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► **To cite this version:**

Colette Anné, Marwa Balti, Nabila Torki-Hamza. m-ACCRETIVE LAPLACIAN ON A NON SYMMETRIC GRAPH. *Indagationes Mathematicae*, 2020, 31 (2), pp.277-293. 10.1016/j.indag.2020.01.005 . hal-02158296v2

HAL Id: hal-02158296

<https://hal.science/hal-02158296v2>

Submitted on 6 Dec 2020

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m-ACCRETIVE LAPLACIAN ON A NON SYMMETRIC GRAPH

COLETTE ANNÉ, MARWA BALTI, AND NABILA TORKI-HAMZA

ABSTRACT. We consider a non self-adjoint Laplacian on a directed graph with non-symmetric weights on edges. We give a criterion for the m -accretiveness and the m -sectoriality of this Laplacian. Our results are based on a comparison of this operator with its symmetric part for which we can apply different results concerning essential self-adjointness of a symmetric Laplace operator on an infinite graph. This gives results on the heat operator related to our non-symmetric Laplacian.

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INTRODUCTION

Some properties for linear unbounded non self-adjoint operators, such as accretiveness, maximal accretiveness (or m -accretiveness) and m -sectoriality are very important for physical and technical problems. They are subjects of

Date: December 6, 2020, *File:* ABTH-3.tex.

2010 *Mathematics Subject Classification.* 47B44, 47H06, 47A12, 47A10, 47B25.

Key words and phrases. Directed graph, Graph Laplacian, Non self-adjoint operator, Numerical range, m -accretive Laplacian, m -sectorial Laplacian.

special attention in view of later applications to analytic and asymptotic perturbation theory [Kat76], [L79], [ELZ83], [H11], [Ou05], [Kh13], [MT15]. The main importance of accretive operators is their appearance in the Hille-Yosida and Lumer-Phillips Theorems: an operator A is maximally accretive if and only if $-A$ is the generator of a contraction semigroup. Also, we shall focus on m -sectorial operators: their spectrum lies in a sector and their resolvent satisfies a certain estimate. The opposite of generators of bounded holomorphic semigroups holomorphic on a sector are m -sectorial operators.

We consider a directed infinite graph and we investigate the associated non-symmetric Laplacian Δ under a Kirchhoff's Assumption [Bal17]. This class of operators can be considered as a generalization of lower semibounded or positive symmetric operators [T-H10], [KL12]. The purpose of the present paper is to give a criterion for the m -accretiveness and the m -sectoriality of the discrete Laplacian.

After the preliminaries in Section 1, we have four sections. In Section 2, we give a general condition on the graph using the notion of χ -completeness introduced in [AT-H15] to establish the m -accretiveness of the non-symmetric Laplacian Δ . Section 3 is devoted to the study of the relations between our non symmetric Laplacian and its symmetrized part. Section 4 deals with the m -sectoriality of Δ as a generalisation of [ABT-H19]. Section 5 presents properties induced by the m -accretivity of our operator.

1. PRELIMINARIES

In this section we have gathered the notations we use and the basic definitions we need in the subsequent sections, see also [B17].

1.1. Notion of Graphs. A directed weighted graph is a triple $G := (V, \vec{E}, b)$, where V is the countable set of the vertices, \vec{E} is the set of directed edges and $b : V \times V \rightarrow [0, \infty)$ is a weight satisfying the following conditions:

- $b(x, x) = 0$ for all $x \in V$ (no loops)
- $b(x, y) > 0$ iff $(x, y) \in \vec{E}$

In addition, we consider a measure on V given by a nonnegative real function

$$m : V \rightarrow (0, \infty).$$

The weighted graph is *symmetric* if for all $x, y \in V$, $b(x, y) = b(y, x)$, as a consequence $(x, y) \in \vec{E} \Rightarrow (y, x) \in \vec{E}$.

The graph is called *simple* if the weights m and b are constant equal to 1 on V and \vec{E} respectively.

On a non-symmetric graph we have several notions of *connexity*.

We fix the following notations:

- The set of *undirected edges* is defined by

$$E = \left\{ \{x, y\}, (x, y) \in \vec{E} \text{ or } (y, x) \in \vec{E} \right\}.$$

- for $x \in V$, $V_x^+ = \{y \in V; (x, y) \in \vec{E}\}$
- for $x \in V$, $V_x^- = \{y \in V; (y, x) \in \vec{E}\}$
- for $x \in V$, $V_x = V_x^+ \cup V_x^- = \{y \in V; \{x, y\} \in E\}$.

1.2. **Definitions.** The degree of a vertex x is denoted by $\text{deg}(x)$ and defined by:

$$\text{deg}(x) = \#V_x.$$

- A chain from the vertex x to the vertex y in G is a finite set of undirected edges $\{x_1, y_1\}; \{x_2, y_2\}; \dots; \{x_n, y_n\}$, $n \geq 1$

$$x_1 = x, y_n = y \text{ and } x_i = y_{i-1} \quad \forall 2 \leq i \leq n.$$

- A path between two vertices x and y in V is a finite set of directed edges $(x_1, y_1); (x_2, y_2); \dots; (x_n, y_n)$, $n \geq 1$ such that

$$x_1 = x, y_n = y \text{ and } x_i = y_{i-1} \quad \forall 2 \leq i \leq n$$

- G is called *weakly connected* if two vertices are always related by a chain.
- G is called *connected* if two vertices are always related by a path.
- G is called *strongly connected* if there is for all vertices x, y a path from x to y and one from y to x .

We assume in the following that the graph under consideration is weakly connected, locally finite and satisfy:

$$\text{for all } x \in V, \sum_{y \in V} b(x, y) > 0.$$

1.3. **Functional spaces.** Let us introduce the following function spaces associated to the graph G .

The space of functions on the graph G is considered as the space of complex functions on V and is denoted by

$$\mathcal{C}(V) = \{f : V \rightarrow \mathbb{C}\}.$$

We denote by $\mathcal{C}_c(V)$ its subset of finite supported functions. We consider for a measure m , the space

$$\ell^2(V, m) = \left\{ f \in \mathcal{C}(V), \sum_{x \in V} m(x) |f(x)|^2 < \infty \right\},$$

which is a Hilbert space when equipped with the scalar product given by

$$\langle f, g \rangle = \sum_{x \in V} m(x) f(x) \overline{g(x)}.$$

The associated norm is given by:

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

1.4. Laplacians and Kirchhoff's Assumption. For a locally finite connected graph without loops, we introduce the weighted Laplacian Δ defined on $\mathcal{C}_c(V)$ by:

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V} b(x, y) (f(x) - f(y)).$$

Kirchhoff's Assumption (β): This assumption says that at each vertex the incoming conductance equals the outgoing conductance. If the Kirchhoff's Assumption is satisfied, the non-symmetric operator Δ enjoys parts of the self-adjoint theory. Defining for $x \in V$, $\beta^-(x) = \sum_{y \in V} b(y, x)$ and $\beta^+(x) = \sum_{y \in V} b(x, y)$, we will suppose in the sequel of this work that

$$(\beta) \quad \forall x \in V, \beta^+(x) = \beta^-(x).$$

With this assumption, the formal adjoint of Δ has a simple expression:

1.4.1. Proposition. *The formal adjoint Δ' of the operator Δ is defined on $\mathcal{C}_c(V)$ by:*

$$\Delta' f(x) = \frac{1}{m(x)} \sum_{y \in V} b(y, x) (f(x) - f(y)).$$

In this situation, we have established, see [B17], an explicit Green formula associated to the non-symmetric Laplacian Δ .

1.4.2. Proposition. *(Green Formula) Let f and g be two functions of $\mathcal{C}_c(V)$. They satisfy*

$$\langle \Delta f, g \rangle + \langle \Delta' f, g \rangle = \sum_{(x, y) \in \vec{E}} b(x, y) (f(x) - f(y)) (\overline{g(x) - g(y)}).$$

Proof:

We have for f and g in $\mathcal{C}_c(V)$

$$\begin{aligned} \langle \Delta f, g \rangle + \langle \Delta' f, g \rangle &= \sum_{(x, y) \in \vec{E}} b(x, y) (f(x) - f(y)) \overline{g(x)} + \sum_{(y, x) \in \vec{E}} b(y, x) (f(x) - f(y)) \overline{g(x)} \\ &= \sum_{(x, y) \in \vec{E}} b(x, y) (f(x) - f(y)) \overline{g(x)} + \sum_{(x, y) \in \vec{E}} b(x, y) (f(y) - f(x)) \overline{g(y)} \\ &= \sum_{(x, y) \in \vec{E}} b(x, y) (f(x) - f(y)) (\overline{g(x) - g(y)}). \end{aligned}$$

□

2. M-ACCRETIVENESS OF THE LAPLACIAN

2.1. First properties. The Hilbert space theory of accretive operators was motivated by the semi-group theory and the Cauchy problem for systems of hyperbolic partial differential equations. It is an important property for operators which cannot be studied in the framework of selfadjointness, [Mi98], [Sch12], [Ok75].

We establish conditions for the m-accretiveness (maximal accretiveness, see Definition 2.1.6) of Δ .

2.1.1. Definition. *The numerical range of an operator A with domain $D(A)$, denoted by $W(A)$ is the non-empty subset of \mathbb{C} defined by*

$$W(A) = \{\langle Af, f \rangle, \quad f \in D(A), \quad \|f\| = 1\}.$$

2.1.2. Definition. *Let \mathcal{H} be a Hilbert space, an operator $A : D(A) \rightarrow H$ is said to be accretive if for each $f \in D(A)$,*

$$\Re \langle Af, f \rangle \geq 0.$$

2.1.3. Lemma. [Ou05, Lem.1.47] *Let A be a densely defined accretive operator on \mathcal{H} . Then A is closable, its closure \overline{A} is accretive, and for every $\lambda > 0$, the range $\text{Im}(\lambda + A)$ is dense in $\text{Im}(\lambda + \overline{A})$.*

2.1.4. Proposition. *Let $A : D(A) \rightarrow \mathcal{H}$ be a closed, densely defined and accretive operator, then*

$$\|(A + \lambda)f\| \geq \Re(\lambda)\|f\|, \quad \forall f \in D(A), \quad \Re(\lambda) > 0$$

and $\text{Im}(A + \lambda)$ is closed.

Proof:

Let $\Re(\lambda) > 0, \forall f \in D(A)$ we have

$$\begin{aligned} \|(A + \lambda)f\|\|f\| &\geq |\langle (A + \lambda)f, f \rangle| \geq \Re \langle (A + \lambda)f, f \rangle \\ &\geq \Re \langle Af, f \rangle + \Re \langle \lambda f, f \rangle \\ &\geq \Re(\lambda)\|f\|^2. \end{aligned}$$

Hence $\forall f \in D(A), \|(A + \lambda)f\| \geq \Re(\lambda)\|f\|$ and $\text{Im}(A + \lambda)$ is closed. □

2.1.5. Proposition. *In our situation, the Laplacian Δ with domain $D(\Delta) = \mathcal{C}_c(V)$ is accretive, closable and for any scalar $\Re(\lambda) > 0$, $\text{Im}(\Delta + \lambda)$ is dense in $\text{Im}(\overline{\Delta} + \lambda)$.*

Proof:

From the Green formula, Proposition 1.4.2, we have for any $f \in \mathcal{C}_c(V)$

$$\begin{aligned} \Re \langle \Delta f, f \rangle &= \frac{1}{2} \left(\langle \Delta f, f \rangle + \overline{\langle \Delta f, f \rangle} \right) \\ &= \frac{1}{2} \left(\langle \Delta f, f \rangle + \langle \Delta' f, f \rangle \right) \\ &= \frac{1}{2} \sum_{(x,y) \in \vec{E}} b(x,y) |f(x) - f(y)|^2 \\ &\geq 0. \end{aligned}$$

Therefore Δ is accretive. We deduce from Lemma 2.1.3 that Δ is closable and $\text{Im}(\lambda + \overline{\Delta}) = \overline{\text{Im}(\lambda + \Delta)}$. □

We introduce the following notations (already introduced in [ABT-H19]): let

$$H = \frac{1}{2}(\Delta + \Delta') \quad B = \frac{1}{2}(\Delta - \Delta') \quad (1)$$

be the symmetric and the skewsymmetric parts of Δ , acting on the space of functions with finite support.

Then, thanks to the assumption (β) , the operator H is the Laplacian on the symmetric graph with an edge $\{x, y\}$ weighted by the symmetric weight defined by

$$b'(x, y) = \frac{b(x, y) + b(y, x)}{2} \quad \text{for all } x, y \in E. \quad (2)$$

2.1.6. Definition. *An accretive operator $A : D(A) \rightarrow \mathcal{H}$ is said to be m-accretive if the left open half-plane is contained in the resolvent set $\rho(A)$ and we have for $\Re(\lambda) > 0$,*

$$\|(A + \lambda)^{-1}\| \leq \frac{1}{\Re(\lambda)}.$$

An m-accretive operator A is maximal accretive, in the sense that A is accretive and has no proper accretive extension, [Kat76].

In the following we give sufficient conditions for Δ to be m-accretive, based on previous works applied to the real part of Δ . First, we recall a relating result, in the symmetric case, essential selfadjointness to graphs with constant weights on V .

2.1.7. Theorem (Theorem 3.1 of [T-H10]). *Let (G, m, b') be an infinite weighted graph with a constant weight m on V . Then the Laplacian H is essentially self-adjoint.*

From the definition of the adjoint operator Δ^* , we can deduce:

$$D(\Delta^*) = \{f \in \ell^2(V, m), \quad \Delta' f \in \ell^2(V, m)\}.$$

Using an idea in the proof of Theorem 3.1 of [T-H10] and the following Lemma, we prove the Proposition 2.1.9 for the non-symmetric Laplacian.

2.1.8. Lemma. [Ou05, Lem.1.48] *Assume that A is accretive. Then \bar{A} is m-accretive if and only if there exists $\lambda > 0$ such that $\lambda + A$ has dense range.*

2.1.9. Proposition. *Let (G, m, b) be an infinite weighted graph with the constant weight m on V . Then the Laplacian $\bar{\Delta}$ is m-accretive.*

Proof:

First, we show that $\Delta^* + 1$ is injective: Let $g \in D(\Delta^*)$ be a function satisfying

$$\Delta^* g + g = 0, \text{ then } g = 0.$$

Let us assume that there is a vertex x_0 such that $g(x_0) > 0$. The equality $\Delta^* g + g = 0$ implies that there exists at least one neighboring vertex x_1 for which $g(x_0) < g(x_1)$. We repeat the procedure with $x_1 \dots$. Hence we build a strictly increasing sequence of strictly positive real numbers $(g(x_n))_n$. We deduce that the function g is not in $\ell^2(V, m)$. A similar way is used to have the same conclusion when we take the assumption $g(x_0) < 0$.

Second, the injectivity of $\Delta^* + 1$ implies that the range $\text{Im}(\Delta + 1)$ is dense.

Finally, from Lemma 2.1.8, we deduce that $\bar{\Delta}$ is m-accretive.

□

2.1.10. Remark. *If Δ is symmetric, $\bar{\Delta}$ is m-accretive if and only if Δ is essentially self-adjoint.*

The property of essential self-adjointness was extensively studied in the symmetric case and many tools related to completeness were introduced to assure this property. In [T-H10], one of us proved that essential self-adjointness followed from completeness for a certain metric of the graph with bounded degree. In [HKMW13] the condition is related on completeness for an *intrinsic metric*. In [AT-H15] we introduced the notion of χ -completeness.

2.2. χ -completeness. We have introduced this notion in [AT-H15] in the symmetric case. It assures the Laplacian (and even the Gauß-Bonnet operator) to be essentially selfadjoint. We suppose in this section that the graph is χ -complete for the symmetric conductance b' defined in (2). It means that there exists an increasing sequence of finite sets $(B_n)_{n \in \mathbb{N}}$ such that $V = \dot{\cup} B_n$ and there exist related functions χ_n satisfying the following three conditions:

- (i) $\chi_n \in \mathcal{C}_c(V)$, $0 \leq \chi_n \leq 1$
- (ii) $v \in B_n \Rightarrow \chi_n(v) = 1$
- (iii) $\exists C > 0, \forall n \in \mathbb{N}, x \in V, \frac{1}{m(x)} \sum_{y \in V_x} b'(x, y) |\chi_n(x) - \chi_n(y)|^2 \leq C.$

2.2.1. Theorem. *Suppose that the graph $G = (V, m, b)$ is χ -complete for the symmetric conductance b' , and that the asymmetry is controlled in the following way*

$$\exists C > 0, \forall x \in V, \frac{1}{m(x)} \sum_{y \in V_x} \frac{|b(x, y) - b(y, x)|^2}{b'(x, y)} \leq C \quad (3)$$

then the non-symmetric Laplacian $\bar{\Delta}$ is m-accretive.

Proof:

We can suppose that the constants for χ -completeness and for (3) are the same. By Lemma 2.1.3 $\bar{\Delta}$ is accretive and by Proposition 2.1.4, its range is closed. Suppose that $\bar{\Delta}$ is not m-accretive, it means that the deficiency of $\bar{\Delta}$, which is constant on the left halfplane, is not 0. For instance at -1 it gives

$$\exists v \in \ell^2(V, m), \forall x \in V, (\Delta' + 1)v(x) = 0.$$

We remark that the operator Δ' is real so we can suppose that v is real. Let $\chi \in \mathcal{C}_c(V)$, such that $0 \leq \chi \leq 1$, and calculate $\langle \chi v, (\Delta' + 1)(\chi v) \rangle$.

First we remark that, as $\chi v \in \mathcal{C}_c(V)$ and has real values:

$$\begin{aligned} \langle \chi v, (\Delta' + 1)(\chi v) \rangle &= \langle (\Delta + 1)(\chi v), \chi v \rangle = \langle \chi v, (\Delta + 1)(\chi v) \rangle = \\ &= \langle \chi v, (H + 1)(\chi v) \rangle \geq \|\chi v\|^2 \end{aligned}$$

On the other hand, using the equation satisfied by v we have:

$$\begin{aligned} (\Delta' + 1)(\chi v)(x) &= \\ \frac{1}{m(x)} \sum_{y \in V} b(y, x) (\chi(x)v(x) - \chi(y)v(y)) - \chi(x) \frac{1}{m(x)} \sum_{y \in V} b(y, x) (v(x) - v(y)) \\ &= \frac{1}{m(x)} \sum_{y \in V} b(y, x) v(y) (\chi(x) - \chi(y)) \end{aligned}$$

it gives

$$\begin{aligned} \langle \chi v, (\Delta' + 1)(\chi v) \rangle &= \sum_{x \in V} \chi(x) v(x) \sum_{y \in V} b(y, x) v(y) (\chi(x) - \chi(y)) \\ &= \frac{1}{2} \sum_{x \in V, y \in V} v(x) v(y) (\chi(x) - \chi(y)) (b(y, x) \chi(x) - b(x, y) \chi(y)) \\ &= \frac{1}{2} \sum_{x \in V, y \in V} v(x) v(y) (b(y, x) \chi(x)^2 + b(x, y) \chi(y)^2 - \chi(x) \chi(y) (b(x, y) + b(y, x))) \\ &= \frac{1}{2} \sum_{x \in V, y \in V} v(x) v(y) (b(y, x) \chi(x)^2 + b(x, y) \chi(y)^2 - 2\chi(x) \chi(y) b'(x, y)). \end{aligned}$$

We use then that $2|v(x)v(y)| \leq v(x)^2 + v(y)^2$, it gives

$$\begin{aligned} \langle \chi v, (\Delta' + 1)(\chi v) \rangle &\leq \\ &\frac{1}{2} \sum_{x \in V} v(x)^2 \sum_{y \in V} |b(y, x)\chi(x)^2 + b(x, y)\chi(y)^2 - 2\chi(x)\chi(y)b'(x, y)|. \end{aligned}$$

We see that

if $\chi(x) = \chi(y) = 1$, then $(b(y, x)\chi(x)^2 + b(x, y)\chi(y)^2 - 2\chi(x)\chi(y)b'(x, y)) = 0$.

Moreover

$$\begin{aligned} b(y, x)\chi(x)^2 + b(x, y)\chi(y)^2 - 2\chi(x)\chi(y)b'(x, y) &= \\ b'(x, y)(\chi(x) - \chi(y))^2 + \frac{b(y, x) - b(x, y)}{2}(\chi(x)^2 - \chi(y)^2) \end{aligned}$$

We remark that

$$|\chi(x)^2 - \chi(y)^2| = |\chi(x) - \chi(y)| \cdot (\chi(x) + \chi(y)) \leq 2|\chi(x) - \chi(y)|,$$

which implies

$$\begin{aligned} \sum_{y \in V} \left| \frac{b(y, x) - b(x, y)}{2} (\chi(x)^2 - \chi(y)^2) \right| &\leq \\ &\sqrt{\sum_{y \in V} \frac{|b(y, x) - b(x, y)|^2}{b'(x, y)}} \sqrt{\sum_{y \in V} b'(x, y) |\chi(x) - \chi(y)|^2} \end{aligned}$$

Applying this calculation to $\chi = \chi_n$ we have then, because of the hypothesis (3)

$$\sum_{y \in V} |b(y, x)\chi_n(x)^2 + b(x, y)\chi_n(y)^2 - 2\chi_n(x)\chi_n(y)b'(x, y)| \leq 2Cm(x)$$

and finally

$$\|\chi_n v\|^2 \leq \langle \chi_n v, (\Delta' + 1)(\chi_n v) \rangle \leq 2C \sum_{x \in W_n} m(x)v(x)^2$$

where $W_n = V \setminus V_n$ and $V_n = \{x \in B_n, \forall y \in V_x, \chi_n(y) = 1\}$. Thus $\lim_{n \rightarrow \infty} \sum_{x \in W_n} m(x)v(x)^2 = 0$ (as $v \in \ell^2(V, m)$), we conclude that $v = 0$.

Thus the deficiency of $\overline{\Delta}$ is 0 on the left halfplane, and we conclude by Proposition 2.1.4.

□

2.2.2. Remark. *The results of essential self-adjointness for the Laplacian in the symmetric case give results for H in our case. Indeed the hypothesis of χ -completeness for the weight b' gives that H is essentially selfadjoint.*

2.2.3. **Example.** Let us consider an infinite simple tree T (the weights m and b are constant equal to 1 on V and \vec{E} respectively) with increasing degree, see Fig.1, we suppose that

$$\forall x \in V \quad \#V_x^+ \setminus (V_x^+ \cap V_x^-) = \#V_x^- \setminus (V_x^+ \cap V_x^-) = 1. \quad (4)$$

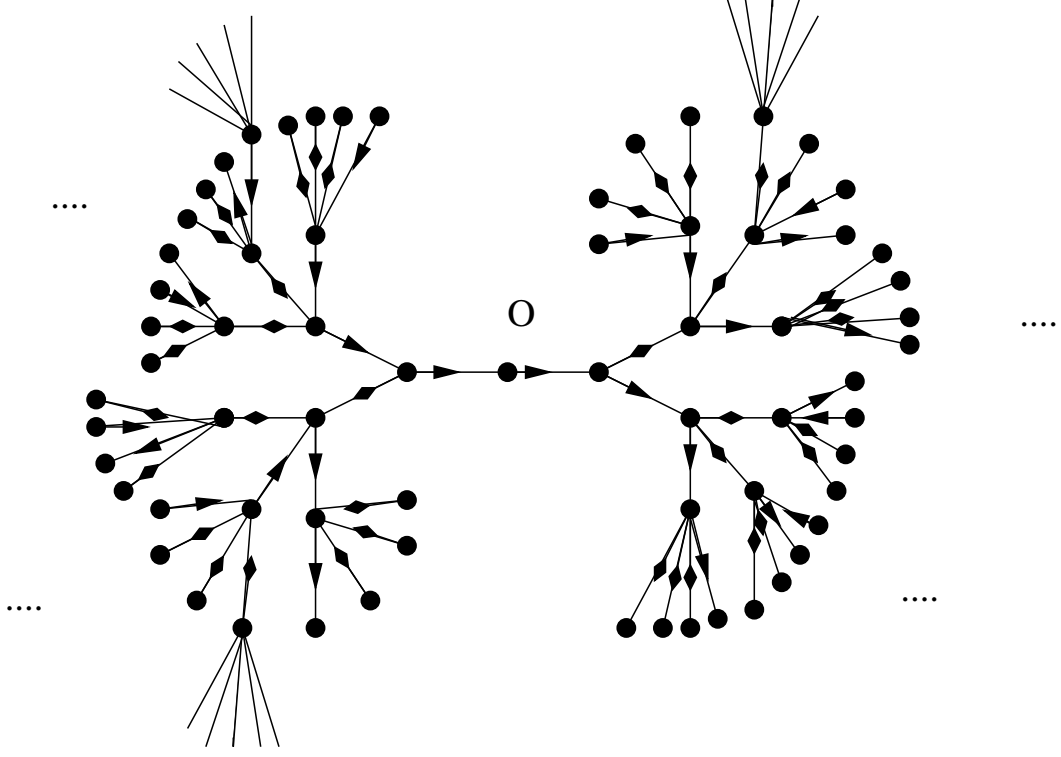


FIGURE 1. a tree with increasing degree

It is shown in [AT-H15, Example 9] that T , considered as a symmetric graph, is χ -complete. The hypothesis (4) assures clearly the Kirchoff law (β) ($\#V_x^+ = \#V_x^-$), we can see also that the property (3) is satisfied, in fact we have for all $x \in V$

$$\begin{aligned} \frac{1}{m(x)} \sum_{y \in V_x} \frac{|b(x, y) - b(y, x)|^2}{b(x, y)} &= \sum_{y \in V_x^+ \setminus (V_x^+ \cap V_x^-)} + \sum_{y \in V_x^- \setminus (V_x^+ \cap V_x^-)} \\ &= \#V_x^+ \setminus (V_x^+ \cap V_x^-) + \#V_x^- \setminus (V_x^+ \cap V_x^-) \\ &= 2. \end{aligned}$$

More generally, we can suppose that

$$\#V_x^+ \setminus (V_x^+ \cap V_x^-) = \#V_x^- \setminus (V_x^+ \cap V_x^-) \leq M.$$

But, using the fact that the degree is not bounded, we can also construct a graph with $\#V_x^+ \setminus (V_x^+ \cap V_x^-) = \#V_x^- \setminus (V_x^+ \cap V_x^-)$ not bounded. This gives a χ -complete graph which does not satisfy the property (3).

2.2.4. Remark. In [BGJ19] the authors give different criteria for χ -completeness on weighted graphs. Let us consider a χ -complete graph (for the symmetrized weight b') following Proposition 5.7 or Theorem 5.11 of [BGJ19]. To obtain the m -accretiveness of $\overline{\Delta}$, it is then sufficient that the non-symmetric graph satisfies moreover the Kirchhoff Assumption (β) and the property (3). This is assured if we suppose for instance that $\forall x \in V, y \in V_x^+ \cap V_x^-, b(x, y) = b(y, x)$ and

$$\exists M > 0, \forall x \in V, \sum_{y \in V_x^+ \setminus (V_x^+ \cap V_x^-)} b(x, y) = \sum_{y \in V_x^- \setminus (V_x^+ \cap V_x^-)} b(y, x) \leq M.$$

2.2.5. Example. Let us consider the following infinite weighted graph G , see Fig.2, with (almost) constant degree. We denote the origin by x_0 and by S_n the spheres for the combinatoric distance of the symmetric underlying graph:

$$d_{\text{comb}}(x_0, x) = \inf\{k; \exists \gamma = (x_0, \dots, x_k) \text{ a chain such that } x_k = x\}$$

So $S_n = \{x \in V, d_{\text{comb}}(x_0, x) = n\} = \{x_n, y_n\}$.

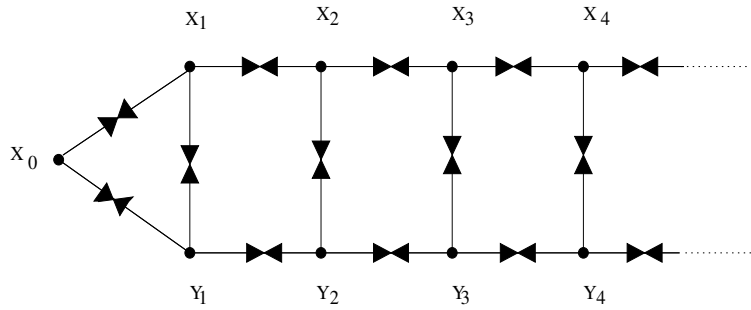


FIGURE 2. a graph with almost constant degree

To define the weights, we take $k \geq 0$ and fix

- $b(x_0, x_1) = b(y_1, x_0) = k + 2$ and $b(x_0, y_1) = b(x_1, x_0) = k$ and for $n \geq 1$
- $b(x_n, x_{n+1}) = (n + 1)^2 + (n + 1)$ and $b(x_{n+1}, x_n) = (n + 1)^2 - (n + 1)$
- $b(y_n, y_{n+1}) = (n + 1)^2 - (n + 1)$ and $b(y_{n+1}, y_n) = (n + 1)^2 + (n + 1)$
- $b(x_n, y_n) = n - 1$ and $b(y_n, x_n) = n + 1$
- $m(x_0) = 1, m(x_n) = m(y_n) = \sqrt{n}$.

We can see that the property (3) is satisfied, in fact for all $x \in S_n = \{x_n, y_n\}$

$$\begin{aligned} \frac{1}{m(x)} \sum_{y \in V_x} \frac{|b(x, y) - b(y, x)|^2}{b'(x, y)} &= \frac{1}{\sqrt{n}} \left(\sum_{y \in S_{n+1} \cap V_x} 4 + \sum_{y \in S_{n-1} \cap V_x} 4 + \sum_{y \in S_n \cap V_x} \frac{2^2}{n} \right) \\ &\leq \frac{12}{\sqrt{n}} \leq 12. \end{aligned}$$

We show now that G is χ -complete from the criterion given in Theorem 5.11 of [BGJ19]: we remark that the sets S_n^+ and S_n^- introduced in [BGJ19] coincide

with S_n and for $x \in S_n$ the weighted degree is constant and for $n \geq 2$:

$$\begin{aligned} a_n^+ &= \sup_{x \in S_n} \frac{1}{\sqrt{n}} \left(\sum_{y \in S_{n+1}} b'(x, y) \right) = \frac{(n+1)^2}{\sqrt{n}}, \quad a_n^- = \sup_{x \in S_n} \frac{1}{\sqrt{n}} \left(\sum_{y \in S_{n-1}} b'(x, y) \right) = n^{\frac{3}{2}} \\ &\Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{a_n^+ + a_{n+1}^-}} \geq \sum_{n=2}^{\infty} \frac{1}{\sqrt{2(n+1)^{\frac{3}{2}}}} = \infty. \end{aligned}$$

Thus, this graph satisfies the hypothesis of Theorem 2.2.1.

3. RELATIONS BETWEEN Δ AND H

We study here the relation between the two hypothesis: m -accretiveness for $\overline{\Delta}$ and essential selfadjointness for H .

3.1. From H to Δ .

3.1.1. Theorem. *Let (G, m, b) be an infinite weighted graph and $\Delta = H + B$ the decomposition of the combinatorial Laplacian of G decomposed in symmetric and skewsymmetric part as in (1). Then if H is essentially selfadjoint and if B is bounded, then $\overline{\Delta}$ is m -accretive.*

Proof:

As B is bounded, $\overline{\Delta}$ and \overline{H} have the same domain and

$$\text{on } \mathcal{D}(\overline{\Delta}) = \mathcal{D}(\overline{H}) : \overline{\Delta} = \overline{H} + B.$$

As H is non-negative and essentially selfadjoint, for any scalar λ , $\Re(\lambda) > 0$, the operator $\overline{H} + \lambda$ is invertible on $\mathcal{D}(\overline{H})$ and

$$\|(\overline{H} + \lambda)^{-1}\| \leq \frac{1}{\Re(\lambda)}.$$

Now, let λ , $\Re(\lambda) > 0$, we write

$$\overline{\Delta} + \lambda = \overline{H} + \lambda + B = (I + B(\overline{H} + \lambda)^{-1})(\overline{H} + \lambda)$$

But $\|B(\overline{H} + \lambda)^{-1}\| \leq \frac{\|B\|}{\Re(\lambda)}$ so $\Re(\lambda) > \|B\| \Rightarrow (\overline{\Delta} + \lambda)$ invertible and

$$\|(\overline{\Delta} + \lambda)^{-1}\| \leq \frac{1}{(1 - \|B\|/\Re(\lambda))\Re(\lambda)} = \frac{1}{\Re(\lambda) - \|B\|}.$$

So we have that the set of $-\lambda$, $\Re(\lambda) > \|B\|$ is included in the resolvent set of $\overline{\Delta}$ but, on the other hand $\overline{\Delta}$ is accretive which implies, by Theorem V.3.2 of [Kat76, p.268], that its deficiency is constant on the set of λ , $\Re(\lambda) < 0$, as a conclusion this deficiency is zero on this set. Finally, for all λ , $\Re(\lambda) > 0$ and $f \in C_c(V)$, because the real part of Δ is non-negative

$$\Re(\lambda)\langle f, f \rangle \leq |\langle (\lambda + \Delta)f, f \rangle| \leq \|(\lambda + \Delta)f\| \cdot \|f\|$$

and this is also true on $\mathcal{D}(\overline{\Delta})$, then, as we already know that $(\lambda + \overline{\Delta})$ is invertible,

$$\|(\lambda + \overline{\Delta})^{-1}\| \leq \frac{1}{\Re(\lambda)}.$$

□

3.1.2. Remark. We have introduced in [ABT-H19] a hypothesis (γ) that assures B to be bounded (and Δ to be sectorial), namely

Assumption (γ) :

$$\exists M > 0, \forall x \in V, \sum_{y \in V} |b(x, y) - b(y, x)| \leq Mm(x)$$

We see easily that if the assumption (γ) is satisfied then (3) is also satisfied: as the weight b is non-negative, we have always

$$|b(x, y) - b(y, x)| \leq (b(x, y) + b(y, x)) = 2b'(x, y)$$

and thus

$$\begin{aligned} \sum_{y \in V_x} \frac{|b(x, y) - b(y, x)|^2}{b'(x, y)} &\leq 2 \sum_{y \in V_x} |b(x, y) - b(y, x)| \\ &\leq 2Mm(x). \end{aligned}$$

3.1.3. Example. The graph considered in the example 2.2.5 satisfies the property (3) and does not satisfy (γ) . In fact we have for all $x \in S_n$

$$\begin{aligned} \sum_{y \in V_x} |b(x, y) - b(y, x)| &= \sum_{y \in S_n} |b(x, y) - b(y, x)| + \sum_{y \in S_{n+1}} |b(x, y) - b(y, x)| \\ &\quad + \sum_{y \in S_{n-1}} |b(x, y) - b(y, x)| \\ &= 2 + 2(n+1) + 2n \end{aligned}$$

which cannot be controlled by $m(x) = \sqrt{n}$.

3.1.4. Remark. The last theorem can be extended in a situation more general than sectoriality (see the definition in Section 4), namely when B is bounded in H -norm with a relative norm sufficiently small. More precisely we suppose that there exist two constants $C > 0$ and $0 < a < 1/2$ such that

$$\forall f \in C_c(V) \quad \|B(f)\| \leq C\|f\| + a\|H(f)\|$$

Then for a real $\lambda > 0$ we have

$$\begin{aligned} \forall f \in C_c(V) \quad \|B(\overline{H} + \lambda)^{-1}f\| &\leq (C + a\lambda)\|(\overline{H} + \lambda)^{-1}f\| + a\|f\| \\ &\Rightarrow \|B(\overline{H} + \lambda)^{-1}\| \leq \frac{C + a\lambda}{\lambda} + a \leq \frac{C}{\lambda} + 2a \end{aligned}$$

This can be made smaller than 1 for λ large enough and then, by the same argument, the deficiency of $\bar{\Delta}$ must be zero on all the left halfspace (notice that under these hypotheses B is also Δ -bounded, with relative norm $\frac{a}{1-a}$).

3.1.5. Theorem. *Let (G, m, b) be an infinite weighted graph and $\Delta = H + B$ the decomposition of the combinatorial Laplacian of G decomposed in symmetric and skewsymmetric part as in (1). If H is essentially selfadjoint and if B is relatively bounded with respect to H with relative norm smaller than $1/2$, then $\bar{\Delta}$ is m -accretive.*

3.1.6. Remark. *The hypothesis (3) gives that B is relatively bounded with respect to H . Indeed for any $f \in C_c(V)$*

$$\begin{aligned} \|B(f)\|^2 &= \sum_{x \in V} \frac{1}{m(x)} \left| \sum_{y \in V} \frac{b(x, y) - b(y, x)}{2} (f(x) - f(y)) \right|^2 \\ &\leq \sum_{x \in V} \frac{1}{m(x)} \sum_{y \in V} \frac{|b(x, y) - b(y, x)|^2}{4b'(x, y)} \sum_{y \in V} b'(x, y) |f(x) - f(y)|^2 \\ &\leq \sum_{x \in V} \frac{C}{4} \sum_{y \in V} b'(x, y) |f(x) - f(y)|^2 = \frac{C}{2} \langle H(f), f \rangle \\ &\leq \left(\frac{C^2}{4} \|f\|^2 + \frac{1}{4} \|H(f)\|^2 \right). \end{aligned}$$

Thus Theorem 2.2.1 is a corollary of Theorem 3.1.5.

3.1.7. Remark. *If $\bar{\Delta}$ is m -accretive then, by definition, the set of $\lambda, \Re(\lambda) < 0$ is included in the resolvent set of $\bar{\Delta}$, we have thus*

$$\bar{\Delta} \text{ m-accretive} \Rightarrow \sigma(\bar{\Delta}) \subset \{\lambda \in \mathbb{C}, \Re(\lambda) \geq 0\}$$

We can study if the hypothesis "H is essentially selfadjoint" is necessary.

3.2. From Δ to H .

3.2.1. Proposition. *Let (G, m, b) be an infinite weighted graph and $\Delta = H + B$ the decomposition of the combinatorial Laplacian of G decomposed in symmetric and skewsymmetric part as in (1). If B is bounded and $\bar{\Delta}$ is m -accretive, then $\bar{\Delta}'$ is m -accretive and H is essentially selfadjoint.*

Proof:

On $C_c(V)$ we have that $\Delta' = \Delta - 2B$. The operator Δ' is accretive, as B is bounded $\mathcal{D}(\bar{\Delta}) = \mathcal{D}(\bar{\Delta}')$ and for any $\lambda, \Re(\lambda) > 0$

$$\Delta' + \lambda = \Delta + \lambda - 2B = (I - 2B(\Delta + \lambda)^{-1})(\Delta + \lambda)$$

then $\bar{\Delta}' + \lambda$ is invertible for $\Re(\lambda)$ large enough, so $\bar{\Delta}'$ is m -accretive. In the same way we have, as $H = \Delta - B$ so $\mathcal{D}(\bar{H}) =$

$\mathcal{D}(\overline{\Delta})$ and

$$H + \lambda = \Delta - B + \lambda = (I - B(\Delta + \lambda)^{-1})(\Delta + \lambda),$$

that $(\overline{H} + \lambda)$ is invertible for λ real large enough and H is essentially selfadjoint.

□

3.2.2. Remark. *In the same way as in Remark 3.1.4 we can extend this result for B bounded in Δ -norm. We obtain that if $\overline{\Delta}$ is m -accretive and B is Δ -bounded with a relative norm strictly smaller than $1/2$ then H is essentially selfadjoint.*

4. SECTORIALITY

In [ABT-H19] we have studied the sectoriality of Δ , we generalize here these results. It was McIntosh, see [Mc86], who initiated and developed a theory of functional calculus for a less restricted large class of operators, namely sectorial operators.

4.1. Definition. *Let \mathcal{H} be a Hilbert space, an operator $A : D(A) \rightarrow \mathcal{H}$ is said to be sectorial if $W(A)$ lies in a sector*

$$S_{a,\theta} := \{z \in \mathbb{C}, |\Im(z)| \leq \tan \theta (\Re z - a)\} := \{z \in \mathbb{C}, |\arg(z - a)| \leq \theta\}$$

for some $a \in \mathbb{R}$, called vertex of $S_{a,\theta}$, and $\theta \in [0, \frac{\pi}{2})$, called semi-angle of $S_{a,\theta}$ (thus $A - a$ is accretive). The operator A is said to be m -sectorial, if it is sectorial and if $A - a$ is m -accretive.

We have used in [ABT-H19] that under the assumption (γ) (see Remark 3.1.2) the Laplacian is sectorial. More generally, we have

4.2. Proposition. *Let (G, m, b) be an infinite weighted graph and $\Delta = H + B$ the decomposition of the combinatorial Laplacian of G decomposed in symmetric and skewsymmetric part as in (1). If the asymmetry of the weight b satisfies the property (3) then Δ is sectorial.*

Proof:

Let $f \in \mathcal{C}_c(V)$ with $\|f\| = 1$, using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
2|(Bf, f)| &= \left| \sum_{x \in V} f(x) \sum_{y \in V} (b(x, y) - b(y, x))(f(x) - f(y)) \right| \\
&= \left| \sum_{x \in V} f(x) \sum_{y \in V} \frac{b(x, y) - b(y, x)}{\sqrt{b'(x, y)}} \sqrt{b'(x, y)} (f(x) - f(y)) \right| \\
&\leq \sum_{x \in V} |f(x)| \left(\sum_{y \in V} \frac{|b(x, y) - b(y, x)|^2}{b'(x, y)} \right)^{\frac{1}{2}} \left(\sum_{y \in V} b'(x, y) |f(x) - f(y)|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{C} \left(\sum_{x \in V} m(x) |f(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{x \in V} \sum_{y \in V} b'(x, y) |f(x) - f(y)|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{C} \|f\| (Hf, f)^{\frac{1}{2}} \\
&\leq 1 + \frac{C}{4} (Hf, f).
\end{aligned}$$

□

4.3. Proposition. *Suppose that the graph (G, m, b) is χ -complete for the symmetric conductance b' , and that the asymmetry of the weight b satisfies the property (3) then the non-symmetric Laplacian $\bar{\Delta}$ is m -sectorial.*

5. THE HEAT SEMIGROUP

The property of m -accretivity can be used to generate strongly continuous semigroups. We recall the theorem of Hille-Yosida. It gives, on Banach spaces, a complete characterization of generators of semigroups with at most exponential growth, we refer here to [RR93], [HP57], [Yo78] and [AE12].

5.1. Existence of a heat semigroup.

5.1.1. Theorem (Hille-Yosida). *Let A be an operator in the Banach space X . Then A is the infinitesimal generator of a C_0 semigroup $T(t)$ satisfying $\|T(t)\| \leq M \exp(wt)$ if and only if the following two conditions hold:*

- $D(A)$ is dense and A is closed.
- Every real number $\lambda > w$ is in the resolvent set of A and

$$\|(A - \lambda)^{-n}\| \leq \frac{M}{(\lambda - w)^{-n}}, \text{ for every } n \in \mathbb{N}.$$

The assumptions of the Hille-Yosida Theorem are easier to achieve when $M = 1$, then the semigroup is said to be quasicontractive (and contractive if we can take $w = 0$). It is known as the Lumer-Phillips Theorem.

5.1.2. Theorem (Lumer-Phillips). *Let A be a linear operator on a Hilbert space \mathcal{H} . If*

- (1) $D(A)$ is dense
- (2) $\Re(x, Ax) \leq w(x, x)$ for $x \in D(A)$
- (3) there exists $\lambda_0 > w$ such that $A - \lambda_0 I$ is onto.

Then A is the generator of a strongly continuous one-parameter quasicontraction semigroup and $\|\exp(tA)\| \leq \exp(tw)$.

We can apply this to $A = -\overline{\Delta}$:

5.1.3. Theorem. *Let (G, m, b) be an infinite weighted graph and $\overline{\Delta}$ its combinatorial Laplacian. If $\overline{\Delta}$ is m-accretive, then $-\overline{\Delta}$ is the generator of a strongly continuous one-parameter contraction semigroup (i.e. $\|\exp(-t\overline{\Delta})\| \leq 1$).*

Using the results of [Z08] we have in the same way

5.1.4. Theorem. *Let (G, m, b) be an infinite weighted graph and $\overline{\Delta}$ its combinatorial Laplacian. If $\overline{\Delta}$ is m-sectorial with angle θ and vertex a , then $-\overline{\Delta}$ is the generator of a holomorphic semigroup on a sector with angle $\pi/2 - \theta$ and vertex 0 .*

Moreover, on this situation one can apply the preceding result on $(\overline{\Delta} - a)$.

5.2. Fast contractivity. For a non-symmetric graph G , we can estimate bounds on the real part of the numerical range of Δ in terms of a Cheeger constant. We restrict here in the case where the weight on vertices is constant equal to 1 and consider the definition of the Cheeger constant given by Dodziuk, applied on the symmetrized graph $(G, 1, b')$.

5.2.1. Definition ([D05]). *Let us consider a weighted symmetric graph $(G, 1, b')$, the Cheeger constant $h(V)$ is defined by*

$$h(V) = \inf_{\substack{U \subset V \\ \text{finite}}} \frac{\sum_{\substack{x \in U, y \in V \setminus U \\ y \in V_x}} \sqrt{b'(x, y)}}{\#U}.$$

The following theorem is a consequence of Theorem 3.7 of [Ba17] and Theorem 3.1 of [D05] (where \deg is the combinatorial degree).

5.2.2. Theorem. *Suppose that $\sup_{x \in V} \deg(x) = M < \infty$. Then the real part of $W(\Delta)$ satisfies*

$$\inf_{z \in W(\Delta)} \Re(z) \geq \frac{h^2(V)}{2M} = \lambda_0.$$

5.2.3. Proposition. *Let G be a graph with bounded degree, satisfying the property (3) and $m = 1$ on V . If $h(V) > 0$, then Δ is m-sectorial with vertex $a \geq 0$.*

Indeed $\overline{\Delta}$ is m-accretive because of Proposition 2.1.9. We remark that $\overline{\Delta} - \lambda_0$ is also m-accretive.

5.2.4. **Example.** Consider the graph of Example 2.2.5, but now with constant weight on vertices: $m = 1$. We define $V_n = \{x_0, x_1, y_1, \dots, x_n, y_n\}$ and remark that

$$\begin{aligned} h(V) &= \inf_{n \in \mathbb{N}} \frac{\sum_{\substack{x \in \{x_n, y_n\} \\ y \in \{x_{n+1}, y_{n+1}\}}} \sqrt{b'(x, y)}}{\#U} \\ &= \inf_{n \in \mathbb{N}} \frac{\sqrt{b'(x_n, x_{n+1})} + \sqrt{b'(y_n, y_{n+1})}}{\#U} \\ &= \inf_{n \in \mathbb{N}} \frac{2(n+1)}{2n+1} \\ &= 1. \end{aligned}$$

Hence

$$\inf_{z \in W(\Delta)} \Re(z) \geq \frac{1}{6}.$$

Then, applying Proposition 5.2.3, there is $\theta \in (0, \pi/2)$ such that the numerical range of Δ lies in the sector $\{z \in \mathbb{C}, |\arg(z)| \leq \theta\}$ and $\bar{\Delta}$ is m -sectorial.

But we can say also that $(\bar{\Delta} - \frac{1}{6})$ is m -accretive, it gives

$$\|\exp(-t\bar{\Delta})\| \leq e^{-\frac{t}{6}}.$$

Acknowledgments: The author Marwa Balti enjoyed a financial support from the Program *DéfiMaths* of the Federation of Mathematical Research of the "Pays de Loire" during her visits to the Laboratory of Mathematics Jean Leray of Nantes (LMJL). Also, the three authors would like to thank the Laboratory of Mathematics Jean Leray of Nantes (LMJL) and the research unity (UR/13ES47) of Faculty of Sciences of Bizerta (University of Carthage) for their continuous financial support.

REFERENCES

- [AT-H15] C. Anné, N. Torki-Hamza *The Gauss-Bonnet operator of an infinite graph*. Anal. Math. Phys. **5**, (2015), 137–159.
- [ABT-H19] C. Anné, M. Balti, N. Torki-Hamza *Sectoriality and essential spectrum of non symmetric graph Laplacians*. Complex Analysis and Operator Theory, **13**, 3, (2019), 967-983.
- [AE12] W. Arendt, A.F.M. ter Elst. *From forms to semigroups*. Spectral theory, mathematical system theory, evolution equations, differential and difference equations, Oper. Theory Adv. Appl., **221**, Birkhäuser/Springer Basel AG, Basel, (2012), 47–69.
- [BGJ19] H. Baloudi, S. Golenia, Aref Jeribi. *The adjacency matrix and the discrete Laplacian acting on forms*. Math.Phys.Anal.Geo., **22**:9, (2019).
- [Bal17] M. Balti. *Laplaciens non auto-adjoints sur un graphe orienté*. Thèse de Doctorat, Université de Carthage et Université de Nantes (2017).
- [Ba17] M. Balti. *On the eigenvalues of non symmetric directed graphs*. Complex analysis and operator theory, **11**, (2017), 1387–1406.

- [B17] M. Balti. *Non self-adjoint Laplacians on a directed graph*. Filomat, **18**, (2017), 5671–5683.
- [D05] J. Dodziuk. *Elliptic operators on infinite graphs*. World Sci. Publ., Hackensack, (2006), 353–368.
- [ELZ83] W. D. Evans, R. T. Lewis, A. Zettl. *Non self-adjoint operators and their essential spectra. In form local times to global geometry, control and physics*, D. Ellworthy, ed., Differential Equation and Operators, Lecture Notes in Mathematics, **1032**, (1983), 123–160.
- [H11] B. Helffer. *Spectral theory and its applications*. Cambridge University Press, (2011).
- [HP57] E. Hille, R. S. Phillips. *Functional Analysis and Semi-Groups*. Revised edition. Amer. Math. Soc., Providence, Rh. I. , (1957).
- [HKMW13] X. Huang, M. Keller, J. Masamune, R.K. Wojciechowski. *A note on self-adjoint extensions of the Laplacian on weighted graphs*, J. Funct. Anal. **265** no. 8 (2013), 1556–1578.
- [Kat76] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, Heidelberg and New York, (1976).
- [KL12] M. Keller, D. Lenz. *Dirichlet forms and stochastic completeness of graphs and subgraphs*. J. Reine Angew. Math. **666**, (2012), 189–223.
- [Kh13] M. Khanalizadeh. *Sectorial forms and m -sectorial operators*. Seminararbeit zum fach funktionalanalysis, Technische Universität Berlin, (2013).
- [L79] R. T. Lewis. *Applications of a comparison Theorem for quasi-accretive operators in a Hilbert space*. In: Everitt W., Sleeman B. (eds) Ordinary and Partial Differential Equations. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg. **964**, (1982), 422–434.
- [Mc86] A. McIntosh . *Operators which have an \mathcal{H}^∞ Functional Calculus*. Proceedings of the centre for mathematical analysis, Australian National University. **14**, (1986), 210–231.
- [Mi98] M. Miklavcic . *Applied functional analysis and partial differential equations*. Word Scientific publishing Co. Pte. Ltd., (1998).
- [MT15] O. Milatovic, F. Truc. *Maximal accretive extensions of Schrödinger operators on vector bundles over infinite graphs*. Integral Equations Operator Theory, **1**, (2015), 35–52.
- [Ok75] N. Okazawa. *Remarks on linear m -accretive operators in a Hilbert space*. J. Math. Soc. Japan, **27**, (1975), 160–165.
- [Ou05] E.M. Ouhabaz. *Analysis of Heat Equations on Domains*. Princeton University Press, 41 William street, (2005).
- [RR93] M. Renardy, R. C. Rogers. *An Introduction to Partial Differential Equations*. Springer-Verlag New York, Inc., (1993).
- [Sch12] K. Schmüdgen. *Unbounded Self-adjoint Operators on Hilbert Space*. Graduate Text in Mathematics, Springer, (2012).
- [T-H10] N. Torki-Hamza. *Laplaciens de graphes infinis. I: Graphes métriquement complets*. Confluentes Math. **2** (2010), 333–350.
Translated to: Essential self-adjointness for combinatorial Schrödinger operators I- Metrically complete graphs. arXiv:1201.4644v1.
- [Yo78] K. Yosida. *Functional analysis*. Die Grundlehren der math. Wissenschaften, 123, Springer-Verlag, Berlin and New York, (1978).
- [Z08] V. A. Zagrebnev. *Quasi-sectorial contractions*. J. Funct. Anal. **254** (2008), n° 9, 2503–2511.

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