}$ on X^α, where $L_e : D(\Delta_D) \mapsto L^p(\Omega)$ is the linear operator defined by

$$L_e v = \Delta_D v + f'_u(x, e(x), \nabla e(x))v + f'_{\nabla u}(x, e(x), \nabla e(x)).\nabla v \ .$$

The operator $-L_e$ is a sectorial operator and a Fredholm operator with compact resolvent. Therefore, the spectrum of L_e consists of a sequence of isolated eigenvalues of finite multiplicity, the norms of which converge to infinity. Since the resolvent of $L_e : X \to X$ is compact, the linear C_0-semigroup $e^{L_e t}$ from X into X is compact and its spectrum consists of a sequence of isolated eigenvalues of finite multiplicity converging to 0. By [49, Chapter 2, Theorem 2.4], μ is an eigenvalue of e^{L_e} if and only if $\mu = e^\lambda$, where λ is an eigenvalue of L_e.

Definition 3.1. *The equilibrium point e is said* **simple** *if 1 does not belong to the spectrum of e^{L_e}. The equilibrium point e is* **hyperbolic** *if e^{L_e} has no spectrum on the unit circle $S^1 \equiv \{z \in \mathbb{C} \mid |z| = 1\}$.*

In the case of the equation (1.1), we may equivalently say that the equilibrium point e is simple if and only if 0 is not an eigenvalue of L_e and that it is hyperbolic if and only if L_e has no eigenvalue with zero real part.

The **Morse index** *$i(e)$ is the (finite) number of eigenvalues of e^{L_e} of norm strictly larger than 1 (counted with their multiplicities) or equivalently the number of eigenvalues of L_e with positive real part.*

Let $p(t)$ be a periodic solution of the scalar parabolic equation (1.1) with period $\omega > 0$. This periodic solution describes the **periodic orbit** $\Gamma = \{p(t) \,|\, t \in [0, \omega)\}$. The linearization of the dynamical system $S(t)$ along $p(t)$ is given by the evolution operator $\Pi_{f,p}(t, s) : v_s \in X^\alpha \mapsto v(t) \in X^\alpha$, $t \geqslant s$, where $v(\tau)$ solves the non-autonomous equation

$$\begin{cases} \partial_\tau v(x, \tau) = \Delta v(x, \tau) + f'_u(x, p, \nabla p)v(x, \tau) + f'_{\nabla u}(x, p, \nabla p)\nabla v(x, \tau) \\ v(x, s) = v_s(x) \ . \end{cases} \tag{3.1}$$

The operator $\Pi_{f,p}(\omega, 0)$ is called the (corresponding) **period map**. One remarks that $\Pi_{f,p}(t + \omega, t) = \Pi_{f,p}(t + m\omega, t + (m-1)\omega)$ for any $t \geqslant 0$ and any $m \in \mathbb{N}$. Notice that $\partial_t p(t)$ is a solution of (3.1) and thus that 1 is an eigenvalue of $\Pi_{f,p}(\omega, 0)$ with eigenvector $\partial_t p(0)$. We emphasize that, due to the smoothing properties in finite positive time of the parabolic equation (3.1), the operator $\Pi_{f,p}(t, s) : X^\alpha \to X^\alpha$, $t > s$, is compact. Therefore, the spectrum of $\Pi_{f,p}(t + \omega, t)$ consists of a sequence of isolated eigenvalues of finite multiplicity, converging to 0. As for the linearized operator e^{L_e} at the equilibrium point e, 0 is the only point where the spectrum of $\Pi_{f,p}(t + \omega, t)$ accumulates. Actually, by the backward uniqueness property, 0 is not an eigenvalue neither of e^{L_e}, nor of $\Pi_{f,p}(t+\omega, t)$. By [31, Lemma 7.2.2], the spectrum $\sigma(\Pi_{f,p}(t + \omega, t))$ of $\Pi_{f,p}(t + \omega, t)$ is independent of $t \in [0, +\infty)$. For this reason, the following definition makes sense.

To simplify the notation, when there is no confusion, we will simply write $\Pi(t, s)$ instead of $\Pi_{f,p}(t, s)$.

Definition 3.2. *A periodic solution $p(t)$ of period ω is* **simple** *or* **non-degenerate** *if the number 1 is a simple (isolated) eigenvalue of $\Pi_{f,p}(\omega, 0)$.*

The periodic solution $p(t)$ is **hyperbolic** *if $\Pi_{f,p}(\omega, 0)$ has no spectrum on the unit circle S^1 except the eigenvalue one, which is simple and isolated.*

Since $\Pi_{f,p}(\omega, 0)$ is a compact operator, the periodic solution $p(t)$ is hyperbolic if and only if 1 is a simple, isolated eigenvalue of $\Pi_{f,p}(\omega, 0)$ and is the only eigenvalue on the unit circle.

The **Morse index** *$i(p)$ of $p(\cdot)$, or the* **Morse index** *$i(\Gamma)$ of Γ, is the (finite) number of eigenvalues of $\Pi_{f,p}(\omega, 0)$ of norm strictly larger than 1 (counted with their multiplicities).*

In what follows, we will sometimes say that the periodic orbit $\Gamma = \{p(t) \,|\, t \in [0, \omega)\}$ is simple (resp. hyperbolic), instead of saying that $p(t)$ is simple (resp. hyperbolic).

A first important consequence of the simplicity property is the persistence of equilibrium points and periodic orbits under perturbations.

Theorem 3.3. *Let $r \geqslant 2$ be given and let $f_0 \in \mathfrak{C}^r$.*

1) Let e_0 be a simple equilibrium point of (1.1) with $f = f_0$. There exist a neighborhood $\mathcal{N}$ of f_0 in $\mathfrak{C}^r$ and a neighborhood $\mathcal{U}$ of e_0 in X^α such that, for any $f \in \mathcal{N}$, there exists a unique equilibrium point $e(f)$ in $\mathcal{U}$. This equilibrium depends continuously on $f \in \mathfrak{C}^r$. In addition, the eigenvalues of $L_{e(f)}$ continuously depend on $f \in \mathfrak{C}^r$.

Moreover, if e_0 is hyperbolic, the neighborhoods $\mathcal{N}$ and $\mathcal{U}$ can be chosen small enough so that $e(f)$ is also hyperbolic and so that the Morse index $i(e)$ is equal to $i(e_0)$.

2) Let $p_0(t)$ be a simple periodic solution with period (resp. minimal period) ω_0 of (1.1) for $f = f_0$. There exist a neighborhood $\mathcal{N}$ of f_0 in $\mathfrak{C}^r$, a positive number η and a neighborhood $\mathcal{U}$ of $\Gamma_0 = \{p_0(t) \,|\, t \in [0, \omega_0)\}$ in X^α such that, for any $f \in \mathcal{N}$, there exists a unique periodic orbit $\Gamma(f) = \{p(f)(t) \,|\, t \in [0, \omega(f))\}$ in $\mathcal{U}$, of period (resp. minimal period) $\omega(f)$ with $|\omega(f) - \omega_0| \leqslant \eta$. The period $\omega(f)$ and the periodic orbit $\Gamma(f)$ continuously depend on f. In addition, the eigenvalues of $\Pi_{f,p(f)}(\omega(f), 0)$ continuously depend on $f \in \mathfrak{C}^r$.

Moreover, if f_0 is hyperbolic, the neighborhoods $\mathcal{N}$ and $\mathcal{U}$ and $\eta > 0$ can be chosen small enough so that the periodic solution $p(f)(t)$ is hyperbolic and so that the Morse index $i(p(f))$ is equal to the Morse-index $i(p_0)$.

Proof: The first statement about the persistence of simple equilibria e_0 is very classical. Assume that $\|e_0\|_{L^\infty} \leqslant m$ and $\|\nabla e_0\|_{L^\infty} \leqslant m$. Then, applying the implicit function theorem or the fixed point theorem of strict contraction (see the proof [7, Lemma 4.c.2]), one shows that there exist a neighborhood $\mathbf{N}_0$ of f_0 in $\mathcal{C}^r(\overline{\Omega} \times [-2m, 2m] \times [-2m, 2m]^d)$ and a neighborhood $\mathcal{U}$ of e_0 in X^α such that for any $f \in \mathbf{N}_0$, there exists a unique equilibrium point $e(f)$ in $\mathcal{U}$. This equilibrium depends continuously of $f \in \mathbf{N}_0$ and, moreover, all the other properties of the first statement hold. Using the restriction mapping R of Section 2.1, we conclude that there exists

a neighborhood $\mathcal{N}$ of f_0 in $\mathfrak{C}^r$ such that, for any $f \in \mathcal{N}$, there exists a unique equilibrium point $e(f)$ in $\mathcal{U}$ and that all the other properties of the first statement hold.

Let $p_0(t)$ be a simple periodic solution of period $\omega_0 > 0$ of (1.1) for $f = f_0$. Assume that $\sup_{t\in[0,\omega_0)} \|p_0(t)\|_{L^\infty} \leqslant m$ and $\sup_{t\in[0,\omega_0)} \|\nabla p_0(t)\|_{L^\infty} \leqslant m$. The statement of the persistence of a simple periodic solution $p_f(t)$ near $p_0(t)$ with period ω_f close to ω_0 and also of the uniqueness (up to a time translation) of this periodic solution, if f belongs to a small enough neighborhood of f_0 in $\mathcal{C}^r(\overline{\Omega}\times[-2m,2m]\times[-2m,2m]^d)$, is a direct consequence of [31, Theorem 8.3.2]; it is proved by using the method of Poincaré sections and the implicit function theorem or the fixed point theorem of strict contraction (for further results in the case where the perturbations are less regular, see also [23] and [24]). One concludes like in the proof of the statement 1) by using the restriction mapping R of Section 2.1.

The continuous dependence of the eigenvalues of $L_{e(f)}$ or of $\Pi_{f,p(f)}(\omega(f),0)$ with respect to $f \in \mathfrak{C}^r$ is a consequence of the proof of the continuity results of Kato (see [40, Theorems IX.24, IV.31, IV.3.18]) and of the properties of the restriction mapping R. Detailed proofs of continuity of the point spectrum can also be found in [22, Section 3]. □

Notice that a periodic solution $p(t)$ of period ω can be simple, whereas the same periodic solution $p(t)$, considered as periodic solution of period $n\omega$ can be non-simple. This is the case when the spectrum of $\Pi_{f,p}(\omega,0)$ contains a n-th root of 1. Thus, in the statement 2) of Theorem 3.3, when $p_0(t)$ is a simple periodic solution of period ω_0 of (1.1) for $f = f_0$, we do not know if $\Gamma(f) = \{p(f)(t)\,|\,t \in [0,\omega(f))\}$ is the unique periodic orbit of (1.1) in the neighborhood $\mathcal{U}$ of Γ_0 if f belongs to $\mathcal{N}$. Indeed,if the spectrum of $\Pi_{f_0,p_0}(\omega_0,0)$ contains a n-th root of unity, then it is possible that new periodic orbits of period close to $n\omega_0$ are created (in the case where $n = 2$, it is the famous "period-doubling bifurcation").

Of course, when $p_0(t)$ is hyperbolic, no such new periodic solutions can be created and $\Gamma(f)$ is still isolated in the set of periodic orbits. Hyperbolicity is a notion independent of the chosen period.

3.2 Stable and unstable manifolds

We recall that a **critical element** means either an equilibrium point or a periodic orbit of (1.1).

Definition 3.4. *Let $\mathcal{C}$ be a critical element of* (1.1). *The* **global stable and unstable sets** *of $\mathcal{C}$ are respectively defined as*

$$W^s(\mathcal{C}) = \{u_0 \in X^\alpha \,|\, S_f(t)u_0 \xrightarrow[t\to+\infty]{} \mathcal{C}\}\ ,$$
$$W^u(\mathcal{C}) = \{u_0 \in X^\alpha \,|\, \forall\, t \leqslant 0,\ S_f(t)u_0 \text{ is well defined and } S_f(t)u_0 \xrightarrow[t\to-\infty]{} \mathcal{C}\}\ .$$

Likewise, if $U_{\mathcal{C}}$ is a neighborhood of $\mathcal{C}$ in X^α, we introduce the **local stable and**

unstable sets *of* $\mathcal{C}$ *defined as*

$$W^s(\mathcal{C}, U_{\mathcal{C}}) \equiv W^s_{loc}(\mathcal{C}) \equiv \{u_0 \in U_{\mathcal{C}} \,|\, S_f(t)u_0 \in U_{\mathcal{C}}, t \geqslant 0\} \ ,$$
$$W^u(\mathcal{C}, U_{\mathcal{C}}) \equiv W^u_{loc}(\mathcal{C}) \equiv \{u_0 \in U_{\mathcal{C}} \,|\, \forall\, t \leqslant 0,\ S_f(t)u_0 \text{ is well defined and stays in } U_{\mathcal{C}}\}.$$

If we need to specify the dependence with respect to the non-linearity f, we will denote these manifolds as $W^s(\mathcal{C}, U_{\mathcal{C}}, f)$ and $W^u(\mathcal{C}, U_{\mathcal{C}}, f)$ or as $W^s_{loc}(\mathcal{C}, f)$ and $W^u_{loc}(\mathcal{C}, f)$.

Let e_0 be an equilibrium point of (1.1) and let $(D_u S(t)e_0) = e^{L_{e_0}t}$ be the corresponding linearized operator around e_0. We denote by P_u (resp. P_s) the projection in X^α onto the space generated by the (generalized) eigenfunctions of $e^{L_{e_0}}$ corresponding to the eigenvalues with modulus strictly larger than 1 (resp. with modulus strictly smaller than 1). Let $X^\alpha_u = P_u(X^\alpha)$ and $X^\alpha_s = P_s(X^\alpha)$. We have seen that, in the case of the parabolic equation (1.1), the Morse index of every hyperbolic equilibrium point is finite, which implies that $P_u(X) = P_u(X^\alpha)$.

The following theorem states the existence of the local stable and unstable manifolds near hyperbolic equilibrium points. The result is very classical. In the case of a vector field on a finite-dimensional compact manifold, we refer the reader to [1], [47], [35] for example, and in the infinite dimensional case, we refer to [31], [26], [25], [11], [59].

Theorem 3.5. *Let f_0 be given in $\mathfrak{C}^r$, $r \geqslant 2$, and let e_0 be a hyperbolic equilibrium point of $S_{f_0}(t)$. Then there is a neighborhood U_0 of e_0 such that the local unstable manifold $W^u(e_0, U_0)$ (resp. the local stable manifold $W^s(e_0, U_0)$) is a $\mathcal{C}^r$-submanifold of dimension $i(e_0)$ (resp. codimension $i(e_0)$), which is tangent to X^α_u (resp. X^α_s) at e_0.*

More precisely, there exist a neighborhood U_0 of e_0 in X^α, two mappings $h_u(f_0) \equiv h^0_u : P_u X^\alpha \to P_s X^\alpha$ and $h_s(f_0) \equiv h^0_s : P_s X^\alpha \to P_u X^\alpha$ of class $\mathcal{C}^r$ such that $h^0_u(0) = 0$, $Dh^0_u(0) = 0$, $h^0_s(0) = 0$, $Dh^0_s(0) = 0$ and

$$\begin{aligned} W^u_{loc}(e_0, f_0) &\equiv W^u(e_0, U_0, f_0) \\ &= \{v \in U_0 \,|\, v = e_0 + P_u(v - e_0) + h^0_u(P_u(v - e_0))\} \\ W^s_{loc}(e_0, f_0) &\equiv W^s(e_0, U_0, f_0) \\ &= \{v \in U_0 \,|\, v = e_0 + P_s(v - e_0) + h^0_s(P_s(v - e_0))\} \ . \end{aligned} \tag{3.2}$$

Furthermore, the convergence rates to the origin are exponential. More precisely, there are positive constants k_1, k_2 and constants $0 < \gamma_2 < 1 < \gamma_1$, such that,

$$\begin{aligned} \|S_{f_0}(t)x\|_X &\leqslant k_1\gamma_1^t\,, \quad \forall\, t \leqslant 0\,, \quad \forall\, x \in W^u(e_0, U_0)\ , \\ \|S_{f_0}(t)x\|_X &\leqslant k_2\gamma_2^t\,, \quad \forall\, t \geqslant 0\,, \quad \forall\, x \in W^s(e_0, U_0)\ . \end{aligned} \tag{3.3}$$

In addition, the local stable and unstable manifolds "continuously" depend of the nonlinear map f. More precisely, there exists $\rho > 0$ and, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{N}$ of f_0 in $\mathfrak{C}^r$ such that, for any $f \in \mathcal{N}$, $S_f(t)$ has a unique equilibrium point $e(f)$ in the ball $B_{X^\alpha}(e_0, \rho)$ of center e_0 and radius ρ in X^α, and $\|e(f) - e_0\|_{X^\alpha} \leqslant \varepsilon$. Moreover, the corresponding local unstable and local stable manifolds of $e(f)$ are

given by

$$\begin{aligned} W^u_{loc}(e(f),f) &\equiv W^u(e(f),U_0,f) \\ &= \{v\in U_0 \,|\, v = e(f) + P_u(v-e(f)) + h_u(f)(P_u(v-e(f)))\} \\ W^s_{loc}(e(f),f) &\equiv W^s_{loc}(e(f),U_0,f) \\ &= \{v\in U_0 \,|\, v = e(f) + P_s(v-e(f)) + h_s(f)(P_s(v-e(f)))\}\ , \end{aligned}$$

where $h_u(f) : P_uX^\alpha \to P_sX^\alpha$ *and* $h_s(f) : P_sX^\alpha \to P_uX^\alpha$ *are maps of class* $\mathcal{C}^r$ *such that* $h_u(f)(0) = 0$, $h_s(f)(0) = 0$ *and* $\|h_u(f) - h_u^0\|_{\mathcal{C}^r} \leqslant \varepsilon$ *and* $\|h_s(f) - h_s^0\|_{\mathcal{C}^r} \leqslant \varepsilon$. *Finally, for any* $f \in \mathcal{N}$, *the above constants* k_i, γ_i *are independent of* f.

Proof: We refer to [31, Theorems 5.2.1. and 5.2.2] for the existence of the local stable and unstable manifolds in the case of a hyperbolic equilibrium point of a parabolic equation. To obtain the last part of the Theorem, that is the smooth dependence with respect to f, we simply use a fixed point theorem with parameter. Indeed, the proof of Theorem 5.2.1 of [31] consists in constructing the mappings h_u and h_s as fixed points of suitable contraction mappings. These maps depend smoothly on f and thus remain contractions mappings for f close to f_0 and their fixed points $h_u(f)$ and $h_s(f)$ depend smoothly on f. Notice that in general $Dh_u(f)(0)$ and $Dh_s(f)(0)$ do not vanish, but are only small of order ε. □

Let $p(x,t)$ be a hyperbolic periodic solution of (1.1) of minimal period $\omega > 0$, let $\Gamma = \{p(t) \,|\, t \in [0,\omega)\}$ be the associated orbit and let $\Pi(t,0) : X^\alpha \to X^\alpha$, be the associated evolution operator defined by the linearized equation (3.1). We denote μ_i, $i \in \mathbb{N}$, the eigenvalues of the period map $\Pi(\omega,0)$. Since $p(x,t)$ is a hyperbolic periodic solution, the intersection of the spectrum of $\Pi(\omega,0)$ with the unit circle S^1 of $\mathbb{C}$ reduces to the eigenvalue 1, which is a simple (isolated) eigenvalue. We recall that, if $p(a)$, $a \in [0,\omega)$, is another point of the periodic orbit, the spectrum of $D_u(S_f(\omega,0)p(a))$ coincides with the one of $\Pi(\omega,0)$ whereas the corresponding eigenfunctions depend on the point $p(a)$.

We denote $P_u(a)$ (resp. $P_c(a)$, resp. $P_s(a)$) the projection in X^α onto the space generated by the (generalized) eigenfunctions of $D_u(S_f(\omega,0)p(a))$ corresponding to the eigenvalues with modulus strictly larger than 1 (resp. equal to 1, resp. with modulus strictly smaller than 1).

Since a hyperbolic periodic orbit is a particular case of a normally hyperbolic $\mathcal{C}^1$ manifold, we may apply, for example, the existence results of [5], [34], [35] or [59, Theorem 14.2 and Remark 14.3] and thus, we may state the following theorem. Other methods of proofs are also given in [1], [35], [26], [25] and [47].

Theorem 3.6. *Let* f_0 *be given in* $\mathfrak{C}^r$, $r \geqslant 2$, *and let* $\Gamma_0 = \{p_0(t) \,|\, t \in [0,\omega_0)\}$ *be a hyperbolic periodic orbit of Eq.* (1.1) *of minimal period* $\omega_0 > 0$.

1) There exists a small neighborhood U_{Γ_0} *of* Γ_0 *in* X^α *such that the local unstable and stable sets*

$$\begin{aligned} W^u_{loc}(\Gamma_0) \equiv W^u(\Gamma_0,U_{\Gamma_0}) &= \{u_0 \in X^\alpha \,|\, S_{f_0}(t)u_0 \in U_{\Gamma_0}\ ,\ \forall t \leqslant 0\} \\ W^s_{loc}(\Gamma_0) \equiv W^s(\Gamma_0,U_{\Gamma_0}) &= \{u_0 \in X^\alpha \,|\, S_{f_0}(t)u_0 \in U_{\Gamma_0}\ ,\ \forall t \geqslant 0\} \end{aligned}$$

are (embedded) $\mathcal{C}^1$-submanifolds of X^α of dimension $i(\Gamma_0)+1$ and codimension $i(\Gamma_0)$ respectively.

2) Moreover, $W^s_{loc}(\Gamma_0)$ and $W^u_{loc}(\Gamma_0)$ are fibrated by the local strongly stable (resp. unstable) manifolds at each point $p_0(a)\in\Gamma_0$, that is,

$$W^s_{loc}(\Gamma_0)=\cup_{a\in[0,\omega_0)}W^{ss}_{loc}(p_0(a))\ ,\quad W^u_{loc}(\Gamma_0)=\cup_{a\in[0,\omega_0)}W^{su}_{loc}(p_0(a))\ ,$$

where there exist positive constants $\tilde r_0$, κ_0 and κ_0^* such that

$$\begin{aligned}W^{ss}_{loc}(p_0(a))=&\{u_0\in X^\alpha\,|\,\|S_{f_0}(t)u_0-p_0(a+t)\|_{X^\alpha}<\tilde r_0\ ,\ \forall t\geqslant 0\,,\\ &\lim_{t\to\infty}e^{\kappa_0 t}\|S_{f_0}(t)u_0-p_0(a+t)\|_{X^\alpha}=0\}\ ,\\ W^{su}_{loc}(p_0(a))=&\{u_0\in X^\alpha\,|\,\|S_{f_0}(t)u_0-p_0(a+t)\|_{X^\alpha}<\tilde r_0\ ,\ \forall t\leqslant 0\,,\\ &\lim_{t\to-\infty}e^{\kappa_0^* t}\|S_{f_0}(t)u_0-p_0(a+t)\|_{X^\alpha}=0\}\ .\end{aligned}\tag{3.4}$$

For any $a\in[0,\omega_0)$, $W^{su}_{loc}(p_0(a))$ (resp. $W^{ss}_{loc}(p_0(a))$) is a $\mathcal{C}^r$-submanifold of X^α of dimension $i(\Gamma)$ (resp. of codimension $i(\Gamma)+1$) tangent at $p_0(a)$ to $P_u(a)X^\alpha$ (resp. $P_s(a)X^\alpha$).

3) Finally, the local stable and unstable manifolds of the periodic orbit continuously depend on the nonlinear map $f\in\mathfrak{C}^r$.

We have seen that the local stable and unstable manifolds are $\mathcal{C}^r$ graphs over P_sX^α and P_uX^α respectively. In general, the global stable and unstable manifolds are not embedded submanifolds of X^α.

Adapting the proof of [31, Theorem 6.1.9], one easily shows the following result.

Theorem 3.7. *Let $f\in\mathfrak{C}^r$, $r\geqslant 2$, be given.*

1) Let e_0 be a hyperbolic equilibrium point of (1.1). Then, the global unstable set $W^u(e_0)$ (resp. global stable set $W^s(e_0)$) is an injectively immersed invariant manifold of class $\mathcal{C}^r$ in X^α of dimension (resp. of codimension) $i(e_0)$.

2) Likewise, let $\Gamma_0=\{p_0(t)\,|\,t\in[0,\omega_0]\}$ be a hyperbolic periodic orbit of minimal period $\omega_0>0$. Then, the global unstable set $W^u(\Gamma_0)$ (resp. global stable set $W^s(\Gamma_0)$) is an injectively immersed invariant manifold of class $\mathcal{C}^r$ in X^α of dimension $i(\Gamma_0)+1$ (resp. of codimension $i(\Gamma_0)$).

Proof: We will give the proof in the case of a hyperbolic equilibrium e_0, since the proof is very similar in the case of a hyperbolic periodic orbit.
Proof for the unstable manifold: For every $m\in\mathbb{N}$, we introduce the open set

$$U_0(m)=\{x\in U_0\,|\,S_f(t)x\text{ is well defined},0\leqslant t\leqslant m\}\ ,$$

where U_0 is the neighborhood of e_0, in which the local stable and unstable manifolds are given as graphs (see Theorem 3.5). By Proposition 2.1, $U_0(m)$ is an open subset of U_0 and thus $W^u_{loc}(e_0)\cap U_0(m)$ is an open subset of $W^u_{loc}(e_0)$. We readily check that

$$W^u(e_0)=\cup_{m=0}^{+\infty}S_f(m)(W^u_{loc}(e_0)\cap U_0(m))\ .\tag{3.5}$$

Moreover, since $W^u_{loc}(e_0)$ is negatively invariant, we have, for any $m \in \mathbb{N}$,

$$S_f(m)(W^u_{loc}(e_0) \cap U_0(m)) \subset S_f(m+1)(W^u_{loc}(e_0) \cap U_0(m+1)) \ .$$

By Corollary 2.6, $S_f(m)$ is an injective map from $U_0(m)$ into X^α. Moreover, by Proposition 2.5, for any $x \in U_0(m)$, $D_u S_f(t)x$ is an injective map from X^α into itself, thus $S_f(m)_{|U_0(m)}$ is an injective $\mathcal{C}^r$-immersion. By Theorem 3.5, $W^u_{loc}(e_0)$ is the image of an injective $\mathcal{C}^r$-map H_u from the open ball $B_{\mathbb{R}^k}(0,1)$ of center 0 and radius 1 of $\mathbb{R}^k$ into X^α, where $k = i(e_0)$. Moreover, the derivative $DH_u(y)$ has rank k at each point $y \in B_{\mathbb{R}^k}(0,1)$. We recall that $H_u^{-1}(W^u_{loc}(e_0) \cap U_0(m))$ is an open subset $V(k,m)$ of $B_{\mathbb{R}^k}(0,1)$. It follows that $S_f(m)W^u_{loc}(e_0) \cap U_0(m))$ is the image of the injective $\mathcal{C}^r$-immersion $S_f(m) \circ H_u : V(k,m) \to X^\alpha$ and thus is a $\mathcal{C}^r$-submanifold of dimension k. Since the invariance is obvious, Statement 1) is proved.
Proof for the stable manifold: We first remark that

$$W^s(e_0) = \cup_{m=0}^{+\infty} S_f(m)^{-1}(W^s_{loc}(e_0)) \ . \tag{3.6}$$

Moreover, since $W^s_{loc}(e_0)$ is positively invariant, we have, for any $m \in \mathbb{N}$,

$$S_f(m)^{-1}(W^s_{loc}(e_0)) \subset S_f(m+1)^{-1}(W^s_{loc}(e_0)) \ .$$

As a consequence of the property (3.2) in Theorem 3.5, where h^0_s is a $\mathcal{C}^r$-map of $P_s X^\alpha$ into the k-dimensional space $P_u X^\alpha$ and where $Dh^0_s(0) = 0$, $W^s_{loc}(e_0)$ is actually represented as the set $\{v \in U_0 \,|\, g(v) = 0\}$, where $g : x \in U_0 \mapsto g(x) \in \mathbb{R}^k$ is a map of class $\mathcal{C}^r$ and $Dg(v)$ has constant rank k at every point $v \in g^{-1}(0)$. By [31, Theorem 7.3.3], $DS_f(m)u$ has dense range at every point $u \in X^\alpha$ at which $S_f(m)u$ exists if $(DS_f(m)u)^*$ is injective. By Proposition 2.5, the adjoint equation (2.14) also satisfies the backward uniqueness property. Thus $DS_f(m)u$ has dense range at every point $u \in S_f(-m)W^s_{loc}(e_0)$, which implies that, at every point $u \in (g \circ S_f(m))^{-1}(0)$, $D(g(S_f(m))u)$ has rank k. In other terms, the mapping $v \to g(S_f(m)v)$ is a submersion of constant rank k at every point $u \in (g \circ S_f(m))^{-1}(0)$. By a theorem on Page 12 of [44] for example, $(g \circ S_f(m))^{-1}(0)$ is a $\mathcal{C}^r$-submanifold of X^α of codimension k. Thus, since $S_{f(m)}$ is injective, $W^s(e_0)$ is an injectively immersed manifold of codimension k. Since the invariance is obvious, Statement 2) is proved. □

3.3 Transversality of connecting orbits

We use here the above concepts of stable and unstable manifolds of hyperbolic equilibrium points or periodic orbits. The definitions related to Theorem 1.1 are as follows.

Definition 3.8. *Let $\mathcal{C}^\pm$ be two hyperbolic critical elements. We say that $W^u(\mathcal{C}^-)$ and $W^s(\mathcal{C}^+)$* **intersect transversally** *(or are* **transverse***) and we denote it by*

$$W^u(\mathcal{C}^-) \pitchfork W^s(\mathcal{C}^+) \ ,$$

if, at each intersection point $u_0 \in W^u(\mathcal{C}^-) \cap W^s(\mathcal{C}^+)$, $T_{u_0}W^u(\mathcal{C}^-)$ splits, that is, contains a closed complement of $T_{u_0}W^s(\mathcal{C}^+)$ in X^α.

It is important to notice that, in this paper, the complement of $T_{u_0}W^s$ in X^α is always closed since $T_{u_0}W^u(\mathcal{C}^-)$ is finite-dimensional. Also note that, by definition, manifolds which do not intersect are transverse.

Definition 3.9. *Let $\mathcal{C}^- \neq \mathcal{C}^+$ be two different hyperbolic critical elements. A trajectory $u(t)$ of $S(t)$ is a* **heteroclinic orbit** *connecting $\mathcal{C}^-$ to $\mathcal{C}^+$ if $u(t) \in W^u(\mathcal{C}^-) \cap W^s(\mathcal{C}^+)$.*

Let $\mathcal{C}$ be a hyperbolic critical element. A trajectory $u(t)$ of $S(t)$ is a **homoclinic orbit** *to $\mathcal{C}$ if $u(t) \in W^u(\mathcal{C}) \cap W^s(\mathcal{C})$.*

A heteroclinic or homoclinic orbit is **transverse** *if the above intersections of stable and unstable manifolds are transverse.*

4 Singular nodal sets for linear parabolic equations with parameter

In this section, we consider a general linear parabolic equation with parameter

$$\partial_t v(x,t,\tau) = \Delta v(x,t,\tau) + a(x,t,\tau)v(x,t,\tau) + b(x,t,\tau).\nabla_x v(x,t,\tau) \ , \qquad (4.1)$$

in a domain Ω of $\mathbb{R}^d$.

We are interested in the singular nodal set of v, that is the points (x,t,τ) where v and $\nabla_x v$ both vanish. To this end, we use techniques coming from [29]. The singular nodal set of solutions of the parabolic equations, with coefficients independent of the parameter τ, has already been studied in [28] and in [10]. Notice that we assume that v is smooth in the variables $(x,t) \in \Omega \times \mathbb{R}$, but this is not a restriction since this property holds in the applications, that we have in mind (see Section 5).

Theorem 4.1. *Let I and J be open intervals of $\mathbb{R}$. Let $a \in \mathcal{C}^\infty(\Omega \times I \times J, \mathbb{R})$ and $b \in \mathcal{C}^\infty(\Omega \times I \times J, \mathbb{R}^d)$ be bounded coefficients. Let v be a strong solution of (4.1) with Dirichlet boundary conditions. Let $r \geqslant 1$ and assume that v is of class $\mathcal{C}^r$ with respect to τ and of class $\mathcal{C}^\infty$ with respect to x and t. Assume moreover that there are no time $t \in I$ and no parameter $\tau \in J$ such that $v(.,t,\tau) \equiv 0$. Then,*

1) $M = \{(x,t,\tau) \in \Omega \times I \times J \,|\, v(x,t,\tau) = 0 \ , \ \nabla_x v(x,t,\tau) = 0\}$ is contained in a countable union of $\mathcal{C}^r-$manifolds of dimension d,

- *either parametrized by t, τ and $d-2$ components of x,*
- *or parametrized by τ and $d-1$ components of x.*

2) the set

$$(TNS) = \{(x_0,t_0) \in \Omega \times I \,|\, \nexists \tau \in J \text{ such that } (v(x_0,t_0,\tau), \nabla v(x_0,t_0,\tau)) = (0,0)\}$$

is generic in $\Omega \times I$.

Proof: We introduce the set

$$M_q = \{(x,t,\tau) \in \Omega \times I \times J \text{ such that for all } |\alpha| \leqslant q \ , \ \ D_x^\alpha v(x,t,\tau) = 0, \\ \text{and there exists } \alpha, \text{ so that } |\alpha| = q+1, \ D_x^\alpha v(x,t,\tau) \neq 0 \ \} \ .$$

By Proposition 2.10, if $v(x,t,\tau)$ vanishes at infinite order in x, then $v(.,t,\tau)$ identically vanishes in Ω. By assumption, this is precluded. Thus, $M = \cup_{q\geqslant 1} M_q$. And, without loss of generality, we can replace M by M_q in Property 1) of Theorem 4.1.

Let $q \geqslant 1$ and $(x_0,t_0,\tau_0) \in \Omega \times I \times J$. Let us first prove that there exists $\rho_{0,q} > 0$ such that Property 1) of Theorem 4.1 holds with $\Omega \times I \times J$ replaced by the ball $B((x_0,t_0,\tau_0),\rho_{0,q})$ and M replaced by M_q. Assume that $(x_0,t_0,\tau_0) \in M_q$ (otherwise the property is trivial). There exists a multi-index β with $|\beta| = q-1$ such that $\mathrm{Hess}(D_x^\beta v(x_0,t_0,\tau_0)) \neq 0$. In particular, there exist i,j, $1 \leqslant i,j \leqslant d$, such that the derivative $D^2_{x_ix_j}(D^\beta v(x_0,t_0,\tau_0)) \neq 0$. We next consider the D_x^β derivative of the equation (4.1). Since v vanishes at order $|\beta|+1$ at (x_0,t_0,τ_0), we obtain the equality

$$\frac{d}{dt} D^\beta v(x_0,t_0,\tau_0) = \Delta_x (D^\beta v(x_0,t_0,\tau_0)) \ .$$

Now two cases can occur:

- Either $\frac{d}{dt} D^\beta v(x_0,t_0,\tau_0) = 0$ and thus $\sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}(D^\beta v(x_0,t_0,\tau_0)) = 0$. In this case, if $\frac{\partial^2}{\partial x_k^2}(D^\beta v(x_0,t_0,\tau_0)) = 0$ for all k, then there exist $i \neq j$, such that $D^2_{x_ix_j}(D^\beta v(x_0,t_0,\tau_0)) \neq 0$. By considering their i^{th} and j^{th} components, we see that $\nabla_x D_{x_i}(D^\beta v(x_0,t_0,\tau_0))$ and $\nabla_x D_{x_j}(D^\beta v(x_0,t_0,\tau_0))$ are linearly independent. If, on the contrary, there exists i such that $\frac{\partial^2}{\partial x_i^2}(D^\beta v(x_0,t_0,\tau_0)) \neq 0$, then there also exists $j \neq i$ such that

  $$\frac{\partial^2}{\partial x_i^2}(D^\beta v(x_0,t_0,\tau_0)) \times \frac{\partial^2}{\partial x_j^2}(D^\beta v(x_0,t_0,\tau_0)) < 0 \ .$$

  By considering their i^{th} and j^{th} components, we notice again that the vectors $\nabla_x D_{x_i}(D^\beta v(x_0,t_0,\tau_0))$ and $\nabla_x D_{x_j}(D^\beta v(x_0,t_0,\tau_0))$ are linearly independent. To summarize, in all the cases, there exist i and j, such that the vectors $\nabla_x D_{x_i}(D^\beta v(x_0,t_0,\tau_0))$ and $\nabla_x D_{x_j}(D^\beta v(x_0,t_0,\tau_0))$ are linearly independent. This implies that there exists $\rho_{0,q} > 0$ such that

  $$B((x_0,t_0,\tau_0),\rho_{0,q}) \cap (D_{x_i} D^\beta v)^{-1}(0) \cap (D_{x_j} D^\beta v)^{-1}(0)$$

  is an embedded $\mathcal{C}^r-$submanifold $M_q(x_0,t_0,\tau_0)$ in $\mathbb{R}^{d+2}$ of dimension d which contains all of $B((x_0,t_0,\tau_0),\rho_{0,q}) \cap M_q$. This submanifold can be written as

  $$\begin{aligned} M_q(x_0,t_0,\tau_0) = \{&(x,t,\tau) \in B((x_0,t_0,\tau_0),\rho_{0,q}) \text{ such that} \\ &(x_i,x_j) = (\Phi_i((x_k)_{k\neq i,j},t,\tau),\Phi_j((x_k)_{k\neq i,j},t,\tau))\} \ . \end{aligned}$$

- Or $\frac{d}{dt} D^\beta v(x_0,t_0,\tau_0) \neq 0$, then there exists i such that $D^2_{x_i} D^\beta v(x_0,t_0,\tau_0) \neq 0$. Notice that, since $D_{x_i} D^\beta v(x_0,t_0,\tau_0) = 0$, $(D_{x_i},D_t) D^\beta v(x_0,t_0,\tau_0)$ and $(D_{x_i},D_t)(D_{x_i} D^\beta v(x_0,t_0,\tau_0))$ are linearly independent. Thus, there exists $\rho_{0,q} > 0$ such that

  $$B((x_0,t_0,\tau_0),\rho_{0,q}) \cap (D_{x_i} D^\beta v)^{-1}(0) \cap (D^\beta v)^{-1}(0)$$

  is an embedded $\mathcal{C}^r-$submanifold $M_q(x_0,t_0,\tau_0)$ in $\mathbb{R}^{d+2}$ of dimension d, which contains all of $B((x_0,t_0,\tau_0),\rho_{0,q}) \cap M_q$. This submanifold can be written as

  $$\begin{aligned} M_q(x_0,t_0,\tau_0) = \big\{&(x,t,\tau) \in B((x_0,t_0,\tau_0),\rho_{0,q}) \text{ such that} \\ &(x_i,t) = (\Phi_i((x_k)_{k\neq i},\tau),\Phi((x_k)_{k\neq i},\tau))\big\} \ . \end{aligned}$$

To finish the proof of the first part of Theorem 4.1, notice that, since $\Omega \times I \times J$ is separable, for any $q \geqslant 1$, we can find a countable number of points $(x_{n,q}, t_{n,q}, \tau_{n,q})_{n\geqslant 1}$ such that $\Omega \times I \times J = \cup_n B((x_{n,q}, t_{n,q}, \tau_{n,q}), \rho_{n,q})$ and therefore we have $M \subset \cup_{q\geqslant 1} \cup_{n\geqslant 1} M_{n,q}$ with $M_{n,q} = M_q(x_{n,q}, t_{n,q}, \tau_{n,q})$.

Let $P : (x, t, \tau) \mapsto (x, t)$ be the canonical projection. Obviously, (TNS) is the complementary of PM. To prove the second part of Theorem 4.1, it is thus sufficient to show that the projections of the manifolds $M_{n,q}$ obtained above have an image which is contained in a closed set of empty interior. For any n and q, $P_{|M_{n,q}}$ is a $\mathcal{C}^r-$ (and a fortiori a $\mathcal{C}^1-$) map defined from a smooth manifold of dimension d into $\Omega \times I \subset \mathbb{R}^{d+1}$. By the Sard theorem (see for example [1, page 41]), the set of regular values of this map is an open dense subset of $\Omega \times I$ (without loss of generality, we may restrict the size of $B((x_{n,q}, t_{n,q}, \tau_{n,q}), \rho_{n,q})$ in order to prove the openness property). Obviously, the derivative of $P_{|M_{n,q}}$ is never surjective and thus the regular values of this projection map are not in its image. Hence, $P(M_{n,q})$ is contained in a closed set of empty interior, and property 2) of Theorem 4.1 follows from the inclusion $M \subset \cup_{q\geqslant 1} \cup_{n\geqslant 1} M_{n,q}$. ▫

Corollary 4.2. *Assume that the hypotheses of Theorem 4.1 hold. Assume moreover that a and b and v do not depend on τ. Then the set*

$$(NS) = \{x_0 \in \Omega \,|\, \textit{there does not exist } t \in I \textit{ such that } (v(x_0, t), \nabla v(x_0, t)) = (0, 0)\}$$

is generic in Ω.

Proof: Since the problem is now independent of τ, Property 1) of Theorem 4.1 becomes: $M = \{(x, t) \in \Omega \times I \,|\, v(x, t) = 0 \ , \ \nabla_x v(x, t) = 0\}$ is contained in a countable union of manifolds of dimension $d-1$, either parametrized by t and $d-2$ components of x, or parametrized by $d-1$ components of x. Then, Corollary 4.2 follows from a use of the Sard theorem like in the proof of Theorem 4.1. ▫

5 One-to-one properties for global solutions

In this section, we use the properties of the singular nodal sets of the linearized equation (4.1) of Section 4 in order to prove one-to-one properties for bounded complete solutions of the parabolic equation (1.1). We recall that, in Section 2.4, we had deduced the backward uniqueness property of (1.1) from the backward uniqueness property of the linearized parabolic equation (2.11) with coefficients a and b given respectively by (2.12) and (2.13), where u_1 and u_2 are two solutions of (1.1) (see the proposition 2.5 and the corollary 2.6).

Our first result concerns the periodic orbits p. It states that, for almost every point $(x_0, t_0) \in \Omega \times \mathbb{R}$, the value $(x_0, p(x_0, t_0), \nabla p(x_0, t_0))$ is not taken twice during a period. Notice that if Ω is the circle S^1, this property holds for all the points (x_0, t_0), see [37].

Proposition 5.1. *Let $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $p(t)$ be a periodic solution of (1.1) with minimal period $\omega > 0$. Then there exists a dense open set of points $(x_0, t_0) \in \Omega \times \mathbb{R}$ such that*

$$\begin{aligned} &i)\ \ (p_t(x_0,t_0), \nabla p_t(x_0,t_0)) \neq (0,0) \\ &ii)\ \ (p(x_0,t_0), \nabla p(x_0,t_0)) \neq (p(x_0,t), \nabla p(x_0,t)) \quad \text{if} \quad t \notin t_0 + \mathbb{Z}\omega \end{aligned}$$

Proof: First, since f is of class $\mathcal{C}^\infty$ and p is a bounded complete solution, Proposition 2.2 implies that $p \in \mathcal{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. We already noticed that p_t satisfies (2.11) with coefficients a and b given by (2.12). Since f and p are of class $\mathcal{C}^\infty$, the coefficients a and b are also of class $\mathcal{C}^\infty$. Moreover, by Proposition 2.5, there exists no time s such that $p_t(s) = 0$. Thus, Corollary 4.2 implies that there is a generic set of points $x_0 \in \Omega$ such that $(p_t(x_0,t), \nabla p_t(x_0,t)) \neq (0,0)$, for any $t \in \mathbb{R}$.

Next, we set $v(x,t,\tau) = p(x,t) - p(x,t+\tau)$, which solves (2.11) with coefficients given by (2.13). Again, we notice that v, a and b are infinitely differentiable with respect to x, t and τ. Moreover, if there exist $t_1 \in \mathbb{R}$ and $0 < \tau_1 < \omega$ so that $v(.,t_1,\tau_1) \equiv 0$, then by the backward uniqueness property of Corollary 2.6, $v(.,t,\tau_1) \equiv 0$, which means that $p(t)$ is periodic of period $\tau_1 < \omega$ and contradicts the fact that ω is the minimal period. Thus, we can apply Theorem 4.1 to v with $I = \mathbb{R}$ and $J = (0,\omega)$ to obtain a generic set of points $(x_0,t_0) \in \Omega \times \mathbb{R}$ such that the condition ii) holds. Therefore, both conditions i) and ii) are satisfied in a generic, and a fortiori dense, subset of $\Omega \times \mathbb{R}$.

It remains to prove the openness. We consider the variable t modulo the period ω, that is we work on $S = \mathbb{R}/(\mathbb{Z}\omega)$. Let $(x_0,t_0) \in \Omega \times S$ satisfying i) and ii). There is an open neighborhood $\mathcal{U}$ of (x_0,t_0) in which i) holds everywhere in $\mathcal{U}$. Moreover, since i) holds, we may assume that for any (x,t) and (x,t') in $\mathcal{U}$, $t \neq t'$, $(p(x,t), \nabla p(x,t)) \neq (p(x,t'), \nabla p(x,t'))$. The set of values $\{(p(x_0,t), \nabla p(x_0,t)), (x_0,t) \notin \mathcal{U}\}$ is compact and does not contain $(p(x_0,t_0), \nabla p(x_0,t_0))$ due to property ii). Hence, this set of values is at positive distance of the value $(p(x_0,t_0), \nabla p(x_0,t_0))$. Therefore, there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ of (x_0,t_0) such that, for any $(x_1,t_1) \in \mathcal{V}$, $(p(x_1,t_1), \nabla p(x_1,t_1))$ is not contained in $\{(p(x_1,t), \nabla p(x_1,t)), (x_1,t) \notin \mathcal{U}\}$. This shows that ii) holds in $\mathcal{V}$ and concludes the proof of the proposition. □

We also need to separate a periodic orbit from any other (bounded) complete solution.

Proposition 5.2. *Let $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $p(t)$ be a periodic orbit of (1.1) of minimal period ω or an equilibrium point, in which case we adopt the convention that p is a periodic solution with minimal period $\omega = 0$. Let $u(t)$ be a bounded complete solution of (1.1), such that, $p(t) \neq u(s)$, for any $(t,s) \in \mathbb{R}^2$. Then there exists a dense open set of points $(x_0,t_0) \in \Omega \times \mathbb{R}$ such that $(u(x_0,t_0), \nabla u(x_0,t_0)) \neq (p(x_0,t), \nabla p(x_0,t))$ for all $t \in \mathbb{R}$.*

Proof: The proof is very similar to the one of Proposition 5.1 and thus the details are left to the reader. We emphasize only a few arguments. Since f is of class $\mathcal{C}^\infty$ and u, p are bounded complete solutions, Proposition 2.2 implies that p and u belong to the space $\mathcal{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$. To prove the genericity of the points $(x_0,t_0) \in$

$\Omega \times \mathbb{R}$ such that $(u(x_0,t_0), \nabla u(x_0,t_0)) \neq (p(x_0,t), \nabla p(x_0,t))$ for all $t \in \mathbb{R}$, we apply Theorem 4.1 to $v(x,t,\tau) = u(x,t) - p(x,t+\tau)$, with $I = J = \mathbb{R}$. The function v satisfies the hypotheses of Theorem 4.1 and, in particular, due to the assumption of the proposition, there are no times t and τ such that $v(.,t,\tau) \equiv 0$. To show the openness of the set of the points $(x_0,t_0) \in \Omega \times \mathbb{R}$ such that $(u(x_0,t_0), \nabla u(x_0,t_0)) \neq (p(x_0,t), \nabla p(x_0,t))$ for all $t \in \mathbb{R}$, one proceeds like in the proof of Proposition 5.1 by using the compactness of the set $\{(p(x_0,t), \nabla p(x_0,t)), t \in \mathbb{R}\}$ (but here the proof is even simpler, since we do not need to introduce the quotient S) □

As a particular case of the previous proposition, notice that we obtain the following result of separation of periodic orbits. In the case where Ω is the circle S^1, the arguments of [13] show that this property holds for all the points (x_0,t_0) (and not only for a dense open subset). The generalization to higher dimension is as follows.

Proposition 5.3. *Let $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $p_1(t)$ and $p_2(t)$ be two periodic solutions of (1.1) of minimal periods ω_1 and ω_2. Assume that they do not correspond to the same periodic orbit, that is that $p_1(t) \neq p_2(s)$ for all $(t,s) \in \mathbb{R}^2$. Then there exists a dense open set of points $(x_0,t_0) \in \Omega \times \mathbb{R}$ such that $(p_1(x_0,t_0), \nabla p_1(x_0,t_0)) \neq (p_2(x_0,t), \nabla p_2(x_0,t))$ for all $t \in \mathbb{R}$.*

The main dynamical result of this paper concerns heteroclinic and homoclinic orbits. We will need the following result.

Proposition 5.4. *Let $f \in \mathcal{C}^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$. Let $p_-(t)$ and $p_+(t)$ be two periodic solutions of (1.1) of minimal periods ω_- and ω_+ respectively. These periodic solutions may coincide or each one may be reduced to an equilibrium point, in which case we adopt the convention that the minimal period ω is equal to 0. Let $u(t)$ be a global solution of (1.1) connecting $p_-(t)$ and $p_+(t)$, that is,*

$$u(t) - p_\pm(t) \xrightarrow[t \longrightarrow \pm\infty]{} 0 \ .$$

Then there exists a dense open set of points $(x_0,t_0) \in \Omega \times \mathbb{R}$ such that

$$\begin{aligned}
&i) \quad (\partial_t u(x_0,t_0), \nabla \partial_t u(x_0,t_0)) \neq (0,0) \\
&ii) \quad (u(x_0,t_0), \nabla u(x_0,t_0)) \neq (u(x_0,t), \nabla u(x_0,t)) \ \forall \ t \neq t_0 \\
&iii) \quad (u(x_0,t_0), \nabla u(x_0,t_0)) \neq (p_\pm(x_0,t), \nabla p_\pm(x_0,t)) \ \forall \ t \in \mathbb{R}
\end{aligned}$$

Proof: Once again, the proof is very similar to the one of Proposition 5.1. We apply Theorem 4.1 to $v(x,t,\tau) = u(x,t) - u(x,t+\tau)$ with $\tau < 0$ and $\tau > 0$ to prove the density of Property ii); and to $v(x,t,\tau) = u(x,t) - p_\pm(x,t+\tau)$ for the density of Property iii). To prove the openness of Properties ii) and iii), we fix a point (x_0,t_0) such that i)-iii) hold. Due to i), there exists a neighborhood $\mathcal{U} = B(x_0,\rho) \times (t_0 - \delta, t_0 + \delta)$ of (x_0,t_0) such that $(u(x,t), \nabla u(x,t))$ is injective in $\mathcal{U}$. Then we use the compactness of $\{(u(x_0,t), \nabla u(x_0,t)), t \in (-\infty, t_0 - \delta] \cup [t_0 + \delta, +\infty)\} \cup \{(p_-(x_0,t), \nabla p_-(x_0,t)), t \in \mathbb{R}\} \cup \{(p_+(x_0,t), \nabla p_+(x_0,t)), t \in \mathbb{R}\}$ with arguments similar to the ones of the proof of Proposition 5.1. □

6 Generic transversality of connecting orbits

To obtain the transversality of a connecting orbit as stated in Theorem 1.1, we need to show that we can perturb any parabolic semiflow $S_f(t)$ to another one, for which the considered stable and unstable manifolds intersect transversally. The construction of a suitable perturbation $f+\varepsilon g$ of f is the main difficulty in this task. Indeed, the global dynamical framework is classical and well understood in finite dimension. In Section 3, we have seen that the infinite dimension of X^α does not really affect this framework. The main novelty in this paper lies in the construction of a suitable perturbation $f+\varepsilon g$ of f because we will need all the accurate PDE results proved in Sections 4 and 5.

6.1 A perturbation to make an orbit transverse

The first step consists in constructing a suitable perturbation g, which acts on a heteroclinic or homoclinic orbit $u(t)$ in a localized time interval only. In the following result, the one-to-one properties proved in Section 5 are crucial.

Proposition 6.1. *Let $f \in \mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$ and let $u(t)$ be a bounded complete solution connecting $p_-(t)$ to $p_+(t)$ where $p_\pm(t)$ are two periodic solutions of minimal periods $\omega_\pm$. Notice that $p_-=p_+$ is possible and that $p_\pm$ could be equilibrium points in which case we use the convention $\omega_\pm=0$. Let E be a compact subset of $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d$ with non empty interior, let $\mathcal{U}$ be an open subset of $\Omega\times\mathbb{R}$ and let $\psi\in\mathcal{C}^0(\mathcal{U},\mathbb{R})$. Assume that there exists $(x_0,t_0)\in\mathcal{U}$ such that $(x_0,u(x_0,t_0),\nabla u(x_0,t_0))$ belongs to the interior of E and $\psi(x_0,t_0)\neq 0$.*

Then, there exists a function $h\in\mathcal{C}^\infty(\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d,\mathbb{R})$ such that

(i) the function $h:\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ has a compact support contained in E,

(ii) the function $h\circ u:(x,t)\in\overline{\Omega}\times\mathbb{R}\longmapsto h(x,u(x,t),\nabla u(x,t))\in\mathbb{R}$ has a support contained in $\mathcal{U}$,

(iii) we have $\int_{\Omega\times\mathbb{R}}\psi(x,t)h(x,u(x,t),\nabla u(x,t))\,dxdt\neq 0$.

Proof: Since $\psi(x_0,t_0)\neq 0$ and $(x_0,t_0)\in\mathcal{U}$, without loss of generality, by choosing $\mathcal{U}$ smaller, we may assume that ψ does not vanish in $\mathcal{U}$. We set

$$\begin{aligned}K=&\{(x,u(x,t),\nabla u(x,t)),\ (x,t)\notin\mathcal{U}\}\cup\{(x,p_-(x,t),\nabla p_-(x,t)),\ (x,t)\in\overline{\Omega}\times\mathbb{R}\}\\&\cup\{(x,p_+(x,t),\nabla p_+(x,t)),\ (x,t)\in\overline{\Omega}\times\mathbb{R}\}\ .\end{aligned}$$

Proposition 5.4 shows that there is a dense open set of points $(\tilde{x},\tilde{t})\in\mathcal{U}$ such that $(\tilde{x},u(\tilde{x},\tilde{t}),\nabla u(\tilde{x},\tilde{t}))$ does not belong to K. Up to perturbing our reference point, we can thus assume in addition that $(x_0,u(x_0,t_0),\nabla u(x_0,t_0))$ does not belongs to K. Notice that $(x_0,u(x_0,t_0),\nabla u(x_0,t_0))$ still belongs to the interior of E if our perturbation is small enough. Since K is compact, $(x_0,u(x_0,t_0),\nabla u(x_0,t_0))$ is in the interior of $E\backslash K$. Hence, we claim that it is sufficient to choose h non-negative, with compact support in $E\backslash K$ and such that $h(x_0,u(x_0,t_0),\nabla u(x_0,t_0))>0$.

Property (i) holds by construction. For all $(x,t)\notin\mathcal{U}$, $(x,u(x,t),\nabla u(x,t))\in K$ and thus $h(x,u(x,t),\nabla u(x,t))\equiv 0$, showing (ii). Moreover, $\psi(x,t)h(x,u,\nabla u)$ is not

zero at (x_0, t_0) and its sign is constant in $\mathcal{U}$. These properties together with (ii) show that (iii) holds. □

Using this perturbation g, we are able to perturb a non-transversal connecting orbit to a transversal one.

Proposition 6.2. *Let $f_0 \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and let $\mathcal{N}_0$ be any small open neighborhood of f_0 in the $\mathfrak{C}^r$-Whitney topology ($r \geqslant 2$). Let $\Gamma_\pm \equiv \{p_\pm(t) \,|\, t \in [0, \omega_\pm)\}$ be two hyperbolic periodic orbits of minimal periods $\omega_\pm \geqslant 0$ of $S_{f_0}(t)$, which may be not distinct and may be equilibrium points if $\omega_\pm = 0$.*

Then there exists a function $f \in \mathcal{N}_0$ such that Γ_- and Γ_+ are still hyperbolic periodic orbits for $S_f(t)$ and the unstable manifold $W^u(\Gamma_-, f)$ of Γ_- intersects transversally the local stable manifold $W^s_{loc}(\Gamma_+, f) \equiv W^s_{loc}(\Gamma_+, f_0)$ of Γ_+.

Proof: We will prove the existence of a function $f \in \mathcal{N}_0$ satisfying the properties of Proposition 6.2 by applying the transversal density Theorem B.3 in Appendix B.

First, notice that the larger the regularity r is, the more difficult is the result. Thus, without loss of generality we assume $r > dim\, W^u(\Gamma_-) - codim\, W^s(\Gamma_+)$ in the remaining part of the proof.

In what follows, E will be a regular compact subset of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ with non-empty interior. We denote by $\mathcal{C}^r_0(E)$ the subset of functions $g \in \mathcal{C}^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, which identically vanish outside E; in fact, we identify $\mathcal{C}^r_0(E)$ with the space of functions in $\mathcal{C}^r(E, \mathbb{R})$, for which the first r derivatives vanish on ∂E. We recall that the topology induced in $\mathcal{C}^r_0(E)$ by the Whitney topology coincides with the classical $\mathcal{C}^r$ topology and thus that $\mathcal{C}^r_0(E)$ is actually a Banach space.

The proof splits in several steps.

First step: construction of particular neighborhoods
By theorems 3.5 and 3.6 and the remarks following both theorems, there exist two neighborhoods $\tilde{\mathcal{N}}_\pm$ of $\Gamma_\pm$, for which the local stable and local unstable manifolds $W^s(\Gamma_\pm, \tilde{\mathcal{N}}_\pm, f_0)$ and $W^u(\Gamma_\pm, \tilde{\mathcal{N}}_\pm, f_0)$ of $\Gamma_\pm$ are well defined and such that $\overline{\tilde{\mathcal{N}}_-} \cap \overline{\tilde{\mathcal{N}}_+} = \varnothing$ if $\Gamma_+ \neq \Gamma_-$. In the case where $\Gamma_+ = \Gamma_-$, $\tilde{\mathcal{N}}_+ = \tilde{\mathcal{N}}_-$ can be chosen so that $\overline{W^u(\Gamma_+, \tilde{\mathcal{N}}_+, f_0)} \cap \overline{W^s(\Gamma_+, \tilde{\mathcal{N}}_+, f_0)} = \Gamma_+$.

We would like to perturb f_0 to deform the global unstable manifold $W^u(\Gamma_-, f_0)$ without changing the dynamics in $\tilde{\mathcal{N}}_\pm$. By construction, the part of $W^u(\Gamma_-, f_0)$ outside $\overline{\tilde{\mathcal{N}}_-} \cup \overline{\tilde{\mathcal{N}}_+}$ is a non-empty open subset of $W^u(\Gamma_-, f_0)$. The difficulty is that the nonlinearity f sees the phase space X^α only through the projections by the evaluation map

$$\mathrm{Ev} \;:\; (x, \varphi) \in \Omega \times X^\alpha \;\longmapsto\; (x, \varphi(x), \nabla\varphi(x)) \in \Omega \times \mathbb{R} \times \mathbb{R}^d \;. \tag{6.1}$$

We need to be sure that for all $u(t)$ connecting Γ_- to Γ_+, not only $u(t)$ goes outside $\tilde{\mathcal{N}}_- \cup \tilde{\mathcal{N}}_+$ but also $\mathrm{Ev}(u(t))$ goes outside $\mathrm{Ev}(\tilde{\mathcal{N}}_- \cup \tilde{\mathcal{N}}_+)$.

The local unstable manifold $W^u(\Gamma_-, \tilde{\mathcal{N}}_-, f_0)$ is an embedded finite dimensional manifold and its boundary $\Sigma^u_- = \partial W^u(\Gamma_-, \tilde{\mathcal{N}}_-, f_0)$ is a compact set such that, for all trajectory $\tilde{u}(t)$ belonging to the global unstable manifold $W^u(\Gamma_-, f_0) \backslash \Gamma_-$, there

exists a time $\tilde{t}_0 \in \mathbb{R}$ such that $\tilde{u}(\tilde{t}_0) \in \Sigma^u_-$. Let $\sigma \in \Sigma^u_-$ and consider the trajectory $u_\sigma(t) = S_{f_0}(t)\sigma$, solution of (1.1) with initial data $u_\sigma(t=0) = \sigma$ and nonlinearity $f = f_0$. For all $t < 0$, $u_\sigma(t)$ belongs to the local unstable manifold $W^u(\Gamma_-, \tilde{\mathcal{N}}_-, f_0)$. Moreover, due to Proposition 5.2, there exists $(x_\sigma, t_\sigma) \in \Omega \times \mathbb{R}_+$ such that $(u_\sigma(x_\sigma,t_\sigma), \nabla u_\sigma(x_\sigma,t_\sigma)) \neq (p_\pm(x_\sigma,t), \nabla p_\pm(x_\sigma,t))$ for all $t \in \mathbb{R}$, or equivalently

$$\{(x_\sigma, u_\sigma(x_\sigma,t_\sigma), \nabla u_\sigma(x_\sigma,t_\sigma))\} \ \cap \ \mathrm{Ev}\big(\{x_\sigma\} \times (\Gamma_- \cup \Gamma_+)\big) \ = \ \varnothing \ .$$

Since $\{(x_\sigma, u_\sigma(x_\sigma,t_\sigma), \nabla u_\sigma(x_\sigma,t_\sigma)\}$ and $\{x_\sigma\} \times (\Gamma_- \cup \Gamma_+)$ are compact sets and since Ev is continuous because X^α is continuously embedded in $\mathcal{C}^1(\Omega)$, we can find $r_\sigma > 0$ and $\rho_\sigma > 0$ and neighborhoods $\mathcal{N}_{\sigma,\pm} \subset \tilde{\mathcal{N}}_\pm$ of $\Gamma_\pm$ in X^α such that

$$\mathcal{U}_\sigma \ := \ B_\Omega(x_\sigma, r_\sigma) \times B_{\mathbb{R}^{d+1}}((u_\sigma(x_\sigma,t_\sigma), \nabla u_\sigma(x_\sigma,t_\sigma)), \rho_\sigma)$$

and $\mathcal{N}_{\sigma,\pm}$ satisfy

$$\min \ \big\{\|\xi_1 - \xi_2\|_{\Omega\times\mathbb{R}^{d+1}} \mid \xi_1 \in \mathcal{U}_\sigma \text{ and } \xi_2 \in \mathrm{Ev}\big(B_\Omega(x_\sigma,r_\sigma) \times (\mathcal{N}_{\sigma,-} \cup \mathcal{N}_{\sigma,+})\big\} \ > \ 0 \ .$$

By continuity of Ev and of the flow $S_f(t)$ with respect to the initial data and with respect to $f \equiv f_0 + g$, there are a neighborhood $\mathcal{V}_\sigma$ of σ in X^α and a neighborhood $\mathcal{W}_\sigma$ of 0 in $\mathfrak{C}^r$ such that for any $\sigma' \in \mathcal{V}_\sigma$ and $g \in \mathcal{W}_\sigma$, the trajectory $S_{f_0+g}(t)\sigma'$ has a projection $\mathrm{Ev}(\{x_0\} \times S_{f_0+g}(t)\sigma')$ contained in $\mathcal{U}_\sigma$ for a non-empty open lapse of time.

We can proceed as above for any point $\sigma \in \Sigma^u_-$. By compactness of Σ^u_-, it can be covered by a finite collection $\mathcal{V}_{\sigma_1}, \ldots, \mathcal{V}_{\sigma_N}$ of neighborhoods of points $\sigma_1, \ldots, \sigma_N$. We set $\mathcal{N}_\pm = \cap_n \mathcal{N}_{\sigma_n,\pm}$ and $E = \cup_n \overline{\mathcal{U}_{\sigma_n}}$. Notice that E is a finite union of closed balls. Thus, $\mathcal{C}^r_0(E)$ is a well-defined Banach subspace of $\mathfrak{C}^r$ and we set $\mathcal{W}^r = \cap_n \mathcal{W}_{u_n} \cap \mathcal{C}^r_0(E)$.

To summarize, our construction satisfies the following properties (see Figure 2):

1. The neighborhoods $\mathcal{N}_\pm$ are small enough such that the local stable and local unstable manifolds $W^s(\Gamma_+, \mathcal{N}_+, f_0)$ and $W^u(\Gamma_-, \mathcal{N}_-, f_0)$ are well defined. Moreover, these local manifolds do not intersect if $\Gamma_+ \neq \Gamma_-$, or have an intersection reduced to Γ if $\Gamma_+ = \Gamma_- = \Gamma$.

2. For any $f = f_0 + g$ where $g \in \mathcal{W}^r$ (in particular g is supported in the set E), the flow $S_{f_0+g}(t)$ is equal to the flow of $S_{f_0}(t)$ in $\mathcal{N}_\pm$. In particular, we have $W^s(\Gamma_+, \mathcal{N}_+, f_0) = W^s(\Gamma_+, \mathcal{N}_+, f_0 + g)$ and $W^u(\Gamma_-, \mathcal{N}_-, f_0) = W^u(\Gamma_-, \mathcal{N}_-, f_0 + g)$ and the properties of 1. still hold when f_0 is perturbed to $f = f_0 + g$.

3. For any $f = f_0 + g$ where $g \in \mathcal{W}^r$, for any global trajectory $u(t) = S_{f_0+g}(t)u(0)$ of the unstable manifold of Γ_- (Γ_- excluded), there exists $(x_0, t_0) \in \Omega \times \mathbb{R}$ and $r > 0$ such that for all $(x,t) \in B_{\Omega\times\mathbb{R}}((x_0,t_0), r)$, $(x, u(x,t), \nabla u(x,t))$ belongs to the interior of E (which is the set where the perturbations $g \in \mathcal{W}^r$ can be constructed) and not in $\mathrm{Ev}(\{x\} \times \mathcal{N}_\pm)$.

Second step: Application of the Sard-Smale transversality Theorem B.3
If $f = f_0 + g$, where g is close to 0 in $\mathcal{C}^r_0(E)$, then f is close to f_0 in $\mathfrak{C}^r$ (equipped

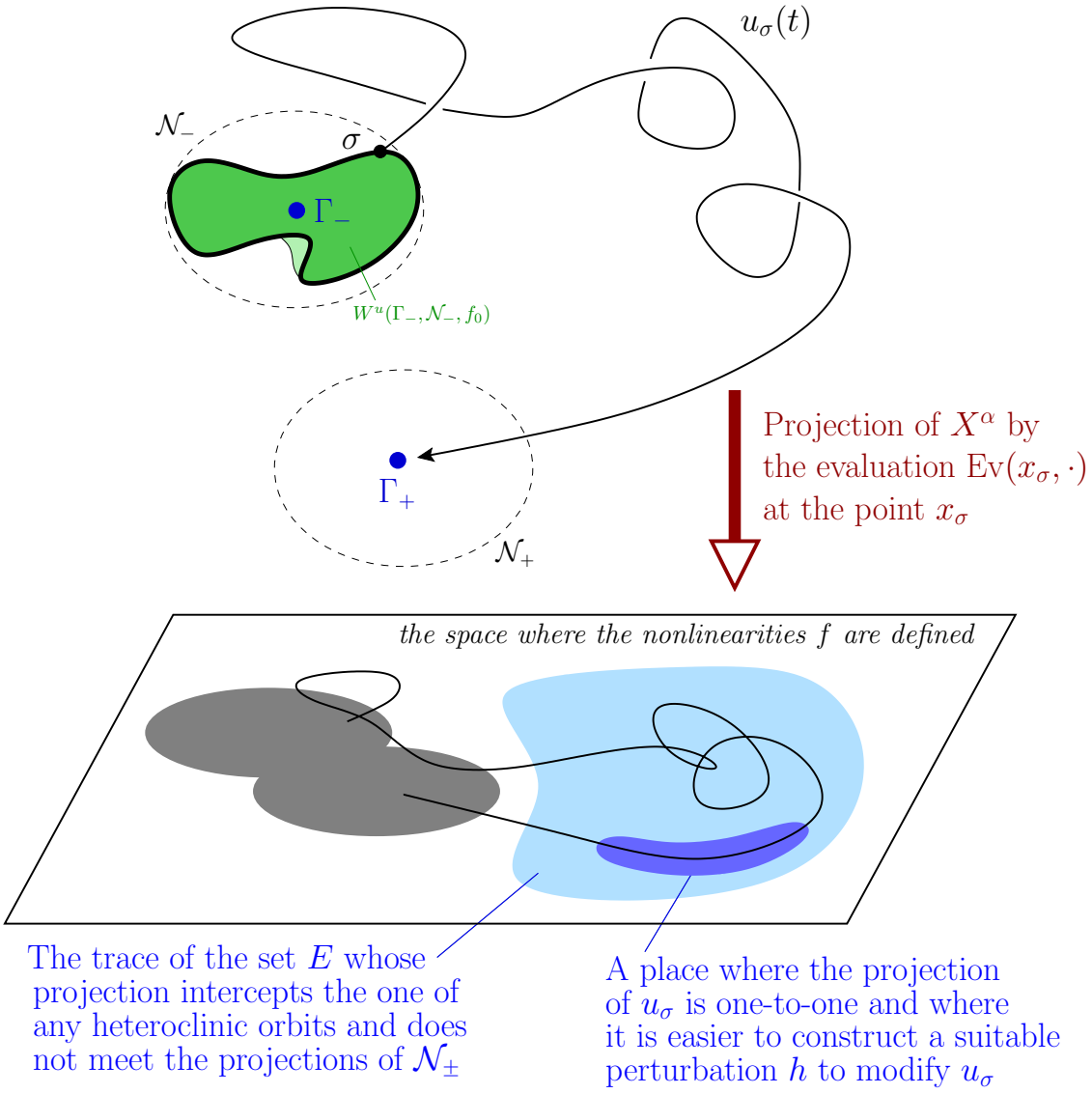

Figure 2: *A figure illustrating the proof of Proposition 6.2. In the phase space, $\mathcal{N}_\pm$ are small enough to define local dynamics and are disjoints in the heteroclinic case. The nonlinearity sees the dynamics only via the projections* $\mathrm{Ev}(x_\sigma, \cdot)$ *by evaluating $(u_\sigma, \nabla u_\sigma)$ at a point x_σ. In the first step, we construct a set E whose projections do not meet the ones of the neighborhoods $\mathcal{N}_\pm$ of the closed orbits and such that, for all connecting orbit u_σ, there is a point x_σ such that the evaluation of $u_\sigma(t)$ at this point enters in E for an open lapse of times. The perturbation g of the nonlinearity f_0 will be supported on this set E to be able to modify any connecting orbits without modifying the closed orbits. Moreover, in the final step of our proof, we will also localize the perturbation in the place where the projection of $u_\sigma(t)$ has no self-intersection and where the modification of $u_\sigma(t)$ by a perturbation of the nonlinearity is easier to understand.*

with the Whitney topology). Moreover, by construction, for any $f = f_0 + g$ with $g \in \mathcal{W}^r$, $S_f(t)$ has the same dynamics as $S_{f_0}(t)$ in the neighborhoods $\mathcal{N}_\pm$ of $\Gamma_\pm$. Therefore, Proposition 6.2 holds if we can find a function $g \in \mathcal{W}^r \subset \mathcal{C}^r_0(E)$ as close to 0 as wanted such that $W^u(\Gamma_-, f_0 + g)$ intersects $W^s(\Gamma_+, \mathcal{N}_+, f_0)$ transversally.

We recall that we did not assume global existence of solutions and thus the solutions in the unstable manifold may blow up. To overcome this technical problem, for all $m \geqslant 1$, we introduce the sets

$$\mathcal{N}_-^m = \{u_0 \in \mathcal{N}_- \ / \ \forall g \in \mathcal{W}^r \ , \ S_{f_0+g}(t)u_0 \text{ is well defined for all } t \in [0, m]\} \ .$$

The global orbit Γ_- is obviously contained in $\mathcal{N}_-^m$ and we recall that ii) of Proposition 2.1 implies that $\mathcal{N}_-^m$ is open, in other words $\mathcal{N}_-^m$ is a neighborhood of Γ_- contained in $\mathcal{N}_-$. Moreover, we have

$$\forall g \in \mathcal{W}^r \ , \quad W^u(\Gamma_-, f_0 + g) = \cup_{m\in\mathbb{N}} S_f(m) W^u(\Gamma_-, \mathcal{N}_-^m, f_0 + g) \ .$$

To prove Proposition 6.2, it is sufficient to show that for any $m \in \mathbb{N}$, there exists a generic subset of functions $g \in \mathcal{W}^r$ such that $S_{f_0+g}(m)W^u(\Gamma_-, \mathcal{N}_-^m, f_0)$ intersects $W^s(\Gamma_+, \mathcal{N}_+, f_0)$ transversally. Indeed the intersection of all these generic subsets is generic and hence dense in $\mathcal{W}^r$ and consists in functions $f = f_0 + g$ such that $W^u(\Gamma_-, f_0 + g)$ intersects $W^s_{loc}(\Gamma_+, f_0 + g)$ transversally.

To show this property, we are going to use the Sard-Smale transversality theorem B.3 in Appendix as follows. Let $m \geqslant 1$, let $\mathcal{M} = W^u(\Gamma_-, \mathcal{N}_-^m, f_0)$, $Y = X^\alpha$ and $\mathcal{W} = W^s(\Gamma_+, \mathcal{N}_+, f_0)$. Let $\Lambda = \mathcal{W}^r$ and $\hat{\Lambda} = \mathcal{C}^\infty_0(E) \cap \mathcal{W}^r$. We define the mapping

$$\Phi \ : \ \left(\begin{array}{ccc} \mathcal{M} \times \Lambda & \longrightarrow & Y \\ (u_0, g) & \longmapsto & S_{f_0+g}(m)u_0 \end{array} \right)$$

Notice that $S_{f_0+g}(m)W^u(\Gamma_-, \mathcal{N}_-^m, f_0)$ intersects $W^s(\Gamma_+, \mathcal{N}_+, f_0)$ transversally if and only if $\Phi(., g)$ intersects $W^s(\Gamma_+, \mathcal{N}_+, f_0)$ transversally. Thus, due to the above discussions, the conclusion of Theorem B.3 in this framework will complete the proof of Proposition 6.2. Hypothesis i) of Theorem B.3 is a consequence of the assumption $r > dim\, W^u(\Gamma_-) - codim\, W^s(\Gamma_+)$ made at the beginning of this proof and of the regularity of the parabolic flow with respect to the parameters. Thus, Hypothesis ii) is the only assumption which remains to be verified.

Third step: checking Hypothesis ii) of Theorem B.3
Let $u_0 \in W^u(\Gamma_-, \mathcal{N}_-^m, f_0) \backslash \Gamma_-$ and $f = f_0 + g$, where $g \in \mathcal{W}^r$. If $S_f(m)u_0$ does not belong to $W^s(\Gamma_+, \mathcal{N}_+, f_0)$, then ii) is trivially satisfied. If $S_f(m)u_0$ belongs to $W^s(\Gamma_+, \mathcal{N}_+, f_0)$, we set $u(t) = S_f(t)u_0$ and we remark that, since $W^s(\Gamma_+, \mathcal{N}_+, f_0) = W^s(\Gamma_+, \mathcal{N}_+, f)$, $u(t)$ is a global solution and $u(t) \in W^s(\Gamma_+, \mathcal{N}_+, f)$ for all $t \geqslant m$.

It remains to show that Φ is transversal to $\mathcal{W}$ in X^α at the point u_0, we have to compute

$$D\Phi(u_0, g).(v_0, h) \ = \ D_u\Phi(u_0, g).v_0 \ + \ D_g\Phi(u_0, g).h \ .$$

Let us consider the second term and let $v(t)$ be the derivative of $u(t)$ with respect to a variation h of the nonlinearity g. By differentiating Equation (1.1), we have that v solves

$$\partial_t v = \Delta v + h(x, u, \nabla u) + f'_u(x, u, \nabla u).v + f'_{\nabla u}(x, u, \nabla u).\nabla v$$

with $v(t=0)=0$. We denote by $U(t,s)$ the family of evolution operators generated by the equation (2.11) with coefficients given by (2.12), which is the linearization of the nonlinear equation along the trajectory $u(t)$. Using the variation of constants formula, we get

$$D_g\Phi(u_0,g).h = \int_0^m U(m,s)h(.,u(.,s),\nabla u(.,s))\ ds\ . \tag{6.2}$$

In a similar way, we obtain that $D_u\Phi(u_0,g).v_0 = U(m,0)v_0$ whose range is the tangent space $T_{u(m)}W^u(\Gamma_-,f)$.

We claim that the image of $D_g\Phi(u_0,g)$ is dense in X^α and we postpone the proof of this density in a final step below. Assuming this property, let us check Hypothesis ii) of Theorem B.3 using Definition B.2. First notice that $T_{u(m)}\mathcal{W} = T_{u(m)}W^s(\Gamma_+,\mathcal{N}_+,f)$ is a closed subspace with finite codimension (see Theorem 3.5). To show that the image of $D\Phi(u_0,g)$ contains a closed complementary subspace of $T_{u(m)}\mathcal{W}$ in X^α, it is sufficient to reach a given finite number of independent vectors $\phi_1,\dots,\ \phi_p$ outside $T_{u(m)}\mathcal{W}$. This is obviously implied by the density of the image of $D_g\Phi(u_0,g)$ in X^α. Since $\mathrm{span}(\phi_1,\dots,\phi_p)\oplus T_{u(m)}\mathcal{W} = X^\alpha$, we have that $T_{u_0,g}\mathcal{M}\times\Lambda = D\Phi(u_0,g)^{-1}(T_{u(m)}\mathcal{W})\oplus\mathrm{span}(\psi_1,\dots,\psi_p)$ where $D\Phi(u_0,g).\psi_j = \phi_j$. By continuity, we directly have that $D\Phi(u_0,g)^{-1}(T_{u(m)}\mathcal{W})$ is closed and its complementary space is also closed because of its finite-dimensionality.

Fourth step: the image of $D_g\Phi(u_0,g)$ is dense in X^α
The operator $(-\Delta_D)^\alpha$ is a homeomorphism from X^α into X. Hence, it is sufficient to show that for any non-zero $\psi_m\in X^*$, there exists $h\in\mathcal{C}_0^\infty(E)$ such that

$$\langle\,\psi_m\,|\,(-\Delta_D)^\alpha D_g\Phi(u_0,g).h\,\rangle_{X^*,X}\neq 0\ .$$

Hence, using the expression of $D_g\Phi(u_0,g).h$ given by (6.2), we have to find a function $h\in\mathcal{C}_0^\infty(E)$ such that

$$\int_0^m \langle\,U(m,s)^*((-\Delta_D)^\alpha)^*\psi_m\,|\,h(.,u(.,s),\nabla u(.,s))\,\rangle_{X^{\alpha,*},X^\alpha}\ ds\neq 0\ .$$

Now, we use Proposition 2.4: $\psi(s)=U(m,s)^*((-\Delta_D)^\alpha)^*\psi_m$ is well defined in X^* and is a solution in $\mathcal{C}^0((0,m),\mathcal{C}^1(\overline{\Omega}))$ of (2.14) with a and b as in (2.12). In particular, ψ satisfies the unique continuation property stated in Proposition 2.8: in any open set of $\Omega\times(0,m)$, there exists (x,t) such that $\psi(x,t)\neq 0$.

By considering the constructions made during the first step (see the third of the properties recalled at the end), we know that there exists a non-empty open set $\mathcal{U}\subset\Omega\times\mathbb{R}$ such that for all $(x_0,t_0)\in\mathcal{U}$, $(x_0,u(x_0,t_0),\nabla u(x_0,t_0))$ belongs to the interior of the set E and is not in $\mathrm{Ev}(\{x\}\times\mathcal{N}_\pm)$. In particular, $u(x_0,t_0)$ cannot belongs to $\mathcal{N}_\pm$ and thus $t_0\in(0,m)$ because we have already noticed that $u(t)\in W^s(\Gamma_+,\mathcal{N}_+,f)$ for all $t\geqslant m$ and because $u(t)\in W^u(\Gamma_-,\mathcal{N}_-,f)$ for all $t\leqslant 0$ by definition of Φ and u. We now apply Proposition 6.1, noticing that the unique continuation property for ψ yields the existence of $(x_0,t_0)\in\mathcal{U}$ such that $\psi(x_0,t_0)\neq 0$. We obtain a function $h\in\mathcal{C}_0^\infty(E)$ such that

$$\int_{\mathbb{R}}\int_\Omega \psi(x,s)h(x,u(x,s),\nabla u(x,s))\,dxds\ \neq\ 0\ .$$

It remains to notice Proposition 6.1 guarantees that $h \circ u$ is supported in $\mathcal{U}$ and that the above discussion shows that $\mathcal{U} \subset \Omega \times (0, m)$. Thus, for any $\psi_m \in X^*$, we may replace the domain $\mathbb{R} \times \Omega$ by $[0, m] \times \Omega$ in the above integral and, in conclusion, we have obtained h such that

$$\langle \psi_m \,|\, (-\Delta_D)^\alpha D_g\Phi(u_0, g).h \rangle_{X^*,X} \;=\; \int_0^m \langle \psi(s) \,|\, h(., u(., s), \nabla u(., s)) \rangle_{X^*,X}\, ds \;\neq\; 0 \;.$$

which implies that the image of $D_g\Phi(u_0, g)$ is dense in X^α. ▫

6.2 Proof of Theorem 1.1

The proof of our main theorem easily follows from the perturbation result of Proposition 6.2.

Let $f_0 \in \mathfrak{C}^r$ be given and let $\mathcal{C}_0^\pm$ be two hyperbolic critical elements. By Theorems 3.3, 3.5 and 3.7, there exists a neighborhood $\mathcal{O}$ of f_0 such that $\mathcal{C}_0^\pm$ are associated with two families $\mathcal{C}^\pm(f)$ of hyperbolic critical elements depending smoothly on f. Moreover, the corresponding local stable and unstable manifolds $W^u_{loc}(\mathcal{C}^-(f))$ and $W^s_{loc}(\mathcal{C}^+(f))$ also depend smoothly on f.

Let $m \in \mathbb{N}$ be given and let

$$\begin{aligned} W^u_m(\mathcal{C}^-(f)) = \{&u \in X^\alpha \text{ such that } \|u\|_{X^\alpha} < m \text{ and there exists} \\ &t \in [0, m] \text{ and } \; u_0 \in W^u_{loc}(\mathcal{C}^-(f)) \text{ such that } u = S_f(t)u_0\} \;. \end{aligned}$$

The set $W^u_m(\mathcal{C}^-(f))$ is a bounded open subset of the global unstable manifold $W^u(\mathcal{C}^-(f))$ and an immersed manifold of X^α. Also notice that $W^u_m(\mathcal{C}^-(f))$ depends smoothly on f. We consider the sets

$$\mathfrak{G}_m = \{f \in \mathcal{O} \mid W^u_m(\mathcal{C}^-(f)) \pitchfork W^s_{loc}(\mathcal{C}^+(f))\} \;.$$

The smooth dependences yield that $\mathfrak{G}_m$ are open subsets of $\mathcal{O}$ (see Appendix A to understand what these smooth dependences mean with respect to the Whitney topology). We claim that the sets $\mathfrak{G}_m$ are also dense. Indeed, X^α is embedded in $\mathcal{C}^1$ and so its ball $\{u \mid \|u\|_{X^\alpha} \leqslant m\}$ provides values $(x, u(x), \nabla u(x))$ uniformly bounded by some constant $C(m)$. For any $f \in \mathcal{O}$, we may perturb f to $\tilde{f}$ such that $\tilde{f}$ is of class $\mathcal{C}^\infty$ in the ball of radius $C(m)$ and equal to f outside the ball of radius $C(m) + 1$. In this way, $\tilde{f}$ is as close as wanted to f in the $\mathfrak{C}^r$ Whitney topology. Moreover, any solution u in $W^u_m(\mathcal{C}^-(\tilde{f}))$ stays in the place where $\tilde{f}$ is a $\mathcal{C}^\infty-$non-linearity. Applying Proposition 6.2, we may perturb $\tilde{f}$ to obtain a non-linearity in $\mathfrak{G}_m$.

Since the sets $\mathfrak{G}_m$ are open and dense in $\mathcal{O}$, by setting $\mathfrak{G} = \cap_m \mathfrak{G}_m$, we obtain the generic set of Theorem 1.1.

7 Further generalizations of the generic transversality stated in Theorem 1.1

Our above arguments are not exactly specific to Equation (1.1). We may easily check the following generalizations.

Other geometries
Dirichlet boundary conditions are not mandatory, we may choose Neumann ones or Robin ones. We may also consider other flat geometries such as Ω being a torus or a cylinder.

We may also add coefficients to the Laplacian operator Δ, typically considering the Laplace-Beltrami operator $\frac{1}{\sqrt{g}}\mathrm{div}(\sqrt{g}g_{ij}\nabla\cdot)$ associated to a metric g. However, notice that part of our results, e.g. Theorem 4.1, require smooth coefficients and thus g needs to be smooth. Thus, we may generalize Theorem 1.1 in the case where Ω is a bounded $\mathcal{C}^\infty-$submanifold of $\mathbb{R}^n$, as a sphere for example.

Systems of parabolic equations
Instead of considering the scalar parabolic equation (1.1), we consider a system of n parabolic equations as follows. We keep the same space $X = L^p(\Omega)$, $p > d$, and the same Δ_D Laplacian operator with homogeneous Dirichlet boundary conditions. Like in the introduction, we keep $\alpha \in (1/2 + d/2p, 1)$, so that $X^\alpha = D((-\Delta_D)^\alpha \hookrightarrow W^{2\alpha,p}(\Omega)$ is compactly embedded in $\mathcal{C}^1(\overline{\Omega})$. Let $n \in \mathbb{N}$, $n \geqslant 1$. We consider the system of parabolic equations

$$\left\{\begin{array}{ll} U_t(x,t) = \Delta U(x,t) + F(x,U(x,t),\nabla U(x,t)), & (x,t) \in \Omega \times (0,+\infty) \\ U(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty) \\ U(x,0) = U_0(x) \in X_n^\alpha \equiv (X^\alpha)^n, & \end{array}\right. \tag{7.1}$$

where $F \equiv (f_1, f_2, \ldots, f_n) \in \mathcal{C}^r(\overline{\Omega}\times\mathbb{R}^n\times\mathbb{R}^{nd}, \mathbb{R}^n)$, $r \geqslant 2$, and where $U \equiv (u_1(x,t), u_2(x,t), \ldots, u_n(x,t)$ belongs to $\mathbb{R}^n$. As in the case $n = 1$, the system (7.1) generates a local dynamical system $S_n(t) \equiv S_{n,F}(t)$ on X_n^α. This (local) dynamical system $S_{n,F}(t)$ satisfies all the smoothing properties of Section 2 as well as the dynamical systems properties given in Section 3. The strong unique continuation property of Proposition 2.10 still holds and is proved in [10, Theorem 2.2] (see also [28]). The singular nodal sets properties as given in Theorem 4.1 and its Corollary 4.2 are still true and are proved with the same arguments (see also [10, Theorem 2.3]). These facts allow us to generalize Theorem 1.1 to the system (7.1).

Genericity for other topologies
We have chosen here to consider the genericity in $\mathfrak{C}^r$ by endowing $\mathcal{C}^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ with the Whitney topology (see the precise definition in Appendix A). Indeed this topology seems to be the most usual one for this kind of question concerning generic dynamics. Moreover, it also seems to be the most delicate topology since it has only a few nice properties (for example the closed sets are not the sequentially closed sets and in particular $\mathfrak{C}^r$ is not a metric space). However, Theorem 1.1 also holds if we endow $\mathcal{C}^r(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ with other reasonable topology. We may for example consider $\mathcal{C}^r_b(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$, the set of bounded $\mathcal{C}^r-$functions on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ endowed with the supremum $\mathcal{C}^r$-norm. We may also extend the previous metric by considering unbounded $\mathcal{C}^r-$functions but defining their neighborhoods with bounded perturbation only (in other words, we may say that if $f - g$ or one of its r first derivatives is unbounded, then f and g are at infinite distance). In any case, the conclusions of Theorem 1.1 remain valid since, in the proofs, we in fact only consider non-linearities via a bounded set of $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$, where all these

topologies are equivalent (see Appendix A).

Some open problems
To conclude, let us mention cases where the generalization is not straightforward and remains an open problem.

We may wonder if Theorem 1.1 is still true for systems of parabolic equations if, instead of considering mappings $F(x, U, \nabla U)$ in the set $\mathcal{C}^r(\overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}^{nd}, \mathbb{R}^n)$, one considers only mappings $F(x, U) \in \mathcal{C}^r(\overline{\Omega} \times \mathbb{R}^n, \mathbb{R}^n)$ depending only on x and of the value of U. Since the Hausdorff dimension of the nodal set is, in general, larger by 1 than the dimension of the singular nodal set (see a simple example in [10, Section 9]), the one-to-one properties of global trajectories, as given in Section 5, can be false if $F = F(x, U)$ and are no longer consequences of Theorem 4.1 (see [10, Section 9]).

We can also wonder if one can extend Theorem 1.1 to the case where the Laplacian operator is replaced by a $2m$-th order homogeneous elliptic linear operator. In this case, in Equation (1.1), we replace the non-linearity $f(x, u, \nabla u)$ by a non-linearity $f^*(x, u, D_x u, D_x^2 u, ..., D_x^{2m-1} u)$ depending on the values of u, $D_x u$, ...,$D_x^{2m-1} u$. If the strong unique continuation property of Proposition 2.10 holds, then, arguing exactly as in the proof of Theorem 4.1, one shows that the statement of this theorem is still true provided we replace the singular nodal set by

$$(TNS) = \{(x_0, t_0) \in \Omega \times I \,|\, \text{there does not exist } \tau \in J \text{ such that } (v, D_x v, D_x^2 v, ..., D_x^{2m-1} v)(x_0, t_0, \tau) = (0, 0, ..., 0)\}$$

Unfortunately, the strong unique continuation property for the parabolic equation with higher order elliptic operators is not always true (concerning the elliptic equation, see [27] and [52] for example). For this reason, we cannot state here a generalization of Theorem 1.1 for higher-order parabolic equation.

A Appendix: The Whitney topology

If we want to prove generic properties for the parabolic equation (1.1) with respect to the non-linearity f, we need to equip the space of nonlinear functions f with a topology. Let $E \subset \mathbb{R}^n$, $n \geqslant 1$, by $f \in \mathcal{C}^r(E, \mathbb{R})$, we mean that f is r times differentiable in the set E and that these derivatives are continuous. We do not a priori endow $\mathcal{C}^r(E, \mathbb{R})$ with any topology and we do not assume that f or its derivatives are bounded.

In this article, we consider $E = \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$ which is unbounded in $\mathbb{R}^{2d+1}$. Since we do not want to exclude unbounded non-linearities, we cannot equip $\mathcal{C}^r(E, \mathbb{R})$ with the classical $\mathcal{C}^r$-topology.

Definition A.1. *For any $r \in \mathbb{N}$, we denote by $\mathfrak{C}^r \equiv \mathfrak{C}^r(E, \mathbb{R})$ the space $\mathcal{C}^r(E, \mathbb{R})$ endowed with the Whitney topology, that is the topology generated by the neighborhoods*

$$\{g \in \mathcal{C}^r(E, \mathbb{R}) \mid |D^i f(y) - D^i g(y)| \leqslant \delta(y) \ , \quad \forall i \in \{0, 1, \ldots, r\} \ , \quad \forall y \in E\} \ ,$$

where f is any function in $\mathcal{C}^r(E, \mathbb{R})$ and δ is any positive continuous function.

We emphasize that, if E is bounded, then the Whitney topology coincides with the classical $\mathcal{C}^r$-topology and thus $\mathfrak{C}^r(E,\mathbb{R})$ is a Banach space equipped with the classical norm $\|f\| = \sup_{i=0,1,\dots,r} \|f^{(i)}\|_{L^\infty}$. However, if $E = \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d$, the neighborhoods of a function f in the Whitney topology cannot be generated by a countable number of them. As a consequence, this topology is not metrizable and open or closed sets cannot be characterized by sequences. In order to give an idea about the uncountable conditions imposed by the Whitney topology, we recall that a sequence of functions (f_n) converges to a function f in the Whitney topology if and only if there is a compact set $K \subset E$ such that $f_n \equiv f$ in $E \backslash K$ for any $n \in \mathbb{N}$, but for a finite number of them, and such that (f_n) converges to f in the space $\mathcal{C}^r(K,\mathbb{R})$, equipped with the classical topology of uniform convergence of the functions together with their derivatives up to order r. This means that the Whitney topology imposes an uncountable number of conditions of proximity outside compact sets and thus a sequence has to be constant there in order to be convergent.

As already written in Section 7, we could have chosen a simpler topology, but the Whitney topology seems to be the most usual one. In order to overcome several technical problems due to this topology, we make more precise some arguments in this appendix. We omit the corresponding problems during the main proofs of this paper to avoid too heavy proofs. However, if all the technical details are written, the interested reader will notice that we easily deal with the fact that the Whitney topology does not generate a Banach space as follows.

Genericity and Baire property: The main purpose of this paper is to obtain the genericity of the transversality of heteroclinic and homoclinic orbits. The notion of generic sets, that are sets containing a countable intersection of dense open sets, is important because it provides a nice notion of large subset. However, the acceptance of this notion is mainly related to the Baire property, that is the fact that the countable intersection of generic sets is generic. A space satisfying the Baire property is called a Baire space. Complete spaces, and in particular Banach ones, are Baire spaces. But when E is unbounded, $\mathfrak{C}^r(E,\mathbb{R})$ with its Whitney topology is even not metrizable. Thus, it is important to emphasize that it is at least a Baire space, implying that the genericity is still a meaningful concept (see [19] or [33] for example).

Smooth dependences, open or dense subsets and other abuses of notations: When E is unbounded, since $\mathfrak{C}^r(E,\mathbb{R})$ is not metrizable, we can speak about continuous dependence on $f \in \mathfrak{C}^r(E,\mathbb{R})$ but not about smooth dependence, even not about derivatives with respect to f. We sometimes use the following abuse of notation. Consider K a compact subset of E and define P as the canonical projection from $\mathfrak{C}^r(E,\mathbb{R})$ onto $\mathfrak{C}^r(K,\mathbb{R})$, that is $Pf := f_{|K}$ is the restriction of f to K. Now, as already noticed, $\mathfrak{C}^r(K,\mathbb{R})$ endowed with the Whitney topology is equivalent to the Banach space $\mathcal{C}^r(K,\mathbb{R})$ endowed with the classical $\mathcal{C}^r-$norm. Consider a function Φ depending on f via the values in K only. We may thus associate with Φ defined in $\mathfrak{C}^r(E,\mathbb{R})$ a function $\tilde{\Phi}$ defined in $\mathfrak{C}^r(K,\mathbb{R})$ and then it is relevant to say that $\tilde{\Phi}$ depends smoothly on Pf. In this case, we may use an abuse of notations by saying that Φ depends smoothly on f instead of saying that $\tilde{\Phi}$ depends smoothly on Pf

(notice that, rigorously, we should not even say that Pf depends smoothly on f).

At this point, it is important to notice that, the restriction operator

$$P : \mathfrak{C}^r(E, \mathbb{R}) \to \mathfrak{C}^r(K, \mathbb{R}) \text{ with } K \subset E \text{ compact and } E \subset \mathbb{R}^n$$

is continuous, open and surjective. Continuity is clear and surjectivity follows from the Whitney extension theorem (see [1]), or a simpler result if $r = 0$ or K is a regular subdomain for which the extension is easily constructed. Openness follows from the following argument: consider $g \in \mathfrak{C}^r(K, \mathbb{R})$ close to 0, extend g to $f \in \mathfrak{C}^r(E, \mathbb{R})$ and truncate f by multiplying it by a smooth function χ with $0 \leqslant \chi \leqslant 1$, $\chi_{|K} \equiv 1$ and $\chi \equiv 0$ outside a small neighborhood of K. This provides a function $\chi f \in \mathfrak{C}^r(E, \mathbb{R})$ with $P(\chi f) = g$ and χf as close to 0 in $\mathfrak{C}^r(E, \mathbb{R})$ as wanted as soon as g is small enough. Thus, the image by P of any neighborhood of 0 contains a neighborhood of 0.

The surjectivity of P enables to define the above functional $\tilde{\Phi}$ in $\mathfrak{C}^r(K, \mathbb{R})$ because to each function $g \in \mathfrak{C}^r(K, \mathbb{R})$ indeed corresponds a class of equivalence of functions $f \in \mathfrak{C}^r(E, \mathbb{R})$ with $Pf = g$. The openness is useful to show that a property is open in $\mathfrak{C}^r(E, \mathbb{R})$ if this property depends on the value of f in K only: if the property is open in $\mathfrak{C}^r(K, \mathbb{R})$ with the above abuse of notation, then it is open in $\mathfrak{C}^r(E, \mathbb{R})$. Together, these properties show that, with the abuse of notation, if a property is open and dense (resp. generic) in $\mathfrak{C}^r(K, \mathbb{R})$ then it is open and dense (resp. generic) in $\mathfrak{C}^r(E, \mathbb{R})$.

Notice that the above tricks have already been widely used in previous articles (see [7] for instance). Finally, for a further study of the Whitney topology and the comparison with the weak topology, we refer the reader to [19] or [33] for example.

B Appendix: Sard Theorem and Sard-Smale transversality theorems

The Sard theorem ([61]) and the transversality theory (which goes back to Thom [71]) are very useful tools for proving the genericity of a given property in finite dimension. In [67], Smale has shown how to use Fredholm theory to generalize the transversality theorems to infinite-dimensional Banach spaces. There exist different version of this kind of transversality theorems (often called Sard-Smale theorems or Thom theorems) with slight changes in the hypotheses, depending on the framework, in which they are used. We recall here the general framework and the version used in this paper.

Let $\mathcal{M}$ and $\mathcal{N}$ be two differentiable Banach manifolds and let $f : \mathcal{M} \longrightarrow \mathcal{N}$ be a differentiable map. We say that $x \in \mathcal{M}$ is a regular point of f if $Df(x) : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$ is surjective and its kernel splits (that is, has a closed complement in $T_x\mathcal{M}$). A point $y \in \mathcal{N}$ is a regular value of f if any $x \in \mathcal{M}$ such that $f(x) = y$ is a regular point of f. The points of $\mathcal{N}$ which are not regular values are said critical values. The classical theorem of Sard is as follows.

Theorem B.1. *If U is an open set of $\mathbb{R}^p$ and if $f : U \longrightarrow \mathbb{R}^q$ is of class $\mathcal{C}^s$ with $s > max(p-q, 0)$, then, the set of critical values of f in $\mathbb{R}^q$ is of Lebesgue measure zero.*

Using Fredholm operators and a Lyapounov-Schmidt method, Smale has generalized Sard Theorem to infinite-dimensional spaces (for introduction to Fredholm operators, see [6] for example). As a consequence of Smale theorem in [67], many versions of Sard-Smale theorems can be obtained, see [1] and [32] for examples. The versions involving a functional formulation have been used since the pioneer work of Robbin [58] and are very useful in the PDE context where the geometrical arguments may be too difficult to perform, see Theorem B.4 below and [7, 8, 36, 37, 38]. In this article, the transversality of connecting orbits may be proved with a more geometrical version of Sard-Smale theorems. Indeed, we only need to perturb an unstable manifold, which is finite-dimensional, and we may do it far from the periodic orbit, so that the basic framework does not depend on the parameter (see Section 6). This kind of geometrical setting is more difficult to use if we want to prove generic hyperbolicity as discussed in Appendix C below.

We recall the following definition (see [1] for more details).

Definition B.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{C}^1$ Banach manifolds and let $f \in \mathcal{C}^1(\mathcal{M}, \mathcal{N})$. Let $\mathcal{W}$ be a $\mathcal{C}^1$ submanifold of $\mathcal{N}$. The function f is said to be* **transversal** *to $\mathcal{W}$ at a point $x \in \mathcal{M}$ if either $f(x) \notin \mathcal{W}$ or $f(x) \in \mathcal{N}$ and*

i) $D_x f^{-1}(T_{f(x)}\mathcal{W})$ is a closed subspace of $T_x\mathcal{M}$ which admits a closed complementary space,

ii) $D_x f(T_x\mathcal{M})$ contains a closed complement to $T_{f(x)}\mathcal{W}$ in $T_{f(x)}\mathcal{N}$.

We need in this article a slight improvement of Theorem 19.1 of [1]. The idea of replacing the condition on Λ by a condition on a dense subset $\hat{\Lambda}$ only has been already used in [7, 8, 36] for example.

Theorem B.3. *Let $r \geqslant 1$. Let $\mathcal{M}$ be a $\mathcal{C}^r$ separable manifold of dimension n. Let $\mathcal{W}$ be a $\mathcal{C}^r$ manifold of codimension m in a Banach space Y. Let Λ be an open subset of a separable Banach space and let $\hat{\Lambda}$ be a dense subset of Λ. Let $\Phi \in \mathcal{C}^r(\mathcal{M} \times \Lambda, Y)$. Assume that*

i) $r > n - m$,

ii) Φ is transversal to $\mathcal{W}$ at any point $(x, \lambda) \in \mathcal{M} \times \hat{\Lambda}$.

Then, there is a generic set of parameters $\lambda \in \Lambda$ such that the map $x \mapsto \Phi(x, \lambda)$ is everywhere transversal to $\mathcal{W}$.

Proof: Theorem B.3 is proved as Theorem 19.1 of [1]. The only difference is that hypothesis ii) is assumed here only for a dense set of parameters λ. To obtain this improvement from the classical version where ii) is assumed everywhere, we argue as follows. Since $\mathcal{M}$ is separable and finite dimensional, we can find a countable sequence of open subsets $(\mathcal{M}_k)$ such that $\mathcal{M} = \cup \mathcal{M}_k$ and $\overline{\mathcal{M}}_k$ is contained in $\mathcal{M}$ and is compact. Let $\lambda_0 \in \hat{\Lambda}$. Let (λ_p) be a sequence converging to λ_0. Assume that there is a point $x_p \in \overline{\mathcal{M}}_k$ such that Φ is not transversal to $\mathcal{W}$ at (x_p, λ_p). By the compactness property, one may assume that (x_p) converges to $x_0 \in \overline{\mathcal{M}}_k$. Since Φ

is $\mathcal{C}^1$, Φ is not transversal to $\mathcal{W}$ at (x_0, λ_0) which is absurd. Thus, there exists a neighborhood $\mathcal{U}$ of λ_0 such that ii) holds for any $(x, \lambda) \in \mathcal{M}_k \times \mathcal{U}$. By applying [1, Theorem 19.1], we obtain a generic subset $\mathcal{U}_k \subset \mathcal{U}$ such that for any $\lambda \in \mathcal{U}_k$, the map $x \mapsto \Phi(x, \lambda)$ is transversal to $\mathcal{W}$ for any $x \in \mathcal{M}_k$. Since $\hat{\Lambda}$ is dense in Λ, we have a generic subset $\tilde{\mathcal{U}}_k \subset \Lambda$ such that for any $\lambda \in \tilde{\mathcal{U}}_k$, the map $x \mapsto \Phi(x, \lambda)$ is transversal to $\mathcal{W}$ for any $x \in \mathcal{M}_k$. The generic set of parameters appearing in the conclusion of Theorem B.3 is then $\cap_k \tilde{\mathcal{U}}_k$. □

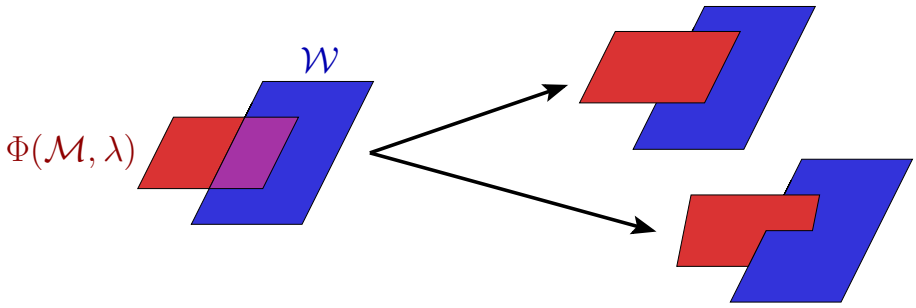

Figure 3: *The geometric idea behind Sard-Smale theorems as Theorem B.3: if perturbing the parameter λ provides enough freedom, a non-transversal intersection between $\Phi(\mathcal{M}, \lambda)$ and $\mathcal{W}$ is generically perturbed into either an empty, and thus transversal, intersection or a non-empty transversal intersection.*

For brief discussions in Appendix C and for the curious reader, we finish by a brief recall of one of the simplest version of Sard-Smale theorem with a functional formulation (see for example [32] for other versions or proofs). Let us recall that a continuous linear map $f : E \longrightarrow F$ between two Banach spaces is a Fredholm map if its image is closed and if the dimension of its kernel and the codimension of its image are finite.

Theorem B.4. *Let $k \geqslant 1$ and let $\mathcal{M}$, $\mathcal{N}$ and Λ be three $\mathcal{C}^k$ Banach manifolds. Let $y \in \mathcal{N}$ and let $\Phi \in \mathcal{C}^k(\mathcal{M} \times \Lambda, \mathcal{N})$. Assume that:*

i) for any $(x, \lambda) \in \Phi^{-1}(\{y\})$, $D_x\Phi(x, \lambda) : T_x\mathcal{M} \to T_y\mathcal{N}$ is a Fredholm map of index i strictly less than k,

ii) for any $(x, \lambda) \in \Phi^{-1}(\{y\})$, $D\Phi(x, \lambda) : T_x\mathcal{M} \times T_\lambda\Lambda \to T_y\mathcal{N}$ is surjective,

iii) $\mathcal{M}$ is separable.

Then, there is a generic set of parameters $\lambda \in \Lambda$ such that for all $x \in \mathcal{M}$ such that $(x, \lambda) \in \Phi^{-1}(\{y\})$, $D_x\Phi(x, \lambda)$ is surjective.

As in Theorem B.3, a similar result holds if Λ is replaced by a dense subset $\hat{\Lambda} \subset \Lambda$ and if Λ is separable (see [7]).

C Appendix: discussion about proving the generic hyperbolicity of periodic orbits

The purpose of this section is unusual. To obtain the genericity of the Kupka-Smale property for the parabolic equation (1.1), it remains to prove the genericity of hyperbolicity of equilibrium points and periodic orbits. The generic hyperbolicity of equilibrium points is proved in [37]. We tried to obtain the generic hyperbolicity of periodic orbits but failed to get a complete proof. In this section, we would like to present some ideas and to point out where there is still a gap in the proof. Maybe this discussion could inspire a motivated reader.

The first proofs of generic hyperbolicity of periodic orbits appeared in [41, 65]. Peixoto in [51] introduced a nice recursion argument, which has been modified in [1] and [43]. Basically, the recursion is as follows. We introduce the sets

$$\mathcal{G}_1(K) = \{f \in \mathfrak{C}^r \mid \text{ any equilibrium point } e \text{ of (1.1) with } \|e\|_{X^\alpha} \leqslant K \text{ is hyperbolic}\}$$

$$\begin{aligned}\mathcal{G}_{3/2}(A,K) = \{f \in \mathcal{G}_1(K) \mid & \text{ any non-constant periodic solution } p(t) \text{ of (1.1)} \\ & \text{with period } T \in (0,A] \text{ such that } \sup_{t\in\mathbb{R}}\|p(t)\|_{X^\alpha} \leqslant K \text{ is non-degenerate}\}\ .\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_2(A,K) = \{f \in \mathcal{G}_1(K) \mid & \text{ any non-constant periodic solution } p(t) \text{ of (1.1)} \\ & \text{with period } T \in (0,A] \text{ such that } \sup_{t\in\mathbb{R}}\|p(t)\|_{X^\alpha} \leqslant K \text{ is hyperbolic}\}\ .\end{aligned}$$

The slightly strange above notation comes from the fact that $\mathcal{G}_1$ and $\mathcal{G}_2$ are the sets originally introduced by Peixoto, whereas the set $\mathcal{G}_{3/2}$ has been introduced later.

We know from the arguments of the second part of Section 3 of [37] that $\mathcal{G}_1(K)$ is a dense open subset of $\mathfrak{C}^r$. The idea of the recursion argument is that there exists $\varepsilon > 0$ small enough, such that $\mathcal{G}_2(\varepsilon, K) = \mathcal{G}_1(K)$ due to the absence of periodic orbits of small period. Then, the method of Peixoto would consist in proving, like in [43], that $\mathcal{G}_2(A,K) \cap \mathcal{G}_{3/2}(3A/2,K)$ is dense in $\mathcal{G}_2(A,K)$ and that $\mathcal{G}_2(3A/2,K)$ is dense in $\mathcal{G}_{3/2}(3A/2,K)$. By this way, we obtain a chain of dense inclusions

$$\ldots \mathcal{G}_2(9\varepsilon/4,K) \underset{\text{dense}}{\subset} \mathcal{G}_{3/2}(9\varepsilon/4,K) \underset{\text{dense}}{\subset} \mathcal{G}_2(3\varepsilon/2,K) \underset{\text{dense}}{\subset} \mathcal{G}_{3/2}(3\varepsilon/2,K) \underset{\text{dense}}{\subset} \mathcal{G}_2(\varepsilon,K) = \mathcal{G}_1(K)$$

which shows the density of the hyperbolicity of periodic orbits in $\mathcal{G}_1$. The openness of these sets is rather simple and similar to the finite-dimensional case considered in [51]. This scheme of proof has been exactly performed in [43] and in [1]. The difficulty lies in the proofs of density.

We claim that the following density holds.

Proposition C.1. *For any positive A and K, $\mathcal{G}_{3/2}(3A/2,K) \cap \mathcal{G}_2(A,K)$ is dense in $\mathcal{G}_2(A,K)$.*

Proof: We give here very brief arguments since this proposition is only an auxiliary result in the whole proof of generic hyperbolicity, which is unfortunately not yet completed.

The proof of Proposition C.1 is very similar to the one of Proposition 6.2. We apply a suitable version of Sard-Smale theorem (similar to Theorem B.4) to the map

$$\Phi \ : \ (T, u_0, g) \ \longmapsto \ S_{f_0+g}(T)u_0 - u_0 \ .$$

As usual, the main difficulty is to obtain a surjectivity as required by Hypothesis ii) of Theorem B.4. We skip the details, but simply notice that checking this property is very similar to the end of the proof of Proposition 6.2: we have to find for any solution φ^* of the adjoint equation along a periodic orbit p, a perturbation g of f such that

$$\int_\Omega \int_0^T g(x, p(x,s), \nabla p(x,s))\varphi^*(x,s) ds dx \neq 0 \ .$$

This is achieved by constructing a function as in Proposition 6.1 by using Proposition 5.1. □

The proof of the genericity of the Kupka-Smale property would be obtained if we could prove the following result.

Conjecture C.2. *For any $A > 0$ and K, $\mathcal{G}_2(3A/2, K)$ is dense in $\mathcal{G}_{3/2}(3A/2, K) \cap \mathcal{G}_2(A, K)$.*

To prove this conjecture, we only need to know how to make hyperbolic a given simple periodic orbit in the following sense.

Conjecture C.3. *Let $f \in \mathcal{C}^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and let $\mathcal{N}$ be any small open neighborhood of f in $\mathcal{C}^r$. Let p be a simple periodic solution of (1.1) with minimal period $\omega > 0$ and such that $\sup_{t\in[0,\omega]} \|p(t)\|_{X^\alpha} \leqslant \tilde{K}$, where $\tilde{K} > 0$. Then, there exists a function $\tilde{f} \in \mathcal{N}$ such that p is a hyperbolic periodic solution of (1.1) with non-linearity $\tilde{f}$.*

Once again, the usual strategy would be to apply a Sard-Smale theorem (similar to Theorem B.4) to an appropriate functional Φ and then to check a surjectivity hypothesis as ii) of Theorem B.3. If we try the most natural way, we will have to find a perturbation g of f satisfying

$$\mathrm{Re} \int_0^\omega \int_\Omega (D_u g, D_{\nabla u} g)(x, p(x,t), \nabla p(x,t)) \, . \, \psi^*(x,t)(\phi, \nabla\phi)(x,t) \ dxdt \ \neq \ 0 \quad \text{(C.1)}$$

where p is the considered simple periodic orbit, ϕ a solution of the linearized equation associated to an eigenvalue λ with modulus $|\lambda| = 1$ and ψ^* a solution of the adjoint equation. Notice in (C.1) the presence of the real part Re since the spectrum of a periodic orbit has complex eigenvalues. To obtain this perturbation g, we may use a construction as follows.

Proposition C.4. *Let $f \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and let $p \in \mathcal{C}^\infty(\Omega \times \mathbb{R}, \mathbb{R})$ be a periodic solution of (1.1) with minimal period ω. Let $V \in \mathcal{C}^\infty(\Omega \times [0, \omega], \mathbb{R}^{d+1})$ be a function, which is not everywhere colinear to $(p_t(x,t), \nabla p_t(x,t))$. Then, there exists a function $g \in \mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ such that*

$$\begin{aligned} &i) \quad g(x, p(x,t), \nabla p(x,t)) = 0 \quad \forall (x,t) \in \Omega \times \mathbb{R}, \\ &ii) \quad \int_0^\omega \int_\Omega (D_u g, D_{\nabla u} g)(x, p(x,t), \nabla p(x,t)).V(x,t) \ dxdt \neq 0 \ . \end{aligned}$$

Proof: To simplify the notations, we denote by U the variable $(u, \nabla u) \in \mathbb{R}^{d+1}$ and we set $P(x,t) = (p(x,t), \nabla p(x,t)) \in \mathbb{R}^{d+1}$.

By assumption, there is an open set $\mathcal{U}$ with $\overline{\mathcal{U}} \subset \Omega \times (0,\omega)$ such that V is never colinear to P_t on $\mathcal{U}$. Notice that, in particular $P_t(x,t) \neq 0$ for all $(x,t) \in \mathcal{U}$. Due to Proposition 5.1, restricting $\mathcal{U}$, we can assume that, for all $(x_0,t_0) \in \mathcal{U}$, the map $(x,t) \in \Omega \times [0,\omega) \mapsto (x, P(x,t)) \in \Omega \times \mathbb{R}^{d+1}$ reaches the value $(x_0, P(x_0,t_0))$ at (x_0,t_0) only.

Let $(x_0,t_0) \in \mathcal{U}$. We complete (P_t, V) to a basis of $\mathbb{R}^{d+1}$: let W_1,...,W_{d-1} be $d-1$ vectors of $\mathbb{R}^{d+1}$ such that $(P_t(x_0,t_0),\ V(x_0,t_0),\ W_1,\ \ldots,\ W_{d-1})$ is a basis of $\mathbb{R}^{d+1}$. Restricting again $\mathcal{U}$, we can assume that $(P_t(x,t),\ V(x,t),\ W_1,\ \ldots,\ W_{d-1})$ is a basis of $\mathbb{R}^{d+1}$ for all $(x,t) \in \mathcal{U}$. Let $\mathcal{V} = \mathcal{U} \times \mathcal{W}$ where $\mathcal{W} \subset \mathbb{R}^d$ is a neighborhood of 0. We define $h : \mathcal{V} \to \Omega \times \mathbb{R}^{d+1}$ by

$$h(x,t,\tau,s_1,\ldots,,s_{d-1}) = (x, P(x,t) + \tau V(x,t) + s_1 W_1 + ... + s_{d-1} W_{d-1}) \ .$$

Up to choosing $\mathcal{V}$ smaller, the local inversion theorem shows that h is a $\mathcal{C}^\infty$-diffeomorphism into its image. We recall that for all $(x_0,t_0) \in \mathcal{U}$, the map $\Omega \times [0,\omega) \ni (x,t) \mapsto (x, P(x,t)) \in \Omega \times \mathbb{R}^{d+1}$ takes the value $(x_0, P(x_0,t_0))$ at (x_0,t_0) only. Due to the compactness of the graph of this map, we can restrict $\mathcal{W}$ such that $(x, P(x,t))$ belongs to $h(\mathcal{V})$ if and only if (x,t) belongs to $\mathcal{U}$. Let $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^{d+1}, \mathbb{R})$ be a function with compact support in $\mathcal{V}$, which will be made more precise later. We set $\theta(x,t,\tau,s_1,\ldots,,s_{d-1}) = \chi(x,t,\tau,s_1,\ldots,,s_{d-1})\tau$. We define the function $g : h(\mathcal{V}) \to \mathbb{R}$ by $g(x,u,\nabla u) = g(x,U) = \theta \circ h^{-1}(x,U)$. We can extend g by 0 outside $h(\mathcal{V})$ to obtain a function in $\mathcal{C}^\infty(\overline{\Omega} \times \mathbb{R}^{d+1})$. By construction, for all $(x,t) \notin \mathcal{U}$, $g(x,P(x,t)) = 0$ and $D_U g(x,P(x,t)) = 0$. Moreover, for all $(x,t) \in \mathcal{U}$, $g(x,P(x,t)) = \theta(x,t,0,0,...,0) = 0$ and

$$\begin{aligned}
\partial_U g(x,P(x,t)).V(x,t) &= D\theta(h^{-1}(x,P(x,t))).\left(\partial_U h^{-1}(x,P(x,t)).V(x,t)\right) \\
&= D\theta(x,t,0,...,0).\left(\partial_U h^{-1}(h(x,t,0,...,0)).\partial_\tau h(x,t,0,...,0)\right) \\
&= D\theta(x,t,0,...,0).\partial_\tau(h^{-1} \circ h)(x,t,0,...,0) \\
&= \partial_\tau \theta(x,t,0,...,0) \\
&= \chi(x,t,0,...,0)
\end{aligned}$$

Thus, Property i) of Proposition C.4 holds and moreover

$$\int_0^\omega \int_\Omega \partial_U g(x,P(x,t)).V(x,t)\ dxdt = \int_{\mathcal{U}} \chi(x,t,0,...,0)\ dxdt \ .$$

Therefore, we can easily choose χ such that Property ii) of Proposition C.4 also holds. □

The final problem lies in checking that the real part of $\psi^*(x,t)(\phi, \nabla\phi)$ in (C.1) is not everywhere colinear to $(p_t, \nabla p_t)$. This is true if we only consider real functions (see Proposition C.5 below), but we consider here complex solutions ψ^* and ϕ and thus the real part of $\psi^*(\phi, \nabla\phi)$ correspond to a combination of two real solutions of the linearized equation: the real and the imaginary parts of ϕ. Even if this colinearity

would be very strange and holds surely in very rare cases only (remember that we may break potential symmetries by perturbing f), we found no rigorous argument to avoid it.

We finish with a statement of non-colinearity which could be inspiring.

Proposition C.5. *Let I be an open interval of $\mathbb{R}$ and Ω and open subset of $\mathbb{R}^d$. Let $a \in \mathcal{C}^\infty(\Omega \times I, \mathbb{R})$ and $b \in \mathcal{C}^\infty(\Omega \times I, \mathbb{R}^d)$ be bounded coefficients. Let v_1 and v_2 be two solutions of the real equation*

$$\partial_t v(x,t) = \Delta v(x,t) + a(x,t)v(x,t) + b(x,t).\nabla_x v(x,t) \ . \tag{C.2}$$

Assume that $(v_1, \nabla v_1)$ is colinear to $(v_2, \nabla v_2)$ at each points (x,t), meaning that there exists real values $\alpha(x,t)$ and $\beta(x,t)$ such that for all $(x,t) \in \Omega \times I$,

$$\alpha(x,t)(v_1, \nabla v_1)(x,t) + \beta(x,t)(v_2, \nabla v_2)(x,t) = 0. \tag{C.3}$$

Then v_1 and v_2 are colinear to v_2 as solutions, that is that (C.3) *holds with real constants α and β.*

Proof: If $v_i \equiv 0$ for $i = 1$ or $i = 2$ the conclusion is trivial. By the unique continuation properties of Section 2, up to choose I and Ω smaller, we may thus assume that $(v_i, \nabla v_i)$ are not zero and thus that $\alpha(x,t)$ and $\beta(x,t)$ are smooth non-zero functions. Moreover, we may fix the normalization $\alpha^2(x,t) + \beta^2(x,t) = 1$. Fix (x_0, t_0) and set $(\tilde{\alpha}, \tilde{\beta}) = (\alpha(x_0,t_0), \beta(x_0,t_0))$. We notice that the value $(\tilde{\alpha}, \tilde{\beta})$ is taken by $(\alpha(x,t), \beta(x,t))$ in a submanifold $\mathcal{M}$ of dimension $d' \geqslant d$ of $\Omega \times I$ because the possible values of the function lie in the circle S^1 which is one-dimensional. The function $w = \tilde{\alpha} v_1 + \tilde{\beta} v_2$ is also a solution of (C.2) and by construction $(w, \nabla w)$ vanishes in the submanifold $\mathcal{M}$ of dimension d'. We now apply Theorem 4.1 with families independent of $\tau \in J = \mathbb{R}$. The singular nodal set of $w(x,t,\tau)$ is $\mathcal{M} \times J$ of dimension $d' + 1 \geqslant d + 1$. Thus $w \equiv 0$ which concludes the proof. ▫

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