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A UNIVERSAL HOCHSCHILD-KOSTANT-ROSENBERG THEOREM

TASOS MOULINOS, MARCO ROBALO, AND BERTRAND TOËN

Abstract. In this work we study the failure of the HKR theorem over rings of positive and mixed characteristic. For this we construct a *filtered circle* interpolating between the usual topological circle and a formal version of it. By mapping to schemes we produce this way an interpolation, realized in practice by the existence of a natural filtration, from Hochschild and (a filtered version of) cyclic homology to derived de Rham cohomology. In particular, we show that this recovers the filtration of Antieau [Ant19] and Bhatt–Morrow–Scholze [BMS19]. The construction of our filtered circle is based on the theory of affine stacks and affinization introduced by the third author, together with some facts about schemes of Witt vectors.

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1. Introduction

The purpose of the present paper is to investigate the failure of the Hochschild-Kostant-Rosenberg theorem in positive and mixed characteristic situations. For this, we construct a *filtered circle* $\mathsf{S}^1_{\mathsf{Fil}}$, an object of an algebro-homotopical nature, which interpolates between the usual homotopy type of a topological circle and a degenerate version of it called the *formal circle*. Given an arbitrary derived scheme X, we take the mapping stack, in the sense of derived algebraic geometry, from $\mathsf{S}^1_{\mathsf{Fil}}$ to X; this provides an interpolation between Hochschild and (a filtered version of) cyclic homology on one side and derived de Rham cohomology of X on the other. The existence of such an interpolation, realized concretely in terms of a filtration, is the main content of this work.

1.1. **Circle action and de Rham differential in characteristic zero.** Over any commutative ring k, the HKR theorem [HKR62] identifies $\Omega^*_{\mathsf{dR}}(X)$ - the graded commutative algebra of differential forms on a smooth k-scheme $X = \mathsf{Spec}\, A$, with $\mathsf{HH}_*(A)$ - the graded algebra of Hochschild homology. When k is of characteristic zero this lifts to the level of chain complexes, identifying the de Rham complex of differential forms $\mathsf{DR}(A)$ with the Hochschild complex $\mathsf{HH}(A)$. More is true; the de Rham complex comes equipped with a natural differential which arises, via this identification, from the natural S^1 action on the Hochschild complex. A precise implementation of this fact requires a further enhanced version of the HKR theorem, combining the intervention of the homotopical circle action and the multiplicative structure on the Hochschild complex on one side, and the full derived de Rham algebra with its natural grading and de Rham differential on the other. This is established in [TV11] as a consequence of an equivalence of symmetric monoidal ∞-categories

$$\mathsf{S}^1 - \mathsf{Mod}_k^{\otimes} \simeq \epsilon - \mathsf{Mod}_k^{\otimes}, \tag{1}$$

where $\mathsf{S}^1 - \mathsf{Mod}_k^{\otimes}$ denotes equivariant k-modules and $\epsilon - \mathsf{Mod}_k^{\otimes}$ is the category of mixed complexes. Here ϵ is a generator of homological degree 1 that makes $k[\epsilon] \simeq k \oplus k[1] \simeq \mathrm{H}_*(\mathsf{S}^1, k)$. On both sides we have symmetric monoidal structures: on the l.h.s using the diagonal of S^1 and on the r.h.s using the co-multiplication given by $\epsilon \mapsto \epsilon \otimes 1 + 1 \otimes \epsilon$.

Via this identification, one obtains a multiplicative equivalence

$$\mathsf{HH}(A) = A \otimes_k \mathsf{S}^1 \simeq \mathsf{Sym}_{\mathrm{A}}(\mathbb{L}_{\mathrm{A}/k}[1]) = \mathsf{DR}(A), \tag{2}$$

and the agreement of the symmetric monoidal structures in (1) guarantees that the circle action on the left matches the de Rham differential on the right. Geometrically as in [BZN12], this can also be interpreted as an identification of the derived stack of free loops on an affine k-scheme $X = \mathsf{Spec}(\mathrm{A})$, $\mathsf{L}X := \mathrm{Map}(\mathsf{S}^1, X)$, with the shifted tangent stack $\mathsf{T}[-1]\,X := \mathrm{Map}(\mathsf{Spec}(k[\eta], X)$ where this time $k[\eta] := k \oplus k[-1] \simeq \mathrm{H}^*(\mathsf{S}^1, k)$ is the differential graded algebra given by the cohomology of the circle. Here η is a generator of homological degree -1. In this language, the HKR theorem reads as an equivalence of derived stacks

$$\mathsf{L}X \simeq \mathsf{T}[-1]\,X \tag{3}$$

and passing to global functions, recovers the isomorphism in (2).

Our first observation is that (1) is no longer a symmetric monoidal equivalence when we abandon the hypothesis that k be a field of characteristic zero; indeed, the proof of (1) uses two essential facts about $\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}$ - the classifying stack of the group $\mathbb{G}_{\mathsf{a}\,k}$:

A) In any characteristic, the stack $\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}$ is equivalent to $\mathsf{Spec}^{\Delta}(\mathsf{Sym}_k^{\mathsf{co}\Delta}(k[-1]))$ where $\mathsf{Sym}_k^{\mathsf{co}\Delta}(k[-1])$ is the free cosimplicial commutative k-algebra over one generator in (cosimplicial) degree 1 (see Notation **1.2.10** below and [Toe06, Lemma 2.2.5]). This can be checked at the level of the functor of points. But when k is a field of characteristic zero, since the cohomology of the symmetric groups with coefficients in k vanishes, we recover an equivalence of commutative differential graded algebras

$$\mathsf{Sym}_k^{\mathsf{co}\Delta}(k[-1]) \simeq k \oplus k[-1] := k[\eta]$$

where on the r.h.s we have the split square zero extension. In particular, we have

$$\mathrm{Map}(\mathsf{B}\mathbb{G}_{\mathsf{a}\,,k}, X) \simeq \mathrm{Map}(\mathsf{Spec}(k[\eta]), X) =: \mathsf{T}[-1]\,X$$

B) For any ring k the complex of singular cochains $\mathsf{C}^*(\mathsf{S}^1, k)$ is given by $k \oplus k[-1]$. The canonical map of groups $\mathbb{Z} \to \mathbb{G}_{\mathsf{a}\,k}$ produces a map of group stacks $\mathsf{S}^1 := \mathsf{B}\mathbb{Z} \to \mathsf{B}\mathbb{G}_{\mathsf{a}\,k}$. As in A), because the cohomology of symmetric groups with coefficients in a field of characteristic zero vanishes, the pullback map in cohomology $\mathsf{C}^*(\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}, \mathscr{O}) \to \mathsf{C}^*(\mathsf{S}^1, k)$ is an equivalence. This fact exhibits the abelian group stack $\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}$ as the *affinization* of the constant group stack S^1 in the sense of [Toe06] (see also [Lur11b], [BZN12, Lemma 3.13] and our Review **3.2.6**). It follows from the universal property of affinization and from Zariski descent that

$$\mathrm{Map}(\mathsf{S}^1, X) \simeq \mathrm{Map}(\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}, X)$$

The accident that allows A) and B) in characteristic zero also makes the equivalences

$$\mathsf{Aff}(\mathsf{S}^1) \underbrace{\simeq}_{B)} \mathsf{B}\mathbb{G}_{\mathsf{a}} \underbrace{\simeq}_{A)} \mathsf{Spec}(k \oplus k[-1]) \tag{4}$$

compatible with the group structures(*). From here it is easy to recover the symmetric monoidal equivalence (1): $k[\epsilon] \simeq \mathrm{H}_*(\mathsf{S}^1, k)$ is dual to $k[\eta] := \mathrm{H}^*(\mathsf{S}^1, k) \simeq \mathsf{Sym}_k^{\mathsf{co}\Delta}(k[-1]))$ and this gives us a symmetric monoidal equivalence

$$\epsilon - \mathsf{Mod}_k^{\otimes} \simeq k[\eta] - \mathrm{CoMod}_k^{\otimes} \tag{5}$$

where on the r.h.s we now have the tensor product induced by convolution with the Hopf algebra structure on $k[\eta] = \mathrm{H}^*(\mathsf{S}^1, k)$ induced by the group structure on the circle. Finally the equivalences of groups in (4) gives a symmetric monoidal equivalence

$$k[\eta] - \mathrm{CoMod}^{\otimes} \simeq \mathsf{S}^1 - \mathsf{Mod}_k^{\otimes} \tag{6}$$

1.2. **What we do in this paper.** Away from characteristic zero, equivalence (1) still holds at the level of ∞-categories. However, the compatibility of the two symmetric monoidal structures fails dramatically. As a consequence, we can no longer identify the circle action with the de Rham differential, the latter which fundamentally requires a notion of a mixed complex.

A key observation we make in this paper is that although these symmetric monoidal structures are not equivalent, there is a natural degeneration between the two. More precisely, we equip the symmetric monoidal category of complexes with a circle action with a filtration, whose "associated graded" is the symmetric monoidal category of

(*)Essentially, by the uniqueness of the abelian group structure on the stack $\mathsf{B}\mathbb{G}_{\mathsf{a}\,k}$. See Corollary **3.4.18** for a similar argument in the context of this paper.

mixed graded complexes, by which we mean complexes equipped with a *strict* co-action of the trivial square zero extension $k \oplus k[-1]$. See Proposition **4.2.3** and Remark **4.2.5**.

As a glimpse to our construction, we remark that the two copies of $\mathsf{B}\mathbb{G}_{\mathsf{a}\,,k}$ appearing in A) and B) play distinct roles. What we propose, working over $\mathbb{Z}_{(p)}$, is a construction that interpolates between the two. It is inspired by an idea of [Toe06] of using the group scheme W_{p^∞} of p-typical Witt vectors as a natural extension of the additive group $\mathbb{G}_{\mathsf{a}}$. The group W_{p^∞} is an abutment of infinitely many of copies of $\mathbb{G}_{\mathsf{a}}$ and comes canonically equipped with a Frobenius map Frob_p. The abelian subgroup Fix of fixed points of the Frobenius map has a natural filtration whose associated graded is the kernel of the Frobenius, Ker. After base change from $\mathbb{Z}_{(p)}$ to $\mathbb{Q}$ both Fix and Ker are isomorphic to $\mathbb{G}_{\mathsf{a}}$ (see the Remark **2.1.2** below) but over $\mathbb{Z}_{(p)}$ they are very different. Without further ado, our first main theorem is the following:

Theorem 1.2.1 (See Proposition **3.3.2** and Theorem **3.4.17**)**.**

(i) The abelian group stack BFix *is the affinization of* S^1 *over* $\mathbb{Z}_{(p)}$*.*

(ii) The abelian group stack BKer *has cohomology ring given the (cosimplicial) split square zero extension (see Notation **3.4.14**)* $\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ *given by the cohomology of the circle.*

(iii) The group stack BFix *is equipped with a* filtration*, compatible with the group structure, whose associated graded stack is* BKer*.*

(iv) After base-change along $\mathsf{Spec}(\mathbb{Q}) \to \mathsf{Spec}(\mathbb{Z}_{(p)})$*, we have*

$$\mathsf{BFix} \otimes \mathbb{Q} \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}\,,\mathbb{Q}}$$

Moreover, the filtration splits and we have

$$(\mathsf{BFix})^{\mathsf{gr}}_{\mathbb{Q}} \simeq \mathsf{BKer} \otimes \mathbb{Q} \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}\,,\mathbb{Q}}$$

Definition 1.2.2. The group stack BFix, equipped with the *filtration* of Theorem **1.2.1** - (iii), will be called the *filtered circle* and denoted as $\mathsf{S}^1_{\mathsf{Fil}}$.

Remark 1.2.3 (Filtrations)**.** In order to define filtrations on stacks, we will follow the point of view of C. Simpson [Sim91, Lemma 19] and [Sim97] which identifies filtered objects with objects over the stack $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ - see [Mou19] and Definition **2.2.11**. The content of Theorem **1.2.1** and Definition **1.2.2** can then be reformulated as the construction of an abelian group stack $\mathsf{S}^1_{\mathsf{Fil}}$ over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ whose fiber at 0 has the property in A); at 1 has the property in B); and whose pullback to $\mathbb{Q}$ is the constant family with values $\mathsf{B}\mathbb{G}_{\mathsf{a}\,,\mathbb{Q}}$. The construction of $\mathsf{S}^1_{\mathsf{Fil}}$ is the subject of Section **2**, after reviewing the basics of Witt vectors in Guide **2.1.1**. The proof that $\mathsf{S}^1_{\mathsf{Fil}}$ satisfies (i) and (ii) will be discussed later in Section **3.4** and Section **3.3**. The proof of (iv) is explained in the Remark **2.1.2**.

Having in mind the well known interpretation of cyclic homology in terms of derived loop spaces, our main theorem can now be stated as follows:

Theorem 1.2.4 (See Theorem **5.1.3** and Theorem **5.4.1**)**.** *Let $X = \mathsf{Spec}(\mathrm{A})$ be a derived affine scheme over $\mathbb{Z}_{(p)}$. Then:*

(i) We have canonical equivalences of derived mapping stacks

$$\mathrm{Map}(\mathsf{S}^1, X) \simeq \mathrm{Map}(\mathsf{BFix}, X) \quad \textit{and} \quad \mathrm{T}[-1]X \simeq \mathrm{Map}(\mathsf{BKer}, X)$$

In particular, the derived mapping stack $\mathrm{Map}(\mathsf{S}^1, X)$ admits the structure of a filtration compatible with an action of the filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ and whose associated graded is $\mathsf{T}X[-1]$;

(ii) Passing to global functions, (i) produces a filtration on $\mathsf{HH}(A)$, compatible with the circle action and the multiplicative structure, and whose associated graded is the derived de Rham algebra $\mathsf{DR}(A)$.

(iii) Since the filtration is compatible with the circle action, by taking homotopy fixed points of $\mathsf{HH}(A)$ seen as a filtered object, get a new filtered object which we call $\mathsf{HC}^-_{\mathsf{Fil}}(A)$. It has a canonical map to the usual fixed points

$$\mathsf{HC}^-_{\mathsf{Fil}}(A) \to \mathsf{HC}^-(A) := \mathsf{HH}(A)^{\mathrm{h}\mathsf{S}^1} \tag{7}$$

and the associated graded pieces of $\mathsf{HC}^-_{\mathsf{Fil}}(A)$ are the truncated complete derived de Rham complexes $\mathbb{L}\widehat{\mathsf{DR}}^{\geq p}(A/k)$.

Remark 1.2.5. As we will see later (Construction **5.2.2**) the map (7) measures the difference between the underlying object of fixed points of a filtration and the fixed points of the underlying object. When A is discrete and quasi-smooth, the two processes coincide - see [Ant19, Lemma 4.10] and Theorem **5.4.1**-(d).

In Section **5.5** we show that the point (iii) extends previously known filtrations constructed in [Ant19] for discrete rings and in Bhatt-Morrow-Scholze in [BMS19] for p-adic rings. The construction presented in our work is very different in nature.

The main emphasis of our result is that this filtration exists on the algebraic circle itself, before the HKR theorem.

In Section **6** we will discuss several applications, generalizations and possible research directions suggested by Theorem **1.2.1** and Theorem **1.2.4**.

Application 1.2.6 (Shifted Symplectic Structures in positive characteristic)**.** In Section **6.1** we discuss the extension of the notion of shifted symplectic structures of [PTVV13a] to derived stacks in positive characteristic. For this we use our HKR theorem in order to produce certain classes in the second layer of the filtration induced on negative cyclic homology. This is achieved by analyzing the Chern character map at the first two graded pieces of the filtration on $\mathsf{HC}^-_{\mathsf{Fil}}$. We also suggest a possible definition of n-shifted symplectic structures and show that the universal 2-shifted symplectic structure on $\mathsf{B}G$ exists essentially over any base ring k. By the techniques developed in [PTVV13a] we obtain this way extensions of various previously known n-shifted symplectic structures over non-zero characteristic bases.

Application 1.2.7 (Generalized Cyclic Homology and Formal groups)**.** The application discussed in Section **6.3** comes from the observation that the degeneration from Fix to Ker of Theorem **1.2.1**-(iii) is Cartier dual to the degeneration of the multiplicative formal group $\widehat{\mathbb{G}_{\mathrm{m}}}$ to the additive formal group $\widehat{\mathbb{G}_{\mathsf{a}}}$ (see Proposition **6.3.3**). In Section **6.3** we will discuss how to generalize Theorem **1.2.1** and Theorem **1.2.4** replacing $\widehat{\mathbb{G}_{\mathrm{m}}}$ by a more general formal group law E, in particular, one associated to an elliptic curve.

Application 1.2.8 (Topological and q-analogues)**.** In Section **6.4** we briefly present topological and q-deformed possible generalizations of our filtered circle. We investigate two related ideas, a first one that predicts the existence of a topological, non-commutative version of $\mathsf{S}^1_{\mathsf{Fil}}$ as a filtered object in spectra. A second one, along the same spirit, predicting the existence of a q-deformed filtered circle $\mathsf{S}^1_{\mathsf{Fil}}(q)$ possibly related to q-deformed de Rham complex (see for instance [Aom00]) in a similar fashion that $\mathsf{S}^1_{\mathsf{Fil}}$ is related to de Rham theory. Again, such a *quantum circle* can only exist if one admits non-commutative objects in some sense.

Relation with other works: The object $\mathsf{S}^1_{\mathsf{Fil}}$ and the constructions behind it can be placed in a general context. For instance, this construction is of homotopical significance, as the underlying object of $\mathsf{S}^1_{\mathsf{Fil}}$ is the affinization of the topological circle over $\mathbb{Z}_{(p)}$ in the sense of [Toe06]. In [Toë19] the third author studies the integral version of the affinization and its relation to our group scheme Fix. The result there may be used in order to extend our HKR theorem over $\mathbb{Z}$. Concerning, the filtration, we believe that our construction is much more general and that for any finite CW homotopy type X the affinization $(X \otimes \mathbb{Z}) = \mathsf{Spec}\,\mathsf{C}^*(X, \mathbb{Z})$ comes equipped with a canonical filtration whose associated graded is $\mathsf{Spec}\,\mathsf{H}^*(X, \mathbb{Z})$ (at least when X has torsion free cohomology groups). This is in a way the canonical filtration that degenerates a homotopy type over $\mathbb{Z}$ to a formal homotopy type.

In a different direction, we would like to mention the work [Arp20], in which the author also constructs a filtered circle, as filtered Hopf algebra object in a suitable ∞-category of *derived commutative rings*. As far as the authors understand [Arp20], this Hopf algebra is a model for the Hopf algebra of functions on our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$.

Acknowledgements 1.2.9. The idea of a filtered circle has been inspired by the construction of the Hodge filtration in Carlos Simpson's non-abelian Hodge theory, and we thank Carlos Simpson, Tony Pantev, Gabriele Vezzosi and Mauro Porta for several conversations on these themes along the years. We also thank Arpon Raksit for discussions related to his approach to the filtered circle. We are very thankful to Pavel Etingof for having pointed to us the possible existence of the q-deformed circle (presented in Section **6.4**), this has also lead the authors to realize the existence of various circle associated to any commutative formal group law.

We are also thankful to the anonymous referee for encouraging us to include a new section comparing our filtration with the one of Antieau.

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Notation 1.2.10 (Simplicial, Cosimplicial and $\mathsf{E}_\infty^\otimes$)**.** Unless mentioned otherwise, all higher categorical notations are borrowed from [Lur17, Lur09]. Let k be a discrete commutative ring. Throughout the paper we will denote by

a) We use homological conventions. We write Mod_k for the ∞-category of chain complexes of k-modules; $\mathsf{Mod}_k^{\geq 0}$, resp. $\mathsf{Mod}_k^{\leq 0}$ the categories of connective and coconnective complexes.

b) The notation CAlg will always be used to denote $\mathsf{E}_\infty^\otimes$-algebras. In particular CAlg_k will denote $\mathsf{E}_\infty^\otimes$-algebras in Mod_k; $\mathsf{CAlg}_k^{\mathsf{cn}} \simeq \mathsf{CAlg}(\mathsf{Mod}_k^{\geq 0})$ the full subcategory of connective algebras [Lur17, 2.2.1.3, 2.2.1.8, 7.1.3.10] and $\mathsf{CAlg}_k^{\mathsf{ccn}}$ the category of $\mathsf{E}_\infty^\otimes$-algebras in $\mathsf{Mod}_k^{\leq 0}$ for the symmetric monoidal structure induced from the fact $\tau_{\leq 0}$ is a monoidal localization. In particular, as the inclusion $\mathsf{Mod}_k^{\leq 0} \subseteq \mathsf{Mod}_k$ is lax monoidal, we have an induced map at the level of algebras $\mathsf{CAlg}_k^{\mathsf{ccn}} \to \mathsf{CAlg}_k$

c) SCR_k the ∞-category of simplicial commutative rings over k. This is the sifted completion of the discrete category of polynomial algebras $\mathsf{N}(\mathsf{Poly}_k)$. See [Lur18, 25.1.1.5] and [Lur09, 5.5.9.3]. The universal property of sifted completion gives us the normalized Dold-Kan functor $\theta : \mathsf{SCR}_k \to \mathsf{CAlg}_k^{\mathsf{cn}}$. By [Lur18, 25.1.2.2, 25.1.2.4] this is both monadic and comonadic and if k is of characteristic zero it is an equivalence. Given $A \in \mathsf{SCR}_k$, $\theta(A)$ will be called the *underlying* $\mathsf{E}_\infty^\otimes$*-algebra of* A.

d) By Sym we will always mean the simplicial version Sym^Δ as a monad in $\mathsf{Mod}_k^{\geq 0}$;

e) coSCR_k the ∞-category of cosimplicial commutative rings over k (See Definition 3.1.1 below.) We also denote by $\theta : \mathsf{coSCR}_k \to \mathsf{CAlg}_k$ the conormalized Dold-Kan construction (see [Toe06, §2.1]). This functor is conservative, commutes with tensor products and preserves limits by the discussion in Section 3.1. It can be factored by a functor θ^{ccn}

$$\mathsf{coSCR}_k \xrightarrow{\theta^{\mathsf{ccn}}} \mathsf{CAlg}_k^{\mathsf{ccn}} \subseteq \mathsf{CAlg}_k$$

where θ^{ccn} is the co-dual Dold-Kan construction of [Kŏ3].

f) By $\mathsf{Sym}^{\mathsf{co}\Delta}$ we will mean the free cosimplicial commutative algebra on $\mathsf{Mod}_k^{\leq 0}$.

g) St_k the ∞-category of stacks over the site of discrete commutative k-algebras and dSt_k the ∞-category of derived stacks, ie, stacks over SCR_k.

h) dSch_k the ∞-category of derived schemes and $\mathsf{Spec} : \mathsf{SCR}_k^{\mathsf{op}} \to \mathsf{dSch}_k$ the affine derived scheme associated to of a simplicial commutative ring.

i) $\mathsf{Spec}^{\Delta} : \mathsf{coSCR}_k^{\mathsf{op}} \to \mathsf{St}_k$ the ∞-functor sending an object $\mathrm{A} \in \mathsf{coSCR}_k$ to the (higher) stack which sends a classical commutative ring B to the mapping space $\mathrm{Map}_{\mathsf{coSCR}_k}(\mathrm{A}, \mathrm{B})$. See also Review **3.2.6**

Notation 1.2.11. We assume k as in Notation **1.2.10**. $\mathbb{G}_{\mathrm{m}\,k}$ and $\mathbb{A}_k^1$ will always denote the flat versions of the multiplicative group and affine line over k.

Notation 1.2.12 (Quasi-coherent sheaves on stacks and t-structures)**.** Again, assume k as in Notation **1.2.10**.

a) QCoh will always denote the ∞-category of quasi-coherent sheaves in the sense of [Lur18, 6.2.2.1, 6.2.2.7, 6.2.3.4], ie, $\mathsf{QCoh}(X) := \lim_{\mathsf{Spec}A \to X} \mathsf{Mod}_A$ with $\mathsf{Spec}(A)$ a derived affine scheme with $A \in \mathsf{SCR}_k$ and Mod_A the category of modules in spectra of the connective $\mathsf{E}_\infty^{\otimes}$-ring $\theta(A)$.

b) Under some mild conditions, QCoh admits a t-structure where, by definition, $F \in \mathsf{QCoh}(X)$ is connective if the pullback to every affine is connective. See [Lur18, 6.2.5.7, 6.2.5.8, 6.2.5.9, 6.2.3.4-(3)]. We will see this in Notation **1.2.12**. In particular, if for any map $f : X \to Y$, the pullback $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is right t-exact (ie, preserves ≥ 0) and the right adjoint f_* is left t-exact (preserves ≤ 0) [Lur17, 1.3.3.1].

c) In the light of the previous item and particularly useful in this paper, is the condition that a stack X over k is presented by a simplicial scheme $\mathsf{Spec}(\mathrm{A}^\bullet)$, with A a cosimplicial commutative algebra $\mathrm{A}^\bullet$ with $\mathrm{A}^0 = k$, each A^m flat and discrete over A^0 and such that all boundary maps $\mathrm{A}^m \to \mathrm{A}^n$ are flat. Then we can proceed as in [Lur11b, 4.5.2] and [Lur18, 6.2.5.7, 6.2.5.8, 6.2.5.9, 6.2.3.4-(3)] and prescribe a left-complete t-structure on $\mathsf{QCoh}(X)$ compatible with the limit decomposition by descent

$$\mathsf{QCoh}(X) \simeq \lim_{[n]\in \mathsf{N}(\Delta^{op})} \mathsf{Mod}_{\mathrm{A}^n}$$

Since the transition maps $\mathrm{A}^m \to \mathrm{A}^n$ are flat, the extension of scalars $\mathsf{Mod}_{\mathrm{A}^m} \to \mathsf{Mod}_{\mathrm{A}^n}$ are both left and right t-exact and we can define

$$\mathsf{QCoh}(X)_{\geq 0} := \lim_{[n]\in \mathsf{N}(\Delta^{op})} (\mathsf{Mod}^{\geq 0}_{\mathrm{A}^n})$$

$$\mathsf{QCoh}(X)_{\leq n} \simeq \lim_{[m]\in \mathsf{N}(\Delta^{op})} \mathsf{Mod}^{\leq n}_{\mathrm{A}^m}$$

In particular, an object $\mathrm{M} \in \mathsf{QCoh}(X)$ connective or coconnective if it is so after pullback along the atlas and is in the heart if and only if its image along the first projection $\mathsf{QCoh}(X) \to \mathsf{Mod}_k$ is in $\mathsf{Mod}_k^{\heartsuit}$ - the classical abelian category of k-modules. We conclude that $\mathscr{O}_X$ is in the heart.

d) For a (derived) stack X over a ring k, we will denote by $\mathsf{C}^*(X, \mathscr{O})$ the $\mathsf{E}_\infty^{\otimes}$-ring in Mod_k given by the quasi-coherent pushforward of $\mathscr{O}_X$ along the structure map $X \to \mathsf{Spec}\, k$.

2. Filtrations, Fixed Points and Kernel of Frobenius on Witt vectors

2.1. **Review of Witt-Vectors.** We start by reviewing the materials concerning Witt vectors that will be used in the paper. We follow [Hes15, §1] closely. Other references are [HL13, KN, Mum66, Haz09, Haz12, Ill79, DG70].

Guide 2.1.1 (Witt Vectors)**.**

1) Throughout this paper we fix p a prime and unless mentioned otherwise, we work with $\mathbb{Z}_{(p)}$-algebras;
2) $\mathsf{W}_{p^\infty} : \mathsf{CRings}_{\mathbb{Z}_{(p)}} \to \mathsf{AbGrp}$ the commutative group scheme of *p-typical Witt vectors*, evaluated on $\mathbb{Z}_{(p)}$-algebras. The underlying scheme of W_{p^∞} is the infinite product $\prod_{n\in S} \mathbb{A}^1$ where $S = \{1, p, p^2, ...\}$.
3) Ghost : $\mathsf{W}_{p^\infty} \to \prod_{n\in S} \mathbb{G}_{\mathsf{a}}$, the map implemented by the Ghost coordinates, defined via $(\lambda_n)_{n\in S} \mapsto (\omega_n)_{n\in S}$ where $\omega_n := \sum_{d|n} d.\lambda_d^{\frac{n}{d}}$. In particular $\omega_1 = \lambda_1$. The group structure on W_{p^∞} of (i) is uniquely determined by the requirement that Ghost defines a map of groups, where on the r.h.s we have the degrewise additive group structure of $\mathbb{G}_{\mathsf{a}}$. See [Hes15, Prop. 1.2]. The group unit is the Witt vector $(1, 0, ..) \in \mathsf{W}_{p^\infty}$.

4) The map Ghost is an isomorphism after base change to $\mathbb{Q}$. See [KN, B.3(2)].

5) $\mathsf{W}_{p^\infty}^{(m)}$ the group scheme of truncated p-typical Witt vectors of length m. Let $S_m := \{1, p, ..., p^{m-1}\}$. The underlying scheme of $\mathsf{W}_{p^\infty}^{(m)}$ is $\prod_{n\in S_m} \mathbb{A}^1$. As a group, $\mathsf{W}_{p^\infty}^{(m)}$ is again defined under the requirement that the truncated Ghost coordinates $\mathsf{W}_{p^\infty}^{(m)} \to \prod_{n\in S_m} \mathbb{G}_{\mathsf{a}}$ define a map of groups. By the same argument as in [KN, B.3(2)]

$$\mathsf{W}_{p^\infty}^{(m)} \otimes \mathbb{Q} \simeq \prod_{\{1,p,...,p^{m-1}\}} \mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \tag{8}$$

6) There are restriction maps $\mathsf{W}_{p^\infty}^{(m+1)} \to \mathsf{W}_{p^\infty}^{(m)}$ The $\mathsf{W}_{p^\infty}^{(1)}$ is canonically isomorphic to $\mathbb{G}_{\mathsf{a}}$ and there are exact sequences

$$0 \to \mathbb{G}_{\mathsf{a}} \to \mathsf{W}_{p^\infty}^{(m+1)} \to \mathsf{W}_{p^\infty}^{(m)} \to 0 \tag{9}$$

Passing to the limit we get a pro-structure $\mathsf{W}_{p^\infty} \simeq \mathsf{lim}_m \mathsf{W}_{p^\infty}^{(m)}$ as group schemes.

7) The group scheme W_{p^∞} comes naturally equipped with a Frobenius endomorphism $\mathrm{Frob}_p : \mathsf{W}_{p^\infty} \to \mathsf{W}_{p^\infty}$. This is uniquely defined as a map of abelian groups under the requirement that on Ghost coordinates it acts by

$$(\omega_1, \omega_p, \omega_{p^2}, ...) \mapsto (\omega_p, \omega_{p^2}, ...)$$

See [Hes15, Lemma 1.4]. In terms of the pro-structure it decomposes as maps

$$\mathrm{Frob}_p : \mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m-1)}$$

After base change to $\mathbb{F}_p$, Frob_p is the standard Frobenius on each coordinate, namely, $(\lambda_n)_{n\in S} \mapsto (\lambda_n^p)_{n\in S}$. See [Hes15, Lemma 1.8].

8) We write Fix for the group scheme given by the kernel of the map $\mathrm{Frob}_p - \mathsf{id} : \mathsf{W}_{p^\infty} \to \mathsf{W}_{p^\infty}$ and Ker for the kernel of $\mathrm{Frob}_p : \mathsf{W}_{p^\infty} \to \mathsf{W}_{p^\infty}$.

To illustrate how Witt vectors will be used in our HKR theorem, let us start with the following observation of what happens in characteristic zero:

Remark 2.1.2. Let A be a $\mathbb{Q}$-algebra. Then the explicit formula for Frob_p on $\mathsf{W}_{p^\infty}(\mathrm{A})$ in terms of the Ghost coordinates tells us that the fixed points for the Frobenius are given by the diagonal embedding

$$\mathsf{Fix}(\mathrm{A}) \simeq \Delta \subseteq \prod_{i\geq 0} \mathbb{G}_{\mathsf{a}}(\mathrm{A})$$

Another easy computation in Ghost coordinates, also tells us that the Kernel of the Frobenius is given by the inclusion of the first coordinate:

$$\mathsf{Ker}(\mathrm{A}) \simeq (\mathbb{G}_{\mathsf{a}}(\mathrm{A}), 0, 0, 0, 0, ...) \subseteq \prod_{i \geq 0} \mathbb{G}_{\mathsf{a}}(\mathrm{A})$$

In other words, as group-schemes we obtain

$$\mathsf{Fix}_{|\mathbb{Q}} \simeq \mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \quad \text{and} \quad \mathsf{Ker}_{|\mathbb{Q}} \simeq \mathbb{G}_{\mathsf{a}\,\mathbb{Q}}$$

The Remark **2.1.2** shows that for $\mathbb{Q}$-algebras, the additive group scheme $\mathbb{G}_{\mathsf{a}}$ can be defined abstractly via Witt vectors, either as Frobenius fixed points or as the kernel. In this paper we utilize this feature to understand the HKR theorem in positive characteristic. Away from $\mathbb{Q}$-algebras, the fixed points and the kernel of Frob_p on W_{p^∞} do not agree. However, we shall see that there is a natural degeneration from the first to the second, or more precisely, a filtration on Fix whose associated graded is Ker. The delooping of this filtration will be, by definition, our filtered circle. For this purpose we will need to explain what is a filtration on a stack. Before addressing that question, let us be precise about the linear versions of filtrations and gradings used in this paper:

2.2. **Graded and filtered objects.** We start this section by recalling the definitions of filtered and graded objects and how t-structures give rise to filtered objects. We conclude by discussing Simpson point of view on filtrations.

Construction 2.2.1. [Lur15] Let C be a cocomplete stable ∞-category . The category of filtered objects in C is the ∞-category of diagrams $\mathsf{Fil}(\mathsf{C}) := \mathsf{Fun}(\mathsf{N}(\mathbb{Z})^{\mathsf{op}}, \mathsf{C})$, with $\mathsf{N}(\mathbb{Z})$ the nerve of the category associated to the poset $(\mathbb{Z}, \leq)$. The category of $\mathbb{Z}$-graded objects in C is the ∞-category of diagrams $\mathsf{C}^{\mathbb{Z}-\mathrm{gr}} := \mathsf{Fun}(\mathbb{Z}^{\mathrm{disc},\mathsf{op}}, \mathsf{C}) \simeq \prod_{i \in \mathbb{Z}} \mathsf{C}$ where $\mathbb{Z}^{\mathrm{disc}}$ is the $\mathbb{Z}$ seen as a discrete category. If C has a symmetric monoidal structure with unit 1, both categories are endowed with symmetric monoidal structures given by Day convolution. The unit of $\mathsf{C}^{\mathbb{Z}-\mathrm{gr}}$ is given by the graded object $(\cdots, \underbrace{0}_{1}, \underbrace{1}_{0}, \underbrace{0}_{-1}, \underbrace{0}_{-2}, \cdots)$. The unit of $\mathsf{Fil}(\mathsf{C})$ is the filtered object $\cdots \underbrace{0}_{1} \to \underbrace{1}_{0} = \underbrace{1}_{-1} = \underbrace{1}_{-2} = \cdots$. Following [Lur15, §3.1 and 3.2], the construction of the associated graded object, respectively, the underlying object, are implemented by symmetric monoidal functors

$$\mathrm{gr} : \mathsf{Fil}(\mathsf{C}) \to \mathsf{C}^{\mathbb{Z}-\mathrm{gr}} \quad \text{and} \quad \mathsf{colim} : \mathsf{Fil}(\mathsf{C}) \to \mathsf{C}.$$

Remark 2.2.2. A map in $\mathsf{Fil}(\mathsf{C})$ is an equivalence if and only if both associated graded and underlying maps are equivalences. Indeed, since $\mathsf{Fil}(\mathsf{C})$ is stable, passing to cofibers, this amounts to show that a filtered object E is zero if and only both ita associated graded and underlying objects are zero. But if the associated graded is zero, we see that all maps $\mathrm{E}_{n+1} \to \mathrm{E}_n$ are equivalences and therefore the filtered object is constant. If moreover its colimit is zero, then the object itself is zero.

Definition 2.2.3. Assume furthermore that C has all limits. Let $\mathrm{F} \in \mathsf{Fil}(\mathsf{C})$ be a filtered object. We say that F is *complete* if $\lim \mathrm{F} \simeq 0$. We denote by $\widehat{\mathsf{Fil}}(\mathsf{C})$ the full subcategory of $\mathsf{Fil}(\mathsf{C})$ spanned by complete filtered objects.

Remark 2.2.4. For any filtered object F and for every $i \in \mathbb{Z}$ we have a cofiber sequence in C

$$\lim \mathrm{F} \underbrace{\simeq}_{cofinal} \lim_{j>i} \mathrm{F}_j \to \mathrm{F}_i \to \mathrm{F}_i/(\lim_{j>i} \mathrm{F}_j) \simeq \lim_{j>i} \mathrm{F}_i/\mathrm{F}_j \tag{10}$$

where the last equivalence follows because $\mathsf{Fil}(\mathsf{C})$ is stable. Therefore, a filtered object is complete if and only if the canonical map $\mathrm{F}_i \to \lim_{j\geq i} \mathrm{F}_i/\mathrm{F}_j$ is an equivalence for all i. Inspire by this, we define a new filtered object $\widehat{\mathrm{F}}$ by the formula $(\widehat{\mathrm{F}})_n := \mathrm{F}_n/(\lim \mathrm{F})$ and by construction, it is complete and comes with a natural map of filtered objects $\mathrm{F} \to \widehat{\mathrm{F}}$ that is an equivalence precisely when F is complete. Following [GP18, Proposition 2.14, Lemma 2.15], the assignment $\mathrm{F} \mapsto \widehat{\mathrm{F}}$ provides a left adjoint $\widehat{(-)} : \mathsf{Fil}(\mathsf{C}) \to \widehat{\mathsf{Fil}}(\mathsf{C})$ to the inclusion of the full subcategory $\widehat{\mathsf{Fil}}(\mathsf{C}) \subseteq \mathsf{Fil}(\mathsf{C})$ and presents $\widehat{\mathsf{Fil}}(\mathsf{C})$ as a Bousfield localization of $\mathsf{Fil}(\mathsf{C})$ with respect to the class of maps of filtered objects $\mathrm{E} \to \mathrm{F}$ whose associated graded $\mathsf{gr}(\mathrm{E}) \to \mathsf{gr}(\mathrm{F})$ is an equivalence of graded modules. This follows from the two fiber sequences

$$\lim \mathrm{F}/\lim \mathrm{E} \to \mathrm{F}_i/\mathrm{E}_i \to \widehat{\mathrm{F}}_i/\widehat{\mathrm{E}}_i \quad \text{and} \quad \mathrm{F}_{i+1}/\mathrm{E}_{i+1} \to \mathrm{F}_i/\mathrm{E}_i \to \mathsf{gr}(\mathrm{F}/\mathrm{E})_i$$

In particular, the inclusion of $\widehat{\mathsf{Fil}}(\mathsf{C}) \subseteq \mathsf{Fil}(\mathsf{C})$ is stable under all limits.

Construction 2.2.5. Let C be stable ∞-category with a t-structure $(\mathsf{C}_{\geq 0}, \mathsf{C}_{\leq 0})$. Then we have an associated Whitehead tower ∞-functor

$$\tau_{\geq} : \mathsf{C} \to \mathsf{Fil}(\mathsf{C}) \ , \text{ sending } X \mapsto \tau_{\geq} X := [\cdots \to \tau_{\geq n+1} X \to \tau_{\geq n} X \to \cdots]$$

If the t-structure is right complete, then $\tau_{\geq}$ admits a left inverse by extracting the underlying object $\mathsf{colim} : \mathsf{Fil}(\mathsf{C}) \to \mathsf{C}$. The essential image of $\tau_{\geq}$ consists of those filtered objects E such that $\pi_i(\mathrm{E}_n) = 0$ if $i < n$ and such that the canonical maps $\pi_i(\mathrm{E}_n) \to \pi_i(\mathsf{colim}\,\mathrm{E})$ are equivalences for every $i \geq n$. In particular, if $\mathrm{F} \in \mathsf{C}$, the graded pieces are given by $\mathsf{gr}^n(\tau_{\geq}\mathrm{F}) \simeq \pi_n(\mathrm{F})[n] \in \mathsf{C}^{\heartsuit}[n]$.

Construction 2.2.6. Let C be stable presentable ∞-category with a left-complete presentable t-structure (in the sense of [Lur17, 1.2.1.17, 1.2.1.19]) and such that $\mathsf{C}_{\geq 0}$ is stable under countable products (such as Sp or Mod_k). Denote by $\mathsf{Fil}^{\dagger}(\mathsf{C})$ the full subcategory of $\mathsf{Fil}(\mathsf{C})$ spanned by those filtered objects E such that for every i, $\mathrm{E}_i \in \mathsf{C}_{\geq i}$. Then we have

$$\mathsf{Fil}^{\dagger}(\mathsf{C}) \subseteq \widehat{\mathsf{Fil}}(\mathsf{C})$$

Indeed, we argue that the limit $(\mathsf{lim}\,\mathrm{E}) \in \mathsf{C}_{\geq i}$ for all $i \in \mathbb{Z}$, so that $(\mathsf{lim}\,\mathrm{E}) \in \bigcap_i \mathsf{C}_{\geq i} = \{0\}$ because of left-completeness. To show that $(\mathsf{lim}\,\mathrm{E}) \in \mathsf{C}_{\geq i}$ we see that by cofinality we have

$$\mathsf{lim}\,\mathrm{E} \simeq \mathsf{lim}_{n \geq i}\,(\cdots \to \mathrm{E}_{i+3} \to \mathrm{E}_{i+2} \to \mathrm{E}_{i+1})$$

This sequential limit can be computed using the cotensorization of C over the ∞-category of spaces via the pullback diagrams in C

$$\begin{tikzcd}
\prod_{n\geq i+1} \mathrm{E}_n[-1] \arrow[r] \arrow[d] & \mathsf{lim}_{n\geq i}\,(\cdots \to \mathrm{E}_{i+3} \to \mathrm{E}_{i+2} \to \mathrm{E}_{i+1}) \arrow[r] \arrow[d] & \prod_{n\geq i+1} \mathrm{E}_n^{\Delta^1} \arrow[d] \\
0 \arrow[r] & \prod_{n\geq i+1} \mathrm{E}_n \arrow[r] & \prod_{n\geq i+1} \mathrm{E}_n \times \mathrm{E}_n
\end{tikzcd}$$

and we obtain

$$\mathsf{lim}\,\mathrm{E} \simeq \mathrm{cofiber}\,(\prod_{n\geq i+1} \mathrm{E}_n[-1] \to \prod_{n\geq i+1} \mathrm{E}_n[-1])$$

Since, by assumption $\mathsf{C}_{\geq i}$ is stable under countable products, we have $\prod_{n\geq i+1} \mathrm{E}_n[-1] \in \mathsf{C}_{\geq i}$. Since $\mathsf{C}_{\geq i}$ is stable under all colimits [Lur17, 1.2.1.6] we conclude that $\mathsf{lim}\,\mathrm{E} \in \mathsf{C}_{\geq i}$.

Remark 2.2.7. The discussion in Construction **2.2.6** also shows that whenever the t-structure on C is left complete, the Whitehead filtration $\tau_{\geq}X$ of Construction **2.2.5** is complete so that $\tau_{\geq}$ factors as

$$\tau_{\geq} : \mathsf{C} \to \mathsf{Fil}^{\dagger} \subseteq \widehat{\mathsf{Fil}}(\mathsf{C}) \subseteq \mathsf{Fil}(\mathsf{C})$$

.

We will also need the notion of graded and filtered categories:

Definition 2.2.8. We define a *graded category* to be a stable presentable ∞-category endowed with a structure of object in $\mathsf{Mod}_{\mathsf{Sp}^{\mathbb{Z}-\mathrm{gr},\otimes}}(\mathsf{Pr}^{\mathsf{L}})$. Similarly, a *filtered category* is an object in $\mathsf{Mod}_{\mathsf{Fil}(\mathsf{C})^{\otimes}}(\mathsf{Pr}^{\mathsf{L}})$.

We will need to use the point of view on filtrations and gradings given by the Rees construction of Simpson [Sim91, Lemma 19] where the following geometric objects play a central role:

Construction 2.2.9. Let $\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$ be the classifying stack of the flat multiplicative abelian group scheme over the sphere spectrum and $e : \mathsf{Spec}(\mathbb{S}) \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$ the canonical atlas. Consider also the canonical geometric action of $\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$ on the flat affine line $\mathbb{A}^1_{\mathbb{S}}$ of weight 1, ie, $(\lambda, a) \mapsto \lambda^1.a$. Algebraically, this is the action for each the variable t is of weight (-1). Form the quotient stack $[\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}]$. This lives canonically as a stack over $\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$ via a map

$$\pi : [\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$$

The inclusion of the zero point $0 : \mathsf{Spec}(\mathbb{S}) \to \mathbb{A}^1_{\mathbb{S}}$ provides a section of the canonical projection $[\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$

$$0 : \mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}} \to [\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}]$$

The inclusion of $\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$ in $\mathbb{A}^1_{\mathbb{S}}$ also passes to the quotient and provides a map

$$[\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}] \leftarrow \mathbb{G}_{\mathrm{m}\,\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}} \simeq * : 1$$

The Rees construction gives us a geometric interpretation of filtered objects and gradings when $\mathsf{C} = \mathsf{Sp}$ is the ∞-category of spectra, in terms of objects over $[\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}]$ and $\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}}$. The proof of the following result appears in [Mou19]:

Theorem 2.2.10. *There exists symmetric monoidal equivalences*

$$\mathsf{Sp}^{\mathbb{Z}-\mathrm{gr},\otimes} \simeq \mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}})^{\otimes} \tag{11}$$

$$\mathrm{Rees} : \mathsf{Fil}(\mathsf{Sp})^{\otimes} \simeq \mathsf{QCoh}([\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}])^{\otimes} \tag{12}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{S}})^{\otimes} \arrow[d, "\sim"'] & \mathsf{QCoh}([\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_{\mathrm{m}\,\mathbb{S}}])^{\otimes} \arrow[l, "0^*"'] \arrow[r, "1^*"] \arrow[d, "\sim"'] & \mathsf{QCoh}(\mathsf{Spec}(\mathbb{S}))^{\otimes} = \mathsf{Sp}^{\otimes} \arrow[d, equal] \\
\mathsf{Sp}^{\mathbb{Z}-\mathrm{gr},\otimes} & \mathsf{Fil}(\mathsf{Sp})^{\otimes} \arrow[l, "\mathsf{gr}"'] \arrow[r, "\mathsf{colim}"] & \mathsf{Sp}^{\otimes}
\end{tikzcd}$$

Moreover, after base change along $\mathsf{Spec}(\mathbb{Z}) \to \mathsf{Spec}(\mathbb{S})$ *we recover analogues of these comparisons for filtered and graded objects in* $\mathsf{Mod}_{\mathbb{Z}}$ *the* ∞*-category derived category of abelian groups and quasi-coherent sheaves on* $\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{Z}}$ *and* $[\mathbb{A}^1_{\mathbb{Z}}/\mathbb{G}_{\mathrm{m}\,\mathbb{Z}}]$ *via the Rees construction (see for instance* [Sim97]*).*

In view of the Theorem **2.2.10** the following definition becomes natural.

Definition 2.2.11. We define a graded stack to be a stack over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$ and a filtered stack to be a stack over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. Let $X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ be a filtered stack. The *associated graded* of X, denoted X^{gr}, is the base change of X along the map $0 : \mathsf{B}\mathbb{G}_{\mathrm{m}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. By abuse of notation we will also write X^{gr} to denote the further pullback along the atlas $* \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$, endowed with its canonical $\mathbb{G}_{\mathrm{m}}$-action.

The *underlying stack*, X^u, is the base-change along $1 : * \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

Definition 2.2.12. Let $\pi : X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ be a (derived) stack over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. We use the notation

$$\mathscr{O}^{\mathsf{fil}}(X) := \pi_* \mathscr{O}_X$$

for the push-forward of the structure sheaf; this is an object of the ∞-category $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ of filtered k-modules.

Remark 2.2.13. Let X be a filtered (resp. graded) stack. Then $\mathsf{QCoh}(X)$ is a filtered (resp. graded) category in the sense of Definition **2.2.8** via the symmetric monoidal pullback along the structure map to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ (resp. $\mathsf{B}\mathbb{G}_{\mathrm{m}}$).

Remark 2.2.14. Let $\pi : X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ be a filtered stack and consider the pullback diagram

$$\begin{tikzcd} X^{\mathsf{gr}} \arrow[r, "\tilde{0}"] \arrow[d, "\pi^{\mathsf{gr}}"'] & X \arrow[d, "\pi"] & X^u \arrow[l, "\tilde{1}"'] \arrow[d, "\pi^u"] \\ \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "0"] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & * \arrow[l, "1"'] \end{tikzcd} \tag{13}$$

In this paper we will need to deal with different situations under which the base change property for quasi-coherent sheaves of $\mathscr{O}_X$-modules along the two pullback diagrams, holds. Namely, the Beck-Chevalley transformations

$$0^*\pi_* \to (\pi^{\mathsf{gr}})_*\tilde{0}^* \quad , \quad 1^*\pi_* \to (\pi^u)_*\tilde{1}^* \qquad \text{on } \mathsf{QCoh}$$

are equivalences.

(i) Following [HLP14, A.1.3 (1)] (see Notation **1.2.12**-c) and [Lur18, 6.2.5.9]) this holds for 1 whenever X admits a flat hypercover by affines with flat transition maps and $F \in \mathsf{QCoh}(X)_{<\infty}$ (homologically bounded above). Indeed, in this case the argument in the proof of [HLP14, A.1.3 (1)] applies since 1 is an open immersion, ergo flat, ergo of finite tor-dimension.

(ii) Also following [HLP14, A.1.3 (1)], the Beck-Chevalley transformation is an isomorphism for 0 whenever X admits a flat atlas by affines with flat transition maps (see Notation **1.2.12**-c) and [Lur18, 6.2.5.9]) and $F \in \mathsf{QCoh}(X)$. This is possible since 0 is an lci closed immersion and therefore of finite tor-dimension and finite.

(iii) Whenever the map π is of finite cohomological dimension. See different analysis in [HLP14, A.1.4, A.1.5, A.1.9], [Lur18, 9.1.5.7, 9.1.5.8, 9.1.5.3], [Lur18, 6.3.4.1] or [BZFN10, Prop. 3.10].

Under these assumptions, the global sections $\pi_*(\mathscr{O}_X) = \mathscr{O}^{\mathsf{fil}}(X)$ admit the structure of an $\mathsf{E}_\infty^\otimes$-algebra in $\mathsf{Fil}(\mathsf{Sp})$ which can be interpreted as a filtration on the $\mathsf{E}_\infty^\otimes$-algebra $(\pi^u)_*\mathscr{O} = \mathsf{C}^*(X^u, \mathscr{O})$, whose associated graded is $(\pi^{\mathsf{gr}})_*\mathscr{O} = \mathsf{C}^*(X^{\mathsf{gr}}, \mathscr{O})$. By the same mechanics, given a graded stack, $Y \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$, the $\mathsf{E}_\infty^\otimes$-algebra $\mathsf{C}^*(Y, \mathscr{O})$ carries a grading compatible with the $\mathsf{E}_\infty^\otimes$-structure.

Construction 2.2.15. Let X be a stack with a $\mathbb{G}_{\mathrm{m}}$-action and consider the stacky quotient $[X/\mathbb{G}_{\mathrm{m}}]$ which lives canonically over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$. We defined the filtered stack associated to X with the $\mathbb{G}_{\mathrm{m}}$-action to be the fiber product of stacks

$$\begin{tikzcd} X^{\mathsf{Fil}} := X/\mathbb{G}_{\mathrm{m}} \times_{\mathsf{B}\mathbb{G}_{\mathrm{m}}} [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \arrow[r] \arrow[d] & {[X/\mathbb{G}_{\mathrm{m}}]} \arrow[d] \\ {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[r] & \mathsf{B}\mathbb{G}_{\mathrm{m}} \end{tikzcd}$$

We have cartesian squares

$$\begin{tikzcd} (X^{\mathsf{Fil}})^{\mathsf{gr}} = {[X/\mathbb{G}_{\mathrm{m}}]} \arrow[r] \arrow[d] & X^{\mathsf{Fil}} \arrow[d] & (X^{\mathsf{Fil}})^u \arrow[l] \arrow[d] \\ \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "0"] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & * \arrow[l, "1"'] \end{tikzcd}$$

Moreover, the cartesian diagram

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "h"] \arrow[d, "p"'] & * \arrow[d, "e"] \\ {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[r, "\pi"] & \mathsf{B}\mathbb{G}_{\mathrm{m}} \end{tikzcd} \tag{14}$$

tells us that the pullback of X^{Fil} to $\mathbb{A}^1$ is isomorphic to $\mathbb{A}^1 \times X$ and exhibits X^{Fil} as the quotient $[(X \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}}]$ for the product of the $\mathbb{G}_{\mathrm{m}}$-action on X and the canonial action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^1$. Moreover, this also tells us that $(X^{\mathsf{Fil}})^u \simeq X$.

The filtered stacks obtained from $\mathbb{G}_{\mathrm{m}}$-equivariant stacks using the previous construction are precisely the *split filtered stacks*.

2.3. **Kernel and fixed points of the Frobenius.** We now turn back to Witt vectors and the interpolation between fixed points and kernel of the Frobenius. The starting ingredient is the following natural grading:

Construction 2.3.1. The abelian group of p-typical Witt vectors $\mathsf{W}_{p^\infty}(\mathrm{A})$ carries an action of the underlying multiplicative monoid of A: for a given $a \in \mathrm{A}^\times$, we define $[a] : \mathsf{W}_{p^\infty} \to \mathsf{W}_{p^\infty}$ by the formula $(\lambda_n)_{n\in S} \mapsto (a^n.\lambda_n)_{n\in S}$ where S is as in the Guide **2.1.1**. It is an easy exercice to check that in terms of the Ghost coordinates this becomes $(\omega_n)_{n\in S} \mapsto (a^n.\omega_n)_{n\in S}$ which is coordinatewise a map of abelian groups under addition. By the unique characterization of the group structure on W_{p^∞} re-engineered from Ghost coordinates (see the Guide **2.1.1**), the map $[a] : \mathsf{W}_{p^\infty} \to \mathsf{W}_{p^\infty}$ is a map of abelian groups.

This defines an action of the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$ on the group scheme W_{p^∞} and makes it a graded group scheme.

We consider the quotient stack $[\mathsf{W}_{p^\infty}/\mathbb{G}_{\mathrm{m}}]$ which lives canonically as an abelian group stack over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$.

Remark 2.3.2. The operation Frob_p is compatible with the action of the multiplicative monoid of $\mathbb{A}^1$ on W_{p^∞} in the following sense: given $a \in \mathrm{A}$ and denoting by $[a] : \mathsf{W}_{p^\infty}(\mathrm{A}) \to \mathsf{W}_{p^\infty}(\mathrm{A})$ the action by a, we have

$$\mathrm{Frob}_p \circ [a] = [a^p] \circ \mathrm{Frob}_p$$

Construction 2.3.3. We let $\mathsf{W}^{\mathsf{Fil}}_{p^\infty}$ be the output of the Construction **2.2.15** applied to the action of $\mathbb{G}_{\mathrm{m}}$ on W_{p^∞} of the Construction **2.3.1**. As $[\mathsf{W}_{p^\infty}/\mathbb{G}_{\mathrm{m}}]$ is a group stack over $\mathsf{B}\mathbb{G}_{\mathrm{m}}$, it follows that $\mathsf{W}^{\mathsf{Fil}}_{p^\infty}$ is a group stack over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. Explicitly,

$$\mathsf{W}^{\mathsf{Fil}}_{p^\infty} \simeq [(\mathsf{W}_{p^\infty} \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}}]$$

the quotient of trivial family $\mathsf{W}_{p^\infty} \times \mathbb{A}^1$ by the diagonal action of $\mathbb{G}_{\mathrm{m}}$.

We will now explain how to use the trivial family of the Construction **2.3.3** to construct a new family that interpolates between Frobenius fixed points and the kernel.

Construction 2.3.4. Consider the trivial group scheme $\mathsf{W}_{p^\infty} \times \mathbb{A}^1$ over $\mathbb{A}^1$. For each A over $\mathbb{Z}_{(p)}$, consider the endomorphism of abelian groups

$$\mathscr{G}_p : \mathsf{W}_{p^\infty}(\mathrm{A}) \times \mathrm{A} \to \mathsf{W}_{p^\infty}(\mathrm{A}) \times \mathrm{A} \quad \text{given by} \quad (f, a) \mapsto (\mathrm{Frob}_p(f) - [a^{p-1}](f),\, a)$$

This is functorial in A and defines a morphism of abelian group schemes over $\mathbb{A}^1$

$$\mathscr{G}_p : \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \to \mathsf{W}_{p^\infty} \times \mathbb{A}^1$$

Remark 2.3.5. The Remark **2.3.2** is equivalent to the statement that $\mathscr{G}_p$ is $\mathbb{G}_{\mathrm{m}}$-equivariant with respect to the diagonal action of $\mathbb{G}_{\mathrm{m}}$ on the source and the twist by $(-)^p : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ on the target. This implies that the inclusion

$$\ker \mathscr{G}_p \subseteq \mathsf{W}_{p^\infty} \times \mathbb{A}^1$$

is $\mathbb{G}_{\mathrm{m}}$-equivariant. The fiber of $\ker \mathscr{G}_p$ over 0 is Ker (Guide **2.1.1**) and is closed under the $\mathbb{G}_{\mathrm{m}}$-action. The fiber over any $\lambda \in \mathrm{A}^\times$, $(\ker \mathscr{G}_p)_\lambda$ is isomorphic to $(\ker \mathscr{G}_p)_1 \simeq \mathsf{Fix}$ via the isomorphism sending $((\lambda_n)_{n\in S}, \lambda) \mapsto ([\frac{1}{\lambda}]((\lambda_n)_{n\in S}), 1)$.

Lemma 2.3.6. *The morphism $\mathscr{G}_p : \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \to \mathsf{W}_{p^\infty} \times \mathbb{A}^1$ is a cover for the fpqc topology*[(†)]*. In particular, it is fpqc-locally surjective and we have a short exact sequence of abelian group-stacks over $\mathbb{A}^1$*

$$0 \longrightarrow \ker \mathscr{G}_p \longrightarrow \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \xrightarrow{\mathscr{G}_p} \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \longrightarrow 0$$

In particular, $\ker \mathscr{G}_p$ is a flat group scheme over $\mathbb{A}^1$.

Proof. As discussed in the Guide **2.1.1**, the group scheme W_{p^∞} is the inverse limit of the system $\mathsf{W}_{p^\infty}^{(m)}$ of m-truncated p-typical Witt vectors and each restriction map $\mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m-1)}$ is isomorphic to a projection $\mathsf{W}_{p^\infty}^{(m-1)} \times \mathbb{A}^1 \to \mathsf{W}_{p^\infty}^{(m-1)}$. Each restriction map is a flat surjection between affine schemes, and therefore, an fpqc cover. In this case (see for instance [Sta19, Lemma 05UU]), in order to show that $\mathscr{G}_p$ is an fpqc cover, it is enough to show that each composition

$$\mathsf{W}_{p^\infty} \times \mathbb{A}^1 \xrightarrow{\mathscr{G}_p} \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \longrightarrow \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{A}^1 \tag{15}$$

is an fpqc cover.

As remarked in the Guide **2.1.1**, the Frobenius map on m-truncated Witt vectors factors as $\mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m-1)}$. The $\mathbb{G}_{\mathrm{m}}$-action on the contrary is defined levelwise $[x] : \mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m)}$. By composing with the truncation maps $[x] : \mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m)} \to \mathsf{W}_{p^\infty}^{(m-1)}$ we obtain a system of maps that after passing to the inverse limit, it recovers the $\mathbb{G}_{\mathrm{m}}$-action on W_{p^∞}. In this case, the composition (15) factors as

$$\begin{tikzcd} \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \arrow[r, "\mathscr{G}_p"] \arrow[d] & \mathsf{W}_{p^\infty} \times \mathbb{A}^1 \arrow[d] \\ \mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{A}^1 \arrow[r] & \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{A}^1 \end{tikzcd} \tag{16}$$

So that (as in [Sta19, Lemma 090N]) to show that each composition (15) is an fpqc cover, it is enough to show that each truncated map is an fpqc cover

(†)ie, surjective on points and flat

$$\mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{A}^1 \longrightarrow \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{A}^1 \tag{17}$$

This morphism is a map of smooth group schemes that commutes with the projections to $\mathbb{A}^1 := \mathbb{A}^1_{\mathbb{Z}_{(p)}} \simeq \mathbb{A}^1_{\mathbb{Z}} \times \mathsf{Spec}(\mathbb{Z}_{(p)})$ and therefore, as now for each n the map is of finite presentation, to check that it is a fpqc cover, it is enough (by a local criterion for flatness [Sta19, Lemma 039D], [Sta19, Section 03NV], and since surjectivity in the topological sense can be tested on closed points [Sta19, Section 0485] and therefore on fields) to check that it is so after base change to any field valued point $\mathsf{Spec}K \to \mathbb{A}^1_{\mathbb{Z}} \times \mathsf{Spec}(\mathbb{Z}_{(p)})$. As both projections to $\mathbb{A}^1_{\mathbb{Z}_{(p)}}$ are compatible with the $\mathbb{G}_{\mathrm{m}}$-action, it is enough to test the statement for the four different points

$$(0, \mathbb{Q}),\ (1, \mathbb{Q}),\ (0, \mathbb{F}_p),\ (1, \mathbb{F}_p)$$

ie, the four maps

$$\mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{Q} \xrightarrow{\mathrm{Frob}_p} \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{Q} \qquad \mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{Q} \xrightarrow{\mathrm{Frob}_p - \mathsf{id}} \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{Q}$$

$$\mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{F}_p \xrightarrow{\mathrm{Frob}_p} \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{F}_p \qquad \mathsf{W}_{p^\infty}^{(m+1)} \times \mathbb{F}_p \xrightarrow{\mathrm{Frob}_p - \mathsf{id}} \mathsf{W}_{p^\infty}^{(m)} \times \mathbb{F}_p$$

Using the Ghost components for truncated p-typical Witt vectors of (8), the first two maps becomes isomorphic to, respectively, the projection away from the first coordinate and a linear projection

$$\prod_{i \in S_{m+1}} \mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \longrightarrow \prod_{i \in S_m} \mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \qquad \prod_{i \in S_{m+1}} \mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \longrightarrow \prod_{i \in S_m} \mathbb{G}_{\mathsf{a}\,\mathbb{Q}}$$

both being clearly surjective and flat.

After base-change to $\mathbb{F}_p$, Frob_p acts by the standard power p Frobenius on $\mathbb{A}^1$ (see Guide **2.1.1**) and the morphisms over $\mathbb{F}_p$ become, respectively, the compositions

$$\mathrm{Frob}_p : \prod_{i \in S_{m+1}} \mathbb{A}^1 \to \prod_{i \in S_{m+1}} \mathbb{A}^1 \to \prod_{i \in S_m} \mathbb{A}^1$$

$$(\lambda_1, \lambda_p, .., \lambda_{p^{m-1}}, \lambda_{p^m}) \mapsto (\lambda_1^p, \lambda_p^p .., \lambda_{p^{m-1}}^p, \lambda_{p^m}^p) \mapsto (\lambda_1^p, .., \lambda_{p^{m-1}}^p)$$

$$\mathrm{Frob}_p - \mathsf{id} : \prod_{i \in S_{m+1}} \mathbb{A}^1 \to \prod_{i \in S_{m+1}} \mathbb{A}^1 \to \prod_{i \in S_m} \mathbb{A}^1$$

$$(\lambda_1, \lambda_p, ..., \lambda_{p^{m-1}}, \lambda_{p^m}) \mapsto (\lambda_1^p - \lambda_1, .., \lambda_{p^{m-1}}^p - \lambda_{p^{m-1}}, \lambda_{p^m}^p - \lambda_{p^m}) \mapsto (\lambda_1^p - \lambda_1, .., \lambda_{p^{m-1}}^p - \lambda_{p^{m-1}})$$

Both projections are fpqc covers. We conclude using the fact that the standard power p Frobenius on $\mathbb{A}^1$, $(-)^p$, is fpqc in our situation (see for instance [Liu02, §4, Exercice 3.13]) and each $(-)^p - \mathsf{id}$ is the Artin-Schreier isogeny well known to be an étale cover in characteristic p.

□

Definition 2.3.7. We define a filtered stack $\mathsf{H}_{p^\infty} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ to be stack over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ given by the quotient

$$\mathsf{H}_{p^\infty} := [(\ker \mathscr{G}_p)/\mathbb{G}_{\mathrm{m}}] \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$$

Remark 2.3.8. It follows from Remark **2.3.5** that the underlying stack $\mathsf{H}^u_{p^\infty}$ is Fix and the associated graded $\mathsf{H}^{\mathsf{gr}}_{p^\infty}$ is the quotient $[\mathsf{Ker}/\mathbb{G}_{\mathrm{m}}]$ under the $\mathbb{G}_{\mathrm{m}}$-action of the Construction **2.3.1** and Remark **2.3.5**.

3. The Filtered Circle

3.1. **Hopf algebras and comodules.** As discussed in the introduction, a key feature that any reasonable construction of a "filtered circle" should satisfy is that it must carry the structure of an abelian group stack. This will manifest itself in the form of a Hopf algebra structure on its cohomology, which on the one hand specializes to the group structure on S^1, and on the other to a strict square-zero multiplication.

Before we proceed, we first set straight what we mean by the ∞-category of cosimplicial commutative k-algebras. Recall that the ordinary category of cosimplicial commutative k-algebras is defined to be $\mathsf{Fun}(\Delta, \mathsf{CAlg}^0_k)$, the category of cosimplicial objects in (discrete) commutative k-algebras. By [Toe06, Théorème 2.1.2], this may be equipped with a closed (simplicial) model structure, with weak equivalences maps with induce isomorphisms on cohomology and fibrations being the level-wise surjections.

Definition 3.1.1. We define the ∞-category of cosimplicial commutative k-algebras coSCR_k to be the localization of the category $\mathsf{Fun}(\Delta, \mathsf{CAlg}^0_k)$, equipped with the model category structure described above.

There exists a functor $\theta : \mathsf{coSCR}_k \to \mathsf{CAlg}^{\mathsf{ccn}}_k$, which arises as the cosimplicial version of the Dold-Kan construction; to be more precise, we remark that the model structure

on coSCR_k is right induced by that of the ∞-category Mod_k^{Δ} of cosimplicial objects in (discrete) k-modules via the forgetful functor; this in turn obtains a model category structure from Mod_R^{ccn} via the adjunction

$$N : \mathsf{Mod}_R^{\Delta} \to \mathsf{Mod}_R^{\mathsf{ccn}} \quad D : \mathsf{Mod}_R^{\mathsf{ccn}} \to \mathsf{Mod}_R^{\Delta}$$

making it into a Quillen equivalence. By the classical Eilenberg-Zilberg theorem, $N(-)$ commutes with tensor products (see [Toe06, 2.1]), so the composition with the forgetful functor, $N \circ U : \mathsf{coSCR}_k \to \mathsf{Mod}_K^{\mathsf{ccn}}$ factors through $\mathsf{CAlg}_k^{\mathsf{ccn}}$, giving us θ. Moreover, this functor preserves homotopy limits as it is the composition of the forgetful functor

$$\mathsf{coSCR} \to \mathsf{Mod}_R^{\mathsf{ccn}}$$

composed with the functor $N : \mathsf{Mod}_R^{\Delta} \to \mathsf{Mod}_R^{\mathsf{ccn}}$ which is a right adjoint functor.

Construction 3.1.2. We will always consider the ∞-category coSCR_k equipped with its cocartesian monoidal structure. Explicitly, this is given by taking coproducts between cofibrant replacements in the model category of cosimplicial commutative algebras. In this case coproducts coincide with usual tensor products of cosimplicial commutative algebras.

Construction 3.1.3. Let C be a presentable ∞-category endowed with the cartesian symmetric monoidal structure. We denote by $\mathsf{Gr}(\mathsf{C}) := \mathsf{Mon}^{\mathsf{gp}}_{\mathsf{E}_1^{\otimes}}(\mathsf{C})$ the category of group objects in C. Following [Lur17, 5.2.6.6, 4.1.2.11, 2.4.2.5.], an explicit model is given by category of diagrams $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathsf{C})$ satisfying the Segal conditions. The colimit functor

$$\mathsf{B} : \mathsf{Gr}(\mathsf{C}) \to \mathsf{C}_* \tag{18}$$

lands in the category of pointed objects in C and admits a right adjoint, sending a pointed object $* \to X$ to its nerve.
There is a dual version of this given by cogroup objects $\mathsf{coGr}(\mathsf{C})$; these will be given by the category of cosimplicial objects $\mathsf{Fun}(\Delta, \mathsf{C})$ of C satisfying the relevant Segal conditions in the opposite category. One has an adjunction

$$\mathsf{lim}_{\Delta} : \mathsf{coGr}(\mathsf{C}) \rightleftarrows \mathsf{C} : \mathsf{CoNerve}$$

Definition 3.1.4. We define, following [Toe06, Definition 3.1.4], a H_∞*-Hopf algebra* to be a cogroup object in the ∞-category coSCR of cosimplicial (commutative) algebras.

Similarly, a *graded H_∞-Hopf algebra* will be a cogroup object in the category $\mathsf{coSCR}^{\mathrm{gr}}$ of graded cosimplicial commutative rings.

$$\mathsf{co}^{\Delta}\mathsf{Hopf} := \mathsf{coGr}(\mathsf{coSCR}) \tag{19}$$

$$\mathsf{co}^{\Delta}\mathsf{Hopf}^{\mathrm{gr}} := \mathsf{coGr}(\mathsf{coSCR}^{\mathrm{gr}}) \tag{20}$$

Remark 3.1.5. Equivalently, this is the data of a cosimplicial object $\mathsf{H}^\bullet$ in coSCR_k with the following properties:

- $\mathsf{H}^0 \simeq k$,
- the natural map $\mathsf{H}^1 \otimes_k ... \otimes_k \mathsf{H}^1 \simeq \mathsf{H}^n$ is an equivalence.

As the symmetric monoidal structure on coSCR_k is cocartesian (see Construction **3.1.2**), this is precisely the data of a cogroup object in this ∞-category .

We now introduce a notion of *homotopy coherent comodules* over a fixed H_∞-Hopf algebra $\mathsf{H}^\bullet$.

Definition 3.1.6. Given $\mathsf{H} = \mathsf{H}^\bullet \in \mathsf{co}^{\Delta}\mathsf{Hopf}$, we define the ∞-category of H-comodules as

$$\mathsf{CoMod}_{\mathsf{H}} := \mathsf{QCoh}(\mathsf{BSpec}^{\Delta}(\mathsf{H}^\bullet)) \simeq \mathsf{lim}_{\Delta}\mathsf{QCoh}(\mathsf{Spec}^{\Delta}(H^\bullet)).$$

Remark 3.1.7. We explain the idea behind this definition a bit further. Since $\mathsf{Spec}^{\Delta} : \mathsf{coSCR}_k \to \mathsf{St}_k$ is fully faithful, a cogroup object $\mathsf{H}^\bullet$ in coSCR gives rise to a group object $G_\bullet$ in St_k, which we may think of as the nerve of an affine group stack $G_1 = \mathsf{Spec}^{\Delta}\mathsf{H}_1$. Then $\mathsf{CoMod}_{\mathsf{H}}$ may be thought of as the ∞-category of $G_1 = \mathsf{Spec}^{\Delta}\mathsf{H}_1$ representations

$$\mathsf{QCoh}(\mathsf{B}G) \simeq \mathsf{lim}_{\Delta}\mathsf{QCoh}(G_\bullet)$$

Remark 3.1.8. This notion will be important to us later on, when we identify $\mathsf{CoMod}_{C^*(\mathsf{Ker},\mathscr{O})^\bullet}$ with the underlying ∞-category of the category of strict dg-comodules over a particular dga.

3.2. **Affinization and Cohomology of stacks.** At this point, after the constructions in the previous section, we have in fact proved Theorem **1.2.1**-(iii). We absorb it in the following definition:

Definition 3.2.1. The *filtered circle* (local at p) is the filtered stack given by the classifying stack (relatively to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ and with respect to the fpqc-topology) of the filtered abelian group stack H_{p^∞}:

$$\mathsf{S}^1_{\mathsf{Fil}} := \mathsf{B}\mathsf{H}_{p^\infty} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$$

Construction 3.2.2. The filtered stack $\mathsf{S}^1_{\mathsf{Fil}}$, being the classifying stack of a filtered abelian group stack is again a filtered abelian group stack. In other words, it carries a canonical abelian group structure compatible with the filtration. In particular, we can take its classifying stack

$$\mathsf{B}\mathsf{S}^1_{\mathsf{Fil}} \simeq \mathsf{K}(\mathsf{H}_{p^\infty}, 2)$$

obtained as the fpqc quotient $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]/\mathsf{S}^1_{\mathsf{Fil}}$.

Remark 3.2.3. If follows from the Lemma **2.3.6** and Definition **2.3.7** that H_{p^∞} is a flat group stack relatively to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ and also, relatively affine. In particular, both $\mathsf{S}^1_{\mathsf{Fil}}$ and $\mathsf{B}\mathsf{S}^1_{\mathsf{Fil}}$ are flat group stacks over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. This implies that both admit a flat hypercover by affines with flat transition maps. For $\mathsf{S}^1_{\mathsf{Fil}}$ this can be seen directly using the bar construction for H_{p^∞} relatively to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$

$$\begin{tikzcd}
\cdots & \mathsf{H}_{p^\infty}^{\times_2} \arrow[r, shift left, shift left, shift left] \arrow[r, shift left, shift left] \arrow[r, shift left] \arrow[dr] & \mathsf{H}_{p^\infty} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[r] \arrow[dl] & \mathsf{S}^1_{\mathsf{Fil}} \arrow[dll] \\
& & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & &
\end{tikzcd}$$

For $\mathsf{B}\mathsf{S}^1_{\mathsf{Fil}}$, we use the fact that $\mathsf{N}(\Delta^{\mathsf{op}})$ is sifted, to exhibit $\mathsf{B}\mathsf{S}^1_{\mathsf{Fil}}$ as the geometric realization of the diagonal of the double bar construction:

$$\begin{tikzcd}
\cdots & \mathsf{H}_{p^\infty}^{\times_4} \arrow[r, shift left, shift left, shift left] \arrow[r, shift left, shift left] \arrow[r, shift left] \arrow[dr] & \mathsf{H}_{p^\infty} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[r] \arrow[dl] & \mathsf{B}\mathsf{S}^1_{\mathsf{Fil}} \arrow[dll] \\
& & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & &
\end{tikzcd} \tag{21}$$

Construction 3.2.4. The discussion in Remark **3.2.3** implies that both $\mathsf{QCoh}(\mathsf{S}^1_{\mathsf{Fil}})$ and $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ admit left-complete t-structures as described in Notation **1.2.12**-c). Concretely, for $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$, it is induced by transferring the t-structure on $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ by pullback along the atlas $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \to \mathsf{BS}^1_{\mathsf{Fil}}$. Here the t-structure on $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ is itself induced by the standard t-structure on $\mathsf{QCoh}(\mathbb{A}^1)$ via the same argument as in Construction **3.2.4** using the flat simplicial atlas encoding the geometric action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^1$ of weight 1 (see Construction **2.2.9**).

Under the equivalence $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) \simeq \mathsf{Fil}(\mathsf{Sp})$ of Theorem **2.2.10** this t-structure corresponds to the levelwise t-structure on filtered spectra (see [Mou19, Proof of 6.1]).

Remark 3.2.5. The conclusion of the Remark **3.2.3** is also particularly important in the context of the base change property of Remark **2.2.14** with $\pi : \mathsf{BS}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$

$$\begin{tikzcd} (\mathsf{BS}^1_{\mathsf{Fil}})^{\mathrm{gr}} \arrow[r, "0_a"] \arrow[d, "\pi^{\mathrm{gr}}"'] & \mathsf{BS}^1_{\mathsf{Fil}} \arrow[d, "\pi"] & (\mathsf{BS}^1_{\mathsf{Fil}})^{\mathrm{u}} \arrow[l, "1_a"'] \arrow[d, "\pi^{\mathrm{u}}"] \\ \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[r, "0"] & [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] & * \arrow[l, "1"'] \end{tikzcd} \tag{22}$$

Indeed, using Remark **3.2.3** we conclude that that base-change holds for π in the following contexts:

- from the Remark **2.2.14**-(ii): the Beck-Chevalley transformation

$$0^* \ \pi_* \ \to (\pi^{\mathrm{gr}})_* \ (0_a)^* \tag{23}$$

  is an equivalence for any $\mathrm{F} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$.

- from the Remark **2.2.14**-(i): we can only guarantee that the Beck-Chevalley transformation

$$1^* \ \pi_* \ \to (\pi^{\mathrm{u}})_* \ (1_a)^* \tag{24}$$

  is an equivalence for homologically bounded above objects, ie, in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{<\infty}$ (for the t-structure of Notation **1.2.12**).

One of the claims in Theorem **1.2.1**-(i) is that our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ is related to the topological circle S^1 via the notion of *affinization* of [Toe06]. *Affine stacks* were introduced in [Toe06, Def. 2.2.4] (see also [Lur11b] where these are called *Coaffine*).

Informally, an (higher) stack X is affine if it can be recovered from its cohomology of global sections. More precisely:

Review 3.2.6 (Affine Stacks)**.** See Notation **1.2.10**. By [Toe06, 2.2.3] the ∞-functor Spec^{Δ} : $\mathsf{coSCR}^{\mathsf{op}}_{\mathbb{Z}_{(p)}} \to \mathsf{St}_{\mathbb{Z}_{(p)}}$ is fully faithful and admits a left adjoint $\mathsf{C}^*_{\Delta}(-,\mathscr{O})$ that enhances the standard $\mathsf{E}^{\otimes}_{\infty}$-algebra structure of cohomology of global sections $\mathsf{C}^*(-,\mathscr{O})$ with a structure of cosimplicial commutative algebra, namely, it provides a lifting in $\widehat{\mathsf{Cat}_{\infty}}$

$$\begin{tikzcd} & \mathsf{coSCR}_{\mathbb{Z}_{(p)}} \arrow[d, "\theta"] \\ \mathsf{St}_{\mathbb{Z}_{(p)}} \arrow[ur, "\mathsf{C}^*_{\Delta}(-,\mathscr{O})"] \arrow[r, "\mathsf{C}^*(-,\mathscr{O})"'] & \mathsf{CAlg}_{\mathbb{Z}_{(p)}} \end{tikzcd}$$

We say that $X \in \mathsf{St}_{\mathbb{Z}_{(p)}}$ is affine if it lives in the essential image of Spec^{Δ}. More generally, given $X \in \mathsf{St}_{\mathbb{Z}_{(p)}}$, we define its affinization as the stack $\mathsf{Spec}^{\Delta}(\mathsf{C}^*_{\Delta}(X,\mathscr{O}))$ [Toe06, 2.3.2].

Proposition 3.2.7. *Both* $\mathsf{S}^1_{\mathsf{Fil}}$ *and* $\mathsf{BS}^1_{\mathsf{Fil}}$ *are relatively affine stacks over* $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ *in the sense of* [Toe06]*. By stability of affine stacks under base-change* [‡] *so are all the stacks* $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$*,* $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$*,* $\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ *and* $\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$*.*

Proof. Let us start by showing that $\mathsf{S}^1_{\mathsf{Fil}}$ is relatively affine over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. Using the atlas $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ this is equivalent to showing that $\mathsf{B}\ker\mathscr{G}_p$ is an affine stack over $\mathbb{A}^1 = \mathsf{Spec}(\mathbb{Z}_{(p)}[T])$. Applying B to the short exact sequence in the Lemma **2.3.6**, we get a fiber sequence of fpqc sheaves over $\mathbb{A}^1$

$$\begin{tikzcd} \mathsf{B}\ker\mathscr{G}_p \arrow[r] \arrow[d] & \mathsf{BW}_{p^\infty} \times \mathbb{A}^1 \arrow[d, "\mathsf{B}\mathscr{G}_p \times \mathsf{id}"] \\ \mathbb{A}^1 \arrow[r] & \mathsf{BW}_{p^\infty} \times \mathbb{A}^1 \end{tikzcd}$$

By [Toe06, 2.2.7] the class of affine stacks over any commutative ring is closed under limits. Therefore, to conclude that $\mathsf{B}\ker\mathscr{G}_p$ is affine it is enough to show that $\mathsf{BW}_{p^\infty} \times \mathbb{A}^1$ is affine. But the group scheme W_{p^∞} can be written as a limit $\mathsf{lim}\mathsf{W}^{(m)}_{p^\infty}$ where each projection $\mathsf{W}^{(m+1)}_{p^\infty} \to \mathsf{W}^{(m)}_{p^\infty}$ is a smooth epimorphism of affine groups with fiber $\mathbb{G}_{\mathsf{a}}$. We claim that this fact together with the fact with the Witt schemes are truncated, implies that the limit decomposition of W_{p^∞} induces a limit decomposition

[‡] see [Toe06, 2.2.7, 2.2.9, Remarque p.49]

$$\mathsf{BW}_{p^\infty} \simeq \lim\, \mathsf{BW}^{(m)}_{p^\infty}. \tag{25}$$

Indeed, the Milnor sequences (see for instance [GJ09, 2.2.9]) tell us that the obstructions for the limit decomposition (25) are given by the groups $\mathsf{lim}^1 \pi_i \operatorname{Map}_{fpqc}(X, \mathsf{W}^{(m)}_{p^\infty})$ for $i \geq 0$ and X affine classical. For $i \geq 1$ these groups vanish because the mapping spaces are discrete. For $i \geq 0$ we have $\pi_i \operatorname{Map}_{fpqc}(X, \mathsf{W}^{(m)}_{p^\infty}) \simeq \mathsf{H}^i_{fpqc}(X, \mathsf{W}^{(m)}_{p^\infty})$, which vanishes for $i > 0$ because X is affine.

In fact, more generally, we also have the same decomposition for the iterated construction

$$\mathsf{B}^j\mathsf{W}_{p^\infty} \simeq \lim\, \mathsf{B}^j\mathsf{W}^{(m)}_{p^\infty}. \tag{26}$$

This follows again because the mapping spaces are discrete and because of the vanishing of the higher cohomology groups

$$\pi_0 \operatorname{Map}_{fpqc}(X, \mathsf{B}^j\mathsf{W}^{(m)}_{p^\infty}) = \mathsf{H}^j_{fpqc}(X, \mathsf{W}^{(m)}_{p^\infty}) = 0 \quad j \geq 1 \tag{27}$$

for X affine classical. One can see this by induction using the long exact sequences extracted from the fact $\mathsf{W}^{(m+1)}_{p^\infty}$ is an extension of $\mathsf{W}^{(m)}_{p^\infty}$ by $\mathbb{G}_{\mathsf{a}}$,

$$0 \to \mathbb{G}_{\mathsf{a}} \to \mathsf{W}^{(m+1)}_{p^\infty} \to \mathsf{W}^{(m)}_{p^\infty} \to 0 \tag{28}$$

The result is true for $\mathbb{G}_{\mathsf{a}}$ and as $\mathsf{W}^{(1)}_{p^\infty} = \mathbb{G}_{\mathsf{a}}$, by induction, it is true all for m.

Finally, knowing (25), by [Toe06, 2.2.7] it becomes enough to show that each $\mathsf{BW}^{(m)}_{p^\infty}$ is affine. But now, each of the group extensions (28) is classified by a map of group stacks $\mathsf{W}^{(m)}_{p^\infty} \to \mathrm{K}(\mathbb{G}_{\mathsf{a}}, 1)$, which we can write as a map $\mathsf{BW}^{(m)}_{p^\infty} \to \mathrm{K}(\mathbb{G}_{\mathsf{a}}, 2)$. By definition of this map, we have a pullback square

$$\begin{tikzcd}
\mathsf{BW}^{(m+1)}_{p^\infty} \arrow[r] \arrow[d] & * \arrow[d] \\
\mathsf{BW}^{(m)}_{p^\infty} \arrow[r] & \mathsf{K}(\mathbb{G}_{\mathsf{a}}, 2)
\end{tikzcd}$$

Each $\mathsf{K}(\mathbb{G}_{\mathsf{a}}, n)$ is known to be affine [Toe06, 2.2.5]. An induction argument concludes the proof.

To prove the claim for $\mathsf{BS}^1_{\mathsf{Fil}}$ it is enough to show that $\mathsf{B}(\mathsf{B}\ker \mathscr{G}_p)$ is affine over $\mathbb{A}^1$. The argument runs the same, using the iterated formula (26).

□

Remark 3.2.8. The proof of the Proposition **3.2.7** shows that more generally, the stacks $\mathsf{K}(\mathsf{H}_{p^\infty}, n)$ are affine.

3.3. **The Underlying Stack of $\mathsf{S}^1_{\mathsf{Fil}}$.** As we now know, by the Proposition **3.2.7**, the underlying stack $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ is affine. We would like, in order to establish Theorem **1.2.1**-(i), to identify it with the affinization of S^1 over $\mathbb{Z}_{(p)}$.

Construction 3.3.1. Recall that $\mathsf{S}^1_{\mathsf{Fil}} = \mathsf{BH}_{p^\infty}$ (Definition **2.3.7**). Let $\mathbb{Z}$ denote the constant group scheme with value $\mathbb{Z}$. There is a canonical morphism of group schemes $\mathbb{Z} \to \mathsf{W}_{p^\infty}$ given by $1 \mapsto (1, 0, 0, ...) \in \mathsf{W}_{p^\infty}(R)$. This Witt vector is fixed by the Frobenius [(§)] so clearly the map factors through $\mathsf{H}^u_{p^\infty} = \mathsf{Fix}$. By passing to classifying stacks we obtain a morphism of stacks

$$u : \mathsf{S}^1 = \mathsf{B}\mathbb{Z} \to (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}} = \mathsf{BFix} \tag{29}$$

with affine target.

The main result of this section is the following:

Proposition 3.3.2. *The map (29) displays* $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}} = \mathsf{BFix}$ *as the affinization of* S^1 *over* $\mathsf{Spec}\mathbb{Z}_{(p)}$. *By* [Toe06, Corollaire 2.3.3]*, this is equivalent to say that (29) induces an equivalence on cosimplicial cochain algebras*

$$\mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \simeq \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{Z}_{(p)}) \tag{30}$$

We will establish below in the Lemma **3.3.10** a local criterion for affinization: in order to prove that the map (29) is the affinization of S^1 over $\mathbb{Z}_{(p)}$ it is enough to know that when base-changed to $\mathbb{Q}$ and $\mathbb{F}_p$, the maps

$$\mathsf{S}^1 \to \mathsf{BFix}_{|\mathbb{Q}} := \mathsf{BFix} \times_{\mathsf{Spec}(\mathbb{Z}_{(p)})} \mathsf{Spec}(\mathbb{Q}) \tag{31}$$

$$\mathsf{S}^1 \to \mathsf{BFix}_{|\mathbb{F}_p} := \mathsf{BFix} \times_{\mathsf{Spec}(\mathbb{Z}_{(p)})} \mathsf{Spec}(\mathbb{F}_p) \tag{32}$$

are affinizations of S^1, respectively, over $\mathbb{Q}$ and $\mathbb{F}_p$.

For the moment let us describe the targets of the maps (31) and (32). By the Remark **2.1.2**, we already know that $\mathsf{BFix}_{|\mathbb{Q}} \simeq \mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathbb{Q}}$. It remains to work over $\mathbb{F}_p$:

[(§)] Can easily be checked using $\mathrm{Ghost}(1, 0, ...) = (1, 1, 1, ..)$

Lemma 3.3.3. *There is an an equivalence*

$$(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}_{|\mathbb{F}_p} \simeq \mathsf{B}\mathbb{Z}_p \tag{33}$$

where $\mathsf{B}\mathbb{Z}_p$ *is the classifying stack of the proconstant group scheme with values in the* p*-adic integers.*

Proof. As discussed in Guide **2.1.1**, for an $\mathbb{F}_p$-algebra A the map Frob_p on $\mathsf{W}_{p^\infty}(\mathrm{A})$ takes the form

$$\mathrm{Frob}_p(\lambda_1, \lambda_p, ..\lambda_{p^k}, ...) = (\lambda_1^p, \lambda_p^p, ...\lambda_{p^k}^p, ...)$$

so that levelwise it coincides with the standard Frobenius of A. Now, for each n we have Artin-Schreier-Witt exact sequences [Ill79, Proposition 3.28]

$$0 \longrightarrow (\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathsf{W}^{(m)}_{p^\infty|_{\mathbb{F}_p}} \xrightarrow{\mathrm{Frob}_p - \mathsf{id}} \mathsf{W}^{(m)}_{p^\infty|_{\mathbb{F}_p}} \longrightarrow 0 \tag{34}$$

where we consider $(\mathbb{Z}/p^n\mathbb{Z})$ as the constant-valued group scheme over $\mathbb{F}_p$. By passing to the limit over n we obtain exact sequences

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathsf{W}_{p^\infty|_{\mathbb{F}_p}} \xrightarrow{\mathrm{Frob}_p - \mathsf{id}} \mathsf{W}_{p^\infty|_{\mathbb{F}_p}} \longrightarrow 0 \tag{35}$$

from where we can conclude the identification $\mathsf{Fix}_{|\mathbb{F}_p} \simeq \mathbb{Z}_p$. □

The following is a key computation:

Proposition 3.3.4. [Toe06, Corollaire 2.5.3 and Lemma 2.5.2] *The affinization of* S^1 *over* $\mathbb{Q}$ *is* $\mathsf{B}\mathbb{G}_\mathsf{a}$ *and over* $\mathbb{F}_p$ *is* $\mathsf{B}\mathbb{Z}_p$ *.*

Let us now work our local criterion for affinization over $\mathbb{Z}_{(p)}$. We start with a simple remark:

Remark 3.3.5. Let M be an object in $\mathsf{Mod}_{\mathbb{Z}_{(p)}}(\mathsf{Sp})$. Suppose that both base changes $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ and $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ are zero. Then $M \simeq 0$. Indeed, as $\mathbb{Q}$ is obtained from $\mathbb{Z}_{(p)}$ by inverting p, $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ is obtain via the filtered colimit of the diagram given by multiplication by p

$$... \xrightarrow{.p} M \xrightarrow{.p} M \xrightarrow{.p} M \xrightarrow{.p} ... \tag{36}$$

At the same time, we know that $\mathbb{F}_p$ is obtained from $\mathbb{Z}_{(p)}$ via an exact sequence

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{.p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{F}_p \longrightarrow 0$$

In particular, we have a cofiber-fiber sequence

$$\begin{tikzcd} M \simeq M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)} \arrow[r, "\cdot p"] \arrow[d] & M \simeq M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)} \arrow[d] \\ 0 \arrow[r] & M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{tikzcd} \tag{37}$$

It follows that $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq 0$ if and only if the multiplication by p is an equivalence of M. In that case, the colimit of the diagram (36), meaning $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$, is equivalent to M. But the assumption $M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq 0$ concludes that $M \simeq 0$.

In particular, given $f : E \to F$ a morphism of chain complexes over $\mathbb{Z}_{(p)}$, if the two base changes to $\mathbb{Q}$ and $\mathbb{F}_p$ are equivalences, then so is f.

Lemma 3.3.6. *Consider the pullback diagrams:*

$$\begin{tikzcd} \mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \simeq \mathsf{BFix}_{|\mathbb{Q}} \arrow[r, "J"] \arrow[d, "f_{\mathbb{Q}}"] & \mathsf{BFix} \arrow[d, "f"] & \mathsf{BFix}_{|\mathbb{F}_p} \simeq \mathsf{B}\mathbb{Z}_p \arrow[l, "I"'] \arrow[d, "f_p"] \\ \mathsf{Spec}(\mathbb{Q}) \arrow[r, "j"] & \mathsf{Spec}(\mathbb{Z}_{(p)}) & \mathsf{Spec}(\mathbb{F}_p) \arrow[l, "i"'] \end{tikzcd}$$

Then we have equivalences of cosimplicial commutative k-algebras

$$\mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathsf{C}^*_\Delta(\mathsf{BFix}_{|\mathbb{Q}}, \mathscr{O}) \quad \textit{and} \quad \mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq \mathsf{C}^*_\Delta(\mathsf{BFix}_{|\mathbb{F}_p}, \mathscr{O}) \tag{38}$$

Proof. To show this lemma one could evoke directly the two cases in the Remark **3.2.5** and Remark **2.2.14**: one deals with $\mathbb{Q}$ the other with the lci closed immersion i.

We shall unfold the argument in more detail. Let us start with the rational equivalence. We write $A^\bullet$ for the co-simplicial object

$$[n] \mapsto \mathsf{C}^*(\mathsf{Fix}, \mathscr{O})^{\otimes n}$$

which is a diagram of discrete and flat modules over $\mathbb{Z}_{(p)}$ with flat transition maps

$$A^\bullet : \mathsf{N}(\Delta) \to \mathsf{Mod}^{\leq 0}_{\mathbb{Z}_{(p)}}$$

Let $\mathrm{Tot}(A^\bullet)$ denote the totalization of this co-simplicial object which we can read as the homotopy limit $\mathsf{lim}_\Delta A^\bullet$. The fact that the cosimplicial object is levelwise discrete implies that the associated spectral sequence is in the third quadrant and that the homotopy limit is also in $\mathsf{Mod}^{\leq 0}_{\mathbb{Z}_{(p)}}$ - use the dual of [Lur17, 1.2.4.5]. In particular, $\mathsf{lim}_\Delta A^\bullet$

satisfies the homological bounded condition of the Remark **3.2.5**-(ii). The claim in the lemma is that the comparison map

$$(\mathsf{lim}_\Delta A^\bullet) \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q} \to \mathsf{lim}_\Delta (A^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q})$$

is an equivalence. Therefore, it is enough to show that for every $n \geq 0$ the map induces an isomorphism on cohomology groups

$$\mathrm{H}^n((\mathsf{lim}_\Delta A^\bullet) \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q}) \to \mathrm{H}^n(\mathsf{lim}_\Delta (A^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q}))$$

For the l.h.s. we have a chain of isomorphisms

$$\mathrm{H}^n((\mathsf{lim}_\Delta A^\bullet) \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q}) \simeq \mathrm{H}^n(\mathsf{lim}_\Delta A^\bullet) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathrm{H}^n(\mathsf{lim}_{\Delta_{\leq n+1}} A^\bullet) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$$

the first because $\mathbb{Q}$ is discrete flat over $\mathbb{Z}_{(p)}$; the second follows from [Lur17, 1.2.4.5(5)].

For the r.h.s, we have

$$\mathrm{H}^n(\mathsf{lim}_\Delta (A^\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}} \mathbb{Q})) \simeq \mathrm{H}^n(\mathsf{lim}_{\Delta_{\leq n+1}} (A^\bullet \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q})) \simeq \mathrm{H}^n([\mathsf{lim}_{\Delta_{\leq n+1}} A^\bullet] \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q})$$

where first we used again [Lur17, 1.2.4.5(5)] and in the second isomorphism we used the fact that since the homotopy limit is now finite, it commutes with derived tensor products. Finally, the comparison between the r.h.s and the l.h.s follows from the isomorphism

$$\mathrm{H}^n(\mathsf{lim}_{\Delta_{\leq n+1}} A^\bullet) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathrm{H}^n([\mathsf{lim}_{\Delta_{\leq n+1}} A^\bullet] \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q})$$

which holds because $\mathbb{Q}$ is discrete and flat over $\mathbb{Z}_{(p)}$.

Let us now show the second equivalence, over $\mathbb{F}_p$. The Lemma **2.3.6** gives us a short exact sequence of fpqc sheaves of groups

$$0 \longrightarrow \mathsf{Fix} \longrightarrow \mathsf{W}_{p^\infty} \xrightarrow{\mathrm{Frob}_p - \mathsf{id}} \mathsf{W}_{p^\infty} \longrightarrow 0$$

where the map $\mathrm{Frob}_p - \mathsf{id}$ is flat. This implies that Fix is a flat group scheme over $\mathsf{Spec}(\mathbb{Z}_{(p)})$ and that the square

$$\begin{tikzcd} \mathsf{Fix} \arrow[r] \arrow[d] & \mathsf{W}_{p^\infty} \arrow[d, "\mathrm{Frob}_p - \mathsf{id}"] \\ \mathsf{Spec}(\mathbb{Z}_{(p)}) \arrow[r, "0"] & \mathsf{W}_{p^\infty} \end{tikzcd} \tag{39}$$

is actually a derived fiber product and so the diagram is

$$\begin{tikzcd} \mathsf{Fix}_{|\mathbb{F}_p} \arrow[r] \arrow[d] & \mathsf{Fix} \arrow[d] \\ \mathsf{Spec}(\mathbb{F}_p) \arrow[r] & \mathsf{Spec}(\mathbb{Z}_{(p)}) \end{tikzcd} \tag{40}$$

The derived base change [Lur18, 2.5.3.3, 2.5.4.5] formula applied to the last square tells us that

$$\mathsf{C}^*_\Delta(\mathsf{Fix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq \mathsf{C}^*_\Delta(\mathsf{Fix}_{|\mathbb{F}_p}, \mathscr{O}) \tag{41}$$

where $\mathsf{C}^*_\Delta(\mathsf{Fix}, \mathscr{O})$ is the Hopf-algebra of functions on the affine group scheme Fix. To show that formula (41) implies formula (38) over $\mathbb{F}_p$ we use the description of the classifying stack BFix as the geometric realization of the simplicial object

$$\cdots \ \mathsf{Fix} \times \mathsf{Fix} \ \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \ \mathsf{Fix} \ \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \ \mathsf{Spec}(\mathbb{Z}_{(p)})$$

which exhibits

$$\mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \simeq \lim_{[n]\in\Delta} \mathsf{C}^*_\Delta(\mathsf{Fix}, \mathscr{O})^{\otimes n} \tag{42}$$

Here, because Fix is affine, the Kunneth formulas for the cohomology of its cartesian powers are automatic. The same argument also tells us that

$$\mathsf{C}^*_\Delta(\mathsf{BFix}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \lim_{[n]\in\Delta} \mathsf{C}^*_\Delta(\mathsf{Fix}_{|\mathbb{F}_p}, \mathscr{O})^{\otimes n} \tag{43}$$

But now we know that, as in the Remark **3.3.5**, $\mathsf{C}^*(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ is the cofiber of multiplication by p

$$\begin{tikzcd} \mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \arrow[r, "\cdot p"] \arrow[d] & \mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \arrow[d] \\ 0 \arrow[r] & \mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{tikzcd} \tag{44}$$

As multiplication by p is actually happening levelwise in (42), we deduce that the square (44) is obtained from the squares

$$\begin{tikzcd} \mathsf{C}^*_\Delta(\mathsf{Fix},\mathscr{O})^{\otimes_n} \arrow[r, "\cdot p"] \arrow[d] & \mathsf{C}^*_\Delta(\mathsf{Fix},\mathscr{O})^{\otimes_n} \arrow[d] \\ 0 \arrow[r] & \mathsf{C}^*_\Delta(\mathsf{Fix},\mathscr{O})^{\otimes_n} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{tikzcd} \tag{45}$$

by passing to the limit in Δ. Now, formula (38) over $\mathbb{F}_p$ follows from the comparison (41) applied to each entry of the cartesian square (45), upon passing to the limit and using formula (43).

□

In fact we can show something more general than Lemma **3.3.6** that will be useful to us later:

Proposition 3.3.7. *The structure map* $\mathsf{BFix} \to \mathsf{Spec}\,\mathbb{Z}_{(p)}$ *is of finite cohomological dimension. In particular, by* [HLP14, A.1.5, A.1.6,A.1.9] *(see also* [Lur18, 9.1.5.6]*), it verifies base-change against any map* $Y \to \mathsf{Spec}\,\mathbb{Z}_{(p)}$.

The Proposition **3.3.7** follows from the following more general analysis:

Lemma 3.3.8. *Let* G *be an affine group scheme over* $\mathbb{Z}_{(p)}$. *Assume that* G *is flat and that both base change maps*

$$f_{\mathbb{Q}} : \mathsf{B}G_{|\mathbb{Q}} \to \mathsf{Spec}\,\mathbb{Q} \quad \textit{and} \quad f_{\mathbb{F}_p} : \mathsf{B}G_{|\mathbb{F}_p} \to \mathsf{Spec}\,\mathbb{F}_p$$

are of finite cohomological dimension d. *Then* $f : \mathsf{B}G \to \mathsf{Spec}\mathbb{Z}_{(p)}$ *is of finite cohomological dimension* $d+1$.

Proof. To check that f is of finite cohomological dimension it is enough (see [HLP14, A.1.4]) to see that f_* sends $\mathsf{QCoh}(\mathsf{B}G)^\heartsuit$ to $\mathsf{Mod}^{\geq -d}_{\mathbb{Z}_{(p)}}$ for some $d > 0$. Let us now consider the pullback squares

$$\begin{tikzcd} \mathsf{B}G_{|\mathbb{Q}} \arrow[r, "J"] \arrow[d, "f_{\mathbb{Q}}"] & \mathsf{B}G \arrow[d, "f"] & \mathsf{B}G|_{\mathbb{F}_p} \arrow[l, "I"'] \arrow[d, "f_p"] \\ \mathsf{Spec}(\mathbb{Q}) \arrow[r, "j"] & \mathsf{Spec}(\mathbb{Z}_{(p)}) & \mathsf{Spec}(\mathbb{F}_p) \arrow[l, "i"'] \end{tikzcd} \tag{46}$$

Let $M \in \mathsf{QCoh}(\mathsf{B}G)^\heartsuit$. In this case we can use the cases in [HLP14, A.1.3]:

- one to conclude that tbe Beck-Chevalley transformation

$$i^* f_* M \to (f_p)_* I^* M$$

is an equivalence since i is an lci closed immersion and therefore finite of finite Tor-dimension.

- the other to conclude that the Beck-Chevalley transformation

$$j^* f_* M \to (f_{\mathbb{Q}})_* J^* M$$

is also an equivalence since M is in the heart and therefore homologically bounded above.

Since pullbacks are right t-exact, it follows that

$$I^*(M) \in \mathsf{QCoh}(\mathsf{B}G_{|_{\mathbb{F}_p}})_{\geq 0} \quad \text{and} \quad J^*(M) \in \mathsf{QCoh}(\mathsf{B}G_{|_{\mathbb{Q}}})_{\geq 0}$$

Now we use our assumption that both $\mathsf{B}G_{|_{\mathbb{Q}}}$ and $\mathsf{B}G_{|_{\mathbb{F}_p}}$ are of finite cohomological dimension d, combined with [HLP14, A.1.6], to deduce that

$$(f_p)_* I^*(M) \in \mathsf{Mod}_{\mathbb{F}_p}^{\geq -d} \quad \text{and} \quad (f_{\mathbb{Q}})_* J^*(M) \in \mathsf{Mod}_{\mathbb{Q}}^{\geq -d}$$

Our final goal is to show that $f_* M$ is in $\mathsf{Mod}_{\mathbb{Z}_{(p)}}^{\geq -(d+1)}$. Using the fact that the Beck-Chevalley transformations above are equivalences, we are therefore reduced to prove the following statement: let $N \in \mathsf{Mod}_{\mathbb{Z}_{(p)}}$ and assume that

$$i^*(N) \in \mathsf{Mod}_{\mathbb{F}_p}^{\geq -d} \quad \text{and} \quad j^*(N) \in \mathsf{Mod}_{\mathbb{Q}}^{\geq -d}$$

Then $N \in \mathsf{Mod}_{\mathbb{Z}_{(p)}}^{\geq -(d+1)}$, ie, $\pi_i(N) = 0$ for $i < -(d+1)$ (homological notation). To see this we use the cofiber sequence Equation (37)

$$\begin{tikzcd} N \arrow[r, "\cdot p"] \arrow[d] & M \arrow[d] \\ 0 \arrow[r] & N \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \end{tikzcd} \tag{47}$$

and the induced long exact sequence

$$\cdots \to \pi_i(N) \to \pi_i(N) \to \pi_i(N \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p) \to \pi_{i-1}(N) \to \cdots \tag{48}$$

But by assumption we have $\pi_i(N \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p) = 0$ for $i < -d$. In particular, for $i \leq -d-2$ the long exact sequence (48) gives isomorphisms

$$\pi_i(N) \xrightarrow[\sim]{.p} \pi_i(N) \tag{49}$$

Finally, using the description $\mathbb{Q} = \mathbb{Z}_{(p)}[p^{-1}]$ and the fact that this localization is flat over $\mathbb{Z}_{(p)}$, we find

$$\pi_i(N) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \pi_i(N \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}) \tag{50}$$

for all i and by assumption we have

$$\pi_i(N) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq 0 \tag{51}$$

for $i < -d$. At the same time, for each i we can write

$$\pi_i(N) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \xrightarrow{\sim} \mathsf{colim}\,(\pi_i(N) \xrightarrow[.p]{} \pi_i(N) \xrightarrow[.p]{} \cdots) \tag{52}$$

So that for $i \leq -d-2$, the combination of (49) and (51) gives

$$0 \simeq \pi_i(N) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \xrightarrow{\sim} \mathsf{colim}\,(\pi_i(N) \xrightarrow[.p]{\sim} \pi_i(N) \xrightarrow[.p]{\sim} \cdots) \xrightarrow{\sim} \pi_i(N), \tag{53}$$

concluding the proof.

□

*Proof of Proposition **3.3.7**.* By Lemma **3.3.8**, to prove the statement in Proposition **3.3.7**, we only need to show that both $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \simeq \mathsf{BFix} \times_{\mathsf{Spec}\,\mathbb{Z}_{(p)}} \mathsf{Spec}\,\mathbb{Q} \to \mathsf{Spec}\,\mathbb{Q}$ and $\mathsf{B}\mathbb{Z}_p \simeq \mathsf{BFix} \times_{\mathsf{Spec}\,\mathbb{Z}_{(p)}} \mathsf{Spec}\,\mathbb{F}_p \to \mathsf{Spec}\,\mathbb{F}_p$ are maps of finite cohomological dimension. For $\mathsf{B}\mathbb{G}_{\mathsf{a}}$ over a field of characteristic zero, this is well-known [Lur18, 9.1.5.4]. More generally, we can use the fact that if H is an affine group scheme over a field k and M is an object in the heart of the category of $\mathscr{O}(H)$-comodules, then

$$\mathsf{H}^i(\mathsf{B}H, \mathrm{M}) \simeq \mathrm{Map}_{\mathsf{QCoh}(\mathsf{B}H)}(\mathscr{O}, \mathrm{M}[i]) \simeq \mathrm{Map}_{\mathsf{St}_k/\mathsf{B}H}(\mathsf{B}H, \mathsf{K}(\mathrm{M}, i)) \simeq \mathrm{Ext}^i_{H-\mathrm{rep}^\heartsuit}(k, \mathrm{M}) \tag{54}$$

where the Ext-groups are computed in the classic abelian category of $\mathscr{O}(H)$-comodules. See for instance [Toe06, Lemme 1.5.1]. We can now use this to show that $\mathsf{B}\mathbb{Z}_p$ is of cohomological dimension 1 over $\mathbb{F}_p$. See [Ser79, Chapter XIII, §1, Corollary] and [Ser94, §3.2, Example]. □

Remark 3.3.9. For a fixed discrete commutative ring R, the $\mathsf{E}_\infty^\otimes$ (resp. cosimplicial) algebra of singular cochains $\mathsf{C}^*(\mathsf{S}^1, R)$ (resp. $\mathsf{C}^*_\Delta(\mathsf{S}^1, R)$) can be defined as the cotensor

R^{S^1} in the ∞-category $\mathsf{CAlg}(\mathsf{Mod}_R)$ (resp. coSCR_R) which can be explicitely described as the limit of the constant diagram with value R, $\mathsf{lim}_{\mathsf{S}^1} R$. Because S^1 has a finite model as a simplicial set, this limit is finite. Moreover, these two constructions coincide under the functor θ^{ccn}. It follows that for any map of rings $R \to R'$, the derived base change $\mathsf{C}^*_\Delta(\mathsf{S}^1, R) \otimes_R R' \to \mathsf{C}^*_\Delta(\mathsf{S}^1, R')$ is an equivalence of algebras. In particular, we have equivalences

$$\mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{F}_p) \quad \text{and} \quad \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{Z}_{(p)}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{Q})$$

We are now ready to prove our local criterion for affinization:

Lemma 3.3.10. *If the two maps (31) and (32) are affinizations of* S^1*, respetively over* $\mathbb{Q}$ *and* $\mathbb{F}_p$*, then the map (29) is an affinization over* $\mathbb{Z}_{(p)}$*.*

Proof. The combination of the Remark **3.3.9** and the Lemma **3.3.6** with [Toe06, Corollaire 2.3.3] tells us that the statement in the lemma is equivalent to the following: to deduce that the map (30) is an equivalence, it is enough to check that both maps

$$\mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \to \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{Q}) \quad \text{and} \quad \mathsf{C}^*_\Delta(\mathsf{BFix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \to \mathsf{C}^*_\Delta(\mathsf{S}^1, \mathbb{F}_p) \tag{55}$$

are equivalences. Formulated this way, the lemma is immediate from the Remark **3.3.5**. □

This concludes the proof of Proposition **3.3.2**.

Remark 3.3.11. The results of this section, combined with the Remark **3.2.8**, show that, more generally, the stacks $\mathsf{K}(\mathsf{Fix}, n)$ are the affinization of $\mathsf{K}(\mathbb{Z}, n)$ over $\mathbb{Z}_{(p)}$. This was left open in [Toe06].

3.4. **The associated graded of $\mathsf{S}^1_{\mathsf{Fil}}$.** Our next order of business is to prove Theorem **1.2.1**-(ii) concerning the associated graded $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} \simeq \mathsf{BKer}$. As in the previous section, we reduce the problem to computations over $\mathbb{Q}$ and $\mathbb{F}_p$.

Lemma 3.4.1. *We have canonical equivalences of commutative cosimplicial algebras*

$$\mathsf{C}^*_\Delta(\mathsf{BKer}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq \mathsf{C}^*_\Delta(\mathsf{BKer}_{|\mathbb{Q}}, \mathscr{O}) \quad \textit{and} \quad \mathsf{C}^*_\Delta(\mathsf{BKer}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq \mathsf{C}^*_\Delta(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O}) \tag{56}$$

Proof. As in the proof of Lemma **3.3.6**, we use base change together with the observation that the exact sequence

$$0 \longrightarrow \mathsf{Ker} \longrightarrow \mathsf{W}_{p^\infty} \xrightarrow{\mathrm{Frob}_p} \mathsf{W}_{p^\infty} \longrightarrow 0$$

exhibits Ker as a flat group scheme over $\mathbb{Z}_{(p)}$. From here the proof goes as in Lemma **3.3.6**. □

Construction 3.4.2. Consider the composition

$$\mathsf{Ker} \subseteq \mathsf{W}_{p^\infty} \xrightarrow{\mathrm{Ghost}} \prod_{n\in S} \mathbb{G}_{\mathsf{a}} \xrightarrow{\mathrm{proj}_{1\in S}} \mathbb{G}_{\mathsf{a}} \tag{57}$$

and the induced map

$$u : \mathsf{BKer} \to \mathsf{B}\mathbb{G}_{\mathsf{a}} \tag{58}$$

By definition of $\mathsf{B}\mathbb{G}_{\mathsf{a}}$, the map u (58) corresponds to an element $u \in \mathsf{H}^1(\mathsf{C}^*(\mathsf{BKer}, \mathscr{O}))$, $u : \mathbb{Z}_{(p)}[-1] \to \mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$. One can check using explicit formulas for the Ghost map (see Guide **2.1.1**) that the composition (57) is compatible with the $\mathbb{G}_{\mathrm{m}}$-actions, where on the l.h.s we have the action of the Construction **2.3.1** and Remark **2.3.5** and on the r.h.s we have the standard $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{G}_{\mathsf{a}}$. In particular, (58) is $\mathbb{G}_{\mathrm{m}}$-equivariant and the element u is realized as a map in $\mathsf{Mod}^{\mathbb{Z}-\mathsf{gr}}_{\mathbb{Z}_{(p)}}$; $\mathbb{Z}_{(p)}[-1]$ is concentrated in weight 1 by definition. At the same time we consider the canonical element $1 : \mathbb{Z}_{(p)} \to \mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$ in $\mathsf{H}^0(\mathsf{C}^*(\mathsf{BKer}, \mathscr{O}))$. Because the structure map $\mathsf{BKer} \to \mathsf{Spec}(\mathbb{Z}_{(p)})$ is $\mathbb{G}_{\mathrm{m}}$-equivariant for the trivial action on the target, 1 also defines a graded map, with $\mathbb{Z}_{(p)}$ sitting in weight 0.

The sum of the graded maps u and 1 give us a map

$$\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1] \to \mathsf{C}^*(\mathsf{BKer}, \mathscr{O}) \quad \text{in} \quad \mathsf{Mod}^{\mathbb{Z}-\mathsf{gr}}_{\mathbb{Z}_{(p)}} \tag{59}$$

This map becomes an equivalence after base change to $\mathbb{Q}$ (Remark **2.1.2**).

Proposition 3.4.3. *The map of graded complexes (59) is an equivalence after tensoring with $\mathbb{F}_p$. By the Lemma **3.4.1** and the Remark **3.3.5**, it is also an equivalence over $\mathbb{Z}_{(p)}$. In particular, the grading on the complex $\mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$ coincides with the cohomological grading.*

We will establish the proof of Proposition **3.4.3** by computing the underlying complex of global sections of the structure sheaf of $\mathsf{BKer}_{|\mathbb{F}_p}$.

Remark 3.4.4. Notice that $\mathsf{Ker}_{|\mathbb{F}_p} = \mathsf{Spec}(\mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O}))$ is an affine abelian group scheme over $\mathbb{F}_p$. Therefore, both BKer and Ker are graded and because these are abelian groups we have an equivalence of cosimplicial graded Hopf algebras

$$\mathsf{C}^*_\Delta(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{lim}_\Delta\, \mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O})^{\otimes n} \tag{60}$$

In order to understand the underlying complex of $\mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O})$ we will characterize its category of representations as an Hopf algebra.

Construction 3.4.5. The group scheme $\mathsf{Ker}_{|\mathbb{F}_p}$ has a natural pro-group structure induced from the decompositon $\mathsf{W}_{p^\infty} \simeq \mathsf{lim}\, \mathsf{W}^{(m)}_{p^\infty}$ by defining $\mathsf{Ker}_{|\mathbb{F}_p}$ to be the kernel of the exact sequence

$$0 \longrightarrow \mathsf{Ker}^{(m)}_{|\mathbb{F}_p} \longrightarrow (\mathsf{W}^{(m)}_{p^\infty})_{|\mathbb{F}_p} \xrightarrow{\mathrm{Frob}_p} (\mathsf{W}^{(m)}_{p^\infty})_{|\mathbb{F}_p} \longrightarrow 0$$

We obtain $\mathsf{Ker}_{|\mathbb{F}_p} \simeq \mathsf{lim}\, \mathsf{Ker}^{(m)}_{|\mathbb{F}_p}$ and therefore a colimit of Hopf algebras

$$\mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{colim}_m\, \mathsf{C}^*_\Delta(\mathsf{Ker}^{(m)}_{|\mathbb{F}_p}, \mathscr{O}) \tag{61}$$

Definition 3.4.6. Let us denote by α_{p^m} the affine scheme over $\mathbb{F}_p$ given by $\mathsf{Spec}(\mathbb{F}_p[T]/(T^{p^m}))$. Its functor of points is given by $R \mapsto \{r \in R : r^{p^m} = 0\}$ classifying p^m-roots of zero. This is an abelian affine group scheme under the additive law over $\mathbb{F}_p$.

Lemma 3.4.7. *For each $m \geq 1$, the Cartier dual of the group scheme α_{p^m} is the algebraic group $\mathsf{Ker}^{(m)}_{|\mathbb{F}_p}$. In particular, it follows from Cartier duality that we have an equivalence of strict abelian 1-categories*

$$\mathsf{CoMod}^\heartsuit_{\mathsf{C}^*_\Delta(\mathsf{Ker}^{(m)}_{|\mathbb{F}_p}, \mathscr{O})} \simeq \mathsf{Mod}^\heartsuit_{\mathbb{F}_p[T]/(T^{p^m})} \tag{62}$$

Proof. This is [Oor66, II.10.3, Remark][¶] (see also [Dem86, III §4]). See also [Sul78] for the equivalence of categories. □

[¶] Formula $L^D_{m,n} = L_{n,m}$

Remark 3.4.8. If we forget the group structures, the affine scheme α_{p^m} is isomorphic to the underlying scheme of $\mu_{p^m} = \mathsf{Spec}(\mathbb{F}_p[U]/(U^{p^m}-1))$ of p^m-roots of unity under the change of coordinates $T \mapsto (U-1)$. This induces an equivalence of strict abelian 1-categories

$$\mathsf{Mod}^{\heartsuit}_{\mathbb{F}_p[U]/(U^{p^m}-1)} \simeq \mathsf{Mod}^{\heartsuit}_{\mathbb{F}_p[T]/(T^{p^m})} \tag{63}$$

Furthermore, we know that μ_{p^m} is Cartier dual to the group scheme $\mathbb{Z}/p^m\mathbb{Z}$ [Oor66, I.2.12, Lemma 2.15] and this gives us an equivalence of strict abelian 1-categories

$$\mathsf{Mod}^{\heartsuit}_{\mathbb{F}_p[T]/(T^{p^m})} \simeq \mathsf{CoMod}^{\heartsuit}_{\mathsf{C}^*_\Delta(\mathbb{Z}/p^m\mathbb{Z},\mathscr{O})} \tag{64}$$

Notice moreover that the isomorphisms of schemes $\mu_{p^m} \simeq \alpha_{p^m}$ are compatible for different m's under inclusions.

Proof of Proposition 3.4.3. Composing the equivalences (62), (63) and (64), the Barr-Beck theorem for strict abelian 1-categories gives us an equivalence of coalgebras

$$\mathsf{C}^*_\Delta(\mathsf{Ker}^{(m)}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{C}^*_\Delta(\mathbb{Z}/p^m\mathbb{Z}, \mathscr{O}) \tag{65}$$

Because Cartier duality is functorial, these equivalences are now compatible under the restriction maps and therefore, the equivalence extends to the filtered colimit of coalgebras

$$\mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{colim}_m\, \mathsf{C}^*_\Delta(\mathsf{Ker}^{(m)}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{colim}_m\, \mathsf{C}^*_\Delta(\mathbb{Z}/p^m\mathbb{Z}, \mathscr{O}) \tag{66}$$

But now the r.h.s is by definition the coalgebra of the group scheme $\mathbb{Z}_p$, and we get an equivalence of coalgebras

$$\mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{C}^*_\Delta(\mathbb{Z}_p, \mathscr{O}) \tag{67}$$

Finally, using the formula (60), we find an equivalence of cosimplicial coalgebras

$$\mathsf{C}^*_\Delta(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathsf{lim}_\Delta\, \mathsf{C}^*_\Delta(\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O})^{\otimes_n} \simeq \mathsf{lim}_\Delta\, \mathsf{C}^*_\Delta(\mathbb{Z}_p, \mathscr{O})^{\otimes_n} \simeq \mathsf{C}^*_\Delta(\mathsf{B}\mathbb{Z}_p, \mathscr{O}) \tag{68}$$

Here, the last equivalence uses the fact that $\mathbb{Z}_p$ is an affine group scheme over $\mathbb{F}_p$ [(*)].

After applying the co-dual Dold-Kan construction (see Notation **1.2.10**) and using the Proposition **3.3.4** we deduce an equivalence in $\mathsf{Mod}_{\mathbb{F}_p}$

[(*)] It is affine of infinite type, being a projective limit of finite constant groups schemes which are affine.

$$\mathsf{C}^*(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O}) \simeq \mathbb{F}_p \oplus \mathbb{F}_p[-1] \tag{69}$$

which one can check is implemented by the underlying map of (59).

□

As in Proposition **3.3.7** we have

Lemma 3.4.9. *The structure map* $\mathsf{BKer} \to \mathsf{Spec}\,\mathbb{Z}_{(p)}$ *is of finite cohomology dimension. In particular, by* [HLP14, B.15, B.16]*, it verifies base-change against any map* $Y \to \mathsf{Spec}\,\mathbb{Z}_{(p)}$.

Proof. As in the proof of the Proposition **3.3.7**, it is now enough to argue that both $\mathsf{BKer}_{|\mathbb{F}_p} \to \mathsf{Spec}\,\mathbb{F}_p$ and $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathbb{Q}} \to \mathsf{Spec}\,\mathbb{Q}$ are of finite cohomological dimension. The case of $\mathsf{B}\mathbb{G}_{\mathsf{a}\,\mathbb{Q}}$ has already been discussed. It remains to discuss $\mathsf{BKer}_{|\mathbb{F}_p}$. The case of $\mathsf{Ker}_{|\mathbb{F}_p}$ follows from the equivalence of coalgebras (67) which implies that the categories of comodules are equivalent. □

The combination of Proposition **3.3.7** and Lemma **3.4.9** also allow us to conclude that

Corollary 3.4.10. *The structure map* $\mathsf{S}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ *is of finite cohomolodical dimension. In particular it verifies the base change formula against any map.*

Proof. The argument boils down to an open-closed complement reduction as in the proof of Proposition **3.3.7**. Namely, we argue that to show that $\mathsf{S}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ is of finite cohomological dimension, it is enough to have both fibers

$$\begin{tikzcd}
\mathsf{BFix} \arrow[r] \arrow[d] & \mathsf{S}^1_{\mathsf{Fil}} \arrow[d] & \mathsf{BKer}/\mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[l] \arrow[d] \\
\mathsf{Spec}\,\mathbb{Z}_{(p)} \arrow[r, "j"] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[l, "i"']
\end{tikzcd}$$

of finite cohomological dimension (which we now know to be true thanks to Proposition **3.3.7** and Lemma **3.4.9**).

To confirm the claim, we use the fact that the t-structure on $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, is determined by its flat atlas $\mathbb{A}^1 \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ as discussed in the Notation **1.2.12**-c). Therefore, by flat base change, we are reduced to discuss the same situation over the open-closed pair obtained by pullback

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}} \arrow[r, "j"] \arrow[d] & \mathbb{A}^1 \arrow[d] & 0 \arrow[l, "i"'] \arrow[d] \\
{[\mathbb{G}_{\mathrm{m}}/\mathbb{G}_{\mathrm{m}}]} \simeq \mathsf{Spec}\,\mathbb{Z}_{(p)} \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[l]
\end{tikzcd}$$

But here, the proof runs exactly as for the pair $\mathsf{Spec}\,\mathbb{Q} \subseteq \mathsf{Spec}\,\mathbb{Z}_{(p)} \supseteq \mathsf{Spec}\,\mathbb{F}_p$ in the proof Proposition **3.3.7** since the situation amounts to the inversion of the coordinate t in $\mathbb{A}^1$ (for the open immesion) and the quotient by t (for the closed).

□

We now discuss both the $\mathsf{E}_\infty^\otimes$ and cosimplicial algebra structure on $\mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$. We start by the $\mathsf{E}_\infty^\otimes$-structure:

Lemma 3.4.11. *Let* $\mathrm{M} := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1] \in \mathsf{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}-\mathsf{gr}}$ *denote the graded complex of the Construction* ***3.4.2****. Then, the space of* $\mathsf{E}_\infty^\otimes$*-algebra structures on* M *compatible with the grading is equivalent to the set of classical commutative graded algebra structures on its cohomology* $\mathsf{H}^*(\mathrm{M})$*. In particular, it is homotopically discrete.*

Proof. The data of an $\mathsf{E}_\infty^\otimes$-algebra structure on a given object $\mathrm{M} \in \mathsf{Mod}_{\mathbb{Z}_{(p)}}$ is the data of a map of ∞-operads from $\mathsf{E}_\infty^\otimes$ to the ∞-operad of endomorphisms of M, for which we shall write $\mathrm{End}(\mathrm{M})^\otimes$. In our case, the object $\mathrm{M} := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ is endowed with a structure of object in $\mathsf{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}-\mathsf{gr}}$ of the Construction **3.4.2**, so in fact, we are interested in the ∞-operad of endomorphisms of M in this ∞-category $\mathrm{End}_{\mathsf{gr}}(\mathrm{M})^\otimes$. Its space of n-ary operations is given by the mapping space $\mathrm{Map}_{\mathsf{Mod}_{\mathbb{Z}_{(p)}}^{\mathbb{Z}-\mathsf{gr}}}(\mathrm{M}^{\otimes n}, \mathrm{M})$ where the powers $\mathrm{M}^{\otimes n}$ are taken with respect to the graded tensor product. It follows from the formula for the Day convolution (see Construction **2.2.1** and the references to [Lur15]) that for every $n \geq 1$ the piece of weight 0 in $\mathrm{M}^{\otimes n}$ is $\mathbb{Z}_{(p)}$ and the piece in weight 1 is given by $\bigoplus_{i=1}^n \mathbb{Z}_{(p)}[-1]$. In particular, we get

$$\mathrm{End}_{\mathsf{gr}}(\mathrm{M})^\otimes(n) \simeq \mathrm{Map}_{\mathsf{Mod}_{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \times \mathrm{Map}_{\mathsf{Mod}_{\mathbb{Z}_{(p)}}}(\bigoplus_{i=1}^n \mathbb{Z}_{(p)}[-1], \mathbb{Z}_{(p)}[-1]) \simeq \bigoplus_{i=0}^n \mathbb{Z}_{(p)}$$

which is a discrete space. This implies that the space of maps of ∞-operads

$$\mathrm{Map}_{\mathsf{Op}_\infty}(\mathsf{E}_\infty^\otimes, \mathrm{End}_{\mathsf{gr}}(\mathrm{M}))^\otimes$$

is discrete. But more is true: consider the cohomology $\mathsf{H}^*(\mathrm{M})$ as a (classical) graded $\mathbb{Z}_{(p)}$-vector space and $\mathrm{End}_{\mathsf{gr},\mathsf{cl}}(\mathsf{H}^*(\mathrm{M}))^{\otimes})$ its classical operad of (graded) endomorphisms. As H^* is lax monoidal, we get a map

$$\mathrm{Map}_{\mathsf{Op}_\infty}(\mathsf{E}_\infty^{\otimes}, \mathrm{End}_{\mathsf{gr}}(\mathrm{M})^{\otimes}) \to \mathrm{Map}_{\mathsf{Op}_\infty}(\mathsf{E}_\infty^{\otimes}, \mathrm{End}_{\mathsf{gr},\mathsf{cl}}(\mathsf{H}^*(\mathrm{M}))^{\otimes})$$

The computation above applied to the classical graded version shows that this map is actually an equivalence of spaces.

□

Remark 3.4.12. The argument in the proof of the Lemma **3.4.11** also shows that there exists a unique commutative $\mathsf{E}_\infty^{\otimes}$-algebra structure on the object $\mathrm{M} := \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ seen as an object of $\mathsf{Mod}^{\leq 0,\mathsf{gr}}_{\mathbb{Z}_{(p)}}$ with the symmetric monoidal structure of Notation **1.2.10**.

We now discuss the co-simplicial multiplicative structure.

Notation 3.4.13. Let us consider graded versions $\mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}}$ and $\mathsf{CAlg}^{\mathsf{ccn},\mathsf{gr}}_{\mathbb{Z}_{(p)}} := \mathsf{CAlg}(\mathsf{Mod}^{\leq 0,\mathsf{gr}}_{\mathbb{Z}_{(p)}})$ of respectively, cosimplicial and $\mathsf{E}_\infty^{\otimes}$-algebras and $\theta^{\mathsf{ccn}} : \mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}} \to \mathsf{CAlg}^{\mathsf{ccn},\mathsf{gr}}_{\mathbb{Z}_{(p)}}$ denote the graded version of the dual Dold-Kan construction (see Notation **1.2.10**).

Notation 3.4.14. We denote by $\mathbb{Z}_{(p)}[\eta]$ the trivial square zero extension structure on the complex $\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(p)}[-1]$ (η of degree 1 cohomological) as an object in $\mathsf{coSCR}_{\mathbb{Z}_{(p)}}$. We will use the same notation for its underlying $\mathsf{E}_\infty^{\otimes}$-algebra under the functor θ of Notation **1.2.10**.

Corollary 3.4.15 (of Lemma **3.4.11** and Remark **3.4.12**)**.** *The map (59) extends as an equivalence of graded $\mathsf{E}_\infty^{\otimes}$-algebras in $\mathsf{Mod}^{\leq 0}_{\mathbb{Z}_{(p)}}$ compatible with the augmentations to $\mathbb{Z}_{(p)}$:*

$$\mathbb{Z}_{(p)}[\eta] \to \mathsf{C}^*(\mathsf{BKer}, \mathscr{O}) \tag{70}$$

In particular, $\mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$ is formal as a graded $\mathsf{E}_\infty^{\otimes}$-algebra.

We now claim that (70) actually lifts via θ^{ccn} to an equivalence of graded cosimplicial commutative algebras. In order to prove this we use the limit formula (60) to reduce to

an argument about the discrete graded abelian group scheme Ker. We will need some preliminaries:

Construction 3.4.16. We consider cogroup objects in the categories $\mathsf{C} = \mathsf{coSCR}^{\mathsf{gr}}$ and $\mathsf{C} = \mathsf{CAlg}^{\mathsf{ccn,gr}}$ as in the Construction **3.1.3**. Namely,

$$\mathsf{coGr}(\mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}}) \underset{\mathsf{CoNerve}}{\overset{\mathsf{lim}_\Delta}{\rightleftarrows}} (\mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}})_{./\mathbb{Z}_{(p)}} \qquad \mathsf{coGr}(\mathsf{CAlg}^{\mathsf{ccn,gr}}_{\mathbb{Z}_{(p)}}) \underset{\mathsf{CoNerve}}{\overset{\mathsf{lim}_\Delta}{\rightleftarrows}} (\mathsf{CAlg}^{\mathsf{ccn,gr}}_{\mathbb{Z}_{(p)}})_{./\mathbb{Z}_{(p)}} \tag{71}$$

Both adjunctions commute with $\theta^{\mathsf{ccn}} : \mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}} \to \mathsf{CAlg}^{\mathsf{cnn,gr}}_{\mathbb{Z}_{(p)}}$ (that θ^{ccn} commutes with $\mathsf{CoNerve}$ follows from the fact that it commutes with tensor products) (Notation **1.2.10**):

$$\begin{tikzcd}
\mathsf{coGr}(\mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}}) \arrow[r, shift left, "\mathsf{lim}_\Delta"] \arrow[d, "\theta^{\mathsf{ccn}}"] & (\mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}})_{./\mathbb{Z}_{(p)}} \arrow[l, shift left, "\mathsf{CoNerve}"] \arrow[d, "\theta^{\mathsf{ccn}}"] \\
\mathsf{coGr}(\mathsf{CAlg}^{\mathsf{ccn,gr}}_{\mathbb{Z}_{(p)}}) \arrow[r, shift left, "\mathsf{lim}_\Delta"] & (\mathsf{CAlg}^{\mathsf{ccn,gr}}_{\mathbb{Z}_{(p)}})_{./\mathbb{Z}_{(p)}} \arrow[l, shift left, "\mathsf{CoNerve}"]
\end{tikzcd} \tag{72}$$

Our main computation is the following:

Theorem 3.4.17. *The equivalence of graded $\mathsf{E}^{\otimes}_{\infty}$-algebras of the Corollary **3.4.15** can be promoted to an equivalence of graded commutative cosimplicial algebras*

$$\mathbb{Z}_{(p)}[\eta] \simeq \mathsf{C}^*_\Delta(\mathsf{BKer}, \mathscr{O}) \tag{73}$$

In particular, as BKer *is an affine stack (Proposition **3.2.7**) we can write*

$$\mathsf{BKer} \simeq \mathsf{Spec}^\Delta(\mathbb{Z}_{(p)}[\eta]) \tag{74}$$

Proof. By construction, the cosimplicial object $\mathsf{C}^*_\Delta(\mathsf{Ker}, \mathscr{O})^\bullet$ in the formula (60) defines an object in $\mathsf{coGr}(\mathsf{coSCR}^{\mathsf{gr}})$ and in the terminology of the Construction **3.4.16** the equivalence (60) reads as an equivalence in $\mathsf{coSCR}^{\mathsf{gr}}_{./\mathbb{Z}_{(p)}}$:

$$\mathsf{C}^*_\Delta(\mathsf{BKer}, \mathscr{O}) \simeq \mathsf{lim}_\Delta[\mathsf{C}^*_\Delta(\mathsf{Ker}, \mathscr{O})^\bullet]$$

Because the stack BKer is affine (Proposition **3.2.7**) we know that $\mathsf{BKer} \simeq \mathsf{Spec}^\Delta(\mathsf{C}^*_\Delta(\mathsf{BKer}, \mathscr{O}))$. Now, using the fact Spec^Δ is a right adjoint, we deduce an equivalence in $\mathsf{coGr}(\mathsf{coSCR}^{\mathsf{gr}})$

$$\mathsf{C}^*_\Delta(\mathsf{Ker},\mathscr{O})^\bullet \simeq \mathsf{coNerve}[\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O})]$$

where we see $\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O})$ augmented over $\mathbb{Z}_{(p)}$ via the atlas. We deduce that

$$\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O}) \simeq \mathsf{lim}_\Delta \circ \ \mathsf{coNerve}\ [\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O})]$$

in $\mathsf{coSCR}^{\mathsf{gr}}_{./\mathbb{Z}_{(p)}}$.

The commutativity of the diagram (72) tells us that a similar formula holds for the graded coconnective $\mathsf{E}^{\otimes}_\infty$-version. Therefore, the equivalence of the Corollary **3.4.15** tells us that a similar formula holds for the graded coconnective $\mathsf{E}^{\otimes}_\infty$-version of $\mathbb{Z}_{(p)}[\eta]$, ie, an equivalence in $\mathsf{CAlg}^{\mathsf{ccn},\mathsf{gr}}_{./\mathbb{Z}_{(p)}}$

$$\theta^{\mathsf{ccn}}(\mathbb{Z}_{(p)}[\eta]) \simeq \mathsf{lim}_\Delta \circ \ \mathsf{coNerve}\ [\theta^{\mathsf{ccn}}(\mathbb{Z}_{(p)}[\eta])]$$

In particular, we have an equivalence in $\mathsf{coGr}(\mathsf{CAlg}^{\mathsf{ccn},\mathsf{gr}})$

$$\mathsf{coNerve}\ [\theta^{\mathsf{ccn}}((\mathbb{Z}_{(p)}[\eta])] \simeq \mathsf{coNerve}\ [\theta^{\mathsf{ccn}}((\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O}))] \simeq \theta^{\mathsf{ccn}}((\mathsf{C}^*_\Delta(\mathsf{Ker},\mathscr{O})^\bullet) \tag{75}$$

Finally, we remark that Ker is a classical affine scheme, so that its global sections are discrete in the co-simplicial direction. In particular, as the functor $\theta^{\mathsf{ccn}} : \mathsf{coSCR}^{\mathsf{gr}}_{\mathbb{Z}_{(p)}} \to \mathsf{CAlg}^{\mathsf{ccn},\mathsf{gr}}_{\mathbb{Z}_{(p)}}$ induces an equivalence on discrete algebras, the equivalence (75) lifts to an equivalence in $\mathsf{coGr}(\mathsf{coSCR}^{\mathsf{gr}})$

$$\mathsf{coNerve}\ [\mathbb{Z}_{(p)}[\eta]] \simeq \mathsf{coNerve}\ [\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O})] \tag{76}$$

and therefore an equivalence in $\mathsf{coSCR}^{\mathsf{gr}}_{./\mathbb{Z}_{(p)}}$

$$\mathbb{Z}_{(p)}[\eta] \simeq \mathsf{lim}_\Delta \circ \ \mathsf{coNerve}\ [\mathbb{Z}_{(p)}[\eta]] \simeq \mathsf{lim}_\Delta \circ \ \mathsf{coNerve}\ [\mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O})] \simeq \mathsf{C}^*_\Delta(\mathsf{BKer},\mathscr{O}) \tag{77}$$

□

Corollary 3.4.18. *The stack* $\mathsf{Spec}^\Delta(\mathbb{Z}_{(p)}[\eta])$ *admits a unique* $\mathsf{E}^{\otimes}_\infty$ *group structure compatible with the grading and the natural base point* $* \to \mathsf{Spec}^\Delta(\mathbb{Z}_{(p)}[\eta])$ *induced from the augmentation* $\mathsf{Spec}^\Delta(\mathbb{Z}_{(p)}[\eta]) \to \mathbb{Z}_{(p)}$ *. In particular the equivalence of (74) is compatible with the group structures.*

Proof. We already know that $\mathsf{Spec}^{\Delta}(\mathbb{Z}_{(p)}[\eta])$ is of the form BKer. The corollary thus follows from the observation that, for any sheaf of abelian groups A, over any Grothendieck site, the stack $\mathsf{B}A$ carries a unique $\mathsf{E}_{\infty}^{\otimes}$-group structure (up to an equivalence) compatible with its natural base point $* \to \mathsf{B}A$. Indeed, suppose that BA is endowed with a $\mathsf{E}_{\infty}^{\otimes}$-group structure with unit $* \to \mathsf{B}A$. The stack $\Omega_*\mathsf{B}A \simeq A$ comes equipped with an induced $\mathsf{E}_{\infty}^{\otimes}$-group structure which is now compatible with the abelian group structure given on A. By the Eckmann-Hilton argument we know that these two group structures on A must be equal. We have an adjunction morphism of of stacks $\mathsf{B}(\Omega_*\mathsf{B}A) \longrightarrow \mathsf{B}A$, which also compatible with $\mathsf{E}_{\infty}^{\otimes}$-group structures on both sides. This shows that any $\mathsf{E}_{\infty}^{\otimes}$-group structure on $\mathsf{B}A$ compatible with the natural base point is canonically equivalent to the standard $\mathsf{E}_{\infty}^{\otimes}$-group structure induced by the abelian group structure of A itself.

The corollary is then obtained by considering $A = \mathsf{Ker}$, as a sheaf over the big site of affine schemes over the stack $\mathsf{B}\mathbb{G}_{\mathrm{m}}$. □

3.5. **Quasi-coherent sheaves on the filtered circle.** To conclude this section we describe the categories of representations of Ker and Fix. More precisely, we understand how the filtered category $\mathsf{QCoh}(\mathsf{S}^1_{\mathsf{Fil}})$ endows a filtration on $\mathsf{QCoh}(\mathsf{BFix})$ with associated graded given by $\mathsf{QCoh}(\mathsf{BKer})$ and we conclude that as categories, this filtration in fact is split.

Remark 3.5.1. If follows as a consequence of Theorem **3.4.17** and Proposition **3.3.2** that the filtration on $\mathsf{C}^*(\mathsf{S}^1_{\mathsf{Fil}}, \mathscr{O})$ splits when regarded as an $\mathsf{E}_1^{\otimes}$-algebra. That same is not true when regarded as an $\mathsf{E}_{\infty}^{\otimes}$-algebra.

The main result of this section is the following:

Lemma 3.5.2. *Let G be a flat affine group scheme over $\mathbb{Z}_{(p)}$ and assume that:*

*(i) $\mathsf{B}G$ is of finite cohomological dimension (in the sense of the Lemma **3.3.8**);*

(ii) Both base changes $G_{|_{\mathbb{Q}}}$ and $G_{|_{\mathbb{F}_p}}$ are unipotent groups.

Then the category $\mathsf{QCoh}(\mathsf{B}G)$ is compactly generated by the tensor unit $\mathscr{O}_{\mathsf{B}G}$ and in particular, we have

$$\mathsf{QCoh}(\mathsf{B}G) \simeq \mathsf{Mod}_{\mathsf{C}^*(\mathsf{B}G,\mathscr{O})} \tag{78}$$

Proof. The equivalence of categories follows directly from compact generation by [Lur17, 7.1.2.1]. To prove the claim that the tensor unit is a compact generator we use [Lur18,

9.1.5.3 -(1)-(2)] to show that $\mathscr{O}_{\mathsf{B}G}$ is a compact object. To show that it is a generator we have to show that if $\mathrm{Map}_{\mathsf{QCoh}(\mathsf{B}G)}(\mathscr{O}, E) \simeq 0$ then E vanishes. If we denote by $f : \mathsf{B}G \to \mathsf{Spec}\mathbb{Z}_{(p)}$ the projection, this is equivalent to show that if $f_*(E)$ vanishes then E has to vanish. But via the Remark **3.3.5**, $f_*(E)$ vanishes if and only if both base changes $f_*(E) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p$ and $f_*(E) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}$ vanish. But since f is assumed to be of finite cohomological dimension, by [HLP14, A.1.5] (see also the Remark **2.2.14**-(iii)), we have

$$f_*(E) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \simeq (f_p)_* I^*(E) \simeq \mathbb{R}\,\mathrm{Hom}(\mathscr{O}_{\mathsf{B}G_{|\mathbb{F}_p}}, I^*(E))$$

and

$$f_*(E) \otimes_{\mathbb{Q}} \mathbb{F}_p \simeq (f_{\mathbb{Q}})_* J^*(E) \simeq \mathbb{R}\,\mathrm{Hom}(\mathscr{O}_{\mathsf{B}G_{|\mathbb{Q}}}, J^*(E))$$

(with the notations of (46)). But, since we are assuming the base changes to fields to be unipotent groups, this means by definition that the tensor units $\mathscr{O}_{\mathsf{B}G_{|\mathbb{F}_p}}$ and $\mathscr{O}_{\mathsf{B}G_{|\mathbb{Q}}}$ given by the trivial representations, are generators. This implies now that both $J^*(E)$ and $I^*(E)$ vanish. But now we can again use the Remark **3.3.5** to conclude that E vanishes. Indeed, it sufficies, if $e : \mathsf{Spec}\mathbb{Z}_{(p)} \to \mathsf{B}G$ denotes the atlas, and since the functor $e^* : \mathsf{QCoh}(\mathsf{B}G) \to \mathsf{Mod}_{\mathbb{Z}_{(p)}}$ is conservative, it suffices to use the commutativivity of the diagram

$$\begin{tikzcd}
\mathsf{Spec}(\mathbb{Q}) \arrow[r, "j"] \arrow[d] & \mathsf{Spec}\mathbb{Z}_{(p)} \arrow[d, "e"] & \mathsf{Spec}(\mathbb{F}_p \arrow[l, "i"'] \arrow[d] \\
\mathsf{B}G_{|\mathbb{Q}} \arrow[r, "J"] & \mathsf{B}G & \mathsf{B}G_{|\mathbb{F}_p} \arrow[l, "I"']
\end{tikzcd}$$

and the induced commutative diagram of pullbacks

$$\begin{tikzcd}
\mathsf{Mod}_{\mathbb{Q}} & \mathsf{Mod}_{\mathbb{Z}_{(p)}} \arrow[l, "j^*"'] \arrow[r, "i^*"] & \mathsf{Mod}_{\mathbb{F}_p} \\
\mathsf{QCoh}(\mathsf{B}G_{|\mathbb{Q}}) \arrow[u] & \mathsf{QCoh}(\mathsf{B}G) \arrow[l, "j^*"'] \arrow[r, "I^*"] \arrow[u, "e^*"] & \mathsf{QCoh}(\mathsf{B}G_{|\mathbb{F}_p}) \arrow[u]
\end{tikzcd}$$

□

Corollary 3.5.3. *We have canonical equivalences of ∞-categories*

$$\mathsf{QCoh}(\mathsf{BFix}) \simeq \mathsf{Mod}_{\mathsf{C}^*(\mathsf{BFix},\mathscr{O})} \tag{79}$$

$$\mathsf{QCoh}(\mathsf{BKer}) \simeq \mathsf{Mod}_{\mathsf{C}^*(\mathsf{BKer},\mathscr{O})} \tag{80}$$

*Moreover, since by the Remark **3.5.1**, $\mathsf{C}^*(\mathsf{BFix},\mathscr{O})$ and $\mathsf{C}^*(\mathsf{BKer},\mathscr{O})$ are equivalent as $\mathsf{E}_1^{\otimes}$-algebras we conclude*

$$\mathsf{QCoh}(\mathsf{BFix}) \simeq \mathsf{QCoh}(\mathsf{BKer}) \tag{81}$$

Proof. The equivalence (79) is a consequence of Lemma **3.5.2** and Proposition **3.3.7** for $G = \mathsf{Fix}$. The equivalence (80) is a consequence of Lemma **3.5.2** and Lemma **3.4.9** for $G = \mathsf{Ker}$. □

4. Representations of the Filtered Circle

Our goal in this section is to describe the category $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ of representations of our filtered $\mathsf{S}^1_{\mathsf{Fil}}$. For this purpose turn our attention to the classifying stack $\mathsf{BS}^1_{\mathsf{Fil}}$ (Construction **3.2.2**); note that by Proposition **3.2.7** this is an affine stack relatively to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

4.1. **The cohomology of $\mathsf{BS}^1_{\mathsf{Fil}}$.** By the Remark **2.2.14**, the cohomology ring $\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}},\mathscr{O})$ admits a natural structure as a filtered $\mathsf{E}_\infty^{\otimes}$-algebra.

As we shall see, the ∞-category of $\mathsf{S}^1_{\mathsf{Fil}}$-representations will coincide with that of mixed complexes; but this identification will not preserve the relevant symmetric monoidal structures. Our first evidence of this is the following

Proposition 4.1.1. *There is an equivalence of filtered $\mathsf{E}_1^{\otimes}$-algebras*

$$\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}},\mathscr{O}) \simeq \mathbb{Z}_{(p)}[u]$$

where u sits in degree 2 (cohomological) and has a split filtration induced by the canonical grading for which u is of weight -1.

The following construction is a key ingredient in the proof of Proposition **4.1.1**:

Construction 4.1.2. We denote by $\mathbb{Z}_{(p)}(-1)$, the copy of $\mathbb{Z}_{(p)}$ seen as a graded module pure of weight (-1). This is an invertible object with respect to the tensor product of graded objects (Construction **2.2.1**) and we denote by $\mathscr{O}(-1)$ the line bundle given by its pullback to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ along the map $q : [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$. We construct a morphism

$$u : \mathscr{O}(-1)[-2] \to \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}},\mathscr{O}) \tag{82}$$

in the ∞-category $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$, as follows: Consider again the composition $\mathsf{W}_{p^\infty} \to \mathbb{G}_{\mathsf{a}}$ of (57). As discussed in the Construction **3.4.2**, this is a map of groups compatible with the $\mathbb{G}_{\mathrm{m}}$-actions. This induces a map of filtered abelian group stacks

$$\mathsf{H}_{p^\infty} := [(\ker \mathscr{G}_p)/\mathbb{G}_{\mathrm{m}}] \to \mathsf{W}^{\mathsf{Fil}}_{p^\infty} \underbrace{:=}_{\textit{Construction } \mathbf{2.3.3}} [(\mathsf{W}_{p^\infty} \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}}] \to [(\mathbb{G}_{\mathsf{a}} \times \mathbb{A}^1)/\mathbb{G}_{\mathrm{m}}] =: \mathbb{G}_{\mathsf{a}}(1) \tag{83}$$

where $\mathbb{G}_{\mathsf{a}}(1)$ can also be described as $\mathbb{G}_{\mathsf{a}}(1) = \mathsf{Spec}(\mathsf{Sym}(\mathscr{O}(-1))$. Geometrically, $\mathbb{G}_{\mathsf{a}}(1)$ corresponds to the filtered group scheme with filtration induced by the geometric action pure weight 1 action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{G}_{\mathsf{a}}$ (see Construction **2.2.9**). We can now take B^2 to obtain a morphism of stacks $\mathsf{BS}^1_{\mathsf{Fil}} \to \mathsf{B}^2\mathbb{G}_{\mathsf{a}}(1)$. We define $u \in \mathsf{H}^2(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}(1))$ to be the resulting class induced by pullback in cohomology, incarnated as a map (82).

Since by definition $\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \simeq \pi_*(\mathscr{O})$ where $\pi : \mathsf{BS}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, by adjunction the map u can also read as a map

$$\pi^*(\mathscr{O}(-1))[-2] \to \mathscr{O} \tag{84}$$

in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$.

As $\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})$ may be viewed as a filtered $\mathsf{E}^{\otimes}_{\infty}$-algebra, and in particular a filtered $\mathsf{E}^{\otimes}_1$-algebra, there is a morphism of filtered $\mathsf{E}^{\otimes}_1$-algebras induced by the universal property

$$\mathbb{Z}_{(p)}[u] \to \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \quad \text{in} \quad \mathsf{Alg}_{\mathsf{E}^{\otimes}_1}(\mathsf{QCoh}([\mathbb{A}^1_{\mathbb{Z}_{(p)}}/\mathbb{G}_{\mathrm{m}\,\mathbb{Z}_{(p)}}])) \tag{85}$$

where $\mathbb{Z}_{(p)}[u]$ is the free $\mathsf{E}^{\otimes}_1$-algebra on $\mathscr{O}(-1)[-2]$.

Remark 4.1.3. Notice in particular that since the filtration splits for $\mathbb{Z}_{(p)}[u]$ as a filtered $\mathsf{E}^{\otimes}_1$-algebra (see Construction **2.2.15**), one has an equivalence of the underlying $\mathsf{E}^{\otimes}_1$-algebras

$$(\mathbb{Z}_{(p)}[u])^{\mathsf{und}} \simeq (\mathbb{Z}_{(p)}[u])^{\mathsf{ass-gr}} \tag{86}$$

where on the r.h.s we forget the grading.

*Proof of Proposition **4.1.1**:* We now show that the map of $\mathsf{E}_1^{\otimes}$-algebras constructed in (85) is an equivalence. As in the Lemma **2.3.6** (see also Remark **2.2.2**), it will be enough to show that (85) is an equivalence after base-change to the field-valued points

$$(0,\mathbb{Q}),(1,\mathbb{Q}),(0,\mathbb{F}_p),(1,\mathbb{F}_p).$$

It will then be enough to test the underlying maps of complexes, forgetting the grading and the algebra structures.

Let us first deal with the associated-graded, ie, the base change of (85) along 0. As a consequence of the base change formula (23) in the Remark **3.2.5** we obtain

$$\begin{tikzcd}
(\mathbb{Z}_{(p)}[u])^{\mathsf{ass-gr}} \arrow[r] \arrow[dr] & \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}},\mathscr{O})^{\mathsf{ass-gr}} \arrow[d, "\sim"', "(23)"] \\
& \mathsf{C}^*(\mathsf{B}^2\mathsf{Ker},\mathscr{O})
\end{tikzcd} \quad \text{in} \quad \mathsf{Alg}_{\mathsf{E}_1^{\otimes}}(\mathsf{QCoh}([\mathsf{B}\mathbb{G}_{\mathrm{m}\,\mathbb{Z}_{(p)}}])) \tag{87}$$

First we test the base change of (87) to $\mathbb{Q}$. In this case we get

$$\begin{tikzcd}
(\mathbb{Z}_{(p)}[u])^{\mathsf{ass-gr}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \arrow[r] \arrow[dd, "\mathbb{Z}_{(p)}[u]\text{ free filtered } \mathsf{E}_1^{\otimes}-\textit{algebra}"', "\sim"] & \mathsf{C}^*(\mathsf{B}^2\mathsf{Ker},\mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \arrow[d, "\sim"', "\text{as in Lemma } \mathbf{3.4.1} \text{ using diagonal res.}"] \\
& \mathsf{C}^*(\mathsf{B}^2\mathsf{Ker}_{\mathbb{Q}},\mathscr{O}) \arrow[d, "\sim"', "\text{Remark } \mathbf{2.1.2}"] \\
& \mathsf{C}^*(\mathsf{B}^2\mathbb{G}_{\mathsf{a}\,\mathbb{Q}},\mathscr{O}) \arrow[d, "\sim"', "\text{Proposition } \mathbf{3.3.4}"] \\
\mathbb{Q}[u]^{\mathrm{und}} \arrow[r] & \mathsf{C}^*(\mathsf{K}(\mathbb{Z},2),\mathbb{Q})
\end{tikzcd} \tag{88}$$

Finally, the bottom map obtained by clockwise composition around the diagram (88), is an equivalence: indeed, after passing to cohomology groups we get the isomorphism of commutative graded algebras

$$\mathbb{Q}[u] \to \mathsf{H}^*(\mathsf{K}(\mathbb{Z},2),\mathbb{Q}) \tag{89}$$

Now we test the base change of (87) to $\mathbb{F}_p$. We get

$$\begin{tikzcd}
(\mathbb{Z}_{(p)}[u])^{\mathsf{ass-gr}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \arrow[r] \arrow[d, "\mathbb{Z}_{(p)}[u] \text{ free filtered } \mathsf{E}_1^{\otimes}-algebra"', "\sim"] & \mathsf{C}^*(\mathsf{B}^2\mathsf{Ker}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p \arrow[d, "\sim"', "\text{as in Lemma } \mathbf{3.4.1} \text{ using diagonal res.}"] \\
\mathbb{F}_p[u]^{\mathrm{und}} \arrow[r] & \mathsf{C}^*(\mathsf{B}^2\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O})
\end{tikzcd} \tag{90}$$

and we have to explain why the bottom map obtained by clockwise composition in (90) is an equivalence. Thanks to (68), we know that $\mathsf{C}^*(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O})$ and $\mathsf{C}^*(\mathsf{B}\mathbb{Z}_p, \mathscr{O})$ are equivalent as coalgebras, and in particular, as complexes. Moreover, as both $\mathsf{BKer}_{|\mathbb{F}_p}$ and $\mathsf{B}\mathbb{Z}_p$ are affine stacks over $\mathbb{F}_p$ (resp. Proposition **3.2.7** and Proposition **3.3.4**), we deduce that

$$\mathsf{C}^*(\mathsf{BKer}_{|\mathbb{F}_p}^{\times^n}, \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{BKer}_{|\mathbb{F}_p}, \mathscr{O})^{\otimes^n} \qquad \text{and} \qquad \mathsf{C}^*(\mathsf{B}\mathbb{Z}_p^{\times^n}, \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{B}\mathbb{Z}_p, \mathscr{O})^{\otimes^n} \tag{91}$$

But then, by passing to the limit, we obtain

$$\mathsf{C}^*(\mathsf{B}^2\mathsf{Ker}_{|\mathbb{F}_p}, \mathscr{O}) = \mathsf{C}^*(\mathsf{K}(\mathsf{Ker}_{|\mathbb{F}_p}, 2), \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{K}(\mathbb{Z}_p, 2), \mathscr{O}) \tag{92}$$

Finally, we get

$$(\mathbb{F}_p[u])^{\mathsf{und}} = (\mathbb{F}_p[u])^{\mathsf{ass-gr}} \to \mathsf{C}^*(\mathsf{K}(\mathsf{Ker}_{|\mathbb{F}_p}, 2), \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{K}(\mathbb{Z}_p, 2), \mathscr{O}) \underbrace{\simeq}_{\text{Proposition } \mathbf{3.3.4}} \mathsf{C}^*(\mathsf{K}(\mathbb{Z}, 2), \mathbb{F}_p) \tag{93}$$

and it is now clear that (93) produces an isomorphism after passing to cohomology

$$\mathbb{F}_p[u] \simeq \mathsf{H}^*(\mathsf{K}(\mathbb{Z}, 2), \mathbb{F}_p) \tag{94}$$

Let us now deal with the underlying objects, ie, the base change of (85) along 1. The Beck-Chevalley transformation of (24) in the Remark **3.2.5** a priori only gives the possibility of a composition

$$\begin{tikzcd}
(\mathbb{Z}_{(p)}[u])^{\mathsf{und}} \arrow[r] \arrow[dr] & \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})^{\mathrm{und}} \arrow[d, "(24)"] \\
& \mathsf{C}^*((\mathsf{BS}^1_{\mathsf{Fil}})^{\mathrm{und}}, \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{B}^2\mathsf{Fix}, \mathscr{O})
\end{tikzcd} \quad \text{in} \quad \mathsf{Alg}_{\mathsf{E}_1^{\otimes}}(\mathsf{Mod}_{\mathbb{Z}_{(p)}}) \tag{95}$$

But, since the structure sheaf of $\mathsf{BS}^1_{\mathsf{Fil}}$ is in the heart of the t-structure (see Notation **1.2.12**), the map (24) is in fact an equivalence.

As before, first we test the base change of (95) to $\mathbb{Q}$. We get

$$\begin{tikzcd}
(\mathbb{Z}_{(p)}[u])^{\mathsf{und}} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \arrow[r] \arrow[dd, "\mathbb{Z}_{(p)}[u] \text{ free filtered } \mathsf{E}_1^{\otimes}-\textit{algebra}", "\sim"'] & \mathsf{C}^*(\mathsf{B}^2\mathsf{Fix}, \mathscr{O}) \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \arrow[d, "\sim", "\text{as in Lemma } \mathbf{3.3.6} \text{ using res. of (21)}"'] \\
& \mathsf{C}^*(\mathsf{B}^2\mathsf{Fix}_{\mathbb{Q}}, \mathscr{O}) \arrow[d, "\sim", "\text{Remark } \mathbf{2.1.2}"'] \\
& \mathsf{C}^*(\mathsf{B}^2\mathbb{G}_{\mathsf{a}\,\mathbb{Q}}, \mathscr{O}) \arrow[d, "\sim", "\text{Proposition } \mathbf{3.3.4}"'] \\
\mathbb{Q}[u]^{\mathrm{und}} \arrow[r] & \mathsf{C}^*(\mathsf{K}(\mathbb{Z}, 2), \mathbb{Q})
\end{tikzcd} \tag{96}$$

After passing to cohomology we recover the isomorphism in (89).

Finally, concerning the base change to $\mathbb{F}_p$, we get:

$$(\mathbb{F}_p[u])^{\mathsf{und}} \to \mathsf{C}^*(\mathsf{B}^2\mathsf{Fix}_{|\mathbb{F}_p}, \mathscr{O}) \underbrace{\simeq}_{(33)} \mathsf{C}^*(\mathsf{K}(\mathbb{Z}_p, 2), \mathscr{O}) \underbrace{\simeq}_{\text{Proposition } \mathbf{3.3.4}} \mathsf{C}^*(\mathsf{K}(\mathbb{Z}, 2), \mathbb{F}_p) \tag{97}$$

which induces recovers the isomorphism (94) after passing to cohomology.

□

4.2. **Representations of $\mathsf{S}^1_{\mathsf{Fil}}$.** Before we proceed with the ramifications of Proposition **4.1.1**, we recall in more detail the notion of mixed complexes.

Definition 4.2.1. Let $\Lambda := \mathbb{Z}_{(p)}[\epsilon] = \mathsf{H}_*(\mathsf{S}^1, \mathbb{Z}_{(p)})$ be the graded associative dg-algebra freely generated with ϵ in degree 1 (homological) and $\epsilon^2 = 0$. The dga Λ is graded, with ϵ sitting in weight 1. Alternatively, Λ can be characterized as the dual graded Hopf algebra of $\mathsf{H}^*(\mathsf{BKer}, \mathscr{O})$ [(‖)].

[(‖)] Notice that $\mathsf{H}^*(\mathsf{BKer}, \mathscr{O}) \simeq \mathsf{C}^*(\mathsf{BKer}, \mathscr{O})$ by the formality in Corollary **3.4.15** of graded $\mathsf{E}_\infty^{\otimes}$-algebras

By giving Λ the split filtration corresponding to its grading, we may consider Λ as a filtered $\mathsf{E}_1^{\otimes}$-algebra.

Construction 4.2.2. We let Mod_Λ denote the ∞-category associated to *strict* graded Λ-modules: concretely, this is obtained by localizing the ordinary symmetric monoidal category of (strict) graded Λ-modules with respect to quasi-isomorphisms. This carries a symmetric monoidal structure $\otimes_k$ due to the fact that Λ is a strict graded dg-Hopf algebra. As a category of modules over a filtered algebra it acquires the structure of a *filtered* ∞-category, which we will denote as $(\mathsf{Mod}_\Lambda)_{\mathsf{Fil}}$.

Together with the symmetric monoidal structures, it becomes a sheaf of $\mathsf{E}_\infty^{\otimes}$-categories over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

It turns out that the induced filtration on the categories of representations $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ is split, when no tensor structure is involved (see Remark **4.2.5**). In proving the HKR theorem we will need a finer result which describes the symmetric monoidal structures on $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}})$ and $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})$. The following is the main result of this section:

Proposition 4.2.3. *(i) The pullback along the map of $\mathbb{Z}_{(p)}$-stacks $u : \mathsf{S}^1 \longrightarrow (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ of (29) induces a symmetric monoidal equivalence*

$$u^* : \mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}) \xrightarrow[\sim]{} \mathsf{QCoh}(\mathsf{BS}^1) \tag{98}$$

(ii) There exist a natural symmetric monoidal equivalence

$$\mathsf{Mod}_\Lambda^{\otimes} \simeq \mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})^{\otimes} \tag{99}$$

*compatible with the graduations on both sides. Here the symmetric monoidal monoidal structure on the l.h.s is the one of Construction **4.2.2**. In particular, an action of $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ is given by a strict square zero differential.*

*Proof of Proposition **4.2.3**:* We start with the proof of (i). To show that (98) is an equivalence, we use presentation of both $\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ and BS^1 as the geometric realizations of simplicial diagrams

$$[n] \mapsto ((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}})^{\times_n} \simeq \mathsf{BFix}^{\times_n}$$

respectively,

$$[n] \mapsto (\mathsf{S}^1)^{\times_n}$$

Since (98) is induced by a map of groups $\mathsf{S}^1 \to \mathsf{BFix}$, to show that it is fully faithful, it is enough to check that levelwise the induced pullback functors

$$u_n^* : \mathsf{QCoh}(\mathsf{BFix}^{\times_n}) \to \mathsf{QCoh}((\mathsf{S}^1)^{\times_n}) \tag{100}$$

are fully faithful.

Using the equivalence (79), combined with the Proposition **3.3.2**, we therefore reduced to show that the pullback functors

$$u_n^* : \mathsf{QCoh}(\mathsf{BFix}^{\times_n}) \underbrace{\simeq}_{(79)} \mathsf{Mod}_{\mathsf{C}^*(\mathsf{BFix},\mathscr{O})^{\otimes n}} \underbrace{\simeq}_{3.3.2} \mathsf{Mod}_{\mathsf{C}^*(\mathsf{S}^1,\mathbb{Z}_{(p)})^{\otimes n}} \to \mathsf{QCoh}((\mathsf{S}^1)^{\times_n}) \tag{101}$$

are fully faithful for every n. But this follows from the fact that k is a compact object in $\mathsf{QCoh}((\mathsf{S}^1)^{\times_n}) \simeq \mathsf{Fun}((\mathsf{S}^1)^{\times_n}, \mathsf{Mod}_k)$ (under this equivalence k denotes with constant diagram with value k) since the spaces $(\mathsf{S}^1)^{\times_n}$ are finite CW-complexes.

To check this, we observe that the functor u_n^* admits a right adjoint

$$(u_n)_* : \mathsf{Fun}((\mathsf{S}^1)^{\times_n}, \mathsf{Mod}_k) \to \mathsf{Mod}_{\mathsf{C}^*((\mathsf{S}^1)^{\times_n},\mathbb{Z}_{(p)})}$$

given by taking the homotopy limit of diagrams. The fact that k is compact says that this functor commutes with all colimits. So does u_n^*. Therefore, in order to show that the unit of this adjuntion

$$N \to (u_n)_* u_n^*(N) \tag{102}$$

is an equivalence for every $N \in \mathsf{Mod}_{\mathsf{C}^*((\mathsf{S}^1)^{\times_n},\mathbb{Z}_{(p)})}$ we can use the fact that the category $\mathsf{Mod}_{\mathsf{C}^*((\mathsf{S}^1)^{\times_n},\mathbb{Z}_{(p)})}$ is generated under colimits by $\mathsf{C}^*((\mathsf{S}^1)^{\times_n}, \mathbb{Z}_{(p)})$ and that u_n^* preserves colimits. Using these two facts, it is enough to check that the unit map (102) is an equivalence when $N = \mathsf{C}^*((\mathsf{S}^1)^{\times_n}, \mathbb{Z}_{(p)})$.

$$\mathsf{C}^*((\mathsf{S}^1)^{\times_n}, \mathbb{Z}_{(p)}) \to (u_n)_* u_n^*(\mathsf{C}^*((\mathsf{S}^1)^{\times_n}, \mathbb{Z}_{(p)})) \tag{103}$$

But finally, since u_n^* is monoidal, we get that $u_n^*(\mathsf{C}^*((\mathsf{S}^1)^{\times_n}, \mathbb{Z}_{(p)}))$ is the constant diagram with values $k = \mathbb{Z}_{(p)}$ and therefore we get that (103) is an equivalence. This concludes the proof that the u_n^* are fully faithful.

To show that (98) is essentially surjective we observe that since u^* is a limit of fully faithful functors on each degree n

$$\mathsf{lim}_\Delta \mathsf{QCoh}(\mathsf{BFix}^{\times_n}) \to \mathsf{lim}_\Delta \mathsf{QCoh}((\mathsf{S}^1)^{\times_n})$$

an object (E_n) in the r.h.s is in the essential image of the limit functor if and only if levelwise each E_n is in the essential image of the functor u_n^*. For $n = 0$, u_0^* is an equivalence, and for $n > 0$, each E_n is obtained from E_0 by pullback from $\mathsf{Spec}(\mathbb{Z}_{(p)})$.

We now address the proof of (ii). The proof is essentially the same as for (1), except that we have to produce a symmetric monoidal ∞-functor

$$\mathsf{Mod}_\Lambda \longrightarrow \mathsf{QCoh}(\mathsf{B}^2\mathsf{Ker}).$$

where Λ is seen as a graded algebra as in Definition **4.2.1**. Such a functor is obtained as follows. We can consider $\mathsf{H}^*(\mathsf{BKer}, \mathscr{O})$ as a strict commutative dg-Hopf algebra. It is a consequence of Corollary **3.4.18** that $\mathsf{H}^*(\mathsf{BKer}, \mathscr{O})$ and $\mathsf{H}^*(\mathsf{S}^1, \mathbb{Z}_{(p)})$ are isomorphic as graded Hopf algebras.

We consider a strict symmetric monoidal category of comodules in chain complexes $\mathsf{CoMod}^{\mathsf{strict}}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})}$ defined as the limit internal to strict symmetric monoidal 1-categories

$$\mathsf{CoMod}^{\mathsf{strict}}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})} := \mathsf{lim}_\Delta \mathsf{Mod}^{\mathsf{strict}}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^{\otimes n}} \tag{104}$$

It follows from the Definition **4.2.1** that Λ, is the dual graded Hopf algebra of $\mathsf{H}^*(\mathsf{BKer}, \mathscr{O})$. It follows in this case that we have an equivalence of strict symmetric monoidal 1-categories of comodules (resp. modules) on chain complexes

$$\mathsf{CoMod}^{\mathsf{strict},\otimes}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})} \simeq \mathsf{Mod}^{\mathsf{strict},\otimes}_\Lambda \tag{105}$$

given by the identity on objects. It can now be localized on both sides along quasi-isomorphism to produce an equivalence of symmetric monoidal ∞-categories.

$$\mathsf{CoMod}^{\mathsf{strict},\otimes}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})}[q.iso^{-1}] \simeq \mathsf{Mod}^{\mathsf{strict},\otimes}_\Lambda[q.iso^{-1}] \underset{Construction\ \mathbf{4.2.2}}{=:} \mathsf{Mod}^{\otimes}_\Lambda \tag{106}$$

We now claim that we are able to construct a symmetric monoidal functor

$$\mathsf{CoMod}^{\mathsf{strict},\otimes}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})}[q.iso^{-1}] \to \mathsf{QCoh}(\mathsf{B}^2\mathsf{Ker}) \tag{107}$$

Indeed, after inverting quasi-isomorphisms we get a naturally defined symmetric monoidal ∞-functor

$$\begin{array}{ccc} \mathsf{Mod}^{\otimes}_{\Lambda} \simeq (\mathsf{lim}_{n\in\Delta}\mathsf{Mod}^{\mathsf{strict}}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^{\otimes n}})[q.iso^{-1}] & \longrightarrow & \mathsf{lim}_{n\in\Delta}(\mathsf{Mod}^{\mathsf{strict}}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^{\otimes n}}[q.iso^{-1}]) \\ & & \downarrow \sim \\ & & \mathsf{lim}_{n\in\Delta}\mathsf{Mod}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^{\otimes n}} \end{array} \tag{108}$$

On the other hand, the symmetric monoidal ∞-category $\mathsf{QCoh}(\mathsf{B}^2\mathsf{Ker})$ is obtained as a limit of symmetric monoidal ∞-categories (by descent from BKer to $\mathsf{B}^2\mathsf{Ker}$)

$$\mathsf{QCoh}(\mathsf{B}^2\mathsf{Ker}) \simeq \mathsf{lim}_{n\in\Delta}\mathsf{QCoh}(\mathsf{BKer}^{\times_n}) \tag{109}$$

We are therefore reduced to the construction of symmetric monoidal functors

$$\mathsf{Mod}_{\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^n} \to \mathsf{QCoh}(\mathsf{BKer}^{\times_n}) \tag{110}$$

compatible with the transition maps in Δ. But this follows again because of formality (Corollary **3.4.15**) as

$$\mathsf{H}^*(\mathsf{BKer},\mathscr{O})^n \simeq \mathsf{C}^*(\mathsf{BKer},\mathscr{O})^n$$

and there is always a symmetric monoidal functor from

$$\mathsf{Mod}_{\mathsf{C}^*(\mathsf{BKer},\mathscr{O})^n} \to \mathsf{QCoh}(\mathsf{BKer}^{\times_n})$$

from modules over global sections to quasi-coherent sheaves, which in fact we showed is an equivalence in (80).

We obtain in this manner the desired symmetric monoidal ∞-functor (107). The fact that (107) is an equivalence on the underlying categories is a consequence of Proposition **4.1.1** together with the following Lemma **4.2.4**, which implies that the functor sends Λ to the compact generator of $\mathsf{QCoh}(\mathsf{B}^2\mathsf{Ker})$. By construction this equivalence is clearly compatible with the action of $\mathbb{G}_m$ on both sides.

□

Lemma 4.2.4. *Let B be a strict (bi)-commutative dg Hopf algebra which is strictly dualizable as a k-module. Set $G = \mathsf{Spec}(B)$ and let*

$$\Psi : \mathsf{CoMod}^{\mathsf{strict},\otimes}_{B}[q.iso^{-1}] \to \mathsf{QCoh}(\mathsf{B}G) \simeq \mathsf{lim}_{\mathsf{Mod}_{B^{\otimes\bullet}}}$$

be the functor as constructed above, between strict and homotopically coherent B-comodules. Then $\Psi(B)$ is a compact generator for the right hand side above.

Proof. We show that the functor

$$F \mapsto \mathrm{Map}_{\mathsf{QCoh}(\mathsf{B}G)}(\Psi(B), -),$$

corepresented by $\Psi(B)$ coincides with the forgetful functor

$$\mathsf{lim}\,\mathsf{Mod}_{B^n} \to \mathsf{Mod}_k;$$

this itself is just the projection to the zeroth component in the cosimplicial diagram of stable ∞-categories above. For this we view $\mathsf{QCoh}(\mathsf{B}G)$ as a cosimplicial ∞-category. By construction, $\Psi(B)$ is represented, in each cosimplicial degree, by the object $B^\wedge \otimes B^{\otimes n}$, where $B^\wedge$ is the k-linear dual of B. Moreover, for any $(F_n) \in \mathsf{lim}_{\mathsf{Mod}_{B^n}}$ there will be equivalences

$$\mathrm{Map}_{\mathsf{Mod}_B^{\otimes n}}(B^\wedge \otimes B^{\otimes n}, F^n) \simeq \mathrm{Map}_{\mathsf{Mod}_k}(B^\wedge, F^n) \simeq \mathrm{Map}_{\mathsf{Mod}_k}(k, B \otimes_k F^n) \simeq B \otimes_k F^n$$

The limit of all the $B \otimes_k F^n$ is the standard cofree resolution of F as B-comodule. Therefore, this colimit is naturally equivalent to the underlying k-module of $F^\bullet$, ie. F^0. Moreover, as this is just the data of the projection of F to the zeroth level of the cosimplicial diagram, this computes the left-adjoint forgetful functor

$$\mathsf{lim}\,\mathsf{Mod}_{B^\bullet} \to \mathsf{Mod}_{B^0} \simeq \mathsf{Mod}_k.$$

The fact that this functor is conservative and commutes with colimits implies that $\Psi(B)$ is a compact generator. □

Remark 4.2.5. Using arguments similar to the ones of Proposition **4.2.3** and the main computation of Proposition **4.1.1** one can show that there exists an equivalence of filtered ∞-categories

$$(\mathsf{Mod}_\Lambda)_{\mathsf{Fil}} \simeq \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$$

between filtered mixed complexes (Construction **4.2.2**) and $\mathsf{S}^1_{\mathsf{Fil}}$-representations. As we do not use this equivalence it in the sequel we do not provide its proof. This equivalence means that the filtration on the underlying ∞-category $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ canonically splits, since the filtration on filtered mixed complexes comes from a grading. A key point however, is that, although this is an equivalence of ∞-categories; the symmetric monoidal structure is not preserved. Hence this is not a multiplicative splitting. Over characteristic zero this can be promoted to a symmetric monoidal equivalence, for more see Section **1.1** and [TV11].

5. The HKR Theorem

We are now ready to prove the results listed in Theorem **1.2.4** for a general simplicial commutative k-algebra $A \in \mathsf{SCR}_k$ (and more generally for derived schemes or stacks by gluing) where k is a fixed commutative $\mathbb{Z}_{(p)}$-algebra.

The filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ sits over $\mathbb{Z}_{(p)}$ and we pull-it back over $\mathsf{Spec}\, k$ to make it a filtered group stack over k. We will continue to denote it by $\mathsf{S}^1_{\mathsf{Fil}}$.

5.1. **The filtered loop stack.**

Definition 5.1.1. Let $X = \mathsf{Spec}\, A$ be an affine derived scheme over k. The *filtered loop space* of X (over k) is defined as the relative mapping stack

$$\mathsf{L}_{\mathsf{Fil}} X := \mathrm{Map}_{/[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]}(\mathsf{S}^1_{\mathsf{Fil}}, X \times [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]).$$

This is a derived scheme over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ equipped with a canonical action of the group stack $\mathsf{S}^1_{\mathsf{Fil}}$. As in Definition **2.2.12** we denote by

$$\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}} X) := p_* \mathscr{O}_{\mathsf{L}_{\mathsf{Fil}} X}$$

the object in $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$.

Proposition 5.1.2. *Let $X = \mathsf{Spec}\, A$ be an affine derived scheme over k and $\mathsf{L}_{\mathsf{Fil}} X$ its filtered loop space. Then, the derived stack $\mathsf{L}_{\mathsf{Fil}} X$ is representable by an affine derived scheme relative to $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.*

Proof. We start noticing that the truncation of $\mathsf{L}_{\mathsf{Fil}} X$ is simply the truncation of $X \times [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, and thus the truncation is a relativitely affine scheme. In order to prove that $\mathsf{L}_{\mathsf{Fil}} X$ is relativitely affine it remains to check that $\mathsf{L}_{\mathsf{Fil}} X$ has a global cotangent complex, admits an obstruction theory and is nil-complete (see [TV08] Appendix C). But this follows easily from the fact that it is a mapping derived stack from BH_{p^∞}, the classifying stack of a filtered group scheme. □

Theorem 5.1.3. *Let $X = \mathsf{Spec}\, A$ be an affine derived scheme over k. Then, the filtered loop stack exhibits a filtration on the usual derived loop stack $\mathsf{L} X$ whose associated graded in the shifted tangent stack $\mathrm{T}[-1]X$. More precisely, we have:*

(1) The composition with the natural map (29) : $\mathsf{S}^1 \longrightarrow (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}_k = \mathsf{BFix}_k$. *induces an isomorphism of derived schemes:*

$$(\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}} := \mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}_k, X) \longrightarrow \mathrm{Map}_k(\mathsf{S}^1, X) =: \mathsf{L}X \tag{111}$$

(2) There is a natural equivalence of graded derived schemes

$$(\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{gr}} := \mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k, X) \simeq \mathsf{Spec}\,\mathsf{Sym}^{\Delta}_A(\mathbb{L}_{A/k}[1]) =: \mathrm{T}[-1]X \tag{112}$$

where the action of $\mathbb{G}_{\mathrm{m}}$ *on* $\mathrm{T}[-1]X$ *is the geometric action of weight (-1), ie,* $\mathbb{L}_{A/k}[1]$ *is in weight 1.*

Proof. We first analyse the statement (1). For this purpose we show that the map (111) is an equivalence as functor of points, ie, that for every simplicial commutative algebra B the induced map

$$(\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}}(B) := \mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}_k \times \mathsf{Spec}\,B, X) \longrightarrow (\mathsf{L}X)(B) := \mathrm{Map}_k(\mathsf{S}^1 \times \mathsf{Spec}\,B, X) \tag{113}$$

is an equivalence. To show this we use the fact that every derived affine scheme can be written as a limit of copies of the affine line $\mathbb{A}^1_k$ (see [Lur11a, 4.1.9]). In this case, we are reduced to showing the statement for $X = \mathbb{A}^1_k$ and use the fact that for any derived stack F, the mapping space $\mathrm{Map}_k(F, \mathbb{A}^1_k)$ is the Dold-Kan construction of the complex of global sections on F, $\mathscr{O}(F)$. In this case, on the l.h.s we have

$$\mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}_k \times \mathsf{Spec}\,B, \mathbb{A}^1_k) \simeq \mathsf{DK}(\mathscr{O}(\mathsf{BFix} \times \mathsf{Spec}\,B)) \tag{114}$$

and on the r.h.s we have

$$\mathrm{Map}_k(\mathsf{S}^1 \times \mathsf{Spec}\,B, X) \simeq \mathsf{DK}(\mathscr{O}(\mathsf{S}^1 \times \mathsf{Spec}\,B)) \tag{115}$$

It remains to show that the induced pullback map

$$\mathscr{O}(\mathsf{BFix} \times \mathsf{Spec}\,B) \to \mathscr{O}(\mathsf{S}^1 \times \mathsf{Spec}\,B)$$

is a quasi-isomorphism of complexes. In this case, we can use the fact that BFix is of finite cohomological dimension (Proposition **3.3.7**) to apply the base change formula [HLP14, A.1.5-(2)][(**)] to establish an quasi-isomorphism between the underlying complexes

[(**)] See also [Lur18, 9.1.5.6-(c) and 9.1.5.7-(c)]

$$\mathscr{O}(\mathsf{BFix} \times \mathsf{Spec}\, B) \simeq \mathscr{O}(\mathsf{BFix}) \otimes B \tag{116}$$

(††)

In the same way, the finite cohomological dimension for S^1 with the base change formula, gives

$$\mathscr{O}(\mathsf{S}^1 \times \mathsf{Spec}\, B) \simeq \mathscr{O}(\mathsf{S}^1) \otimes B \tag{117}$$

Finally, the fact that BFix is the affinization of S^1 (Proposition **3.3.2**) concludes the proof.

We now deal with statement (2). We start constructing the map by noticing the existence of the following commutative square of graded affine stacks

$$\begin{array}{ccc} \mathsf{Spec}\, k[\epsilon] & \longrightarrow & \mathsf{Spec}\, k \\ \downarrow & & \downarrow \\ \mathsf{Spec}\, k & \longrightarrow & (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k . \end{array}$$

with ϵ in degree 0 and weigth 1. Such a commutative square of stacks is given by an element of $\mathsf{Ker}(k[\epsilon])$ interpreted as a map of graded affine schemes

$$\mathsf{Spec} k[\epsilon] \to \mathsf{Ker} \simeq \Omega\, (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$$

which simply is the p-typical Witt vector

$$\underline{\epsilon} := (\epsilon, 0, 0, 0, ...) \tag{118}$$

whose Ghost components are $(\epsilon, \epsilon^p, \epsilon^{p^2}, ..)$. As explained in the Guide **2.1.1**, the action of Frob_p on W_{p^∞} is determined by what it does on the Ghost coordinates, in this case, $(\epsilon, \epsilon^p, \epsilon^{p^2}, ..) \mapsto (\epsilon^p, \epsilon^{p^2}, ..)$. Reverse-engineering the Ghost cordinates we obtain $\mathrm{Frob}_p(\epsilon, 0, 0, 0, ...) = (\epsilon^p, 0, 0, ..)$. The fact that $\epsilon^p = 0$ tells us that the p-typical Witt vector $\underline{\epsilon}$ is in the kernel of Frob_p.
This square induces a commutative diagram of graded derived affine schemes, obtained by mapping to X

(††)If $i : \mathsf{Spec}\, B \to \mathsf{Spec}\, k$ and $\pi : \mathsf{BFix} \to \mathsf{Spec}\, k$, with projections $p_1 : \mathsf{BFix} \times_{\mathsf{Spec}\, k} \mathsf{Spec}\, B \to \mathsf{BFix}$, $p_2 : \mathsf{BFix} \times_{\mathsf{Spec}\, k} \mathsf{Spec}\, B \to \mathsf{Spec}\, B$, then the r.h.s is $i^*\pi_*$ and the l.h.s is $(p_2)_* (p_1)^*$.

$$\begin{tikzcd} \mathsf{Spec}\,\mathsf{Sym}_A(\mathbb{L}_A) & X \arrow[l] \\ X \arrow[u] & \mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k, X). \arrow[l] \arrow[u] \end{tikzcd} \tag{119}$$

This in turn produces a natural morphism of derived stacks

$$\mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k, X) \longrightarrow X \times_{\mathsf{Spec}\,\mathsf{Sym}_A(\mathbb{L}_A)} X \simeq \mathsf{Spec}\,\mathsf{Sym}_A(\mathbb{L}_A[1]) \tag{120}$$

To check its an equivalence we proceed as before by reducing to $X = \mathbb{A}^1_k$ using the fact that the construction $A \mapsto \mathsf{Sym}_A(\mathbb{L}_A[1])$ is compatible with colimits (using the fact that the cotangent complex is left adjoint) and that BKer is of finite cohomological dimension (Lemma **3.4.9**): indeed, for $X = \mathbb{A}^1_k$ the l.h.s of (120) we find the functor of points

$$B \mapsto \mathsf{DK}(\mathscr{O}(\mathsf{BKer} \times \mathsf{Spec}\, B)) \simeq \mathsf{DK}(\mathscr{O}(\mathsf{BKer}) \otimes B) \simeq \mathsf{DK}(B \oplus B[-1])$$

where the last equivalence follows from Theorem **3.4.17**. At the same time, on the r.h.s we find

$$B \mapsto \mathsf{DK}(B \times_{B[\epsilon]} B) \simeq \mathsf{DK}(B \oplus B[-1])$$

where $B[\epsilon]$ is with ϵ in degree 0. By the choice of the canonical element (118) in $\mathsf{Ker}(k[\epsilon])$ used to produce the diagram (119), we conclude that the map from the r.h.s to the l.h.s sends the generator in degree -1 to a generator in degree -1.

□

5.2. **The action of the filtered circle on filtered loops and its fixed points.**

Construction 5.2.1. Consider the filtered loops stack $p : \mathsf{L}_{\mathsf{Fil}}X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ as in Definition **5.1.1**. Since the action of $\mathsf{S}^1_{\mathsf{Fil}}$ is compatible with the filtration, the map p is $\mathsf{S}^1_{\mathsf{Fil}}$-equivariant and descends to the quotients

$$a : \mathsf{L}_{\mathsf{Fil}}X/\mathsf{S}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]/\mathsf{S}^1_{\mathsf{Fil}} \simeq \mathsf{B}\mathsf{S}^1_{\mathsf{Fil}}$$

and we have a pullback diagram of filtered stacks

$$\begin{tikzcd} \mathsf{L}_{\mathsf{Fil}}X \arrow[r, "\rho"] \arrow[d, "p"] & \mathsf{L}_{\mathsf{Fil}}X/\mathsf{S}^1_{\mathsf{Fil}} \arrow[d, "a"] \\ [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \arrow[r, "e"] & \mathsf{BS}^1_{\mathsf{Fil}} \end{tikzcd}$$

where ρ and e are the canonical atlases. Since a is an affine map (Proposition **5.1.2**), we can use [Lur18, 9.1.5.8, 9.1.5.3, 9.1.5.7] (see the Remark **2.2.14**-(iii)) to deduce the base change property for the diagram

$$\begin{tikzcd} \mathsf{QCoh}(\mathsf{L}_{\mathsf{Fil}}X) \arrow[d, "p_*"] & \mathsf{QCoh}(\mathsf{L}_{\mathsf{Fil}}X)^{\mathsf{S}^1_{\mathsf{Fil}}-eq} \arrow[l, "\rho^*"'] \arrow[d, "a_*"] \\ \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) & \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \arrow[l, "e^*"] \end{tikzcd} \tag{121}$$

where ρ^* and e^* corresponding to the functors forgetting the action. We find that that the object $\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X) = p_*\mathscr{O}_{\mathsf{L}_{\mathsf{Fil}}X}$ of Definition **5.1.1** carries a canonical action of $\mathsf{S}^1_{\mathsf{Fil}}$. We will sometimes write $\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X)$ to denote its lift as an object in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. The result of this construction can be exhibited as an ∞-functor which we will suggestively[(‡‡)] denote as

$$\mathsf{HH}_{\mathsf{Fil}} : \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \quad , \quad \mathsf{HH}_{\mathsf{Fil}}(A) := \mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}\mathsf{Spec}A) \tag{122}$$

As a consequence of Proposition **4.2.3**-(i), the associated underlying object is an $\mathsf{E}^{\otimes}_{\infty}$-algebra with a compatible $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$-action, and by Proposition **4.2.3**-(ii), the associated graded is a graded mixed $\mathsf{E}^{\otimes}_{\infty}$-algebra with a compatible $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$-action.

Construction 5.2.2. Consider the filtered loops stack $p : \mathsf{L}_{\mathsf{Fil}}X \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ as in Definition **5.1.1**. We want to understand the operation of extracting fixed points of $\mathsf{S}^1_{\mathsf{Fil}}$-action on the filtered object $\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X) \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. Concretely this operation is given by pushforward along the structure map $\pi : \mathsf{BS}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

We want to understand how this is compatible with taking associated graded and underlying objects. To that end, we start by considering the following commutative diagram of pullback squares

(‡‡)This choice of notation will become clear with the Theorem **5.4.1**-(a) below.

$$
\begin{tikzcd}
 & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{gr}} \arrow[rr, "\tilde{0}"] \arrow[dl, "\rho^{\mathrm{gr}}"'] \arrow[dd, "p^{\mathrm{gr}}"] & & \mathsf{L}_{\mathsf{Fil}}X \arrow[dl, "\rho"] \arrow[dd, "p"] & & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}} \arrow[ll, "\tilde{1}"'] \arrow[dl, "\rho^{\mathrm{u}}"'] \arrow[dd, "p"] \\
(\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{gr}}/(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{gr}} \arrow[rr, "\tilde{0}_a"] \arrow[dd, "a^{\mathrm{gr}}"] & & \mathsf{L}_{\mathsf{Fil}}X/\mathsf{S}^1_{\mathsf{Fil}} \arrow[dd, "a"] & & (\mathsf{L}_{\mathsf{Fil}}X)^{\mathrm{u}}/(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}} \arrow[ll, "\tilde{1}_a"'] \arrow[dd, "a^{\mathrm{u}}"] & \\
 & \mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[rr, "0"] \arrow[dl, "e^{\mathrm{gr}}"'] & & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[dl, "e"] \arrow[dl, "(j)"'] & & * \arrow[ll, "1"'] \arrow[dl, "e^{\mathrm{u}}"'] \\
\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} \arrow[rr, "0_a"] \arrow[d, "\pi^{\mathrm{gr}}"] & & \mathsf{B}\mathsf{S}^1_{\mathsf{Fil}} \arrow[d, "\pi"] & & \mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}} \arrow[ll, "1_a"'] \arrow[d, "\pi^{\mathrm{u}}"] & \\
\mathsf{B}\mathbb{G}_{\mathrm{m}} \arrow[rr, "0"] & & {[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} & & * \arrow[ll, "1"'] &
\end{tikzcd}
\tag{123}
$$

and observe that the Beck-Chevalley transformations

$$(0_a)^*\ a_*\ \mathscr{O} \to (a^{\mathrm{gr}})_*\ (\tilde{0}_a)^*\ \mathscr{O} \tag{124}$$

and

$$(1_a)^*\ a_*\ \mathscr{O} \to (a^{\mathrm{u}})_*\ (\tilde{1}_a)^*\ \mathscr{O} \tag{125}$$

are equivalences. As in the Construction **5.2.1**, this follows because the map a is an affine map, and we can use [Lur18, 9.1.5.8, 9.1.5.3, 9.1.5.7] (see the Remark **2.2.14**-(iii)) to deduce the base change property against all maps.
Finally, using Remark **3.2.5** we conclude that that base-change holds for π in the context of Remark **2.2.14**-(i) and Remark **2.2.14**-(ii), namely:

- The combination of (124) and (23) implies that the Beck-Chevalley transformation

$$([\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X)]^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}})^{\mathrm{gr}} \simeq 0^*\ \pi_*\ a_*\ \mathscr{O}\ \to (\pi^{\mathrm{gr}})_*\ (0_a)^*\ a_*\ \mathscr{O} \simeq [(\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X))^{\mathrm{gr}}]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{gr}}} \tag{126}$$

  is an equivalence, or that, in order words, taking fixed points commutes with taking the associated graded.

- The failure of (24) to be an equivalence in general tells that we cannot guarantee that the natural map induced by (125)

$$([\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X)]^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}})^{\mathrm{u}} \simeq 1^*\ \pi_*\ a_*\ \mathscr{O}\ \to (\pi^{\mathrm{u}})_*\ (1_a)^*\ a_*\ \mathscr{O} \simeq [(\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X))^{\mathrm{u}}]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}}} \tag{127}$$

is an equivalence in general, or, in other words, that taking the underlying object of the fixed points of the filtration is the fixed points of the original underlying object. Notice that via the actions of $\mathsf{BS}^1 \simeq \mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ of Proposition **4.2.3**-(i), and the base change (125), the r.h.s of (127) recovers the definition of negative cyclic homology

$$[(\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X))^{\mathrm{u}}]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathrm{u}}} \simeq [\mathsf{HH}(X)]^{\mathrm{h}\mathsf{S}^1} \simeq \mathsf{HC}^-(X)$$

.

However, since 1_a is an open immersion, the Remark **2.2.14**-(i) guarantees that that the Beck-Chevaley transformation (127) is an equivalence for objects in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{<\infty}$, ie, the commutativity of

$$\begin{array}{ccc} \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{<\infty} & \xrightarrow{(-)^{\mathrm{u}}} & \mathsf{QCoh}(\mathsf{BS}^1) \\ \downarrow{\scriptstyle (-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}} & & \downarrow{\scriptstyle (-)^{\mathrm{h}\mathsf{S}^1}} \\ \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) & \xrightarrow{(-)^{\mathrm{u}}} & \mathsf{Mod}_k \end{array} \tag{128}$$

is established.

Due to the failure of (127) to be an equivalence in general, we introduce the following definition:

Definition 5.2.3. Consider the filtered loop space as in Definition **5.1.1**. We define a filtered version of negative cyclic homology as the filtered object

$$\mathsf{HC}^-_{\mathsf{Fil}}(X) := [\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X)]^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \simeq 1^*\ \pi_*\ a_*\ \mathscr{O}\ \in\ \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) \tag{129}$$

By the Construction **5.2.2**, the underlying object of this filtration comes canonically equipped with a map to usual negative Hochschild homology

$$\mathsf{HC}^-_{\mathsf{Fil}}(X)^{\mathrm{u}} \to \mathsf{HC}^-(X) \tag{130}$$

5.3. **De Rham algebra functor.**

Construction 5.3.1. Recall that any $A \in \mathsf{SCR}_k$ possesses a derived de Rham algebra $\mathsf{DR}(A/k)$. For us this is a graded mixed $\mathsf{E}_\infty^\otimes$-algebra constructed as follows. When A is smooth over k, $\mathsf{DR}(A/k)$ simply is the strictly commutative dg-algebra $\mathsf{Sym}_A(\Omega^1_{A/k})$ endowed with its de Rham differential. This is a strictly commutative monoid inside the strict category of graded mixed complexes, and thus, by the Proposition **4.2.3**-(ii), can be considered as a $\mathsf{E}_\infty^\otimes$-algebra object inside $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})$, the symmetric monoidal ∞-category of graded mixed complexes (here over k).

This produces an ∞-functor from smooth algebras over k, and thus from polynomial k-algebras, to graded mixed $\mathsf{E}_\infty^\otimes$-algebras. By left Kan extension we get the de Rham functor

$$\mathsf{DR} : \mathsf{SCR}_k \longrightarrow \mathsf{CAlg}(\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})).$$

Remark 5.3.2. We will see below that the functor DR as defined above with values in $\mathsf{E}_\infty^\otimes$-algebras in $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})$ can be refined to take in values "simplicial commutative mixed graded algebras". More precisely, what this means is that we can exhibit it as a functor with valued in derived affine schemes over $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$, which contains slightly more information.

Construction 5.3.3. The functor of global sections produces a symmetric lax-monoidal ∞-functor

$$\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}) \longrightarrow \mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}}).$$

which formally corresponds to taking homotopy fixed points with respect to the action of the graded group stack $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$. We will denote it by $(-)^{\mathsf{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}$. The composition

$$\mathsf{SCR}_k \xrightarrow{\mathsf{DR}} \mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}) \longrightarrow \mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}})$$

sends A to the derived de Rham complex of A

$$\mathbb{L}\widehat{\mathsf{DR}}(A/k) := \mathsf{DR}(A/k)^{\mathsf{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}} \in \mathsf{CAlg}(\mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}})).$$

This is a graded $\mathsf{E}_\infty^\otimes$-algebra whose piece of weight i will be denoted by $\mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(A/k)$. One can provide an explicit description of these graded pieces. Indeed, by means of

the equivalence $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}) \simeq \mathsf{Mod}_\Lambda$ of Proposition **4.2.3**-(ii), the functor $(-)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}$ can be identified with the functor

$$\mathsf{Mod}_\Lambda \to \mathsf{Mod}_k^{\mathbb{Z}-\mathsf{gr}} \quad , \quad \mathrm{M} \mapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{RHom}_{\mathsf{Mod}_\Lambda}(\mathbb{Z}_{(p)}(i), \mathrm{M})$$

where the graded object $\mathbb{Z}_{(p)}(i)$ is seen as a graded Λ-module with a trivial action of the element ϵ. In [PTVV13b, Prop. 1.3] the authors prescribe an explicit resolution of $\mathbb{Z}_{(p)}(i)$ as a Λ-module, given by $Q(i) := \bigoplus_{n\geq 0} \Lambda(i+n)[2n]$ so that the graded piece of weight i in $(\mathrm{M})^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}$ is given by $\prod_{n\geq 0} \mathsf{gr}^{i+n}\mathrm{M}[-2n]$. In the case of the de Rham complex, we obtain for $i \geq 0$ that $\mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(A/k)$ should be understood as the complex $\prod_{n\geq 0}(\wedge^{i+n}\mathbb{L}_{A/k}[i-n]) \simeq \prod_{q\geq i}(\wedge^q \mathbb{L}_{A/k}[2i-q])$ endowed with the total differential $d+\mathsf{d}_{\mathsf{dR}}$, where d is the cohomological differential induced from A and d_{dR} is the de Rham differential. See for instance [Toë14, §5] or [PTVV13a, §1.2]. For $i \leq 0$ the graded pieces of weight i are all equivalent to a shifted copy of the piece of weight 0, namely, $\mathbb{L}\widehat{\mathsf{DR}}^{\geq 0}(A/k)[2i]$, thus reflecting the 2-periodic shape of the cohomology of the graded circle of Proposition **4.1.1**.

Remark 5.3.4. When $A \in \mathsf{SCR}$ is a smooth discrete algebra, for $i \geq 0$, the graded pieces $\mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(A/k)$ of Construction **5.3.3** are given by $\Omega_A^{\geq i}[2i]$ where $\Omega_A^{\geq i}$ is the chain complex

$$\cdots \to 0 \to \Omega_A^i \to \Omega_A^{i+1} \to \Omega_A^{i+2} \to \cdots$$

where Ω_A^i is in homological degree $-i$ and the maps are given by the de Rham differential d_R. In this case when $i \leq 0$, we have

$$\mathsf{gr}^i(\mathsf{DR}(A/k)^{\mathsf{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}) \simeq \Omega_A^{\geq 0}[2i] \tag{131}$$

5.4. **Main theorem.** We are now ready to prove the main results in this paper.

Theorem 5.4.1. *Let $X = \mathsf{Spec}\, A$ be an affine derived scheme over k and $\mathsf{L}_{\mathsf{Fil}}X$ its filtered loop space as in Definition **5.1.1**. Then, the cohomology $\mathsf{E}_\infty^\otimes$-algebra $\mathscr{O}^{\mathsf{fil}}(\mathsf{L}_{\mathsf{Fil}}X)$ endowed with its natural $\mathsf{S}^1_{\mathsf{Fil}}$-action is such that:*

(a) Its underlying object is naturally equivalent to $\mathsf{HH}(A/k)$*, the Hochschild homology of* A *over* k*, together with its natural* S^1*-action.*

(b) Its associated graded is naturally equivalent to $\mathsf{DR}(A/k)$ *as a graded mixed* $\mathsf{E}_\infty^\otimes$*-algebra.*

(c) Being compatible with the action of the filtered circle, the filtration descends to fixed points in the sense of Definition ***5.2.3*** *and makes* $\mathsf{HC}^-_{\mathsf{Fil}}(A)$ *a filtered algebra, whose*
- *associated graded pieces are the truncated complete derived de Rham complexes* $\mathbb{L}\widehat{\mathsf{DR}}^{\geq p}(A/k)$*;*
- *underlying object comes equipped with a canonical map to usual negative cyclic homology* (130)*:* $\mathsf{HC}^-_{\mathsf{Fil}}(X)^{\mathrm{u}} \to \mathsf{HC}^-(X)$

(d) If A *is discrete and smooth, then the above map* (130) *is an equivalence, hence it endows* $\mathsf{HC}^-(X)$ *with an exhaustive filtration in the sense of* [BMS19, Definition 5.1].

Proof. The claim (a) is direct a re-interpretation of the base change (125) established in the Construction **5.2.2**.

To prove (b) we have to compare the two ∞-functors

$$\mathsf{SCR}_k \longrightarrow \mathsf{CAlg}(\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k))$$

The first one given by $\mathsf{DR}(-/k)$, the second given by

$$A \mapsto \mathscr{O}(\mathrm{Map}_k((\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}_k, \mathsf{Spec}\, A))$$

For this, we first notice that if we forget the action of the group $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ (i.e. the mixed structure), then these two ∞-functors are equivalent and given by $A \mapsto \mathsf{Sym}_A(\mathbb{L}_A[1])$. As the functor $A \mapsto \mathbb{L}_A$ is obtained by extension under sifted colimits from polynomial rings (see Notation **1.2.10**), this shows that the same is true for the two functors to be shown to be equivalent. In other words, we can restrict these to the category of polynomial k-algebras.

We then observe that for any polynomial k-algebra A the space of graded mixed structures on the graded $\mathsf{E}_\infty^\otimes$-algebra $\mathsf{Sym}_A(\Omega^1_{A/k}[1])$ is a discrete space. Indeed, this follows from the fact that the space of graded $\mathsf{E}_\infty^\otimes$-endomorphisms is itself discrete, because the weight grading coincide with the cohomological grading (as in the proof of Lemma **3.4.11**). As a consequence, in order to show that the two above ∞-functors

are equivalent it is enough to show that for a fixed polynomial k-algebra A, the natural isomorphism of graded algebras

$$\mathsf{DR}(A/k) \simeq \mathscr{O}(\mathsf{L}^{\mathsf{gr}}_{\mathsf{Fil}}(X))$$

intertwine the two graded mixed structures. We can even be more precise, the compatible graded mixed structures on the graded $\mathsf{E}^{\otimes}_{\infty}$-algebra $\mathsf{DR}(A/k)$ form a discrete space which embeds into the set of k-linear derivations $A \longrightarrow \Omega^1_A$.

As a result, we are reduced to proving that, by the above identification, the differential obtain from the $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$-action on the right hand side

$$d : \pi_0(\mathscr{O}(\mathsf{L}^{\mathsf{gr}}_{\mathsf{Fil}}(X)) \simeq A \longrightarrow \pi_1(\mathscr{O}(\mathsf{L}^{\mathsf{gr}}_{\mathsf{Fil}}(X))) \simeq \Omega^1_{A/k}$$

is indeed equal to the standard de Rham differential. For this, we can of course assume that $k = \mathbb{Z}_{(p)}$, as the general case would be obtained by base change. But in this case all complexes involved are torsion free; one may then simply base change to $\mathbb{Q}$ to check the mixed structure above is the de Rham differential. But the result is well known in characteristic zero (see [TV11]).

Parts (c) is a direct consequence of the base change formula (126) in Construction Construction **5.2.2**, Construction **5.3.3** and point (b).

It remains to prove (d). Recall that the map (130) is a Beck-Chevalley transformation of (127). To conclude that it is an equivalence, following the commutativity of (128), it is enough to show that when A is smooth and discrete, $\mathsf{HH}_{\mathsf{Fil}}(A) \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{<\infty}$. By the Construction **3.2.4**, this means that as an object $\mathsf{HH}_{\mathsf{Fil}}(A) \in \mathsf{Fil}(\mathsf{Mod}_k)$, each level of the filtration, $\mathsf{HH}_{\mathsf{Fil}}(A)_i$ is in $\mathsf{Mod}_k^{<\infty}$. We show that this is indeed the case for A smooth and discrete. By (a) we know that the underlying object of $\mathsf{HH}_{\mathsf{Fil}}(A)$ is $\mathsf{HH}(A)$. By (b), in the smooth discrete case the associated graded is $\bigoplus_{i\geq 0} \Omega^i_A[i]$ where the Ω^i_A are concentrated in a single degree, zero. Moreover, if A is smooth of finite dimension d, by the classical HKR theorem [HKR62] we have an identification of abelian groups

$$\pi_i(\mathsf{HH}(A)) \simeq \begin{cases} \Omega^i_A, & \text{for } 0 \leq i \leq d \\ 0, & \text{for } i < 0 \text{ and } i > d \end{cases}$$

Therefore, we have

$$\mathsf{gr}^i(\mathsf{HH}_{\mathsf{Fil}}(A)) \simeq 0 \text{ for } i < 0 \text{ and } i > d$$

Using the cofiber sequences

$$\begin{tikzcd} \mathsf{HH}_{\mathsf{Fil}}(A)_{i+1} \arrow[r] \arrow[d] & \mathsf{HH}_{\mathsf{Fil}}(A)_i \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^i\,\mathsf{HH}_{\mathsf{Fil}}(A) \end{tikzcd}$$

we have

$$\mathsf{HH}_{\mathsf{Fil}}(A)_0 \simeq \mathsf{HH}(A) \quad , \quad \lim \mathsf{HH}_{\mathsf{Fil}}(A) \simeq \mathsf{HH}_{\mathsf{Fil}}(A)_{d+1} \simeq 0$$

and $\mathsf{HH}_{\mathsf{Fil}}(A)_i \in \mathsf{Mod}_k^{\leq d}$. This concludes the proof.

□

5.5. **Comparison with the filtration of Antieau.** In this section we compare the filtered object $\mathsf{HC}^-_{\mathsf{Fil}}(A)$ (Definition **5.2.3** and Theorem **5.4.1**) with the filtered object $\mathsf{F}_{\mathsf{B}}\mathsf{HC}^-(A)$ constructed by Antieau in [Ant19, Thm 1.1] which we will revise below. Our comparison neglects the compatibility with the algebra structures and focuses only on underlying filtered complexes.

The strategy consists of two main steps:

(i) We compare the two filtrations on smooth polynomial algebras (Proposition **5.5.18**)

(ii) We use the fact that SCR_k is the sifted completion of the discrete category of polynomial algebras $\mathsf{N}(\mathsf{Poly}_k)$ (see [Lur18, 25.1.1.5] and [Lur09, 5.5.9.3]) to Kan extend the comparison to all simplicial commutative algebras (Corollary **5.5.34**).

Let us start by setting some terminology and collecting some facts.

Definition 5.5.1. We say that an object $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ is *complete* if its underlying filtered module obtained by pullback along the canonical atlas $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \to \mathsf{BS}^1_{\mathsf{Fil}}$

$$\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \to \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) \simeq \mathsf{Fil}(\mathsf{Mod}_k)$$

is complete (in the sense of Definition **2.2.3**). We denote by $\widehat{\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})}$ the full subcategory of complete filtered modules with an $\mathsf{S}^1_{\mathsf{Fil}}$-action. Moreover, as in Construction **2.2.6**, we will denote by $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$ the full subcategory of $\widehat{\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})}$ spanned by those representations whose underlying filtered object is in $\mathsf{Fil}^\dagger(\mathsf{Mod}_k)$. In particular, $\mathscr{O} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$ by definition of the tensor unit in filtered objects (see Construction **2.2.1**).

Remark 5.5.2. The proof of Theorem **5.4.1**-(d) shows that when A is smooth discrete, we have a canonical equivalence in $\mathsf{Fil}(\mathsf{Mod}_k)$, $\mathsf{HH}_{\mathsf{Fil}}(A) \simeq \tau_{\geq}\mathsf{HH}(A)$, where $\tau_{\geq}$ is the Whitehead tower of Construction **2.2.5** for the standard t-structure on Mod_k. Indeed, this follows from the discussion in loc.cit and the characterization of the essential image of $\tau_{\geq}$. In particular, since the t-structure on Mod_k is left complete [Lur17, 7.1.1.13], by the Construction **2.2.6**, the filtration $\mathsf{HH}_{\mathsf{Fil}}(A)$ is complete and $\mathsf{HH}_{\mathsf{Fil}}(A) \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger}$.

Proposition 5.5.3. *The ∞-functor $\mathsf{HH}_{\mathsf{Fil}} : \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ preserves sifted colimits. In particular it is determined by its restriction*

$$\mathsf{N}(\mathsf{Poly}_k) \subseteq \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \tag{132}$$

Proof. Since the functor extracting the underlying filtered module $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \to \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ preserves all colimits and is conservative, to conclude the proof of the proposition it is enough to look at the composition

$$\mathsf{SCR}_k \xrightarrow[\sim]{\mathsf{Spec}} \mathsf{dAff}^{\mathsf{op}} \xrightarrow{\mathrm{Map}(\mathsf{S}^1_{\mathsf{Fil}},-)} (\mathsf{dSt}^{\mathrm{rel-aff}}_{/[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]_k})^{\mathsf{op}} \xrightarrow{\mathscr{O}^{\mathsf{Fil}}} \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$$

where $(\mathsf{dSt}^{\mathrm{rel-aff}}_{/[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]_k})^{\mathsf{op}}$ denotes the category of derived stacks over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ that are relatively affine (Proposition **5.1.2**).
The functor Spec, being an equivalence, sends sifted colimits to sifted limits of affine schemes. By definition (and Yoneda), the functor $\mathrm{Map}(\mathsf{S}^1_{\mathsf{Fil}}, -)$ commutes with all limits. Finally, since the derived loop spaces are relatively affine over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, by base-change along the atlas $\mathbb{A}^1_k \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ one deduces that the functor of (relative) global sections also commutes with all limits. The last claim in the proposition follows the description of SCR_k as a completion of $\mathsf{N}(\mathsf{Poly}_k)$ under sifted colimits. □

Remark 5.5.4. The inclusion $\mathsf{Fil}^{\dagger}(\mathsf{Mod}_k) \subseteq \mathsf{Fil}(\mathsf{Mod}_k)$ is stable under all colimits. Indeed, since colimits in $\mathsf{Fil}(\mathsf{Mod}_k)$ are computed levelwise, this amounts to the fact that the full subcategory $\mathsf{Mod}_k^{\geq i} \subseteq \mathsf{Mod}_k$ of i-connected objects is stable under all colimits (see [Lur17, 1.2.1.6]). This implies that the inclusion $\mathsf{Fil}^{\dagger}(\mathsf{Mod}_k) \subseteq \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)$ of Construction **2.2.6** also commutes with colimits, where on the r.h.s colimits are computed using the completion functor.
The same argument shows that both inclusions $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger} \subseteq \widehat{\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})}$ and $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger} \subseteq \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ commute with all colimits.

Proposition 5.5.5. *The ∞-functor $\mathsf{HH}_{\mathsf{Fil}} : \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ has image in the subcategory $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$. Moreover, the factorization*

$$\mathsf{HH}_{\mathsf{Fil}} : \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger \tag{133}$$

is compatible with sifted colimits.

Proof. This is now a consequence of the fact SCR_k is the sifted completion of $\mathsf{N}(\mathsf{Poly}_k)$, the Proposition **5.5.3**, the fact that the restriction of $\mathsf{HH}_{\mathsf{Fil}}$ along $\mathsf{N}(\mathsf{Poly}_k) \subseteq \mathsf{SCR}_k$ factors through $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$ (Remark **5.5.2**) and finally the fact the inclusion $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger \subseteq \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ commute with all colimits (Remark **5.5.4**).

□

Remark 5.5.6. We denote by $\tau^{\mathsf{Fil}}_{\geq}$ the Whitehead tower functor for the left complete t-structure on $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ of Construction **3.2.4**. The argument used in loc. cit. also explains the existence of left complete t-structures both on $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}})$ and $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})$:

(i) Since the map $1_a : \mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}} \to \mathsf{BS}^1_{\mathsf{Fil}}$ in (123) is an open immersion, the pullback $1_a^* : \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \to \mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}})$ is t-exact. The characterization of the two t-structures via the atlas guarantees that under the equivalence of Proposition **4.2.3**-(i)

$$\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}) \simeq \mathsf{QCoh}(\mathsf{BS}^1) \simeq \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_k)$$

the t-structure on $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}})$ becomes the t-structure on $\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_k)$ induced from the *standard* t-structure on Mod_k via the forgetful functor $\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_k) \to \mathsf{Mod}_k$. Therefore, the associated Whitehead tower functors of Construction **2.2.5** commute:

$$\begin{array}{ccc} \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) & \xrightarrow{1_a^*} & \mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_k) \\ \tau^{\mathsf{Fil}}_{\geq} \downarrow & & \downarrow \tau^{\mathsf{std}}_{\geq} \\ \mathsf{Fil}(\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})) & \xrightarrow{1_a^* \circ -} & \mathsf{Fil}(\mathsf{Fun}(\mathsf{BS}^1, \mathsf{Mod}_k)) \end{array} \tag{134}$$

Notice that forgetting the circle actions and under the equivalence of Theorem **2.2.10**, the discussion in Construction **3.2.4** shows that the t-exactness of 1_a^* amounts to the t-exactness of the colim functor $\mathsf{colim} : \mathsf{Fun}(\mathsf{N}(\mathbb{Z}^{\mathsf{op}}), \mathsf{Mod}_k) \to \mathsf{Mod}_k$ where

on the l.h.s we have the levelwise standard t-structure of [Lur17, Proposition 1.4.3.6]. In this form, the statement amounts to the fact that homology groups commute with filtered colimits (see [Lur17, Proposition 1.3.5.21]).

(ii) The same argument of Construction **3.2.4** using the atlas, shows that under the equivalence $\mathsf{QCoh}(\mathsf{B}\mathbb{G}_{\mathrm{m}}) \simeq \mathsf{Mod}_k^{\mathbb{Z}-\mathsf{gr}}$ of Theorem **2.2.10**, the t-structure on $\mathsf{QCoh}(\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}})$ is the levelwise t-structure. In particular, since the pullback along the map $0_a : \mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} \to \mathsf{BS}^1_{\mathsf{Fil}}$ of (123) is right t-exact (Notation **1.2.12**-b)), implying that the canonical maps

$$\tau^{\mathsf{gr}}_{\geq i}(\,(0_a)^* \circ \tau^{\mathsf{Fil}}_{\geq i}\,) \longrightarrow (\,(0_a)^* \circ \tau^{\mathsf{Fil}}_{\geq i}\,) \tag{135}$$

are equivalences.

Remark 5.5.7. If $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$ then for each $n \in \mathbb{Z}$ we have $\tau^{\mathsf{Fil}}_{\geq n}\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$. Indeed, as explained in the Construction **3.2.4**, the underlying filtered object of $\tau^{\mathsf{Fil}}_{\geq n}\mathrm{E}$ is the sequence

$$\cdots \to \tau_{\geq n}\mathrm{E}_3 \to \tau_{\geq n}\mathrm{E}_2 \to \tau_{\geq n}\mathrm{E}_1 \to \tau_{\geq n}\mathrm{E}_0 \to \cdots$$

where $\tau_{\geq n}$ is the truncation in Mod_k. In particular, the fact that $\mathrm{E}_i \in \mathsf{Mod}_k^{\geq i}$ implies, as explained in the Construction **2.2.6**, that $\tau_{\geq n}\mathrm{E}_i \in \mathsf{Mod}_k^{\geq i}$ so that $\tau^{\mathsf{Fil}}_{\geq n}\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger$. In particular, since the Whitehead tower provides a complete filtered object (see the conclusion of Construction **2.2.6**) we find a commutative diagram

$$\begin{tikzcd}
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger \arrow[r, hook] \arrow[d, dashed] & \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \arrow[d, "\tau^{\mathsf{Fil}}_{\geq}"] \\
\widehat{\mathsf{Fil}}(\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^\dagger) \arrow[r, hook] & \widehat{\mathsf{Fil}}(\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}))
\end{tikzcd}$$

Remark 5.5.8. The construction of homotopy fixed points $(-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}$ preserves complete filtrations. Indeed, taking fixed points corresponds to the pushforward functor along the projection $\mathsf{BS}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$:

$$(-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} : \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \to \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) \simeq \mathsf{Fil}(\mathsf{Mod}_k)$$

By construction this functor preserves limits and therefore by the Definition **5.5.1**, it preserves complete filtrations. In particular, we have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})} \arrow[r, "(-)^{\mathrm{hS}^1_{\mathsf{Fil}}}"] \arrow[d, hook] & \widehat{\mathsf{Fil}}(\mathsf{Mod}_k) \arrow[d, hook] \\
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \arrow[r, "(-)^{\mathrm{hS}^1_{\mathsf{Fil}}}"] & \mathsf{Fil}(\mathsf{Mod}_k)
\end{tikzcd}$$

We finally have all the ingredients to construct the commutative diagram that will allow us to compare our construction to that of Antieau [Ant19] in the smooth polynomial case:

Construction 5.5.9. Notice that restriction to polynomial algebras

$$\mathsf{HH}_{\mathsf{Fil}} : \mathsf{N}(\mathsf{Poly}_k) \subseteq \mathsf{SCR}_k \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger}$$

takes values in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}$ (see the proof of Theorem **5.4.1**-(d)). Therefore, the combination of Remark **5.5.6**-(i), Remark **5.5.7**, Remark **5.5.8** and the commutativity of (128) allows us to establish the commutativity of

$$\begin{tikzcd}
\mathsf{N}(\mathsf{Poly}_k) \arrow[d, "\mathsf{HH}_{\mathsf{Fil}}"] \arrow[dr, "\mathsf{HH}"] & & \\
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty} \arrow[r, "(-)^{\mathrm{u}}"] \arrow[d, "\tau^{\mathsf{Fil}}_{\geq}"] & \mathsf{QCoh}(\mathsf{BS}^1)^{<\infty} \arrow[d, "\tau^{\mathrm{std}}_{\geq}"] \\
\widehat{\mathsf{Fil}}[\ \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}\ ] \arrow[r, "(-)^{\mathrm{u}}_{lvl}"] \arrow[d, "(-)^{\mathrm{hS}^1_{\mathsf{Fil}}}_{lvl}"] & \widehat{\mathsf{Fil}}[\ \mathsf{QCoh}(\mathsf{BS}^1)\ ] \arrow[d, "(-)^{\mathrm{hS}^1}_{lvl}"] \\
\widehat{\mathsf{Fil}}[\ \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)\ ] \arrow[r, "(-)^{\mathrm{u}}_{lvl}"] & \widehat{\mathsf{Fil}}[\ \mathsf{Mod}_k\ ]
\end{tikzcd} \tag{136}$$

Here the functor $(-)^{\mathrm{hS}^1}$ applied levelwise preserves complete filtrations because it is a right adjoint and therefore commutes with all limits.

Notation 5.5.10. As in [Ant19, §3] we denote the composition $(-)^{\mathrm{h}\mathsf{S}^1}_{lvl} \circ \tau^{\mathrm{std}}_{\geq} \circ \mathsf{HH}$ by $\mathrm{F}^{\bullet}_{\mathrm{HKR}}\mathsf{HC}(-)^{-}$.

Notation 5.5.11. We shall denote by $\mathsf{Fil}^{\mathrm{const}\geq 0}(\mathsf{C})$ the full subcategory of $\mathsf{Fil}(\mathsf{C})$ spanned by those filtered objects E such that the maps $\mathrm{E}_0 \to \mathrm{E}_{-1} \to \mathrm{E}_{-2} \to \cdots$ are equivalences. In particular, if $\mathrm{E} \in \mathsf{Fil}^{\mathrm{const}\geq 0}(\mathsf{C})$, its colimit is E_0. Moreover, for any functor $F : \mathsf{C} \to \mathsf{D}$, by functoriality $\mathsf{Fil}^{\mathrm{const}\geq 0}(\mathsf{C})$ is sent to $\mathsf{Fil}^{\mathrm{const}\geq 0}(\mathsf{D})$ and is compatible with the extraction of underlying objects.

Remark 5.5.12. We can complete the diagram (136) to the left. As explained in the Remark **5.5.2**, when A is a polynomial algebra, as filtered objects we have $\mathsf{HH}_{\mathsf{Fil}}(A) \simeq \tau_{\geq}\,\mathsf{HH}(A)$. In particular, the computations in Theorem **5.4.1**-(d) and the definition of the t-structure in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ show that the restriction $\mathsf{HH}_{\mathsf{Fil}} : \mathsf{N}(\mathsf{Poly}_k) \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}$ factors through the full subcategory $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0}$.
Notice that if $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0}$ then the associated Whitehead tower $\tau^{\mathsf{Fil}}_{\geq}\,\mathrm{E}$ is a $\mathbb{Z}^{\mathsf{op}}$-diagram which is constant starting from level 0. In particular, it follows that

$$\mathsf{colim}_{i\in\mathbb{Z}^{\mathsf{op}}}\,(\tau^{\mathsf{Fil}}_{\geq i}\,\mathrm{E}) = \mathrm{E}$$

In other words, we have commutativity for the diagram

$$\begin{tikzcd}
 & \mathsf{N}(\mathsf{Poly}_k) \arrow[d, "\mathsf{HH}_{\mathsf{Fil}}"] \\
 & \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0} \arrow[d, "\tau^{\mathsf{Fil}}_{\geq}"] \arrow[dl, equal] \\
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0} & \widehat{\mathsf{Fil}}^{\mathrm{const}\geq 0}[\,\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0}\,] \arrow[l, "\mathsf{colim}"']
\end{tikzcd} \tag{137}$$

where the colimit is taken with respect to the Whitehead filtration.

Remark 5.5.13. We observe that we can extend the commutative diagram (137) to the extraction of fixed points for the circle action. More precisely, the Notation **5.5.11** guarantees the commutativity of the diagram

$$\begin{tikzcd} \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0} \arrow[d, "(-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}"] & \widehat{\mathsf{Fil}}^{\mathrm{const}\geq 0}[\ \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0}\ ] \arrow[l, "\mathsf{colim}"'] \arrow[d, "(-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{levelwise}"] \\ \widehat{\mathsf{Fil}}(\mathsf{Mod}_k) & \widehat{\mathsf{Fil}}^{\mathrm{const}\geq 0}[\ \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)\ ] \arrow[l, "\mathsf{colim}"'] \end{tikzcd} \tag{138}$$

where the bottom map is the colimit with respect to the exterior filtration on the bottom right corner.

We now recall the construction of the Beilinson t-structure on filtered objects:

Theorem 5.5.14. *(Beilison t-structure* [BMS19, Thm 5.4]*) The category of filtered objects $\mathsf{Fil}(\mathsf{Mod}_k)$ admits a t-structure t_B where $\mathsf{Fil}(\mathsf{Mod}_k)_{\geq_B 0}$ is the full subcategory spanned by those filtered objects E such that $\mathsf{gr}^n\mathrm{E} \in \mathsf{Mod}_k^{\geq -n}$. If $\tau^B_{\geq n}\mathrm{E}$ denotes the truncation, its associated graded pieces are given by*

$$\mathsf{gr}^i\tau^B_{\geq n}\mathrm{E} \simeq \tau_{\geq n-i}\,\mathsf{gr}^i(\mathrm{E}) \tag{139}$$

where on the r.h.s we have the standard t-structure on Mod_k. The heart of this t-structure $\mathsf{Fil}(\mathsf{Mod}_k)^{\heartsuit_B}$ is equivalent to the abelian category of chain complexes via the construction sending a filtered complex $\mathrm{E} \in \mathsf{Fil}(\mathsf{Mod}_k)^{\heartsuit_B}$ to the chain complex

$$[\cdots \to \underbrace{\mathrm{H}_n(\mathsf{gr}^0\mathrm{E})}_{0} \to \underbrace{\mathrm{H}_{n-1}(\mathsf{gr}^1\mathrm{E})}_{-1} \to \underbrace{\mathrm{H}_{n-2}(\mathsf{gr}^2\mathrm{E})}_{-2} \to \cdots] \tag{140}$$

and the maps are the boundary maps of the cofiber sequences

$$\mathsf{gr}^{i+1}(\mathrm{E}) = \mathrm{E}_{i+1}/\mathrm{E}_{i+2} \to \mathrm{E}_i/\mathrm{E}_{i+2} \to \mathrm{E}_i/\mathrm{E}_{i+1} = \mathsf{gr}^i(\mathrm{E})$$

Notation 5.5.15. We denote by $\tau^B_{\geq} : \mathsf{Fil}(\mathsf{Mod}_k) \to \mathsf{Fil}(\mathsf{Fil}(\mathsf{Mod}_k)$ the Whitehead tower for the Beilinson t-structure.

Notice that since the Beilinson t-structure is not left complete (see [BMS19, 5.3]), the Beilison-Whitehead tower is not automatically complete. However, by [Ant19, Lemma 3.2], if $\mathrm{E} \in \widehat{\mathsf{Fil}}$ then each truncation $\tau^B_{\geq n}\,\mathrm{E}$ is also complete and since complete filtered objects are stable under limits (Remark **2.2.4**) so is the filtered object $\lim\limits_n \tau^B_{\geq n}\,\mathrm{E}$.

Therefore, we have a well-defined functor $\tau^B_{\geq} : \widehat{\mathsf{Fil}} \to \mathsf{Fil}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))$.

Notice also that if E is in (Notation **5.5.11**) $\widehat{\mathsf{Fil}}^{\mathrm{const}\geq 0}(\mathsf{Mod}_k)$, then the formula (139) implies that every truncation $\tau^B_{\geq n}\mathrm{E}$ is also in $\widehat{\mathsf{Fil}}^{\mathrm{const}\geq 0}(\mathsf{Mod}_k)$.

Remark 5.5.16. In [Ant19] Antieau computes the Whitehead tower of $\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-$ for A smooth, (see Notation **5.5.10**) with respect to the Beilinson t-structure, showing that under the equivalence between $\mathsf{Fil}(\mathsf{Mod}_k)^{\heartsuit_B}$ and chain complexes (see [BMS19, Thm 5.4-(3)] for this equivalence) given by the formula (140), for $n \geq 0$, the filtered object $\pi^B_{2n}(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$ corresponds to the chain complex $\Omega^{\geq n}_A$ of the Remark **5.3.4**. Moreover, $\pi^B_n(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$ is zero for positive odd n's. For $n < 0$ one can use the formula (140) to show that $\pi^B_n(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-) = \pi^B_0(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$ if n is even, and zero if n is odd (see also [Ant19, Example 2.4] or the projection formula for invariants on the trivial circle action discussed in the Remark **5.5.20** below for $\mathsf{BS}^1 \simeq \mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$.) For this reason one is allowed to run the Whitehead tower for the Beilison t-structure at a double-speed, since $\tau^B_{\geq 2n+2}(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-) \to \tau^B_{\geq 2n+1}(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$ are equivalences $\forall n$.

Although the Beilinson-Whitehead tower is not complete for a general filtered object (see Notation **5.5.15**), it is in the case that concerns us:

Proposition 5.5.17. *Let $A \in \mathsf{N}(\mathsf{Poly}_k)$. Then, the double-speed Beilinson-Whitehead tower of the filtered object $\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-$,*

$$[\cdots \to \underbrace{\tau^B_{\geq 2n}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-}_{n} \to \cdots \to \underbrace{\tau^B_{\geq 2}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-}_{1} \to \underbrace{\tau^B_{\geq 0}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-}_{0} \to \cdots] \quad (141)$$

is complete.

Proof. The discussion in the Remark **5.5.16** shows that the objects $\pi^B_n(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$ are non-zero only for $n = 0, 2, 4, ..., 2d$, where $d = dim\, A$. This implies that the double-speed Beilison-Whitehead tower (141) is constant for $n \geq d+1$. Therefore its limit coincides with the filtered object $\tau^B_{\geq 2d+2}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-$. We argue that this object is zero in $\mathsf{Fil}(\mathsf{Mod})$. Since $\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-$ is complete (see Notation **5.5.10**), by the discussion in the Notation **5.5.15**, $\tau^B_{\geq 2d+2}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-$ is complete. Therefore by the discussion in Remark **2.2.4**, to prove that it is zero it is enough to show all its graded pieces vanish. Using (139) we have

$$\mathsf{gr}^i\tau^B_{\geq 2d+2}\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^- \simeq \tau_{\geq 2d+2-i}\mathsf{gr}^i(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)$$

But the construction of $\mathrm{F}^{\bullet}_{\mathrm{HKR}}\mathsf{HC}(A)^{-}$ (see Notation **5.5.10** and [Ant19, Example 2.4]) shows that

$$\mathsf{gr}^i(\mathrm{F}^{\bullet}_{\mathrm{HKR}}\mathsf{HC}(A)^{-}) = \begin{cases} (\Omega^i_A[i])^{\mathrm{h}\mathsf{S}^1} \simeq \Omega^i_A[i] \otimes \mathsf{C}^*(\mathsf{BS}^1, k) \simeq \bigoplus_{j\geq 0} \Omega^i_A[i-2j], & \text{if } 0 \leq i \leq d. \\ 0\,, & \text{otherwise} \end{cases}$$

Therefore, all graded pieces are automatically zero outside the range $0 \leq i \leq d$. Inside the range, we find $2d + 2 - i \geq i + 2 > i$ so that the truncations also vanish. This concludes the proof.

□

The following is our first comparison result: the smooth polynomial case.

Proposition 5.5.18. *Consider the commutative diagram (136):*

$$\begin{tikzcd} \mathsf{N}(\mathsf{Poly}_k) \arrow[d, "(\tau^{\mathsf{Fil}}_{\geq}\,\mathsf{HH}_{\mathsf{Fil}})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{lvl}"'] \arrow[dr, "\mathrm{F}^{\bullet}_{\mathrm{HKR}}\mathsf{HC}(-)^{-}"] & \\ \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k)) \arrow[r, "\mathsf{Fil}(\mathsf{colim})"] & \widehat{\mathsf{Fil}}(\mathsf{Mod}_k) \end{tikzcd} \tag{142}$$

Then the universal property of the double-speed Beilinson-Whitehead tower $\tau^{B}_{\geq 2*} : \mathsf{Fil}(\mathsf{Mod}_k) \to \mathsf{Fil}(\mathsf{Fil}(\mathsf{Mod}_k))$ *renders the commutativity of*

$$\begin{tikzcd} \mathsf{N}(\mathsf{Poly}_k) \arrow[d, "(\tau^{\mathsf{Fil}}_{\geq}\,\mathsf{HH}_{\mathsf{Fil}})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{lvl}"'] \arrow[dr, "\mathrm{F}^{\bullet}_{\mathrm{HKR}}\mathsf{HC}(-)^{-}"] & \\ \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k)) \arrow[d, "\mathrm{twist}"', "\sim"] & \widehat{\mathsf{Fil}}(\mathsf{Mod}_k) \arrow[dl, "\tau^{B}_{\geq 2*-}"'] \\ \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k)) & \end{tikzcd} \tag{143}$$

where the equivalence $\mathsf{Fil}(\mathsf{Fil}) \simeq \mathsf{Fil}(\mathsf{Fil})$ *is the twist interchanging the order of the filtrations.*

Proof. By the universal property of the Whitehead tower, it is enough to show that for each polynomial algebra A, the composition

$$\mathrm{twist} \circ (\tau^{\mathsf{Fil}}_{\geq}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{lvl}$$

has the properties described in Construction **2.2.5** in $\mathsf{Fil}(\mathsf{Fil}(\mathsf{Mod}_k))$. The bifiltered object $(\tau^{\mathsf{Fil}}_{\geq}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{lvl}$ is given by the sequence of filtered objects

$$[\cdots \to (\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \to \cdots \to (\tau^{\mathsf{Fil}}_{\geq 1}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \to (\tau^{\mathsf{Fil}}_{\geq 0}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} = \cdots] \tag{144}$$

To describe its image under the twist operation, let us denote by F_n the filtered object obtained by extracting the level n in (144):

$$\mathrm{F}_n := [\cdots \to [(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}]_n \to \cdots \to [(\tau^{\mathsf{Fil}}_{\geq 1}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}]_n \to [(\tau^{\mathsf{Fil}}_{\geq 0}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}]_n = \cdots]$$

Then, the construction twist $\circ\ (\tau^{\mathsf{Fil}}_{\geq}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}_{lvl}$ is given by the bifiltered object F

$$\mathrm{F} := [\cdots \to \mathrm{F}_{n+1} \to \mathrm{F}_n \cdots \to \mathrm{F}_1 \to \mathrm{F}_0 \to \mathrm{F}_{-1} \to \cdots] \tag{145}$$

Our goal is to show that (145) is equivalent to the double-speed Beilinson-Whitehead tower (141). We start with the computation of the graded pieces $\mathsf{gr}^n(\mathrm{F})$. This is a filtered object and unfolding the construction of F we find that its i-th level is given by

$$\mathsf{gr}^n(\mathrm{F})_i \simeq \mathsf{gr}^n[(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}]$$

Furthermore, the base-change formula of (23) tells us that

$$\mathsf{gr}[(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}] \simeq [\mathsf{gr}(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}$$

The compatibility with truncations of (135) in Remark **5.5.6**-(ii), tells us that

$$[\mathsf{gr}(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}} \simeq [\tau^{\mathsf{gr}}_{\geq i}\,\mathsf{gr}(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A))]^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}$$

and one now observes that the canonical map $\tau^{\mathsf{Fil}}_{\geq i}\mathsf{HH}_{\mathsf{Fil}}(A) \to \mathsf{HH}_{\mathsf{Fil}}(A)$ induces an equivalence

$$\tau^{\mathsf{gr}}_{\geq i}\ \big(\mathsf{gr}(\tau^{\mathsf{Fil}}_{\geq i}\,\mathsf{HH}_{\mathsf{Fil}}(A)\big) \simeq \tau^{\mathsf{gr}}_{\geq i}\ (\mathsf{gr}(\mathsf{HH}_{\mathsf{Fil}}(A))$$

Indeed, using the Theorem **5.4.1**-(b) and the Remark **5.5.2** for the smooth polynomial case, this map reads as

$$\tau^{\mathsf{gr}}_{\geq i}\left(\bigoplus_{j\geq i}\Omega^j_A[j](j)\right) \to \tau^{\mathsf{gr}}_{\geq i}\left(\bigoplus_{j\geq 0}\Omega^j_A[j](j)\right)$$

which we can check to be an equivalence using the definition of $\tau^{\mathsf{gr}}_{\geq i}$ as weight-wise truncations, togehter with the fact that for a smooth polynomial algebra, the Ω^i_A are all concentrated in homological degree 0. All summarized, we now have obtained

$$\mathsf{gr}^n(\mathrm{F})_i \simeq \mathsf{gr}^n\left[\left(\tau^{\mathsf{gr}}_{\geq i}\mathsf{DR}(A)\right)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}\right]$$

with $\tau^{\mathsf{gr}}_{\geq i}\mathsf{DR}(A) \simeq \bigoplus_{j\geq 0}\tau_{\geq i}\Omega^j[j] \simeq \bigoplus_{j\geq i}\Omega^j[j]$, and we find

$$\mathsf{gr}^n(\mathrm{F})_i \simeq \mathsf{gr}^n\left[\left(\bigoplus_{j\geq i}\Omega^j[j]\right)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}\right]$$

Let us first deal with the case $n \geq 0$.

We find that for $i \leq 0$, $\mathsf{gr}^n(\mathrm{F})$ is constant and that for $i = 0$, we have $\mathsf{gr}^n(\mathrm{F})_0 \simeq \mathsf{gr}^n[\mathsf{DR}(A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \simeq \Omega^{\geq n}_A[2n]$ (see Remark **5.3.4** and Theorem **5.4.1**-(c)). We compute $\mathsf{gr}^n(\mathrm{F})_1$ using the cofiber sequence of graded objects

$$\begin{tikzcd}
\mathsf{gr}^n(\mathrm{F})_1 \simeq \mathsf{gr}^n[(\bigoplus_{j\geq 1}\Omega^j[j])^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \arrow[r] \arrow[d] & \mathsf{gr}^n[(\bigoplus_{j\geq 0}\Omega^j[j])^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \simeq \Omega^{\geq n}_A[2n] \arrow[d] \\
0 \arrow[r] & \mathsf{gr}^n[(\Omega^0_A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}]
\end{tikzcd}$$

where Ω^0_A is seen as a graded module pure of weight 0 with a trivial action of $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$. Using the projection formula discussed in the Remark **5.5.20**, we obtain $(\Omega^0_A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}} \simeq \Omega^0_A \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})^{\mathsf{gr}}$. After the Proposition **4.1.1**, we find $\Omega^0_A \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})^{\mathsf{gr}} \simeq \bigoplus_{k\geq 0}\Omega^0_A(-k)[-2k]$ so that

$$\mathsf{gr}^n[(\Omega^0_A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] = \begin{cases} 0\,, & \text{if } n > 0. \\ \Omega^0_A(n)[2n], & \text{if } n \leq 0 \end{cases}$$

and

$$\mathsf{gr}^n(\mathrm{F})_1 = \begin{cases} \mathsf{gr}^n(\mathrm{F})_0 \simeq \Omega^{\geq n}_A[2n]\,, & \text{if } n > 0. \\ \mathrm{fiber}(\,\Omega^{\geq 0}_A \to \Omega^0_A) \simeq \Omega^{\geq 1}_A\,, & \text{if } n = 0 \end{cases}$$

Recall that by Theorem **5.1.3**-(2) $\Omega^1_A[1]$ is in weight 1 in $\mathsf{DR}(A)$. A similar computation as above shows that for a fixed $i \geq 0$, and the graded object of pure weight i, $\Omega^i_A[i](i)$ with a trivial action of $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ we find

$$\mathsf{gr}^n[(\Omega^i_A[i](i))^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \simeq (\Omega^i_A[i](i)) \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})^{\mathsf{gr}} = \begin{cases} 0\,, & \text{if } n > i. \\ \Omega^i_A[-i+2n], & \text{if } 0 \leq n \leq i \end{cases}$$

By induction on i, using the cofiber sequences

$$\begin{tikzcd} \mathsf{gr}^n(\mathrm{F})_{i+1} \simeq \mathsf{gr}^n[(\bigoplus_{j\geq i+1} \Omega^j[j])^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \arrow[r] \arrow[d] & \mathsf{gr}^n[(\bigoplus_{j\geq i} \Omega^j[j])^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^n[(\Omega^i_A[i](i))^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}] \end{tikzcd} \tag{146}$$

for $n \geq 0$, we find the filtered object

$$\mathsf{gr}^n(\mathrm{F}) = [\cdots \to \underbrace{\Omega^{\geq n+2}_A[2n]}_{n+2} \to \underbrace{\Omega^{\geq n+1}_A[2n]}_{n+1} \to \underbrace{\Omega^{\geq n}_A[2n]}_{n} = \underbrace{\Omega^{\geq n}_A[2n]}_{n-1} = \cdots = \underbrace{\Omega^{\geq n}_A[2n]}_{0} = \cdots]$$

Following the discussion in the Remark **5.5.16**, we see that we have an equivalence of filtered objects

$$\mathsf{gr}^n(\mathrm{F}) \simeq \pi^B_{2n}(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)[2n] \quad \text{if } n \geq 0$$

For $n < 0$ we have

$$\mathsf{gr}^n(\mathrm{F}) \simeq \pi^B_0(\mathrm{F}^\bullet_{\mathrm{HKR}}\mathsf{HC}(A)^-)[2n] \simeq \Omega^{\geq 0}_A[2n] \quad \text{if } n < 0$$

Indeed, this follows from the use of the same cofiber sequences (146), but this time starting from the formula (131) for the negative graded pieces and comparing with the discussion in the Remark **5.5.16**.

Since both (145) and (141) are complete, this shows they are the same double-speed Beilinson-Whithead tower.

□

Lemma 5.5.19. *Let* $\pi : \mathsf{BS}^1_{\mathsf{Fil}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ *be the canonical projection and let* $\mathrm{E} \in \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])^{<\infty}$ *(homologically bounded above). Then the canonical map*

$$(\pi^*(\mathrm{E}))^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \simeq \pi_*\,\pi^*(\mathrm{E}) \to \pi_*(\mathscr{O}) \otimes \mathrm{E} \simeq \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \otimes \mathrm{E} \tag{147}$$

is an equivalence in $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$.

Proof. Following Notation **1.2.12**-(b), π_* is left-t-exact. Moreover, since the t-structure in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ is determined by pullback along the atlas $e : [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \to \mathsf{BS}^1_{\mathsf{Fil}}$ (see Construction **3.2.4**), we have $e^* \circ \pi^* = \mathrm{id}$ which implies that π^* is t-exact.
Moreover, since for every N, $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])_{\leq N} \subseteq \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ is stable under filtered colimits (because homology groups commute with filtered colimits), so is $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{\leq N} \subseteq \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. This shows that $\pi^* : \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])_{\leq N} \to \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{\leq N}$ preserves filtered colimits. Finally, using [HLP14, A.1.3 - part 2] (evoking our flat atlas by affines for $\mathsf{BS}^1_{\mathsf{Fil}}$ of Construction **3.2.4**) we deduce that $\pi_* : \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{\leq N} \to \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])_{\leq N}$ also commutes with filtered colimits. In parallel, is is easy to notice that the operation $\mathrm{E} \mapsto \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \otimes \mathrm{E}$ is well-defined as a functor

$$\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])_{\leq N} \to \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])_{\leq N}$$

and also commutes with filtered colimits. Since the family $\mathscr{O}(n) \in \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$ generate $\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])$, we are reduced to show that (147) is an equivalence when $\mathrm{E} = \mathscr{O}(n)$. But since the $\mathscr{O}(n)$ are invertible objects (therefore dualizable), we are reduced to show the claim for $\mathrm{E} = \mathscr{O}$, where this is a tautology.

□

Remark 5.5.20. The arguments and conclusion of Lemma **5.5.19** apply mutatis-mutantis to the two projections $\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} \to \mathsf{B}\mathbb{G}_{\mathrm{m}}$ and $\mathsf{B}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}} \to \mathsf{Spec}\,\mathbb{Z}_{(p)}$.

This concludes the comparison of the two constructions in the smooth polynomial case. We now embark on a new digression to explain how to extend this comparison to all SCR_k. This is not a straightforward consequence of Proposition **5.5.18** because none of the functors used in the diagram (142), when Kan extended, give rise to the correct construction, as the following remark intends to illustrate:

Remark 5.5.21. The composition

$$\mathsf{HC}^-_{\mathsf{Fil}} : \mathsf{SCR}_k \xrightarrow{\mathsf{HH}_{\mathsf{Fil}}} \widehat{\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})} \xrightarrow{(-)^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}} \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)$$

does not commute with sifted colimits in general. Indeed, recall from the Remark **2.2.4** that $\widehat{\mathsf{Fil}}(\mathsf{Mod}_k)$ is a presentable localization of $\mathsf{Fil}(\mathsf{Mod}_k)$ with a left adjoint to the inclusion $\widehat{\mathsf{Fil}}(\mathsf{Mod}_k) \subseteq \mathsf{Fil}(\mathsf{Mod}_k)$ given by the completion functor $\widehat{(-)}$. Let then $A : K \to \mathsf{SCR}_k$ be a sifted diagram with $B := \mathsf{colim}_{k\in K} A_k$. Then we a canonical map

$$\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)} \mathsf{HC}^-_{\mathsf{Fil}}(A_k) \to \mathsf{HC}^-_{\mathsf{Fil}}(B)$$

But by definition of a presentable localization, l.h.s is computed as the completion of the colimit in $\mathsf{Fil}(\mathsf{Mod}_k)$

$$\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \widehat{\mathsf{Fil}}(\mathsf{Mod}_k)} \mathsf{HC}^-_{\mathsf{Fil}}(A_k) \simeq (\widehat{\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \mathsf{Fil}(\mathsf{Mod}_k)} \mathsf{HC}^-_{\mathsf{Fil}}(A_k)})$$

In other words, the commutation with sifted colimits is measured by fact the canonical map in $\mathsf{Fil}(\mathsf{Mod}_k)$

$$(\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \mathsf{Fil}(\mathsf{Mod}_k)} \mathsf{HC}^-_{\mathsf{Fil}}(A_k)) \to \mathsf{HC}^-_{\mathsf{Fil}}(B) \tag{148}$$

becomes an isomorphism after completion. But again using the Remark **2.2.4** we know that the map (148) becomes an equivalence after completion if and only if the induced map between the associated graded objects is an equivalence:

$$[\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \mathsf{Fil}(\mathsf{Mod}_k)} \mathsf{HC}^-_{\mathsf{Fil}}(A_k))]^{\mathsf{gr}} \to \mathsf{HC}^-_{\mathsf{Fil}}(B)^{\mathsf{gr}} \tag{149}$$

Even better, since gr commutes with all colimits, we only have to test the map of graded objects

$$\underbrace{\mathsf{colim}_{k\in K}}_{\text{in } \mathsf{Mod}_k^{\mathbb{Z}-\mathsf{gr}}} \mathsf{HC}^-_{\mathsf{Fil}}(A_k))^{\mathsf{gr}} \to \mathsf{HC}^-_{\mathsf{Fil}}(B)^{\mathsf{gr}} \tag{150}$$

But this poses a problem: in light of Theorem **5.4.1**-(c), both sides are computed in terms of the completed truncated de Rham complex which as explained in Construction **5.3.3** contains an infinite product of wedge powers of the cotangent complex. But infinite products do not commute with sifted colimits in general.

The idea to solve the problem posed in the Remark **5.5.21** is to consider an extra filtration that disassembles the components of the infinite product found in the formula (150).

Construction 5.5.22. (Skeletal Filtration) Consider the map (84) of Construction **4.1.2**:

$$u : \pi^*(\mathscr{O}(-1))[-2] \to \mathscr{O} \tag{151}$$

in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. By abuse of notation, we will write again $\mathscr{O}(-1)[-2] := \pi^*(\mathscr{O}(-1))[-2]$. Since $\mathscr{O}(-1)$ is a line bundle, u can also be seen as a map

$$\mathscr{O} \to \mathscr{O}(1)[2]$$

in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. In what follows, we will consider the $\mathbb{Z}^{\mathsf{op}}$-sequence $\widetilde{\mathscr{O}}$ in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ given by

$$\cdots \longrightarrow \widetilde{\mathscr{O}}_1 = \mathscr{O} \xrightarrow{=} \widetilde{\mathscr{O}}_0 = \mathscr{O} \xrightarrow{u} \widetilde{\mathscr{O}}_{-1} = \mathscr{O}(1)[2] \xrightarrow{u} \widetilde{\mathscr{O}}_{-2} = \mathscr{O}(2)[4] \xrightarrow{u} \cdots$$

Since we have a canonical identification

$$(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} := \pi_*(\mathrm{E}) \simeq \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\mathscr{O}, E)$$

we can use $\widetilde{\mathscr{O}}$ to obtain a lift $\widetilde{\pi}_*$ of π_*:

$$\begin{tikzcd} \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \arrow[r, "\pi_*"] \arrow[d, "\widetilde{\pi}_*"] & \mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]) \\ \mathsf{Fil}(\mathsf{QCoh}([\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}])) \arrow[ur, "\mathsf{colim}"] & \end{tikzcd} \tag{152}$$

Informally, this lift sends $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$ to

$$[\cdots \to \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\mathscr{O}(2)[4], E) \to \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\mathscr{O}(1)[2], E) \to = (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} = (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} = \cdots]$$

Moreover, since $\mathscr{O}(t)[2t]$ are line bundles, we have

$$\mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\mathscr{O}(t)[2t], E) = (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \otimes \mathscr{O}(-t)[-2t] =: (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t]$$

for $t \geq 0$, and the object $\widetilde{\pi}_*(E)$ becomes

$$[\cdots \to \underbrace{(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-2)[-4]}_{2} \to \underbrace{(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-1)[-2]}_{1} \to = \underbrace{(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}}_{0} = \underbrace{(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}}_{-1} = \cdots] \tag{153}$$

Construction 5.5.23. The extra filtration of the Construction **5.5.22** is our version for the filtered circle of the usual skeletal filtration of the topological space BS^1 by complex projective spaces $\mathbb{CP}^n$ used in [Ant19, §4] and [Lur15]. Indeed, we have $(\mathbb{C})^{\mathrm{h}\mathsf{S}^1} \simeq \mathsf{C}^*(\mathsf{BS}^1, \mathbb{C}) \simeq \mathbb{C}[u]$ where u is a generator in homological degree (-2). In particular, we find cofiber sequences

$$\begin{tikzcd}
\mathbb{C}[u][-2t] \arrow[r, "u"] \arrow[d] & \mathbb{C}[u] = (\mathbb{C})^{\mathrm{h}\mathsf{S}^1} \arrow[d] \\
0 \arrow[r] & (\mathbb{C})^{\mathrm{h}\Omega\,\mathbb{CP}^{t-1}} \simeq \mathbb{C}[0] \oplus \mathbb{C}[-2] \oplus \cdots \oplus \mathbb{C}[-2(t-1)]
\end{tikzcd}$$

More generally, if $\mathrm{M} \in \mathsf{QCoh}(\mathsf{BS}^1)$, we define the skeletal filtration of $(\mathrm{M})^{\mathrm{h}\mathsf{S}^1}$ to be the $\mathbb{Z}^{op}$ sequence $\mathrm{F}_\bullet(\mathrm{M})$ whose level $t \geq 0$ is given by

$$\mathrm{F}_t(\mathrm{M}) := \mathrm{fiber}\left((\mathrm{M})^{\mathrm{h}\mathsf{S}^1} \to (\mathrm{M})^{\mathrm{h}\Omega\mathbb{CP}^{t-1}}\right)$$

and is constant equal to $(\mathrm{M})^{\mathrm{h}\mathsf{S}^1}$ for $t \leq 0$. This construction provides a lifting of $(-)^{\mathrm{h}\mathsf{S}^1} : \mathsf{QCoh}(\mathsf{BS}^1) \to \mathsf{Mod}_k$ to filtered objects

$$\begin{tikzcd}
\mathsf{QCoh}(\mathsf{BS}^1) \arrow[r, "(-)^{\mathrm{h}\mathsf{S}^1}"] \arrow[d, "\widetilde{(-)^{\mathrm{h}\mathsf{S}^1}}"'] & \mathsf{Mod}_k \\
\mathsf{Fil}(\mathsf{Mod}_k) \arrow[ur, "\mathsf{colim}"'] &
\end{tikzcd} \tag{154}$$

Remark 5.5.24. The Construction **5.5.23** and the Construction **5.5.22** are compatible under the extraction of underlying objects. Indeed, the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{BS}^1_{\mathsf{Fil}} \arrow[d, "\pi"] & \mathsf{BS}^1 \arrow[l, "1_a"'] \arrow[d, "\pi^{\mathrm{u}}"] \\
{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} \arrow[d, "q"] & \mathsf{Spec}\,\mathbb{Z}_{(p)} \arrow[l, "1"'] \arrow[dl, "e"] \\
\mathsf{B}\mathbb{G}_{\mathrm{m}} &
\end{tikzcd}$$

implies that

$$(1_a)^*\,(\mathscr{O}(t)[2t]) \simeq (1_a)^*\,\pi^*\,q^*(\mathbb{Z}_{(p)}(t)[2t]) \simeq (\pi^{\mathrm{u}})^*\,e^*(\mathbb{Z}_{(p)}(t)[2t])$$

but since e is the atlas, e^* amounts to forgetting the grading, so that

$$(\pi^{\mathrm{u}})^*\,e^*(\mathbb{Z}_{(p)}(t)[2t]) \simeq \mathscr{O}[2t]$$

.

It follows, since 1^* is symmetric monoidal, that

$$((\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t])^{\mathrm{u}} \simeq ((\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}})^{\mathrm{u}}[-2t]$$

As in (128), if we assume that $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{<\infty}$, we find

$$((\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t])^{\mathrm{u}} \simeq ((\mathrm{E}^{\mathrm{u}})^{\mathrm{h}\mathsf{S}^1}[-2t]$$

In summary, this discussion establishes the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{<\infty} \arrow[r, "1_a^*"] \arrow[d, "\widetilde{\pi_*}"] & \mathsf{QCoh}(\mathsf{BS}^1) \arrow[d, "\widetilde{(-)^{\mathrm{h}\mathsf{S}^1}}"] \\
\mathsf{Fil}(\mathsf{QCoh}(\mathsf{S}^1_{\mathsf{Fil}})) \arrow[r, "\mathsf{Fil}(\,1_a^*)"] & \mathsf{Fil}(\mathsf{Mod}_k)
\end{tikzcd} \tag{155}$$

Notation 5.5.25. We denote by $\mathsf{HC}^-_{\mathsf{BiFil}}$ the composition

$$\mathsf{SCR}_k \xrightarrow{\mathsf{HH}_{\mathsf{Fil}}} \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}}) \xrightarrow{\widetilde{\pi_*}} \mathsf{Fil}(\mathsf{Fil}(\mathsf{Mod}_k))$$

providing $\mathsf{HC}^-_{\mathsf{Fil}}$ with the extra skeletal filtration of Construction **5.5.22**.

Proposition 5.5.26. *The skeletal filtration of Construction* ***5.5.22*** *is complete for any* $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{>-\infty}$ *(homologically bounded below). In particular, the functor* $\mathsf{HC}^-_{\mathsf{BiFil}}$ *factors through bi-complete filtered modules:*

$$\mathsf{HC}^-_{\mathsf{BiFil}} : \mathsf{SCR}_k \to \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))$$

Proof. Let $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})_{>-\infty}$. We want to show that the limit of the tower (153) is zero. Let us denote by C_t the fiber in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$

$$\begin{tikzcd} C_t \arrow[r] \arrow[d] & 0 \arrow[d] \\ \mathscr{O} \arrow[r, "u^t"] & \mathscr{O}(t)[2t] \end{tikzcd}$$

Unfolding the definitions, this gives us cofiber sequences

$$\begin{tikzcd} (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t] \arrow[r, "u^t"] \arrow[d] & (\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} \arrow[d] \\ 0 \arrow[r] & \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathrm{E}) \end{tikzcd} \tag{156}$$

so that by stability, the limit of (153) vanishes if and only the induced map

$$(\mathrm{E})^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}} = \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\mathscr{O}, \mathrm{E}) \to \lim_t \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathrm{E}) \simeq \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(\operatorname{colim}_t C_t, \mathrm{E}) \tag{157}$$

is an equivalence. Using the assumption that E is homologically bounded below we are reduced to show the statement for E in the heart of the structure of $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$. Indeed, since the t-structure is left-t-complete (see Notation **1.2.12**)-(c)), every object E is the limit of its Postnikov tower $\mathrm{E} \simeq \lim_n \tau^{\mathsf{Fil}}_{\leq n}\mathrm{E}$. In particular, if we assume that E is bounded below, there exists $N \in \mathbb{Z}$ such that $\mathrm{E} \simeq \lim_{n \geq N} \tau^{\mathsf{Fil}}_{\leq n}\mathrm{E}$ with $\tau^{\mathsf{Fil}}_{\leq N}\mathrm{E} = (\pi^{\mathsf{Fil}}_N \mathrm{E})[N]$. Since both sides of (157) preserve limits in E, and by the nature of the Postnikov tower, by induction we are reduced to show that (157) is an equivalence when E is in $\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\heartsuit}$. But in this case the action of $\mathsf{S}^1_{\mathsf{Fil}}$ must be trivial and we can use the Lemma **5.5.19** and the fact that $\otimes$ is compatible with colimits separetely in each variable, to establish that the cofiber sequence (156) is equivalent to

$$\begin{tikzcd} \mathrm{E} \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})(-t)[-2t] \arrow[r, "u^t"] \arrow[d] & \mathrm{E} \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \arrow[d] \\ 0 \arrow[r] & \mathrm{E} \otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathscr{O}) \end{tikzcd} \tag{158}$$

in filtered objects. Therefore, to conclude the proof it will be enough to show that if $\mathrm{E} \in \mathsf{Fil}(\mathsf{Mod}_k)^{\heartsuit_{\mathrm{std}}}$ (the levelwise standard t-structure), then the the canonical map of filtered objects

$$\mathrm{E} \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O}) \to \lim_t \mathrm{E} \otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathscr{O}) \tag{159}$$

is an equivalence of filtered objects. For this purpose we start with the observation that the since the standard t-structure in Mod_k is both left and right complete (see [Lur17, 7.1.1.13]), so is the levelwise t-structure on $\mathsf{Fil}(\mathsf{Mod}_k)$. Therefore, if we denote by π_i^{Fil} the homotopy groups with respect to the levelwise t-structure in $\mathsf{Fil}(\mathsf{Mod}_k)$, to show that (159) is an equivalence in $\mathsf{Fil}(\mathsf{Mod}_k)$ it is enough to show that it for every i the induced map

$$\pi_i^{\mathsf{Fil}}(\mathrm{E} \otimes \mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})) \to \pi_i^{\mathsf{Fil}}(\lim_t \mathrm{E} \otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathscr{O})) \tag{160}$$

is an isomorphism. By the conclusion of Proposition **4.1.1**, we know that as a filtered object $\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})$ decomposes as a direct sum of filtered objects

$$\mathsf{C}^*(\mathsf{BS}^1_{\mathsf{Fil}}, \mathscr{O})) \simeq \bigoplus_{p \geq 0} \mathbb{Z}_{(p)}(-p)[-2p] \tag{161}$$

This formula, together with the cofiber-sequence (159) when $\mathrm{E} = \mathscr{O}$, tells us that, as filtered objects, we have a direct sum decomposition

$$\mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t, \mathscr{O})) \simeq \bigoplus_{p \geq 0}^{t-1} \mathbb{Z}_{(p)}(-p)[-2p] \tag{162}$$

Combining the formulas (161) and (162) and the compatibility of the Day tensor products with direct sums in $\mathsf{Fil}(\mathsf{Mod}_k)$, the morphism (159) can be written as

$$\bigoplus_{p \geq 0} \mathrm{E}(-p)[-2p] \to \lim_t \bigoplus_{p \geq 0}^{t-1} \mathrm{E}(-p)[-2p] \tag{163}$$

By definition of π_i^{Fil} and the assumption that $\mathrm{E} = \pi_0^{\mathsf{Fil}}(\mathrm{E})$, we find

$$\pi_i^{\mathsf{Fil}}(\bigoplus_{p\geq 0}\mathrm{E}(-p)[-2p]) \simeq \bigoplus_{p\geq 0}\pi_i^{\mathsf{Fil}}(\mathrm{E}(-p)[-2p])) \simeq \begin{cases}\mathrm{E}(-p) & \text{for } i=-2p\\ 0, & \text{otherwise}\end{cases} \tag{164}$$

To conclude we have to compute the objects $\pi_i^{\mathsf{Fil}}(\lim_t \bigoplus_{p\geq 0}^{t-1}\mathrm{E}(-p)[-2p])$. For this purpose we use the limit spectral sequence. Since $\mathsf{Fil}(\mathsf{Mod}_k)$ is a stable ∞-category and the sequence

$$t \mapsto \lim_t \mathrm{E}\otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t,\mathscr{O}))$$

vanishes (by definition) for $t<0$, we can use a dual version of [Lur17, 1.2.2.14] providing a spectral sequence with first page given by

$$\pi_{t+q}^{\mathsf{Fil}}(\mathrm{fiber}\,[\ \mathrm{E}\otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t,\mathscr{O})) \to \mathrm{E}\otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_{t-1},\mathscr{O})\ ]\,)$$

and converging to $\pi_{t+q}^{\mathsf{Fil}}(\lim_t \bigoplus_{p\geq 0}^{t-1}\mathrm{E}(-p)[-2p]))$. But an immediate computation with the formula (162) shows that

$$\mathrm{fiber}\,[\mathrm{E}\otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_t,\mathscr{O})) \to \mathrm{E}\otimes \mathrm{RHom}_{\mathsf{BS}^1_{\mathsf{Fil}}}(C_{t-1},\mathscr{O})] \ \simeq \mathrm{E}(-t)[-2t]$$

and the spectral sequence becomes

$$\pi_{t+q}^{\mathsf{Fil}}(\mathrm{E}(-t)[-2t]) \implies \pi_{t+q}^{\mathsf{Fil}}(\lim_t \bigoplus_{p\geq 0}^{t-1}\mathrm{E}(-p)[-2p]))$$

But since E is in the heart, we have

$$\pi_{t+q}^{\mathsf{Fil}}(\mathrm{E}(-t)[-2t]) \simeq \begin{cases}\mathrm{E}(-t) & \text{for } t+q=-2t\\ 0, & \text{otherwise}\end{cases}$$

and therefore the spectral sequence degenerates at the first page and we find

$$\pi_i^{\mathsf{Fil}}(\lim_t \bigoplus_{p\geq 0}^{t-1}\mathrm{E}(-p)[-2p])) = \begin{cases}\mathrm{E}(-t) & \text{for } i=-2t\\ 0, & \text{otherwise}\end{cases}$$

thus coinciding with (164). This concludes the proof.

□

Remark 5.5.27. The argument in the proof of Proposition **5.5.26** can also be used to show that the functor $\widetilde{(-)}^{\mathrm{hS}^1}$ in Construction **5.5.23** also lands in complete filtrations. See the proof of [Ant19, Lemma 4.2].

We will now compute the bigraded pieces $\mathsf{HC}^-_{\mathsf{BiFil}}$. We start with a simple observation:

Remark 5.5.28. The two ways of extracting the associated bigraded object from a biltered object, coincide. More precisely, we have a commutative diagram

$$\begin{tikzcd} \mathsf{Fil}(\,\mathsf{Fil}(\mathsf{Mod}_k)\,) \arrow[r, "\mathsf{Fil}(\,\mathsf{gr}\,)"] \arrow[d, "\mathsf{gr}(\,-\,)"] & \mathsf{Fil}(\,\mathsf{Mod}_k^{\mathbb{Z}-\mathsf{gr}}\,) \arrow[d, "\mathsf{gr}(\,-\,)"] & \\ (\,\mathsf{Fil}(\mathsf{Mod}_k)\,)^{\mathbb{Z}-\mathsf{gr}} \arrow[r, "(\,\mathsf{gr}\,)^{\mathbb{Z}-\mathsf{gr}}"] & (\,\mathsf{Mod}_k^{\mathbb{Z}-\mathsf{gr}}\,)^{\mathbb{Z}-\mathsf{gr}} \arrow[r, "\sim"] & \mathsf{Mod}_k^{\mathbb{Z}\times\mathbb{Z}-\mathsf{gr}} \end{tikzcd} \tag{165}$$

To check this it is enough to see what happens for each bigraded piece (i,n): if $\mathrm{E} \in \mathsf{Fil}(\mathsf{Fil}(\mathsf{Mod}_k))$ with i the index of exterior filtration and n for the interior, we find

$$\text{down-left composition} = \mathrm{cofiber}(\mathrm{E}_i^{n+1}/\mathrm{E}_{i+1}^{n+1} \to \mathrm{E}_i^n/\mathrm{E}_{i+1}^n)$$

$$\text{up-right composition} = \mathrm{cofiber}(\mathrm{E}_{i+1}^n/\mathrm{E}_{i+1}^{n+1} \to \mathrm{E}_i^n/\mathrm{E}_i^{n+1})$$

It is an easy exercice to use concatenation of pushouts in Mod_k to check that the two agree canonically.

Remark 5.5.29. Let us compute the graded pieces of the skeletal filtration of Construction **5.5.22**. Given $\mathrm{E} \in \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})$, the bifiltered object $\widetilde{\pi}_*(\mathrm{E})$ is given by a diagram of filtered objects

$$[\cdots \to \underbrace{\mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}}(-t-1)[-2(t+1)]}_{t+1} \to \underbrace{\mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}}(-t)[-2t]}_{t} \to \cdots \to \underbrace{\mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}}}_{0} = \mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}} = \cdots]$$

In particular, the associated graded piece of degree t with respect to the skeletal filtration $\mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E}))$, is the filtered object given by the cofiber in $\mathsf{Fil}(\mathsf{Mod}_k)$

$$\begin{tikzcd} \mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}}(-t-1)[-2(t+1)] \arrow[r] \arrow[d] & \mathrm{E}^{\mathrm{hS}^1_{\mathsf{Fil}}}(-t)[-2t] \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E})) \end{tikzcd} \tag{166}$$

The filtered object $\mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E}))$ has its own internal associated graded $\mathsf{gr}(\mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E})))$ that can be computed as follows: since $\mathsf{gr} : \mathsf{Fil}(\mathsf{C}) \to \mathsf{C}$ commutes with all colimits, the cofiber sequence (166) produces a cofiber sequence in graded objects

$$\begin{tikzcd} \mathsf{gr}(\mathrm{E}^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t-1)[-2(t+1)]) \arrow[r] \arrow[d] & \mathsf{gr}(\mathrm{E}^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t]) \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}(\mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E}))) \end{tikzcd} \tag{167}$$

Finally, we notice that we have a canonical equivalence in graded objects

$$\mathsf{gr}(\mathrm{E}^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}(-t)[-2t]) \simeq \mathsf{gr}(\mathrm{E}^{\mathrm{h}\mathsf{S}^1_{\mathsf{Fil}}}) \otimes \mathsf{gr}(\mathscr{O}(-t)[-2t])) \simeq \mathsf{gr}(E)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}(-t)[-2t]$$

that follows from the base-change formula (23) together with the fact that gr is identified with the pullback along $0 : \mathsf{B}\mathbb{G}_{\mathrm{m}} \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ (see Theorem **2.2.10**) so that $\mathsf{gr}(\mathscr{O}(-t)[-2t]) \simeq 0^*(q^*(\mathbb{Z}_{(p)}(-t)[-2t]) = \mathrm{Id}^*(\mathbb{Z}_{(p)}(-t)[-2t]) = \mathbb{Z}_{(p)}(-t)[-2t]$ using the notations in Construction **4.1.2**.
By definition of the Day tensor product on graded objects and since $\mathbb{Z}_{(p)}(-1)$ is pure of weight -1, we find

$$\mathsf{gr}^i(\mathsf{gr}(E)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}}(-t][-2t]) \simeq \mathsf{gr}^{i+t}(\mathsf{gr}(E)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}})[-2t]$$

and the cofiber sequence (167) becomes

$$\begin{tikzcd} \mathsf{gr}^{i+t+1}(\mathsf{gr}(E)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}})[-2t-2] \arrow[r] \arrow[d] & \mathsf{gr}^{i+t}(\mathsf{gr}(E)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}})[-2t] \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^i(\mathsf{gr}^t_{\mathrm{sk}}(\widetilde{\pi}_*(\mathrm{E}))) \end{tikzcd} \tag{168}$$

Proposition 5.5.30. *Let $A \in \mathsf{SCR}_k$. Then the graded pieces of the bifiltered object $\mathsf{HC}^-_{\mathsf{BiFil}}(A)$ (Notation **5.5.25**) are given by*

$$\mathsf{gr}^i(\mathsf{gr}^t_{\mathrm{sk}}(\mathsf{HC}^-_{\mathsf{BiFil}}(A))) \simeq \bigwedge^{i+t} \mathbb{L}_A[i-t] \tag{169}$$

Proof. Using the Theorem **5.4.1**-(b), the cofiber sequence (168) becomes

$$\begin{tikzcd} \mathsf{gr}^{i+t+1}(\mathsf{DR}(A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}})[-2t-2] \arrow[r] \arrow[d] & \mathsf{gr}^{i+t}(\mathsf{DR}(A)^{\mathrm{h}(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}})[-2t] \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^i(\mathsf{gr}^t_{\mathrm{sk}}(\mathsf{HC}^-_{\mathsf{BiFil}}(A))) \end{tikzcd} \tag{170}$$

and using Theorem **5.4.1**-(c), it becomes

$$\begin{tikzcd} \mathbb{L}\widehat{\mathsf{DR}}^{\geq i+t+1}(A/k)[-2t-2] \arrow[r] \arrow[d] & \mathbb{L}\widehat{\mathsf{DR}}^{\geq i+t}(A/k)[-2t] \arrow[d] \\ 0 \arrow[r] & \mathsf{gr}^i(\mathsf{gr}^t_{\mathrm{sk}}(\mathsf{HC}^-_{\mathsf{BiFil}}(A))) \end{tikzcd} \tag{171}$$

Finally, using the explicit formula for the derived completed truncated de Rham complex of Construction **5.3.3**, we obtain

$$\mathsf{gr}^i(\mathsf{gr}^t_{\mathrm{sk}}(\mathsf{HC}^-_{\mathsf{BiFil}}(A))) \simeq \bigwedge^{i+t} \mathbb{L}_A[2(i+t)-(i+t)][-2t] \simeq \bigwedge^{i+t} \mathbb{L}_A[i-t]$$

□

Proposition 5.5.31. *The functor* $\mathsf{HC}^-_{\mathsf{BiFil}} : \mathsf{SCR}_k \to \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))$ *(see Proposition **5.5.26**) preserves sifted colimits. In particular it is the left Kan extension of its restriction to polynomial algebras.*

Proof. This is now a consequence of the fact that a map of bi-complete, bi-filtered complexes is an equivalence if and only the associated bi-graded map is an equivalence - this follows from applying the discussion in the Remark **2.2.4** to filtered objects in filtered objects. We then proceed as in the Remark **5.5.21** but this time it is enough to check that the colimit map (150) on the bi-graded pieces is compatible with sifted colimits. But this exactly where the formula in the Proposition **5.5.30** comes into play: the bi-graded pieces $\bigwedge^{i+t} \mathbb{L}_A[i-t]$ commute with sifted colimits. This can be deduced for instance as a combination of the Proposition **5.5.5** and the fact the functor gr commutes with all colimits and the graded pieces of $\mathsf{HH}_{\mathsf{Fil}}(A)$ are precisely what we need by the Theorem **5.4.1**-(b). □

We are now in position to establish the comparison result for all simplicial commmutative algebras.

Construction 5.5.32. By incorporating the skeletal filtration of Construction **5.5.22** in the diagrams (136), (137), (138) and (155), we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathsf{N}(\mathsf{Poly}_k) \arrow[d, "\mathsf{HH}_{\mathsf{Fil}}"] \arrow[dr, "\mathsf{HH}"] & \\
 & \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0} \arrow[r, "(-)^{\mathrm{u}}"] \arrow[d, "\tau^{\mathsf{Fil}}_{\geq}"] & \mathsf{QCoh}(\mathsf{BS}^1)^{<\infty} \arrow[d, "\tau^{\mathrm{std}}_{\geq}"] \\
\mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0} \arrow[ur, equal] \arrow[d, "\widetilde{\pi_*}"] & \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\ \mathsf{QCoh}(\mathsf{BS}^1_{\mathsf{Fil}})^{\dagger,<\infty}_{\geq 0}\ ] \arrow[l, "\mathsf{colim}_{\mathrm{Wth}}"] \arrow[r, "(-)^{\mathrm{u}}_{lvl}"] \arrow[d, "(\widetilde{\pi_*})_{lvl}"] & \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\ \mathsf{QCoh}(\mathsf{BS}^1)\ ] \arrow[d, "((\widetilde{-})^{\mathrm{hS}^1})_{lvl}"] \\
\widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathsf{S}^1_{\mathsf{Fil}}}{\mathsf{Fil}}}(\mathsf{Mod}_k)\ ] & \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\ \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathsf{S}^1_{\mathsf{Fil}}}{\mathsf{Fil}}}(\mathsf{Mod}_k)\ ]\ ] \arrow[l, "\mathsf{colim}_{\mathrm{Wth}}"] \arrow[r, "\mathsf{colim}_{\mathsf{S}^1_{\mathsf{Fil}}}"] \arrow[d, "\sim\ \mathrm{twist}"] & \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\ \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\mathsf{Mod}_k]\ ] \arrow[d, "\sim\ \mathrm{twist}_{\mathrm{sk},\mathrm{Wth}}"] \\
 & \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathsf{S}^1_{\mathsf{Fil}}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}(\mathsf{Mod}_k)\ ]\ ] \arrow[ul, "\mathsf{colim}_{\mathrm{Wth}}"] \arrow[r, "\mathsf{colim}_{\mathsf{S}^1_{\mathsf{Fil}}}"] & \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\mathsf{Mod}_k]\ ]
\end{tikzcd} \tag{172}$$

where the subscript indicates the origin of the filtration: Whitehead tower for the standard t-structures, Wth, skeletal sk and our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$. Finally, the comparison result of Proposition **5.5.18** (and our Notation **5.5.25**) provides commutativity for

$$\begin{tikzcd}
 & \mathsf{N}(\mathsf{Poly}_k) \arrow[dl, "\mathsf{HC}^-_{\mathsf{BiFil}}"'] \arrow[d] \arrow[dr] & \\
\widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathsf{S}^1_{\mathsf{Fil}}}{\mathsf{Fil}}}(\mathsf{Mod}_k)\ ] & \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathsf{S}^1_{\mathsf{Fil}}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}(\mathsf{Mod}_k)\ ]\ ] \arrow[l, "\mathsf{colim}_{\mathrm{Wth}}"] & \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}^{\mathrm{const}\geq 0}[\mathsf{Mod}_k]\ ] \arrow[l, "\widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \tau^B_{\geq 2*}\ ]"]
\end{tikzcd} \tag{173}$$

Notation 5.5.33. As in [Ant19, §4], we denote the right-diagonal arrow in the diagram (173), $\mathsf{N}(\mathsf{Poly}_) \to \widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \widehat{\underset{\mathrm{Wht}}{\mathsf{Fil}}}[\mathsf{Mod}_k]\ ]$, by $\mathrm{F}_{\mathrm{HKR}}\mathrm{F}_{\mathrm{CW}}\mathsf{HC}^-$ and we denote by $\mathrm{F}^{\bullet}_B\mathsf{HC}^-$ the composition

$$\mathrm{F}^{\bullet}_B\mathsf{HC}^- := (\mathsf{colim}_{\mathrm{Wth}}) \circ (\widehat{\underset{\mathrm{sk}}{\mathsf{Fil}}}[\ \tau^B_{\geq 2*}\ ]) \circ \mathrm{F}_{\mathrm{HKR}}\mathrm{F}_{\mathrm{CW}}\mathsf{HC}^-$$

In [Ant19, §4] the author defines the value of $\mathrm{F}^{\bullet}_{B}\mathsf{HC}^{-}$ for objects in SCR_k by Kan extension from polynomial algebras, under sifted colimits. Let us denote this extension by $\mathrm{LKE}(\mathrm{F}^{\bullet}_{B}\mathsf{HC}^{-})$.

Corollary 5.5.34. *The two ∞-functors*

$$\mathrm{F}^{\bullet}_{B}\mathsf{HC}^{-}, \mathsf{HC}^{-}_{\mathsf{BiFil}} : \mathsf{N}(\mathsf{Poly}_k) \to \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))$$

*are naturally equivalent under the commutativity of (173). Moreover, since $\mathsf{HC}^{-}_{\mathsf{BiFil}}$ preserves sifted colimits (Proposition **5.5.31**), both ∞-functors*

$$\mathrm{LKE}(\mathrm{F}^{\bullet}_{B}\mathsf{HC}^{-}), \mathsf{HC}^{-}_{\mathsf{BiFil}} : \mathsf{N}(\mathsf{Poly}_k) \to \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))$$

are naturally equivalent via the equivalence provided by restriction along $\mathsf{N}(\mathsf{Poly}_k) \subseteq \mathsf{SCR}_k$:

$$\mathsf{Fun}^{\mathrm{sifted}}(\mathsf{SCR}_k, \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k))) \xrightarrow{\sim} \mathsf{Fun}(\mathsf{N}(\mathsf{Poly}_k), \widehat{\mathsf{Fil}}(\widehat{\mathsf{Fil}}(\mathsf{Mod}_k)))$$

6. Applications and complements

6.1. **Towards shifted symplectic structures in non-zero characteristics.** Let k be a commutative $\mathbb{Z}_{(p)}$-algebra. We assume that $p \neq 2$.

For a commutative simplicial k-algebra A, Theorem **5.4.1** provides a filtered version of negative cyclic homology complex $\mathsf{HC}^{-}_{\mathsf{Fil}}(A/k)$ and tells us that the graded pieces are canonically given by

$$\mathsf{gr}^i\mathsf{HC}^{-}_{\mathsf{Fil}}(A/k) \simeq \mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(A/k).$$

The ∞-functors $A \mapsto \mathsf{HC}^{-}_{\mathsf{Fil}}(A/k)$ and $A \mapsto \mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(A/k)$ are extended by descent to all derived stacks $\mathscr{Y}$:

$$\mathsf{HC}^{-}_{\mathsf{Fil}}(\mathscr{Y}/k) = \mathsf{lim}_{\mathsf{Spec}\, A \to \mathscr{Y}}\mathsf{HC}^{-}_{\mathsf{Fil}}(A/k)$$

and this comes equiped with a canonical filtration whose graded pieces are $\mathbb{L}\widehat{\mathsf{DR}}^{\geq i}(\mathscr{Y}/k)$, also defined by left Kan extension. The natural generalization of the notion of shifted symplectic structures of [PTVV13a] is the following definition.

Definition 6.1.1.

(i) For a derived stack $\mathscr{Y}$ over k, we define the *complex of closed q-forms on $\mathscr{Y}$* to be

$$\mathscr{A}^{cl,q}(\mathscr{Y}/k) := \mathsf{gr}^q\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k)[-q] \simeq \mathbb{L}\widehat{\mathsf{DR}}^{\geq q}(\mathscr{Y}/k)[-q].$$

(ii) A closed 2-form ω of degree n on a derived stack $\mathscr{Y}$ is *non-degenerate* if the underlying element in $\mathsf{H}^n(\mathscr{Y}, \wedge^2\mathbb{L}_{\mathscr{Y}/k})$ is non-degenerate in the sense of [PTVV13a].

The above definition is a rather naive notion, as we believe that there may exist more subtle versions. For instance, it is very natural to ask for a shifted symplectic structure to lift to an element in the second level of the filtration $F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k)$. We thus define the notion of *enhanced shifted symplectic structures* as follows.

Definition 6.1.2. For a derived stack $\mathscr{Y}$ over k, an *enhanced shifted symplectic structure of degree n on $\mathscr{Y}$* consists of the data of an element

$$\omega \in H^n(F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k)[-2])$$

such that its image by the canonical map

$$F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k) \longrightarrow \mathsf{gr}^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k)$$

defines a non-degenerate closed 2-form of degree n on $\mathscr{Y}$.

In characteristic zero, the HKR theorem implies that there is always a canonical lift, but outside of this case lifts might not even exist (and if it does, it may not be canonical). The data of such a lifting seems to be of some importance to us, in particular for questions concerning quantization. As a first example of existence of this type of lift we show below that most shifted symplectic structures constructed in nature do possesses such lifts, by means of the Chern character construction.

Reminder 6.1.3 (Chern Character)**.** Recall the existence of a Chern character $\mathsf{Ch} : \mathsf{K}^{\mathsf{c}}(A) \longrightarrow \mathsf{HC}^-(A/k)$, which is here considered as a map of spectra (see for instance [TV15]). Note also that K^{c} stands here for the space of connective K-theory of A. This map can be enhanced into a morphism of stacks of spaces on the site of derived affine

schemes over k

$$\mathsf{Ch} : \underline{\mathsf{K}^{\mathsf{c}}} \longrightarrow \mathsf{HC}^-$$

($\underline{\mathsf{K}^{\mathsf{c}}}$ is the stack associated to $A \mapsto \mathsf{K}^{\mathsf{c}}(A)$, note that $A \mapsto \mathsf{HC}^-(A/k)$ is itself already a stack because HH itselt is a stack in the étale topology [WG91]).

When $\mathscr{Y}$ is a smooth Artin stack over k, theorem 5.4.1 (d) implies that the canonical map $\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k) \longrightarrow \mathsf{HC}^-(\mathscr{Y}/k)$ is an equivalence, and thus the Chern character produces a well defined map

$$Ch : \mathsf{K}_0(\mathscr{Y}) \longrightarrow \mathrm{H}^0(\mathsf{HC}^-_{\mathsf{Fil}}(\mathscr{Y}/k)).$$

This applies, in particular, when $\mathscr{Y}$ is of the form $\mathsf{B}G$ for G a smooth group scheme over k.

We now consider the stack BSL_n, and contemplate the following splitting statement.

Lemma 6.1.4. *We have a canonical splitting*

$$\mathrm{H}^0(\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \simeq \mathrm{H}^0(F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \oplus \mathrm{H}^0(\mathsf{gr}^0\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)).$$

Moreover, the unit in the graded ring $\mathrm{H}^0(\mathsf{gr}^*\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n))$ *induces an isomorphism*

$$k \simeq \mathrm{H}^0(\mathsf{gr}^0\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)).$$

Proof. First of all, by 5.4.1 (c), $\mathsf{gr}^0\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)$ computes the algebraic de Rham complex of the stack BSL_n. Therefore, the unit map

$$k \longrightarrow \mathrm{H}^0(\mathsf{gr}^0\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n))$$

is indeed an isomorphism. Moreover, we see by the same argument that all the complexes $\mathsf{gr}^p\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)$ are cohmologically concentrated in non-negative degrees. As a result, we have a short exact sequence of k-modules

$$0 \longrightarrow \mathrm{H}^0(F^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \longrightarrow \mathrm{H}^0(\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \longrightarrow k.$$

This sequence splits on the right because of the unit map $k \to \mathrm{H}^0(\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n))$.

In order to prove the lemma it thus remains to show that the natural map

$$\mathrm{H}^0(F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \longrightarrow \mathrm{H}^0(F^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n))$$

is bijective.

Because of the exact sequence

$$0 \longrightarrow \mathrm{H}^0(F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \longrightarrow \mathrm{H}^0(F^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \longrightarrow \mathrm{H}^0(\mathsf{gr}^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)),$$

it only remains to show that $\mathrm{H}^0(\mathsf{gr}^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) = 0$. By the theorem 5.4.1 (c) we now that $\mathsf{gr}^1\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n) \simeq \mathbb{L}\widehat{\mathsf{DR}}^{\geq 1}(\mathsf{BSL}_n/k)$. We claim that $\mathrm{H}^0(\mathbb{L}\widehat{\mathsf{DR}}^{\geq 1}(\mathsf{BSL}_n/k)) \simeq (\mathfrak{sl}_n^*)^{\mathsf{SL}_n}$ is the module of SL_n-invariants in the dual of the Lie algebra $\mathfrak{sl}_n$, and thus is indeed zero. To see this we use the Hodge filtration on the derived de Rham complex, which here consists of an exact triangle

$$\mathbb{L}\widehat{\mathsf{DR}}^{\geq 2}(\mathsf{BSL}_n/k)[-2] \longrightarrow \mathbb{L}\widehat{\mathsf{DR}}^{\geq 1}(\mathsf{BSL}_n/k) \longrightarrow \Gamma(\mathsf{BSL}_n, \mathbb{L}_{\mathsf{BSL}_n})[1].$$

The cotangent complex of BSL_n is given by $\mathfrak{sl}_n^*[-1]$, with its natural action of SL_n. Therefore, $\Gamma(\mathsf{BSL}_n, \mathbb{L}_{\mathsf{BSL}_n})[1] \simeq (\mathfrak{sl}_n^*)^{h\mathsf{SL}_n}$ is the homotopy fixed point of coadjoint representation. We have $\mathrm{H}^0((\mathfrak{sl}_n^*)^{h\mathsf{SL}_n}) \simeq (\mathfrak{sl}_n^*)^{\mathsf{SL}_n} \simeq 0$. □

The lemma implies that the Chern character of the tautological rank n bundle on BSL_n defines an element

$$Ch_{\geq 2} \in \mathrm{H}^0(F^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)).$$

The image of this element in $\mathrm{H}^0(\mathsf{gr}^2\mathsf{HC}^-_{\mathsf{Fil}}(\mathsf{BSL}_n)) \simeq (Sym^2(\mathfrak{sl}_n)^*)^{\mathsf{SL}_n}$ is the quadratic form

$$\mathfrak{sl}_n \otimes \mathfrak{sl}_n \longrightarrow k$$

sending (A, B) to $Tr(AB)$ and is thus non-degenerate. In other words $Ch_{\geq 2}$ defines an enhanced shifted symplectic structure of degree 2 on BSL_n.

As a result, any vector bundle on a derived stack X, equipped wit a trivialization of its determinant line bundle, defines an element

$$\omega \in \mathrm{H}^0(F^2\mathsf{HC}^-_{\mathsf{Fil}}(X/k))$$

by pull-back of $Ch_{\geq 2}$ by the classifying map $X \longrightarrow \mathsf{BSL}_n$. This construction can be used in order to construct enhancements of the shifted symplectic structures on moduli of SL_n-bundles on Calabi-Yau varieties constructed in [PTVV13a].

6.2. **Filtration on Hochschild cohomology.** As a second example of possible applications of the filtered circle, we explain here how it can also provide interesting filtrations on Hochschild cohomology. For this, we will have to consider $\mathsf{S}^1_{\mathsf{Fil}}$ not as a group anymore but as a cogroup object inside filtered stacks. It is even more, as it carries a $\mathsf{E}_2^{\otimes}$-cogroupoid structure over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ which can be exploited to get a filtration on Hochschild cohomology compatible with its natural E_2-structure. Moreover, all the constructions in this part make sense over the sphere spectrum, and so provide filtrations on topological Hochschild cohomology as well.

As a start we consider the natural closed embedding of stacks

$$0 : \mathsf{B}\mathbb{G}_{\mathrm{m}} \hookrightarrow [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}].$$

The direct image of the structure sheaf defines a commutative cosimplicial algebra over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. Let us denote it by $\mathscr{E}$, and let $\mathscr{O} \longrightarrow \mathscr{E}$ be the unit map. The nerve of this map produces a groupoid object inside commutative cosimplicial algebras over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, which is denoted by $\mathscr{E}^{(1)}$. In the same manner, we can consider the nerve of $\mathscr{O} \longrightarrow \mathscr{E}^{(1)}$ to get an $\mathsf{E}_2^{\otimes}$-groupoid object (that is a groupoid object inside groupoid objects) $\mathscr{E}^{(2)}$ and so on and so forth.

We define this way an $\mathsf{E}_n^{\otimes}$-groupoid object $\mathscr{E}^{(n)}$ inside the ∞-category of commutative cosimplicial algebras over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$.

Definition 6.2.1. The *filtered n-sphere* is defined to be $\mathsf{Spec}^{\Delta}\,\mathscr{E}^{(n+1)}$. It is an $\mathsf{E}_{n+1}^{\otimes}$-cogroupoid object in affine stacks over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$. It is denoted by $\mathsf{S}^n_{\mathsf{Fil}}$.

Note that $\mathsf{S}^n_{\mathsf{Fil}}$ possesses an underlying object of $(1, ..., 1)$-morphisms. Explicitly, this is given by $\mathsf{Spec}\,\mathsf{H}^*(\mathsf{S}^n, \mathbb{Z}_{(p)})$, and is called the formal or graded sphere. When $n = 1$ we recover our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ as a filtered affine stack, but now it comes equiped with an $\mathsf{E}_2^{\otimes}$-cogroupoid structure rather than a group structure.

We now consider $\mathscr{L}^{(n)}_{Fil}X = \mathrm{Map}(\mathsf{S}^n_{\mathsf{Fil}}, X)$, for a derived affine scheme $X = \mathsf{Spec}\,A$. The cogroupoid structure on $\mathsf{S}^n_{\mathsf{Fil}}$ endows $\mathscr{L}^{(n)}_{Fil}X$ with an $\mathsf{E}_{n+1}^{\otimes}$-groupoid structure acting on X. Passing to functions and taking linear duals we get a filtered $\mathsf{E}_{n+1}^{\otimes}$-algebra over $\mathbb{Z}_p$ whose underlying object is $\mathsf{HH}^*_{\mathsf{E}_n^{\otimes}}(A)$, the n-the iterated Hochschild cohomology of A, and the associated graded is $\mathsf{Sym}_A(\mathbb{L}_A[n])^{\vee}$, the dual of shifted differential forms, which can be defined as shifted polyvector fields over X. In summary, we expect the following proposition:

Proposition 6.2.2. *Let k be a commutative $\mathbb{Z}_p$-algebra, and A a commutative simplicial k-algebra. The iterated Hochschild cohomology $\mathsf{HH}^*_{\mathsf{E}_{n+1}^{\otimes}}(A/k)$, carries a canonical filtration compatible with its $\mathsf{E}_{n+1}^{\otimes}$-multiplicative structure, whose associated graded is the $\mathsf{E}_{\infty}^{\otimes}$-algebra of n-shifted polyvectors on X.*

The last proposition can possibly be used in order to define singular supports of coherent sheaves, or of sheaves of linear categories, over any base scheme. For instance, in the context of bounded coherent sheaves, this could allows one to extend the notion and construction of [AG15].

6.3. **Generalized cyclic homology and formal groups.** The filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ we have constructed in this paper is part of a much more general framework that associates a circle S^1_E to any reasonable abelian formal group E. To be more precise:

Construction 6.3.1. We can start by an abelian formal group E over some base commutative ring k, and assume that E is formally smooth and of relative dimension 1 over k. The Cartier dual G_E of E is a flat abelian group scheme over $\mathsf{Spec}\, k$, obtained as $\mathsf{Spec}\, \mathscr{O}(E)^\vee$, where $\mathscr{O}(E)^\vee$ is the commutative and cocommutative Hopf algebra of distributions on E. Because E is smooth and of relative dimension 1, $\mathscr{O}(E)^\vee$ is a flat commutative k-coalgebra which is locally for the Zariski topology on k isomorphic to $k[X]$ with the standard comultiplication $\Delta(X^n) = \sum_{i+j=n} \binom{n}{i} X^i \otimes X^j$. The *$E$-circle* is defined as the group stack over k defined by

$$\mathsf{S}^1_E := \mathsf{B}G_E.$$

Under reasonable assumptions on E the stack S^1_E is an affine stack over k. Moreover, its ∞-category of representations, $\mathsf{QCoh}(\mathsf{BS}^1_E)$ is naturally equivalent to the ∞-category of mixed complexes over k, at least locally on $\mathsf{Spec}\, k$. To be more precise, if we denote by ω_E the line bundle of relative 1-forms on G_E, the ∞-category $\mathsf{QCoh}(\mathsf{BS}^1_E)$ is equivalent to ω_E-twisted mixed complexes, namely comodules over the k-coalgebra $k \oplus \omega_E[-1]$. However, the symmetric monoidal structure on $\mathsf{QCoh}(\mathsf{BS}^1_E)$ corresponds to a non-standard monoidal structure on mixed complexes that depends on the formal group structure on E.

The filtration on $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$ whose associated graded is $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ seems to also exists in some interesting examples of formal group laws.

We recover the results in this paper when E is either the additive or the multiplivative formal group $\widehat{\mathbb{G}_{\mathsf{a}}}$, resp. $\widehat{\mathbb{G}_{\mathrm{m}}}$:

Construction 6.3.2. Let k be a commutative ring. There exists a filtered group deforming $\mathbb{G}_{\mathrm{m}\,k}$ to $\mathbb{G}_{\mathsf{a}\,k}$. Namely, given $\lambda \in k$ take $\mathbb{G}^\lambda_{\mathrm{m}\,k} =: \mathsf{Spec}(k[T, \frac{1}{1+\lambda T}])$. This is a group scheme under the multiplicative rule $T \mapsto 1 \otimes T + T \otimes 1 + \lambda . T \otimes T$ and unit $T \mapsto 0$. When $\lambda = 0$ we get $\mathbb{G}_{\mathsf{a}\,k}$ and for $\lambda = 1$ we get $\mathbb{G}_{\mathrm{m}\,k}$. Taking formal completions this deforms $\widehat{\mathbb{G}_{\mathrm{m}\,k}}$ to $\widehat{\mathbb{G}_{\mathsf{a}\,k}}$.

Proposition 6.3.3. *Let $k = \mathbb{Z}_{(p)}$. Then*

$$\mathsf{S}^1_{\widehat{\mathbb{G}_{\mathsf{a}}}} := \mathsf{B}G_{\widehat{\mathbb{G}_{\mathsf{a}}}} \simeq (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}} \quad \textit{and} \quad \mathsf{S}^1_{\widehat{\mathbb{G}_{\mathrm{m}}}} := \mathsf{B}G_{\widehat{\mathbb{G}_{\mathrm{m}}}} \simeq (\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{u}}$$

Moreover, the filtration on Fix *is Cartier dual to the filtration on* $\widehat{\mathbb{G}_{\mathrm{m}\,k}}$ *of Construction* ***6.3.2****.*

Proof. The proposition is equivalent to the claims that:

(i) Fix is Cartier dual to $\widehat{\mathbb{G}_{\mathrm{m}}}$;
(ii) Ker is Cartier dual to $\widehat{\mathbb{G}_{\mathsf{a}}}$;
(iii) the filtrations are Cartier dual

This is precisely the content of [SS01, Theorem]. See [Haz12, 37.3.4] for Cartier duality. □

Let E be an abelian formal group over k as before and S^1_E the corresponding E-circle. For any derived affine k-scheme X we define the E-loop space $\mathscr{L}_E X := \mathrm{Map}(\mathsf{S}^1_E, X)$, that comes equiped with an S^1_E-action. The E-Hochschild homology of X over k is by definition the complex of functions $\mathscr{O}(\mathscr{L}_E X)$. It is denoted by $\mathsf{HH}(X, E)$. The S^1_E-action on $\mathsf{HH}^E(X)$ induces a mixed structure on $\mathsf{HH}(X, E)$ whose total complex computes the S^1_E-equivariant cohomology and is called by definition the negative cyclic E-homology $\mathsf{HC}^-(X, E)$. When a filtration exists on E, then there is an HKR-type filtration on $\mathsf{HC}^-(X, E)$ whose associated graded is again derived de Rham cohomology.

Of course, the results of this work are recovered when E is taken to be the multiplicative formal group law and we recover an isomorphism of filtered group schemes $\mathsf{H}_{p^\infty} = G_{E_T}$. See [SS01].

An example of particular interest is when E comes, by completion, from an elliptic curve. The corresponding Hochscild and cyclic homology can be called *elliptic Hochschild and cyclic homology.*

6.4. **Topological and q-analogues.** The filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$ constructed in this work possesses at least two extensions, both of quantum /non-commutative nature: one as a non-commutative group stack over the sphere spectrum and a second extension as filtered group stack over $\mathbb{Z}[q, q^{-1}]$.

6.4.1. ***q-analogue.*** As a start, we relax the restriction of working p locally and attempt to make sense of these constructions over $\mathbb{Z}$. A first possibility is simply to use big Witt vectors and define the filtered group scheme H as the intersection of all kernels of the endomorphisms $\mathscr{G}_p$ for all primes p. There is however a second possible description, which has the merit of showing the natural q-deformed version, which we now describe.

We start by the filtered formal group $\mathbb{G}$, interpolating between the formal multiplicative and the formal additive group over $\mathbb{Z}$. The corresponding formal group over $\mathbb{A}^1$ is given by $X + Y + \lambda XY$ where λ is the coordinate on the affine line. The underlying formal group is $\widehat{\mathbb{G}_m}$ whereas the associated graded is $\widehat{\mathbb{G}_a}$ together with its natural graduation given the natural action of $\mathbb{G}_m$. The algebra of distributions of the filtered formal scheme $\mathbb{G}$ defines a filtered commutative and cocommutative Hopf algebra $\mathscr{R}$. This algebra can be described explicitely as being the algebra of integer valued polynomials, that is the subring of $\mathbb{Q}[X]$ formed by all polynomials P such that $P(\mathbb{Z}) \subset \mathbb{Z}$. The filtration is then induced the degree of polynomials.

The associated graded to this filtration is the ring $Gr\mathscr{R}$ of divided powers over $\mathbb{Z}$. This is the subring of $\mathbb{Q}[X]$ generated by the $\frac{X^n}{n!}$.

An integral version of the filtered group scheme H_{p^∞}, and of the filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$, can then be defined as $\mathsf{H}_{\mathbb{Z}} := \mathsf{Spec}(Rees(\mathscr{R}))$, where $Rees(\mathscr{R})$ is the Rees construction associated to the filtered Hopf algebra $\mathscr{R}$. The integral version of the filtered circle is then defined to be

$$\mathsf{S}^1_{\mathsf{Fil},\mathbb{Z}} := \mathsf{BH}_{\mathbb{Z}}.$$

It is a pleasant exercise to show that when restricted over $\mathsf{Spec}\,\mathbb{Z}_p$ this recovers our filtered circle $\mathsf{S}^1_{\mathsf{Fil}}$. We believe that all the statement proved in this work can be extended over $\mathbb{Z}$, but some of the strategies of proof we use do not obviously extend to the situation where we deal with an infinite number of primes.

One advantage of the above presentation using integer valued polynomial algebras is the striking fact that these admit natural q-deformed versions. The q-deformed version $\mathscr{R}_q$ of the ring $\mathscr{R}$ is introduced and studied in [HH17], and is essentially the Cartan part $\mathbf{U}^0(\mathfrak{sl}_2)$ of the divided power quantum group of Lusztig (see [HH17] end of section 4). In particular, we think that the filtered Hopf algebra $\mathscr{R}$ possesses a q-deformed version $\mathscr{R}_q$, which is a commutative and cocommutative filtered Hopf algebra over $\mathbb{Z}[q, q^{-1}]$, recovering $\mathscr{R}$ when $q = 1$. The spectrum of this provides a q-deformed version of $\mathsf{H}_{\mathbb{Z}}$ that we denote by $\mathsf{H}_{\mathbb{Z},q}$. Its classifying stack is by definition the q-deformed filtered circle.

Definition 6.4.1. The *q-deformed filtered circle* is the filtered stack $\mathsf{S}^1_{\mathsf{Fil},\mathbb{Z}}(q) := \mathsf{BH}_{\mathbb{Z},q}$. It is a stack over $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \times \mathsf{Spec}\,\mathbb{Z}[q, q^{-1}]$.

As in Theorem **5.4.1**, by considering the derived mapping stack $\mathrm{Map}(\mathsf{S}^1_{\mathsf{Fil},\mathbb{Z}}(q), X)$, it is then possible to define q-analogues of Hochschild and cyclic homology of a scheme X,

together with a filtration whose associated graded may be used to define a notion of a q-deformed derived de Rham cohomology; this should be compared to various notions of q-deformed de Rham complexes appearing the literature (see for instance [Aom00] and [Man92]).

However, to make the above definition precise requires some extra work. For instance, it seems to us that the associated graded of $\mathsf{S}^1_{\mathsf{Fil},\mathbb{Z}}(q)$ can not truly exist as a naive commutative object and requires to work over some braided monoidal base category associated to $\mathbb{Z}[q,q^{-1}]$, as this is done for instance in the theory of Ringel-Hall algebras, see for instance [LZ00]. In fact, we expect the associated graded of $\mathsf{S}^1_{\mathsf{Fil},\mathbb{Z}}(q)$ to be of the form $\mathsf{B}K(q)$, where $K(q)$ is the spectrum of the Ringel-Hall algebra over the one point Quiver.

6.4.2. ***Topological Analogue.*** Let us mention yet another extension of the filtered circle, now over the sphere spectrum. We do not believe that the filtered stack $\mathsf{S}^1_{\mathsf{Fil}}$ can exist as a spectral stack in any sense, as the associated graded $(\mathsf{S}^1_{\mathsf{Fil}})^{\mathsf{gr}}$ probably cant exist over the sphere spectrum. However, it is possible to construct a non-commutative version of this object, using the 2-periodic sphere spectrum of [Lur15]. As shown in [Lur15] there exists a filtered $\mathsf{E}_2^{\otimes}$-algebra whose underlying object is $\mathbb{S}^{K(\mathbb{Z},2)}$ (so is $\mathsf{E}_\infty^{\otimes}$) but its associated graded is a 2-periodic version of the sphere spectrum $\mathbb{S}[\beta,\beta^{-1}]$. This 2-periodic sphere spectrum is known not to exist as an $\mathsf{E}_\infty^{\otimes}$-ring. However, we can consider the natural augmentation

$$\mathbb{S}^{K(\mathbb{Z},2)} \longrightarrow \mathbb{S}$$

and consider the spectrum

$$A := \mathbb{S} \otimes_{\mathbb{S}^{K(\mathbb{Z},2)}} \mathbb{S}$$

As a mere spectrum, this is equivalent to the group ring over the circle $A \simeq \mathbb{S}[K(\mathbb{Z},1)]$. However, the E_2-filtration on $\mathbb{S}^{K(\mathbb{Z},2)}$ induces a structure of a filtered bialgebra on A, which should be considered as a non-commutative analogue of the filtered circle.

More precisely, the idea is to study the dual filtered bialgebra $B = A^*$ and consider $\mathsf{Spec}B$ in some sense to produce a topological version of the filtered circle. We however do not know currently know how to exploit the existence of B and invite the interested reader to pursue this approach further.

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