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Some qualitative properties of branching multiplicities
 N. Ressayre

Abstract. Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . We consider the multiplicities $c_{G,\hat{G}}$ as a function from the set of pairs of dominant weights to the set of integers. We recall that this function is piecewise quasipolynomial. Its support is a finitely generated semigroup ; we describe an irredundant list of inequalities determining the cone generated. The relation with the projection of coadjoint orbits for the Lie algebras of the compact forms of G and \hat{G} is also recalled.

We also consider the multiplicities for the fusion products for G . More precisely, we explain how the small quantum cohomology rings of homogeneous spaces G/P allow to parametrize an irredundant set of inequalities determining the multiplicative eigenvalue problem for the compact form K of G .

1. Introduction

Let G be a connected reductive subgroup of a complex connected reductive group \hat{G} . The branching problem consists in

decomposing irreducible representations of \hat{G} as sum of irreducible G -modules.

Fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and \hat{G} . Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. For $\nu \in X(T)^+$, $V_G(\nu)$ denotes the irreducible representation of highest weight ν . Similarly we use notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat{G}}(\hat{\nu})$ relatively to \hat{G} . For any G -module V , the subspace of G -fixed vectors is denoted by V^G . For $\nu \in X(T)^+$ and $\hat{\nu} \in X(\hat{T})^+$, set

$$c_{G,\hat{G}}(\nu, \hat{\nu}) = \dim(V_G(\nu) \otimes V_{\hat{G}}(\hat{\nu}))^G. \quad (1)$$

The branching problem is equivalent to the knowledge of these coefficients since

$$V_{\hat{G}}(\hat{\nu}) = \sum_{\nu \in X(T)^+} c_{G,\hat{G}}(\nu, \hat{\nu}) V_G(\nu)^*, \quad (2)$$

where $V_G(\nu)^*$ is the dual of $V_G(\nu)$.

For G diagonally embedded in $\hat{G} = G \times G$, $V_{\hat{G}}(\hat{\nu})$ is the tensor product of two irreducible representations of G and the coefficient $c_{G,\hat{G}}(\nu, \hat{\nu})$ are the multiplicities for the decomposition of $V_{\hat{G}}(\hat{\nu})$ as a sum of irreducible G -modules. If $G = GL_n(\mathbb{C})$, $X(T)^+$ identifies with the set of non-increasing sequences $\nu = (\nu_1 \geq \dots \geq \nu_n)$ of n integers and the coefficients are the Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$. For $\hat{G} = GL_n(\mathbb{C})$ and $G = \hat{T}$, the multiplicities $c_{G,\hat{G}}(\nu, \hat{\nu})$ are the Kostka coefficients given by the Weyl character formula. In this note, we recall some qualitative results on the function

$$\begin{aligned} c_{G\hat{G}} : X(T)^+ \times X(\hat{T})^+ &\longrightarrow \mathbb{Z}_{\geq 0} \\ (\nu, \hat{\nu}) &\longmapsto c_{G\hat{G}}(\nu, \hat{\nu}). \end{aligned}$$

More precisely:

- (i) There exists a polyhedral convex cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ outside which the multiplicities are zero. A multiplicity corresponding to a point in this cone can also vanish, but it becomes nonzero after scalar multiplication.
- (ii) The cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ is the support of some fan called GIT-fan and the function $c_{G\hat{G}}$ is quasipolynomial on each cone of the fan.
- (iii) In Section 3, we give an explicit description of the irredundant list of linear inequalities of the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$. These inequalities are parametrized by pairs of Schubert classes for some G and \hat{G} homogeneous spaces that satisfy some cohomological condition. This condition is expressed by the Belkale-Kumar product.
- (iv) We also present a description for coefficients on the boundary of the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$: they are equal to similar coefficients for Levi subgroups of G and \hat{G} .
- (v) The support $\text{LR}(G, \hat{G})$ of the function $c_{G\hat{G}}$ is a finitely generated semigroup generating a group $\mathbb{Z}\text{LR}(G, \hat{G})$. We present some results comparing $\text{LR}(G, \hat{G})$ with the set $\mathbb{Z}\text{LR}(G, \hat{G}) \cap \mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ of integral points in the cone. This problem, called the question of saturation is far from being completely solved and we present some conjectures.
- (vi) We present the PRV conjecture and its recent generalizations. These statements allow to produce easily points in the semigroup $\text{LR}(G, \hat{G})$.

Consider the case of the tensor product decomposition, that is when G is diagonally embedded in $\hat{G} = G \times G$. Let K a maximal compact subgroup of G and \mathfrak{k} its Lie algebra. It turns out that the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ identifies with the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint K -orbits in \mathfrak{k} such that the set $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ of sums contains 0. This spectral problem admits a multiplicative analogue: describe the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of conjugacy classes in K such that $\mathcal{O}_1 \cdot \mathcal{O}_2 \cdot \mathcal{O}_3$ contains the unit e of K . By a result of Meinrenken-Woodward, these triples are described by a convex polytope. Here, we describe recent results that give explicitly the minimal list of linear inequalities characterizing this polytope. These inequalities are parametrized by triples of Schubert classes satisfying some conditions expressed using the quantum cohomology. If the additive case is related to tensor product decomposition, the multiplication problem is equivalent to the description of the asymptotic support of the fusion product. In this context, the question of saturation can also be asked: the only known result due to Belkale is in type A.

2. Quasipolynomiality and GIT-fan

2.1. A brief review on Geometric Invariant Theory. Consider a complex irreducible projective variety X acting on by a reductive group G . The

set of G -linearized line bundles on X is a group (for the tensor product) denoted by $\text{Pic}^G(X)$. For $\mathcal{L} \in \text{Pic}^G(X)$, let $H^0(X, \mathcal{L})$ denote the G -module of regular sections of \mathcal{L} and let $H^0(X, \mathcal{L})^G$ denote the subspace of G -invariant sections. We consider the following set of *semi-stable points* for \mathcal{L} :

$$X^{\text{ss}}(\mathcal{L}) = \{x \in X : \exists n > 0 \text{ and } \tau \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ such that } \tau(x) \neq 0\}. \quad (3)$$

Note that this definition of $X^{\text{ss}}(\mathcal{L})$ is NOT standard. Indeed, it is usually agreed that the open subset defined by the non vanishing of τ is affine. Our definition coincides with the usual one if \mathcal{L} is ample. A line bundle \mathcal{L} over X is said to be *semi-ample* if a positive power of \mathcal{L} is base point free.

2.2. The GIT-fan. To simplify, we assume that the rank of the Picard group of X , and hence that of $\text{Pic}^G(X)$ are finite. Otherwise one has to consider the Neron-Severi group (see [DH98]). Since $X^{\text{ss}}(\mathcal{L}) = X^{\text{ss}}(\mathcal{L}^{\otimes n})$ for any $n > 0$, the definition of $X^{\text{ss}}(\mathcal{L})$ extends to the case when $\mathcal{L} \in \text{Pic}^G(X)_{\mathbb{Q}}$. Following [DH98], we say that two points \mathcal{L}_1 and \mathcal{L}_2 in $\text{Pic}^G(X)_{\mathbb{Q}}$ are *GIT-equivalent* if and only if $X^{\text{ss}}(\mathcal{L}_1) = X^{\text{ss}}(\mathcal{L}_2)$. A point $\mathcal{L} \in \text{Pic}^G(X)_{\mathbb{Q}}$ is said to be *G-effective* if $X^{\text{ss}}(\mathcal{L})$ is not empty. Let $\mathcal{C}^+(X)$ denote the cone of $\text{Pic}^G(X)_{\mathbb{Q}}$ generated by the points $\mathcal{L} \otimes 1$ where $\mathcal{L} \in \text{Pic}^G(X)$ is semi-ample. Let $\mathcal{C}^{G,+}(X)$ denote the cone of $\text{Pic}^G(X)_{\mathbb{Q}}$ generated by the points $\mathcal{L} \otimes 1$ where $\mathcal{L} \in \text{Pic}^G(X)$ is semi-ample and G -effective. Then $\mathcal{C}^+(X)$ and $\mathcal{C}^{G,+}(X)$ are convex. A sub-cone of $\mathcal{C}^+(X)$ is said to be *polyhedral in $\mathcal{C}^+(X)$* if it is the intersection of $\mathcal{C}^+(X)$ with finitely many rational and closed half spaces of $\text{Pic}^G(X)_{\mathbb{Q}}$. The geometry of the GIT-classes is described by the following result.

Theorem 2.1. *The cone $\mathcal{C}^{G,+}(X)$ is polyhedral in $\mathcal{C}^+(X)$. There are finitely many GIT-classes. Each GIT-class is the relative interior of some convex cone polyhedral in $\mathcal{C}^+(X)$. The closures of the GIT-classes in $\mathcal{C}^+(X)$ form a fan, called the GIT-fan.*

Let \mathcal{C}_1 and \mathcal{C}_2 be two GIT-classes and fix $\mathcal{L}_1 \in \mathcal{C}_1$ and $\mathcal{L}_2 \in \mathcal{C}_2$. Then $X^{\text{ss}}(\mathcal{L}_1)$ is contained in $X^{\text{ss}}(\mathcal{L}_2)$ if and only if $\overline{\mathcal{C}_1}$ contains \mathcal{C}_2 if and only if $\overline{\mathcal{C}_2}$ is a face of $\overline{\mathcal{C}_1}$.

The points $\mathcal{L} \otimes 1$ for ample $\mathcal{L} \in \text{Pic}^G(X)$ generate an open convex cone $\mathcal{C}^{++}(X)$ in $\mathcal{C}^+(X)$. Theorem 2.1 when $\mathcal{C}^+(X)$ is replaced by $\mathcal{C}^{++}(X)$ is proved in [Res00] following [DH98, Tha96]. The proofs in [Res00] can be applied without changing to get Theorem 2.1.

2.3. Application to branching coefficients. We now explain the geometric interpretation of the branching coefficients allowed by Borel-Weil theorem. Let $X = G/B \times \hat{G}/\hat{B}$. For any pair $(\nu, \hat{\nu})$ in $X(T) \times X(\hat{T})$, there exists a unique $(G \times \hat{G})$ -linearized line bundle $\mathcal{L}(\nu, \hat{\nu})$ on X such that $T \times \hat{T}$ acts on the base-point of X with weight $-(\nu, \hat{\nu})$. Then $\mathcal{L}(\nu, \hat{\nu})$ is semi-ample if and only if ν and $\hat{\nu}$ are dominant. In this case, $H^0(X, \mathcal{L}(\nu, \hat{\nu}))$ is a $G \times \hat{G}$ -module isomorphic to $V_G(\nu)^* \otimes V_{\hat{G}}(\hat{\nu})^*$. In particular $c_{G, \hat{G}}(\nu, \hat{\nu})$ is the dimension of $H^0(X, \mathcal{L}(\nu, \hat{\nu}))^G$ where G acts diagonally.

Consider the morphism $\theta : X(T \times \hat{T})_{\mathbb{Q}} \rightarrow \text{Pic}^G(X)_{\mathbb{Q}}$ that maps $(\nu, \hat{\nu})$ on $\mathcal{L}(\nu, \hat{\nu})$ endowed with the diagonal G -action. The pullback $\theta^{-1}(\mathcal{C}^+(X))$ is the cone $X(T \times \hat{T})_{\mathbb{Q}}^+$ generated by dominant weights, and $\theta^{-1}(\mathcal{C}^{G,+}(X))$ is

$$\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G}) := \{(\nu, \hat{\nu}) \in X(T \times \hat{T})_{\mathbb{Q}}^+ : \exists n > 0 \quad c_{G, \hat{G}}(n\nu, n\hat{\nu}) \neq 0\}.$$

The pullback in $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ of a GIT-class in $\text{Pic}^G(X)_{\mathbb{Q}}$ is called a GIT-class.

Quasipolynomiality. Let \mathfrak{g} and $\hat{\mathfrak{g}}$ denote the Lie algebras of G and \hat{G} . It turns out that $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ has nonempty interior in $X(T \times \hat{T})_{\mathbb{Q}}$ if and only if no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$. Under this assumption, the GIT-classes \mathcal{C} of nonempty interior in $X(T \times \hat{T})_{\mathbb{Q}}^+$ are called the *GIT-chambers*. Their closures $\bar{\mathcal{C}}$ are the maximal cones of the GIT-fan.

Theorem 2.2. *Let \mathcal{C} be a GIT-chamber. There exists a cofinite lattice $\Lambda_{\mathcal{C}}$ of $X(T \times \hat{T})$ and a collection of polynomial functions f_l indexed by $l \in X(T \times \hat{T})/\Lambda_{\mathcal{C}}$ such that*

$$c_{G, \hat{G}}(\nu, \hat{\nu}) = f_l(\nu, \hat{\nu}),$$

for any $(\nu, \hat{\nu}) \in \bar{\mathcal{C}} \cap X(T \times \hat{T})$ congruent to l modulo $\Lambda_{\mathcal{C}}$.

Meinrenken-Sjamaar proved Theorem 2.2 in [MS99] using symplectic geometry. A proof using the Riemann-Roch theorem for singular varieties can be found in [KP13]. Examples of GIT-fans and quasipolynomial functions can be found in [KP13, AH09, Ras04, Kot11, CT04]. Another example is given in Section 3.4.

3. The cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$

Theorems 2.1 or 5.1 imply that $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ is a closed convex polyhedral cone in $X(T \times \hat{T})_{\mathbb{Q}}$. The aim of this section is to describe this cone by an explicit list of inequalities.

3.1. Spectral interpretation. Let K and \hat{K} be two maximal compact subgroups of G and \hat{G} such that $K \subset \hat{K}$. Let \mathfrak{k} and $\hat{\mathfrak{k}}$ denote the Lie algebras of K and \hat{K} . Consider the projection $p : \hat{\mathfrak{k}} \rightarrow \mathfrak{k}$ orthogonal for the Cartan-Killing form of $\hat{\mathfrak{k}}$. We are interested in the projections of adjoint orbits of $\hat{\mathfrak{k}}$.

Up to changing T , we may assume that $H = K \cap T$ is a Cartan subgroup of K . Consider the Lie algebra \mathfrak{h} of H . Any root α of (G, T) induces (by derivation) a linear form (still denoted by α) on the Lie algebra $\text{Lie}(T)$ of T . The Lie algebra $\text{Lie}(H)$ of H identifies with the real Lie sub-algebra of $\xi \in \text{Lie}(T)$ such that $\alpha(\xi) \in \sqrt{-1}\mathbb{R}$ for any root α . Consider the group $X_*(T)$ of one parameter subgroups of T and its pairing $\langle \cdot, \cdot \rangle$ with $X(T)$. The dominant chamber in $X_*(T)_{\mathbb{R}} := X_*(T) \otimes \mathbb{R}$ is

$$X_*(T)_{\mathbb{R}}^+ = \{\lambda \in X_*(T)_{\mathbb{R}} : \langle \lambda, \alpha \rangle \geq 0 \text{ for any simple root } \alpha\}.$$

By derivation, $X_*(T)$ identifies with a sub-lattice of $\text{Lie}(T)$, and hence, $X_*(T)_{\mathbb{R}}$ identifies with $\sqrt{-1}\text{Lie}(H)$. Any adjoint K -orbit in \mathfrak{k} contains a unique element belonging to $\sqrt{-1}X_*(T)_{\mathbb{R}}^+$; for any $\lambda \in X_*(T)_{\mathbb{R}}^+$, we denote by \mathcal{O}_λ the adjoint K -orbit containing $\sqrt{-1}\lambda$. Similarly, we define the adjoint \hat{K} -orbit $\mathcal{O}_{\hat{\lambda}}$ for any $\hat{\lambda} \in X_*(\hat{T})_{\mathbb{R}}^+$. Let w_0 be the longest element of the Weyl group of G . Set

$$\mathcal{C}(K, \hat{K}) = \{(\lambda, \hat{\lambda}) \in X_*(T)_{\mathbb{R}}^+ \times X_*(\hat{T})_{\mathbb{R}}^+ : \mathcal{O}_{-w_0\lambda} \subset p(\mathcal{O}_{\hat{\lambda}})\}.$$

The Kirwan convexity theorem (see [Kir84a]) in symplectic geometry shows that $\mathcal{C}(K, \hat{K})$ is a closed polyhedral cone.

Fix a W -invariant scalar product $(\cdot, \cdot)_T$ on $X_*(T)_{\mathbb{R}}$. Then, for $\lambda \in X_*(T)_{\mathbb{R}}$, $(\lambda, \cdot)_T$ is a linear form on $X_*(T)_{\mathbb{R}}$ and it corresponds to a point in $X^*(T)_{\mathbb{R}}$ for the pairing $\langle \cdot, \cdot \rangle$. Similarly $(\cdot, \cdot)_{\hat{T}}$ is fixed on $X_*(\hat{T})_{\mathbb{Q}}$.

Theorem 3.1. *Let $(\lambda, \hat{\lambda}) \in X_*(T \times \hat{T})_{\mathbb{Q}}^+$. Then $(\lambda, \hat{\lambda}) \in \mathcal{C}(K, \hat{K})$ if and only if $((\lambda, \cdot)_T, (\hat{\lambda}, \cdot)_{\hat{T}}) \in \mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$.*

As it was pointed out by Guillemin-Sternberg [GS82a], Heckman's work [Hec82] implies Theorem 3.1. This result is also a consequence of Kempf-Ness' theorem [KN79].

Example. If G is diagonally embedded in $\hat{G} = G \times G$ then the branching problem is the problem of decomposition of tensor products of 2 irreducible representations of G . The cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ is denoted by $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$. The cone $\mathcal{C}(K, \hat{K}) =: \mathcal{C}(K^3)$ identifies with the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint orbits in \mathfrak{k} such that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ contains 0. A good survey on this case is [Kum13].

3.2. Belkale-Kumar Schubert calculus. It is known since A. Klyachko [Kly98] that the inequalities that characterize the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ are related to the cohomology of flag varieties, that is to Schubert calculus. In 2006, Belkale-Kumar [BK06] defined a new product on the cohomology groups of flag varieties that is useful to parametrize irredundantly the inequalities of $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$.

Let P be a parabolic subgroup of G containing B . Let W and W_P denote respectively the Weyl groups of G and P . The Weyl group W is generated by the simple reflections s_α indexed by the simple roots α . The corresponding length function is denoted by l . Let W^P be the set of minimal length representative in the cosets of W/W_P . For any $w \in W^P$, let X_w be the corresponding Schubert variety (that is, the closure of BwP/P) and let $\sigma_w \in H^{2(\dim(G/P)-l(w))}(G/P, \mathbb{C})$ be its cohomology class. The structure coefficients $c_{w_1 w_2}^{w_3}$ of the cup product are written as

$$\sigma_{w_1} \cdot \sigma_{w_2} = \sum_{w_3 \in W^P} c_{w_1 w_2}^{w_3} \sigma_{w_3}, \quad \forall w_1, w_2 \in W^P. \quad (4)$$

Let L be the Levi subgroup of P containing T and let Z be the neutral component of the center of L . Under the action of Z , the tangent space $\mathcal{T}_{P/P}G/P$ of G/P at

the base point P/P decomposes as

$$\mathcal{T}_{P/P}G/P = \bigoplus_{\chi \in X(Z)} \mathcal{T}^\chi, \quad (5)$$

where Z acts on \mathcal{T}^χ with weight χ . By [ABS90], each \mathcal{T}^χ is an irreducible L -module. For any $w \in W^P$, the tangent space $\mathcal{T}_w := \mathcal{T}_{P/P}w^{-1}X_w$ of the variety $w^{-1}X_w$ at the smooth point P/P also decomposes

$$\mathcal{T}_w = \bigoplus_{\chi \in X(Z)} \mathcal{T}_w^\chi, \quad (6)$$

where $\mathcal{T}_w^\chi = \mathcal{T}^\chi \cap \mathcal{T}_w$. The weights of T in \mathcal{T}_w are the opposite of the elements of the inversion set

$$\Phi(w) = \{\alpha \in \Phi^+ : w\alpha \in -\Phi^+\},$$

where Φ^+ is the set of positive roots of G relatively to B . It is contained in the set $\Phi(G/P)$ of positive roots that are not roots of L . If $w \in W$ then $w \in W^P$ if and only if $\Phi(w)$ is contained in $\Phi(G/P)$. For $\chi \in X(Z)$, denote $\Phi(w, \chi)$ the set of $\alpha \in \Phi(w)$ whose the restriction to Z is $-\chi$. Similarly, define $\Phi(G/P, \chi)$. Since σ_w has degree $2(\#\Phi(G/P) - \#\Phi(w))$ in the graded algebra $H^*(G/P)$, if $c_{w_1w_2}^{w_3} \neq 0$ then

$$\#\Phi(w_1) + \#\Phi(w_2) = \#\Phi(G/P) + \#\Phi(w_3), \quad (7)$$

that is

$$\sum_{\chi \in X(Z)} \left(\#\Phi(w_1, \chi) + \#\Phi(w_2, \chi) \right) = \sum_{\chi \in X(Z)} \left(\#\Phi(G/P, \chi) + \#\Phi(w_3, \chi) \right). \quad (8)$$

The Belkale-Kumar product requires the equality (8) to hold term by term. More precisely, the structure constants $\tilde{c}_{w_1w_2}^{w_3}$ of the Belkale-Kumar product \odot ,

$$\sigma_{w_1} \odot \sigma_{w_2} = \sum_{w_3 \in W^P} \tilde{c}_{w_1w_2}^{w_3} \sigma_{w_3} \quad (9)$$

can be defined as follows (see [RR11, Proposition 2.4]):

$$\tilde{c}_{w_1w_2}^{w_3} = \begin{cases} c_{w_1w_2}^{w_3} & \text{if } \forall \chi \in X(Z) \quad \#\Phi(w_1, \chi) + \#\Phi(w_2, \chi) = \#\Phi(G/P, \chi) + \#\Phi(w_3, \chi), \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Theorem 3.2. [BK06] *The product \odot on $H^*(G/P, \mathbb{C})$ is commutative, associative and satisfies Poincaré duality.*

Denote by \mathfrak{p} the Lie algebra of P and consider the convex cone C in $X(Z)_\mathbb{Q}$ generated by the weights of Z acting on \mathfrak{p} . It is a closed strictly convex polyhedral cone in $X(Z)_\mathbb{Q}$. Consider the partial order \succ on $X(Z)_\mathbb{Q}$ defined by $\alpha \succ \beta$ if and only if $\alpha - \beta$ belongs to C . Then

$$\mathcal{T}^{\succ \alpha} := \bigoplus_{\beta \succ \alpha} \mathcal{T}^\beta \quad (11)$$

is P -stable. Observe that the tangent bundle $\mathcal{T}G/P$ of G/P identifies with the fiber product $G \times_P \mathcal{T}_{P/P}G/P$. Since $\mathcal{T}^{\geq \alpha}$ is P -stable, it induces a G -homogeneous sub-bundle $\mathcal{T}^{\geq \alpha}G/P$ of the tangent bundle $\mathcal{T}G/P$. This family of G -sub-bundles is decreasing: if $\alpha \geq \beta$ then $\mathcal{T}^{\geq \alpha}G/P$ is a sub-bundle of $\mathcal{T}^{\geq \beta}G/P$. It is also integrable in the sense that

$$[\mathcal{T}^{\geq \alpha}G/P, \mathcal{T}^{\geq \beta}G/P] \subset \mathcal{T}^{\geq \alpha+\beta}G/P. \quad (12)$$

This allows us to define a filtration (“à la Hodge”) of the De Rham complex and hence of the algebra $H^*(G/P, \mathbb{C})$ indexed by the group $X(Z)$. We consider the associated graded algebra.

Theorem 3.3. [Res13a] *The $X(Z)$ -graded algebra $GrH^*(G/P, \mathbb{C})$ associated to the $X(Z)$ -filtration is isomorphic to the Belkale-Kumar algebra $(H^*(G/P, \mathbb{C}), \odot)$.*

Sketch of proof. The key point to prove the isomorphism is that the subspaces of the filtration of $H^*(G/P, \mathbb{C})$ are spanned by the Schubert classes $(\sigma_w)_{w \in W^P}$ that it contains. To obtain this result, we use Kostant’s harmonic forms [Kos61]. \square

3.3. A description of $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$. Let λ be a one parameter subgroup of T . The set of $g \in G$ such that $\lim_{t \rightarrow 0} \lambda(t)g\lambda(t^{-1})$ exists in G is a parabolic subgroup of G denoted by $P(\lambda)$. Since λ is also a one parameter subgroup of \hat{G} we have a parabolic subgroup $\hat{P}(\lambda)$ and an embedding $\phi_\lambda : G/P(\lambda) \rightarrow \hat{G}/\hat{P}(\lambda)$. The comorphism ϕ_λ^* in cohomology induces a morphism (see [RR11]) $\phi_\lambda^\odot : (H^*(\hat{G}/\hat{P}(\lambda), \mathbb{C}), \odot) \rightarrow (H^*(G/P(\lambda), \mathbb{C}), \odot)$.

A description of $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$. An indivisible dominant $\lambda \in X_*(T)$ is said to be *special* if the set of weights χ of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$ such that $\langle \chi, \lambda \rangle = 0$ spans an hyperplane of $X(T)_{\mathbb{Q}}$.

Theorem 3.4. *Assume that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$. Let $(\nu, \hat{\nu}) \in X(T \times \hat{T})_{\mathbb{Q}}^+$. Then $(\nu, \hat{\nu})$ belongs to $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ if and only if*

$$\langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle \leq 0 \quad (13)$$

for any special $\lambda \in X_*(T)$, for any $w \in W^{P(\lambda)}$ and $\hat{w} \in \hat{W}^{\hat{P}(\lambda)}$ such that

$$\phi_\lambda^\odot(\sigma_{\hat{w}}) \odot \sigma_w = \sigma_e. \quad (14)$$

Moreover, this system of inequalities is irredundant.

Sketch of proof. Consider the action of G on $X = G/B \times \hat{G}/\hat{B}$. As explained in Section 2.3, it remains to determine $\mathcal{C}^{G,+}(X)$.

Let \mathcal{L} be a semi-ample G -linearized line bundle on X . Let x be a point in X and $\lambda : \mathbb{C}^* \rightarrow G$ be a one parameter subgroup. Since X is complete, $\lim_{t \rightarrow 0} \lambda(t)x$ exists; let z denote this limit. The image of λ fixes z , and hence, acts via λ on the

fiber \mathcal{L}_z . There exists an integer denoted by $\mu^{\mathcal{L}}(x, \lambda)$ such that for all $t \in \mathbb{C}^*$ and $\tilde{z} \in \mathcal{L}_z$ we have:

$$\lambda(t).\tilde{z} = t^{-\mu^{\mathcal{L}}(x, \lambda)}\tilde{z}.$$

The integers $\mu^{\mathcal{L}}(x, \lambda)$ are used in [MFK94] to give a numerical criterion (namely the Hilbert-Mumford criterion) for stability with respect to an ample \mathcal{L} . With definition (3) of semi-stability, the Hilbert-Mumford theorem admits the following direct generalization.

Lemma 3.5 (See e.g. [Res10a, Lemma 2]). *Recall that \mathcal{L} is semi-ample. Then, x is semi-stable for \mathcal{L} if and only if $\mu^{\mathcal{L}}(x, \lambda) \leq 0$ for any one parameter subgroup λ of G .*

Assume now that λ is a dominant one parameter subgroup of T . Fix an irreducible component C of the fixed point set X^λ of λ in X . Denote by C^+ the Białinicki-Birula cell of points $x \in X$ such that $\lim_{t \rightarrow 0} \lambda(t)x \in C$. Let $\mathcal{L} \in \mathcal{C}^+(X)$. The integer $\mu^{\mathcal{L}}(x, \lambda)$ does not depend on x in C^+ ; let $\mu^{\mathcal{L}}(C, \lambda)$ denote this integer. A key observation is the following lemma.

Lemma 3.6. *If $G.C^+$ is dense in X and $X^{\text{ss}}(\mathcal{L})$ is not empty then $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.*

Proof. Since $X^{\text{ss}}(\mathcal{L})$ is open, it intersects $G.C^+$. Since $X^{\text{ss}}(\mathcal{L})$ is G -stable, it intersects C^+ . Let $x \in C^+ \cap X^{\text{ss}}(\mathcal{L})$. By Lemma 3.5, $\mu^{\mathcal{L}}(C, \lambda) = \mu^{\mathcal{L}}(x, \lambda) \leq 0$. \square

The subvariety C^+ is $P(\lambda)$ -stable and one can form the fibered product $G \times_{P(\lambda)} C^+$. Consider the morphism

$$\begin{aligned} \eta : G \times_{P(\lambda)} C^+ &\longrightarrow X \\ [g : x] &\longmapsto gx. \end{aligned}$$

There exists $(w, \hat{w}) \in W^{P(\lambda)} \times \hat{W}^{\hat{P}(\lambda)}$ such that $C^+ = P(\lambda)w^{-1}B/B \times \hat{P}(\lambda)w^{-1}\hat{B}/\hat{B}$. Using Kleiman's transversality theorem, one can prove that η is dominant if and only if

$$\phi_\lambda^*(\sigma_{\hat{w}}).\sigma_w \neq 0. \tag{15}$$

Since

$$\mu^{\mathcal{L}(\nu, \hat{\nu})}(C, \lambda) = \langle w\lambda, \nu \rangle + \langle \hat{w}\lambda, \hat{\nu} \rangle,$$

this proves that inequalities (13) in the theorem are satisfied by points in $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$.

If we believe the theorem, we just obtained a redundant family of inequalities. We now explain how to prove that the inequalities corresponding to pairs (w, \hat{w}) satisfying condition (14) are sufficient to characterize the cone. We check that η is birational if and only if $\phi_\lambda^*(\sigma_{\hat{w}}).\sigma_w = \sigma_e$. One can prove that the stronger condition (14) is equivalent to the existence of an open G -stable subset Ω in X that intersects C such that the restriction of η to $G \times_{P(\lambda)}(C^+ \cap \Omega)$ is an isomorphism onto Ω . In this case, the pair (C, λ) is said to be *well covering*. The point is that,

given $\mathcal{L} \in \mathcal{C}^+(X) - \mathcal{C}^{G,+}(X)$, Kempf's theory of instability allows to construct a well covering pair (C, λ) such that $\mu^{\mathcal{L}}(C, \lambda) > 0$. Note that Ω can be chosen to be the open Hesselink strata (see [Kir84b]).

Fix a well covering pair (C, λ) , where $\lambda \in X_*(T)$ is dominant and indivisible. Let $\mathcal{F}(C, \lambda)$ denote the set of $\mathcal{L} \in \mathcal{C}^{G,+}(X)$ such that $\mu^{\mathcal{L}}(C, \lambda) = 0$; it is a face of $\mathcal{C}^{G,+}(X)$.

Let $\mathcal{L} \in \mathcal{F}(C, \lambda)$ and $x \in X^{\text{ss}}(\mathcal{L}) \cap C^+$. Let $n > 0$ and $\sigma \in H^0(X, \mathcal{L}^{\otimes n})^G$ such that $\sigma(x) \neq 0$. Set $z = \lim_{t \rightarrow 0} \lambda(t)x$. From $\mu^{\mathcal{L}}(x, \lambda) = 0$, one can deduce that $\sigma(z) = \lim_{t \rightarrow 0} \lambda(t)\sigma(x)$ belongs to the pointed fiber $\mathcal{L}_z - \{z\}$. In particular z is semi-stable. It follows that the neutral component G_z^o of the isotropy of z acts trivially on \mathcal{L}_z . In λ is not special, this implies that \mathcal{L} belongs to a codimension 2 linear subspace of $\text{Pic}^G(X)_{\mathbb{Q}}$. In particular, $\mathcal{F}(C)$ has not codimension one and the inequality $\mu^{\mathcal{L}}(C, \lambda)$ can be removed.

The last step is to prove the irredundancy. Fix a well covering pair (C, λ) with some special λ . We have to prove that $\mathcal{F}(C, \lambda)$ has codimension one. Consider the restriction morphism

$$\rho : \text{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \text{Pic}^{G^\lambda}(C)_{\mathbb{Q}}.$$

An explicit computation shows that ρ is surjective. Moreover, by induction, the dimension of $\mathcal{C}^{G^\lambda,+}(C)$ is equal to $\dim(\text{Pic}^G(X)_{\mathbb{Q}}) - 1$. Let $\mathcal{M} \in \text{Pic}^{G^\lambda}(C)$ such that $H^0(C, \mathcal{M})^{G^\lambda} \neq 0$. It would be sufficient to prove that there exists $\mathcal{L} \in \text{Pic}^G(X)$ such that $H^0(X, \mathcal{L})^G \neq 0$ and $\rho(\mathcal{L}) = \mathcal{M}$. This is not true directly but it is true after a little modification of \mathcal{M} . Let $\mathcal{L} \in \text{Pic}^G(X)$ such that $\rho(\mathcal{L}) = \mathcal{M}$. Fix a nonzero regular G^λ -invariant section τ of \mathcal{M} . Let η and $\Omega \subset X$ be as above. Let E_1, \dots, E_k be the codimension one irreducible components of $X - \Omega$. Using the inverse of η one can prove that $\mathcal{L}|_{\Omega}$ admits a nonzero G -invariant section σ . Then, σ does not necessarily extend to a section of \mathcal{L} on X ; but it certainly extends to a section of $\mathcal{L} \otimes d\mathcal{O}(\sum_i E_i)$ for d big enough. Since no E_i contains C , $\mathcal{L} \otimes d\mathcal{O}(\sum_i E_i)$ belongs to $\mathcal{F}(C, \lambda)$.

By this method, one can produce a family of points in $\mathcal{F}(C, \lambda)$ that generates a codimension one cone. The irredundancy follows. \square

3.4. The case of $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$. We assume that G is semi-simple and simply connected and we consider the cone $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$. The set of non-trivial weights of T acting on $\hat{\mathfrak{g}}/\mathfrak{g}$ is Φ . There are $\text{rk}(G)$ special one parameter subgroups of T ; for any simple root α exactly one λ_α is proportional to the fundamental coweight ϖ_{α^\vee} . The parabolic subgroup $P(\lambda_\alpha)$ is the maximal parabolic subgroup P_α containing B associated to α .

Theorem 3.7. *Let $(\nu_1, \nu_2, \nu_3) \in (X^*(T)_{\mathbb{Q}}^+)^3$. Then $(\nu_1, \nu_2, \nu_3) \in \mathbb{Q}_{\geq 0}\text{LR}(G^3)$ if and only if*

$$\sum_{i=1}^3 \langle w_i \varpi_{\alpha^\vee}, \nu_i \rangle \leq 0 \tag{16}$$

for any simple root α and any $w_1, w_2, w_3 \in W^{P_\alpha}$ such that

$$\sigma_{w_1} \circ \sigma_{w_2} \circ \sigma_{w_3} = \sigma_e. \quad (17)$$

Moreover, this list of inequalities is irredundant.

Theorem 3.7 without the irredundancy is the main result of [BK06].

The Horn case. Let $\mathcal{H}_n(\mathbb{C})$ be the space of Hermitian matrices of size n . For $A \in \mathcal{H}_n(\mathbb{C})$, we denote by $\lambda(A) = (\lambda_1 \geq \dots \geq \lambda_n) \in \mathbb{R}^n$ the spectrum of A . For $G = \mathrm{GL}_n(\mathbb{C})$ diagonally embedded in $\hat{G} = G \times G$, Section 3.1 implies that the cone $\mathbb{R}_{\geq 0}\mathrm{LR}(G, \hat{G})$ identifies with

$$\mathrm{Horn}_{\mathbb{R}}(n) = \left\{ (\lambda(A), \lambda(B), \lambda(C)) \in \mathbb{R}^{3n} : \begin{array}{l} A, B, C \in \mathcal{H}_n(\mathbb{C}) \text{ s.t.} \\ A + B + C = 0 \end{array} \right\}.$$

After an easy reduction, we can apply Theorem 3.7 (with $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C})$) to obtain a description of $\mathrm{Horn}_{\mathbb{R}}(n)$. The homogeneous spaces G/P_α are the Grassmannian variety $\mathbb{G}(r, n)$ of r -dimensional subspaces of \mathbb{C}^n . The Schubert classes $(\sigma_I)_{I \in \mathcal{S}(r, n)}$ of $\mathbb{G}(r, n)$ are indexed by the set $\mathcal{S}(r, n)$ of subsets of $\{1, \dots, n\}$ with r elements. Since $\mathbb{G}(r, n)$ is cominuscule, the product \circ coincide with the ordinary one.

Theorem 3.8. *Let (λ, μ, ν) be a triple of non-increasing sequences of n real numbers. Then $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \quad (18)$$

and

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0, \quad (19)$$

for any $r \in \{1, \dots, n-1\}$, for any $I, J, K \in \mathcal{S}(r, n)$ such that

$$\sigma_I \cdot \sigma_J \cdot \sigma_K = \sigma_e. \quad (20)$$

Moreover, this list of inequalities is irredundant.

An example: $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$. The symmetric group S_3 acts on $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$ by permuting the three copies of G . Since $(V_1 \otimes V_2 \otimes V_3)^G$ and $(V_1^* \otimes V_2^* \otimes V_3^*)^G$ have the same dimension the group $\mathbb{Z}/2\mathbb{Z}$ also acts on $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$. Finally, we get an action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$. For $\mathrm{SL}_3(\mathbb{C})$, we use the base of fundamental weights to identify $X(T)^+$ with $\mathbb{Z}_{\geq 0}^2$ and $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ with a cone in \mathbb{Z}^6 . Let (ν_1, ν_2, ν_3) be three dominant weights corresponding to $(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{Z}_{\geq 0}^6$. If $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$ is not a multiple of 3, then $c_{\mathrm{SL}_3}(\nu_1, \nu_2, \nu_3) = 0$ (to check this, one can consider the action of the center of $\mathrm{SL}_3(\mathbb{C})$). Assume now that 3 divides $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$ and set $\delta = \frac{1}{3}(x_1 + y_1 - z_2 + 2(x_2 + y_2 - z_1))$. Then $c_{\mathrm{SL}_3}(\nu_1, \nu_2, \nu_3)$ is equal to the Littlewood-Richardson coefficient $c_{x_1+x_2 \geq x_2, y_1+y_2 \geq y_2}^{z_1+z_2+\delta \geq x_2 \delta \geq \delta}$.

There are two Grassmannians homogeneous under $\mathrm{SL}_3(\mathbb{C})$, \mathbb{P}^2 and its dual. In \mathbb{P}^2 , the identity $[\mathbb{P}^1].[\mathbb{P}^1].[\mathbb{P}^2] = [\mathrm{pt}]$ gives the inequality $x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2 \geq 0$ and the relation $[\mathbb{P}^2].[\mathbb{P}^2].[\mathrm{pt}] = [\mathrm{pt}]$ gives the inequality $x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2 \geq 0$. Using the action of $S_3 \times \mathbb{Z}/2\mathbb{Z}$, we get 12 inequalities. Adding the inequalities of dominance, we get that $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ has 18 facets.

Consider the point $b = (1, 1, 1, 1, 1, 1) \in \mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$. The cones generated by b and one of the 18 facets of $\mathbb{Q}_{\geq 0}\mathrm{LR}(\mathrm{SL}_3(\mathbb{C})^3)$ are the maximal cones of the GIT-fans. Assuming that 3 divides $x_1 + y_1 + z_1 + 2(x_2 + y_2 + z_2)$, the Littlewood-Richardson coefficient $c_{x_1+x_2 \geq x_2, y_1+y_2 \geq y_2}^{z_1+z_2+\delta \geq x_2 \delta \geq \delta}$ is given on the cones corresponding to the inequalities $x_1 \geq 0$, $x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2 \geq 0$ and $x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2 \geq 0$ respectively by the polynomials $1 + x_1$, $1 + \frac{1}{3}(x_1 - x_2 + y_1 - y_2 + z_1 + 2z_2)$ and $1 + \frac{1}{3}(x_1 + 2x_2 + y_1 + 2y_2 - 2z_1 - z_2)$.

Theorem 3.8 has a rich and long story starting with H. Weyl [Wey12] who proved, in 1912, inequalities (19) for $G/P_\alpha = \mathbb{P}^{n-1}$. In 1998, A. Klyachko [Kly98] made an important step proving the theorem is true (without the irredundancy) if condition (20) is replaced by

$$\sigma_I \cdot \sigma_J \cdot \sigma_K = d\sigma_e, \quad \text{for some positive integer } d. \quad (21)$$

In 2000, Belkale [Bel01] proved that Klyachko's condition (21) can be replaced by condition (20). The irredundancy was first proved by Knutson-Tao-Woodward in [KTW04] using the Honeycomb model for Littlewood-Richardson coefficients.

Let us now explain Horn's contribution. For $I = \{i_1 < \dots < i_r\} \in \mathcal{S}(r, n)$, set $\tau^I = (i_r - r, \dots, i_1 - 1)$ and $I^\vee = \{n+1-i_r < \dots < n+1-i_1\}$. In 1962, Horn conjectured that Theorem 3.8 is true if one replaces condition (20) by

$$(\tau^I, \tau^J, \tau^K - (n-r)^r) \in \mathrm{Horn}(r), \quad (22)$$

where $(n-r)^r = (n-r, \dots, n-r)$ in \mathbb{R}^r . By the classical Lesieur's result (see [Les47]), $\sigma_I \cdot \sigma_J \cdot \sigma_K = d\sigma_e$ is equivalent to $c_{\tau^I \tau^J}^{\tau^{K^\vee}} = d$. In particular, Klyachko's condition is equivalent to

$$c_{\tau^I \tau^J}^{\tau^{K^\vee}} > 0, \quad (23)$$

whereas condition (22) is equivalent to

$$\exists k > 0 \quad c_{k\tau^I \ k\tau^J}^{k\tau^{K^\vee}} > 0. \quad (24)$$

The equivalence between conditions (23) and (24) is called saturation (see Section 5) and was first proved by Knutson-Tao in [KT99].

Horn's conjecture has the advantage to be inductive and elementary (without cohomology or representation theory). Theorem 3.8 has the advantage to give the minimal list of inequalities. In [Res11], we get the two advantages by giving an inductive algorithm to decide if a given Littlewood-Richardson coefficient is equal to one or not.

3.5. Some inequalities for nonzero Kronecker coefficients.. If $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_e \geq 0)$ is a partition, we set $|\alpha| = \sum_i \alpha_i$ in such a way α is a partition of $|\alpha|$. Consider the symmetric group \mathcal{S}_n on n letters. The irreducible representations of \mathcal{S}_n are parametrized by the partitions of n , see *e.g.* [Mac95, I. 7]. Let $[\alpha]$ denote the representation of $\mathcal{S}_{|\alpha|}$ corresponding to α . The Kronecker coefficients $g_{\alpha\beta\gamma}$, depending on three partitions α, β , and γ of the same integer n , are defined by

$$[\alpha] \otimes [\beta] = \sum_{\gamma} g_{\alpha\beta\gamma} [\gamma]. \quad (25)$$

The length $l(\alpha)$ of the partition α is the number of nonzero parts α_i .

Theorem 3.9 (see [Res12]). *Let e and f be two positive integers and $j \in \{2, \dots, f+1\}$. Let α, β , and γ be three partitions of the same integer n such that*

$$l(\alpha) \leq e+1, \quad l(\beta) \leq f+1, \quad \text{and} \quad l(\gamma) \leq e+f+1. \quad (26)$$

Let $0 < r < e$, $0 < s < f$, $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that

$$c_{\tau^I \tau^J}^{\tau^K} = 1. \quad (27)$$

If $g_{\alpha\beta\gamma} \neq 0$ then

$$n + \sum_{i \in I} \alpha_{i+1} - \alpha_1 + \sum_{j \in J} \beta_{j+1} - \beta_1 \geq \sum_{k \in K} \gamma_{k+1} - \gamma_1. \quad (28)$$

Sketch of proof. Given a complex vector space V and a partition α such that $l(\alpha) \leq \dim(V)$, let $S^\alpha V$ denote the irreducible $\mathrm{GL}(V)$ -representation of highest weight α . Fix two complex vector spaces V_1, V_2 of dimension $e+1$ and $f+1$. The Schur-Weyl duality implies that

$$S^\gamma(V_1 \otimes V_2) = \bigoplus_{\alpha\beta} g_{\alpha\beta\gamma} S^\alpha V_1 \otimes S^\beta V_2.$$

We consider the action of $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$ on the product X of the manifolds of complete flags in V_1 and V_2 and some partial flag manifold on $V_1 \otimes V_2$. The coefficient $g_{\alpha\beta\gamma}$ is the dimension of the space of G -invariant sections of some line bundle on X . Then, we use techniques similar to those used to prove Theorem 3.4. \square

3.6. Relations between cones $\mathbb{Q}_{\geq 0}\mathrm{LR}(G^3)$ for various G . In this section, G is assumed to be simple, simply connected and of simply-laced type. Consider an automorphism σ of the Dynkin diagram of G . It induces an automorphism, still denoted by σ , of G that stabilizes a maximal torus T , a Borel subgroup B and a compact form K of G . The fixed point set G^σ is a simple group with maximal torus T^σ , Borel subgroup B^σ and compact form K^σ . The inclusion $T^\sigma \subset T$ induces an immersion $X_*(T^\sigma)_\mathbb{R} \subset X_*(T)_\mathbb{R}$ satisfying $X_*(T^\sigma)_\mathbb{R}^+ = X_*(T^\sigma)_\mathbb{R} \cap X_*(T)_\mathbb{R}^+$.

Theorem 3.10. *We have*

$$\mathcal{C}((K^\sigma)^3) = \mathcal{C}(K^3) \cap X_*(T^\sigma)_{\mathbb{R}}^3.$$

Sketch of proof. The proof uses Theorem 3.4 and compares conditions (14) in G and G^σ -homogeneous spaces. Beyond this general principle, the proof is case by case according to the following complete list:

- (i) $(\mathrm{SL}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C}))$, $n \geq 2$;
- (ii) $(\mathrm{SL}_{2n+1}(\mathbb{C}), \mathrm{SO}_{2n+1}(\mathbb{C}))$, $n \geq 2$;
- (iii) $(\mathrm{Spin}_{2n}(\mathbb{C}), \mathrm{Spin}_{2n-1}(\mathbb{C}))$, $n \geq 4$;
- (iv) $(\mathrm{Spin}_8(\mathbb{C}), G_2)$;
- (v) (E_6, F_4) .

The two first cases was proved in [BK10a] (see also [Sot10] for a simplification in some key argument). Case (iii) is proved in Braley's thesis [Bra12]. The two remaining cases are proved in Lee's thesis [Lee12]. \square

Remark 3.11. As a consequence of Theorem 3.10, it is proved in [Res11] that condition (14) in Theorem 3.4 in the cohomology of symplectic and odd orthogonal Grassmannians are equivalent to similar conditions for ordinary Grassmannians.

4. Reduction for coefficients on the boundary

In this section, we are interested in the coefficients $c_{G, \hat{G}}(\nu, \hat{\nu})$ when $(\nu, \hat{\nu})$ belongs to the boundary of $\mathrm{LR}(G, \hat{G})$. Indeed, such multiplicities are equal to analogous numbers for Levi subgroups of G and \hat{G} . The results could be obtained by applying results of type “quantification commutes with reduction” in symplectic geometry (see [GS82b]). Our proof is more direct. Indeed, it remains to prove that two spaces have the same dimension: we find an explicit and natural isomorphism.

Theorem 4.1. *Let $X = G/P \times \hat{G}/\hat{P}$ be a flag manifold for the group $G \times \hat{G}$. Let λ be a one-parameter subgroup of G and C be an irreducible component of the fixed point set X^λ of λ in X . Let G^λ be the centralizer of the image of λ in G . We assume that (C, λ) is a well covering pair. Let \mathcal{L} be a G -linearized line bundle on X generated by its global sections such that λ acts trivially on the restriction $\mathcal{L}|_C$. Then the restriction map induces an isomorphism*

$$H^0(X, \mathcal{L})^G \longrightarrow H^0(C, \mathcal{L}|_C)^{G^\lambda},$$

between the spaces of invariant sections of \mathcal{L} and $\mathcal{L}|_C$.

Sketch of proof. Consider the closure $\overline{C^+}$ of the Białynicki-Birula cell C^+ . The morphism

$$\begin{aligned} \overline{\eta} : G \times_{P(\lambda)} \overline{C^+} &\longrightarrow X \\ [g : x] &\longmapsto gx \end{aligned}$$

that is proper and birational, induces a G -equivariant isomorphism

$$H^0(X, \mathcal{L}) \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L})).$$

In particular

$$H^0(X, \mathcal{L})^G \simeq H^0(G \times_{P(\lambda)} \overline{C^+}, \overline{\eta}^*(\mathcal{L}))^G \simeq H^0(\overline{C^+}, \mathcal{L}|_{\overline{C^+}})^{P(\lambda)}.$$

On the other hand, since λ acts trivially on $\mathcal{L}|_C$, [Res10a, Lemma 5] proves that

$$H^0(C^+, \mathcal{L}|_{C^+})^{P(\lambda)} \simeq H^0(C, \mathcal{L}|_C)^{G^\lambda}. \quad (29)$$

Then we have to prove that

$$H^0(\overline{C^+}, \mathcal{L}|_{\overline{C^+}})^{P(\lambda)} \simeq H^0(C^+, \mathcal{L}|_{C^+})^{P(\lambda)};$$

that is, that any regular $P(\lambda)$ -invariant section σ of \mathcal{L} on C^+ extends to $\overline{C^+}$. Using the SL_2 -theory, one can check that such a section has no pole along the divisors of $\overline{C^+} - C$. We conclude that such a section extends to $\overline{C^+}$ by normality of $\overline{C^+}$ that is a Schubert variety. \square

Let \mathcal{F} be a face of $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$. Assume that \mathcal{F} is regular, that is that it contains pairs $(\nu, \hat{\nu})$ of regular dominant weights. If S is a torus in G and H is a subgroup of G containing S , H^S denotes the centralizer of S in H . By [Res10b], the regular face \mathcal{F} corresponds to a pair (S, \hat{w}) where S is a subtorus of T and $\hat{w} \in \hat{W}$ such that

$$\hat{G}^S \cap \hat{w}\hat{B}\hat{w}^{-1} = \hat{B}^S, \quad (30)$$

and the span of \mathcal{F} is the set of pairs $(\nu, \hat{\nu}) \in (X(T) \times X(\hat{T})) \otimes \mathbb{Q}$ such that

$$\nu|_S + \hat{w}\hat{\nu}|_S = 0 \in X(S) \otimes \mathbb{Q}. \quad (31)$$

Corollary 4.2. *Let $(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+$ be a pair of dominant weights. Assume that $(\nu, \hat{\nu})$ belongs to the span of \mathcal{F} (equivalently that it satisfies condition (31)). Then*

$$c_{\nu, \hat{\nu}}(G, \hat{G}) = c_{\nu, \hat{w}\hat{\nu}}(G^S, \hat{G}^S).$$

Sketch of proof. The corollary is obtained by applying Theorem 4.1 with $X = G/B \times \hat{G}/\hat{B}$ and $C = G^S B/B \times \hat{G}^S \hat{w}\hat{B}/\hat{B}$. \square

Several particular cases of Theorems 4.1 and its corollary was known before. If $G = T$ is a maximal torus of $G = \text{GL}_n(\mathbb{C})$, our theorem is equivalent to [KTT07,

Theorem 5.8]. If $\hat{G} = G \times G$ (or more generally $\hat{G} = G^s$ for some integer $s \geq 2$) and G is diagonally embedded in \hat{G} then $c_{\nu, \hat{\nu}}(G, \hat{G})$ (resp. $c_{\nu, \hat{\nu}}(G^S, \hat{G}^S)$) are tensor product multiplicities for the group G (resp. G^S). This case was recently proved independently by Derksen and Weyman in [DW11, Theorem 7.4] and King, Tollu and Toumazet in [KTT09, Theorem 1.4] if $G = GL_n(\mathbb{C})$ and for any reductive group by Roth in [Rot11]. If ν is regular then Theorem 4.1 can be obtained applying [Bri99, Theorem 3] and [Res10a]. Similar reductions can be found in [Bri93, Man97, Mon96].

Remark 4.3. In Section 3.4, we seen that the multiplicities corresponding to the points in the boundary of the cone $\mathbb{Q}_{\geq 0}\text{LR}(\text{SL}_3(\mathbb{C})^3)$ are equal to one. This agrees with Corollary 4.2, since the tensor product of two irreducible $\text{SL}_2(\mathbb{C})$ -modules is multiplicity free.

5. The question of saturation

5.1. The branching semigroup. Consider the set $\text{LR}(G, \hat{G})$ of pairs $(\nu, \hat{\nu})$ of dominant weights such that $c_{G, \hat{G}}(\nu, \hat{\nu}) \neq 0$.

Theorem 5.1 (Brion-Knop (see [É92])). *The set $\text{LR}(G, \hat{G})$ is a finitely generated semigroup in $X(T)^+ \times X(\hat{T})^+$.*

Proof. Start with Frobenius' decomposition of $\mathbb{C}[\hat{G}]$ as a $\hat{G} \times \hat{G}$ -module:

$$\mathbb{C}[\hat{G}] = \bigoplus_{\hat{\nu} \in X(\hat{T})^+} V_{\hat{G}}(\hat{\nu}) \otimes V_{\hat{G}}(\hat{\nu})^*.$$

Let U and \hat{U}^- denote the unipotent radicals of B and \hat{B}^- . Consider the algebra

$$\mathbb{C}[\hat{G}]^{U \times \hat{U}^-} = \bigoplus_{\hat{\nu} \in X(\hat{T})^+} V_{\hat{G}}(\hat{\nu})^U \otimes (V_{\hat{G}}(\hat{\nu})^*)^{\hat{U}^-}.$$

Observe that \hat{T} acts on the line $(V_{\hat{G}}(\hat{\nu})^*)^{\hat{U}^-}$ by the weight $-\hat{\nu}$ and that $V_G(\nu)$ is a submodule of $V_{\hat{G}}(\hat{\nu})$ if and only if $V_{\hat{G}}(\hat{\nu})^U$ contains a T -eigenvector of weight ν . Then $(\nu, \hat{\nu})$ belongs to $\text{LR}(G, \hat{G})$ if and only if $(\nu, -\hat{\nu})$ is a weight of $T \times \hat{T}$ acting on the algebra $\mathbb{C}[\hat{G}]^{U \times \hat{U}^-}$. This implies that $\text{LR}(G, \hat{G})$ is a semigroup.

To prove that $\text{LR}(G, \hat{G})$ is finitely generated it is sufficient to prove that $\mathbb{C}[\hat{G}]^{U \times \hat{U}^-}$ is. Recall that $\mathbb{C}[G]^U$ is finitely generated. But

$$\mathbb{C}[\hat{G}]^{U \times \hat{U}^-} = (\mathbb{C}[G]^U \otimes \mathbb{C}[\hat{G}]^{\hat{U}^-})^G.$$

Since $\mathbb{C}[G]^U$ and $\mathbb{C}[\hat{G}]^{\hat{U}^-}$ are finitely generated and G is reductive the Hilbert theorem implies that $\mathbb{C}[\hat{G}]^{U \times \hat{U}^-}$ is finitely generated. \square

The subgroup of $X(T \times \hat{T})$ generated by $\text{LR}(G, \hat{G})$ is denoted by $\mathbb{Z}\text{LR}(G, \hat{G})$. We already described the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$. The following statement describes the group $\mathbb{Z}\text{LR}(G, \hat{G})$.

Proposition 5.2 (see [PR13, Bri12]). *Assume that no ideal of \mathfrak{g} is an ideal of $\hat{\mathfrak{g}}$. Let \hat{Z} denote the center of \hat{G} .*

Then the group $\mathbb{Z}\text{LR}(G, \hat{G})$ is the set of pairs $(\nu, \hat{\nu}) \in X(T \times \hat{T})$ such that

$$\nu(t) \cdot \hat{\nu}(t) = 1$$

for any $t \in \hat{Z} \cap G$.

The semigroup is said to be saturated if it can be recovered from the knowledge of the cone $\mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G})$ and the group $\mathbb{Z}\text{LR}(G, \hat{G})$:

Definition 5.3. The semigroup $\text{LR}(G, \hat{G})$ is said to be *saturated* if

$$\text{LR}(G, \hat{G}) = \mathbb{Q}_{\geq 0}\text{LR}(G, \hat{G}) \cap \mathbb{Z}\text{LR}(G, \hat{G}). \quad (32)$$

5.2. The case of tensor product decomposition. In this section, we review results on the saturation of $\text{LR}(G^3)$ for G is simple and simply connected. Observe that $\mathbb{Z}\text{LR}(G, \hat{G})$ is the set $(\nu_1, \nu_2, \nu_3) \in X^*(T)^3$ such that $\nu_1 + \nu_2 + \nu_3$ belongs to the root lattice Λ_R .

Theorem 5.4 (Knutson-Tao). *The semigroup $\text{LR}(G^3)$ is saturated for $G = \text{SL}_n(\mathbb{C})$.*

The first proof [KT99] of Theorem 5.4 due to Knutson and Tao uses a combinatorial model for Littlewood-Richardson coefficients called honeycombs. Belkale reproved [Bel06] this theorem using intersection theory. Derksen and Weyman reproved [DW00] this result using representations of quivers and Kapovich and Millson obtained a proof [KM08] using the geometry of Bruhat-Tits buildings.

The best known uniform generalization of Theorem 5.4 to any simple group G is

Theorem 5.5 (Kapovich-Millson [KM08]). *Let ν_1, ν_2 , and ν_3 be three dominant weights of the simple group G . Let k be the square of the least common multiple of the coefficients of the highest root of G written in terms of simple roots.*

If $(V_G(N\nu_1) \otimes V_G(N\nu_2) \otimes V_G(N\nu_3))^G \neq \{0\}$ for some positive integer N and $\nu_1 + \nu_2 + \nu_3 \in \Lambda_R$, then $(V_G(k\nu_1) \otimes V_G(k\nu_2) \otimes V_G(k\nu_3))^G \neq \{0\}$.

Observe that for $G = \text{SL}_n(\mathbb{C})$, $k = 1$. Belkale and Kumar [BK10b] and Sam [Sam12] obtained better constants than k for classical groups.

Two important conjectures in the topic are still open. The first one asserts that tensor product decompositions for simply-laced groups satisfy the saturation property. It was checked by explicit computations for $G = \text{Spin}_8(\mathbb{C})$ in [KKM09]. Observe that, for G of type E_8 , the constant k in Theorem 5.5 is equal to 3600. The second conjecture asserts that Theorem 5.5 is satisfied with $k = 1$ for any G if the weights ν_i are regular.

5.3. Other examples. If $G = \hat{T}$, $(\nu, \hat{\nu})$ belongs to the group $\mathbb{Z}\text{LR}(G, \hat{G})$ if and only if $\hat{\nu} - \nu$ belongs to the root lattice $\hat{\Lambda}_R$. The multiplicities $c_{G, \hat{G}}(\nu, \hat{\nu})$ are given by the character of $V_{\hat{G}}(\hat{\nu})$. The saturation is well known in this case.

Others examples of semigroups have been determined explicitly in [PR13]. A consequence of these computations is the following list of examples of saturated semigroups.

Theorem 5.6. *For $(G, \hat{G}) = (\text{Spin}_{2n-1}(\mathbb{C}), \text{Spin}_{2n}(\mathbb{C})), (\text{SL}_3(\mathbb{C}), G_2), (G_2, \text{Spin}_7(\mathbb{C})), (\text{Spin}_9(\mathbb{C}), F_4), (F_4, E_6), (\text{Sp}_4(\mathbb{C}), \text{SL}_4(\mathbb{C})), (\text{Sp}_6(\mathbb{C}), \text{SL}_6(\mathbb{C})), (\text{Sp}_8(\mathbb{C}), \text{SL}_8(\mathbb{C}))$ and $(\text{Sp}_{10}(\mathbb{C}), \text{SL}_{10}(\mathbb{C}))$ the semigroup $\text{LR}(G, \hat{G})$ is saturated.*

6. PRV

6.1. The classical case. Recall that $X(T)^+$ is a fundamental domain for the action of W on $X(T)$; for any $\nu \in X(T)$, we denote by $\bar{\nu}$ the unique dominant element in the orbit $W\nu$. Parthasarathy-Ranga Rao-Varadarajan conjectured in the sixties [PRRV67] (a weaker version of) the following

PRV conjecture. Let $V_G(\mu)$ and $V_G(\nu)$ be two irreducible G -modules with highest weights μ and ν respectively. Then, for any $w \in W$, the irreducible G -module $V_G(\overline{\mu + w\nu})$ with extremal weight $\mu + w\nu$, occurs with multiplicity at least one in $V_G(\mu) \otimes V_G(\nu)$.

This conjecture was proved independently by S. Kumar in [Kum88] and O. Mathieu in [Mat89].

6.2. A double generalization. The homogeneous space \hat{G}/G is said to be *spherical* if it contains an open \hat{B} -orbit. It is said to be *spherical of minimal rank* if there exists a T -fixed point in \hat{G}/\hat{B} whose the G -orbit is open. This condition is very strong and was classified in [Res10c]. The pairs (G, \hat{G}) such that \hat{G}/G is spherical of minimal rank reduces to the following list $(G, G \times G), (\text{Sp}_{2n}(\mathbb{C}), \text{SL}_{2n}(\mathbb{C})), (\text{Spin}_{2n-1}(\mathbb{C}), \text{Spin}_{2n}(\mathbb{C})), (G_2, \text{Spin}_7(\mathbb{C}))$ and (F_4, E_6) .

Let $\rho : X(\hat{T}) \rightarrow X(T)$ be the restriction morphism. Let Δ (resp $\hat{\Delta}$) denote the set of simple roots of G and \hat{G} . By [Res10c, Lemma 4.6], $\rho(\hat{\Delta}) = \Delta$. Moreover, for any $\alpha \in \Delta$, we have the following alternative:

- (i) there exists a unique $\hat{\alpha}_0 \in \hat{\Delta}$ such that $\rho(\hat{\alpha}_0) = \alpha$; or
- (ii) there exist exactly two simple roots $\hat{\alpha}_1$ and $\hat{\alpha}_2$ in $\hat{\Delta}$ such that $\rho(\hat{\alpha}_1) = \rho(\hat{\alpha}_2) = \alpha$.

The set of simple roots satisfying the second assertion is denoted by Δ_2 . For $\alpha \in \Delta_2$, let \hat{W}_α denote the subgroup of \hat{W} generated by $s_{\hat{\alpha}_1}$ and $s_{\hat{\alpha}_2}$. Then \hat{W}_α is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Theorem 6.1. [MPR11b, MPR11a] *Fix two connected reductive groups $G \subset \hat{G}$ such that \hat{G}/G is spherical of minimal rank. Let $\alpha \in \Delta_2$ and $\hat{w} \in \hat{W}$.*

Let $\hat{\nu}$ be a dominant weight of \hat{G} . Let $\hat{w}_1 \in \hat{W}_\alpha \hat{w}$ be such that $(\rho(\hat{w}_1 \hat{\nu}), \alpha)$ is maximal.

Fix an integer k such that $0 \leq k \leq \min(\langle \rho(s_{\hat{\alpha}_1} \hat{w}_1 \hat{\nu}), \hat{\alpha}_1^\vee \rangle, \langle \rho(s_{\hat{\alpha}_2} \hat{w}_1 \hat{\nu}), \hat{\alpha}_2^\vee \rangle)$. Then, if $\nu = \rho(\hat{w}_1 \hat{\nu}) - k\alpha$ is dominant, the irreducible G -module $V_G(\nu)$ occurs with multiplicity at least one in $V_{\hat{G}}(\hat{\nu})$.

For $\hat{G} = G \times G$ and k extremal in the interval, Theorem 6.1 implies the PRV conjecture.

Sketch of proof. Denote the \hat{G} -linearized line bundle $\mathcal{L}_{\hat{\nu}}$ on \hat{G}/\hat{B} such that $H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}})$ is isomorphic to $V_{\hat{G}}(\hat{\nu})^*$. Set $\hat{v} = s_{\hat{\alpha}_1} \hat{w} \in \hat{W}$, $X^\circ(\hat{v}) = G \cdot \hat{v} \hat{B}/\hat{B}$. Let $X(\hat{v})$ denote the closure of $X^\circ(\hat{v})$. By [Bri01, Corollary 8], the restriction map

$$H^0(\hat{G}/\hat{B}, \mathcal{L}_{\hat{\nu}}) \longrightarrow H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}})$$

is surjective. Then, it is sufficient to prove the existence of a section $\sigma \in H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}})$ that is an eigenvector of weight $-\nu$ for B^- .

We first construct a section $\tau \in H^0(X(\hat{v}), \mathcal{L}_{\hat{\nu}}^{\otimes n})$ that is an eigenvector of weight $-n\nu$ for B^- , for some positive integer n . Consider $X = G/B^- \times \hat{G}/\hat{B}$, the neutral component $S \subset T$ of the kernel of α , the centralizer G^S of S in G , the fixed point set X^S of S in X and the irreducible component C of X^S containing $(B^-/B^-, \hat{v} \hat{B}/\hat{B})$. Let $\mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}$ be the $(G \times \hat{G})$ -linearized line bundle on X such that $H^0(X, \mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}) \simeq V_G(\nu) \otimes V_{\hat{G}}(\hat{\nu})^*$. Actually, G^S is isomorphic to $(\text{P})\text{SL}_2(\mathbb{C})$ and C is isomorphic to $(\mathbb{P}^1)^3$. We can deduce that C contains points semi-stable for the action of G^S relatively to $\mathcal{L}^\nu \otimes \mathcal{L}_{\hat{\nu}}$. Then, a Luna's theorem (see [Lun75, Corollary 2 and Remark 1]) shows that C contains points semi-stable for the action of G . The existence of τ and n follows.

Secondly, we prove that $\dim(H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)^{-\nu}}) = 1$. Observe that the stabilizer $G_{\hat{v} \hat{B}/h_B}$ contains T . Then, Forbenius' theorem allows to embed $H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)^{-\nu}}$ in $V_G(\hat{\nu})^{(T)^{\rho(\hat{v}\hat{\nu})}}$. Since $\rho(\hat{v}\hat{\nu})$ belongs to $\nu + \mathbb{Z}\alpha$ this space has dimension at most one (it is, once again, a consequence of the SL_2 -theory). From the existence of τ , we deduce that $H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)^{-\nu}}$ and $V_G(\hat{\nu})^{(T)^{\rho(\hat{v}\hat{\nu})}}$ have dimension one.

Finally, fix a nonzero element $\sigma \in H^0(X^\circ(\hat{v}), \mathcal{L}_{\hat{\nu}})^{(B^-)^{-\nu}}$. By unicity $\sigma^{\otimes n}$ and the restriction $\tau|_{X^\circ(\hat{v})}$ must coincide (up to scalar multiplication). In particular, $\sigma^{\otimes n}$ extends to a section of $\mathcal{L}_{\hat{\nu}}^{\otimes n}$ on $X(\hat{v})$. Since $X(\hat{v})$ is normal (see [Bri03, Theorem 1]), σ itself extends to a section of $\mathcal{L}_{\hat{\nu}}$ on $X(\hat{v})$. This ends the proof. \square

It is natural to ask if the $\text{LR}(\text{Sp}_{2n}(\mathbb{C}), \text{SL}_{2n}(\mathbb{C}))$ is saturated for any n .

7. The multiplicative Horn problem

In this section, we assume that G is simple and simply-connected.

7.1. The Meinrenken-Woodward polytope. Recall from Section 3.1 that $\mathcal{C}(K^3)$ identifies with the set of triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint orbits in \mathfrak{k}

such that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ contains 0. We now consider a multiplicative analogous of $\mathcal{C}(K^3)$. Let \mathcal{O}_1 and \mathcal{O}_2 be two conjugacy classes in K . Then the product $\mathcal{O}_1 \cdot \mathcal{O}_2 = \{k_1 k_2 : k_1 \in \mathcal{O}_1 \text{ and } k_2 \in \mathcal{O}_2\}$ is stable by conjugacy.

What conjugacy classes are contained in $\mathcal{O}_1 \cdot \mathcal{O}_2$? (†)

Let θ be the longest root of G . The fundamental alcove in $X_*(T)_{\mathbb{R}}$ is

$$\mathcal{A}_* = \left\{ \lambda \in X_*(T)_{\mathbb{R}} : \begin{cases} \langle \lambda, \alpha \rangle \geq 0 & \forall \alpha \in \Delta \\ \langle \lambda, \theta \rangle \leq 1 \end{cases} \right\}.$$

Consider the exponential map

$$\begin{aligned} \exp : \text{Lie}(H) &\longrightarrow H \\ \mu &\longmapsto \exp(\mu). \end{aligned}$$

Any conjugacy class in K contains a unique element of the form $\exp(\sqrt{-1}\lambda)$ for some $\lambda \in \mathcal{A}_*$ (see e.g. [Bou05, Chapter IX. §5]); we denote by \mathcal{O}_{λ}^K the conjugacy class containing $\exp(\sqrt{-1}\lambda)$.

To answer the question (†) we want to describe the set

$$\mathcal{P}_K = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{A}_*^3 : \mathcal{O}_{\lambda_1}^K \cdot \mathcal{O}_{\lambda_2}^K \cdot \mathcal{O}_{\lambda_3}^K \ni e\},$$

where e is the unit element of K . According to the convexity theorem proved by Meinrenken-Woodward [MW98], \mathcal{P}_K is a convex polytope of nonempty interior in \mathcal{A} . The aim of this section is to describe the minimal list of inequalities that characterize \mathcal{P}_K .

7.2. The fusion product. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((z)) \oplus \mathbb{C}c$ be the affine Lie algebra with c central in $\tilde{\mathfrak{g}}$ and

$$[x \otimes f, y \otimes g] = [x, y] \otimes fg + (x, y) \text{Res}_{z=0}(gdf)c,$$

for $x, y \in \mathfrak{g}$ and $f, g \in \mathbb{C}((z))$. Set $\tilde{\mathfrak{g}}_+ = \mathfrak{g} \otimes z\mathbb{C}[[z]]$. Fix a positive integer and set $X(T)_{\mathfrak{l}}^+ = \{\nu \in X(T)^+ : \langle \nu, \theta^{\vee} \rangle \leq \mathfrak{l}\}$. For any $\nu \in X(T)_{\mathfrak{l}}^+$, there exists a unique $\tilde{\mathfrak{g}}$ -module $V_{\tilde{\mathfrak{g}}}(\nu, \mathfrak{l})$ such that c acts on by multiplication by \mathfrak{l} and the subspace of $V_{\tilde{\mathfrak{g}}}(\nu, \mathfrak{l})$ annihilated by $\tilde{\mathfrak{g}}_+$ is isomorphic to $V_{\mathfrak{g}}(\nu)$ as a \mathfrak{g} -module.

Consider now the projective line \mathbb{P}^1 with four distinct marked points $\{0, p_1, p_2, p_3\}$. Set $U = \mathbb{P}^1 - \{0\}$ and consider the ring $\mathcal{O}(U)$ of regular functions on U identified with $\mathbb{C}[z^{-1}]$. Then $\mathfrak{g} \otimes \mathcal{O}(U)$ is a sub-algebra of $\tilde{\mathfrak{g}}$ and acts on $V_{\tilde{\mathfrak{g}}}(0, \mathfrak{l})$. For each point p_i , consider the evaluation map $ev_i : \mathcal{O}(U) \rightarrow \mathbb{C}$ at p_i and the associated morphism (still denoted ev_i) from $\mathfrak{g} \otimes \mathcal{O}(U)$ to \mathfrak{g} . Fix three weights $\nu_1, \nu_2, \nu_3 \in X(T)_{\mathfrak{l}}^+$. Consider the action of $\mathfrak{g} \otimes \mathcal{O}(U)$ on $V_{\tilde{\mathfrak{g}}}(0, \mathfrak{l}) \otimes V_{\mathfrak{g}}(\nu_1) \otimes V_{\mathfrak{g}}(\nu_2) \otimes V_{\mathfrak{g}}(\nu_3)$ given by

$$\begin{aligned} \xi \cdot (v_0 \otimes v_1 \otimes v_2 \otimes v_3) = & (\xi v_0) \otimes v_1 \otimes v_2 \otimes v_3 + v_0 \otimes (ev_1(\xi)v_1) \otimes v_2 \otimes v_3 \\ & + v_0 \otimes v_1 \otimes (ev_2(\xi)v_2) \otimes v_3 \\ & + v_0 \otimes v_1 \otimes v_2 \otimes (ev_3(\xi)v_3). \end{aligned}$$

The space of conformal blocks $V_{\mathbb{P}^1}^\dagger(\nu_1, \nu_2, \nu_3)$ can be defined (see [Bea96, Corollary 3.5]) as the space of $\mathfrak{g} \otimes \mathcal{O}(U)$ -invariant linear forms on $V_{\mathfrak{g}}(0, \ell) \otimes V_{\mathfrak{g}}(\nu_1) \otimes V_{\mathfrak{g}}(\nu_2) \otimes V_{\mathfrak{g}}(\nu_3)$. It is finite dimensional and the fusion multiplicities are

$$N^\ell(\nu_1, \nu_2, \nu_3) = \dim(V_{\mathbb{P}^1}^\dagger(\nu_1, \nu_2, \nu_3)).$$

The fusion product \otimes_ℓ on $\oplus_{\nu \in X(T)_\ell^+} \mathbb{Z}V(\nu)$ is defined by

$$V(\nu_1) \otimes_\ell V(\nu_2) = \sum_{\nu_3 \in X(T)_\ell^+} N^\ell(\nu_1, \nu_2, \nu_3) V(-w_0\nu_3),$$

for any $\nu_1, \nu_2 \in X(T)_\ell^+$. The product \otimes_ℓ is associative and commutative (see e.g. [Bea96]).

The fundamental alcove in $X^*(T)_\mathbb{Q}$ is

$$\mathcal{A}_\mathbb{Q}^* = \{ \lambda \in X^*(T)_\mathbb{Q} : \begin{cases} \langle \lambda, \alpha^\vee \rangle \geq 0 & \forall \alpha \in \Delta \\ \langle \lambda, \theta^\vee \rangle \leq 1 \end{cases} \}.$$

The multiplicative analogous to $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$ is

$$\mathcal{P}_\otimes = \{ (\nu_1, \nu_2, \nu_3) \in (\mathcal{A}_\mathbb{Q}^*)^3 : \begin{array}{l} \ell\nu_1, \ell\nu_2, \ell\nu_3 \in X(T)_\ell^+ \text{ and } N^\ell(\nu_1, \nu_2, \nu_3) \neq 0, \\ \text{for some positive } \ell. \end{array} \}.$$

Theorem 7.1 (see [TW03]). *Let $(\nu_1, \nu_2, \nu_3) \in (\mathcal{A}_\mathbb{Q}^*)^3$. Then $(\nu_1, \nu_2, \nu_3) \in \mathcal{P}_\otimes$ if and only if $((\nu_1, \cdot)_T, (\nu_2, \cdot)_T, (\nu_3, \cdot)_T) \in \mathcal{P}_K$.*

7.3. Relation with $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$. The fusion multiplicities are related to the tensor product multiplicities by

$$\lim_{\ell \rightarrow \infty} N^\ell(\nu_1, \nu_2, \nu_3) = c_{G^3}(\nu_1, \nu_2, \nu_3). \quad (33)$$

A trivial consequence for the cones is the following.

Proposition 7.2. *The cone in $X^*(T^3)_\mathbb{R}$ generated by \mathcal{P}_\otimes is $\mathbb{Q}_{\geq 0}\text{LR}(G^3)$.*

7.4. Quantum Belkale-Kumar Schubert calculus. Fix a simple root α and consider the associated maximal parabolic subgroup P_α containing B . Let L_α be its Levi-subgroup containing T . The Picard group $\text{Pic}(G/P_\alpha)$ identifies with $H^2(G/P_\alpha, \mathbb{Z}) = \mathbb{Z}\sigma_{s_\alpha}$. We denote by $\sigma_{s_\alpha}^*$ the element of $\text{Hom}(H^2(G/P_\alpha, \mathbb{Z}), \mathbb{Z})$ mapping σ_{s_α} to 1.

Let $\gamma : \mathbb{P}^1 \rightarrow G/P_\alpha$ be a curve. Identifying the group $\text{Pic}(\mathbb{P}^1)$ to \mathbb{Z} (by mapping ample line bundles on positive integers), the pullback of line bundles induces an element of $\text{Hom}(H^2(G/P_\alpha, \mathbb{Z}), \mathbb{Z})$ called the degree of γ and denoted by $\underline{d}(\gamma)$. By construction $\underline{d}(\gamma) \in \mathbb{Z}_{\geq 0}\sigma_{s_\alpha}^*$.

Let ρ and ρ^{L_α} denote the half sum of positive roots of G and L_α respectively. Set

$$n_\alpha = \langle \alpha^\vee, 2(\rho - \rho^{L_\alpha}) \rangle. \quad (34)$$

Fix a nonnegative integer d . Let $\overline{M}_{0,3}(G/P, \underline{d})$ be the moduli space of stable maps of degree $d\sigma_{s_\alpha}^*$ with 3 marked points into G/P_α . It is a projective variety and

$$\dim(\overline{M}_{0,3}(G/P, \underline{d})) = \dim(G/P_\alpha) + dn_\alpha.$$

It comes equipped with 3 evaluation maps $ev_i : \overline{M}_{0,3}(G/P, \underline{d}) \rightarrow G/P_\alpha$. The Gromov-Witten invariant associated to three Schubert classes (corresponding to $w_i \in W^{P_\alpha}$) and a degree $\underline{d} = d\sigma_{s_\alpha}^*$ is then the intersection number

$$GW(w_1, w_2, w_3; d) = \int_{\overline{M}_{0,3}(G/P, \underline{d})} ev_1^*(\sigma_{w_1}) \cdot ev_2^*(\sigma_{w_2}) \cdot ev_3^*(\sigma_{w_3}).$$

Introduce a variable q and consider the group

$$\begin{aligned} QH^*(G/P_\alpha, \mathbb{Z}) &:= H^*(G/P, \mathbb{Z}) \otimes \mathbb{Z}[q] \\ &= \bigoplus_{w \in W^{P_\alpha}} \mathbb{Z}[q] \sigma_w. \end{aligned}$$

The $\mathbb{Z}[q]$ -linear quantum product \star on $QH^*(G/P_\alpha, \mathbb{Z})$ is defined by, for any $w_1, w_2 \in W^{P_\alpha}$,

$$\sigma_{w_1} \star \sigma_{w_2} = \sum GW(w_1, w_2, w_3; d) q^d \sigma_{w_3}^\vee,$$

where the sum runs over $w_3 \in W^{P_\alpha}$ and over nonnegative integers d .

The grading on $H^*(G/P_\alpha, \mathbb{Z})$ extends to the quantum setting by setting $\deg(q) = 2n_\alpha$. In particular $GW(w_1, w_2, w_3; \underline{d}) \neq 0$ implies

$$l(w_1) + l(w_2) + l(w_3) + dn_\alpha = 2 \dim(G/P_\alpha). \quad (35)$$

Condition (35) can be rewritten like

$$\#\Phi(w_1) + \#\Phi(w_2) + \#\Phi(w_3) + dn_\alpha = 2 \#\Phi(G/P_\alpha). \quad (36)$$

Set $h = d\alpha^\vee$. Since $2(\rho - \rho^{L_\alpha}) = \sum_{\beta \in \Phi(G/P)} \beta$, condition (36) can be rewritten like

$$\sum_{\chi \in X^*(Z)} \left(\sum_{i=1}^3 \#\Phi(w_i, \chi) + \sum_{\beta \in \Phi(G/P, \chi)} \langle h, \beta \rangle \right) = 2 \sum_{\chi \in X^*(Z)} \#\Phi(G/P, \chi). \quad (37)$$

The Belkale-Kumar quantum product requires the equality (37) to hold term by term. More precisely, set

$$\sigma_{w_1} \otimes \sigma_{w_2} = \sum_{\substack{w_3 \in W^P \\ d \in \mathbb{Z}_{\geq 0}}} \widehat{GW}(w_1, w_2, w_3; d) q^d \sigma_{w_3}^\vee, \quad (38)$$

where $\widetilde{GW}(w_1, w_2, w_3; d) = GW(w_1, w_2, w_3; d)$ if

$$\forall \chi \in X(Z) \quad \sum_i \# \Phi(w_i, \chi) + \sum_{\beta \in \Phi(G/P_\alpha, \chi)} d \langle \alpha^\vee, \beta \rangle = 2 \# \Phi(G/P, \chi), \quad (39)$$

and $\widetilde{GW}(w_1, w_2, w_3; d) = 0$ otherwise.

Theorem 7.3 (Bekale-Kumar [BK13]). *The product \otimes is associative.*

7.5. Descriptions of \mathcal{P}_K . Let ϖ_α denote the fundamental weight associated to α . To any $(w_1, w_2, w_3) \in (W^{P_\alpha})^3$ and any $d \in \mathbb{Z}_{\geq 0}$, we associate the following linear inequality on points $(\lambda_1, \lambda_2, \lambda_3)$ in $X_*(T)_\mathbb{R}$:

$$\mathcal{I}_\alpha(w_1, w_2, w_3; d) \quad \langle w_1 \varpi_\alpha, \lambda_1 \rangle + \langle w_2 \varpi_\alpha, \lambda_2 \rangle + \langle w_3 \varpi_\alpha, \lambda_3 \rangle \leq d.$$

Here, comes a first description of \mathcal{P}_K .

Theorem 7.4 (Teleman-Woodward (see [TW03])). *Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{A}_*^3$. Then $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_K$ if and only if inequality $\mathcal{I}_\alpha(w_1, w_2, w_3; d)$ is fulfilled for any simple root α , any nonnegative integer d and any $w_1, w_2, w_3 \in W^{P_\alpha}$ such that*

$$GW(w_1, w_2, w_3; d) = 1 \quad (40)$$

in G/P_α .

Recently, Theorem 7.4 was improved as follows.

Theorem 7.5. *Belkale-Kumar [BK13], R. [Res13b]] Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{A}_*^3$. Then $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_K$ if and only if inequality $\mathcal{I}_\alpha(w_1, w_2, w_3; d)$ is fulfilled for any simple root α , any nonnegative integer d and any (w_1, w_2, w_3) such that, in $QH^*(G/P_\alpha)$,*

$$\widetilde{GW}(w_1, w_2, w_3; d) = 1. \quad (41)$$

Now, Teorem 7.5 is optimal.

Theorem 7.6 (Belkale-Kumar [BK13]). *The list of inequalities given by Theorem 7.5 is irredundant.*

Remark 7.7. For G classical, condition (41) can be checked using works of Bertram, Buch, Kresch and Tamvakis or the software qcalc [Buc]. Explicit lists of inequalities can be downloaded on the homepage of the author. For example, the polytope $\mathcal{P}_{\mathbb{S}P_{12}}$ has 43 136 facets and 20 839 vertices.

In this multiplicative context, the question of saturation can be asked for the fusion product multiplicities by analogy with Section 5. The only known result is in type A:

Theorem 7.8. [Bel08] *Let $G = \mathrm{SL}_n(\mathbb{C})$, ℓ be a positive integer and ν_1, ν_2 and ν_3 be three dominant weights in $X(T)_\ell^+$. We assume that $\nu_1 + \nu_2 + \nu_3$ belongs to the root lattice Λ_R .*

Then $N^\ell(\nu_1, \nu_2, \nu_3) \neq 0$ if and only if there exists a positive integer k such that $N^{k\ell}(k\nu_1, k\nu_2, k\nu_3) \neq 0$.

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