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Schrödinger operators with Leray-Hardy potential singular on the boundary

Huyuan Chen*

Laurent Véron †

Abstract

We study the kernel function of the operator $u \mapsto \mathcal{L}_\mu u = -\Delta u + \frac{\mu}{|x|^2} u$ in a bounded smooth domain $\Omega \subset \mathbb{R}_+^N$ such that $0 \in \partial\Omega$, where $\mu \geq -\frac{N^2}{4}$ is a constant. We show the existence of a Poisson kernel vanishing at 0 and a singular kernel with a singularity at 0. We prove the existence and uniqueness of weak solutions of $\mathcal{L}_\mu u = 0$ in Ω with boundary data $\nu + k\delta_0$, where ν is a Radon measure on $\partial\Omega \setminus \{0\}$, $k \in \mathbb{R}$ and show that this boundary data corresponds in a unique way to the boundary trace of positive solution of $\mathcal{L}_\mu u = 0$ in Ω .

Key Words: Hardy Potential, Harnack inequality, Limit set, Radon Measure.

MSC2010: 35J75, 35B44.

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1 Introduction

We denote by \mathcal{L}_μ the Schrödinger operator defined in a domain $\Omega \subset \mathbb{R}^N$ by

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2} u,$$

where μ is a real constant and $N \geq 2$. This operator which is associated to the Hardy inequality has been thoroughly studied in the last thirty years. When the singular point 0 belongs to Ω , it appears a critical value

$$\mu_0 = -\left(\frac{N-2}{2}\right)^2$$

and the range of the parameter μ in which the operator is bounded from below is $[\mu_0, \infty)$. This is linked to the Hardy inequality

$$\int_{\Omega} |\nabla \zeta|^2 + \mu_0 \int_{\Omega} \frac{\zeta^2}{|x|^2} dx \geq 0 \quad \text{for all } \zeta \in C_0^\infty(\Omega).$$

Furthermore this inequality is never achieved if Ω is bounded, in which case a remainder was shown to exist by Brézis and Vázquez [4]. When λ is a Radon measure in Ω , the associated Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied in its full generality in [8] and [9] thanks to the introduction of a notion of very weak solution associated to some specific weight. Thanks to this new formulation an extensive treatment of the associated semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nondecreasing function, is developed in [9].

In this article we assume that the singular point of the potential lies on the boundary of the domain Ω , and we are mainly interested in the two problems:

1- To define a notion of *very weak solution* for the problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is a Radon measure on $\partial\Omega$, and more generally on $\partial\Omega \setminus \{0\}$;

2- To prove the existence of a boundary trace for any positive \mathcal{L}_μ -harmonic function, i.e. solution of $\mathcal{L}_\mu u = 0$ in Ω and to connect it to the problem (1.1).

The model example is $\Omega = \mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$ although it is not a bounded domain. There exists a critical value

$$\mu \geq \mu_1 := -\frac{N^2}{4}. \tag{1.2}$$

This value is fundamental for the operator \mathcal{L}_μ to be bounded from below since there holds,

$$\int_{\mathbb{R}_+^N} |\nabla \zeta|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\zeta^2}{|x|^2} dx \geq 0 \quad \text{for all } \zeta \in C_0^\infty(\mathbb{R}_+^N). \quad (1.3)$$

The analysis of the model case is explicit. Let $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$ be the spherical coordinates in \mathbb{R}_+^N , then, if (1.2) is satisfied, there exist two different types of positive \mathcal{L}_μ -harmonic functions vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$,

$$\gamma_\mu(r, \sigma) = r^{\alpha_+} \psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r, \sigma) = \begin{cases} r^{\alpha_-} \psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1}) \psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \quad (1.4)$$

where $\psi_1(\sigma) = \frac{x_N}{|x|}$ generates $\ker(-\Delta' + (N-1)I)$ in $H_0^1(\mathbb{S}_+^{N-1})$, and where

$$\alpha_+ := \alpha_+(\mu) = \frac{2-N}{2} + \sqrt{\mu + \frac{N^2}{4}} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2-N}{2} - \sqrt{\mu + \frac{N^2}{4}}. \quad (1.5)$$

Put $d\gamma_\mu(x) = \gamma_\mu(x)dx$. We define the γ_μ -dual operator \mathcal{L}_μ^* of \mathcal{L}_μ by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu} \langle \nabla \gamma_\mu, \nabla \zeta \rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}_+^N}), \quad (1.6)$$

and we prove that ϕ_μ is, in some sense, the fundamental solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \mathbb{R}_+^N, \\ u = \delta_0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

since it satisfies

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}_+^N}) \cap C^{1,1}(\mathbb{R}_+^N)$$

such that $\rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N)$, where $c_\mu > 0$ is a normalized constant and $\rho(x) = \text{dist}(x, \partial\Omega)$. Here $\rho(x) = x_N$ when $\Omega = \mathbb{R}_+^N$.

When \mathbb{R}_+^N is replaced by a bounded domain Ω satisfying the condition

$$(C-1) \quad 0 \in \partial\Omega, \quad \Omega \subset \mathbb{R}_+^N \quad \text{and} \quad \langle x, \mathbf{n} \rangle = O(|x|^2) \quad \text{for all } x \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}_x$ is the outward normal vector at x , inequality (1.3) holds but it is never achieved in the Hilbert space $H_0^1(\Omega)$. Note that the last condition in (C-1) holds if Ω is a C^2 domain. It is proved in [5] that under the assumption (C-1) there exists a remainder under the following form:

$$\int_\Omega |\nabla \zeta|^2 + \mu_1 \int_\Omega \frac{\zeta^2}{|x|^2} dx \geq \frac{1}{4} \int_\Omega \frac{\zeta^2}{|x|^2 \ln^2(|x|R_\Omega^{-1})} dx \quad \text{for all } \zeta \in C_c^\infty(\Omega), \quad (1.7)$$

where $R_\Omega = \max_{z \in \Omega} |z|$. Then there holds

$$\ell_\mu^\Omega := \inf \left\{ \int_\Omega \left(|\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx : v \in C_c^1(\Omega), \int_\Omega v^2 dx = 1 \right\} > 0.$$

This first eigenvalue is achieved in $H_0^1(\Omega)$ if $\mu > \mu_1$, or in the space $H(\Omega)$ which is the closure of $C_c^1(\Omega)$ for the norm

$$v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_{\Omega} \left(|\nabla v|^2 + \frac{\mu_1}{|x|^2} v^2 \right) dx},$$

when $\mu = \mu_1$. In the sequel we set

$$H_{\mu}(\Omega) = \begin{cases} H_0^1(\Omega) & \text{if } \mu > \mu_1, \\ H(\Omega) & \text{if } \mu = \mu_1. \end{cases}$$

Moreover, under the assumption (C-1) the imbedding of $H_{\mu}(\Omega)$ in $L^2(\Omega)$ is compact (see e.g. [6]). We denote by γ_{μ}^{Ω} the positive eigenfunction, its satisfies

$$\begin{cases} \mathcal{L}_{\mu} \gamma_{\mu}^{\Omega} = \ell_{\mu}^{\Omega} \gamma_{\mu}^{\Omega} & \text{in } \Omega, \\ \gamma_{\mu}^{\Omega} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \quad (1.8)$$

We prove in Appendix that there exist $c_j = c_j(\Omega, \mu) > 0$, $j = 1, 2$, such that

$$\begin{aligned} (i) \quad & \gamma_{\mu}^{\Omega}(x) = c_1 \rho(x) |x|^{\alpha+1} (1 + o(1)) \quad \text{as } x \rightarrow 0, \\ (ii) \quad & |\nabla \gamma_{\mu}^{\Omega}(x)| \leq c_2 \frac{\gamma_{\mu}^{\Omega}(x)}{\rho(x)} \quad \text{for all } x \in \Omega. \end{aligned} \quad (1.9)$$

This function will play the role of a weight function for replacing γ_{μ} . Next we construct the Poisson kernel $\mathbb{K}_{\mu}^{\Omega}$ of \mathcal{L}_{μ} in $\Omega \times \partial\Omega$. When $\mu \geq 0$ this construction can be made by truncation as in [18], considering for $\epsilon > 0$ and $\lambda \in \mathfrak{M}_+(\partial\Omega)$ the solution u_{ϵ} of

$$\begin{cases} -\Delta u + \frac{\mu}{\max\{\epsilon^2, |x|^2\}} u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

By a more elaborate method, we also construct the Poisson kernel when $\mu_1 \leq \mu < 0$. It is important to notice that when $\mu > 0$ the kernel has the property that

$$K_{\mu}^{\Omega}(x, 0) = 0 \quad \text{for all } x \in \bar{\Omega} \setminus \{0\} \quad (1.10)$$

by [18, Theorem A.1]. Because of (1.10) it is clear that the Poisson kernel cannot be the tool for describing all the positive \mathcal{L}_{μ} -harmonic functions. Our first concern in this article is to clarify the Poisson kernel of \mathcal{L}_{μ} .

We first characterize the positive \mathcal{L}_{μ} -harmonic functions which are singular at 0.

Theorem A *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ and $\mu \geq \mu_1$. If u is a nonnegative \mathcal{L}_{μ} -harmonic function in Ω vanishing on $B_{r_0}(0) \cap (\partial\Omega \setminus \{0\})$ for some $r_0 > 0$, then there exists $k \geq 0$ such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{\rho(x) |x|^{\alpha-1}} = k,$$

if $\mu > \mu_1$ and

$$\lim_{x \rightarrow 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x) \ln |x|} = -k,$$

if $\mu = \mu_1$.

Actually the above convergences hold in a stronger way. In order to prove that such solutions truly exist we construct the *kernel function* ϕ_μ^Ω (see [13] for the denomination) which is the analogue in a bounded domain of the explicit singular solution ϕ_μ defined in \mathbb{R}_+^N .

Theorem B *Let Ω be a C^2 bounded domain satisfying (C-1) and $\mu \geq \mu_1$. Then there exists a positive \mathcal{L}_μ -harmonic function in Ω , which vanishes on $\partial\Omega \setminus \{0\}$, which satisfies*

$$\phi_\mu^\Omega(x) = \rho(x)|x|^{\alpha-1}(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.11)$$

if $\mu > \mu_1$, and

$$\phi_{\mu_1}^\Omega(x) = \rho(x)|x|^{-\frac{N}{2}}(|\ln |x|| + 1)(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.12)$$

if $\mu = \mu_1$.

As in the model case, we define the γ_μ^Ω -dual operator of \mathcal{L}_μ by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu^\Omega} \langle \nabla \gamma_\mu^\Omega, \nabla \zeta \rangle + \ell_\mu^\Omega \zeta \quad \text{for all } \zeta \in C^{1,1}(\Omega).$$

The following commutation formula holds

$$\mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) = \gamma_\mu^\Omega \mathcal{L}_\mu^* \zeta. \quad (1.13)$$

Corollary C *Let Ω be a C^2 bounded domain satisfying (C-1) and $\mu \geq \mu_1$. Then ϕ_μ^Ω is the unique function belonging to $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$, which satisfies*

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = k c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \quad (1.14)$$

where $d\gamma_\mu^\Omega = \gamma_\mu^\Omega dx$, here and in the sequel the test function space

$$\mathbb{X}_\mu(\Omega) = \{\zeta \in C(\overline{\Omega}) : \gamma_\mu^\Omega \zeta \in H_\mu(\Omega) \text{ and } \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\Omega)\}.$$

Furthermore, if u is a nonnegative \mathcal{L}_μ -harmonic function vanishing on $\partial\Omega \setminus \{0\}$, there exists $k \geq 0$ such that $u = k\phi_\mu^\Omega$.

We denote by $\mathfrak{M}(\Omega; \gamma_\mu^\Omega)$ the set of Radon measures ν in Ω such that

$$\sup \left\{ \int_\Omega \zeta d|\lambda| : \zeta \in C_c(\Omega), 0 \leq \zeta \leq \gamma_\mu^\Omega \right\} := \int_\Omega \gamma_\mu^\Omega d|\nu| < +\infty.$$

If $\nu \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$ the measure $\gamma_\mu^\Omega \nu$ is a nonnegative bounded measure in Ω . Put

$$\beta_\mu^\Omega(x) = -\frac{\partial \gamma_\mu^\Omega(x)}{\partial \mathbf{n}_x} = \lim_{t \rightarrow 0^+} \frac{\gamma_\mu^\Omega(x - t\mathbf{n}_x)}{t} = \lim_{t \rightarrow 0^+} \frac{\gamma_\mu^\Omega(x - t\mathbf{n}_x)}{\rho^*(x - t\mathbf{n}_x)}, \quad \forall x \in \partial\Omega \setminus \{0\} \quad (1.15)$$

and from (3.2) and (1.9), we have that

$$c_1|x|^{\alpha+1} \leq \beta_\mu^\Omega(x) \leq c_1c_3|x|^{\alpha+1} \quad \text{for } x \in \partial\Omega \setminus \{0\}. \quad (1.16)$$

As a consequence, the following potential function plays an important role in defining our boundary data. Denote

$$\beta_\mu(x) = |x|^{\alpha+1} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \quad (1.17)$$

The set Radon measures λ in $\partial\Omega \setminus \{0\}$ such that

$$\sup \left\{ \int_{\partial\Omega \setminus \{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial\Omega \setminus \{0\}), 0 \leq \zeta \leq \beta_\mu \right\} := \int_{\partial\Omega \setminus \{0\}} \beta_\mu d|\lambda| < +\infty$$

is denoted by $\mathfrak{M}(\partial\Omega \setminus \{0\}; \beta_\mu)$. The extension of $\lambda \in \mathfrak{M}_+(\partial\Omega \setminus \{0\}; \beta_\mu)$ as a measure $\beta_\mu\lambda$ in $\partial\Omega$ is given by

$$\int_{\partial\Omega} \zeta d(\beta_\mu\lambda) = \sup \left\{ \int_{\partial\Omega} v\beta_\mu d\lambda : v \in C_c(\partial\Omega \setminus \{0\}), 0 \leq v \leq \zeta \right\} \quad \text{for all } \zeta \in C_c(\partial\Omega), \zeta \geq 0$$

and $\beta_\mu\lambda = \beta_\mu\lambda_+ - \beta_\mu\lambda_-$ if λ is a signed measure in $\mathfrak{M}(\partial\Omega \setminus \{0\}; \beta_\mu)$, and this defines the set $\mathfrak{M}(\partial\Omega; \beta_\mu)$ of all such extensions. The Dirac mass at 0 does not belong to $\mathfrak{M}(\partial\Omega; \beta_\mu)$, but it is the limit of sequences of measures in this space in the same way as it is a limit of measures in $\mathfrak{M}_+(\partial\Omega \setminus \{0\}; \beta_\mu)$. In the next result we prove the existence and uniqueness of a solution to

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \quad (1.18)$$

Thanks to (1.7) the Green kernel G_μ^Ω is easily constructible. If $\nu \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ the following expressions are well defined

$$\mathbb{K}_\mu^\Omega[\lambda](x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{and} \quad \mathbb{G}_\mu^\Omega[\nu](x) = \int_{\Omega} G_\mu^\Omega(x, y) d\nu(y).$$

Our main existence result is the following.

Theorem D *Let Ω be a C^2 bounded domain satisfying (C-1) and $\mu \geq \mu_1$. If $\nu \in \mathfrak{M}_+(\Omega; \gamma_\mu^\Omega)$, $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \in \mathbb{R}$, the function*

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda, k)] \quad (1.19)$$

is the unique solution of (1.18) in the very weak sense that $u \in L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ and

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + kc_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \quad (1.20)$$

In the next result we prove that all the positive \mathcal{L}_μ -harmonic functions in Ω are described by formula (1.19) (with $\nu = 0$).

Theorem E *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying (C-1), $\mu \geq \mu_1$ and u be a nonnegative \mathcal{L}_μ -harmonic functions in Ω . Then there exist $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \geq 0$, such that*

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = \mathbb{H}_\mu^\Omega[(0, \lambda, k)].$$

The couple $(\lambda, k\delta_0)$ is called the boundary trace of u .

The rest of this paper is organized as follows. In Section 2, we introduce the distributional identity of \mathcal{L}_μ harmonic function ϕ_μ in \mathbb{R}_+^N . Section 3 is devoted to build the Kato's type inequalities, to construct Poisson kernel and related properties. Section 4 is addressed to classify the boundary isolated singular \mathcal{L}_μ harmonic functions in a bounded domain, i.e. **Theorem A** and to show the existence and related distributional identity in a (C-1) domain: proofs of **Theorem B** and **Corollary C**. We classify the boundary trace for general \mathcal{L}_μ harmonic functions and give the existence of \mathcal{L}_μ harmonic functions with the boundary trace $(\lambda, k\delta_0)$: **Theorem D** and **Theorem E** respectively in Section 5. Finally, we show Estimates (1.9) and (3.2) in Appendix.

In a forthcoming article [10] we develop our observations to study qualitative properties of the semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

when the origin lies on the boundary of Ω and λ is a Radon measure defined in proper way.

2 The half-space setting

Let $\mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$, $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$ be the spherical coordinates in \mathbb{R}_+^N and Δ' is the Laplace Beltrami operator on \mathbb{S}^{N-1} . Then

$$\mathcal{L}_\mu u = -\partial_{rr}u - \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\Delta' u + \frac{\mu}{r^2}u.$$

If $u(r, \sigma) = r^\alpha \phi(\sigma)$ is a (separable) solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$, then ψ_k satisfies

$$\begin{cases} -\Delta' \psi_k = \lambda_k \psi_k & \text{in } \mathbb{S}_+^{N-1} := \mathbb{S}^{N-1} \cap \mathbb{R}_+^N, \\ \psi_k = 0 & \text{in } \partial\mathbb{S}_+^{N-1} \approx \mathbb{S}^{N-2}, \end{cases}$$

where λ_k is a constant which necessarily belongs to the spectrum

$$\sigma_{\mathbb{S}_+^{N-1}}(-\Delta') = \{\lambda_k = k(N+k-2) : k \in \mathbb{N}^*\},$$

and $\alpha = \alpha_{k+}, \alpha_{k-}$ is a root of

$$\alpha^2 + (N-2)\alpha - \lambda_k - \mu = 0. \quad (2.1)$$

The fundamental state corresponds to $k = 1$, in which case since $\lambda_1 = N-1$, existence of real roots of (2.1) necessitates $\mu \geq \mu_1 = -\frac{N^2}{4} = \mu_1$ and we denote $\alpha_{1+} = \alpha_+$ and $\alpha_{1-} = \alpha_-$. Note that this value is connected to the boundary Hardy

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{S}_+^{N-1}).$$

If this condition is fulfilled, the two roots α_+ and α_- corresponding to $k = 1$ and λ_1 are

$$\alpha_+ = \frac{2-N}{2} + \sqrt{\mu - \mu_1} \quad \text{and} \quad \alpha_- = \frac{2-N}{2} - \sqrt{\mu - \mu_1}. \quad (2.2)$$

The corresponding positive separable solutions γ_μ and ϕ_μ of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$ are defined by (1.4). We set $d\gamma_\mu(x) = \gamma_\mu(x)dx$ and define the operator \mathcal{L}_μ^* by (1.6).

Proposition 2.1 *The function ϕ_μ belongs to $L_{loc}^1(\mathbb{R}_+^N, \rho^{-1}d\gamma_\mu)$. It satisfies*

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N), \quad (2.3)$$

where

$$c_\mu = \frac{1}{N} \begin{cases} \sqrt{\mu - \mu_1} |\mathbb{S}^{N-1}| & \text{if } \mu > \mu_1, \\ \frac{1}{2N} |\mathbb{S}^{N-1}| & \text{if } \mu = \mu_1 \end{cases} \quad (2.4)$$

and $\mathbb{X}_\mu(\mathbb{R}_+^N) = \left\{ \zeta \in C_c(\overline{\mathbb{R}_+^N}) : \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N) \right\}$.

Proof. Let $\zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N)$, $\epsilon > 0$ and set $B_\epsilon^+ = B_\epsilon(0) \cap \mathbb{R}_+^N$, $(B_\epsilon^+)^c = B_\epsilon^c(0) \cap \mathbb{R}_+^N$ and $\Gamma_\epsilon^+ = \partial B_\epsilon(0) \cap \overline{\mathbb{R}_+^N}$, by direct compute, we have that

$$\begin{aligned} 0 &= \int_{(B_\epsilon^+)^c} \zeta \gamma_\mu \mathcal{L}_\mu \phi_\mu dx \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \int_{\Gamma_\epsilon^+} \left(-\frac{\partial \phi_\mu}{\partial \mathbf{n}} \zeta \gamma_\mu + \left(\gamma_\mu \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu}{\partial \mathbf{n}} \right) \phi_\mu \right) dS \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \zeta(0) A(\epsilon) \end{aligned}$$

and $\mathbf{n} = -\epsilon^{-1}x$ on Γ_ϵ^+ , and

$$A(\epsilon) = \begin{cases} -2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} x_N^2 dS + O(\epsilon) & \text{if } \mu > \mu_1, \\ -\int_{\mathbb{S}_+^{N-1}} x_N^2 dS + O(\epsilon) & \text{if } \mu = \mu_1, \end{cases} \quad (2.5)$$

together with the fact that $\int_{\mathbb{S}_+^{N-1}} x_N^2 dS = \frac{1}{2N} |\mathbb{S}^{N-1}|$, which implies (2.3)-(2.4). \square

3 The Poisson kernel

In this section we assume that Ω is a bounded C^2 domain included in B_1 (which can always be assumed by scaling) and $0 \in \partial\Omega$.

We let $\sigma_\mu^\Omega \in H_\mu(\Omega)$ be the unique variational solution of

$$\mathcal{L}_\mu u = \frac{\gamma_\mu^\Omega}{\rho^*} \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \quad (3.1)$$

where $\rho^*(x) = \min\{\frac{1}{l_\mu^\Omega}, \rho\}$. We prove in Appendix that there is $c_3 > 1$ such that

$$\gamma_\mu^\Omega \leq \sigma_\mu^\Omega \leq c_3 \gamma_\mu^\Omega \quad \text{in } \Omega. \quad (3.2)$$

Note that $\sigma_\mu^\Omega \in C^2(\bar{\Omega} \setminus \{0\})$ is a positive classical solution of (3.1) with zero Dirichlet boundary condition on $\partial\Omega \setminus \{0\}$, i.e. $\sigma_\mu^\Omega = 0$ on $\partial\Omega \setminus \{0\}$. Moreover, $\frac{\partial\sigma_\mu^\Omega}{\partial\mathbf{n}} < 0$ on $\partial\Omega \setminus \{0\}$. We set

$$\eta = \frac{\sigma_\mu^\Omega}{\gamma_\mu^\Omega} \quad \text{in } \Omega,$$

which satisfies

$$\mathcal{L}_\mu^* \eta = \frac{1}{\rho} \quad \text{in } \Omega. \quad (3.3)$$

This function play a key role in the sequel. Clearly, $\eta \in C^2(\bar{\Omega} \setminus \{0\})$ and $1 \leq \eta \leq c_2$ in $\bar{\Omega} \setminus \{0\}$ by (3.2).

We start with the following identity of commutation valid for all $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $\zeta \in C^{1,1}(\bar{\Omega})$

$$-\int_{\partial\Omega} \frac{\partial(\zeta \gamma_\mu^\Omega)}{\partial\mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) \quad \text{for all } \zeta \in C^{1,1}(\bar{\Omega}), \quad (3.4)$$

where β_μ^Ω is defined in (1.15) with the asymptotic behavior (1.16) near the origin.

The following inequality extends the classical Kato inequality to our framework.

Lemma 3.1 *Assume $N \geq 3$ and $\mu \geq \mu_1$ or $N = 2$ and $\mu > \mu_1$. Then for any $f \in L^1(\Omega, \gamma_\mu^\Omega dx)$, $h \in L^1(\partial\Omega, \beta_\mu^\Omega dx)$, there exists a unique weak solution u to*

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \quad (3.5)$$

in the sense that

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h \zeta d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \quad (3.6)$$

Furthermore, for any $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta \geq 0$, there holds

$$\int_{\Omega} |u| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h| \zeta d\beta_\mu^\Omega \quad (3.7)$$

and

$$\int_{\Omega} u^+ \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}_+(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h^+ \zeta d\beta_\mu^\Omega. \quad (3.8)$$

Proof. *Uniqueness.* Assume that u is a weak solution of (3.5) with $f = h = 0$. Then for any $\zeta \in \mathbb{X}_\mu(\Omega)$ there holds

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = 0.$$

Let $\phi \in C_c^\infty(\Omega)$, and $v \in H_\mu(\Omega)$ be the variational solution of

$$\mathcal{L}_\mu v = \frac{\gamma_\mu^\Omega}{\rho} \phi \quad \text{and} \quad u \in H_\mu(\Omega).$$

Then $v \in C^\infty(\Omega)$, $|v| \leq \|\phi\|_{L^\infty} \sigma_\mu^\Omega$; the equation is satisfied everywhere and in the sense of distributions in Ω . Clearly $w := (\gamma_\mu^\Omega)^{-1} v$ belongs to $C^\infty(\Omega)$ and satisfies

$$\mathcal{L}_\mu^* w := \frac{1}{\rho} \phi.$$

Thus,

$$\int_\Omega \frac{u}{\rho} \phi d\gamma_\mu^\Omega = 0.$$

Since ϕ is arbitrary, we have that $u = 0$.

Existence and estimates. We proceed by approximation as in [8, Prop. 2.1]. We assume that $\{(f_n, h_n)\} \subset C_0^1(\Omega) \times C_0^1(\partial\Omega \setminus \{0\})$ is a sequence which converges to (f, h) in $L^1(\Omega, \gamma_\mu^\Omega dx) \times L^1(\partial\Omega, \beta_\mu^\Omega dx)$. We set $V(x) = |x|^{-2}$, denote by \mathbb{K}_0^Ω the Poisson potential of $-\Delta$ in Ω and consider the approximate problem

$$\begin{cases} \mathcal{L}_\mu w_n = f_n - \mu V \mathbb{K}_0^\Omega[h_n] & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.9)$$

Near 0, we have $V \mathbb{K}_0^\Omega[h_n](x) = O(\frac{\rho(x)}{|x|^2})$, hence, if $N \geq 3$, $V \mathbb{K}_0^\Omega[h_n] \in L^2(\Omega)$. If $N = 2$, the function $x \mapsto \frac{\rho(x)}{|x|^2}$ belongs to the Lorentz space $L^{2,\infty}(\Omega)$ which is the dual of $L^{2,1}(\Omega)$. Since $H_\mu(\Omega) \subset L^{2,1}(\Omega)$ by (1.7), it follows that $H'_\mu(\Omega) \subset L^{2,\infty}(\Omega)$. Hence, by Lax-Milgram theorem there exists a unique $w_n \in H_\mu(\Omega)$ such that (3.9) holds in the variational sense. Then $u_n = w_n + \mathbb{K}_0^\Omega[h_n]$, which has the same regularity as w_n , satisfies

$$\begin{cases} \mathcal{L}_\mu u_n = f_n & \text{in } \Omega, \\ u_n = h_n & \text{on } \partial\Omega. \end{cases} \quad (3.10)$$

For $\sigma > 0$, we set

$$m_\sigma(t) = \begin{cases} |t| - \frac{\sigma}{2} & \text{if } |t| \geq \sigma, \\ \frac{t^2}{2\sigma} & \text{if } |t| < \sigma. \end{cases}$$

Then m_σ is convex, $|m'_\sigma(t)| \leq 1$ and $m'_\sigma(t) \rightarrow \text{sign}_0(t)$ as $\sigma \downarrow 0$. Let $\zeta \in C^{1,1}(\overline{\Omega})$, $\zeta \geq 0$. We have that

$$\begin{aligned} & \int_\Omega (\langle \nabla w_n, \nabla(\zeta m'_\sigma(u_n) \gamma_\mu^\Omega) \rangle + \mu V w_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega) dx \\ &= \int_\Omega \zeta m'_\sigma(u_n) \gamma_\mu^\Omega f_n dx - \mu \int_\Omega V \mathbb{K}_0^\Omega[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx =: \mathbb{R}(\sigma). \end{aligned}$$

By the fact $u_n = w_n + \mathbb{K}_0^\Omega[h_n]$, we have that

$$\begin{aligned}
\mathbb{R}(\sigma) &= \int_{\Omega} \langle \nabla u_n, \nabla(\zeta m'_\sigma(u_n) \gamma_\mu^\Omega) \rangle dx - \int_{\Omega} \langle \nabla \mathbb{K}_0^\Omega[h_n], \nabla(\zeta m'_\sigma(u_n) \gamma_\mu^\Omega) \rangle dx \\
&\quad + \mu \int_{\Omega} V u_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx - \mu \int_{\Omega} V \mathbb{K}_0^\Omega[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx \\
&= \int_{\Omega} |\nabla u_n|^2 m''_\sigma(u_n) \zeta \gamma_\mu^\Omega dx + \int_{\Omega} \langle \nabla m_\sigma(u_n), \nabla(\zeta \gamma_\mu^\Omega) \rangle dx + \mu \int_{\Omega} V u_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}_0^\Omega[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx \\
&\geq - \int_{\Omega} m_\sigma(u_n) \Delta(\zeta \gamma_\mu^\Omega) dx + \int_{\partial\Omega} m_\sigma(h_n) \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} dS + \mu \int_{\Omega} V u_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}_0^\Omega[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx \\
&\geq \int_{\Omega} m_\sigma(u_n) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega - \int_{\partial\Omega} m_\sigma(h_n) d\beta_\mu^\Omega + \mu \int_{\Omega} V (u_n m'_\sigma(u_n) - m_\sigma(u_n)) \gamma_\mu^\Omega \zeta dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}_0^\Omega[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx,
\end{aligned}$$

thus, we obtain that

$$\begin{aligned}
&\int_{\Omega} m_\sigma(u_n) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \mu \int_{\Omega} V (u_n m'_\sigma(u_n) - m_\sigma(u_n)) \zeta d\gamma_\mu^\Omega \\
&\leq \int_{\Omega} \zeta m'_\sigma(u_n) f_n d\gamma_\mu^\Omega + \int_{\partial\Omega} m_\sigma(h_n) \zeta d\beta_\mu^\Omega.
\end{aligned} \tag{3.11}$$

Since m_σ is convex, $u_n m'_\sigma(u_n) - m_\sigma(u_n) \geq 0$. Hence for $\mu \geq 0$, we can let $\sigma \rightarrow 0$ in (3.11) and obtain (3.7).

For $\mu \in [\mu_1, 0)$, we note that

$$0 \leq u_n m'_\sigma(u_n) - m_\sigma(u_n) \leq \frac{|u_n|^2}{2\sigma^2} \chi_{\{|u_n| \leq \sigma\}}$$

and

$$0 \leq \int_{\Omega} V (u_n m'_\sigma(u_n) - m_\sigma(u_n)) \zeta d\gamma_\mu^\Omega \leq \frac{\|\zeta\|_{L^\infty}}{2} \int_{\{|u_n| \leq \sigma\}} |x|^{-1 - \frac{N}{2} + \sqrt{\mu - \mu_1}} dx. \tag{3.12}$$

Hence if $N \geq 3$, or $N = 2$ and $\mu > \mu_1 = -1$, the right-hand side of (3.12) tends to 0 as $\sigma \rightarrow 0$ and then we obtain (3.7). The proof of (3.8) is similar.

Applying estimate (3.7) to $u_n - u_m$, we obtain for all $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta \geq 0$,

$$\int_{\Omega} |u_n - u_m| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} |f_n - f_m| \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h_n - h_m| \zeta d\beta_\mu^\Omega.$$

For test function, we take η , the solution of (3.3), then

$$\int_{\Omega} \frac{|u_n - u_m|}{\rho} d\gamma_{\mu}^{\Omega} \leq \int_{\Omega} |f_n - f_m| d\sigma_{\mu}^{\Omega} + \int_{\partial\Omega} |h_n - h_m| d(\eta\beta_{\mu}^{\Omega}).$$

Therefore $\{u_n\}$ is a cauchy sequence in $L^1(\Omega, \rho^{-1}d\gamma_{\mu}^{\Omega})$ with limit u . Since u_n satisfies (3.10), we let n go to infity in

$$\int_{\Omega} u_n \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} = \int_{\Omega} \zeta \sigma_{\mu}^{\Omega} f_n dx + \int_{\partial\Omega} \zeta \beta_{\mu}^{\Omega} h_n dS$$

and obtain (3.6). \square

Lemma 3.2 *Assume $\lambda \in \mathfrak{M}(\partial\Omega; \beta_{\mu})$ and $\zeta \in \mathbb{X}_{\mu}(\Omega)$, then there holds*

$$\int_{\Omega} \left(\mathcal{L}_{\mu}^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}_0^{\Omega}[\lambda] d\gamma_{\mu}^{\Omega} = \int_{\partial\Omega} \zeta d(\beta_{\mu}^{\Omega} \lambda).$$

Proof. Note that $d(\beta_{\mu}^{\Omega} \lambda)$ is equivalent to $d(\beta_{\mu} \lambda)$ by (1.16). By (1.13) we have almost everywhere in Ω ,

$$\left(\mathcal{L}_{\mu}^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}_0^{\Omega}[\lambda] \gamma_{\mu}^{\Omega} = \left(\mathcal{L}_{\mu}(\gamma_{\mu}^{\Omega} \zeta) - \frac{\mu}{|x|^2} \gamma_{\mu}^{\Omega} \zeta \right) \mathbb{K}_0^{\Omega}[\lambda] = -\Delta(\gamma_{\mu}^{\Omega} \zeta) \mathbb{K}_0^{\Omega}[\lambda].$$

If we assume that λ vanishes in a neighborhood of 0 we derive from (3.4)

$$-\int_{\Omega} \Delta(\gamma_{\mu}^{\Omega} \zeta) \mathbb{K}_0^{\Omega}[\lambda] dx = -\int_{\partial\Omega} \frac{\partial(\zeta \gamma_{\mu}^{\Omega})}{\partial \mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_{\mu}^{\Omega} \lambda).$$

Since $\gamma_{\mu}^{\Omega} \left(\mathcal{L}_{\mu}^* \zeta - \frac{\mu}{|x|^2} \zeta \right)$ is bounded, we obtain the result first if λ is nonnegative by considering the sequence $\{\chi_{B_{\epsilon}} \lambda\}$ and letting $\epsilon \rightarrow 0$, and then for any $\lambda = \lambda^+ - \lambda^-$. \square

We observe also that the existence of the Green kernel follows from Lax-Milgram theorem which gives the existence of a unique variational solution in $H_{\mu}(\Omega)$ to

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by G_{μ}^{Ω} the Green kernel and by $\mathbb{G}_{\mu}^{\Omega}$ the corresponding Green operator defined by $\mathbb{G}_{\mu}^{\Omega}[f](x) = \int_{\Omega} G_{\mu}^{\Omega}(x, y) f(y) dy$.

3.1 Construction of the Poisson kernel when $\mu > 0$

For the sake of completeness, we recall the construction in [18]. For $\epsilon > 0$, we set $V_{\epsilon}(x) = \max\{\epsilon^{-2}, |x|^{-2}\}$ and $V_0(x) = V(x) = |x|^{-2}$, and if $\lambda \in \mathfrak{M}(\partial\Omega)$, let u_{ϵ} be the solution of

$$\begin{cases} -\Delta u + \mu V_{\epsilon} u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

Then

$$u_\epsilon(x) = \int_{\partial\Omega} K_{\mu,\epsilon}^\Omega(x, y) d\lambda(y) = \mathbb{K}_{\mu,\epsilon}^\Omega[\lambda].$$

We obtain by the maximum principle,

$$K_{\mu,\epsilon}^\Omega \leq K_{\mu',\epsilon'}^\Omega \leq K_0^\Omega \quad \text{for all } \mu \geq \mu' \geq 0 \text{ and } \epsilon' \geq \epsilon > 0,$$

where K^Ω is the usual Poisson kernel in Ω and there exists

$$K_\mu^\Omega(x, y) = \lim_{\epsilon \rightarrow 0} K_{\mu,\epsilon}^\Omega(x, y) \quad \text{for all } (x, y) \in \Omega \times \partial\Omega.$$

Therefore we infer, firstly by monotone convergence if $\lambda \geq 0$, and then for any $\lambda \in \mathfrak{M}(\partial\Omega)$, that

$$\lim_{\epsilon \rightarrow 0} u_\epsilon(x) = u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega. \quad (3.13)$$

Since V is finite in $B_\epsilon^c \cap \Omega$, for any $x \in \Omega$, $K_\mu^\Omega(x, y) > 0$ for all $y \in \partial\Omega \setminus \{0\}$. Letting G_0^Ω be the Green kernel in Ω , there holds

$$u_\epsilon(x) + \mu \int_{\Omega} G_0^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy = \int_{\partial\Omega} K_0^\Omega(x, y) d\lambda(y).$$

If $\lambda \geq 0$, we have by Fatou's lemma,

$$\int_{\Omega} G_0^\Omega(x, y) V(y) u(y) dy \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} G_0^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy. \quad (3.14)$$

Combined with (3.13) it yields

$$u(x) + \mu \int_{\Omega} G_0^\Omega(x, y) V(y) u(y) dy \leq \int_{\partial\Omega} K_0^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega.$$

Since the function $u + \mu G_0^\Omega[Vu]$ is nonnegative and harmonic in Ω , it admits a boundary trace which is a nonnegative Radon measure λ^* and there holds

$$u(x) + \mu \int_{\Omega} G_0^\Omega(x, y) V(y) u(y) dy = \int_{\partial\Omega} K_0^\Omega(x, y) d\lambda^*(y) \quad \text{for all } x \in \Omega. \quad (3.15)$$

Because of (3.14), we have that $0 \leq \lambda^* \leq \lambda$ and the measure λ^* is the reduced measure associated to λ . Since (3.15) is equivalent to

$$u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda^*(y),$$

there holds

$$\int_{\partial\Omega} K_\mu^\Omega(x, y) d(\lambda - \lambda^*)(y) = 0.$$

This implies that $\lambda = \lambda^*$ in $\partial\Omega \setminus \{0\}$. With the notations of [18], we recall that

$$\begin{aligned} \mathcal{S}ing_V(\Omega) &:= \{y \in \partial\Omega : \exists x_0 \in \Omega \text{ s.t. } K_\mu^\Omega(x_0, y) = 0\} \\ &\subset Z_V := \left\{ y \in \partial\Omega : \int_\Omega K_0^\Omega(x, y) V(x) \rho(x) dx = \infty \right\}. \end{aligned}$$

Actually, if $y \in \mathcal{S}ing_V(\Omega)$, $K_\mu^\Omega(x_0, y) = 0$ for any $x_0 \in \Omega$ by Harnack inequality. Clearly $0 \in Z_V$ and if $y \neq 0$ the integral term in the definition of Z_V is finite. Hence $\mathcal{S}ing_V(\Omega) \subset Z_V = \{0\}$. Since for any truncated cone $C_{0,\delta} \Subset \Omega$ with vertex 0, there holds

$$\int_{C_{0,\delta}} V(x) \frac{dx}{|x-y|^{N-2}} = \infty,$$

it follows by Ancona's result [18, Theorem A1] that $0 \in \mathcal{S}ing_V(\Omega)$. Finally

$$K_\mu^\Omega(x, 0) = 0 \quad \text{for all } x \in \Omega.$$

3.2 Construction of the Poisson kernel when $\mu_1 \leq \mu < 0$

For $\epsilon > 0$ and $\lambda \in C(\partial\Omega)$, $\lambda \geq 0$ we denote by $w = w_{\epsilon,\lambda}$ the variational solution in $H_\mu(\Omega)$ of

$$\begin{cases} -\Delta w + \mu V_\epsilon w = -\mu V_\epsilon \mathbb{K}_0^\Omega[\lambda] & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $w_{\epsilon,\lambda} \geq 0$ and $u = u_{\epsilon,\lambda} := w_\epsilon + \mathbb{K}_0^\Omega[\lambda]$ satisfies

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases} \quad (3.16)$$

Since $-\Delta u_{\epsilon,\lambda} + \mu V u_{\epsilon,\lambda} \leq 0$, there holds from Lemma 3.1

$$\int_\Omega u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda \zeta d\beta_\mu^\Omega \quad \text{for } \zeta \in \mathbb{X}_\mu(\Omega), \zeta \geq 0$$

and in particular

$$\int_\Omega \frac{u_{\epsilon,\lambda}}{\rho} d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda d(\eta\beta_\mu^\Omega). \quad (3.17)$$

If $\epsilon > \epsilon' > 0$ and $\lambda' > \lambda > 0$ we have

$$-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} = \mu (V_{\epsilon'} - V_\epsilon) \leq 0.$$

Since

$$\begin{aligned} &\int_\Omega \left(-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} \right) (u_{\epsilon,\lambda}^2 - u_{\epsilon',\lambda'}^2)_+ dx \\ &= \int_{\{u_{\epsilon,\lambda} \geq u_{\epsilon',\lambda'}\}} \left(\left| \nabla u_{\epsilon,\lambda} - \frac{u_{\epsilon,\lambda}}{u_{\epsilon',\lambda'}} \nabla u_{\epsilon',\lambda'} \right|^2 + \left| \nabla u_{\epsilon',\lambda'} - \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}} \nabla u_{\epsilon,\lambda} \right|^2 \right) dx, \end{aligned}$$

we deduce that the function $x \mapsto \frac{u_{\epsilon', \lambda'}}{u_{\epsilon, \lambda}}(x)$ is constant on the set $\{x : u_{\epsilon, \lambda}(x) > u_{\epsilon', \lambda'}(x)\}$. If this set is non-empty we get a contradiction since it is strictly included in Ω . Therefore the mapping

$$(\epsilon, \lambda) \mapsto u_{\epsilon, \lambda}$$

is decreasing in ϵ and increasing in λ .

Next we can assume that $\lambda \in \mathfrak{M}_+(\partial\Omega)$ vanishes in $B_\delta \cap \partial\Omega$ and that $\{\lambda_n\} \subset C(\partial\Omega)$ is a sequence of functions which converge to λ in the weak sense of measures. We denote by u_{ϵ, λ_n} the solution of (3.16) with λ replaced by λ_n . Since $\mu < 0$, $\alpha_+ < 1$, $\rho^{-1}\gamma_\mu^\Omega \sim |x|^{\alpha_+-1} \geq R_\omega^{\alpha_+-1}$, where $R_\omega = \max\{|z| : z \in \Omega\}$. Hence

$$\int_\Omega u_{\epsilon, \lambda_n} dx \leq c_8 \int_{\partial\Omega} \lambda_n d(\eta\beta_\mu^\Omega) \leq c_9 \|\lambda\|_{\mathfrak{M}(\partial\Omega)}.$$

Hence u_{ϵ, λ_n} and $V_\epsilon u_{\epsilon, \lambda_n}$ are uniformly bounded in $L^1(\Omega)$. From standard regularity estimates the sequence $\{u_{\epsilon, \lambda_n}\}_{n \in \mathbb{N}}$ is bounded in the Lorentz spaces $L^{\frac{N}{N-1}, \infty}(\Omega)$ and weakly relatively compact in $L^1(\Omega)$ (see e.g. [12]). This implies that, up to a subsequence, u_{ϵ, λ_n} converges in $L^1(\Omega)$ and a.e. in Ω to a weak solution $u_{\epsilon, \lambda}$ of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

that is a function which satisfies

$$\int_\Omega (-u_{\epsilon, \lambda} \Delta \zeta + \mu V_\epsilon u_{\epsilon, \lambda} \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}). \quad (3.18)$$

Furthermore (3.17) holds (with the same notation). For test function ζ in (3.18), we take $\zeta = \theta_1$ be the solution of

$$\begin{cases} -\Delta \theta_1 = 1 & \text{in } \Omega, \\ \theta_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$\int_\Omega u_{\epsilon, \lambda} dx = -\mu \int_\Omega V_\epsilon u_{\epsilon, \lambda} \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

By the monotone convergence theorem we obtain that $V_\epsilon u_{\epsilon, \lambda} \rightarrow V u_\lambda$ in $L^1(\Omega, \theta_1 dx)$ by letting $\epsilon \rightarrow 0$ and

$$\int_\Omega u_\lambda dx = -\mu \int_\Omega V u_\lambda \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

Hence

$$\int_\Omega (-\Delta \zeta + \mu V \zeta) u_\lambda dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}).$$

We also have

$$\int_\Omega u_{\epsilon, \lambda_n} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_\Omega (V - V_\epsilon) u_{\epsilon, \lambda_n} \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta \lambda_n d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Since u_{ϵ, λ_n} converges in $L^1(\Omega)$ we obtain if $\zeta \geq 0$,

$$\int_{\Omega} u_{\epsilon, \lambda} \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} = \mu \int_{\Omega} (V - V_{\epsilon}) u_{\epsilon, \lambda} \zeta d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} \zeta d(\lambda \beta_{\mu}^{\Omega}) \leq \int_{\partial\Omega} \zeta d(\lambda \beta_{\mu}^{\Omega})$$

by using the fact that $\mu(V - V_{\epsilon}) \leq 0$. When $\epsilon \rightarrow 0$, $u_{\epsilon, \lambda}$ increases and converges to some u_{λ} in $L^1(\Omega, \rho^{-1} d\gamma_{\mu}^{\Omega})$, which satisfies

$$\int_{\Omega} u_{\lambda} \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} \leq \int_{\partial\Omega} \zeta d(\lambda \beta_{\mu}^{\Omega}) \quad \text{for all } \zeta \in \mathbb{X}_{\mu}(\Omega), \zeta \geq 0.$$

For $\delta > 0$ denote by ζ_{δ} the solution of

$$\mathcal{L}_{\mu}^* \zeta_{\delta} = \chi_{\Omega_{\delta}} \mathcal{L}_{\mu}^* \zeta \quad \text{where } \Omega_{\delta} = \{x \in \Omega : \rho(x) > \delta\}.$$

As $\zeta \in \mathbb{X}_{\mu}(\Omega)$, $|\mathcal{L}_{\mu}^* \zeta| \leq c_{10} \rho$, hence $\zeta_{\delta} \in \mathbb{X}_{\mu}(\Omega)$, $|\zeta_{\delta}| \leq c_{10} \eta$ and $\zeta_{\delta} \rightarrow \zeta$ when $\delta \rightarrow 0$. Furthermore, since $c_{11}|x|$ is a supersolution for $c_{11} > 0$ large enough, $\eta_{\delta}(x) \leq c_{11}|x|$. Hence

$$\int_{\Omega_{\delta}} u_{\epsilon, \lambda} \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} = \mu \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon, \lambda} \zeta_{\delta} d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} \zeta_{\delta} d(\lambda \beta_{\mu}^{\Omega}).$$

Because $|x|^{-2} u_{\epsilon, \lambda} |\zeta_{\delta}| \leq c_{11} \rho^{-1} u_{\epsilon, \lambda}$ and $u_{\epsilon, \lambda} \rightarrow u_{\lambda}$ in $L^1(\Omega, \rho^{-1} d\gamma_{\mu}^{\Omega})$, we derive that

$$\lim_{\epsilon \rightarrow 0} \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon, \lambda} \zeta_{\delta} d\gamma_{\mu}^{\Omega} = 0,$$

which implies

$$\int_{\Omega_{\delta}} u_{\lambda} \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} = \int_{\partial\Omega} \zeta_{\delta} d(\lambda \beta_{\mu}^{\Omega}).$$

Letting $\delta \rightarrow 0$ we obtain by monotonicity

$$\int_{\Omega} u_{\lambda} \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} = \int_{\partial\Omega} \zeta d(\lambda \beta_{\mu}^{\Omega}). \quad (3.19)$$

Finally, if $\lambda \in \mathfrak{M}_{+}(\partial\Omega, \beta_{\mu})$ we replace it by $\lambda_{\delta} = \chi_{B_{\delta}^c} \lambda$ and denote by $u_{\lambda_{\delta}}$ the weak solution of

$$\begin{cases} -\Delta u + \mu V u = 0 & \text{in } \Omega, \\ u = \lambda_{\delta} & \text{on } \partial\Omega. \end{cases}$$

The mapping $\delta \mapsto u_{\lambda_{\delta}}$ is monotone. Hence, by the monotone convergence theorem $u_{\lambda_{\delta}}$ increases and converges to some u_{λ} in $L^1(\Omega, \rho^{-1} d\gamma_{\mu}^{\Omega})$ and clearly u_{λ} satisfies (3.19) for all $\zeta \in \mathbb{X}_{\mu}(\Omega)$.

4 The singular kernel

In this section we construct the singular kernel ϕ_{μ}^{Ω} and prove that it satisfies estimates (1.11)-(1.12) and it is associated to Dirac mass at 0. Up to a rotation we can assume that the inward normal direction to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0', 1) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Hence the tangent hyperplane to $\partial\Omega$ at 0 is $\partial\mathbb{R}_+^N = \mathbb{R}^{N-1}$. For $R > 0$ set $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$ and $D_R = B'_R \times (-R, R)$. Then there exist $R > 0$ and a C^2 function $\theta : B'_R \mapsto \mathbb{R}$ such that $\partial\Omega \cap D_R = \{x = (x', x_N) : x_N = \theta(x') \text{ for } x' \in B'_R\}$ and $\Omega \cap D_R = \{x = (x', x_N) : \theta(x') < x_N < R\}$. Furthermore $\nabla\theta(0) = 0$.

4.1 Classification of Boundary isolated singularities

We characterize the positive solutions of $\mathcal{L}_\mu u = 0$ which vanish on $\partial\Omega \setminus \{0\}$.

Lemma 4.1 *Let $\mu \geq \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ be a positive solution of $\mathcal{L}_\mu u = 0$ in Ω vanishing on $\partial\Omega \setminus \{0\}$. Then there exist $a > 0$ and $c_{12} > 0$ such that*

$$u(x) \leq c_{12}|x|^{-a-1}\rho(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}. \quad (4.1)$$

Proof. This is a direct consequence of Boundary Harnack inequality [3, Th. 2.7].

Proposition 4.2 *Assume that $\mu \geq \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$, vanishing on $\partial\Omega \setminus \{0\}$ satisfying (4.1) with $a \geq -\alpha_-$. Then the following convergences hold in $C^1(\mathbb{S}_+^{N-1})$:*

(i) *If $\mu > \mu_1$ and $a = -\alpha_-$, there exists $c_{13} \geq 0$ such that*

$$\lim_{r \rightarrow 0} \frac{u(r, \cdot)}{r^{\alpha_-}} = c_{13}\psi_1 \quad \text{as } r \rightarrow 0. \quad (4.2)$$

(ii) *$\mu \geq \mu_1$ and $a > -\alpha_-$ there exist $\tau > a + \alpha_-$ depending on a and μ , and $c_{14} \geq 0$ such that*

$$u(x) \leq c_{14}|x|^{-a-1+\tau}\rho(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}.$$

Proof. *Step 1. Straightening the boundary.* We define the function $\Theta = (\Theta_1, \dots, \Theta_N)$ on D_R by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$ we can assume that Θ is a diffeomorphism from D_R onto $\Theta(D_R)$. We set

$$u(x) = \tilde{u}(y) \quad \text{for all } x \in D_R^+ = B'_R \times [0, R]. \quad (4.3)$$

Then

$$\begin{aligned} u_{x_j x_j} &= \tilde{u}_{y_j y_j} - 2\theta_{x_j} \tilde{u}_{y_j y_N} - \theta_{x_j, x_j} \tilde{u}_{y_N} + \theta_{x_j}^2 \tilde{u}_{y_N y_N} \quad \text{for } 1 \leq j \leq N-1, \\ u_{x_N x_N} &= \tilde{u}_{y_N y_N} \end{aligned} \quad (4.4)$$

and

$$\Delta \tilde{u} + |\nabla \theta|^2 \tilde{u}_{y_N y_N} - 2\langle \nabla \theta, \nabla \tilde{u}_{y_N} \rangle - \tilde{u}_{y_N} \Delta \theta - \frac{\mu}{|\Theta^{-1}(y)|^2} \tilde{u} = 0. \quad (4.5)$$

We use here the spherical coordinates (r, σ) in the variable y and we recall that Δ' is the Laplace-Beltrami operator on \mathbb{S}^{N-1} and ∇' is the tangential gradient on \mathbb{S}^{N-1} identified with the covariant derivative via the isometric imbedding $\mathbb{S}^{N-1} \hookrightarrow \mathbb{R}^N$ which enables the formula

$$\nabla \tilde{u}(y) = \left(\tilde{u}_r \mathbf{n} + \frac{1}{r} \nabla' \tilde{u} \right) (r, \sigma) \quad \text{with } \mathbf{n} = |y|^{-1} y.$$

After a lengthy computation the details of which can be found in [12, p 298-300] we obtain

$$\begin{aligned} & r^2 \tilde{u}_{rr} \left[1 - 2\theta_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla \theta|^2 (\langle \mathbf{n}, \mathbf{e}_N \rangle)^2 \right] \\ & + r \tilde{u}_r \left[N - 1 - r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta \theta + r |\nabla \theta|^2 (\langle \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle), \mathbf{e}_N \rangle - 2\langle \nabla' \theta, \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle) \rangle) \right] \\ & + \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle \left[-r \Delta \theta + 2\theta_r - |\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle \right] + r \langle \nabla' \tilde{u}_r, \mathbf{e}_N \rangle \left[2\theta_r + 2|\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle \right] \\ & - 2\langle \nabla' \tilde{u}, \nabla' \theta \rangle \langle \mathbf{n}, \mathbf{e}_N \rangle + \langle \nabla' (\langle \nabla' \tilde{u}, \mathbf{e}_N \rangle), |\nabla \theta|^2 \mathbf{e}_N - 2r^{-1} \nabla' \theta \rangle + \Delta' \tilde{u} - \frac{\mu}{|\Theta^{-1}(y)|^2} = 0. \end{aligned}$$

Next we set

$$\tilde{u}(r, \sigma) = r^{-a}v(t, \sigma) \quad \text{with } t = \ln r,$$

and we assume that

$$a \neq \frac{N-2}{2}. \quad (4.6)$$

We notice that

$$\begin{aligned} r^2 &= \sum_{j=1}^N y_j^2 = \sum_{j=1}^{N-1} x_j^2 + (x_N - \theta(x'))^2 = |x^2| - 2x_N\theta(x') = |x|^2(1 + O(r)) \quad \text{as } r \rightarrow 0 \\ &= |x|^2(1 + O(e^t)) \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

By a straightforward computation we find that v satisfies the following asymptotically autonomous equation in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$

$$\begin{aligned} (1 + \epsilon_1(t, \cdot))v_{tt} + (N-2-2a + \epsilon_2(t, \cdot))v_t + (a(a+2-N) - \mu + \epsilon_3(t, \cdot))v \\ + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned} \quad (4.7)$$

where the ϵ_j satisfies

$$|\epsilon_j(t, \cdot)| + |\partial_t \epsilon_j(t, \cdot)| + |\nabla' \epsilon_j(t, \cdot)| \leq c_{15}e^t. \quad (4.8)$$

This is due to the fact that $|\theta(x')| = O(|x'|^2)$ near 0.

Step 2. The convergence process. Since v is bounded in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$ and vanishes on $(-\infty, r_0] \times \partial \mathbb{S}_+^{N-1}$ and all the coefficients are continuous functions, we obtain that v is bounded in $W^{2,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ independently of $T \leq r_0 - 2$, for any $q < \infty$. Hence v is bounded in any $C^{1,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$ for any $\tau \in [0, 1)$. Differentiating the equation and using the standard elliptic equations regularity, we obtain that v is bounded in $W^{3,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ and in $C^{2,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$. We consider the negative trajectory of v in $C_0^1(\overline{\mathbb{S}_+^{N-1}})$ defined by

$$\mathcal{T}_-(v) = \bigcup_{t \leq r_0-1} \{v(t, \cdot)\}.$$

By the previous estimates and the Arzela-Ascoli theorem, it is a relatively compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$, hence its limit set at $-\infty$ (or alpha-limit set), denoting $A(\mathcal{T}_-(v))$, is a non-empty connected compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$. Multiplying (4.7) by v_t and integrating on \mathbb{S}_+^{N-1} yields

$$\begin{aligned} &\int_{\mathbb{S}_+^{N-1}} \left(N-2-2a + \epsilon_2 - \frac{1}{2}\partial_t \epsilon_1 \right) v_t^2 dS - \frac{1}{2} \int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\ &= \frac{d}{dt} \left[\int_{\mathbb{S}_+^{N-1}} \left(\frac{1}{2} |\nabla v|^2 - \frac{1}{2} [a(a+2-N) - \mu + \epsilon_3] v^2 - \frac{1}{2} (1 + \epsilon_1) v_t^2 \right) dS \right] \\ &\quad - \int_{\mathbb{S}_+^{N-1}} (\langle \nabla'v, \epsilon_4 \rangle + \langle \nabla'v_t, \epsilon_5 \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6 \rangle) v_t^2 dS. \end{aligned} \quad (4.9)$$

Next we integrate over $(-\infty, r_2)$ for some r_2 large enough so that

$$\left| N - 2 - 2a + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right| \geq \frac{1}{2} |N - 2 - 2a| > 0,$$

here we use the crucial assumption (4.6). Since all the terms on the right-hand side of (4.9) are integrable on $(-\infty, r_2)$ because of (4.8) and the bounds on v , we obtain that

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_t^2 dS < \infty. \quad (4.10)$$

Differentiating (4.7) with respect to t and using the estimates on v and the ϵ_j we obtain (see [12, p. 302] for a similar calculation)

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_{tt}^2 dS < \infty. \quad (4.11)$$

Because v_t and v_{tt} are uniformly continuous on $(-\infty, r_1]$, we infer from (4.10) and (4.11)

$$\lim_{t \rightarrow -\infty} \left(\|v_t(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} + \|v_{tt}(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} \right) = 0.$$

Therefore the set $A(\mathcal{T}_-(v))$ is a compact connected subset of the set of nonnegative solutions of

$$\begin{cases} \Delta' \omega + (a(a+2-N) - \mu) \omega = 0 & \text{in } \mathbb{S}_+^{N-1}, \\ \omega = 0 & \text{on } \partial \mathbb{S}_+^{N-1}. \end{cases}$$

Step 3. The case $a(a+2-N) - \mu = N-1$. The set $A(\mathcal{T}_-(v))$ is a subset of $\ker(-\Delta' - (N-1))I_d$ in $H_0^1(\mathbb{S}_+^{N-1})$ and more precisely $A(\mathcal{T}_-(v)) = \{m\psi_1 : m \in I^*\}$ where I^* is a compact interval of $[0, \infty)$. We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS.$$

Then X satisfies that

$$X''(t) + (N-2-2a)X'(t) + F(t) = 0, \quad (4.12)$$

where

$$\begin{aligned} F(t) = \int_{\mathbb{S}_+^{N-1}} & \left[\epsilon_1(t, \cdot) v_{tt} + \epsilon_2(t, \cdot) v_t + \epsilon_3(t, \cdot) v + \langle \nabla' v, \epsilon_4(t, \cdot) \rangle \right. \\ & \left. + \langle \nabla' v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla' (\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle \right] \psi_1 dS. \end{aligned}$$

Then $|F(t)| \leq c_{16} e^t$. We consider a sequence $\{t_n\}$ converging to $-\infty$ and $c^* \in I^*$ such that $X(t_n) \rightarrow c^*$. Since $X'(t)$ and $X''(t)$ converges to 0 as $t \rightarrow -\infty$, we integrate (4.12) on (t_n, t) and let $n \rightarrow \infty$. Then we get

$$X'(t) + (N-2-2a)(X(t) - c^*) + O(e^t) = 0.$$

Letting $t \rightarrow -\infty$ yields $X(t) \rightarrow c^*$. Hence we have proved that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c^* \psi_1 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}).$$

Step 4. The case $a(a+2-N) - \mu \neq N-1$. Clearly $A(\mathcal{T}_-(v)) = \{0\}$ and

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = 0 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}). \quad (4.13)$$

Furthermore, since we have assumed $a \geq -\alpha_-$, there holds actually $a > -\alpha_-$. We recall that λ_k is the k -th eigenvalue of $-\Delta'$ in $H_0^1(\mathbb{S}_+^{N-1})$ and put

$$H_k = \ker(-\Delta' - \lambda_k Id) = \text{span}\langle \varphi_{k,1}, \varphi_{k,2}, \dots, \varphi_{k,j_k} \rangle \quad \text{and} \quad H_0^1(\mathbb{S}_+^{N-1}) = \bigoplus_{k=1}^{\infty} H_k.$$

We denote

$$P_k(x) = x^2 + (N-2)x - \mu - \lambda_k.$$

Then $P_1(\alpha_-) = 0$ and $P_k(\alpha_-) = \lambda_1 - \lambda_k < 0$ for $k \geq 2$. Since $a(a+2-N) - \mu \neq N-1$ by assumption, we define a partition of \mathbb{N}^* by setting

$$N_1 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k \geq 0\}, \quad N_2 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k < 0\}$$

and

$$W_1 = \bigoplus_{k \in N_1} H_k \quad \text{and} \quad W_2 = \bigoplus_{k \in N_2} H_k.$$

Then

$$-\int_{\mathbb{S}_+^{N-1}} \varphi \Delta' \varphi dS \geq \gamma \int_{\mathbb{S}_+^{N-1}} \varphi^2 dS \quad \text{for all } \varphi \in W_2, \quad (4.14)$$

where

$$\gamma = \mu + \lambda_{k_2} - a(a+2-N) > 0 \quad \text{with } k_2 = \inf N_2.$$

We denote by P_j the orthogonal projector onto W_j in $H_0^1(\mathbb{S}_+^{N-1})$ and set $v = P_1 v + P_2 v = v_1 + v_2$. Then the projection of (4.7) on to W_2 is

$$(v_2)_{tt} + (N-2-2a)(v_2)_t + (a(a+2-N) - \mu)v_2 + \Delta' v_2 = F_2(t, \cdot),$$

where F_2 satisfies the same estimates (4.8) as ϵ_j . Then, using (4.8) and (4.14)

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS + (N-2-2a) \int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS - \gamma \int_{\mathbb{S}_+^{N-1}} v_2^2 dS \geq -c_{17} e^t \left(\int_{\mathbb{S}_+^{N-1}} v_2^2 dS \right)^{\frac{1}{2}}.$$

Put $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$, because

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS = Y'(t)Y(t) \quad \text{and} \quad \int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS \geq Y''(t)Y(t),$$

we obtain the following differential inequality

$$Y'' + (N - 2 - 2a)Y' - \gamma Y \geq -c_{17}e^t \text{ in } \mathcal{D}'(-\infty, r_2).$$

The characteristic roots of the equation $y'' + (N - 2 - 2a)y' - \gamma y = 0$ are

$$\begin{aligned} a_{k_2,-} &= a + \frac{1}{2} \left(2 - N - \sqrt{4\mu + 4\lambda_{k_2} + (N - 2)^2} \right) = \alpha_{k_2,-} + a < 0 \quad \text{and} \\ a_{k_2,+} &= a + \frac{1}{2} \left(2 - N + \sqrt{4\mu + 4\lambda_{k_2} + (N - 2)^2} \right) = \alpha_{k_2,+} + a > 0, \end{aligned} \quad (4.15)$$

where the $\alpha_{k_2,\pm}$ are the roots of equations (2.1) with $k = k_2$. The solutions of

$$z'' + (N - 2 - 2a)z' - \gamma z = -c_{17}e^t \text{ in } \mathcal{D}'(-\infty, r_2).$$

endow the form $z(t) = Ae^{ta_{k_2,-}} + Be^{ta_{k_2,+}} + c_{18}e^t$ if $a_{k_2,+} \neq 1$ or $z(t) = Ae^{ta_{k_2,-}} + Be^t + Cte^t$ if $a_{k_2,+} = 1$, for some explicit constant c_{18} depending on c_{17} and the coefficients in the equation. Since $Y(t) \rightarrow 0$ when $t \rightarrow -\infty$ by (4.13), it follows from the maximum principle that

$$Y(t) \leq c_{19}e^{ta_{k_2,+}} + c_{18}e^t \text{ if } a_{k_2,+} \neq 1, \text{ or } Y(t) \leq c_{20}|t|e^t \text{ if } a_{k_2,+} = 1 \text{ for } t \leq r_2.$$

Then using standard elliptic equations a priori estimates, initially in $L^2(\mathbb{S}_+^{N-1})$, then in $L^p(\mathbb{S}_+^{N-1})$ and finally in $C^r(\overline{\mathbb{S}_+^{N-1}})$, we obtain that for $t \leq r_3$,

$$\|v_2(t, \cdot)\|_{C^1(\mathbb{S}_+^{N-1})} \leq \begin{cases} c_{21}e^{ta_{k_2,+}} + c_{22}e^t & \text{if } a_{k_2,+} \neq 1, \\ c_{23}|t|e^t & \text{if } a_{k_2,+} = 1, \end{cases} \quad (4.16)$$

where $r_3 \leq r_2 - 1$.

For the components in W_1 we have

$$v_1(t, \cdot) = \sum_{k \in N_1} \sum_{1 \leq j \leq j_k} w_{k,j}(t) \varphi_{k,j}(\cdot), \quad (4.17)$$

where the $\varphi_{k,j}$ form an orthorormal basis of H_k . Then

$$w_{k,j}'' + (N - 2 - 2a)w_{k,j}' + (a(a + 2 - N) - \mu - \lambda_k) w_{k,j} = F_{k,j}(t). \quad (4.18)$$

The characteristic roots of equation $z'' + (N - 2 - 2a)z' + (a(a + 2 - N) - \mu - \lambda_k)z = 0$ are given in (4.15) with a general k , $a_{k-} = a + \alpha_{k-}$ and $a_{k+} = a + \alpha_{k+}$ where $\alpha_{k\pm}$ are the roots of (2.1). They have same sign (including 0) since $a(a + 2 - N) - \mu - \lambda_k \geq 0$, furthermore, their sum is positive since $N - 2 - 2a < 0$, as a consequence of $a > -\alpha_-$. By standard calculation the solution of (4.18) has the form

$$w_{k,j}(t) = m_1 e^{ta_{k+}} + m_2 e^{ta_{k-}} - \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds. \quad (4.19)$$

Since $|F_{k,j}(s)| \leq c_{24}e^s$ there holds

$$\left| \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds \right| \leq c_{25} \begin{cases} |t|e^t & \text{if } a_{k-} = 1 \\ \max\{e^t, e^{ta_{k-}}\} & \text{if } a_{k-} \neq 1 \end{cases} \quad (4.20)$$

In particular, if $k_1 = \max N_1$, then $a_{k_1 \pm} = \min\{a_k \pm : k \in N_1\}$.

We assume first that $a_{k_1 -} > 0$. Combining this fact with (4.17) and (4.20) we obtain

$$\|v_1(t, \cdot)\|_{L^\infty(\mathbb{S}_+^{N-1})} \leq c_{26} \begin{cases} |t|e^t & \text{if } a_{k_1 -} = 1 \\ \max\{e^t, e^{ta_{k_1 -}}\} & \text{if } a_{k_1 -} \neq 1 \end{cases} \quad (4.21)$$

Furthermore, because of the explicit formulation and (4.8), the left-hand side of (4.20) can be replaced by $\|v_1(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})}$. Combining (4.16) and (4.20) we obtain the result since $v(t, \cdot) = 0$ on $(-\infty, r_1) \times \partial\mathbb{S}_+^{N-1}$.

Next we suppose that $a_{k_1 -} = 0$. Then for $k = k_1$, (4.19) endows the form

$$w_{k_1, j}(t) = m_1 e^{ta_{k_1 +}} + m_2 - \frac{1}{a_{k_1 +}} \int_t^0 (e^{(t-s)a_{k_1 +}} - 1) F_{k_1, j}(s) ds. \quad (4.22)$$

This implies that

$$w_{k_1, j}(t) \rightarrow m_2 + \frac{1}{a_{k_1 +}} \int_{-\infty}^0 F_{k_1, j}(s) ds := A_{k_1, j} \quad \text{as } t \rightarrow \infty.$$

If $A_{k_1, j} \neq 0$ it would imply that $\sum_{j=1}^{j_k} A_{k_1, j} \varphi_{k_1, j}$ is a nonzero eigenfunction of order $k_1 > 1$, hence it changes sign and it would imply that v changes sign at $-\infty$ (notice that all the other terms $w_{k, j}(t)$ tends to 0 exponentially because of (4.19)-(4.20)). Hence $A_{k_1, j} = 0$ and (4.22) endows the form

$$w_{k_1, j}(t) = m_1 e^{ta_{k_1 +}} - \frac{1}{a_{k_1 +}} \int_t^0 e^{(t-s)a_{k_1 +}} F_{k_1, j}(s) ds - \int_{-\infty}^t F_{k_1, j}(s) ds.$$

Because

$$\int_{-\infty}^t F_{k_1, j}(s) ds = O(e^t) \quad \text{as } t \rightarrow \infty,$$

we conclude that for $k = k_1$, there holds

$$|w_{k_1, j}(t)| \leq c_{27} \begin{cases} |t|e^t & \text{if } a_{k_1, +} = 1, \\ \max\{e^t, e^{ta_{k_1, +}}\} & \text{if } a_{k_1, +} \neq 1 \end{cases}$$

and finally we infer (4.21), which complete the proof. \square

Proof of Theorem A. Assume that $u \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\Omega \setminus \{0\}$.

Case 1: $\mu > \mu_1$. We claim (4.2) holds for some $c_{13} \geq 0$.

By Lemma 4.1, (4.1) holds for some $a > 0$. If $a < -\alpha_-$, then (4.2) holds with $c_{13} = 0$. If $a = -\alpha_-$, then (4.2) holds by Proposition 4.2-(i). Hence we are left with the case $a > -\alpha_-$. As in the proof of Proposition 4.2 we define k_1 and k_2 . By replacing a by $a' = a + \epsilon$, we can assume that $a_{k_2, +} \neq 1$ and $a_{k_1, -} \neq 1$, to avoid the resonance complication in (4.16) and (4.21), hence

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{27} (e^{ta_{k_2, +}} + e^{ta_{k_1, -}} + e^t).$$

Furthermore $k_2 = k_1 + 1$ and

$$a_{k_2+} - a_{k_1-} = \frac{1}{2} \left(\sqrt{4\mu + 4\lambda_{k_1+1} + (N-2)^2} + \sqrt{4\mu + 4\lambda_{k_1} + (N-2)^2} \right) > 0,$$

which yields

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{28} (e^{ta_{k_1,-}} + e^t).$$

This implies that u satisfies

$$u(x) \leq c_{29} (|x|^{\alpha_{k_1,-}} + |x|^{1-a}) \rho(x).$$

We iterate this procedure up to obtain

$$u(x) \leq c_{30} |x|^{\alpha-} \rho(x)$$

and we conclude as in the proof of Proposition 4.2, Step 3. \square

Case 2: $\mu = \mu_1$. In this case, the difficulty comes from the fact that there is no dissipation of energy in (4.9) for $a = -\alpha_- = \frac{N-2}{2}$. But from the above iterative procedure in the *Case 1*, we could obtain could obtain that for some $\delta \in (0, 1)$,

$$u(x) \leq c_{31} |x|^{-\frac{N-2}{2}-\delta} \rho(x).$$

We finally show that there exists $c_{32} \geq 0$ such that

$$\lim_{r \rightarrow 0} r^{\frac{N-2}{2}} \frac{u(r, \cdot)}{\ln r} = -c_{32} \psi_1(\cdot) \quad (4.23)$$

in $C^1(\mathbb{S}_+^{N-1})$ and

$$\lim_{r \rightarrow 0} r^{\frac{N}{2}} \frac{u_r(r, \cdot)}{\ln r} = \frac{(N-2)c_{32}}{2} \psi_1(\cdot) \quad (4.24)$$

uniformly in \mathbb{S}_+^{N-1} .

Note that (4.7) reduces that

$$(1 + \epsilon_1(t))v_{tt} + \epsilon_2(t)v_t + (N-1 + \epsilon_3(t))v + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0,$$

in $(-\infty, r_0) \times \mathbb{S}_+^{N-1}$, vanishes on $(-\infty, r_0) \times \partial\mathbb{S}_+^{N-1}$ and the ϵ_j verify (4.8), and

$$v(t, \sigma) \leq c_{33} e^{-\delta t}.$$

Since the operator involved in the equation is uniformly elliptic we have by standard regularity theory

$$\begin{aligned} \|v\|_{C^{2,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_t\|_{C^{1,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_{tt}\|_{C^\delta([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} \\ \leq c_{34} \|v\|_{L^\infty((T-2, T+2) \times \mathbb{S}_+^{N-1})} \\ \leq c_{35} e^{-\delta T}, \end{aligned}$$

for any $T \leq r_0 + 3$. We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS,$$

then

$$X''(t) + F(t) = 0, \quad (4.25)$$

where

$$F(t) = \int_{\mathbb{S}_+^{N-1}} (\epsilon_1 v_{tt} + \epsilon_2 v_t + \epsilon_3 v + \langle \nabla' v, \epsilon_4(t, \cdot) \rangle + \langle \nabla' v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla' (\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle) \psi_1 dS.$$

Hence

$$|F(t)| \leq c_{36} e^{(1-\delta)t}. \quad (4.26)$$

This implies that $X'(t)$ admits a limit $c_{37} \leq 0$ when $t \rightarrow -\infty$ and

$$\lim_{t \rightarrow -\infty} t^{-1} X(t) = c_{37}.$$

Set

$$W_2 = \bigoplus_{k \geq 2} \ker(\Delta' + \lambda_k Id),$$

and denote by v_2 the orthogonal projection of v onto W_2 . Then

$$v_2 tt + (N-1)v_2 + \Delta' v_2 = F_2(t, \cdot), \quad (4.27)$$

where

$$|F_2(t, \cdot)| \leq c_{38} e^{(1-\delta)t}.$$

Since $\lambda_2 = 2N$, the function $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$ satisfies in $\mathcal{D}'(-\infty, r_1)$

$$Y'' - (N+1)Y \geq -c_{38} e^{(1-\delta)t}.$$

Because $Y(t) = o(e^{-\sqrt{N+1}t})$ when $t \rightarrow -\infty$, it follows by the maximum principle that $Y(t) = O(e^{\sqrt{N+1}t} + e^{(1-\delta)t}) = O(e^{(1-\delta)t})$. Using again the standard regularity estimates for elliptic equations, we derive

$$\|v_2(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} + \|v_2 t(t, \cdot)\|_{C(\overline{\mathbb{S}_+^{N-1}})} \leq c_{39} e^{(1-\delta)t}. \quad (4.28)$$

Combining (4.25) and (4.27) we derive (4.23). Since $v(t, \cdot) = X(t)\psi_1 + v_2(t, \cdot)$ it follows from (4.28) that

$$\lim_{t \rightarrow -\infty} v_t(t, \cdot) = c_{37} \psi_1 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

Thus, the identity $u_r(r, \cdot) = r^{-\frac{N}{2}} \left(\frac{2-N}{2} v(t, \cdot) + v_t(t, \cdot) \right)$ implies (4.23) and (4.24). \square

4.2 Existence and uniqueness

Proof of Theorem B. We recall that Ω satisfies the condition (C-1) and $\partial\mathbb{R}_+^N$ is tangent to $\partial\Omega$ at 0. Let u_ϵ be the solution of

$$\begin{cases} \mathcal{L}_\mu u_\epsilon = 0 & \text{in } \Omega_\epsilon := \Omega \setminus B_\epsilon, \\ u_\epsilon = 0 & \text{on } \partial\Omega \cap \overline{B_\epsilon^c}, \\ u_\epsilon = \phi_\mu & \text{on } \Omega \cap \partial B_\epsilon. \end{cases} \quad (4.29)$$

Since $\Omega \subset \mathbb{R}_+^N$, $u_\epsilon \leq \phi_\mu$ in Ω_ϵ and

$$\left. \frac{\partial u_\epsilon}{\partial \mathbf{n}} \right|_{\Omega \cap \partial B_\epsilon} \leq \left. \frac{\partial \phi_\mu}{\partial \mathbf{n}} \right|_{\Omega \cap \partial B_\epsilon} \leq 0, \quad (4.30)$$

where $\mathbf{n} = \epsilon^{-1}\mathbf{x}$. Furthermore, if $0 < \epsilon' < \epsilon$, $u_{\epsilon'}|_{\Omega \cap \partial B_\epsilon} \leq u_\epsilon|_{\Omega \cap \partial B_\epsilon} = \phi_\mu$, hence $u_{\epsilon'} \leq u_\epsilon$ in Ω_ϵ . There exists $\phi_\mu^\Omega = \lim_{\epsilon \rightarrow 0} u_\epsilon$ and ϕ_μ^Ω is a nonnegative solution of $\mathcal{L}_\mu u = 0$ in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and is smaller than ϕ_μ . Therefore,

$$\limsup_{|x| \rightarrow 0^+} \frac{\phi_\mu^\Omega}{\phi_\mu} \leq 1. \quad (4.31)$$

Let $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta > 0$, then for $\mathbf{n}' = -\frac{x}{|x|} = -\mathbf{n}$, we have that

$$0 = \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu u_\epsilon dx = \int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\partial B_\epsilon \cap \Omega} \left(-\frac{\partial u_\epsilon}{\partial \mathbf{n}'} \zeta \gamma_\mu^\Omega + \left(\zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}'} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}'} \right) u_\epsilon \right) dS.$$

Using (4.29) and (4.30), we obtain

$$\int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega - \left(\zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS.$$

We take $\zeta = 1$, hence $\mathcal{L}_\mu^* \zeta = \ell_\mu^\Omega$ and we get

$$\ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \geq 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1),$$

in the case $\mu > \mu_1$, and

$$\ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \geq \left(\frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1),$$

in the case $\mu = \mu_1$. Since $u_\epsilon \leq \phi_\mu$,

$$u_\epsilon \gamma_\mu^\Omega \leq \gamma_\mu^\Omega \phi_\mu = r^{2-N} \psi_1^2 \in L^1(\Omega).$$

Therefore, by dominated convergence theorem, we conclude that

$$\ell_\mu^\Omega \int_\Omega \phi_\mu^\Omega d\gamma_\mu \geq \begin{cases} 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1\right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1. \end{cases} \quad (4.32)$$

We infer that the function ϕ_μ^Ω is nonzero. It is a positive solution of $\mathcal{L}_\mu \phi_\mu^\Omega = 0$ in Ω which vanishes on $\partial\Omega \setminus \{0\}$. It follows from **Theorem A** that there exists $k \in [0, 1]$ by (4.31) such that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\phi_\mu^\Omega(x)}{\rho(x)|x|^{\alpha-1}} &= k & \text{if } \mu > \mu_1, \\ \lim_{x \rightarrow 0} \frac{\phi_\mu^\Omega(x)}{\rho(x)|x|^{-N/2} \ln|x|} &= -k & \text{if } \mu = \mu_1. \end{aligned}$$

Next we show that $k = 1$. In fact, if $k < 1$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$u_\epsilon \leq \frac{k+1}{2} \phi_\mu,$$

and then

$$\lim_{\epsilon \rightarrow 0^+} \ell_\mu \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega \leq \frac{k+1}{2} \ell_\mu^\Omega \int_{\Omega_\epsilon} \phi_\mu d\gamma_\mu^\Omega < \begin{cases} 2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1\right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1, \end{cases}$$

which contradicts (4.32). Thus, (1.11) and (1.12) hold true. \square

Proof of Corollary C. Identity (1.14). As a consequence of Proposition 4.2, for any $\zeta \in \mathbb{X}_\mu(\Omega)$ and $\epsilon > 0$, we set $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$ and then there holds

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu \phi_\mu^\Omega dx \\ &= \int_{\Omega_\epsilon} \phi_\mu^\Omega \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\Omega \cap \partial B_\epsilon} \left(-\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left(\gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS. \end{aligned}$$

Using **Proposition A.1**, we have that

$$\int_{\Omega \cap \partial B_\epsilon} \left(-\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left(\gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS = -\zeta(0)A(\epsilon)(1 + o(1)),$$

where $A(\epsilon)$ is defined in (2.5).

The uniqueness follows direct from Kato's inequality (3.7). \square

5 The Dirichlet problem

Proof of Theorem D. Note that in Section 3.2 for $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$, problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting $\mathbb{K}_\mu^\Omega(\lambda)$, which verifies the identity

$$\int_\Omega \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Moreover, problem

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting $\mathbb{G}_\mu^\Omega(\nu)$, which verifies the identity

$$\int_\Omega \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d\gamma_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Together with **Corollary C** and the linearity of operator \mathcal{L}_μ , we have that $\mathbb{K}_\mu^\Omega(\lambda) + \mathbb{G}_\mu^\Omega(\nu) + k\phi_\Omega^\mu$ is a weak solution of (1.18) satisfying (1.20) and the uniqueness follows directly from Kato's inequality (3.7). \square

Our final part is to classify the boundary data for nonnegative \mathcal{L}_μ -harmonic function.

Proof of Theorem E. Let Ω be a bounded C^2 domain and u be a nonnegative \mathcal{L}_μ -harmonic function in Ω . We now show that there exists a nonnegative measure λ on $\partial\Omega \setminus \{0\}$ and $k \geq 0$ such that

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega. \quad (5.1)$$

For $\epsilon > 0$, the term $\mu|x|^{-2}$ is bounded in $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$. Hence there exists a nonnegative Radon measure λ_ϵ such that u is the unique solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega_\epsilon, \\ u = \lambda_\epsilon & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Furthermore λ_ϵ is the boundary trace is achieved in *dynamical sense*, see [14] and references therein. Hence for any $\zeta \in C(\overline{\Omega})$ vanishing on B_ϵ , there holds

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \zeta dS = \int_{\partial\Omega \cap B_\epsilon^c} \zeta d\lambda_\epsilon,$$

where $\Sigma_\delta = \{x \in \Omega : \rho(x) = \delta\}$. If we write

$$\lambda_\epsilon = \lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c} + u|_{\Omega \cap \partial B_\epsilon},$$

it proves that for $0 < \epsilon' < \epsilon$, $\lambda_\epsilon|_{\partial\Omega\cap B_\epsilon} = \lambda_{\epsilon'}|_{\partial\Omega\cap B_{\epsilon'}}$. This defines in a unique way a nonnegative Radon λ on $\partial\Omega \setminus \{0\}$ measure such that (5.1) holds for all $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing near 0. Furthermore $\rho u \in L^1(\Omega_\epsilon)$ for any $\epsilon > 0$. Denote by $\mathbb{K}_\mu^{\Omega_\epsilon}$ the Poisson potential of \mathcal{L}_μ in Ω_ϵ . Then

$$u|_{\Omega_\epsilon} = \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega\cap B_\epsilon}] + \mathbb{K}_\mu^{\Omega_\epsilon}[u|_{\Omega\cap\partial B_\epsilon}].$$

For $0 < \epsilon' < \epsilon$, one has that $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega\cap B_{\epsilon'}}]|_{\Omega\cap\partial B_\epsilon} \geq 0$. Therefore $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega\cap B_{\epsilon'}}] \geq \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega\cap B_\epsilon}]$ in Ω_ϵ . Hence

$$\lim_{\epsilon \rightarrow 0} \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega\cap B_\epsilon}] = \mathbb{K}_\mu^\Omega[\lambda] \leq u \quad \text{in } \Omega.$$

Next we aim to characterize the behaviour at 0. By contradiction, we assume that

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega\cap\partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega\cap\partial B_{\epsilon_k}} u d\beta_\mu^\Omega = \infty.$$

Then for any $m > 0$, there exists a sequence $\{\epsilon_{m,k}\} \subset \mathbb{R}_+^*$ tending to 0 and a sequence $\{\ell_{m,k}\} \subset \mathbb{R}_+^*$ tending to ∞ such that

$$\int_{\Omega\cap\partial B_{\epsilon_k}} \min\{u, \ell_{m,k}\} d\beta_\mu^\Omega = m.$$

Set $\tau_{m,k} = \min\{u|_{\Omega\cap\partial B_{\epsilon_k}}, \ell_{m,k}\}$ and set $u_{m,k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\tau_{m,k}\chi_{\Omega\cap\partial B_{\epsilon_k}}]$. Then

$$u_{m,k} \leq u \quad \text{in } \Omega_{\epsilon_k},$$

and we recall that

$$\int_{\Omega\cap\partial B_{\epsilon_k}} \phi_\mu^\Omega d\beta_\mu^\Omega = c_\mu(1 + o(1)),$$

where c_μ is the constant defined in (2.4). Combining the boundary Harnack inequality with the standard Harnack inequality, one infers

$$c_{47} \frac{\rho(x)}{\rho(y)} \leq c_{46} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq \frac{u_{m,k}(x)}{u_{m,k}(y)} \leq c_{44} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq c_{45} \frac{\rho(x)}{\rho(y)} \quad (5.2)$$

for all $x, y \in \Omega$ such that $|x| = |y| \geq 2\epsilon_k$. If we set $\tilde{\phi}_\mu^\Omega(x) = \frac{\phi_\mu^\Omega(x)}{\rho(x)}$ and $\tilde{u}_{m,k}(x) = \frac{u_{m,k}(x)}{\rho(x)}$, then (5.2) becomes

$$c_{47} \leq c_{46} \frac{\tilde{\phi}_\mu^\Omega(x)}{\tilde{\phi}_\mu^\Omega(y)} \leq \frac{\tilde{u}_{m,k}(x)}{\tilde{u}_{m,k}(y)} \leq c_{44} \frac{\tilde{\phi}_\mu^\Omega(x)}{\tilde{\phi}_\mu^\Omega(y)} \leq c_{45}. \quad (5.3)$$

Assume for a while that we have proved that there exists $\theta > 0$, independent of m and k such that for any

$$\int_{\partial B_{2\epsilon_k} \cap \Omega} \tilde{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{u}_{m,k} d(\rho\beta_\mu^\Omega) = \theta m. \quad (5.4)$$

If we assume that for $\delta \leq 2\epsilon_{k_0}$,

$$2c_\mu \geq \int_{\Omega\cap\partial B_\delta} \tilde{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega) \geq \frac{c_\mu}{2},$$

one has for $k \geq k_0$,

$$\int_{\Omega \cap \partial B_{2\epsilon_k}} \tilde{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta m \geq \frac{\theta m}{2c_\mu} \int_{\Omega \cap \partial B_{2\epsilon_k}} \tilde{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega)$$

Since

$$\tilde{\phi}_\mu^\Omega(x) \leq \frac{c_{45}}{c_{46}} \tilde{\phi}_\mu^\Omega(y)$$

and

$$\tilde{u}_{m,k}(x) \geq c_{47} \tilde{u}_{m,k}(y),$$

we derive

$$\frac{1}{c_{47}} \tilde{u}_{m,k}(x) \int_{\partial B_{2\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) \geq \frac{\theta m c_{46}}{2c_\mu c_{45}} \tilde{\phi}_\mu^\Omega(x) \int_{\Omega \cap \partial B_{2\epsilon_k}} d(\rho\beta_\mu^\Omega).$$

Therefore

$$u_{m,k}(x) \geq c_{48} m \phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega \text{ s.t. } |x| = 2\epsilon_k,$$

and $c_{48} > 0$ is independent of m and ϵ_k . This implies by the maximum principle and letting $\epsilon_k \rightarrow 0$,

$$u(x) \geq u_{m,k}(x) \geq c_{48} m \phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega. \quad (5.5)$$

Since m is arbitrary we obtain a contradiction. Hence there holds

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = m_u < \infty. \quad (5.6)$$

Then inequality (5.5) holds without truncation with m replaced by m_u . We recall that

$$w_\epsilon := \mathbb{K}_\mu^{\Omega_\epsilon} [u|_{\Omega \cap \partial B_\epsilon}] = u|_{\Omega_\epsilon} - \mathbb{K}_\mu^{\Omega_\epsilon} [\lambda_\epsilon|_{\partial \Omega \cap B_\epsilon^\epsilon}] \quad \text{in } \Omega_\epsilon. \quad (5.7)$$

Case 1: We first assume that $m_u > 0$. Then (5.4) combined with the maximum principle yields

$$\int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \int_{\partial B_{2\epsilon_k} \cap \Omega} \tilde{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) = \theta m_u (1 + o(1))$$

with $\tilde{w}_\epsilon = \rho^{-1} w_\epsilon$. Inequality (5.3) is replaced by

$$c_{47} \leq c_{46} \frac{\tilde{\phi}_\mu^\Omega(x)}{\tilde{\phi}_\mu^\Omega(y)} \leq \frac{\tilde{w}_\epsilon(x)}{\tilde{w}_\epsilon(y)} \leq c_{44} \frac{\tilde{\phi}_\mu^\Omega(x)}{\tilde{\phi}_\mu^\Omega(y)} \leq c_{45} \quad \text{in } \Omega_{2\epsilon}. \quad (5.8)$$

Therefore, for ϵ_k small enough and $|x| = 2\epsilon_k$,

$$\begin{aligned} \tilde{w}_\epsilon(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\leq c_{45} \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \leq 2c_{45} m_u \\ &\leq \frac{4c_{45} m_u}{c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \leq \frac{4c_{44} c_{45} m_u}{c_\mu c_{47}} \tilde{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega), \end{aligned}$$

which implies

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.9)$$

Hence

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.10)$$

Since w_{ϵ_k} and ϕ_μ are \mathcal{L}_μ -harmonic in $\Omega_{2\epsilon_k}$, and vanishes on $\partial\Omega \cap B_{2\epsilon_k}$ it follows that inequality (5.10) also holds for any $x \in \Omega_{2\epsilon_k}$. By definition $w_{\epsilon_k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u|_{\Omega \cap \partial B_{\epsilon_k}}]$, hence

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \leq c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.11)$$

Next we obtain the estimate from below. From (5.8), with $|x| = 2\epsilon_k$,

$$\begin{aligned} \tilde{w}_{\epsilon_k}(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\geq c_{47} \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}m_u}{2} \\ &\geq \frac{c_{47}m_u}{4c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \tilde{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \tilde{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega). \end{aligned}$$

Hence

$$w_{\epsilon_k}(x) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \phi_\mu^\Omega(x) := c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

It follows that

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \geq c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.12)$$

From (5.7), (5.12) and (5.11) we infer

$$c_{50}m_u \phi_\mu^\Omega \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^c}] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}. \quad (5.13)$$

This implies, by letting $\epsilon_k \rightarrow 0$,

$$c_{50}m_u \phi_\mu^\Omega \leq u - \mathbb{K}_\mu^\Omega[\lambda] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega.$$

Therefore, the function $u - \mathbb{K}_\mu^\Omega[\lambda]$ is \mathcal{L}_μ -harmonic and positive in Ω and it vanishes on $\partial\Omega$. By **Corollary C**, it implies that it coincides with $c\phi_\mu^\Omega$ for some $c \geq 0$ (and in that case $c_{50}m_u \leq c \leq c_{48}m_u$).

Case 2: Assume $m_u = 0$. Following the same inequalities as in Case 1, (5.9) is replaced by: for any $\delta > 0$ there exists $k_0 > 0$ such that for $k \geq k_0$,

$$w_{\epsilon_k}(x) \leq \delta \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

Hence (5.13) is transformed into

$$0 \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^c}] \leq \delta \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}.$$

Letting successively $\epsilon_k \rightarrow 0$ and $\delta \rightarrow 0$ yields $u - \mathbb{K}_\mu^\Omega[\lambda] = 0$ in Ω , which ends the proof. \square

Appendix: Estimates (1.9) and (3.2)

Proposition A.1 *Assume Ω is a bounded C^2 domain such that $0 \in \partial\Omega$ satisfying condition (C-1) and let γ_μ^Ω be defined by (1.8) and normalized by $\|\gamma_\mu^\Omega\|_{L^2(\Omega)} = 1$. Then*

$$\lim_{r \rightarrow 0} r^{1-\alpha_+} \gamma_\mu^\Omega(r, \cdot) = c_1 \psi_1 \quad \text{in } C_{loc}^1(\mathbb{S}_+^{N-1})$$

and

$$\lim_{r \rightarrow 0} r^{-\alpha_+} \gamma_{\mu r}^\Omega(r, \cdot) = c_1 \left(1 - \frac{N}{2}\right) \psi_1 \quad \text{locally uniformly in } \mathbb{S}_+^{N-1}.$$

Proof. Since $\alpha_+ + (N-2)\alpha_+ - \mu + 1 - N = 0$, the function $x \mapsto w(x) := |x|^{\alpha_+}$ satisfies

$$\tilde{\mathcal{L}}_\mu w(x) := \mathcal{L}_\mu w - \ell_\mu^\Omega w = |x|^{\alpha_+-2} (N-1 - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore, $\nabla w \in L_{loc}^2(\mathbb{R}^N)$. Let $R_0 > 0$ such that $N-1 \geq \ell_\mu^\Omega R_0^2$ and $m > 0$ such that $mw \geq \gamma_\mu^\Omega$ on $\Omega \cap B_{R_0}^c$. Then the function $(\gamma_\mu^\Omega - mw)_+$ belongs to $H_\mu(\Omega)$ and satisfies $\tilde{\mathcal{L}}_\mu(\gamma_\mu^\Omega - mw)_+ \leq 0$ in the dual of $H_\mu(\Omega)$. Hence

$$\int_\Omega \left(|\nabla(\gamma_\mu^\Omega - mw)_+|^2 + \left(\frac{\mu}{|x^2|} - \ell_\mu^\Omega \right) (\gamma_\mu^\Omega - mw)_+^2 \right) dx \leq 0.$$

Therefore $(\gamma_\mu^\Omega - mw)_+ \leq 0$, which implies that

$$0 < \gamma_\mu^\Omega(x) \leq m|x|^{\alpha_+} \quad \text{for all } x \in \Omega.$$

Then we proceed as in the proof of Proposition 4.2. We flatten the boundary near 0 and set

$$v(t, \sigma) = r^{-\alpha_+} \tilde{\gamma}_\mu^\Omega(r, \sigma) \quad \text{with } t = \ln r,$$

where the function $\tilde{\gamma}_\mu^\Omega$ is defined similarly as \tilde{u} in (4.3). Then v is bounded in $(-\infty, T_0] \times \mathbb{S}_+^{N-1}$ and it satisfies

$$\begin{aligned} (1 + \epsilon_1(t, \cdot))v_{tt} + (N-2 + 2\alpha_+ + \epsilon_2(t, \cdot))v_t + (\alpha_+(\alpha_+ + N-2) - \mu + \epsilon_3(t, \cdot) + e^{2t}\ell_\mu^\Omega)v \\ + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned}$$

instead of (4.7). It vanishes on $(-\infty, T_0] \times \partial\mathbb{S}_+^{N-1}$ and then ϵ_j satisfies again (4.8).

Case 1: $\mu > \mu_1$. The energy method used in proof of Proposition 4.2 applies with no modification and we infer that there exists $c_{51} \geq 0$ such that

$$v(t, \cdot) \rightarrow c_{51}\psi_1 \quad \text{as } t \rightarrow -\infty$$

in $C^1(\mathbb{S}_+^{N-1})$ and $v_t(t, \cdot) \rightarrow 0$ uniformly in \mathbb{S}_+^{N-1} . If $c_{51} = 0$, we can prove, as in Proposition 4.2-(ii) that there exists $\tau > 0$ such that

$$\gamma_\mu^\Omega(x) \leq c_{52}|x|^{\alpha_++\tau} \quad \text{for all } x \in \Omega. \quad (5.14)$$

Iterating this process, we infer that (5.14) holds for any $\tau > 0$. For $k > 1$, let $\alpha_{k,+}$ be the positive root of (2.1) and put $w_k(x) = |x|^{\alpha_{k,+}}$. Then

$$\tilde{\mathcal{L}}_\mu w_k(x) = |x|^{\alpha_{k,+}-2} (\lambda_k - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since $\lambda_k \rightarrow \infty$, as $k \rightarrow \infty$, we choose k such that $\lambda_k > \ell_\mu^\Omega (\text{diam}(\Omega))^2$. Hence w_k is a supersolution of $\tilde{\mathcal{L}}_\mu$. Because $\gamma_\mu^\Omega(x) = o(w_k(x))$ near $x = 0$, it follows that $\gamma_\mu^\Omega(x) \leq \epsilon o(w_k(x))$ for any $x \in \Omega$. Hence $\gamma_\mu^\Omega = 0$, which is a contradiction. Finally it implies that $c_{51} > 0$, which yields (1.9)-(i). Because the convergence holds in $C^1(S_+^{N-1})$ and $v_t(t, \cdot) \rightarrow 0$, we infer

$$\lim_{r \rightarrow 0} r^{1-\alpha_+} \nabla \tilde{\gamma}_\mu^\Omega(r, \cdot) = c_{51} (\alpha_+ \psi_1 \mathbf{e} + \nabla' \psi_1)$$

where $\mathbf{e} = \frac{x}{|x|}$. This implies the claim.

Case 2: $\mu = \mu_1$. Set $v(t, \cdot) = r^{\frac{N}{2}-1} u(r, \cdot)$ with $t = \ln r$ and $X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS$ and obtain again (4.25), where $F(t, \cdot)$ satisfies (4.26). Since $X'(t) \rightarrow 0$ and X is bounded, it follows that $X(t)$ admits a limit $c_{52} \geq 0$ when $t \rightarrow -\infty$. As in the proof of **Theorem A**, we infer that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c_{52} \psi_1 \quad \text{in } C^1(\mathbb{S}_+^{N-1}) \quad \text{and} \quad \lim_{t \rightarrow -\infty} v_t(t, \cdot) = 0 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

If $c_{52} = 0$ we derive a contradiction as in the first case. \square

Proof of (3.2). Since $\rho^* \leq \frac{1}{\ell_\mu^\Omega}$, then comparison principle implies that $\sigma_\mu^\Omega \geq \gamma_\mu^\Omega$ in Ω . Next we show $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$ in Ω . In fact, we only have to show this inequality holds in a neighborhood of the origin.

Case 1: the boundary is flat at the origin. We first prove above inequality when Ω is flat in a neighborhood of the origin, i.e. $B'_R \times [0, R] \subset \Omega \subset \mathbb{R}_+^N$ for some $R > 0$.

For $\tau \in \mathbb{R}$, denote

$$w_\tau(x) = |x|^\tau x_N \quad \text{and} \quad \tilde{w}_\tau(x) = |x|^\tau x_N^2 \quad \text{in } \mathbb{R}_+^N,$$

and direct calculation shows that

$$\mathcal{L}_\mu w_\tau(x) = [\mu - \tau(\tau + N)] |x|^{\tau-2} x_N, \quad \mathcal{L}_\mu \tilde{w}_\tau(x) = [\mu - \tau(\tau + N + 2)] |x|^{\tau-2} x_N^2 - 2|x|^\tau \quad \text{for } x \in \mathbb{R}_+^N.$$

Let

$$\bar{u}(x) = \begin{cases} w_{\alpha_+-1} - \frac{1}{2} \tilde{w}_{\alpha_+-1} & \text{if } \alpha_+ \geq 0, \\ w_{\alpha_+-1} - \frac{N-2}{2(N+2)} w_{\alpha_+} - \frac{2}{N+2} \tilde{w}_{\alpha_+-1} & \text{if } \alpha_+ < 0, \end{cases}$$

by resetting $R \in (0, 1]$ such that $\bar{u} > 0$ in $\mathbb{R}^{N-1} \times (0, R]$.

When $\alpha_+ \geq 0$, we have that $\mu - (\alpha_+ - 1)(\alpha_+ - 1 + N) = 0$ and

$$\mathcal{L}_\mu \bar{u}(x) = \alpha_+ |x|^{\alpha_+-3} x_N^2 + |x|^{\alpha_+-1} \geq |x|^{\alpha_+-1},$$

thus there exists $t_1 > 0$ such that $t_1 \bar{u} \geq \sigma_\mu^\Omega$ on $\Omega \cap (\mathbb{R}^{N-1} \times \{R\})$ and

$$\mathcal{L}_\mu(t_1 \bar{u})(x) \geq t_1 |x|^{\alpha_+-1} \geq \mathcal{L}_\mu \sigma_\mu^\Omega(x) \quad \text{in } \Omega \cap (\mathbb{R}^{N-1} \times (0, R))$$

By comparison principle, we have that

$$\sigma_\mu^\Omega \leq t_1 \bar{u}$$

which, together with the inequality $\bar{u} \leq 2t_1 w_{\alpha_+-1}$, implies that $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$.

When $\alpha_+ \in [\frac{2-N}{2}, 0)$ if $N \geq 3$,

$$\begin{aligned} \mathcal{L}_\mu \bar{u}(x) &= \frac{N-2}{2(N+2)}(2\alpha_+ + 2 + N)|x|^{\alpha_+-2} x_N + \frac{4}{N+2}\alpha_+ |x|^{\alpha_+-3} x_N^2 + \frac{4}{N+2}|x|^{\alpha_+-1} \\ &\geq \left(\frac{N-2}{2(N+2)}(2\alpha_+ + 2 + N) - \frac{4\alpha_+}{N+2} \right) |x|^{\alpha_+-2} x_N + \frac{4}{N+2}|x|^{\alpha_+-1} \\ &\geq \frac{4}{N+2}|x|^{\alpha_+-1}. \end{aligned}$$

The remaining of the proof is similar to the previous one and we omit it.

Case 2: the boundary is not flat at the origin.

We define the function $\Theta = (\Theta_1, \dots, \Theta_N)$ on D_R by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$, we can assume that Θ is a diffeomorphism from D_R onto $\Theta(D_R)$. We set

$$u_1(x) = \bar{u}(y) \quad \text{for all } y \in D_R^+ = B'_R \times [0, R].$$

Then by (4.4) and (4.5), we have that

$$(-\Delta u_1(x) + \frac{\mu}{|x|^2})u_1(x) = (-\Delta \bar{u}(y) + \frac{\mu}{|y|^2}\bar{u}(y)) + O(|y|) \left(|D^2 \bar{u}(y)| + \frac{\mu}{|y|^2}\bar{u}(y) \right)$$

Then by resetting $R > 0$ small and the calculation in *Case 1*, we have that

$$\mathcal{L}_\mu u_1(x) \geq c_{53}|x|^{\alpha_+-1}, \quad \forall x \in \Theta^{-1}(D_R^+).$$

By Hopf's Lemma, there exists $t_2 > 0$ such that $t_2 u_1 \geq \sigma_\mu^\Omega$ on $\Theta^{-1}(\partial B'_R \times [0, R])$ and by compactness of $\Theta^{-1}(\bar{B}'_R \times \{R\})$, there exists $t_3 > 0$ such that $t_3 u_1 \geq \sigma_\mu^\Omega$ on $\Theta^{-1}(\bar{B}'_R \times \{R\})$. Applying comparison principle, for some $t_4 \geq \max\{t_2, t_3\}$, we have that

$$\sigma_\mu^\Omega \leq t_4 \bar{u} \quad \text{in } \Theta^{-1}(D_R^+)$$

and we have $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$ near the origin. □

Proposition A.2 *Under the assumption of Proposition A.1, there exists $c_{53} > 0$ such that*

$$\lim_{r \rightarrow 0} r^{1-\alpha_+} \sigma_\mu^\Omega(r, \cdot) = c_{53} \psi_1 \quad \text{in } C_{loc}^1(\mathbb{S}_+^{N-1}) \quad (5.15)$$

and

$$\lim_{r \rightarrow 0} r^{-\alpha_+} \sigma_{\mu r}^\Omega(r, \cdot) = c_{53} \left(1 - \frac{N}{2}\right) \psi_1 \quad \text{locally uniformly in } \mathbb{S}_+^{N-1}. \quad (5.16)$$

Proof. We follow the proof of Proposition A.1, flattening the boundary near 0 and defining a new function $\tilde{\sigma}_\mu^\Omega$ as previously. By (3.2), the function

$$v(t, \sigma) = r^{-\alpha_+} \tilde{\sigma}_\mu^\Omega(r, \sigma) \quad \text{with } t = \ln r,$$

is bounded and it satisfies

$$(1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 + 2\alpha_+ + \epsilon_2(t, \cdot))v_t + (\alpha_+(\alpha_+ + N - 2) - \mu + \epsilon_3(t, \cdot))v \\ + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = e^t m(t, \cdot),$$

instead of (4.7), where the function m is bounded as well as its gradient. Then v satisfies the same bounds as the ones in the proof of Proposition 4.2. The only difference is that the energy estimate (4.9) is replaced by

$$\int_{\mathbb{S}_+^{N-1}} \left(N - 2 - 2\alpha_+ + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right) v_t^2 dS - \frac{1}{2} \int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\ = \frac{d}{dt} \left[\int_{\mathbb{S}_+^{N-1}} \left(\frac{1}{2} |\nabla v|^2 - \frac{1}{2} [\alpha_+(\alpha_+ + 2 - N) - \mu + \epsilon_3] v^2 - \frac{1}{2} (1 + \epsilon_1) v_t^2 \right) dS \right] \\ - \int_{\mathbb{S}_+^{N-1}} (\langle \nabla'v, \epsilon_4 \rangle + \langle \nabla'v_t, \epsilon_5 \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6 \rangle) v_t^2 dS + e^t \int_{\mathbb{S}_+^{N-1}} m(t, \cdot) v(t, \cdot) dS.$$

Therefore, if $2\alpha_+ \neq N - 2$, we conclude that (4.10) holds, and (4.11) follows. We infer (5.15) and (5.16) as in the proof of Proposition 4.2. When $2\alpha_+ = N - 2$, the proof of (4.10) and (4.10) in the case $2\alpha_+ = N - 2$ is carried out as in the proof of Proposition 4.2-Step 3. \square

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References

- [1] P. Bauman. Positive solutions of elliptic equations in nondivergence form and their adjoints, *Ark. Mat.* **22** (1984), 153-173.
- [2] G. Barbatis, S. Filippas, A. Tertikas. Sharp Hardy and Hardy-Sobolev inequalities with point singularities on the boundary, *J. Math. Pures Appl.* **117** (2018), 146-184.
- [3] M.F. Bidaut-Véron, R. Borghol, L. Véron. Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations, *Calc. Var. Part. Diff. Eq.* **27(2)** (2006), 159-177.
- [4] H. Brezis, J. L. Vazquez. Blow-up solutions of some nonlinear elliptic equations, *Rev. Mat. Complut.* **10** (1997), 443-469.
- [5] C. Cazacu. On Hardy inequalities with singularities on the boundary, *C. R. Math. Acad. Sci. Paris* **349** (2011), 273-277.

- [6] C. Cazacu. Hardy inequality and Pohozaev identity for operators with boundary singularities: Some applications. *C. R. Math. Acad. Sci. Paris* **349** (2011), 1167-1172.
- [7] C. Cazacu. *Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results*, *J. Funct. Anal.* **263** (2012), 3741-3782.
- [8] H. Chen, A. Quaas, F. Zhou. On nonhomogeneous elliptic equations with the Hardy- Leray potentials, arXiv:1705.08047.
- [9] H. Chen, L. Véron. Weak solutions of semilinear elliptic equations with Leray-Hardy potential and measure data, *Mathematics in Engineering* **1** (2019), 391-418.
- [10] H. Chen, L. Véron. Semilinear elliptic equations with Leray-Hardy potential singular on the boundary and measure data. *In preparation*.
- [11] S. Filippas, A. Tertikas and J. Tidblom. On the structure of Hardy-Sobolev-Mazya inequalities. *J. Eur. Math. Soc.* **11** (2009), 1165-1185.
- [12] A. Gmira, L. Véron. Boundary singularities of solutions of nonlinear elliptic equations, *Duke Math. J.* **64** (1991), 271-324.
- [13] R. A. Hunt and R. L. Wheeden. Positive harmonic functions on Lipschitz domains, *Trans. Amer. Math. Soc.* **147** (1970), 507-527.
- [14] M. Marcus, L. Véron, *Nonlinear Second Order Elliptic Equations Involving Measures*, De Gruyter Series in Nonlinear Analysis and Applications **21** (2013), xiii+pp. 1-262.
- [15] M. Marcus, L. Véron, Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains: the subcritical case, *Ann. Scu. Norm. Sup. Pisa Ser. V* **Vol. X** (2011), 913-984.
- [16] M. Marcus, L. Véron, Boundary trace of positive solutions of supercritical elliptic equations in dihedral domains, *Ann. Scu. Norm. Sup. Pisa Ser. V*, **Vol. XV** (2016), 501-542.
- [17] L. Véron. Elliptic Equations Involving Measures, in *Stationary Partial differential equations* Vol. I, 593-712. Handb. Differ. Equ., North-Holland, Amsterdam (2004).
- [18] L. Véron, C. Yarur. Boundary value problems with measures for elliptic equations with singular potentials, *J. Funct. Anal.* **262** (2012), 733-772.
- [19] L. Véron. Existence and Stability of Solutions of General Semilinear Elliptic Equations with Measure Data, *Advanced Nonlinear Studies* **13** (2013), 447-460.