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Unsteady micropolar fluid flow in a thin domain with Tresca fluid-solid interface law

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Abstract

We consider a micropolar fluid flow in a two-dimensional domain. We assume that the velocity field satisfies a non-linear slip boundary condition of friction type on a part of the boundary while the micro-rotation field satisfies non-homogeneous Dirichlet boundary conditions. We prove the existence and uniqueness of a solution. Then motivated by lubrication problems we assume that the thickness and the roughness of the domain are of order $0 < \varepsilon \ll 1$ and we study the asymptotic behaviour of the flow as ε tends to zero. By using the two-scale convergence technique we derive the limit problem which is totally decoupled for the limit velocity and pressure (v^0, p^0) on one hand and the limit micro-rotation Z^0 on the other hand. Moreover we prove that v^0 , p^0 and Z^0 are uniquely determined via auxiliary well-posed problems.

Keywords: Micropolar fluid, Tresca friction law, asymptotic analysis, two-scale convergence

2010 MSC: 35Q35, 76A05, 35K86, 76D08, 76M50

1. Description of the problem

Several industrial problems involve nowadays complex fluids like polymers, colloidal fluids, ferro-liquids or liquid crystals. Such fluids contain suspensions of rigid particles that undergo rotations and the classical Navier-Stokes theory is inadequate since it does not take into account the effects of the micro-structures. The micropolar fluid model has been introduced by A.C. Eringen in [10] in order to describe the macroscopic behaviour of such fluids under the assumptions that the particles are randomly oriented or spherical and the deformation of the particles is neglected. The unknowns are the fluid velocity $u = (u_1, u_2, u_3)$, the pressure p and the micro-rotation field $\omega = (\omega_1, \omega_2, \omega_3)$ which can be interpreted as the angular velocity field of the micro-particles. Then the equilibrium of

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momentum, of mass and moment of momentum lead to a system of coupled partial differential equations for the triplet (u, p, ω) (see [10] or [2, 3, 11, 15]).

Motivated by lubrication problems we consider a flow in an infinite journal bearing. The cross section of the domain is thus given by the gap between two non-concentric discs which is much smaller than the discs radii. After a radial cut the fluid domain can be represented as

$$\Omega^\varepsilon \times (-\infty, +\infty), \quad \Omega^\varepsilon = \{(z_1, z_2) \in \mathbb{R}^2; 0 < z_1 < L, 0 < z_2 < \varepsilon h^\varepsilon(z_1)\}$$

where $L > 0$ and h^ε is a function of class C^∞ which is L -periodic in z_1 such that

$$0 < h_m \leq h^\varepsilon(z_1) \leq h_M \quad \text{for all } z_1 \in \mathbb{R}.$$

By assuming that the flow and the external excitation fields do not depend on the coordinate along the longitudinal axis of the bearing we obtain a 2D problem given by

$$u_t^\varepsilon - (\nu + \nu_r)\Delta u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = 2\nu_r \text{rot}(\omega^\varepsilon) + f^\varepsilon \quad \text{in } (0, T) \times \Omega^\varepsilon \quad (1)$$

$$\text{div}(u^\varepsilon) = 0 \quad \text{in } (0, T) \times \Omega^\varepsilon \quad (2)$$

$$\omega_t^\varepsilon - \alpha \Delta \omega^\varepsilon + (u^\varepsilon \cdot \nabla)\omega^\varepsilon + 4\nu_r \omega^\varepsilon = 2\nu_r \text{rot}(u^\varepsilon) + g^\varepsilon \quad \text{in } (0, T) \times \Omega^\varepsilon \quad (3)$$

with the initial conditions

$$u^\varepsilon(0, \cdot) = u_0^\varepsilon, \quad \omega^\varepsilon(0, \cdot) = \omega_0^\varepsilon \quad \text{in } \Omega^\varepsilon$$

where $T > 0$, ν , ν_r and α are three positive physical parameters and $u^\varepsilon = (u_1^\varepsilon(t, z), u_2^\varepsilon(t, z))$, $\omega^\varepsilon = \omega_3^\varepsilon(t, z)$. Let us emphasize that the choice $\nu_r = 0$ would allow to decouple the two first equations from the third one and we would recover the Navier-Stokes system for $(u^\varepsilon, p^\varepsilon)$. For this reason ν_r is called the micro-rotation viscosity while ν is called the Newtonian viscosity of the micropolar fluid.

We decompose $\partial\Omega^\varepsilon$ as $\Gamma_0 = \{z \in \partial\Omega^\varepsilon : z_2 = 0\}$, $\Gamma_1^\varepsilon = \{z \in \partial\Omega^\varepsilon : z_2 = \varepsilon h^\varepsilon(z_1)\}$, and Γ_L^ε is the lateral part of the boundary. Due to the original geometry of the flow domain, we have

$$u^\varepsilon, \omega^\varepsilon, p^\varepsilon \quad \text{L-periodic with respect to } z_1.$$

A first study for non-homogeneous Dirichlet conditions on Γ_0 and non-standard free boundary conditions on Γ_1^ε has been proposed in [7]. Nevertheless, experimental studies ([16, 12]) have shown that non-linear slip boundary conditions of friction type are more realistic for such complex fluids. Hence we will consider in this paper non-homogeneous Dirichlet boundary conditions on Γ_1^ε and Tresca friction boundary conditions for the fluid velocity on Γ_0 i.e.

$$u^\varepsilon = (U_0(t), 0) \quad \text{on } (0, T) \times \Gamma_1^\varepsilon$$

where U_0 is a given function of the time variable and

$$u^\varepsilon \cdot n = 0, \quad |\tau \cdot \sigma(u^\varepsilon, p^\varepsilon) \cdot n| \leq k^\varepsilon \quad \text{on } (0, T) \times \Gamma_0$$

with

$$\begin{cases} |\tau \cdot \sigma(u^\varepsilon, p^\varepsilon) \cdot n| < k^\varepsilon \Rightarrow u^\varepsilon \cdot \tau = s_0(t) \\ |\tau \cdot \sigma(u^\varepsilon, p^\varepsilon) \cdot n| = k^\varepsilon \Rightarrow u^\varepsilon \cdot \tau = s_0(t) - \lambda \tau \cdot \sigma(u^\varepsilon, p^\varepsilon) \cdot n, \quad \lambda \geq 0 \end{cases}$$

where k^ε is a positive given function defined on $(0, T) \times \Gamma_0$ and s_0 is a given function of the time variable corresponding to the Tresca friction threshold for the shear stress and the velocity of the inner cylinder of the journal bearing, respectively (see [9]). We have denoted as τ and n the tangent and outward normal unit vectors to the boundary and $\sigma(u^\varepsilon, p^\varepsilon) = (\sigma_{ij}(u^\varepsilon, p^\varepsilon))_{1 \leq i, j \leq 2}$ is the Cauchy stress tensor defined as

$$\sigma_{ij}(u^\varepsilon, p^\varepsilon) = -p^\varepsilon \delta_{ij} + (\nu + \nu_r) \left(\frac{\partial u_i^\varepsilon}{\partial z_j} + \frac{\partial u_j^\varepsilon}{\partial z_i} \right), \quad 1 \leq i, j \leq 2.$$

For the micro-rotation field we assume

$$\omega^\varepsilon = W_0(t) \quad \text{on } (0, T) \times \Gamma_1^\varepsilon, \quad \omega^\varepsilon = 0 \quad \text{on } (0, T) \times \Gamma_0$$

where W_0 is a given function of the time variable.

In order to obtain a variational formulation of the problem we introduce the following functional spaces. Let

$$\tilde{V}^\varepsilon = \{v \in (C^\infty(\overline{\Omega^\varepsilon}))^2 : v \text{ is L-periodic in } z_1, v|_{\Gamma_1^\varepsilon} = 0, v \cdot n|_{\Gamma_0} = 0\},$$

$$H^\varepsilon = \text{closure of } \tilde{V}^\varepsilon \text{ in } (L^2(\Omega^\varepsilon))^2, \quad V^\varepsilon = \text{closure of } \tilde{V}^\varepsilon \text{ in } (H^1(\Omega^\varepsilon))^2,$$

$$V_{div}^\varepsilon = \{v \in V^\varepsilon : \text{div } v = 0 \text{ in } \Omega^\varepsilon\},$$

and for the micro-rotations

$$\tilde{H}^{1,\varepsilon} = \{Z \in C^\infty(\overline{\Omega^\varepsilon}) : Z \text{ is L-periodic in } z_1, Z|_{\Gamma_0 \cup \Gamma_1^\varepsilon} = 0\}$$

$$H^{0,\varepsilon} = \text{closure of } \tilde{H}^{1,\varepsilon} \text{ in } L^2(\Omega^\varepsilon), \quad H^{1,\varepsilon} = \text{closure of } \tilde{H}^{1,\varepsilon} \text{ in } H^1(\Omega^\varepsilon)$$

endowed with the inner products

$$\begin{aligned} [[\bar{v}, \Theta]] &= (\nabla v, \nabla \varphi) + (\nabla Z, \nabla \psi) \quad \text{for all } \bar{v} = (v, Z) \text{ and } \Theta = (\varphi, \psi) \text{ in } V^\varepsilon \times H^{1,\varepsilon}, \\ [\bar{v}, \Theta] &= (v, \varphi) + (Z, \psi) \quad \text{for all } \bar{v} = (v, Z) \text{ and } \Theta = (\varphi, \psi) \text{ in } H^\varepsilon \times H^{0,\varepsilon}, \end{aligned}$$

and the corresponding norms $[[\bar{v}]]$ (resp. $[\bar{v}]$) for all $\bar{v} \in V^\varepsilon \times H^{1,\varepsilon}$ (resp. $H^\varepsilon \times H^{0,\varepsilon}$). We introduce also

$$\begin{aligned} a(\bar{v}, \Theta) &= (\nu + \nu_r)(\nabla v, \nabla \varphi) + \alpha(\nabla Z, \nabla \psi), \\ \mathcal{R}(\bar{v}, \Theta) &= -2\nu_r(\text{rot } Z, \varphi) - 2\nu_r(\text{rot } v, \psi) + 4\nu_r(Z, \psi), \end{aligned}$$

where (\cdot, \cdot) denotes the inner product of $(L^2(\Omega^\varepsilon))^d$ with $d = 1, d = 2$ or $d = 4$ and

$$B(\bar{v}, \bar{u}, \Theta) = b(v, u, \varphi) + b_1(v, w, \psi) = \sum_{i,j=1}^2 \int_{\Omega^\varepsilon} v_i \frac{\partial u_j}{\partial z_i} \varphi_j dz + \sum_{i=1}^2 \int_{\Omega^\varepsilon} v_i \frac{\partial w}{\partial z_i} \psi dz$$

for all $\bar{v} = (v, Z)$, $\bar{u} = (u, w)$ and $\Theta = (\varphi, \psi)$ in $V^\varepsilon \times H^{1,\varepsilon}$.

Then we define new unknown velocity and micro-rotation fields as

$$\begin{aligned} v^\varepsilon(t, z) &= u^\varepsilon(t, z) - U^\varepsilon(t, z_2)e_1 \text{ in } (0, T) \times \Omega^\varepsilon, \\ Z^\varepsilon(t, z) &= \omega^\varepsilon(t, z) - W^\varepsilon(t, z_2) \text{ in } (0, T) \times \Omega^\varepsilon, \end{aligned}$$

where U^ε and W^ε are two extensions of U^0 and W^0 given by

$$\begin{aligned} U^\varepsilon(t, z_2) &= s_0(t) + (U_0(t) - s_0(t))\mathcal{U}^\varepsilon(z_2) = s_0(t) + (U_0(t) - s_0(t))\mathcal{U}\left(\frac{z_2}{\varepsilon}\right), \\ W^\varepsilon(t, z_2) &= W_0(t)\mathcal{W}^\varepsilon(z_2) = W_0(t)\mathcal{W}\left(\frac{z_2}{\varepsilon}\right), \end{aligned}$$

with functions $\mathcal{U} \in \mathcal{C}^\infty(\mathbb{R})$ and $\mathcal{W} \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\mathcal{U}(0) = \mathcal{U}'(0) = 0 = \mathcal{W}(0),$$

and

$$\mathcal{U}(X) = \mathcal{W}(X) = 1 \quad \forall X \in [h_m, h_M].$$

It follows immediately that

$$\begin{aligned} U^\varepsilon(t, 0) &= s_0(t), \quad U^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = U_0(t), \quad \frac{\partial U^\varepsilon}{\partial z_2}(t, 0) = 0, \\ W^\varepsilon(t, 0) &= 0, \quad W^\varepsilon(t, \varepsilon h^\varepsilon(z_1)) = W_0(t), \end{aligned}$$

for all $(t, z_1) \in (0, T) \times (0, L)$. Moreover $\sigma(u^\varepsilon, p^\varepsilon) = \sigma(v^\varepsilon, p^\varepsilon)$ on $(0, T) \times \Gamma_0$.

By multiplying (1) and (3) by test-functions and applying Green's formula we obtain the following variational formulation of the problem:

Problem (P^ε) Find $(v^\varepsilon, p^\varepsilon, Z^\varepsilon)$ such that

$$\bar{v}^\varepsilon = (v^\varepsilon, Z^\varepsilon) \in L^\infty(0, T; H^\varepsilon \times H^{0,\varepsilon}) \cap L^2(0, T; V_{div}^\varepsilon \times H^{1,\varepsilon}),$$

$$p^\varepsilon \in H^{-1}(0, T; L^2_0(\Omega^\varepsilon))$$

and

$$\begin{aligned} &\left\langle \frac{\partial}{\partial t} [\bar{v}^\varepsilon, \Theta], \theta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} + \int_0^T a(\bar{v}^\varepsilon, \Theta) \theta dt + \int_0^T B(\bar{v}^\varepsilon, \bar{v}^\varepsilon, \Theta) \theta dt \\ &+ \int_0^T \mathcal{R}(\bar{v}^\varepsilon, \Theta) \theta dt + j^\varepsilon(\varphi \theta + v^\varepsilon) - j^\varepsilon(v^\varepsilon) \geq \int_0^T (p^\varepsilon, \operatorname{div} \varphi) \theta dt \\ &+ \int_0^T (\mathcal{F}(\bar{v}^\varepsilon), \Theta) \theta dt \quad \forall \Theta = (\varphi, \psi) \in V^\varepsilon \times H^{1,\varepsilon}, \quad \forall \theta \in \mathcal{D}(0, T) \end{aligned}$$

with the initial condition

$$\bar{v}^\varepsilon(0, \cdot) = \bar{v}_0^\varepsilon = (v_0^\varepsilon, Z_0^\varepsilon) = (u_0^\varepsilon - U^\varepsilon(0, \cdot)e_1, \omega_0^\varepsilon - W^\varepsilon(0, \cdot)),$$

where

$$\begin{aligned} (\mathcal{F}(\bar{v}^\varepsilon(t)), \Theta) &= -a(\bar{\xi}^\varepsilon(t), \Theta) - B(\bar{\xi}^\varepsilon(t), \bar{v}^\varepsilon(t), \Theta) - B(\bar{v}^\varepsilon(t), \bar{\xi}^\varepsilon(t), \Theta) \\ &\quad - \mathcal{R}(\bar{\xi}^\varepsilon(t), \Theta) - \left[\frac{\partial \bar{\xi}^\varepsilon}{\partial t}(t), \Theta \right] + [\bar{f}^\varepsilon(t), \Theta] \end{aligned}$$

with $\bar{\xi}^\varepsilon = (U^\varepsilon e_1, W^\varepsilon)$, $\bar{f}^\varepsilon = (f^\varepsilon, g^\varepsilon)$ and

$$j^\varepsilon(w) = \int_0^T \int_{\Gamma_0} k^\varepsilon |w| dz_1 dt \quad \forall w \in L^2(0, T; (L^2(\Gamma_0))^2).$$

Let us emphasize that we identify $\varphi\theta + v^\varepsilon$ and v^ε with their trace on Γ_0 in the definition of $j^\varepsilon(\varphi\theta + v^\varepsilon)$ and $j^\varepsilon(v^\varepsilon)$.

The paper is organized as follows. In Section 2 we prove the existence and uniqueness of a solution to problem (P^ε) for any $\varepsilon > 0$. Then motivated by the asymptotic behaviour of thin flows we study the limit as ε tends to zero. Having in mind that the roughness of the domain depends also on ε , we consider the particular scaling of a roughness in inverse proportion of the thickness as usual in lubrication theory ([6, 7]). In Section 3 we establish some a priori estimates for the v^ε , p^ε and Z^ε and we apply the two-scale convergence technique to get a limit velocity v^0 , pressure p^0 and micro-rotation Z^0 . Finally in Section 4 we derive the limit problem and we prove that v^0 , p^0 and Z^0 are uniquely determined through auxiliary well-posed problems.

2. Existence and uniqueness for (P^ε)

Theorem 1. *Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$, $f^\varepsilon \in (L^2((0, T) \times \Omega^\varepsilon))^2$, $g^\varepsilon \in L^2((0, T) \times \Omega^\varepsilon)$, $k^\varepsilon \in L^\infty(0, T; L^2_+(\Gamma_0))$ and $(v_0^\varepsilon, Z_0^\varepsilon) \in H^\varepsilon \times H^{0, \varepsilon}$. Then problem (P^ε) admits an unique solution.*

PROOF. In order to establish the existence of a solution we replace Tresca functional j^ε by some regularization j^ε_η ($\eta > 0$) and we consider the condition “ $\text{div } v^\varepsilon = 0$ ” as a constraint. Hence we introduce the doubly penalized problem

Problem $(P^\varepsilon_{\eta\delta})$ Find $(v^\varepsilon_{\eta\delta}, Z^\varepsilon_{\eta\delta})$ such that

$$\bar{v}^\varepsilon_{\eta\delta} = (v^\varepsilon_{\eta\delta}, Z^\varepsilon_{\eta\delta}) \in L^\infty(0, T; H^\varepsilon \times H^{0, \varepsilon}) \cap L^2(0, T; V^\varepsilon \times H^{1, \varepsilon})$$

and

$$\begin{aligned}
& \left\langle \frac{\partial}{\partial t} [\bar{v}_{\eta\delta}^\varepsilon, \Theta], \theta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} + \int_0^T a(\bar{v}_{\eta\delta}^\varepsilon, \Theta) \theta dt + \int_0^T B(\bar{v}_{\eta\delta}^\varepsilon, \bar{v}_{\eta\delta}^\varepsilon, \Theta) \theta dt \\
& + \frac{1}{2} \int_0^T (v_{\eta\delta}^\varepsilon \operatorname{div} v_{\eta\delta}^\varepsilon, \varphi) \theta dt + \frac{1}{2} \int_0^T (Z_{\eta\delta}^\varepsilon \operatorname{div} v_{\eta\delta}^\varepsilon, \psi) \theta dt + j_\eta^\varepsilon(v_{\eta\delta}^\varepsilon, \varphi \theta) \\
& = \int_0^T \left(-\frac{1}{\delta} \operatorname{div} v_{\eta\delta}^\varepsilon, \operatorname{div} \varphi \right) \theta dt + \int_0^T (\mathcal{F}(\bar{v}_{\eta\delta}^\varepsilon), \Theta) \theta dt - \int_0^T \mathcal{R}(\bar{v}_{\eta\delta}^\varepsilon, \Theta) \theta dt \\
& \forall \Theta = (\varphi, \psi) \in V^\varepsilon \times H^{1,\varepsilon}, \quad \forall \theta \in \mathcal{D}(0,T)
\end{aligned}$$

with the initial condition

$$\bar{v}_{\eta\delta}^\varepsilon(0) = \bar{v}_0^\varepsilon$$

where

$$j_\eta^\varepsilon(w) = \int_0^T \int_{\Gamma_0} k^\varepsilon \sqrt{\eta^2 + |w|^2} dz_1 dt \quad \forall w \in L^2(0,T; (L^2(\Gamma_0))^2)$$

and $j_\eta^\varepsilon(w, \varphi \theta)$ denotes the Gateaux derivative of j_η^ε at w in the direction $\varphi \theta$.

Let us observe that $-\frac{1}{\delta} \operatorname{div} v_{\eta\delta}^\varepsilon$ ($\delta > 0$) can be interpreted as an approximate pressure while the two terms $\frac{1}{2} \int_0^T (v_{\eta\delta}^\varepsilon \operatorname{div} v_{\eta\delta}^\varepsilon, \varphi) \theta dt$ and $\frac{1}{2} \int_0^T (Z_{\eta\delta}^\varepsilon \operatorname{div} v_{\eta\delta}^\varepsilon, \psi) \theta dt$ are added for technical reasons. Indeed, in this approximate problem the velocity is not anymore divergence free. Hence the trilinear term $B(\bar{v}_{\eta\delta}^\varepsilon, \bar{v}_{\eta\delta}^\varepsilon, \bar{\Theta})$ does not vanish anymore when we choose $\bar{\Theta} = \bar{v}_{\eta\delta}^\varepsilon$ and these two additional terms are introduced in order to cancel with $B(\bar{v}_{\eta\delta}^\varepsilon, \bar{v}_{\eta\delta}^\varepsilon, \bar{\Theta})$ when $\bar{\Theta} = \bar{v}_{\eta\delta}^\varepsilon$.

Then we prove the existence of a solution to Problem $(P_{\eta\delta}^\varepsilon)$ by using a Galerkin method. Indeed let $(\Phi_j)_{j \geq 1}$ and $(\Psi_j)_{j \geq 1}$ be Hilbertian bases of V^ε and $H^{1,\varepsilon}$, respectively such that $(\Phi_j)_{j \geq 1}$ and $(\Psi_j)_{j \geq 1}$ are orthogonal for the inner product of $(H^1(\Omega^\varepsilon))^2$ and $H^1(\Omega^\varepsilon)$ and orthonormal for the inner product of $(L^2(\Omega^\varepsilon))^2$ and $L^2(\Omega^\varepsilon)$, respectively. Then for all $m = (m_1, m_2)$ with $m_1 \geq 1$ and $m_2 \geq 1$ we let

$$v_{\eta\delta m_1}^\varepsilon(t, z) = \sum_{j_1=1}^{m_1} v_{\eta\delta j_1}^\varepsilon(t) \Phi_{j_1}(z), \quad Z_{\eta\delta m_2}^\varepsilon(t, z) = \sum_{j_2=1}^{m_2} Z_{\eta\delta j_2}^\varepsilon(t) \Psi_{j_2}(z)$$

such that $\bar{v}_{\eta\delta m}^\varepsilon = (v_{\eta\delta m_1}^\varepsilon, Z_{\eta\delta m_2}^\varepsilon)$ satisfies

$$\begin{aligned}
& \left[\frac{\partial \bar{v}_{\eta\delta m}^\varepsilon}{\partial t}, \Theta_j \right] + a(\bar{v}_{\eta\delta m}^\varepsilon, \Theta_j) + B(\bar{v}_{\eta\delta m}^\varepsilon, \bar{v}_{\eta\delta m}^\varepsilon, \Theta_j) + \frac{1}{2} (v_{\eta\delta m}^\varepsilon \operatorname{div} v_{\eta\delta m}^\varepsilon, \Phi_{j_1}) \\
& + \frac{1}{2} (Z_{\eta\delta m}^\varepsilon \operatorname{div} v_{\eta\delta m}^\varepsilon, \Psi_{j_2}) + \int_{\Gamma_0} k^\varepsilon \frac{v_{\eta\delta m}^\varepsilon \cdot \Phi_{j_1}}{\sqrt{\eta^2 + |v_{\eta\delta m}^\varepsilon|^2}} dz_1 dt \\
& = \left(-\frac{1}{\delta} \operatorname{div} v_{\eta\delta m}^\varepsilon, \operatorname{div} \Phi_{j_1} \right) + (\mathcal{F}(\bar{v}_{\eta\delta m}^\varepsilon), \Theta_j) - \mathcal{R}(\bar{v}_{\eta\delta m}^\varepsilon, \Theta_j) \\
& \forall \Theta_j = (\Phi_{j_1}, \Psi_{j_2}), \quad \forall (j_1, j_2) \in \{1, \dots, m_1\} \times \{1, \dots, m_2\}
\end{aligned} \tag{4}$$

with the initial condition

$$\bar{v}_{\eta\delta m}^\varepsilon(0) = \bar{v}_{0m}^\varepsilon = (v_{0m_1}^\varepsilon, Z_{0m_2}^\varepsilon) \quad (5)$$

where $v_{0m_1}^\varepsilon$ and $Z_{0m_2}^\varepsilon$ are the orthogonal projections of v_0^ε and Z_0^ε on the finite-dimensional subspaces $\langle \Phi_1, \dots, \Phi_{m_1} \rangle$ and $\langle \Psi_1, \dots, \Psi_{m_2} \rangle$ with respect to the inner product of $(L^2(\Omega^\varepsilon))^2$ and $L^2(\Omega^\varepsilon)$, respectively. The nonlinear differential system (4)-(5) for the scalar functions $(v_{\eta\delta j_1}^\varepsilon, Z_{\eta\delta j_2}^\varepsilon)_{1 \leq j_1 \leq m_1, 1 \leq j_2 \leq m_2}$ admits a unique maximal solution in $(H^1(0, T_m))^{m_1+m_2}$ with $T_m \in (0, T]$.

By choosing $\bar{v}_{\eta\delta m}^\varepsilon$ as test-function we obtain

$$\begin{aligned} & \left[\frac{\partial \bar{v}_{\eta\delta m}^\varepsilon}{\partial t}, \bar{v}_{\eta\delta m}^\varepsilon \right] + a(\bar{v}_{\eta\delta m}^\varepsilon, \bar{v}_{\eta\delta m}^\varepsilon) + \underbrace{\int_{\Gamma_0} k^\varepsilon \frac{v_{\eta\delta m}^\varepsilon \cdot v_{\eta\delta m}^\varepsilon}{\sqrt{\eta^2 + |v_{\eta\delta m}^\varepsilon|^2}} dz_1 dt}_{\geq 0} \\ & = \left(-\frac{1}{\delta} \operatorname{div} v_{\eta\delta m}^\varepsilon, \operatorname{div} v_{\eta\delta m}^\varepsilon \right) + (\mathcal{F}(\bar{v}_{\eta\delta m}^\varepsilon), \bar{v}_{\eta\delta m}^\varepsilon) - \mathcal{R}(\bar{v}_{\eta\delta m}^\varepsilon, \bar{v}_{\eta\delta m}^\varepsilon) \end{aligned}$$

and we recover the same energy inequality as in [7] i.e.

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} (\|v_{\eta\delta m}^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \|Z_{\eta\delta m}^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2) + (\nu + \nu_r) \|\nabla v_{\eta\delta m}^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \\ & + \alpha \|\nabla Z_{\eta\delta m}^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 + \frac{1}{\delta} \|\operatorname{div} v_{\eta\delta m}^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 \leq (\mathcal{F}(\bar{v}_{\eta\delta m}^\varepsilon), \bar{v}_{\eta\delta m}^\varepsilon) - \mathcal{R}(\bar{v}_{\eta\delta m}^\varepsilon, \bar{v}_{\eta\delta m}^\varepsilon). \end{aligned}$$

Therefore we infer that that $T_m = T$ and there exists a constant $C > 0$ independent of m , δ and η such that

$$[\bar{v}_{\eta\delta m}^\varepsilon(t)]^2 + k \int_0^t [[\bar{v}_{\eta\delta m}^\varepsilon(s)]]^2 ds + \frac{2}{\delta} \int_0^t \|\operatorname{div} v_{\eta\delta m}^\varepsilon(s)\|^2 ds \leq C \quad \forall t \in [0, T]$$

where $k = \min\{\nu, \alpha\}$ (see Theorem 2.2 in [7]).

Moreover, for all $\Theta = (\varphi, \psi) \in V^\varepsilon \times H^{1,\varepsilon}$, let $\Theta_m = (\varphi_{m_1}, \psi_{m_2})$ where φ_{m_1} and ψ_{m_2} are the projections of φ and ψ on $\langle \Phi_1, \dots, \Phi_{m_1} \rangle$ and $\langle \Psi_1, \dots, \Psi_{m_2} \rangle$ with respect to the inner product of $(H^1(\Omega^\varepsilon))^2$ and $H^1(\Omega^\varepsilon)$, respectively. Since $(\Phi_j)_{j \geq 1}$ and $(\Psi_j)_{j \geq 1}$ are orthonormal for the inner product of $(L^2(\Omega^\varepsilon))^2$ and $L^2(\Omega^\varepsilon)$ we have

$$\left(\frac{\partial v_{\eta\delta m}^\varepsilon}{\partial t}, \varphi \right) = \left(\frac{\partial v_{\eta\delta m}^\varepsilon}{\partial t}, \varphi_{m_1} \right), \quad \left(\frac{\partial Z_{\eta\delta m}^\varepsilon}{\partial t}, \psi \right) = \left(\frac{\partial Z_{\eta\delta m}^\varepsilon}{\partial t}, \psi_{m_2} \right)$$

and with the same kind of computations as in Theorem 2.2 in [7] we obtain

$$\left\| \frac{\partial v_{\eta\delta m}^\varepsilon}{\partial t} \right\|_{L^{4/3}(0, T; (V^\varepsilon)')} \leq C, \quad \left\| \frac{\partial Z_{\eta\delta m}^\varepsilon}{\partial t} \right\|_{L^{4/3}(0, T; (H^{1,\varepsilon})')} \leq C.$$

where $C > 0$ is a constant independent of m and η (see also Lemma 4.2 in [5]).

Then by using compactness arguments (see Theorem 2.2 in [7] and Theorem 4.4, Theorem 6.1 and Theorem 6.3 in [5]) we pass to the limit as m_1, m_2 tend to $+\infty$, then as δ tends to zero and finally as η tends to zero.

In order to prove uniqueness of the solution to Problem (P^ε) , we use Sobolev's inequalities. Indeed, since $\Omega^\varepsilon \subset \mathbb{R}^2$, there exists a constant $C(\Omega^\varepsilon) > 0$ such that

$$\begin{aligned} |b(u, u, w)| &= |b(u, w, u)| \leq \|u\|_{(L^4(\Omega^\varepsilon))^2}^2 \|w\|_{(H^1(\Omega^\varepsilon))^2} \\ &\leq C(\Omega^\varepsilon) \|u\|_{(L^2(\Omega^\varepsilon))^2} \|u\|_{V^\varepsilon} \|w\|_{V^\varepsilon} \quad \forall (u, w) \in V_{div}^\varepsilon \times V^\varepsilon, \end{aligned}$$

and

$$\begin{aligned} |b_1(u, w, \psi)| &= |b(u, \psi, w)| \leq \|u\|_{(L^4(\Omega^\varepsilon))^2} \|w\|_{L^4(\Omega^\varepsilon)} \|\psi\|_{H^1(\Omega^\varepsilon)} \\ &\leq C(\Omega^\varepsilon) \|u\|_{(L^2(\Omega^\varepsilon))^2}^{1/2} \|u\|_{V^\varepsilon}^{1/2} \|w\|_{L^2(\Omega^\varepsilon)}^{1/2} \|w\|_{H^{1,\varepsilon}}^{1/2} \|\psi\|_{H^{1,\varepsilon}} \\ &\forall (u, w, \psi) \in V_{div}^\varepsilon \times (H^{1,\varepsilon})^2. \end{aligned}$$

Hence $B(\bar{v}^\varepsilon, \bar{v}^\varepsilon, \cdot) \in L^2(0, T; (V^\varepsilon \times H^{1,\varepsilon})')$. By using the same technique as in Lemma 3.1 and Proposition 3.1 in [4] we infer that $\frac{\partial \bar{v}^\varepsilon}{\partial t} \in L^2(0, T; (V_{div}^\varepsilon \times H^{1,\varepsilon})')$ which implies uniqueness of the solution and $\bar{v}^\varepsilon \in \mathcal{C}([0, T]; H^\varepsilon \times H^{1,\varepsilon})$. \blacksquare

Remark 1. Since $\frac{\partial \bar{v}^\varepsilon}{\partial t} \in L^2(0, T; (V_{div}^\varepsilon \times H^{1,\varepsilon})')$ we have

$$\left\langle \frac{\partial}{\partial t} [\bar{v}^\varepsilon, \Theta], \theta \right\rangle_{\mathcal{D}'(0,T), \mathcal{D}(0,T)} = \int_0^T \left\langle \frac{\partial \bar{v}^\varepsilon}{\partial t}, \underbrace{\Theta \theta}_{\Theta} \right\rangle_{(V_{div}^\varepsilon \times H^{1,\varepsilon})', V_{div}^\varepsilon \times H^{1,\varepsilon}} dt$$

for all $\Theta \in V_{div}^\varepsilon \times H^{1,\varepsilon}$ and for all $\theta \in \mathcal{D}(0, T)$. By density of $\mathcal{D}(0, T) \otimes (V_{div}^\varepsilon \times H^{1,\varepsilon})$ into $L^2(0, T; V_{div}^\varepsilon \times H^{1,\varepsilon})$ we may consider any test-function $\tilde{\Theta} \in L^2(0, T; V_{div}^\varepsilon \times H^{1,\varepsilon})$.

3. A priori estimates and two-scale convergence limit

As usual in lubrication theory the roughness of the flow domain depends also on ε and is inversely proportional to its thickness (see [6, 7]). Hence we assume from now on that $h^\varepsilon(z_1) = h\left(z_1, \frac{z_1}{\varepsilon}\right)$ with

$$\frac{L}{\varepsilon} \in \mathbb{N}, \quad h : (z_1, \eta_1) \mapsto h(z_1, \eta_1) \text{ is } L\text{-periodic in } z_1 \text{ and } 1\text{-periodic in } \eta_1$$

and $h \in \mathcal{C}^\infty(\mathbb{R}^2)$ is bounded from above and from below by two positive real numbers h_M and h_m .

Let us observe that the roughness introduces a rapid scale variable $\frac{z_1}{\varepsilon}$ in the horizontal direction. So motivated by the asymptotic behaviour as ε tends to zero we introduce the following scaling

$$y_1 = z_1, \quad y_2 = \frac{z_2}{\varepsilon h^\varepsilon(z_1)}$$

which transforms Ω^ε into the rectangle $\Omega = (0, L) \times (0, 1)$ and we let $\Gamma_1 = [0, L] \times \{1\}$, $\Gamma_L = \{0, 1\} \times [0, 1]$. With the chain rule we obtain

$$\frac{\partial}{\partial z_2} = \frac{1}{\varepsilon h^\varepsilon} \frac{\partial}{\partial y_2}, \quad \frac{\partial}{\partial z_1} = \frac{\partial}{\partial y_1} \frac{\partial y_1}{\partial z_1} + \frac{\partial}{\partial y_2} \frac{\partial y_2}{\partial z_1} = \frac{\partial}{\partial y_1} + \left(-\frac{y_2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y_1} \right) \frac{\partial}{\partial y_2}$$

and we introduce the differential operator $b_\varepsilon \cdot \nabla$ as

$$\frac{\partial}{\partial z_1} = \left(1, -\frac{y_2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y_1} \right) \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix} \stackrel{\text{def}}{=} b_\varepsilon \cdot \nabla$$

We can now consider the unknown velocity, pressure and micro-rotation fields as functions of the time variable t and the original space variable (z_1, z_2) or as functions of t and the new rescaled space variables (y_1, y_2) . For the sake of notational simplicity we will still denote them as v^ε , p^ε and Z^ε whatever we choose as space variables i.e.

$$\begin{aligned} v^\varepsilon(t, z_1, z_2) &= v^\varepsilon(t, y_1, \varepsilon h^\varepsilon(y_1)y_2) \quad \forall (t, z) \in (0, T) \times \Omega^\varepsilon \\ &:= v^\varepsilon(t, y_1, y_2) \quad \forall (t, y) \in (0, T) \times \Omega, \end{aligned}$$

and similarly

$$\begin{aligned} p^\varepsilon(t, z) &:= p^\varepsilon(t, y), & Z^\varepsilon(t, z) &:= Z^\varepsilon(t, y) \\ f^\varepsilon(t, z) &:= f^\varepsilon(t, y), & g^\varepsilon(t, z) &:= g^\varepsilon(t, y). \end{aligned}$$

We obtain the following a priori estimates.

Proposition 1. *Let us assume that $\varepsilon^2 \bar{f}^\varepsilon$ is bounded in $(L^2((0, T) \times \Omega))^3$, $\varepsilon \bar{v}_0^\varepsilon$ is bounded in $(L^2(\Omega))^3$ and $\varepsilon k^\varepsilon$ is bounded in $L^\infty(0, T; L^{\infty}_+(\Gamma_0))$. Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$. Then there exists a constant $C > 0$ independent of ε such that*

$$\|(\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon)\|_{L^2((0, T) \times \Omega)} \leq C, \quad \|(\varepsilon b_\varepsilon \cdot \nabla Z^\varepsilon)\|_{L^2((0, T) \times \Omega)} \leq C, \quad (6)$$

$$\left\| \frac{\partial v_i^\varepsilon}{\partial y_2} \right\|_{L^2((0, T) \times \Omega)} \leq C, \quad \left\| \frac{\partial Z^\varepsilon}{\partial y_2} \right\|_{L^2((0, T) \times \Omega)} \leq C, \quad (7)$$

$$\left\| \frac{\partial v_i^\varepsilon}{\partial y_1} \right\|_{L^2((0, T) \times \Omega)} \leq \frac{C}{\varepsilon}, \quad \left\| \frac{\partial Z^\varepsilon}{\partial y_1} \right\|_{L^2((0, T) \times \Omega)} \leq \frac{C}{\varepsilon}, \quad (8)$$

$$\|v_i^\varepsilon\|_{L^2((0, T) \times \Omega)} \leq C, \quad \|Z^\varepsilon\|_{L^2((0, T) \times \Omega)} \leq C, \quad \varepsilon^2 \|p^\varepsilon\|_{H^{-1}(0, T; L^2(\Omega))} \leq C \quad (9)$$

for $i = 1, 2$ and

$$\|v_1^\varepsilon\|_{L^2((0, T) \times \Gamma_0)} \leq C \quad (10)$$

where v_1^ε is identified with its trace on Γ_0 .

PROOF. In Problem (P^ε) we choose the test-function $\bar{\Theta} = -\bar{v}^\varepsilon \mathbf{1}_{[0,s]}$ with $s \in [0, T]$ and we get:

$$\begin{aligned}
& \int_0^s \left\langle \frac{\partial \bar{v}^\varepsilon}{\partial t}, \bar{v}^\varepsilon \right\rangle_{(V_{div}^\varepsilon \times H^{1,\varepsilon})', V_{div}^\varepsilon \times H^{1,\varepsilon}} dt + \int_0^s a(\bar{v}^\varepsilon, \bar{v}^\varepsilon) dt \\
& + \int_0^s \underbrace{B(\bar{v}^\varepsilon, \bar{v}^\varepsilon, \bar{v}^\varepsilon)}_{=0} dt + \int_0^s \mathcal{R}(\bar{v}^\varepsilon, \bar{v}^\varepsilon) dt + \int_0^s \int_{\Gamma_0} \underbrace{k^\varepsilon |v^\varepsilon|}_{\geq 0} dz_1 dt \\
& \leq \int_0^s \underbrace{(p^\varepsilon, \operatorname{div} v^\varepsilon)}_{=0} dt - \int_0^s a(\bar{\xi}^\varepsilon, \bar{v}^\varepsilon) dt - \int_0^s \underbrace{B(\bar{\xi}^\varepsilon, \bar{v}^\varepsilon, \bar{v}^\varepsilon)}_{=0} dt \\
& - \int_0^s B(\bar{v}^\varepsilon, \bar{\xi}^\varepsilon, \bar{v}^\varepsilon) dt - \int_0^s \mathcal{R}(\bar{\xi}^\varepsilon, \bar{v}^\varepsilon) dt - \int_0^s \left[\frac{\partial \bar{\xi}^\varepsilon}{\partial t}, \bar{v}^\varepsilon \right] dt + \int_0^s [\bar{f}^\varepsilon, \bar{v}^\varepsilon] dt.
\end{aligned}$$

Then we rewrite all these integrals by using the rescaled variables (y_1, y_2) and we apply Young's inequality and Gronwall's lemma to derive (6)-(9). The reader is referred to Proposition 3.2 and Proposition 3.3 in [7] for the detailed computations. Finally by using Poincaré's inequality

$$\|w_1\|_{L^2(\Gamma_0)} \leq \left\| \frac{\partial w_1}{\partial y_2} \right\|_{L^2(\Omega)}$$

for all w in the closure in $(H^1(\Omega))^2$ of \tilde{V} with

$$\tilde{V} = \{v \in (C^\infty(\bar{\Omega}))^2 : v \text{ is } L\text{-periodic in } y_1, v|_{\Gamma_1} = 0, v \cdot n|_{\Gamma_0} = 0\}$$

we obtain (10). ■

In order to keep some information at the limit about the dependence of the flow with respect to the rapid scale oscillations of the boundary we choose to apply the two-scale convergence technique, introduced by G. Nguetseng ([18]) and G. Allaire ([1]) in the framework of elliptic partial differential equations and extended later on to time-dependent problems (see for instance [17, 13, 19, 7]). For the convenience of the reader let us recall the definition and main results of two-scale convergence.

Let \mathcal{O} be an open bounded subset of \mathbb{R}^d with $d \geq 1$, $Y = [0, 1]^d$,

$$\mathcal{C}_\#^\infty(Y) = \{\varphi \in C^\infty(\mathbb{R}^d) : \varphi \text{ is } Y\text{-periodic}\},$$

and

$$H_\#^1(Y) = \text{closure of } \mathcal{C}_\#^\infty(Y) \text{ in } H^1(Y).$$

Definition 1. A sequence $(w^\varepsilon)_{\varepsilon>0}$ of $L^2((0, T) \times \mathcal{O})$ (resp. $H^{-1}(0, T; L^2(\mathcal{O}))$) two-scale converges to $w^0 \in L^2(0, T; L^2(\mathcal{O} \times Y))$ (resp. $w^0 \in H^{-1}(0, T; L^2(\mathcal{O} \times Y))$) if and only if

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathcal{O}} w^\varepsilon(t, y) \varphi\left(y, \frac{y}{\varepsilon}\right) \theta(t) dy dt = \int_0^T \int_{\mathcal{O} \times Y} w^0(t, y, \eta) \varphi(y, \eta) \theta(t) d\eta dy dt$$

for all $\theta \in \mathcal{D}(0, T)$, for all $\varphi \in \mathcal{D}(\mathcal{O}; \mathcal{C}_\#^\infty(Y))$. In such a case we will denote $w^\varepsilon \rightharpoonup w^0$.

Remark 2. We may consider less regular test-functions namely test-functions in $\mathcal{C}([0, T]; \mathcal{C}(\overline{\mathcal{O}}; \mathcal{C}_\#(Y)))$ (resp. $\mathcal{C}_0^1([0, T]; \mathcal{C}(\overline{\mathcal{O}}; \mathcal{C}_\#(Y)))$). With Lemma 1.3 in [1] we obtain immediately that any sequence $(w^\varepsilon)_{\varepsilon>0}$ with $w^\varepsilon(t, y) = w\left(t, y, \frac{y}{\varepsilon}\right)$ for all $(t, y) \in (0, T) \times \mathcal{O}$ and $w \in \mathcal{C}([0, T]; \mathcal{C}(\overline{\mathcal{O}}; \mathcal{C}_\#(Y)))$ two-scale converges to w .

Theorem 2 (Theorem 4.2 in [7]). *Let $(w^\varepsilon)_{\varepsilon>0}$ be bounded in $L^2((0, T) \times \mathcal{O})$ (resp. in $H^{-1}(0, T; L^2(\mathcal{O}))$). There exists $w^0 \in L^2(0, T; L^2(\mathcal{O} \times Y))$ (resp. $w^0 \in H^{-1}(0, T; L^2(\mathcal{O} \times Y))$) such that, possibly extracting a subsequence still denoted $(w^\varepsilon)_{\varepsilon>0}$, we have*

$$w^\varepsilon \rightharpoonup w^0.$$

Then, having in mind the previous a priori estimates for v^ε we obtain

Proposition 2 (Two-scale limit of the velocity). *Let us assume that $\varepsilon^2 \bar{f}^\varepsilon$ is bounded in $(L^2((0, T) \times \Omega))^3$, $\varepsilon \bar{v}_0^\varepsilon$ is bounded in $(L^2(\Omega))^3$ and $\varepsilon k^\varepsilon$ is bounded in $L^\infty(0, T; L^+_\infty(\Gamma_0))$. Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$.*

There exist $v^0 \in (L^2(0, T; L^2(\Omega; H^1_\#(Y))))^2$ such that $\frac{\partial v^0}{\partial y_2} \in (L^2(0, T; L^2(\Omega \times Y)))^2$ and $v^1 \in (L^2(0, T; L^2(\Omega \times (0, 1); H^1_\#(0, 1)_{/\mathbb{R}})))^2$ such that, for $i = 1, 2$:

$$v_i^\varepsilon \rightharpoonup v_i^0, \quad \frac{\partial v_i^\varepsilon}{\partial y_2} \rightharpoonup \frac{\partial v_i^0}{\partial y_2} + \frac{\partial v_i^1}{\partial \eta_2}, \quad \varepsilon \frac{\partial v_i^\varepsilon}{\partial y_1} \rightharpoonup \frac{\partial v_i^0}{\partial \eta_1}.$$

Furthermore v^0 does not depend on η_2 , v^0 is divergence free in the following sense

$$h(y_1, \eta_1) \frac{\partial v_1^0}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1}(y_1, \eta_1) \frac{\partial v_1^0}{\partial y_2} + \frac{\partial v_2^0}{\partial y_2} = 0 \quad \text{in } (0, T) \times \Omega \times (0, 1),$$

and satisfies the following boundary conditions

$$v^0 = 0 \quad \text{on } (0, T) \times \Gamma_1 \times (0, 1), \quad v^0 \cdot n = v_2^0 = 0 \quad \text{on } (0, T) \times \Gamma_0 \times (0, 1).$$

Moreover

$$v_1^\varepsilon \rightharpoonup v_1^0 \quad \text{on } (0, T) \times \Gamma_0 \times (0, 1).$$

PROOF. The first part of the result is obtained as in Proposition 4.3 in [7] and there remains only to establish the properties of v^0 on the boundary.

With (10) we know that the sequence $(v_1^\varepsilon(\cdot, \cdot, 0))_{\varepsilon>0}$ is uniformly bounded in $L^2((0, T) \times \Gamma_0) = L^2((0, T) \times (0, L))$ hence it admits a two-scale limit, denoted as $\xi = \xi(t, y_1, \eta_1)$. We choose $\varphi \in \mathcal{C}^\infty(\bar{\Omega}; \mathcal{C}_\#^\infty(0, 1))$ and $\theta \in \mathcal{D}(0, T)$:

$$\begin{aligned} \int_0^T \int_\Omega \frac{\partial v_1^\varepsilon}{\partial y_2}(t, y) \varphi\left(y, \frac{y_1}{\varepsilon}\right) \theta(t) dy dt &= - \int_0^T \int_\Omega v_1^\varepsilon(t, y) \frac{\partial \varphi}{\partial y_2}\left(y, \frac{y_1}{\varepsilon}\right) \theta(t) dy dt \\ &- \int_0^T \int_0^L v_1^\varepsilon(t, y_1, 0) \varphi\left(y_1, 0, \frac{y_1}{\varepsilon}\right) \theta(t) dy dt. \end{aligned}$$

We pass to the limit as ε tends to zero:

$$\begin{aligned} &\int_0^T \int_{\Omega \times Y} \frac{\partial v_1^0}{\partial y_2}(t, y, \eta_1) \varphi(y, \eta_1) \theta(t) d\eta dy dt \\ &+ \underbrace{\int_0^T \int_{\Omega \times Y} \frac{\partial v_1^1}{\partial \eta_2}(t, y, \eta) \varphi(y, \eta_1) \theta(t) d\eta dy dt}_{=0} \\ &= - \int_0^T \int_{\Omega \times Y} v_1^0(t, y, \eta_1) \frac{\partial \varphi}{\partial y_2}(y, \eta_1) \theta(t) d\eta dy dt \\ &- \int_0^T \int_{(0, L) \times (0, 1)} \xi(t, y_1, \eta_1) \varphi(y_1, 0, \eta_1) \theta(t) d\eta_1 dy_1 dt. \end{aligned}$$

With Green's formula we get

$$v_1^0 = \xi \quad \text{on } (0, T) \times \Gamma_0 \times (0, 1), \quad v_1^0 = 0 \quad \text{on } (0, T) \times \Gamma_1 \times (0, 1).$$

Starting now from $\int_0^T \int_\Omega \frac{\partial v_2^\varepsilon}{\partial y_2}(t, y) \varphi\left(y, \frac{y_1}{\varepsilon}\right) \theta(t) dy dt$ we obtain also

$$v_2^0 = 0 \quad \text{on } (0, T) \times (\Gamma_0 \cup \Gamma_1) \times (0, 1).$$

■

Similarly we have

Proposition 3 (Two-scale limit of the micro-rotation). *Assume that $\varepsilon^2 \bar{f}^\varepsilon$ is bounded in $(L^2((0, T) \times \Omega))^3$, $\varepsilon \bar{v}_0^\varepsilon$ is bounded in $(L^2(\Omega))^3$ and $\varepsilon k^\varepsilon$ is bounded in $L^\infty(0, T; L_+^\infty(\Gamma_0))$. Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$.*

There exist $Z^0 \in L^2(0, T; L^2(\Omega; H_\#^1(Y)))$ such that $\frac{\partial Z^0}{\partial y_2} \in L^2(0, T; L^2(\Omega \times Y))$ and $Z^1 \in L^2(0, T; L^2(\Omega \times (0, 1); H_\#^1(0, 1)_{/\mathbb{R}}))$ such that

$$Z^\varepsilon \rightarrow\rightarrow Z^0, \quad \frac{\partial Z^\varepsilon}{\partial y_2} \rightarrow\rightarrow \frac{\partial Z^0}{\partial y_2} + \frac{\partial Z^1}{\partial \eta_2}, \quad \frac{\partial Z^\varepsilon}{\partial y_1} \rightarrow\rightarrow \frac{\partial Z^0}{\partial \eta_1}.$$

Furthermore Z^0 does not depend on η_2 , and $Z^0 \equiv 0$ on $(0, T) \times (\Gamma_0 \cup \Gamma_1) \times (0, 1)$.

Proposition 4 (Two-scale limit of the pressure). *Let us assume that $\varepsilon^2 \bar{f}^\varepsilon$ is bounded in $(L^2((0, T) \times \Omega))^3$, $\varepsilon \bar{v}_0^\varepsilon$ is bounded in $(L^2(\Omega))^3$ and $\varepsilon k^\varepsilon$ is bounded in $L^\infty(0, T; L^2_+(\Gamma_0))$. Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$. There exists $p^0 \in H^{-1}(0, T; L^2(\Omega \times Y))$ such that, possibly extracting a subsequence still denoted $(p^\varepsilon)_{\varepsilon > 0}$, we have*

$$\varepsilon^2 p^\varepsilon \rightharpoonup p^0.$$

Moreover p^0 depends only on t and y_1 , $p^0 \in H^{-1}(0, T; H^1_\#(0, L))$, $\frac{\partial p^0}{\partial y_1} \in L^2((0, T) \times (0, L))$ and

$$\int_0^L p^0(t, y_1) \left(\int_0^1 h(y_1, \eta_1) d\eta_1 \right) dy_1 = 0 \quad \text{in } \mathcal{D}'(0, T).$$

PROOF. The reader is referred to the proofs of Proposition 4.4 and Proposition 4.5 in [7].

4. The limit problem

There remains to identify the limit problem for (v^0, p^0, Z^0) . Let us recall the definition of the differential operator $b_\varepsilon \cdot \nabla$:

$$b_\varepsilon \cdot \nabla = \left(1, -\frac{y_2}{h^\varepsilon} \frac{\partial h^\varepsilon}{\partial y_1} \right) \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix}.$$

We observe that

$$\begin{aligned} \varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon &= \varepsilon \frac{\partial v_i^\varepsilon}{\partial y_1}(y) - \frac{y_2}{h(y_1, \frac{y_2}{\varepsilon})} \left(\varepsilon \frac{\partial h}{\partial y_1} \left(y_1, \frac{y_2}{\varepsilon} \right) + \frac{\partial h}{\partial \eta_1} \left(y_1, \frac{y_2}{\varepsilon} \right) \right) \frac{\partial v_i^\varepsilon}{\partial y_2}(y) \\ &\rightarrow \frac{\partial v_i^0}{\partial \eta_1}(y, \eta_1) - \frac{y_2}{h(y_1, \eta_1)} \frac{\partial h}{\partial \eta_1}(y_1, \eta_1) \left(\frac{\partial v_i^0}{\partial y_2}(y, \eta_1) + \frac{\partial v_i^1}{\partial \eta_2}(y, \eta) \right) \end{aligned}$$

for $i = 1, 2$. Hence we get a limit differential operator defined as

$$\bar{b} \cdot \nabla = \left(1, -\frac{y_2}{h} \frac{\partial h}{\partial \eta_1} \right) \begin{pmatrix} \frac{\partial}{\partial \eta_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix}$$

such that

$$\varepsilon b_\varepsilon \cdot \nabla v_i^\varepsilon \rightarrow \bar{b} \cdot \nabla v_i^0 - \frac{y_2}{h} \frac{\partial h}{\partial \eta_1} \frac{\partial v_i^1}{\partial \eta_2}$$

and similarly

$$b_\varepsilon \cdot \nabla Z^\varepsilon \rightarrow \bar{b} \cdot \nabla Z^0 - \frac{y_2}{h} \frac{\partial h}{\partial \eta_1} \frac{\partial Z^1}{\partial \eta_2}.$$

The appropriate functional framework for the limit problem is thus given by

$$\tilde{V} = \left\{ \varphi \in (\mathcal{C}^\infty(\bar{\Omega}; \mathcal{C}_\#^\infty(0, 1)))^2 : \varphi \text{ is } L\text{-periodic in } y_1, \varphi|_{\Gamma_1 \times (0, 1)} = 0, \right. \\ \left. \varphi \cdot n|_{\Gamma_0 \times (0, 1)} = 0 \right\},$$

$$\tilde{V}_{div} = \left\{ \varphi \in \tilde{V} : h \frac{\partial \varphi_1}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1} \frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } \Omega \times (0, 1) \right\},$$

$$V_{div} = \text{closure of } \tilde{V}_{div} \text{ in } \left(L^2([0, L]; F) \right)^2,$$

$$\tilde{V}_{0,div} = \{ \varphi \in \tilde{V}_{div} : \varphi|_{\Gamma_0 \times (0, 1)} = 0 \}, \quad V_{0,div} = \text{closure of } \tilde{V}_{0,div} \text{ in } \left(L^2([0, L]; F) \right)^2,$$

for the velocity field and for the micro-rotation field

$$\tilde{H}^1 = \left\{ \psi \in \mathcal{C}^\infty(\bar{\Omega}; \mathcal{C}_\#^\infty(0, 1)) : \psi \text{ is } L\text{-periodic in } y_1, \psi|_{(\Gamma_0 \cup \Gamma_1) \times (0, 1)} = 0 \right\},$$

$$H_{0\#}^1 = \text{closure of } \tilde{H}^1 \text{ in } L^2([0, L]; F),$$

with

$$F = \left\{ \varphi \in L^2((0, 1); H_\#^1(0, 1)) : \frac{\partial \varphi}{\partial y_2} \in L^2((0, 1) \times (0, 1)) \right\}.$$

Then we obtain first

Proposition 5. *Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$ and assume that $\varepsilon \bar{v}_0^\varepsilon$ is bounded in $(L^2(\Omega))^3$. Let us assume moreover that there exist $f \in (\mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_\#(0, 1))))^2$, $g \in \mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_\#(0, 1)))$ and $k \in \mathcal{C}([0, T]; \mathcal{C}([0, L]; \mathcal{C}_\#(0, 1)))$ such that k takes its values in \mathbb{R}_*^+ , f , g and k are L -periodic in y_1 and*

$$\varepsilon^2 f^\varepsilon(t, y) = f\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon^2 g^\varepsilon(t, y) = g\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon k^\varepsilon(t, y_1) = k\left(t, y_1, \frac{y_1}{\varepsilon}\right)$$

for all $(t, y) \in [0, T] \times \bar{\Omega}$. Then v^0 , p^0 and Z^0 satisfy

$$\begin{aligned}
& (\nu + \nu_r) \int_0^T \int_{\Omega \times (0,1)} \sum_{i=1}^2 \left(h(\bar{b} \cdot \nabla v_i^0)(\bar{b} \cdot \nabla \varphi_i) + \frac{1}{h} \frac{\partial v_i^0}{\partial y_2} \frac{\partial \varphi_i}{\partial y_2} \right) \theta \, d\eta_1 \, dy \, dt \\
& + \alpha \int_0^T \int_{\Omega \times (0,1)} \left(h(\bar{b} \cdot \nabla Z^0)(\bar{b} \cdot \nabla \psi) + \frac{1}{h} \frac{\partial Z^0}{\partial y_2} \frac{\partial \psi}{\partial y_2} \right) \theta \, d\eta_1 \, dy \, dt \\
& + \int_0^T \int_{\Omega \times (0,1)} \frac{\partial p^0}{\partial y_1} h \varphi_1 \theta \, d\eta_1 \, dy \, dt \\
& = -(\nu + \nu_r) \int_0^T \int_{\Omega \times (0,1)} \left(h(\bar{b} \cdot \nabla \bar{U})(\bar{b} \cdot \nabla \varphi_1) + \frac{1}{h} \frac{\partial \bar{U}}{\partial y_2} \frac{\partial \varphi_1}{\partial y_2} \right) \theta \, d\eta_1 \, dy \, dt \quad (11) \\
& - \alpha \int_0^T \int_{\Omega \times (0,1)} \left(h(\bar{b} \cdot \nabla \bar{W})(\bar{b} \cdot \nabla \psi) + \frac{1}{h} \frac{\partial \bar{W}}{\partial y_2} \frac{\partial \psi}{\partial y_2} \right) \theta \, d\eta_1 \, dy \, dt \\
& + \int_0^T \int_{\Omega \times (0,1)} f \varphi \theta \, h \, d\eta_1 \, dy \, dt + \int_0^T \int_{\Omega \times (0,1)} g \psi \theta \, h \, d\eta_1 \, dy \, dt, \\
& \forall \theta \in \mathcal{D}(0, T), \quad \forall \Theta = (\varphi, \psi) \in V_{0,div} \times H_{0,\sharp}^1,
\end{aligned}$$

where

$$\begin{aligned}
\bar{U}(t, y_1, y_2, \eta_1) &= s_0(t) + (U_0(t) - s_0(t)) \mathcal{U}(h(y_1, \eta_1) y_2), \\
\bar{W}(t, y_1, y_2, \eta_1) &= W_0(t) \mathcal{W}(h(y_1, \eta_1) y_2),
\end{aligned}$$

for all $(t, y_1, y_2, \eta_1) \in [0, T] \times \bar{\Omega} \times [0, 1]$.

PROOF. Let $\Theta = (\varphi, \psi) \in \tilde{V}_{0,div} \times \tilde{H}^1$ and let $\Theta^\varepsilon = (\varphi^\varepsilon, \psi^\varepsilon)$ with

$$\varphi^\varepsilon(z) = \varphi \left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon} \right) + \frac{z_2}{h^\varepsilon(z_1)} \frac{\partial h}{\partial y_1} \left(z_1, \frac{z_1}{\varepsilon} \right) \varphi_1 \left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon} \right) e_2$$

and $\psi^\varepsilon(z) = \psi \left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon} \right)$ for all $(z_1, z_2) \in \Omega^\varepsilon$. We can check immediately that $\Theta^\varepsilon \in \tilde{V}^\varepsilon \times \tilde{H}^{1,\varepsilon}$ and for any $\theta \in \mathcal{D}(0, T)$ we have

$$j^\varepsilon(\pm \varphi^\varepsilon \theta + v^\varepsilon) - j^\varepsilon(v^\varepsilon) = \int_0^T \int_{\Gamma_0} k^\varepsilon (|\pm \varphi^\varepsilon \theta + v^\varepsilon| - |v^\varepsilon|) \, dz_1 \, dt = 0.$$

We get

$$\begin{aligned}
& - \int_0^T [\bar{v}^\varepsilon, \Theta^\varepsilon] \theta' \, dt + \int_0^T a(\bar{v}^\varepsilon, \Theta^\varepsilon) \theta \, dt + \int_0^T B(\bar{v}^\varepsilon, \bar{v}^\varepsilon, \Theta^\varepsilon) \theta \, dt + \int_0^T \mathcal{R}(\bar{v}^\varepsilon, \Theta^\varepsilon) \theta \, dt \\
& = \int_0^T (p^\varepsilon, \operatorname{div} \varphi^\varepsilon) \theta \, dt + \int_0^T (\mathcal{F}(\bar{v}^\varepsilon), \Theta^\varepsilon) \theta \, dt.
\end{aligned}$$

Hence we may perform the same computations as in Theorem 5.1 in [7]: we rewrite these integrals in terms of (y_1, y_2) , we multiply by ε and we pass to the limit as ε tends to zero. \blacksquare

We may observe that the time variable appears now as a parameter in (11). Moreover by choosing test-functions $\Theta = (0, \psi) \in V_{0,div} \times H_{0,\#}^1$ we obtain a variational equality for Z^0 leading to the following limit problem:

Problem (P_{Z^0}) Find $Z^0 \in L^2(0, T; H_{0,\#}^1)$ such that

$$\begin{aligned} \alpha \int_0^L a_{y_1}(Z^0, \psi) dy_1 &= -\alpha \int_0^L a_{y_1}(\bar{W}(t), \psi) dy_1 \\ + \int_0^L \left(\int_Y g(t, y_1, \cdot, \cdot) h(y_1, \cdot) \psi d\eta_1 dy_2 \right) dy_1 &\quad \forall \psi \in H_{0,\#}^1, \quad \forall \text{a.e. } t \in [0, T] \end{aligned}$$

where a_{y_1} is the bilinear symmetric form defined for all $y_1 \in [0, L]$ by

$$\begin{aligned} a_{y_1}(w, \psi) &= \int_Y \left(h(y_1, \cdot) (\bar{b} \cdot \nabla w) (\bar{b} \cdot \nabla \psi) + \frac{1}{h(y_1, \cdot)} \frac{\partial w}{\partial y_2} \frac{\partial \psi}{\partial y_2} \right) d\eta_1 dy_2 \\ \forall (w, \psi) &\in F^2. \end{aligned}$$

Let us emphasize that we recover the same limit problem for Z^0 as in [7]: this is not surprising since the coupling terms in Problem (P^ε) involve only first order derivatives so they disappear at the limit leading to a decoupled problem for (v^0, p^0) on one hand and Z^0 on the other hand. As a consequence we have:

Proposition 6 (Proposition 5.2 in [7]). *Under the assumptions of Proposition 5 the limit micro-rotation field Z^0 is uniquely given by*

$$Z^0(t, y_1, y_2, \eta_1) = W_0(t) z_{y_1}^1(y_2, \eta_1) + z_{t, y_1}^2(y_2, \eta_1) \quad \text{a.e. in } (0, T) \times \Omega \times (0, 1)$$

where $z_{y_1}^1 \in H_{0, y_1, \#}^1$ and $z_{t, y_1}^2 \in H_{0, y_1, \#}^1$ are the unique solutions of the following auxiliary cell problems:

$$a_{y_1}(z_{y_1}^1, \psi) = -a_{y_1}(\mathcal{W}(h(y_1, \cdot)), \psi) \quad \forall \psi \in H_{0, y_1, \#}^1$$

and

$$a_{y_1}(z_{t, y_1}^2, \psi) = \alpha^{-1} \int_Y g(t, y_1, \cdot, \cdot) h(y_1, \cdot) \psi d\eta_1 dy_2 \quad \forall \psi \in H_{0, y_1, \#}^1$$

with

$$H_{0, y_1, \#}^1 = \text{closure of } \left\{ \psi \in C^\infty([0, 1]; \mathcal{C}_\#^\infty(0, 1)) : \psi|_{\{0, 1\} \times (0, 1)} = 0 \right\} \text{ in } F.$$

Going back to (11) and choosing $\Theta = (\varphi, 0) \in V_{0,div} \times H_{0,\#}^1$ we obtain

$$\begin{aligned} (\nu + \nu_r) \int_0^T \int_{\Omega \times (0, 1)} \sum_{i=1}^2 \left(h(\bar{b} \cdot \nabla v_i^0) (\bar{b} \cdot \nabla \varphi_i) + \frac{1}{h} \frac{\partial v_i^0}{\partial y_2} \frac{\partial \varphi_i}{\partial y_2} \right) \theta d\eta_1 dy dt \\ + \int_0^T \int_{\Omega \times (0, 1)} \frac{\partial p^0}{\partial y_1} h \varphi_1 \theta d\eta_1 dy dt \\ = -(\nu + \nu_r) \int_0^T \int_{\Omega \times (0, 1)} \left(h(\bar{b} \cdot \nabla \bar{U}) (\bar{b} \cdot \nabla \varphi_1) + \frac{1}{h} \frac{\partial \bar{U}}{\partial y_2} \frac{\partial \varphi_1}{\partial y_2} \right) \theta d\eta_1 dy dt \\ + \int_0^T \int_{\Omega \times (0, 1)} f \varphi \theta h d\eta_1 dy dt \quad \forall \theta \in \mathcal{D}(0, T), \quad \forall \varphi \in V_{0,div}. \end{aligned} \tag{12}$$

Here the test-functions do not satisfy the same boundary conditions as v^0 on Γ_0 so (12) is not a variational problem for (v^0, p^0) . Nevertheless, starting from (12) we introduce also well-posed auxiliary cell problems for the velocity. More precisely for all $y_1 \in [0, L]$, we let

$$\tilde{V}_{y_1} = \left\{ \varphi \in (\mathcal{C}^\infty([0, 1]; \mathcal{C}_\#^\infty(0, 1)))^2 : \varphi(1, \cdot) = 0, \varphi_2(0, \cdot) = 0 \text{ on } (0, 1) \right\},$$

$$\tilde{V}_{y_1, div} = \left\{ \varphi \in \tilde{V}_{y_1} : h(y_1, \cdot) \frac{\partial \varphi_1}{\partial \eta_1} - y_2 \frac{\partial h}{\partial \eta_1}(y_1, \cdot) \frac{\partial \varphi_1}{\partial y_2} + \frac{\partial \varphi_2}{\partial y_2} = 0 \text{ in } Y \right\}$$

$$\tilde{V}_{0y_1, div} = \left\{ \varphi \in \tilde{V}_{y_1, div} : \varphi_1(0, \cdot) = 0 \text{ on } (0, 1) \right\},$$

$$V_{y_1, div} = \text{closure of } \tilde{V}_{y_1, div} \text{ in } F^2, \quad V_{0y_1, div} = \text{closure of } \tilde{V}_{0y_1, div} \text{ in } F^2$$

and

$$\bar{a}_{y_1}(w, \varphi) = (\nu + \nu_r) \sum_{i=1}^2 a_{y_1}(w_i, \varphi_i) \quad \forall (w, \varphi) \in V_{y_1, div}.$$

Then, for all $(t, y_1) \in [0, T] \times [0, L]$, we define $w_{y_1}^1, w_{y_1}^2, w_{t, y_1}^3$ in $V_{0y_1, div}$ as the unique solutions of

$$\bar{a}_{y_1}(w_{y_1}^1, \varphi) = - \int_Y h(y_1, \cdot) \varphi_1 d\eta_1 dy_2 \quad \forall \varphi \in V_{0y_1, div}$$

$$\bar{a}_{y_1}(w_{y_1}^2, \varphi) = -\bar{a}_{y_1}(\mathcal{U}(h(y_1, \cdot) \cdot) e_1, \varphi) \quad \forall \varphi \in V_{0y_1, div}$$

and

$$\bar{a}_{y_1}(w_{t, y_1}^3, \varphi) = \int_Y f(t, y_1, \cdot, \cdot) h(y_1, \cdot) \varphi d\eta_1 dy_2 \quad \forall \varphi \in V_{0y_1, div}.$$

By observing that \bar{a}_{y_1} is coercive on $V_{y_1, div}$ uniformly with respect to y_1 (see Proposition 5.2 in [7]), the continuity properties of h, \mathcal{U} and f imply that the mappings $y_1 \mapsto w_{y_1}^1$ and $y_1 \mapsto w_{y_1}^2$ belong to $\mathcal{C}_\#([0, L]; V_{0y_1, div})$ and $(t, y_1) \mapsto w_{t, y_1}^3$ belongs to $\mathcal{C}([0, T]; \mathcal{C}_\#([0, L]; V_{0y_1, div}))$. So we decompose v^0 as

$$v^0(t, y_1, y_2, \eta_1) = \frac{\partial p^0}{\partial y_1}(t, y_1) w_{y_1}^1(y_2, \eta_1) + (U_0(t) - s_0(t)) w_{y_1}^2(y_2, \eta_1) + w_{t, y_1}^3(y_2, \eta_1) + w^*(t, y_1, y_2, \eta_1) \quad \text{a.e. in } (0, T) \times \Omega \times (0, 1) \quad (13)$$

with $w^* \in L^2(0, T; V_{div})$. With the definition of $w_{y_1}^1, w_{y_1}^2$ and w_{t, y_1}^3 we infer that

$$\int_0^L \bar{a}_{y_1}(w^*(t, y_1, \cdot, \cdot), \varphi(y_1, \cdot, \cdot)) dy_1 = 0 \quad \forall \varphi \in V_{0, div}, \quad \forall \text{a.e. } t \in (0, T)$$

which may be interpreted as an orthogonality property with respect to $V_{0,div}$ and we let

$$V_{div}^* = \left\{ w \in V_{div} : \int_0^L \bar{a}_{y_1}(w(y_1, \cdot, \cdot), \varphi(y_1, \cdot, \cdot)) dy_1 = 0 \quad \forall \varphi \in V_{0,div} \right\}.$$

Then we obtain that the limit pressure p^0 satisfies a Reynolds equation. More precisely

Proposition 7. *Under the assumptions of Proposition 5 the limit pressure p^0 can be decomposed as $\bar{p}^0 + \bar{p}^1$ with $\bar{p}^0(t, \cdot)$ and $\bar{p}^1(t, \cdot)$ given as the unique solutions in $H_{\sharp}^1(0, L)_{|\mathbb{R}}$ of the stationary Reynolds problems*

$$\begin{aligned} \int_0^L \frac{\partial \bar{p}^0}{\partial y_1} \frac{\partial \psi}{\partial y_1} \bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1) dy_1 &= -(U_0(t) - s_0(t)) \int_0^L \frac{\partial \psi}{\partial y_1} \bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^2) dy_1 \\ &- \int_0^L \frac{\partial \psi}{\partial y_1} \bar{a}_{y_1}(w_{y_1}^1, w_{t,y_1}^3) dy_1 \end{aligned}$$

and

$$\int_0^L \frac{\partial \bar{p}^1}{\partial y_1} \frac{\partial \psi}{\partial y_1} \bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1) dy_1 = \int_0^L \frac{\partial \psi}{\partial y_1} \left(\int_Y w_1^*(t, y_1, \cdot, \cdot) h(y_1, \cdot) d\eta_1 dy_2 \right) dy_1$$

for all $\psi \in H_{\sharp}^1(0, L)$ such that

$$\int_0^L \bar{p}^i(t, y_1) \left(\int_0^1 h(y_1, \eta_1) d\eta_1 \right) dy_1 = 0 \quad \forall \text{a.e. } t \in (0, T) \quad i = 0, 1.$$

PROOF. Let $\theta \in \mathcal{D}(0, T)$, $\psi \in \mathcal{C}_{\sharp}^{\infty}([0, L])$ and $\psi^{\varepsilon}(z) = \psi(z_1)$ for all $z = (z_1, z_2) \in \Omega^{\varepsilon}$. Recalling that $\text{div}_z v^{\varepsilon} = 0$ in Ω^{ε} we have

$$\begin{aligned} 0 &= \frac{1}{\varepsilon} \int_0^T \int_{\Omega^{\varepsilon}} \left(\frac{\partial v_1^{\varepsilon}}{\partial z_1}(t, z) + \frac{\partial v_2^{\varepsilon}}{\partial z_2}(t, z) \right) \psi^{\varepsilon}(z) \theta(t) dz dt \\ &= -\frac{1}{\varepsilon} \int_0^T \int_{\Omega^{\varepsilon}} v_1^{\varepsilon}(t, z) \frac{\partial \psi^{\varepsilon}}{\partial z_1}(z) \theta(t) dz dt \\ &= -\int_0^T \int_{\Omega} v_1^{\varepsilon}(t, y) \frac{\partial \psi}{\partial y_1}(y_1) h\left(y_1, \frac{y_1}{\varepsilon}\right) \theta(t) dy dt. \end{aligned}$$

By passing to the limit as ε tends to zero we get

$$0 = \int_0^T \int_{\Omega \times (0,1)} v_1^0(t, y, \eta_1) \frac{\partial \psi}{\partial y_1}(y_1) h(y_1, \eta_1) \theta(t) d\eta_1 dy dt. \quad (14)$$

With (13) we infer that

$$\begin{aligned}
& \int_0^L \frac{\partial p^0}{\partial y_1} \frac{\partial \psi}{\partial y_1} \underbrace{\left(\int_Y w_{y_1,1}^1 h(y_1, \cdot) d\eta_1 dy_2 \right)}_{=-\bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1)} dy_1 \\
& + \int_0^L (U_0(t) - s_0(t)) \frac{\partial \psi}{\partial y_1} \underbrace{\left(\int_Y w_{y_1,1}^2 h(y_1, \cdot) d\eta_1 dy_2 \right)}_{=-\bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^2)} dy_1 \\
& + \int_0^L \frac{\partial \psi}{\partial y_1} \underbrace{\left(\int_Y w_{t,y_1,1}^3 h(y_1, \cdot) d\eta_1 dy_2 \right)}_{=-\bar{a}_{y_1}(w_{y_1}^1, w_{t,y_1}^3)} dy_1 \\
& + \int_0^L \frac{\partial \psi}{\partial y_1} \left(\int_Y w_1^*(t, y_1, \cdot, \cdot) h(y_1, \cdot) d\eta_1 dy_2 \right) dy_1 = 0 \quad \forall \text{a.e. } t \in (0, T).
\end{aligned}$$

Moreover let $\varphi_{y_1} \in \tilde{V}_{0y_1,div}$ be given by

$$\varphi_{y_1}(y_2, \eta_1) = \left(\frac{y_2^2(y_2 - 1)}{h(y_1, \eta_1)}, \frac{\partial h}{\partial \eta_1}(y_1, \eta_1) \frac{y_2^2(y_2^2 - y_2)}{h(y_1, \eta_1)} \right) \quad \forall (y_2, \eta_1) \in Y, \quad \forall y_1 \in [0, L].$$

We have

$$\frac{1}{12} = \bar{a}_{y_1}(w_{y_1}^1, \varphi_{y_1}) \leq (\bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1))^{1/2} (\bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1}))^{1/2} \quad \forall y_1 \in [0, L]$$

and the mapping $y_1 \mapsto \bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1})$ is continuous from $[0, L]$ to \mathbb{R}_*^+ . Hence

$$\frac{1}{144 \min_{y_1 \in [0, L]} \bar{a}_{y_1}(\varphi_{y_1}, \varphi_{y_1})} \leq \bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1) \quad \forall y_1 \in [0, L]$$

and we may conclude. ■

Remark 3. We may observe that \bar{p}^1 depends linearly and continuously on w^* i.e. there exists $\mathcal{Q} \in \mathcal{L}_c(V_{div}^*, H_{\sharp}^1(0, L)_{|\mathbb{R}})$ such that $\bar{p}^1 = \mathcal{Q}(w^*)$.

It remains to identify the limit problem for v^0 .

Theorem 3. Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$ and $f \in (\mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_{\sharp}(0, 1)))^2$, $g \in \mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_{\sharp}(0, 1)))$ and $k \in \mathcal{C}([0, T]; \mathcal{C}([0, L]; \mathcal{C}_{\sharp}(0, 1)))$ such that k takes its values in \mathbb{R}_*^+ , f , g and k are L -periodic in y_1 and

$$\varepsilon^2 f^\varepsilon(t, y) = f\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon^2 g^\varepsilon(t, y) = g\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon k^\varepsilon(t, y_1) = k\left(t, y_1, \frac{y_1}{\varepsilon}\right)$$

for all $(t, y) \in [0, T] \times \bar{\Omega}$. Let us assume moreover that $\varepsilon Z_0^\varepsilon$ is bounded in $L^2(\Omega)$ and that u_0^ε satisfies the following compatibility condition: there exists $\mathcal{U} \in C^\infty(\mathbb{R})$ such that

$$\mathcal{U}(0) = \mathcal{U}'(0) = 0, \quad \mathcal{U}(X) = 1 \quad \forall X \in [h_m, h_M]$$

and

$$u_0^\varepsilon = s_0(0)e_1 + (U_0(0) - s_0(0))\mathcal{U}\left(\frac{z_2}{\varepsilon}\right)e_1.$$

Then the limit problem for v^0 is given by

Problem (P_{v^0})

$$\begin{aligned} & \int_0^T \int_0^L \bar{a}_{y_1}(v^0, \varphi) dy_1 dt + \int_0^T \int_0^L \frac{\partial p^0}{\partial y_1} \left(\int_Y h(y_1, \cdot) \varphi_1 d\eta_1 dy_2 \right) dy_1 dt \\ & + \int_0^T \int_{(0,L) \times (0,1)} k(|\varphi + v^0| - |v^0|) d\eta_1 dy_1 dt \geq - \int_0^T \int_0^L \bar{a}_{y_1}(\bar{U}e_1, \varphi) dy_1 dt \\ & + \int_0^T \int_0^L \left(\int_Y fh\varphi d\eta_1 dy_2 \right) dy_1 dt \quad \forall \varphi \in L^2(0, T; V_{div}). \end{aligned}$$

PROOF. With the compatibility condition for u_0^ε we may choose the extension U^ε given by

$$U^\varepsilon(t, z_2) = s_0(t) + (U_0(t) - s_0(t))\mathcal{U}\left(\frac{z_2}{\varepsilon}\right)$$

which implies $v_0^\varepsilon = 0$.

Now let $\varphi \in \tilde{V}_{div}$ and $\Theta^\varepsilon = (\varphi^\varepsilon, 0)$ with

$$\varphi^\varepsilon(z) = \varphi\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon}\right) + \frac{z_2}{h^\varepsilon(z_1)} \frac{\partial h}{\partial y_1}\left(z_1, \frac{z_1}{\varepsilon}\right) \varphi_1\left(z_1, \frac{z_2}{\varepsilon h^\varepsilon(z_1)}, \frac{z_1}{\varepsilon}\right) e_2$$

for all $(z_1, z_2) \in \Omega^\varepsilon$. We have $\Theta^\varepsilon \in \tilde{V}^\varepsilon \times \tilde{H}^{1,\varepsilon}$ and we introduce this test-function in (P^ε) . Then we introduce $\bar{\Theta}^\varepsilon = (-v^\varepsilon, 0)$ as test-function in (P^ε) and we add the two inequalities. For all $\theta \in \mathcal{D}(0, T)$ we get

$$\begin{aligned} & - \int_0^T [\bar{v}^\varepsilon, \Theta^\varepsilon] \theta' dt + \int_0^T a(\bar{v}^\varepsilon, \Theta^\varepsilon) \theta dt + \int_0^T B(\bar{v}^\varepsilon, \bar{v}^\varepsilon, \Theta^\varepsilon) \theta dt \\ & + \int_0^T \mathcal{R}(\bar{v}^\varepsilon, \Theta^\varepsilon) \theta dt - \int_0^T \mathcal{R}(\bar{v}^\varepsilon, (v^\varepsilon, 0)) dt + \underbrace{j^\varepsilon(\varphi^\varepsilon \theta + v^\varepsilon) - j^\varepsilon(v^\varepsilon)}_{\leq j^\varepsilon(\varphi^\varepsilon \theta)} \\ & \geq j^\varepsilon(v^\varepsilon) + \int_0^T (p^\varepsilon, \operatorname{div} \varphi^\varepsilon) \theta dt + \int_0^T (\mathcal{F}(\bar{v}^\varepsilon), \Theta^\varepsilon) \theta dt \\ & - \int_0^T (\mathcal{F}(\bar{v}^\varepsilon), (v^\varepsilon, 0)) dt + (\nu + \nu_r) \int_0^T (\nabla v^\varepsilon, \nabla v^\varepsilon) dt + \underbrace{\frac{1}{2} \|v^\varepsilon(T)\|_{L^2(\Omega^\varepsilon)}^2}_{\geq 0}. \end{aligned}$$

We rewrite the integrals in terms of (y_1, y_2) and we multiply by ε . Then we pass to the limit as ε tends to zero by using the same techniques as in Theorem 5.1 in [7] (see also Proposition 5), except for the quadratic term $\varepsilon \int_0^T (\nabla v^\varepsilon, \nabla v^\varepsilon) dt$ for which we apply Proposition 1.6 in [1], the boundary term $\varepsilon j^\varepsilon(\varphi^\varepsilon \theta)$ which

converges to $\int_0^T \int_{(0,L) \times (0,1)} k(t, y_1, \eta_1) |\varphi_1(y_1, 0, \eta_1) \theta(t)| d\eta_1 dy_1 dt$ thanks to Remark 2 and for the boundary term $\varepsilon j^\varepsilon(v^\varepsilon)$ for which we apply the following lemma.

Lemma 4. *Let $(w^\varepsilon)_{\varepsilon>0}$ be a bounded sequence of $L^2(0, T; L^2(0, L))$ and let us denote as $w^0 \in L^2(0, T; L^2((0, L) \times (0, 1)))$ its two-scale limit. Then*

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_0^L |w^\varepsilon(t, y_1)| dy_1 dt \geq \int_0^T \int_{(0,L) \times (0,1)} |w^0(t, y_1, \eta_1)| d\eta_1 dy_1 dt.$$

PROOF. Let us denote as ϕ the convex, proper and continuous function defined by $\phi(s) = |s|$ for all $s \in \mathbb{R}$. The Yosida approximation of $A = \partial\phi$ is defined by $A_\lambda = \{\nabla\phi_\lambda\}$ with $\phi_\lambda \in C^1(\mathbb{R}; \mathbb{R})$ given by

$$\phi_\lambda(z) = \inf_{s \in \mathbb{R}} \left(\phi(s) + \frac{|s - z|^2}{2\lambda} \right) \quad \forall z \in \mathbb{R}, \quad \forall \lambda > 0.$$

With the usual properties about the subdifferential of convex functions (see for instance Proposition 2.6 and Proposition 2.11 in [8]), we know that ϕ_λ is convex, $\nabla\phi_\lambda$ is $\frac{1}{\lambda}$ -Lipschitz continuous and

$$0 \leq \phi_\lambda(z) \leq \phi(z), \quad \lim_{\lambda \rightarrow 0} \phi_\lambda(z) = \phi(z) \quad \forall z \in \mathbb{R}. \quad (15)$$

Now let $\lambda > 0$. For all $\varepsilon > 0$, by using the convexity of ϕ_λ we have

$$\begin{aligned} & \int_0^T \int_0^L \phi(w^\varepsilon(t, y_1)) dy_1 dt \geq \int_0^T \int_0^L \phi_\lambda(w^\varepsilon(t, y_1)) dy_1 dt \\ & \geq \int_0^T \int_0^L \phi_\lambda\left(w\left(t, y_1, \frac{y_1}{\varepsilon}\right)\right) dy_1 dt \\ & + \int_0^T \int_0^L \nabla\phi_\lambda\left(w\left(t, y_1, \frac{y_1}{\varepsilon}\right)\right) \left(w^\varepsilon(t, y_1) - w\left(t, y_1, \frac{y_1}{\varepsilon}\right)\right) dy_1 dt \end{aligned}$$

for all $w \in \mathcal{D}((0, T) \times (0, L) \times (0, 1))$. Then we pass to the limit as ε tends to zero: the left hand side of the previous inequality is bounded, so it admits a limit inf while the right hand side admits a limit and we obtain

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_0^L \phi(w^\varepsilon(t, y_1)) dy_1 dt \geq \int_0^T \int_{(0,L) \times (0,1)} \phi_\lambda(w(t, y_1, \eta_1)) dy_1 dt \\ & + \int_0^T \int_{(0,L) \times (0,1)} \nabla\phi_\lambda(w(t, y_1, \eta_1)) (w^0(t, y_1, \eta_1) - w(t, y_1, \eta_1)) dy_1 dt. \end{aligned}$$

Using again the convexity of ϕ_λ we infer

$$\begin{aligned} & \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_0^L \phi(w^\varepsilon(t, y_1)) dy_1 dt \geq \int_0^T \int_{(0,L) \times (0,1)} \phi_\lambda(w^0(t, y_1, \eta_1)) dy_1 dt \\ & + \int_0^T \int_{(0,L) \times (0,1)} \left(\nabla\phi_\lambda(w^0(t, y_1, \eta_1)) - \nabla\phi_\lambda(w(t, y_1, \eta_1)) \right) \\ & \quad \times (w(t, y_1, \eta_1) - w^0(t, y_1, \eta_1)) dy_1 dt. \end{aligned}$$

Since $w^0 \in L^2(0, T; L^2((0, L) \times (0, 1))) = L^2((0, T) \times (0, L) \times (0, 1))$ we may consider a sequence $(w_k^0)_{k \in \mathbb{N}} \in \mathcal{D}((0, T) \times (0, L) \times (0, 1))$ which converges strongly to w^0 in $L^2((0, T) \times (0, L) \times (0, 1))$. Thus

$$\liminf_{\varepsilon \rightarrow 0} \int_0^T \int_0^L \phi(w^\varepsilon(t, y_1)) dy_1 dt \geq \int_0^T \int_{(0, L) \times (0, 1)} \phi_\lambda(w^0(t, y_1, \eta_1)) dy_1 dt$$

for all $\lambda > 0$. Finally we pass to the limit as λ tends to zero with Lebesgue's theorem. \blacksquare

We obtain

$$\begin{aligned} & \int_0^T \int_0^L \bar{a}_{y_1}(v^0, \varphi\theta - v^0) dy_1 dt + \int_0^T \int_0^L \frac{\partial p^0}{\partial y_1} \left(\int_Y h(y_1, \cdot) \varphi_1 d\eta_1 dy_2 \right) \theta dy_1 dt \\ & + \int_0^T \int_{(0, L) \times (0, 1)} k(|\varphi\theta| - |v^0|) d\eta_1 dy_1 dt \geq - \int_0^T \int_0^L \bar{a}_{y_1}(\bar{U}e_1, \varphi\theta - v^0) dy_1 dt \\ & + \int_0^T \int_{(0, L) \times Y} f(\varphi\theta - v^0) h d\eta_1 dy_1 dt \quad \forall \theta \in \mathcal{D}(0, T), \quad \forall \varphi \in \tilde{V}_{div} \end{aligned}$$

and we conclude by using the density of $\mathcal{D}(0, T) \otimes \tilde{V}_{div}$ into $L^2(0, T; V_{div})$ and (14). \blacksquare

Remark 4. We may weaken the compatibility condition on u_0^ε by assuming only that $(\varepsilon u_0^\varepsilon)_{\varepsilon > 0}$ converges to zero in $(L^2(\Omega))^2$.

By replacing v^0 by its decomposition (13) we obtain a limit problem for w^* and we have

Proposition 8. *Under the assumptions of Theorem 3, $w^* \in L^2(0, T; V_{div}^*)$ is the unique solution of the variational inequality*

$$\begin{aligned} & b(w^*, \varphi) + \int_0^T \int_{(0, L) \times (0, 1)} k(|\varphi + w^*| - |w^*|) d\eta_1 dy_1 dt + T(\varphi) \geq 0 \\ & \forall \varphi \in L^2(0, T; V_{div}^*) \end{aligned} \quad (16)$$

with

$$b(w, \varphi) = \int_0^T \int_0^L \bar{a}_{y_1}(w, \varphi) dy_1 dt + \int_0^T \int_0^L \frac{\partial \mathcal{Q}(w)}{\partial y_1} \frac{\partial \mathcal{Q}(\varphi)}{\partial y_1} \bar{a}_{y_1}(w_{y_1}^1, w_{y_1}^1) dy_1 dt$$

and

$$\begin{aligned} T(\varphi) &= \int_0^T \int_0^L \frac{\partial \bar{p}^0}{\partial y_1} \left(\int_Y h \varphi_1 d\eta_1 dy_2 \right) dy_1 dt + \int_0^T \int_0^L \bar{a}_{y_1}(\bar{U}e_1, \varphi) dy_1 dt \\ &- \int_0^T \int_0^L \left(\int_Y f h \varphi d\eta_1 dy_2 \right) dy_1 dt \end{aligned}$$

for all $\varphi \in L^2(0, T; V_{div}^*)$ and for all $w \in L^2(0, T; V_{div}^*)$.

PROOF. With straightforward computations we obtain (16) and it remains only to prove that (16) admits a unique solution. Since $\mathcal{Q} \in \mathcal{L}_c(V_{div}^*, H_{\#}^1(0, L)_{|\mathbb{R}})$ it is clear that b is a bilinear continuous mapping on $L^2(0, T; V_{div}^*)$. Moreover \bar{a}_{y_1} is coercive on $V_{y_1, div}$ for all $y_1 \in [0, L]$ uniformly with respect to y_1 (see Proposition 5.2 in [7]) which implies that b is coercive on $L^2(0, T; V_{div}^*)$ and we may conclude by applying for instance Theorem 8.5 in [14]. ■

As a consequence of the uniqueness of w^* , we can state:

Theorem 5. *Let $(U_0, W_0, s_0) \in (H^1(0, T))^3$ and $f \in (\mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_{\#}(0, 1)))^2$, $g \in \mathcal{C}([0, T]; \mathcal{C}(\bar{\Omega}; \mathcal{C}_{\#}(0, 1)))$ and $k \in \mathcal{C}([0, T]; \mathcal{C}([0, L]; \mathcal{C}_{\#}(0, 1)))$ such that k takes its values in \mathbb{R}_*^+ , f , g and k are L -periodic in y_1 and*

$$\varepsilon^2 f^\varepsilon(t, y) = f\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon^2 g^\varepsilon(t, y) = g\left(t, y, \frac{y_1}{\varepsilon}\right), \quad \varepsilon k^\varepsilon(t, y_1) = k\left(t, y_1, \frac{y_1}{\varepsilon}\right)$$

for all $(t, y) \in [0, T] \times \bar{\Omega}$. Let us assume moreover that $\varepsilon Z_0^\varepsilon$ is bounded in $L^2(\Omega)$ and that u_0^ε satisfies the following compatibility condition: there exists $\mathcal{U} \in C^\infty(\mathbb{R})$ such that

$$\mathcal{U}(0) = \mathcal{U}'(0) = 0, \quad \mathcal{U}(X) = 1 \quad \forall X \in [h_m, h_M]$$

and

$$u_0^\varepsilon = s_0(0)e_1 + (U_0(0) - s_0(0))\mathcal{U}\left(\frac{z_2}{\varepsilon}\right)e_1.$$

Then the whole sequences $(\varepsilon^2 p^\varepsilon)_{\varepsilon>0}$, $(v^\varepsilon)_{\varepsilon>0}$ and $(Z^\varepsilon)_{\varepsilon>0}$ satisfy the following convergence:

$$\begin{aligned} \varepsilon p^\varepsilon &\rightarrow \bar{p}^0 + \bar{p}^1 \\ v^\varepsilon &\rightarrow v^0 = \frac{\partial \bar{p}^0}{\partial y_1} w_{y_1}^1 + (U_0 - s_0)w_{y_1}^2 + w_{t, y_1}^3 + \frac{\partial \bar{p}^1}{\partial y_1} w_{y_1}^1 + w^* \\ Z^\varepsilon &\rightarrow Z^0 = W_0 z_{y_1}^1 + z_{t, y_1}^2. \end{aligned}$$

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