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Abstract

We deal with three aspects of the complexity of the problem of finding a maximum matching that minimizes the number of colors in a vertex-colored graph. We first prove that it is W[2]-hard, next that it is hard to approximate in a similar way as the Set Cover problem, and finally that it is fixed-parameter tractable for a suitable (yet meaningful) choice of parameter.

1 Introduction

Graphs are a powerful modelisation tool, whose uses are widespread. But when dealing with complex systems, we often want to use additional information along with the structure they offer. There are many works that deal with label graphs, such as edge-weighted graphs, that add a such new layer of informations on the edges of the graph.

Another natural path would be to add information would be to put it in the edges, and we will focus on that one, by studying graphs where the additional layer of information is given by a coloration of the vertices. This formalism can be used, for example, to modelize the Web, where we complete the underlying graph with a coloration on each vertex to capture the type of content it holds. By choosing a constraint on colors, many new interesting objects and problems emerge.

This work, that focus on the variation of the Maximum Matching problem while minimizing the number of colors, follows a previous study on another variation of that problem where the maximum matching was said to be tropical [2], a notion first introduced in [4].

A *vertex-colored* graph is a couple $G^c = (G, c)$ where $G = (V, E)$ is a simple undirected graph and c a *coloring* on V (i.e. a function giving a color to each vertex in V). Observe that it doesn't need to be a *proper* coloration : Two adjacent vertices can be of the same color.

$H^{c'}$ is said to be a *(vertex-colored) subgraph of* G^c a vertex-colored graph when H is a subgraph of G and c' is c restricted to $V(H)$.

Given the definition, $H^{c'}$ can be alternatively written H^c or H, when it's clearly stated that it is a subgraph of G^c.

We'll also use the following notations concerning vertex-sets and edge-sets when it's convenient :

- For G a graph, $V(G)$ denotes its vertex-set, $E(G)$ its edge-set

- For M a set of edges from G, $V(M)$ denotes the vertex-set of the subgraph induced by M
- Given x and y two vertices, xy denotes the edge, if any, between x and y in G.

While, in a graph G^c where $G = (V, E)$, for x a vertex in V, $c(x)$ is already well-defined, we'll also use the following notations :

- For A a set of vertices, subset of V, $c(A)$ denotes direct image of A by c (the set of the colors of A)
- For H a subgraph of G^c, $c(H)$ denotes the direct image by c of its vertex-set
- For M a set of edges, subset of V, $c(M)$ denotes the direct image by c of the vertex-set of the subgraph of G induced by M

In a vertex-colored graph $(V, E)^c$, a set of vertices $A \subset V$ is said to be *tropical* when the set of colors used on A is exactly the one used on the whole coloration of the graph (ie when $c(A) = c(V)$). By extension, a set of edges M is said to be *tropical* when the vertex-set of its induced subgraph is tropical (ie when $c(M) = c(V)$).

The problem of finding a maximum matching is a classical one, but let's define that properly so we can extend it to the colored case.

A *matching* M is a subset of edges of $E(G^c)$ such that any two edges of the matching have no common incident vertex. The vertices incident to an edge of M are said to be *matched* or *covered* by M. A *maximal* matching is a matching that is maximal (under inclusion), while a *maximum* matching is a matching with highest cardinality among all possible matchings (which is trivially always a maximal matching).

The decision problem associated to this optimisation problem is known to be polynomial [6], but what happens when we add some constraint on the colors to the problem ? For example one could think about the *tropical* version of the problem :

TROPICAL MAXIMUM MATCHING

Input: *A vertex-colored graph* G^c
Output: *A tropical maximum matching* M *of* G^c *if any*

We can observe that a perfect matching is always tropical, consequently the above question is interesting for maximum (not perfect) matchings. In [3], the authors handle this case, giving a polynomial-time algorithm. Using their Theorem 2.2, an immediate corollary is that we still have a polynomial time algorithm when we replace *tropical* with *maximum colored* :

MAXIMUM (VERTEX-)COLORED MAXIMUM MATCHING

Input: *A vertex-colored graph* G^c
Output: *A maximum matching* M *in* G^c *with maximum number of colors*

An other natural variation is to consider the minimization of the number of colors instead of maximizing it :

MINIMUM (VERTEX-)COLORED MAXIMUM MATCHING (MCMM)

Input: *A vertex-colored graph G^c*
Output: *A maximum matching M in G^c with minimum number of colors*

That problem, however, is not as easy to solve, as we will prove. The corresponding decision problem, is indeed NP-hard. More than that, one cannot expect to solve easily instances of MCMM whose solution has few colors, since as we will show, it is, when parametrized by the number of colors of the solution, at least as hard as Minimum Dominating Set parametrized by the size of the solution, as we will prove in Section 2.

For approximate solution to *Minimum edge-colored Maximum Matching*, since finding a regular maximum matching is easy, one could spontaneously think of using the number of colors of such matching as weight to evaluate the quality of a maximum matching as an approximation. As we will prove in Section 3, our problem, parametrized in such fashion, is nearly as hard as the Set Cover problem regarding approximation.

However, as hard as the problem can be with the most natural parameter, there is an other sensible one -the size of a maximum matching- with which the problem becomes FPT (Fixed-Parameter Tractable), as we prove in Section 4.

2 W[2]-hardness of MCMM

In this section we prove the following hardness result.

Theorem 2.1. *Minimum colored maximum matching (MCMM) is $W[2]$-hard on trees considering the total number of colors of the input as parameter.*

The proof of this theorem is based on a reduction from the Dominating Set problem and uses the construction and lemmas below. In particular it will be an immediate consequence of Lemma 2.7.

Given a connected graph $G = (V, E)$, let us define from G a vertex-colored tree T^c as follows :

- $V(T) = \{x_u | u \in V\} \cup \{x_{u,v} | v \in N(u)\} \cup \{x'_0, x_0\}$
- $E(T) = \{x_0 x_u | u \in V\} \cup \{x_u x_{u,v} | v \in N(u)\} \cup \{x'_0 x_0\}$?

Then, we color the vertices of T with $n + 1$ colors so that :

- $c(x'_0) = c(x_0) = 0$
- For every $u \in V$, $c(x_u) = 0$
- For each $(u, v) \in V \times N(u)$, $c(x_{u,v}) = v$

Notice that $|V(T)| = |V| + 2|E| + 2$, and $|E(T)| = |V| + 2|E| + 1$, and that we can build T^c in polynomial time from G. Notice also that there are $|V| + 2$ internal vertices.

In order to facilitate discussions, in the sequel, we let $\mathcal{R}$ denote the function that given G as an input returns T^c. The following series of lemmas explores the properties of $\mathcal{R}$.

Lemma 2.1. $\{x'_0x_0\} \cup \{x_ux_{u,u} | u \in V\}$ *is a maximum matching of* $\mathcal{R}(G)$

Proof. One can easily see that $\{x'_0x_0\} \cup \{x_ux_{u,u} | u \in V\}$ is a matching, and that there is no augmenting path, since all paths that go from an unmatched vertex to another are of length 4. Thus, it is a maximum matching. □

An immediate consequence of the previous lemma is that the size of any maximum matching of $\mathcal{R}(G)$ is $|V| + 1$.

Lemma 2.2. *If* M *is a matching of* $\mathcal{R}(G)$ *and* $M \cap \{x_0x_u | u \in V\} \neq \emptyset$, *then* M *is not a maximum one.*

Proof. Let M be a matching of $\mathcal{R}(G)$.

Assume that $x_0x_u \in M$ for some $u \in V$. Then since M is a matching, x'_0x_0 and $x_ux_{u,u}$ are not in M, so $x'_0, x_0, x_u, x_{u,u}$ is an augmenting path and M is not maximum. □

Lemma 2.3. *Let* v *be a vertex in* G. *Any maximum matching of* $\mathcal{R}(G)$ *uses exactly one edge in* $\{x_vx_{v,v}, x_vx_{v,u} : (v, u) \in E\}$, *and contains edge* x'_0x_0.

Proof. We prove this lemma by contraction. Assume that a maximum matching M of $\mathcal{R}(G)$ has no edge in $\{x_vx_{v,v}, x_vx_{v,u} : (v, u) \in E\}$. Since $x_0x_u \notin M$, by Lemma 2.2, $M \cup \{x_ux_{u,u}\}$ is a matching greater than M, a contradiction to the maximality of M.

By the same argument, x'_0x_0 must be in M, which concludes the proof. □

Let M be a maximum matching of $\mathcal{R}(G)$. We then define g by $g(M) = \{v | \exists u \in V, x_ux_{u,v} \in M\}$.

Lemma 2.4. *If* M *is a maximum matching of* $\mathcal{R}(G)$, *then* $g(M)$ *is a dominating set of* G.

Proof. Let u be a vertex of G. As M is a maximum matching of $\mathcal{R}(G)$, by Lemma 2.3, M has one edge in $\{x_ux_{u,u}, x_ux_{u,v} : vu \in E\}$, say $x_ux_{u,v}$.

By the definition of $g(M)$, $v \in g(M)$, which ensures that u is dominated by v and also by $g(M)$. □

Lemma 2.5. *If* M *is a* $k+1$*-colored maximum matching of* $\mathcal{R}(G)$, *then* $g(M)$ *is a dominating set of* G *of size* k.

Proof. Let M be a maximum matching of $\mathcal{R}(G)$ with $k + 1$ colors.

By Lemma 2.3, Color 0 is always covered by a maximum matching of $\mathcal{R}(G)$, and M covers k other colors in V. If M contains color $v \in V$, then by construction of $\mathcal{R}(G)$, there is some u such that $x_ux_{u,v} \in M$. The definition of function g implies that $v \in g(M)$. Thus $|g(M)| \geq k$.

Conversely, if M does not contain color $v \in V$, by construction of $\mathcal{R}(G)$, there is no vertex u such that $x_ux_{u,v} \in M$. Moreover $v \notin g(M)$ (by definition of g). Thus $|g(M)| \leq k$

We conclude that $|g(M)| = k$, and since $g(M)$ is a dominating set of G by Lemma 2.4, $g(M)$ is then a dominating set of G of size k. □

Lemma 2.6. *Graph* G *admits a dominating set of size* k *if and only if* $\mathcal{R}(G)$ *admits a maximum matching with* $k + 1$ *colors.*

Proof. By Lemma 2.5, if $\mathcal{R}(G)$ admits a maximum matching with $k+1$ colors, G admits a dominating set of size k.

Conversely, Assume that S is a dominating set of size k in G.

Let α be an arbitrary injective valuation on V. For each $u \in V$ we define φ by

$$\varphi(u) = \begin{cases} u, \text{ if } u \in N_G(u) \cap S \\ min_\alpha(N_G(u) \cap S), \text{ otherwise} \end{cases}$$

Since S is a dominating set of G, for each $u \in V, N_G(u) \cap S$ is not empty, and φ is then well-defined.

Then we define a matching $M = \{x'_0 x_0\} \cup \{x_u x_{u,v} | v = \varphi(u)\}$, which is maximum since it is of size $|V|+1$. Furthermore any vertex covered by M is of color either 0 or $u \in S$, and each of those $k+1$ colors appears at least once (if $u \in S$ then by construction $x_u x_{u,u} \in M$ and $x_{u,u}$ has u as a color). Consequently M is then $k+1$-colored, which concludes the proof. □

Lemma 2.7. *$\mathcal{R}$ is a FPT-reduction from the Dominating Set problem with parameter size of the optimal solution to the MCMM problem on trees with parameter number of colors of the optimal solution.*

Proof. Immediate, from Lemma 2.6 and the fact that $\mathcal{R}$ uses polynomial time in the size of the input (and thus, is a FPT-reduction regarding any parameter). □

Thus, Theorem 2.1 holds as an immediate corollary of Lemma 2.7.

3 Hardness of approximating MCMM

We consider as candidates for approximating MCMM any maximum matching, with weight function being the number of colors used. For that definition, we prove the following inapproximability result.

Theorem 3.1. *Minimum colored maximum matching cannot be approximated with approximation ratio better than $log(n+1)(1-o(1))$ (where n is the number of internal vertices) unless $P = NP$.*

The proof of this theorem is based on a reduction from the Set Cover problem, which is known to be non-approximable with approximation ratio better than $log(n)(1-o(1))$ [5].

MINIMUM SET COVER

Input: *A finite set U, and $\mathcal{F} \subset \mathcal{P}(U)$ of subsets of U, such that* $U = \bigcup_{F \in \mathcal{F}} F$

Output: $\Xi \subset \mathcal{F}$ *such that* $U = \bigcup_{F \in \mathcal{F}} F$ *with minimum cardinality*

As it is more convenient for us, we will use the equivalent following form of the problem :

MINIMUM SET COVER (BIPARTITE GRAPH)

Input: *A bipartite graph $G = (U, V, E)$ such that no $u \in U$ is isolated and no two*

v, v' *vertices of* V *have the same neighbours in* U

Output: $\quad \Xi \subset V$ *such that* $U = \bigcup_{v \in \Xi} N(v) \setminus \{v\}$ *with minimum cardinality*

The proof of Theorem 3.1 uses the construction and lemmas below.

Given an instance of Set Cover $G = (U, V, E)$, we define a vertex-colored tree T^c defined as follows :

- $V(T) = \{x'_0, x_0\} \cup \{x_u | u \in U\} \cup \{x_{u,v} | u \in U, uv \in E\}$
- $E(T) = \{x_u x_{u,v} | u \in U, uv \in E\} \cup \{x_u x_0 | u \in U\} \cup \{x'_0 x_0\}$

Then we color the vertices of T with $n + 1$ colors so that :

- $c(x'_0) = 0$, $c(x_0) = 0$, and for each $u \in U$, $c(x_u) = 0$
- For each $uv \in E$, $c(x_{u,v}) = v$

Notice that $|V(T)| = |U| + |E| + 2$, $|E(T)| = |U| + |E| + 1$, and that we can obtain T^c in polynomial time from G.

In order to facilitate discussions, in the sequel, we let $\mathcal{Q}$ denote the function that given G as an input returns T^c. The folowing lemmas explore the properties of $\mathcal{Q}$.

Lemma 3.1. $\{x'_0 x_0\} \cup \{x_u x_{u,v} | u \in U, uv \in E\}$ *is a maximum matching of* G

Proof. One can easily see that there is no augmenting path, since all paths that go from an unmatched vertex to another are of length 4. □

Moreover, no maximum matching can use an edge that doesn't cover a leaf, since this would create an augmenting path.

Lemma 3.2. *If* M *is a matching of* $\mathcal{Q}(G)$ *and* $M \cap \{x_0 x_u | u \in U\} \neq \emptyset$, *then* M *is not a maximum matching.*

Proof. Let M be a matching of $\mathcal{Q}(G)$.

Let's suppose that $x_0 x_u \in M$ for some $u \in U$. Since there is no isolated vertex in G, there exists $v \in V$ such that $uv \in E$. Then since M is a matching, $x'_0 x_0$ and $x_u x_{u,v}$ are not in M, so $x'_0 x_0 x_u x_{u,v}$ is an augmenting chain and M is not maximal therefore not maximum. □

Lemma 3.3. *Let* u *be a element in* U. *Any maximum matching of* $\mathcal{Q}(G)$ *uses exactly one edge in* $\{x_u x_{u,v} | uv \in E\}$, *and contains the edge* $x'_0 x_0$.

Proof. Let M be a maximum matching of $\mathcal{Q}(G)$. Let's suppose that there exists $u \in U$ such that $M \cap \{x_u x_{u,v} | uv \in E\} = \emptyset$. Since $x_0 x_u \notin M$ by Lemma 3.2, $M \cup \{x_u x_{u,v}\}$ would be also a matching of greater size than M, a contraction to the maximality property of M.

By the same argument, $x'_0 x_0$ must be in M, which concludes the proof. □

Given a maximum matching M of $\mathcal{Q}(G)$, we then define g by $g(M) = \{v \in V | \exists u, x_u x_{u,v} \in M\}$.

Lemma 3.4. *If M is a maximum matching of $\mathcal{Q}(G)$, then $g(M) \cup \{0\} = c(M)$ and $g(M) = c(M) \setminus 0$*

Proof. Let M be a maximum matching of $\mathcal{Q}(G)$.

For $v \in g(M)$, by definition of $g(M)$ there exists $u \in U$ such that $x_u x_{u,v} \in M$, thus $v \in c(M)$. Since we have also $0 \in c(M)$ (by Lemma 3.3, $x'_0 x_0 \in M$), $g(M) \cup \{0\} \subset c(M)$.

Conversely, for $v \in c(M) \setminus \{0\}$, there must exist $u \in U$ such that $x_u x_{u,v} \in M$ (since only those edges can touch colors different from 0). By definition of $g(M)$, $v \in g(M)$. Thus, $c(M) \subset g(M) \cup \{0\}$.

Therefore, we have $g(M) \cup \{0\} = c(M)$, and the second equality follows immediately, since $0 \notin g(M)$ by definition. □

Lemma 3.5. *If M a maximum matching of $\mathcal{Q}(G)$, then $g(M)$ is a set cover of G.*

Proof. Let M be a maximum matching of $\mathcal{Q}(G)$ and u be a vertex from U. As M is a maximum matching of $\mathcal{Q}(G)$, by Lemma 3.3 there exists v such that $x_u x_{u,v}$ is in M, which ensures that u is covered by $g(M)$. □

Lemma 3.6. *If M is a $k+1$-colored maximum matching of $\mathcal{Q}(G)$, then $g(M)$ is a set cover of G of size k.*

Proof. Let M be a maximum matching of $\mathcal{Q}(G)$.

By Lemma 3.5, $g(M) = c(M) \setminus 0$, so we have $|g(M)| = |c(M)| - 1 = k$ (since $0 \in c(M)$ by direct corollary of Lemma 3.3). By Lemma 3.5, $g(M)$ is also a set cover, which conclude the proof.

□

Lemma 3.7. *G admits a minimal set cover of size k if and only if $\mathcal{Q}(G)$ admits a minimaly colored maximum matching (ie a matching whose set of colors is minimal, but could not be minimum) with $k+1$ colors.*

Proof. Let α be a choice function on V.

By Lemma 3.6, if $\mathcal{Q}(G)$ admits a minimally-colored maximum matching M with $k+1$ colors, then G admits a set cover $g(M)$ of size k.Assume that $g(M)$ was not minimal, *i.e.* that there exists $v_0 \in g(M)$ such that $g(M) \setminus \{v_0\}$ is a set cover of size $k-1$. For $u \in U$, let us denote $\varphi(u) = \alpha(\{v | uv \in E, v \in g(M) \setminus \{v_0\}\})$ (which is well-defined since $g(M) \setminus \{v_0\}$ is a set cover of G). We can then define $M' = \left\{x_u x_{u,\varphi(u)} | u \in U\right\} \cup \{x_0 x'_0\}$. Notice that M' is a maximum matching since it's a matching of size $|U| + 1$, and by construction its color set is included in $(g(y) \cup \{0\}) \setminus \{v_0\}$, contradiction with the minimality of the color set of M.

Conversely, let S be a minimal set cover of G of size k. For $u \in U$, let us denote $\psi(u) = \alpha(\{v | uv \in E, v \in S\})$ (which is well-defined since S is a set cover of G). Then we define $M = \{x'_0 x_0\} \cup \left\{x_u x_{u,\psi(u)} | u \in U\right\}$. This matching M is of the same size as the one presented in Lemma 3.1. Thus it is a maximum matching with at most $k+1$ colors, since all colors used are in S. It remains to prove that M has $k+1$ colors and is minimally-colored. If it was false, that would mean either that it is not minimally-colored, or that M has not $k+1$ colors. If M was not minimally-colored, there would be a maximum matching M' of $\mathcal{Q}(G)$ such that $c(M') \subsetneq c(M) \subset S \cup \{0\}$. If M had not $k+1$ colors, then we would have $c(M) \subsetneq S \cup \{0\}$. Therefore, in both case, there exists a matching M° such that $c(M^\circ) \subsetneq S \cup \{0\}$ which is equivalent to $g(M^\circ) \subsetneq S$. But $g(M^\circ)$ is a set cover of G of size at most $k-1$ (by Lemma 3.7), a contradiction to the minimality of S.

□

Proof of Theorem 3.1

From every not minimal set cover one can extract in polynomial time a minimal set cover that is smaller than the previous one. Then, without loss of generality, we only consider minimal set covers as approximation candidates for the Minimum Set Cover problem.

Let's suppose that MCMM is approximable on trees, with ratio $\gamma(m)$ where m is the number of internal vertices of the MCMM instance. Given an instance G of the Set Cover problem, then we use $\mathcal{Q}$ in order to compute in polynomial time an instance of MCMM of size less than m. By the above hypothesis we can compute a $\gamma(|V|)$-approximation of that instance of MCMM. Then we can use g to compute in polynomial time a set cover which is, by Lemma 3.7, of same size as the approximate solution to MCMM, that is, at most a $\gamma(|V|)$-approximation of the solution of the Minimum Set Cover on $|G|$.

Then, if $\gamma(n)$ was asymptotically better than $log(n+1)(1-o(1))$, the corresponding approximation ratio for *Set Cover* would be better than $log(n)(1-o(1))$, contradiction unless P=NP [5].

Thus, Theorem 3.1 holds. □

4 MCMM is FTP when parametrized by the maximum size of a matching in the input graph

In this section, we prove the following result :

Theorem 4.1. *Minimum colored maximum matching is FPT with the size of a maximum matching in the input as parameter.*

To show this, we will construct an exploration tree in a much similar way as in [8].

Let G^c be a vertex-colored graph with maximum matching size k.

We consider an arbitrary maximum matching M_0 of G obtained in polynomial time.

Notation. $I_0 = V(G) \setminus V(M_0)$, and $G[M_0]$ be the subgraph induced by $V(M_0)$ in G.

If we consider a minimum-colored maximum matching M^*, we can assert that each edge of M^* has at least one commun vertex extremity with M_0 (otherwise M_0 is not a maximum matching). Thus we can split edges of M^* into two parts, the one included in $G[M_0]$ and the remaining ones.

We are going to use that property to decompose the search for an optimal solution.

In similar way, we are going to use other "natural" splits to decompose the configuration space we want to explore. For the first splits, we will remain exhaustive (as, for those, it doesn't cost much), and then we will only follow arbitrarily some paths when we are sure that thoses choices will produce an optimal solution if there is any in that part of the configuration space.

Then, we will build the exploration tree as follows :

From the root ω_0, we branch, for every possible selection (M, S) where $M \subset E(G[M_0])$ is a matching, $S \subset V(G[M_0]) \setminus V(M)$, and $|M| + |S| = k$, on $\omega_{M,S}$ labelled (M, S, C) with C being the set of colors that are represented in M or in S.

At that point, we have created at most $T_k\binom{2k}{k}$ new leaves, where T_i is the i-th telephon number.
This enumeration can be done in time $O(k \times T(k)2^{2k})$ ($O(k)$ by distinct choice).

Then, for every leaf ω labelled (M, S, C) we branch for every partition Σ of S on ω_Σ labelled (M, Σ, C).

For each previous leaf, we created the $B_{|S|} \leq B_k$ possible partitions of S which can be enumerated in time $O(kB_k)$ ($O(k)$ by distinct partition).

Then, for every leaf ω labelled (M, Σ, C), we branch for every possible choice of partial injective coloration of nonempty parts of Σ by C, Ξ, on ω_Ξ labelled (M, Σ, Ξ). We formally define Ξ as a function $\Xi : \Sigma \to C \uplus \{0\}$ injective on $\Sigma \setminus \Xi^{-1}(0)$.

For each leaf of the previous step, we created at most

$$\sum_{i=0}^{min(|C|,|\Sigma|)} i!\binom{|C|}{i} \leq min(|C|, |\Sigma|)! \times 2^{|C|} \leq k! \times 2^k$$

possible partial injective coloration of Σ (and that much new leaves), which can be enumerated in time $O(k \times k! \times 2^k)$ ($O(k)$ by distinct coloration).

For every leaf produced at the previous step, labelled (M, Σ, Ξ), we compute monochromatic matchings for every part of the partition $s \in \Sigma$:

- If $\Xi(s) \neq 0$, we compute, if any, μ a matching between s and vertices of I_0 of color $\Xi(s)$, and we write $\Gamma(s) = \{\mu\}$. If no such matching exists, $\Gamma(s) = \emptyset$.
- If $\Xi(s) = 0$ then for every color $c_0 \in c(V)$, we compute, if any, μ a matching between s and vertices of I_0 of color c_0, and denote by $\Gamma(s)$ the set of those matchings truncated at $k+1$ (if we have more than $k+1$, we only keep the $k+1$ first matchings computed).

We relabel that leaf (Γ, Σ, Ξ)

For each current leaf, for every color, we computed at most a maximum matching, each one being computed in $O(k^{5/2})$ [1].

Then, for each leaf labelled ω labelled (Γ, Σ, Ξ), we compute a maximum matching γ on the bipartite graph

$$(\Sigma, c(\bigcup_{(s,\mu)\in\Sigma\times\Gamma(s)} V(\mu) \cap I_0), \{sc(\mu)|s \in \Sigma, \mu \in \Gamma(s)\})$$

, where $c(\mu)$ denote the only color in $c(V(\mu) \cap I_0)$

Then, we add a child ω_γ to ω. If $|\gamma| = |\Sigma|$, we define $M_\infty = \bigcup_{sc(\mu)\in\gamma} \mu$ and label ω_γ with $(\gamma, |c(M_\infty)|)$, else we label it with $\perp$.

For a previous leaf, the computation of the auxilliary matching can be done in $O(k^{5/2})$ [1], the following computation of a matching of G takes $O(k^2)$, and finally the computation of its number of colors $|c(M_\infty)|$ in $O(k)$. It is then a $O(k^{5/2})$.

Lemma 4.1. *The exploration tree described above can be computed in time $O(k^4 T_k B_k k! 2^{3k}|V|)$ from a given maximum matching on G.*

Proof. From the analysis boxed between steps of the tree construction we have that the tree can be computed in

$$O(k^{1/2}2^{2k}T_k \times (k + B_k(k + k! \times 2^k(k + k \times |im(c)| \times k^{5/2} + k^{5/2}))$$

Which is then $O(k^4 T_k B_k k! 2^k |V|)$, (by taking $|im(c)| = O(|V|)$). □

Remark. For any $\varepsilon > 0$, the above is $O((\frac{k}{e})^{(3/2+\varepsilon)k}|V|)$.

Lemma 4.2. *There exists a leaf in the research tree which is labelled by a maximum matching whose number of colors is minimal.*

Proof. Let M_{opt} be a minimum colored maximum matching. Let us decompose it relatively to M_0 into $M_{opt} = M_{in} \uplus M_{out}$ where :

- $M_{in} = M_{opt} \cap E(G[V(M_0)])$
- $M_{out} = M_{opt} \backslash M_{in} \subset V(M_0) \times I_0$ (the inclusion come from the fact that there cannot exist an edge between two vertices outside $V(M_0)$ since it would contradict the maximality of M_0)

We define $S_{out} = V(M_{out}) \cap V(M_0)$ and go in the exploration tree to ω, the vertex labelled $(M_{in}, S_{M_{opt}}, c(M_{in}) \cup c(S_{M_{opt}}))$. Which exists since we branched exhaustively.

We compute the following partition of S_{out} :

$\Sigma = \{\{u \in S_{out} | uv \in M_{out}, c(v) = c_0\} \,| c_0 \in c(V(M_{out}) \cap I_0\}$

and search among the children of ω for the child ω' labelled with Σ, which exists since we branched on all the partitions.

Then we define $\Xi(s)$ as the only color in $c(V(\{uv|u \in s, uv \in M_{out}\}) \cap (c(M_{in} \cup c(S_{out}))$ if it is nonempty (there cannot be more than one color in that set from the construction of Σ), and as 0 otherwise. From the construction of $\mathcal{P}$, Ξ is injective on $\Sigma \setminus \Xi^{-1}(0)$. We search for ω'' the child of ω' having Ξ in his labelling, which exists since we branched on all possible partial permutations of already chosen colors on the parts of the partition.

Considering the maximum matching computed at that point in the construction of the exploration tree of the following bipartite graph

$$\Omega = (\Sigma, c(\bigcup_{(s,m)\in\Sigma\times\Xi(s)} V(m) \cap I_0), \{sc(m)|s \in \Sigma, m \in \Gamma(s)\})$$

, where $c(m)$ denote the only color in $c(V(m) \cap I_0)$

We know that the computed maximum matching is of size $|\Sigma|$, since we can construct the following matching :

- For all $s \in \Sigma$ that have k or less edges in Ω, we take the edge corresponding to the color attributed to s in M_{OPT} (that is the only color in $c(\{u|uv \in M_{OPT}, v \in s\})$). That color appears in Ω since we exhaustively enumerated possible colors for $s \in \Sigma$ that had less than k possible color to match to. Let us denote by $n_{matched}$ the number of $s \in \Sigma$ in this situation.
- Then, there are at most $k - n_{matched}$ parts of Σ that still needs to be matched. For every one of those s, $\Gamma(s)$ contains $k+1$ matchings of different colors on the I_0) side, that is for every one of those s, it has edges to $k+1$ colors in Ω, at least $k+1-|n_{matched}| > k-|n_{matched}|$ of them not being already matched. We can then choose greedily a different color to match every remaining $s \in \Sigma$.

The described matching is of size $|\Sigma|$, so the maximum matching computed when constructing the exploration tree must have size $|\Sigma|$. Then, by construction of the exploration tree, the only child of ω'' can't be labelled $\perp$, and is labelled with a maximum matching M_∞. In that matching, the parts of Σ that are matched to colors already in M_{in} or in S_{out} are the same as in M_{OPT} (since there is only one edge from those in Ω). Every other parts of Σ is matched in M_∞ to a different new color (a color not appearing in M_{in} or S_{out}) as it is the case in M_{OPT} by construction of Σ. Thus M_∞ has the same number of colors as M_{OPT}, which conclude the proof of Lemma 4.2. □

Proof of Theorem 4.1

Since we supposed that G admits a maximum matching of size k, by Lemma 4.2, the size of the exploration tree, and its construction time, we conclude that we can compute a Minimum-colored Maximum Matching in $O(|E|\sqrt{|V|}) + O(k^4 T_k B_k k! 2^k |V|)$ (the $O(|E|\sqrt{|V|})$ coming from the construction of the arbitrary maximum matching M_0 [9]) and thus Theorem 4.1 holds. □

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