



HAL
open science

Schrödinger operators with Leray-Hardy potential singular on the boundary

Huyuan Chen, Laurent Veron

► **To cite this version:**

Huyuan Chen, Laurent Veron. Schrödinger operators with Leray-Hardy potential singular on the boundary. 2019. hal-02157156v2

HAL Id: hal-02157156

<https://hal.science/hal-02157156v2>

Preprint submitted on 19 Jun 2019 (v2), last revised 25 Feb 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Schrödinger operators with Leray-Hardy potential singular on the boundary

Huyuan Chen*

Laurent Véron †

Abstract

We study the kernel function of the operator $u \mapsto \mathcal{L}_\mu u = -\Delta u + \frac{\mu}{|x|^2} u$ in a bounded smooth domain $\Omega \subset \mathbb{R}_+^N$ such that $0 \in \partial\Omega$, where $\mu \geq -\frac{N^2}{4}$ is a constant. We show the existence of a Poisson kernel vanishing at 0 and a singular kernel with a singularity at 0. We prove the existence and uniqueness of weak solutions of $\mathcal{L}_\mu u = 0$ in Ω with boundary data $\nu + k\delta_0$, where ν is a Radon measure on $\partial\Omega \setminus \{0\}$, $k \in \mathbb{R}$ and show that this boundary data corresponds in a unique way to the boundary trace of positive solution of $\mathcal{L}_\mu u = 0$ in Ω .

Key Words: Hardy Potential, Harnack inequality, limit set, Radon Measure.

MSC2010: 35B44, 35J75.

Contents

1	Introduction	2
2	The half-space setting	7
3	The Poisson kernel	9
3.1	Construction of the Poisson kernel when $\mu > 0$	12
3.2	Construction of the Poisson kernel when $\mu_1 \leq \mu < 0$	14
4	The singular kernel	16
4.1	Classification of Boundary isolated singularities	16
4.2	Existence and uniqueness	24
5	The Dirichlet problem	26

*Department of Mathematics, Jiangxi Normal University, Nanchang 330022, China. E-mail: chen-huyuan@yeah.net

†Laboratoire de Mathématiques et Physique Théorique, Université de Tours, 37200 Tours, France. E-mail: veronl@univ-tours.fr

1 Introduction

We denote by \mathcal{L}_μ the Schrödinger operator defined in a domain $\Omega \subset \mathbb{R}^N$ by

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2} u,$$

where μ is a real constant and $N \geq 2$. This operator which is associated to the Hardy inequality has been thoroughly studied in the last thirty years. When the singular point 0 belongs to Ω , it appears a critical value

$$\mu_0 = -\left(\frac{N-2}{2}\right)^2$$

and the range of the μ in which the operator is bounded from below is $[\mu_0, \infty)$. This is linked to the Hardy inequality

$$\int_{\Omega} |\nabla \phi|^2 + \mu_0 \int_{\Omega} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Furthermore this inequality is never achieved if Ω is bounded, in which case a remainder was shown to exist by Brézis and Vázquez [4]. When λ is a Radon measure in Ω , the associated Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied in its full generality in [8] and [9] thanks to the introduction of a notion of very weak solution associated to some specific weight. Thanks to this new formulation an extensive treatment of the associated semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function is developed in [9].

In this article we assume that the singular point of the potential lies on the boundary of the domain Ω , and we are mainly interested in the two problems:

1- To define a notion of *very weak solution* for the problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is a Radon measure on $\partial\Omega$, and more generally on $\partial\Omega \setminus \{0\}$;

2- To prove the existence of a boundary trace for any positive \mathcal{L}_μ -harmonic function, i.e. solution of $\mathcal{L}_\mu u = 0$ in Ω and to connect it to the problem (1.1).

The model example is $\Omega = \mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$ although it is not a bounded domain. There exists a critical value

$$\mu \geq \mu_1 := -\frac{N^2}{4}. \tag{1.2}$$

This value is fundamental for the operator \mathcal{L}_μ to be bounded from below since there holds,

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}_+^N). \quad (1.3)$$

The analysis of the model case is explicit. Let $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$ be the spherical coordinates in \mathbb{R}_+^N , then, if (1.2) is satisfied, there exist two different types of positive \mathcal{L}_μ -harmonic functions vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$,

$$\gamma_\mu(r, \sigma) = r^{\alpha_+} \psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r, \sigma) = \begin{cases} r^{\alpha_-} \psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1}) \psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \quad (1.4)$$

where $\psi_1(\sigma) = \frac{x_N}{|x|}$ generates $\ker(-\Delta' + (N-1)I)$ in $H_0^1(\mathbb{S}_+^{N-1})$, and where

$$\alpha_+ := \alpha_+(\mu) = \frac{2-N}{2} + \sqrt{\mu + \frac{N^2}{4}} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2-N}{2} - \sqrt{\mu + \frac{N^2}{4}}. \quad (1.5)$$

Put $d\gamma_\mu(x) = \gamma_\mu(x)dx$. We define the γ_μ -dual operator \mathcal{L}_μ^* of \mathcal{L}_μ by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu} \langle \nabla \gamma_\mu, \nabla \zeta \rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}_+^N}), \quad (1.6)$$

and we prove that ϕ_μ is, in some sense, the fundamental solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \mathbb{R}_+^N, \\ u = \delta_0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

since it satisfies

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}_+^N}) \cap C^{1,1}(\mathbb{R}_+^N)$$

such that $\rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N)$, where $c_\mu > 0$ is a normalized constant and $\rho(x) = \text{dist}(x, \partial\Omega)$. Here $\rho(x) = x_N$ when $\Omega = \mathbb{R}_+^N$.

When \mathbb{R}_+^N is replaced by a bounded domain Ω satisfying the condition

$$(C-1) \quad 0 \in \partial\Omega, \quad \Omega \subset \mathbb{R}_+^N \quad \text{and} \quad \langle x, \mathbf{n} \rangle = O(|x|^2) \quad \text{for all } x \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}_x$ is the outward normal vector at x , inequality (1.3) holds but it is never achieved in the Hilbert space $H_0^1(\Omega)$. Note that the last condition in (C-1) holds if Ω is a C^2 domain. It is proved in [5] that there exists a remainder under the following form:

$$\int_\Omega |\nabla \phi|^2 + \mu_1 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq \frac{1}{4} \int_\Omega \frac{\phi^2}{|x|^2 \ln^2(|x|R_\Omega^{-1})} dx \quad \text{for all } \phi \in C_c^\infty(\Omega), \quad (1.7)$$

where $R_\Omega = \max_{z \in \Omega} |z|$. Under the assumption (C-1), there holds

$$\ell_\mu^\Omega := \inf \left\{ \int_\Omega \left(|\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx : v \in C_c^1(\Omega), \int_\Omega v^2 dx = 1 \right\} > 0.$$

This first eigenvalue is achieved in $H_0^1(\Omega)$ if $\mu > \mu_1$, or in the space $H(\Omega)$ which is the closure of $C_c^1(\Omega)$ for the norm

$$v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_{\Omega} \left(|\nabla v|^2 + \frac{\mu_1}{|x|^2} v^2 \right) dx},$$

when $\mu = \mu_1$. In the sequel we set

$$H_{\mu}(\Omega) = \begin{cases} H_0^1(\Omega) & \text{if } \mu > \mu_1 \\ H(\Omega) & \text{if } \mu = \mu_1. \end{cases}$$

Moreover, under the assumption (C-1) the imbedding of $H_{\mu}(\Omega)$ in $L^2(\Omega)$ is compact (see e.g. [6]). We denote by γ_{μ}^{Ω} the positive eigenfunction, its satisfies

$$\begin{cases} \mathcal{L}_{\mu} \gamma_{\mu}^{\Omega} = \ell_{\mu}^{\Omega} \gamma_{\mu}^{\Omega} & \text{in } \Omega, \\ \gamma_{\mu}^{\Omega} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \quad (1.8)$$

We prove that there exist $c_j = c_j(\Omega, \mu) > 0$, $j=1, 2$, such that

$$\begin{aligned} (i) \quad & \gamma_{\mu}^{\Omega}(x) = c_1 \rho(x) |x|^{\alpha+1} (1 + o(1)) \quad \text{as } x \rightarrow 0, \\ (ii) \quad & |\nabla \gamma_{\mu}^{\Omega}(x)| \leq c_2 \frac{\gamma_{\mu}^{\Omega}(x)}{\rho(x)} \quad \text{for all } x \in \Omega. \end{aligned} \quad (1.9)$$

This function will play the role of a weight function for replacing γ_{μ} . Next we construct the Poisson kernel $\mathbb{K}_{\mu}^{\Omega}$ of \mathcal{L}_{μ} in $\Omega \times \partial\Omega$. When $\mu \geq 0$ this construction can be made by truncation as in [18], considering for $\epsilon > 0$ and $\lambda \in \mathfrak{M}_+(\partial\Omega)$ the solution u_{ϵ} of

$$\begin{cases} -\Delta u + \frac{\mu}{\max\{\epsilon^2, |x|^2\}} u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

By a more elaborate method, we also construct the Poisson kernel when $\mu_1 \leq \mu < 0$. It is important to notice that when $\mu > 0$ the kernel has the property that

$$K_{\mu}^{\Omega}(x, 0) = 0 \quad \text{for all } x \in \bar{\Omega} \setminus \{0\} \quad (1.10)$$

by [18, Theorem A.1]. Because of (1.10) it is clear that the Poisson kernel cannot be the tool for describing all the positive \mathcal{L}_{μ} -harmonic functions. Our first concern in this article is to clarify the Poisson kernel of \mathcal{L}_{μ} .

We first characterize the positive \mathcal{L}_{μ} -harmonic functions which are singular at 0.

Theorem A *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ and $\mu \geq \mu_1$. If u is a nonnegative \mathcal{L}_{μ} -harmonic function vanishing on $B_{r_0}(0) \cap (\partial\Omega \setminus \{0\})$ for some $r_0 > 0$, there exists $k \geq 0$ such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{\rho(x) |x|^{\alpha-1}} = k,$$

if $\mu > \mu_1$ and

$$\lim_{x \rightarrow 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x) \ln |x|} = -k,$$

if $\mu = \mu_1$.

Actually the above convergences hold in a stronger way. In order to prove that such solutions truly exist we construct the *kernel function* ϕ_μ^Ω (see [13] for the denomination) which is the analogue in a bounded domain of the explicit singular solution ϕ_μ defined in \mathbb{R}_+^N .

Theorem B *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying (C-1) and $\mu \geq \mu_1$. Then there exists a positive \mathcal{L}_μ -harmonic function in Ω , which vanishes on $\partial\Omega \setminus \{0\}$ which satisfies,*

$$\phi_\mu^\Omega(x) = \rho(x)|x|^{\alpha-1}(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.11)$$

if $\mu > \mu_1$, and

$$\phi_{\mu_1}^\Omega(x) = \rho(x)|x|^{-\frac{N}{2}}(|\ln|x||+1)(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.12)$$

if $\mu = \mu_1$.

As in the model case, we define the γ_μ^Ω -dual operator of \mathcal{L}_μ by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu^\Omega} \langle \nabla \gamma_\mu^\Omega, \nabla \zeta \rangle + \ell_\mu^\Omega \zeta \quad \text{for all } \zeta \in C^{1,1}(\Omega).$$

The following commutation formula holds

$$\mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) = \gamma_\mu^\Omega \mathcal{L}_\mu^* \zeta. \quad (1.13)$$

Corollary C *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying (C-1) and $\mu \geq \mu_1$. Then ϕ_μ^Ω is the unique function belonging to $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ which satisfies*

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = kc_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \quad (1.14)$$

where and in the sequel the test function space

$$\mathbb{X}_\mu(\Omega) = \{ \zeta \in C(\overline{\Omega}) : \gamma_\mu^\Omega \zeta \in H_\mu(\Omega) \text{ and } \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\Omega) \}.$$

Furthermore, if u is a nonnegative \mathcal{L}_μ -harmonic function vanishing on $\partial\Omega \setminus \{0\}$, there exists $k \geq 0$ such that $u = k\phi_\mu^\Omega$.

We let $\sigma_\mu^\Omega \in H_\mu(\Omega)$ be the unique variational solution of

$$\mathcal{L}_\mu u = \frac{\gamma_\mu^\Omega}{\rho^*} \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.15)$$

where $\rho^*(x) = \min\{\frac{1}{\ell_\mu^\Omega}, \rho\}$. We prove that there is $c_2 > 1$ such that

$$\gamma_\mu^\Omega \leq \sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega \quad \text{in } \Omega. \quad (1.16)$$

Note that $\sigma_\mu^\Omega \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive classical solution of (1.15) with zero Dirichlet boundary condition on $\partial\Omega \setminus \{0\}$, i.e. $\sigma_\mu^\Omega = 0$ on $\partial\Omega \setminus \{0\}$. Moreover, $\frac{\partial\sigma_\mu^\Omega}{\partial\mathbf{n}} < 0$ on $\partial\Omega \setminus \{0\}$. We set

$$\eta = \frac{\sigma_\mu^\Omega}{\gamma_\mu^\Omega} \quad \text{in } \Omega,$$

which satisfies

$$\mathcal{L}_\mu^* \eta = \frac{1}{\rho} \quad \text{in } \Omega, \quad (1.17)$$

play a key role in the sequel. Clearly, $\eta \in C^2(\overline{\Omega} \setminus \{0\})$ and $1 \leq \eta \leq c_2$ in $\overline{\Omega} \setminus \{0\}$ by (1.16). We denote by $\mathfrak{M}(\Omega; \sigma_\mu^\Omega)$ the set of Radon measures ν in Ω such that

$$\sup \left\{ \int_\Omega \zeta d|\lambda| : \zeta \in C_c(\Omega), 0 \leq \zeta \leq \sigma_\mu^\Omega \right\} := \int_\Omega \sigma_\mu^\Omega d|\nu| < +\infty.$$

If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$ the measure $\sigma_\mu^\Omega \nu$ is a nonnegative bounded measure in Ω . Put

$$\beta_\mu^\Omega(x) = -\frac{\partial\gamma_\mu^\Omega(x)}{\partial\mathbf{n}_x} = \lim_{t \rightarrow 0^+} \frac{\gamma_\mu^\Omega(x - tn_x)}{t} = \lim_{t \rightarrow 0^+} \frac{\gamma_\mu^\Omega(x - tn_x)}{\rho^*(x - tn_x)}, \quad \forall x \in \partial\Omega \setminus \{0\} \quad (1.18)$$

and from (1.16) and (1.9), we have that

$$c_1|x|^{\alpha+-1} \leq \beta_\mu^\Omega(x) \leq c_1c_2|x|^{\alpha+-1} \quad \text{for } x \in \partial\Omega \setminus \{0\}. \quad (1.19)$$

As a consequence, the following potential function plays an important role in defining our boundary data. Denote

$$\beta_\mu(x) = |x|^{\alpha+-1} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \quad (1.20)$$

The set Radon measures λ in $\partial\Omega \setminus \{0\}$ such that

$$\sup \left\{ \int_{\partial\Omega \setminus \{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial\Omega \setminus \{0\}), 0 \leq \zeta \leq \beta_\mu \right\} := \int_{\partial\Omega \setminus \{0\}} \beta_\mu d|\lambda| < +\infty$$

is denoted by $\mathfrak{M}(\partial\Omega \setminus \{0\}; \beta_\mu)$. The extension of $\lambda \in \mathfrak{M}_+(\partial\Omega \setminus \{0\}; \beta_\mu)$ as a measure $\beta_\mu \lambda$ in $\partial\Omega$ is given by

$$\int_{\partial\Omega} \zeta d(\beta_\mu \lambda) = \sup \left\{ \int_{\partial\Omega} v \beta_\mu d\lambda : v \in C_c(\partial\Omega \setminus \{0\}), 0 \leq v \leq \zeta \right\} \quad \text{for all } \zeta \in C_c(\partial\Omega), \zeta \geq 0$$

and $\beta_\mu \lambda = \beta_\mu \lambda_+ - \beta_\mu \lambda_-$ if λ is a signed measure in $\mathfrak{M}(\partial\Omega \setminus \{0\}; \beta_\mu)$, and this defines the set $\mathfrak{M}(\partial\Omega; \beta_\mu)$ of all such extensions. The Dirac mass at 0 does not belong to $\mathfrak{M}(\partial\Omega; \beta_\mu)$, but it is the limit of sequences of measures in this space in the same way as it is a limit of measures in $\mathfrak{M}_+(\partial\Omega \setminus \{0\}; \beta_\mu)$. In the next result we prove the existence and uniqueness of a solution to

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \quad (1.21)$$

Thanks to (1.7) the Green kernel G_μ^Ω is easily constructible. If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ the following expressions are well defined

$$\mathbb{K}_\mu^\Omega[\lambda](x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{and} \quad \mathbb{G}_\mu^\Omega[\nu](x) = \int_\Omega G_\mu^\Omega(x, y) d\nu(y).$$

Our main existence result is the following.

Theorem D *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying (C-1) and $\mu \geq \mu_1$. If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$, $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \in \mathbb{R}$, the function*

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda, k)] \quad (1.22)$$

is the unique solution of (1.21) in the very weak sense that $u \in L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ and

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + kc_\mu \zeta(0) \quad (1.23)$$

for all $\zeta \in \mathbb{X}_\mu(\Omega)$.

In the next result we prove that all the positive \mathcal{L}_μ -harmonic functions in Ω are described by formula (1.22) (with $\nu = 0$).

Theorem E *Let Ω be a C^2 bounded domain such that $0 \in \partial\Omega$ satisfying (C-1), $\mu \geq \mu_1$ and u be a nonnegative \mathcal{L}_μ -harmonic functions in Ω . Then there exist $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $k \geq 0$, such that*

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = \mathbb{H}_\mu^\Omega[(0, \lambda, k)].$$

The couple $(\lambda, k\delta_0)$ is called the boundary trace of u .

The rest of this paper is organized as follows. In section 2, we introduce the distributional identity of \mathcal{L}_μ harmonic function ϕ_μ in \mathbb{R}_+^N . Section 3 is devoted to build the Kato's type inequalities, to construct Poisson kernel and related properties. Section 4 is addressed to classify the boundary isolated singular \mathcal{L}_μ harmonic functions in a bounded domain, i.e. **Theorem A** and to show the existence and related distributional identity in a (C-1) domain: proofs of **Theorem B** and **Corollary C**. We classify the boundary trace for general \mathcal{L}_μ harmonic functions and give the existence of \mathcal{L}_μ harmonic functions with the boundary trace $(\lambda, k\delta_0)$: **Theorem D** and **Theorem E** respectively in Section 5. Finally, we show Estimates (1.9) and (1.16) in Appendix.

In a forthcoming article [10] we study the semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

2 The half-space setting

Let $\mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$, $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$ be the spherical coordinates in \mathbb{R}_+^N and Δ' is the Laplace Beltrami operator on \mathbb{S}^{N-1} . Then

$$\mathcal{L}_\mu u = -\partial_{rr} u - \frac{N-1}{r} \partial_r u - \frac{1}{r^2} \Delta' u + \frac{\mu}{r^2} u.$$

If $u(r, \sigma) = r^\alpha \phi(\sigma)$ is a (separable) solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$, then ϕ satisfies

$$\begin{cases} -\Delta' \phi = \lambda_k \phi & \text{in } \mathbb{S}_+^{N-1} := \mathbb{S}^{N-1} \cap \mathbb{R}_+^N, \\ \phi = 0 & \text{in } \partial\mathbb{S}_+^{N-1} \approx \mathbb{S}^{N-2}, \end{cases}$$

where λ_k a constant which necessarily belongs to the spectrum

$$\sigma_{\mathbb{S}_+^{N-1}}(-\Delta') = \{\lambda_k = k(N + k - 2) : k \in \mathbb{N}^*\},$$

and $\alpha = \alpha_{k+}, \alpha_{k-}$ is a root of

$$\alpha^2 + (N - 2)\alpha - \lambda_k - \mu = 0. \quad (2.1)$$

The fundamental state corresponds to $k = 1$, in which case since $\lambda_1 = N - 1$, existence of real roots of (2.1) necessitates $\mu \geq \mu_1 = -\frac{N^2}{4} = \mu_1$ and we denote $\alpha_{1+} = \alpha_+$ and $\alpha_{1-} = \alpha_-$. Note that this value is connected to the boundary Hardy

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(S_+^{N-1}).$$

If this condition is fulfilled, the two roots α_+ and α_- corresponding to $k = 1$ and λ_1 are

$$\alpha_+ = \frac{2 - N}{2} + \sqrt{\mu - \mu_1} \quad \text{and} \quad \alpha_- = \frac{2 - N}{2} - \sqrt{\mu - \mu_1}. \quad (2.2)$$

The corresponding positive separable solutions γ_μ and ϕ_μ of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}_+^N \setminus \{0\}$ are defined by (1.4). We set $d\gamma_\mu(x) = \gamma_\mu(x)dx$ and define the operator \mathcal{L}_μ^* by (1.6).

Proposition 2.1 *The function ϕ_μ belongs to $L_{loc}^1(\mathbb{R}_+^N, \rho^{-1}d\gamma_\mu)$. It satisfies*

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N), \quad (2.3)$$

where

$$c_\mu = \begin{cases} 2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1 \end{cases} \quad (2.4)$$

and $\mathbb{X}_\mu(\mathbb{R}_+^N) = \left\{ \zeta \in C_c(\overline{\mathbb{R}_+^N}) : \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N) \right\}$.

Proof. Let $\zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N)$, $\epsilon > 0$ and set $B_\epsilon^+ = B_\epsilon(0) \cap \mathbb{R}_+^N$, $(B_\epsilon^+)^c = B_\epsilon^c(0) \cap \mathbb{R}_+^N$ and $\Gamma_\epsilon^+ = \partial B_\epsilon(0) \cap \overline{\mathbb{R}_+^N}$

$$\begin{aligned} 0 &= \int_{(B_\epsilon^+)^c} \zeta \gamma_\mu \mathcal{L}_\mu \phi_\mu dx \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \int_{\Gamma_\epsilon^+} \left(-\frac{\partial \phi_\mu}{\partial \mathbf{n}} \zeta \gamma_\mu + \left(\gamma_\mu \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu}{\partial \mathbf{n}} \right) \phi_\mu \right) dS \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \zeta(0) A(\epsilon) \end{aligned}$$

and $\mathbf{n} = -\epsilon^{-1}x$ on Γ_ϵ^+ , and

$$A(\epsilon) = \begin{cases} -2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS + O(\epsilon) & \text{if } \mu > \mu_1, \\ -\int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS + O(\epsilon) & \text{if } \mu = \mu_1, \end{cases} \quad (2.5)$$

which implies (2.3)-(2.4). \square

3 The Poisson kernel

In this section we assume that Ω is a bounded C^2 domain included in B_1 (which can always be assumed by scaling) and $0 \in \partial\Omega$. We start with the following identity of commutation valid for all $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $\zeta \in C^{1,1}(\overline{\Omega})$

$$-\int_{\partial\Omega} \frac{\partial(\zeta\gamma_\mu^\Omega)}{\partial\mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega\lambda) \quad \text{for all } \zeta \in C^{1,1}(\overline{\Omega}), \quad (3.1)$$

where β_μ^Ω is defined in (1.18).

The following inequality extends the classical Kato inequality to our framework.

Lemma 3.1 *Assume $N \geq 3$ and $\mu \geq \mu_1$ or $N = 2$ and $\mu > \mu_1$. Then for any $f \in L^1(\Omega, \sigma_\mu^\Omega dx)$, $h \in L^1(\partial\Omega, \beta_\mu^\Omega dx)$ there exists a unique weak solution u to*

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \quad (3.2)$$

in the sense that

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h \zeta d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \quad (3.3)$$

Furthermore, for any $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta \geq 0$, there holds

$$\int_{\Omega} |u| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h| \zeta d\beta_\mu^\Omega \quad (3.4)$$

and

$$\int_{\Omega} u^+ \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}_+(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h^+ \zeta d\beta_\mu^\Omega. \quad (3.5)$$

Proof. *Uniqueness.* Assume that u is a weak solution of (3.2) with $f = h = 0$. Then for any $\zeta \in \mathbb{X}_\mu(\Omega)$ there holds

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = 0.$$

Let $\phi \in C_c^\infty(\Omega)$, and $v \in H_\mu(\Omega)$ be the variational solution of

$$\mathcal{L}_\mu v = \frac{\gamma_\mu^\Omega}{\rho} \phi \quad \text{and} \quad u \in H_\mu(\Omega).$$

Then $v \in C^\infty(\Omega)$, $|v| \leq \|\phi\|_{L^\infty} \sigma_\mu^\Omega$; the equation is satisfied everywhere and in the sense of distributions in Ω . Clearly $w = (\gamma_\mu^\Omega)^{-1}v$ belongs to $C^\infty(\Omega)$ and satisfies

$$\mathcal{L}_\mu^* w = \frac{1}{\rho} \phi.$$

Thus,

$$\int_\Omega \frac{u}{\rho} \phi d\gamma_\mu^\Omega = 0.$$

Since ϕ is arbitrary, we have that $u = 0$.

Existence and estimates. We proceed by approximation as in [8, Prop. 2.1]. We assume that $\{(f_n, h_n)\} \subset C_0^1(\Omega) \times C_0^1(\partial\Omega \setminus \{0\})$ is a sequence which converges to (f, h) in $L^1(\Omega, \gamma_\mu^\Omega dx) \times L^1(\partial\Omega, \beta_\mu^\Omega dx)$. We set $V(x) = |x|^{-2}$, denote by \mathbb{K}^Ω the Poisson potential of $-\Delta$ in Ω and consider the approximate problem

$$\begin{cases} \mathcal{L}_\mu w_n = f_n - \mu V \mathbb{K}[h_n] & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.6)$$

Near 0, we have $V\mathbb{K}[h_n](x) = O(\frac{\rho(x)}{|x|^2})$, hence, if $N \geq 3$, $V\mathbb{K}[h_n] \in L^2(\Omega)$. If $N = 2$, the function $x \mapsto \frac{\rho(x)}{|x|^2}$ belongs to the Lorentz space $L^{2,\infty}(\Omega)$ which is the dual of $L^{2,1}(\Omega)$. Since $H(\Omega) \subset L^{2,1}(\Omega)$ by (1.7), it follows that $H'(\Omega) \subset L^{2,\infty}(\Omega)$. Hence, by Lax-Milgram theorem there exists a unique $w_n \in H(\Omega)$ such that (3.6) holds in the variational sense. Then $u_n = w_n + \mathbb{K}[h_n]$, which has the same regularity as w_n , satisfies

$$\begin{cases} \mathcal{L}_\mu u_n = f_n & \text{in } \Omega, \\ u_n = h_n & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

For $\sigma > 0$, we set

$$m_\sigma(t) = \begin{cases} |t| - \frac{\sigma}{2} & \text{if } |t| \geq \sigma, \\ \frac{t^2}{2\sigma} & \text{if } |t| < \sigma. \end{cases}$$

The m_σ is convex, $|m'_\sigma(t)| \leq 1$ and $m'_\sigma(t) \rightarrow \text{sign}_0(t)$ as $\sigma \downarrow 0$. Let $\zeta \in C^{1,1}(\overline{\Omega})$, $\zeta \geq 0$. We have that

$$\begin{aligned} & \int_\Omega (\langle \nabla w_n, \nabla(\zeta m'_\sigma(u_n) \gamma_\mu^\Omega) \rangle + \mu V w_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega) dx \\ &= \int_\Omega \zeta m'_\sigma(u_n) \gamma_\mu^\Omega f_n dx - \mu \int_\Omega V \mathbb{K}[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx =: \mathbb{R}(\sigma). \end{aligned}$$

By the fact $u_n = w_n + \mathbb{K}[h_n]$, we have that

$$\begin{aligned}
\mathbb{R}(\sigma) &= \int_{\Omega} \langle \nabla u_n, \nabla(\zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega}) \rangle dx - \int_{\Omega} \langle \nabla \mathbb{K}[h_n], \nabla(\zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega}) \rangle dx \\
&\quad + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&= \int_{\Omega} |\nabla u_n|^2 m''_{\sigma}(u_n) \zeta \gamma_{\mu}^{\Omega} dx + \int_{\Omega} \langle \nabla m_{\sigma}(u_n), \nabla(\zeta \gamma_{\mu}^{\Omega}) \rangle dx + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\geq - \int_{\Omega} m_{\sigma}(u_n) \Delta(\zeta \gamma_{\mu}^{\Omega}) dx + \int_{\partial\Omega} m_{\sigma}(h_n) \zeta \frac{\partial \gamma_{\mu}^{\Omega}}{\partial \mathbf{n}} dS + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\geq \int_{\Omega} m_{\sigma}(u_n) \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} - \int_{\partial\Omega} m_{\sigma}(h_n) d\beta_{\mu}^{\Omega} + \mu \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \gamma_{\mu}^{\Omega} \zeta dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx,
\end{aligned}$$

thus, we obtain that

$$\begin{aligned}
&\int_{\Omega} m_{\sigma}(u_n) \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} + \mu \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \zeta d\gamma_{\mu}^{\Omega} \\
&\leq \int_{\Omega} \zeta m'_{\sigma}(u_n) f_n d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} m_{\sigma}(h_n) \zeta d\beta_{\mu}^{\Omega}.
\end{aligned} \tag{3.8}$$

Since m_{σ} is convex, $u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n) \geq 0$. Hence for $\mu \geq 0$, we can let $\sigma \rightarrow 0$ in (3.8) and obtain (3.4).

For $\mu \in [\mu_1, 0)$, we note that

$$0 \leq u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n) \leq \frac{|u_n|^2}{2\sigma^2} \chi_{\{|u_n| \leq \sigma\}}$$

and

$$0 \leq \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \zeta d\gamma_{\mu}^{\Omega} \leq \frac{\|\zeta\|_{L^{\infty}}}{2} \int_{\{|u_n| \leq \sigma\}} |x|^{-1 - \frac{N}{2} + \sqrt{\mu - \mu_1}} dx. \tag{3.9}$$

Hence if $N \geq 3$, or $N = 2$ and $\mu > \mu_1 = -1$, the right-hand side of (3.9) tends to 0 as $\sigma \rightarrow 0$ and then we obtain (3.4). The proof of (3.5) is similar.

Applying estimate (3.4) to $u_n - u_m$, we obtain for all $\zeta \in \mathbb{X}_{\mu}(\Omega)$, $\zeta \geq 0$,

$$\int_{\Omega} |u_n - u_m| \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} \leq \int_{\Omega} |f_n - f_m| \zeta d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} |h_n - h_m| \zeta d\beta_{\mu}^{\Omega},$$

For test function, we take η , the solution of (1.17), then

$$\int_{\Omega} \frac{|u_n - u_m|}{\rho} d\gamma_{\mu}^{\Omega} \leq \int_{\Omega} |f_n - f_m| d\sigma_{\mu}^{\Omega} + \int_{\partial\Omega} |h_n - h_m| d(\eta \beta_{\mu}^{\Omega}),$$

Therefore $\{u_n\}$ is a cauchy sequence in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ with limit u . Since u_n satisfies (3.7), we let n go to infity in

$$\int_{\Omega} u_n \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta \sigma_\mu^\Omega f_n dx + \int_{\partial\Omega} \zeta \beta_\mu^\Omega h_n dS$$

and obtain (3.3). \square

Lemma 3.2 *Assume $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$ and $\zeta \in \mathbb{X}_\mu(\Omega)$, then there holds*

$$\int_{\Omega} \left(\mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}^\Omega[\lambda] d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

Proof. Note that $d(\beta_\mu^\Omega \lambda)$ is equivalent to $d(\beta_\mu^\Omega \lambda)$ by (1.19). By (1.13) we have almost everywhere in Ω ,

$$\left(\mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}^\Omega[\lambda] \gamma_\mu^\Omega = \left(\mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) - \frac{\mu}{|x|^2} \gamma_\mu^\Omega \zeta \right) \mathbb{K}^\Omega[\lambda] = -\Delta(\gamma_\mu^\Omega \zeta) \mathbb{K}^\Omega[\lambda].$$

If we assume that λ vanishes in a neighborhood of 0 we derive from (3.1)

$$-\int_{\Omega} \Delta(\gamma_\mu^\Omega \zeta) \mathbb{K}^\Omega[\lambda] dx = -\int_{\partial\Omega} \frac{\partial(\zeta \gamma_\mu^\Omega)}{\partial \mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

Since $\gamma_\mu^\Omega \left(\mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right)$ is bounded, we obtain the result first if λ is nonnegative by considering the sequence $\{\chi_{B_\epsilon} \lambda\}$ and letting $\epsilon \rightarrow 0$, and then for any $\lambda = \lambda^+ - \lambda^-$. \square

We observe also that the existence of the Green kernel follows from Lax-Milgram theorem which gives the existence of a unique variational solution in $H(\Omega)$ to

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by G_μ^Ω the Green kernel and by \mathbb{G}_μ^Ω the corresponding Green operator.

3.1 Construction of the Poisson kernel when $\mu > 0$

For the sake of completeness, we recall the construction in [18]. For $\epsilon > 0$ we set $V_\epsilon(x) = \max\{\epsilon^{-2}, |x|^{-2}\}$ and $V_0(x) = V(x) = |x|^{-2}$, and if $\lambda \in \mathfrak{M}(\partial\Omega)$ let u_ϵ be the solution of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

Then

$$u_\epsilon(x) = \int_{\partial\Omega} K_{\mu, \epsilon}^\Omega(x, y) d\lambda(y) = \mathbb{K}_{\mu, \epsilon}^\Omega[\lambda].$$

We obtain by the maximum principle,

$$K_{\mu, \epsilon}^\Omega \leq K_{\mu', \epsilon'}^\Omega \leq K^\Omega \quad \text{for all } \mu \geq \mu' \geq 0 \text{ and } \epsilon' \geq \epsilon > 0,$$

where K^Ω is the usual Poisson kernel in Ω and there exists

$$K_\mu^\Omega(x, y) = \lim_{\epsilon \rightarrow 0} K_{\mu, \epsilon}^\Omega(x, y) \quad \text{for all } (x, y) \in \Omega \times \partial\Omega.$$

Therefore we infer, firstly by monotone convergence if $\lambda \geq 0$, and then for any $\lambda \in \mathfrak{M}(\partial\Omega)$, that

$$\lim_{\epsilon \rightarrow 0} u_\epsilon(x) = u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega. \quad (3.10)$$

Since V is finite in $B_\epsilon^c \cap \Omega$, for any $x \in \Omega$, $K_\mu^\Omega(x, y) > 0$ for all $y \in \partial\Omega \setminus \{0\}$. If G^Ω is the Green kernel in Ω , there holds

$$u_\epsilon(x) + \mu \int_{\Omega} G^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy = \int_{\partial\Omega} K^\Omega(x, y) d\lambda(y)$$

If $\lambda \geq 0$, we have by Fatou's lemma,

$$\int_{\Omega} G^\Omega(x, y) V(y) u(y) dy \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} G^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy. \quad (3.11)$$

Combined with (3.10) it yields

$$u(x) + \mu \int_{\Omega} G^\Omega(x, y) V(y) u(y) dy \leq \int_{\partial\Omega} K^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega.$$

Since the function $u + \mu \mathbb{G}[Vu]$ is nonnegative and harmonic in Ω , it admits a boundary trace which is a nonnegative Radon measure λ^* and there holds

$$u(x) + \mu \int_{\Omega} G^\Omega(x, y) V(y) u(y) dy = \int_{\partial\Omega} K^\Omega(x, y) d\lambda^*(y) \quad \text{for all } x \in \Omega. \quad (3.12)$$

Because of (3.11) $0 \leq \lambda^* \leq \lambda$. The measure λ^* is the reduced measure associated to λ . Since (3.12) is equivalent to

$$u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda^*(y),$$

there holds

$$\int_{\partial\Omega} K_\mu^\Omega(x, y) d(\lambda - \lambda^*)(y) = 0.$$

This implies that $\lambda = \lambda^*$ in $\partial\Omega \setminus \{0\}$. With the notations of [18], we recall that

$$\begin{aligned} \text{Sing}_V(\Omega) &:= \{y \in \partial\Omega : \exists x_0 \in \Omega \text{ s.t. } K_\mu^\Omega(x_0, y) = 0\} \\ &\subset Z_V := \left\{ y \in \partial\Omega : \int_{\Omega} K_0^\Omega(x, y) V(x) \rho(x) dx = \infty \right\}. \end{aligned}$$

Actually, if $y \in \text{Sing}_V(\Omega)$, $K_\mu^\Omega(x_0, y) = 0$ for any $x_0 \in \Omega$ by Harnack inequality. Clearly $0 \in Z_V$ and if $y \neq 0$ the integral term in the definition of Z_V is finite. Hence $\text{Sing}_V(\Omega) \subset Z_V = \{0\}$. Since for any truncated cone $C_{0, \delta} \Subset \Omega$ with vertex 0, there holds

$$\int_{C_{0, \delta}} V(x) \frac{dx}{|x - y|^{N-2}} = \infty,$$

it follows by Ancona's result [18, Theorem A1] that $0 \in \text{Sing}_V(\Omega)$. Finally

$$K_\mu^\Omega(x, 0) = 0 \quad \text{for all } x \in \Omega.$$

3.2 Construction of the Poisson kernel when $\mu_1 \leq \mu < 0$

For $\epsilon > 0$ and $\lambda \in C(\partial\Omega)$, $\lambda \geq 0$ we denote by $w = w_{\epsilon,\lambda}$ the variational solution in $H(\Omega)$ of

$$\begin{cases} -\Delta w + \mu V_\epsilon w = -\mu V_\epsilon \mathbb{K}[\lambda] & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $w_{\epsilon,\lambda} \geq 0$ and $u = u_{\epsilon,\lambda} := w_\epsilon + \mathbb{K}[\lambda]$ satisfies

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Since $-\Delta u_{\epsilon,\lambda} + \mu V u_{\epsilon,\lambda} \leq 0$, there holds from Lemma 3.1

$$\int_{\Omega} u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda \zeta d\beta_\mu^\Omega \quad \text{for } \zeta \in \mathbb{X}_\mu(\Omega), \zeta \geq 0$$

and in particular

$$\int_{\Omega} \frac{u_{\epsilon,\lambda}}{\rho} d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda d(\eta\beta_\mu^\Omega). \quad (3.14)$$

If $\epsilon > \epsilon' > 0$ and $\lambda' > \lambda > 0$ we have

$$-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} = \mu (V_{\epsilon'} - V_\epsilon) \leq 0.$$

Since

$$\begin{aligned} & \int_{\Omega} \left(-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} \right) (u_{\epsilon,\lambda}^2 - u_{\epsilon',\lambda'}^2)_+ dx \\ &= \int_{\{u_{\epsilon,\lambda} \geq u_{\epsilon',\lambda'}\}} \left(\left| \nabla u_{\epsilon,\lambda} - \frac{u_{\epsilon,\lambda}}{u_{\epsilon',\lambda'}} \nabla u_{\epsilon',\lambda'} \right|^2 + \left| \nabla u_{\epsilon',\lambda'} - \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}} \nabla u_{\epsilon,\lambda} \right|^2 \right) dx, \end{aligned}$$

we deduce that the function $x \mapsto \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}}(x)$ is constant on the set $\{x : u_{\epsilon,\lambda}(x) > u_{\epsilon',\lambda'}(x)\}$. If this set is non-empty we get a contradiction since it is strictly included in Ω . Therefore the mapping

$$(\epsilon, \lambda) \mapsto u_{\epsilon,\lambda}$$

is decreasing in ϵ and increasing in λ .

Next we can assume that $\lambda \in \mathfrak{M}_+(\partial\Omega)$ vanishes in $B_\delta \cap \partial\Omega$ and that $\{\lambda_n\} \subset C(\partial\Omega)$ is a sequence of functions which converge to λ in the weak sense of measures. We denote by u_{ϵ,λ_n} the solution of (3.13) with λ replaced by λ_n . Since $\mu < 0$, $\alpha_+ < 1$, $\rho^{-1}\gamma_\mu^\Omega \sim |x|^{\alpha_+-1} \geq R_\omega^{\alpha_+-1}$, where $R_\omega = \max\{|z| : z \in \Omega\}$. Hence

$$\int_{\Omega} u_{\epsilon,\lambda_n} dx \leq c_8 \int_{\partial\Omega} \lambda_n d(\eta\beta_\mu^\Omega) \leq c_9 \|\lambda\|_{\mathfrak{M}(\partial\Omega)}.$$

Hence u_{ϵ,λ_n} and $V_\epsilon u_{\epsilon,\lambda_n}$ are uniformly bounded in $L^1(\Omega)$. From standard regularity estimates the sequence $\{u_{\epsilon,\lambda_n}\}_{n \in \mathbb{N}}$ is bounded in the Lorentz spaces $L^{\frac{N}{N-1}, \infty}(\Omega)$ and weakly relatively

compact in $L^1(\Omega)$ (see e.g. [12]). This implies that, up to a subsequence, u_{ϵ, λ_n} converges in $L^1(\Omega)$ and a.e. in Ω to a weak solution $u_{\epsilon, \lambda}$ of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

that is a function which satisfies

$$\int_{\Omega} (-u_{\epsilon, \lambda} \Delta \zeta + \mu V_\epsilon u_{\epsilon, \lambda} \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}). \quad (3.15)$$

Furthermore (3.14) holds (with the same notation). For test function ζ in (3.15), we take $\zeta = \theta_1$ be the solution of

$$\begin{cases} -\Delta \theta_1 = 1 & \text{in } \Omega, \\ \theta_1 = 0 & \text{on } \partial\Omega \end{cases}$$

Then

$$\int_{\Omega} u_{\epsilon, \lambda} dx = -\mu \int_{\Omega} V_\epsilon u_{\epsilon, \lambda} \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

By the monotone convergence theorem we obtain that $V_\epsilon u_{\epsilon, \lambda} \rightarrow V u_\lambda$ in $L^1(\Omega, \theta_1 dx)$ by letting $\epsilon \rightarrow 0$ and

$$\int_{\Omega} u_\lambda dx = -\mu \int_{\Omega} V u_\lambda \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

Hence

$$\int_{\Omega} (-\Delta \zeta + \mu V \zeta) u_\lambda dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}).$$

We also have

$$\int_{\Omega} u_{\epsilon, \lambda_n} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\Omega} (V - V_\epsilon) u_{\epsilon, \lambda_n} \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta \lambda_n d\beta_\mu^\Omega$$

for all $\zeta \in \mathbb{X}_\mu(\Omega)$. Since u_{ϵ, λ_n} converges in $L^1(\Omega)$ we obtain if $\zeta \geq 0$,

$$\int_{\Omega} u_{\epsilon, \lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\Omega} (V - V_\epsilon) u_{\epsilon, \lambda} \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \leq \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega),$$

since $\mu(V - V_\epsilon) \leq 0$. When $\epsilon \rightarrow 0$, $u_{\epsilon, \lambda}$ increases and converges to some u_λ in $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ which satisfies

$$\int_{\Omega} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \zeta \geq 0.$$

For $\delta > 0$ denote by ζ_δ the solution of

$$\mathcal{L}_\mu^* \zeta_\delta = \chi_{\Omega_\delta} \mathcal{L}_\mu^* \zeta \quad \text{where } \Omega_\delta = \{x \in \Omega : \rho(x) > \delta\}.$$

As $\zeta \in \mathbb{X}_\mu(\Omega)$, $|\mathcal{L}_\mu^* \zeta| \leq c_{10} \rho$, hence $\zeta_\delta \in \mathbb{X}_\mu(\Omega)$, $|\zeta_\delta| \leq c_{10} \eta$ and $\zeta_\delta \rightarrow \zeta$ when $\delta \rightarrow 0$. Furthermore, since $c_{11}|x|$ is a supersolution for $c_{11} > 0$ large enough, $\eta_\delta \leq c_{11}|x|$. Hence

$$\int_{\Omega_\delta} u_{\epsilon, \lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon, \lambda} \zeta_\delta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta_\delta d(\lambda \beta_\mu^\Omega).$$

Because $|x|^{-2}u_{\epsilon,\lambda}|\zeta_\delta| \leq c_{11}\rho^{-1}u_{\epsilon,\lambda}$ and $u_{\epsilon,\lambda} \rightarrow u_\lambda$ in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$, we derive that

$$\lim_{\epsilon \rightarrow 0} \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon,\lambda} \zeta_\delta d\gamma_\mu^\Omega = 0$$

which implies

$$\int_{\Omega_\delta} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta_\delta d(\lambda\beta_\mu^\Omega).$$

Letting $\delta \rightarrow 0$ we obtain by monotonicity

$$\int_{\Omega} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega). \quad (3.16)$$

Finally, if $\lambda \in \mathfrak{M}_+(\partial\Omega, \beta_\mu)$ we replace it by $\lambda_\delta = \chi_{B_\delta^c} \lambda$ and denote by u_{λ_δ} the weak solution of

$$\begin{cases} -\Delta u + \mu V u = 0 & \text{in } \Omega, \\ u = \lambda_\delta & \text{on } \partial\Omega. \end{cases}$$

The mapping $\delta \mapsto u_{\lambda_\delta}$ is monotone. Hence, by the monotone convergence theorem u_{λ_δ} increases and converges to some u_λ in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ and clearly u_λ satisfies (3.16) for all $\zeta \in \mathbb{X}_\mu(\Omega)$.

4 The singular kernel

In this section we construct the singular kernel ϕ_μ^Ω and prove that it satisfies estimates (1.11)-(1.12) and it is associated to Dirac mass at 0. Up to a rotation we can assume that the inward normal direction to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0', 1) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Hence the tangent hyperplane to $\partial\Omega$ at 0 is $\partial\mathbb{R}_+^N = \mathbb{R}^{N-1}$. For $R > 0$ set $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$ and $D_R = B'_R \times (-R, R)$. Then there exist $R > 0$ and a C^2 function $\theta : B'_R \mapsto \mathbb{R}$ such that $\partial\Omega \cap D_R = \{x = (x', x_N) : x_N = \theta(x') \text{ for } x' \in B'_R\}$ and $\Omega \cap D_R = \{x = (x', x_N) : \theta(x') < x_N < R\}$. Furthermore $\nabla\theta(0) = 0$.

4.1 Classification of Boundary isolated singularities

We characterize the positive solutions of $\mathcal{L}_\mu u = 0$ which vanish on $\partial\Omega \setminus \{0\}$.

Lemma 4.1 *Let $\mu \geq \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ be a positive solution of $\mathcal{L}_\mu u = 0$ in Ω vanishing on $\partial\Omega \setminus \{0\}$. Then there exist $a > 0$ and $c_{12} > 0$ such that*

$$u(x) \leq c_{12}|x|^{-a-1}\rho(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}. \quad (4.1)$$

Proof. This is a direct consequence of Boundary Harnack inequality [3, Th. 2.7].

Proposition 4.2 *Assume that $\mu \geq \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$, vanishing on $\partial\Omega \setminus \{0\}$ satisfying (4.1) with $a \geq -\alpha_-$. Then the following convergences hold in $C^1(\mathbb{S}_+^{N-1})$:*

(i) *If $\mu > \mu_1$ and $a = -\alpha_-$, there exists $c_{13} \geq 0$ such that*

$$\lim_{r \rightarrow 0} \frac{u(r, \cdot)}{r^{\alpha_-}} = c_{13}\phi \quad \text{as } r \rightarrow 0. \quad (4.2)$$

(ii) $\mu \geq \mu_1$ and $a > -\alpha_-$ there exist $\tau > a + \alpha_-$ depending on a and μ , and $c_{14} \geq 0$ such that

$$u(x) \leq c_{14}|x|^{-a-1+\tau}\rho(x) \quad \text{for all } x \in \bar{\Omega} \setminus \{0\}.$$

Proof. *Step 1. Straightening the boundary.* We define the function $\Theta = (\Theta_1, \dots, \Theta_N)$ on D_R by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$ we can assume that Θ is a diffeomorphism from D_R onto $\Theta(D_R)$. We set

$$u(x) = \tilde{u}(y) \quad \text{for all } x \in D_R^+ = B'_R \times [0, R]. \quad (4.3)$$

Then

$$\begin{aligned} u_{x_j x_j} &= \tilde{u}_{y_j y_j} - 2\theta_{x_j} \tilde{u}_{y_j y_N} - \theta_{x_j, x_j} \tilde{u}_{y_N} + \theta_{x_j}^2 \tilde{u}_{y_N y_N} \quad \text{for } 1 \leq j \leq N-1, \\ u_{x_N x_N} &= \tilde{u}_{y_N y_N} \end{aligned} \quad (4.4)$$

and

$$\Delta \tilde{u} + |\nabla \theta|^2 \tilde{u}_{y_N y_N} - 2\langle \nabla \theta, \nabla \tilde{u}_{y_N} \rangle - \tilde{u}_{y_N} \Delta \theta - \frac{\mu}{|\Theta^{-1}(y)|^2} \tilde{u} = 0. \quad (4.5)$$

We use here the spherical coordinates (r, σ) in the variable y and we recall that Δ' is the Laplace-Beltrami operator on \mathbb{S}^{N-1} and ∇' is the tangential gradient on \mathbb{S}^{N-1} identified with the covariant derivative via the isometric imbedding $\mathbb{S}^{N-1} \hookrightarrow \mathbb{R}^N$ which enables the formula

$$\nabla \tilde{u}(y) = \left(\tilde{u}_r \mathbf{n} + \frac{1}{r} \nabla' \tilde{u} \right) (r, \sigma) \quad \text{with } \mathbf{n} = |y|^{-1} y.$$

After a lengthy computation the details of which can be found in [12, P 298-300] we obtain

$$\begin{aligned} & r^2 \tilde{u}_{rr} [1 - 2\theta_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla \theta|^2 (\langle \mathbf{n}, \mathbf{e}_N \rangle)^2] \\ & + r \tilde{u}_r [N-1 - r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta \theta + r |\nabla \theta|^2 (\langle \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle), \mathbf{e}_N \rangle - 2\langle \nabla' \theta, \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle) \rangle)] \\ & + \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle [-r \Delta \theta + 2\theta_r - |\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle] + r \langle \nabla' \tilde{u}_r, \mathbf{e}_N \rangle [2\theta_r + 2|\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle] \\ & - 2\langle \nabla' \tilde{u}, \nabla' \theta \rangle \langle \mathbf{n}, \mathbf{e}_N \rangle + \langle \nabla' (\langle \nabla' \tilde{u}, \mathbf{e}_N \rangle), |\nabla \theta|^2 \mathbf{e}_N - 2r^{-1} \nabla' \theta \rangle + \Delta' \tilde{u} - \frac{\mu}{|\Theta^{-1}(y)|^2} = 0. \end{aligned}$$

Next we set

$$\tilde{u}(r, \sigma) = r^{-a} v(t, \sigma) \quad \text{with } t = \ln r,$$

and we assume that

$$a \neq \frac{N-2}{2}. \quad (4.6)$$

We notice that

$$\begin{aligned} r^2 = \sum_{j=1}^N y_j^2 &= \sum_{j=1}^{N-1} x_j^2 + (x_N - \theta(x'))^2 = |x^2| - 2x_N \theta(x') = |x|^2 (1 + O(r)) \quad \text{as } r \rightarrow 0 \\ &= |x|^2 (1 + O(e^t)) \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

By a straightforward computation we find that v satisfies the following asymptotically autonomous equation in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$

$$(1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 - 2a + \epsilon_2(t, \cdot))v_t + (a(a + 2 - N) - \mu + \epsilon_3(t, \cdot))v + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \quad (4.7)$$

where the ϵ_j satisfies

$$|\epsilon_j(t, \cdot)| + |\partial_t \epsilon_j(t, \cdot)| + |\nabla' \epsilon_j(t, \cdot)| \leq c_{15} e^t. \quad (4.8)$$

This is due to the fact that $|\theta(x')| = O(|x'|^2)$ near 0.

Step 2. The convergence process. Since v is bounded in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$ and vanishes on $(-\infty, r_0] \times \partial \mathbb{S}_+^{N-1}$ and all the coefficients are continuous functions, we obtain that v is bounded in $W^{2,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ independently of $T \leq r_0 - 2$, for any $q < \infty$. Hence v is bounded in any $C^{1,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$ for any $\tau \in [0, 1)$. Differentiating the equation and using the standard elliptic equations regularity, we obtain that v is bounded in $W^{3,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ and in $C^{2,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$. We consider the negative trajectory of v in $C_0^1(\overline{\mathbb{S}_+^{N-1}})$ defined by

$$\mathcal{T}_-(v) = \bigcup_{t \leq r_0 - 1} \{v(t, \cdot)\}.$$

By the previous estimates and the Arzela-Ascoli theorem, it is a relatively compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$, hence its limit set at $-\infty$ (or alpha-limit set), denoting $A(\mathcal{T}_-(v))$, is a non-empty connected compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$. Multiplying (4.7) by v_t and integrating on \mathbb{S}_+^{N-1} yields

$$\begin{aligned} & \int_{\mathbb{S}_+^{N-1}} \left(N - 2 - 2a + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right) v_t^2 dS - \frac{1}{2} \int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\ &= \frac{d}{dt} \left[\int_{\mathbb{S}_+^{N-1}} \left(\frac{1}{2} |\nabla v|^2 - \frac{1}{2} [a(a + 2 - N) - \mu + \epsilon_3] v^2 - \frac{1}{2} (1 + \epsilon_1) v_t^2 \right) dS \right] \\ & \quad - \int_{\mathbb{S}_+^{N-1}} (\langle \nabla'v, \epsilon_4 \rangle + \langle \nabla'v_t, \epsilon_5 \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6 \rangle) v_t^2 dS. \end{aligned} \quad (4.9)$$

Next we integrate over $(-\infty, r_2)$ for some r_2 large enough so that

$$\left| N - 2 - 2a + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right| \geq \frac{1}{2} |N - 2 - 2a| > 0,$$

here we use the crucial assumption (4.6). Since all the terms on the right-hand side of (4.9) are integrable on $(-\infty, r_2)$ because of (4.8) and the bounds on v , we obtain that

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_t^2 dS < \infty. \quad (4.10)$$

Differentiating (4.7) with respect to t and using the estimates on v and the ϵ_j we obtain (see [12, p. 302] for a similar calculation)

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_{tt}^2 dS < \infty. \quad (4.11)$$

Because v_t and v_{tt} are uniformly continuous on $(-\infty, r_1]$, we infer from (4.10) and (4.11)

$$\lim_{t \rightarrow -\infty} \left(\|v_t(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} + \|v_{tt}(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} \right) = 0.$$

Therefore the set $A(\mathcal{T}_-(v))$ is a compact connected subset of the set of nonnegative solutions of

$$\begin{cases} \Delta' \omega + (a(a+2-N) - \mu) \omega = 0 & \text{in } \mathbb{S}_+^{N-1}, \\ \omega = 0 & \text{on } \partial \mathbb{S}_+^{N-1}. \end{cases}$$

Step 3. The case $a(a+2-N) - \mu = N-1$. The set $A(\mathcal{T}_-(v))$ is a subset of $\ker(-\Delta' - (N-1)Id)$ in $H_0^1(\mathbb{S}_+^{N-1})$ and more precisely $A(\mathcal{T}_-(v)) = \{m\psi_1 : m \in I^*\}$ where I^* is a compact interval of $(0, \infty)$. We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS.$$

Then X satisfies

$$X''(t) + (N-2-2a)X'(t) + F(t) = 0, \quad (4.12)$$

where

$$F(t) = \int_{\mathbb{S}_+^{N-1}} \left[\epsilon_1(t, \cdot) v_{tt} + \epsilon_2(t, \cdot) v_t + \epsilon_3(t, \cdot) v + \langle \nabla' v, \epsilon_4(t, \cdot) \rangle + \langle \nabla' v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle \right] \psi_1 dS.$$

Then $|F(t)| \leq c_{16}e^t$. We consider a sequence $\{t_n\}$ converging to $-\infty$ and $c^* \in I^*$ such that $X(t_n) \rightarrow c^*$. Since $X'(t)$ and $X''(t)$ converges to 0 as $t \rightarrow -\infty$, we integrate (4.12) on (t_n, t) and let $n \rightarrow \infty$. Then we get

$$X'(t) + (N-2-2a)(X(t) - c^*) + O(e^t) = 0.$$

Letting $t \rightarrow -\infty$ yields $X(t) \rightarrow c^*$. Hence we have proved that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c^* \psi_1 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}).$$

Step 4. The case $a(a+2-N) - \mu \neq N-1$. Clearly $A(\mathcal{T}_-(v)) = \{0\}$ and

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = 0 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}). \quad (4.13)$$

Furthermore, since we have assumed $a \geq -\alpha_-$, there holds actually $a > -\alpha_-$. We recall that λ_k is the k -th eigenvalue of $-\Delta'$ in $H_0^1(\mathbb{S}_+^{N-1})$ and put

$$H_k = \ker(-\Delta' - \lambda_k Id) = \text{span}\langle \phi_{k,1}, \phi_{k,2}, \dots, \phi_{k,j_k} \rangle \quad \text{and} \quad H_0^1(\mathbb{S}_+^{N-1}) = \bigoplus_{k=1}^{\infty} H_k.$$

We denote

$$P_k(x) = x^2 + (N-2)x - \mu - \lambda_k.$$

Then $P_1(\alpha_-) = 0$ and $P_k(\alpha_-) = \lambda_1 - \lambda_k < 0$ for $k \geq 2$. Since $a(a+2-N) - \mu \neq N-1$ by assumption, we define a partition of \mathbb{N}^* by setting

$$N_1 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k \geq 0\}, \quad N_2 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k < 0\}$$

and

$$W_1 = \bigoplus_{k \in N_1} H_k \quad \text{and} \quad W_2 = \bigoplus_{k \in N_2} H_k.$$

Then

$$-\int_{\mathbb{S}_+^{N-1}} \phi \Delta' \phi dS \geq \gamma \int_{\mathbb{S}_+^{N-1}} \phi^2 dS \quad \text{for all } \phi \in W_2, \quad (4.14)$$

where

$$\gamma = \mu + \lambda_{k_2} - a(a+2-N) > 0 \quad \text{with } k_2 = \inf N_2.$$

We denote by P_j the orthogonal projector onto W_j in $H_0^1(\mathbb{S}_+^{N-1})$ and set $v = P_1 v + P_2 v = v_1 + v_2$. Then the projection of (4.7) on to W_2 is

$$(v_2)_{tt} + (N-2-2a)(v_2)_t + (a(a+2-N) - \mu)v_2 + \Delta' v_2 = F_2(t, \cdot),$$

where F_2 satisfies the same estimates (4.8) as ϵ_j . Then, using (4.8) and (4.14)

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS + (N-2-2a) \int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS - \gamma \int_{\mathbb{S}_+^{N-1}} v_2^2 dS \geq -c_{17} e^t \left(\int_{\mathbb{S}_+^{N-1}} v_2^2 dS \right)^{\frac{1}{2}}.$$

Put $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$, because

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS = Y'(t)Y(t) \quad \text{and} \quad \int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS \geq Y''(t)Y(t),$$

we obtain the following differential inequality

$$Y'' + (N-2-2a)Y' - \gamma Y \geq -c_{17} e^t \quad \text{in } \mathcal{D}'(-\infty, r_2).$$

The characteristic roots of the equation $y'' + (N-2-2a)y' - \gamma y = 0$ are

$$\begin{aligned} a_{k_2,-} &= a + \frac{1}{2} \left(2 - N - \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2} \right) = \alpha_{k_2,-} + a < 0 \\ a_{k_2,+} &= a + \frac{1}{2} \left(2 - N + \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2} \right) = \alpha_{k_2,+} + a > 0. \end{aligned} \quad (4.15)$$

where the $\alpha_{k_2,\pm}$ are the roots of equations (2.1) with $k = k_2$. The solutions of

$$z'' + (N-2-2a)z' - \gamma z = -c_{17} e^t \quad \text{in } \mathcal{D}'(-\infty, r_2).$$

endow the form $z(t) = Ae^{ta_{k_2,-}} + Be^{ta_{k_2,+}} + c_{18}e^t$ if $a_{k_2,+} \neq 1$ or $z(t) = Ae^{ta_{k_2,-}} + Be^t + Cte^t$ if $a_{k_2,+} = 1$, for some explicit constant c_{18} depending on c_{17} and the coefficients in the equation. Since $Y(t) \rightarrow 0$ when $t \rightarrow -\infty$ by (4.13), it follows from the maximum principle that

$$Y(t) \leq c_{19} e^{ta_{k_2,+}} + c_{18} e^t \quad \text{if } a_{k_2,+} \neq 1, \quad \text{or } Y(t) \leq c_{20} |t| e^t \quad \text{if } a_{k_2,+} = 1 \quad \text{for } t \leq r_2.$$

Then using standard elliptic equations a priori estimates, initially in $L^2(\mathbb{S}_+^{N-1})$, then in $L^p(\mathbb{S}_+^{N-1})$ and finally in $C^\tau(\overline{\mathbb{S}_+^{N-1}})$, we obtain that for $t \leq r_3$,

$$\|v_2(t, \cdot)\|_{C^1(\mathbb{S}_+^{N-1})} \leq \begin{cases} c_{21}e^{ta_{k_2,+}} + c_{22}e^t & \text{if } a_{k_2,+} \neq 1, \\ c_{23}|t|e^t & \text{if } a_{k_2,+} = 1, \end{cases} \quad (4.16)$$

where $r_3 \leq r_2 - 1$.

For the components in W_1 we have

$$v_1(t, \cdot) = \sum_{k \in N_1} \sum_{1 \leq j \leq j_k} w_{k,j}(t) \phi_{k,j}(\cdot), \quad (4.17)$$

where the $\phi_{k,j}$ form an orthormal basis of H_k . Then

$$w''_{k,j} + (N - 2 - 2a)w'_{k,j} + (a(a + 2 - N) - \mu - \lambda_k)w_{k,j} = F_{k,j}(t) \quad (4.18)$$

The characteristic roots of equation $z'' + (N - 2 - 2a)z' + (a(a + 2 - N) - \mu - \lambda_k)z = 0$ are given in (4.15) with a general k , $a_{k-} = a + \alpha_{k-}$ and $a_{k+} = a + \alpha_{k+}$ where $\alpha_{k\pm}$ are the roots of (2.1). They have same sign (including 0) since $a(a + 2 - N) - \mu - \lambda_k \geq 0$, furthermore, their sum is positive since $N - 2 - 2a < 0$, as a consequence of $a > -\alpha_-$. By standard calculation the solution of (4.18) has the form

$$w_{k,j}(t) = m_1 e^{ta_{k+}} + m_2 e^{ta_{k-}} - \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds. \quad (4.19)$$

Since $|F_{k,j}(s)| \leq c_{24}e^s$ there holds

$$\left| \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds \right| \leq c_{25} \begin{cases} |t|e^t & \text{if } a_{k-} = 1 \\ \max\{e^t, e^{ta_{k-}}\} & \text{if } a_{k-} \neq 1 \end{cases} \quad (4.20)$$

In particular, if $k_1 = \max N_1$, then $a_{k_1\pm} = \min\{a_{k\pm} : k \in N_1\}$.

We assume first that $a_{k_1-} > 0$. Combining this fact with (4.17) and (4.20) we obtain

$$\|v_1(t, \cdot)\|_{L^\infty(\mathbb{S}_+^{N-1})} \leq c_{26} \begin{cases} |t|e^t & \text{if } a_{k_1-} = 1 \\ \max\{e^t, e^{ta_{k_1-}}\} & \text{if } a_{k_1-} \neq 1 \end{cases} \quad (4.21)$$

Furthermore, because of the explicit formulation and (4.8), the left-hand side of (4.20) can be replaced by $\|v_1(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})}$. Combining (4.16) and (4.20) we obtain the result since $v(t, \cdot) = 0$

on $(-\infty, r_1) \times \partial\mathbb{S}_+^{N-1}$.

Next we suppose that $a_{k_1-} = 0$. Then for $k = k_1$, (4.19) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1+}} + m_2 - \frac{1}{a_{k_1+}} \int_t^0 (e^{(t-s)a_{k_1+}} - 1) F_{k_1,j}(s) ds. \quad (4.22)$$

This implies that

$$w_{k_1,j}(t) \rightarrow m_2 + \frac{1}{a_{k_1+}} \int_{-\infty}^0 F_{k_1,j}(s) ds := A_{k_1,j} \quad \text{as } t \rightarrow \infty.$$

If $A_{k_1,j} \neq 0$ it would imply that $\sum_{j=1}^{j_k} A_{k_1,j} \phi_{k_1,j}$ is a nonzero eigenfunction of order $k_1 > 1$, hence it changes sign and it would imply that v changes sign at $-\infty$ (notice that all the other terms $w_{k,j}(t)$ tends to 0 exponentially because of (4.19)-(4.20)). Hence $A_{k_1,j} = 0$ and (4.22) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1,+}} - \frac{1}{a_{k,+}} \int_t^0 e^{(t-s)a_{k,+}} F_{k_1,j}(s) ds - \int_{-\infty}^t F_{k_1,j}(s) ds.$$

Because

$$\int_{-\infty}^t F_{k_1,j}(s) ds = O(e^t) \quad \text{as } t \rightarrow \infty,$$

we conclude that for $k = k_1$, there holds

$$|w_{k_1,j}(t)| \leq c_{27} \begin{cases} |t|e^t & \text{if } a_{k_1,+} = 1, \\ \max\{e^t, e^{ta_{k_1,+}}\} & \text{if } a_{k_1,+} \neq 1 \end{cases}$$

and finally we infer (4.21), which complete the proof. \square

Proof of Theorem A. Assume that $u \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\Omega \setminus \{0\}$.

Case 1: $\mu > \mu_1$. We claim (4.2) holds for some $c_{13} \geq 0$.

By Lemma 4.1, (4.1) holds for some $a > 0$. If $a < -\alpha_-$, then (4.2) holds with $c_{13} = 0$. If $a = -\alpha_-$, then (4.2) holds by Proposition 4.2-(i). Hence we are left with the case $a > -\alpha_-$. As in the proof of Proposition 4.2 we define k_1 and k_2 . By replacing a by $a' = a + \epsilon$, we can assume that $a_{k_2,+} \neq 1$ and $a_{k_1,-} \neq 1$, to avoid the resonance complication in (4.16) and (4.21), hence

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{27} (e^{ta_{k_2,+}} + e^{ta_{k_1,-}} + e^t).$$

Furthermore $k_2 = k_1 + 1$ and

$$a_{k_2,+} - a_{k_1,-} = \frac{1}{2} \left(\sqrt{4\mu + 4\lambda_{k_1+1} + (N-2)^2} + \sqrt{4\mu + 4\lambda_{k_1} + (N-2)^2} \right) > 0,$$

which yields

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{28} (e^{ta_{k_1,-}} + e^t).$$

This implies that u satisfies

$$u(x) \leq c_{29} (|x|^{\alpha_{k_1,-}} + |x|^{1-a}) \rho(x).$$

We iterate this procedure up to obtain

$$u(x) \leq c_{30} |x|^{\alpha_-} \rho(x)$$

and we conclude as in the proof of Proposition 4.2, Step 3. \square

Case 2: $\mu = \mu_1$. In this case, the difficulty comes from the fact that there is no dissipation of energy in (4.9) for $a = -\alpha_- = \frac{N-2}{2}$. But from the above iterative procedure in the *Case 1*, we could obtain that for some $\delta \in (0, 1)$,

$$u(x) \leq c_{31}|x|^{-\frac{N-2}{2}-\delta}\rho(x).$$

We finally show that there exists $c_{32} \geq 0$ such that

$$\lim_{r \rightarrow 0} r^{\frac{N-2}{2}} \frac{u(r, \cdot)}{\ln r} = -c_{32}\psi_1(\cdot) \quad (4.23)$$

in $C^1(\mathbb{S}_+^{N-1})$ and

$$\lim_{r \rightarrow 0} r^{\frac{N}{2}} \frac{u_r(r, \cdot)}{\ln r} = \frac{(N-2)c_{32}}{2}\psi_1(\cdot) \quad (4.24)$$

uniformly in \mathbb{S}_+^{N-1} .

Note that (4.7) reduces that

$$\begin{aligned} (1 + \epsilon_1(t))v_{tt} + \epsilon_2(t)v_t + (N-1 + \epsilon_3(t))v + \Delta'v \\ + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned}$$

in $(-\infty, r_0) \times \mathbb{S}_+^{N-1}$, vanishes on $(-\infty, r_0) \times \partial\mathbb{S}_+^{N-1}$ and the ϵ_j verify (4.8), and

$$v(t, \sigma) \leq c_{33}e^{-\delta t}.$$

Since the operator involved in the equation is uniformly elliptic we have by standard regularity theory

$$\begin{aligned} \|v\|_{C^{2,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_t\|_{C^{1,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_{tt}\|_{C^\delta([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} \\ \leq c_{34}\|v\|_{L^\infty((T-2, T+2) \times \mathbb{S}_+^{N-1})} \\ \leq c_{35}e^{-\delta T} \end{aligned}$$

for any $T \leq r_0 + 3$. We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS,$$

then

$$X''(t) + F(t) = 0 \quad (4.25)$$

where

$$F(t) = \int_{\mathbb{S}_+^{N-1}} (\epsilon_1 v_{tt} + \epsilon_2 v_t + \epsilon_3 v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle) \psi_1 dS.$$

Hence

$$|F(t)| \leq c_{36}e^{(1-\delta)t}. \quad (4.26)$$

This implies that $X'(t)$ admits a limit $c_{37} \leq 0$ when $t \rightarrow -\infty$ and

$$\lim_{t \rightarrow -\infty} t^{-1}X(t) = c_{37}.$$

Set

$$W_2 = \bigoplus_{k \geq 2} \ker(\Delta' + \lambda_k Id),$$

and denote by v_2 the orthogonal projection of v onto W_2 . Then

$$v_2 tt + (N - 1)v_2 + \Delta' v_2 = F_2(t, \cdot), \quad (4.27)$$

where

$$|F_2(t, \cdot)| \leq c_{38}e^{(1-\delta)t}.$$

Since $\lambda_2 = 2N$, the function $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$ satisfies in $\mathcal{D}'(-\infty, r_1)$

$$Y'' - (N + 1)Y \geq -c_{38}e^{(1-\delta)t}.$$

Because $Y(t) = o(e^{-\sqrt{N+1}t})$ when $t \rightarrow -\infty$, it follows by the maximum principle that $Y(t) = O(e^{\sqrt{N+1}t} + e^{(1-\delta)t}) = O(e^{(1-\delta)t})$. Using again the standard regularity estimates for elliptic equations, we derive

$$\|v_2(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} + \|v_{2t}(t, \cdot)\|_{C(\overline{\mathbb{S}_+^{N-1}})} \leq c_{39}e^{(1-\delta)t}. \quad (4.28)$$

Combining (4.25) and (4.27) we derive (4.23). Since $v(t, \cdot) = X(t)\psi_1 + v_2(t, \cdot)$ it follows from (4.28) that

$$\lim_{t \rightarrow -\infty} v_t(t, \cdot) = c_{37}\psi_1 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

Thus, the identity $u_r(r, \cdot) = r^{-\frac{N}{2}} \left(\frac{2-N}{2}v(t, \cdot) + v_t(t, \cdot) \right)$ implies (4.23) and (4.24). \square

4.2 Existence and uniqueness

Proof of Theorem B. We still assume that Ω satisfies the condition (C-1) and $\partial\mathbb{R}_+^N$ is tangent to $\partial\Omega$ at 0. For $\epsilon > 0$ let u_ϵ be the solution of

$$\begin{cases} \mathcal{L}_\mu u_\epsilon = 0 & \text{in } \Omega_\epsilon := \Omega \setminus \overline{B}_\epsilon, \\ u_\epsilon = 0 & \text{on } \partial\Omega \cap \overline{B}_\epsilon^c, \\ u_\epsilon = \phi_\mu & \text{on } \Omega \cap \partial B_\epsilon. \end{cases} \quad (4.29)$$

Since $\Omega \subset \mathbb{R}_+^N$, $u_\epsilon \leq \phi_\mu$ in Ω_ϵ and

$$\frac{\partial u_\epsilon}{\partial \mathbf{n}} \Big|_{\Omega \cap \partial B_\epsilon} \leq \frac{\partial \phi_\mu}{\partial \mathbf{n}} \Big|_{\Omega \cap \partial B_\epsilon} < 0, \quad (4.30)$$

where $\mathbf{n} = \epsilon^{-1}\mathbf{x}$. Furthermore, if $0 < \epsilon' < \epsilon$, $u_{\epsilon'}|_{\Omega \cap \partial B_\epsilon} \leq u_\epsilon|_{\Omega \cap \partial B_\epsilon} = \phi_\mu$, hence $u_{\epsilon'} \leq u_\epsilon$ in Ω_ϵ . There exists $u_0 = \lim_{\epsilon \rightarrow 0} u_\epsilon$ and u_0 is a nonnegative solution of $\mathcal{L}_\mu u = 0$ in Ω which vanishes on $\partial\Omega \setminus \{0\}$ and is smaller than ϕ_μ .

Let $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta > 0$, then, with $\mathbf{n}' = -\frac{x}{|x|} = -\mathbf{n}$,

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu u_\epsilon dx \\ &= \int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\partial B_\epsilon \cap \Omega} \left(-\frac{\partial u_\epsilon}{\partial \mathbf{n}'} \zeta \gamma_\mu^\Omega + \left(\zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}'} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}'} \right) u_\epsilon \right) dS. \end{aligned}$$

Using (4.29) and (4.30) we obtain

$$\int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega - \left(\zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS.$$

We take $\zeta = 1$, hence $\mathcal{L}_\mu^* \zeta = \ell_\mu^\Omega$ and we get

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case $\mu > \mu_1$, and

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left(\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq \left(\frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case $\mu = \mu_1$. Since $u_\epsilon \leq \phi_\mu^\Omega$,

$$u_\epsilon \gamma_\mu^\Omega \leq \gamma_\mu^\Omega \phi_\mu^\Omega = r^{2-N} \psi_1^2 \in L^1(\Omega).$$

Therefore, by dominated convergence theorem, we conclude that

$$\ell_\mu^\Omega \int_{\Omega} u_0 d\gamma_\mu^\Omega \geq \begin{cases} 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1. \end{cases} \quad (4.31)$$

We infer that the function u_0 is nonzero. It is a positive solution of $\mathcal{L}_\mu u_0 = 0$ in Ω which vanishes on $\partial\Omega \setminus \{0\}$. It follows from **Theorem A** that there exists $k \geq 0$ such that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{u(x)}{\rho(x)|x|^{\alpha-1}} &= k & \text{if } \mu > \mu_1, \\ \lim_{x \rightarrow 0} \frac{u(x)}{\rho(x)|x|^{-N/2} \ln|x|} &= k & \text{if } \mu = \mu_1. \end{aligned}$$

Next we next show that $k = 1$. In fact, if $k < 1$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$

$$u_\epsilon \leq \frac{k+1}{2} \phi_\mu^\Omega,$$

and then

$$\lim_{\epsilon \rightarrow 0^+} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega \leq \frac{k+1}{2} \ell_\mu^\Omega \int_{\Omega_\epsilon} \phi_\mu^\Omega d\gamma_\mu^\Omega < \begin{cases} 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1\right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1, \end{cases}$$

which contradicts (4.31). Thus, (1.11) and (1.12) hold true. \square

Proof of Corollary C. *Identity (1.14).* As a consequence of Proposition 4.2, for any $\zeta \in \mathbb{X}_\mu(\Omega)$ and $\epsilon > 0$ we set $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$, and there holds

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu \phi_\mu^\Omega dx \\ &= \int_{\Omega_\epsilon} \phi_\mu^\Omega \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\Omega \cap \partial B_\epsilon} \left(-\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left(\gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS. \end{aligned}$$

Using **Proposition A.1** we have

$$\int_{\Omega \cap \partial B_\epsilon} \left(-\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left(\gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS = -\zeta(0)A(\epsilon)(1 + o(1)),$$

where $A(\epsilon)$ is defined in (2.5).

The uniqueness follows direct from Kato's inequality (3.4). \square

5 The Dirichlet problem

Proof of Theorem D. Note that in section §3.2 for $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu)$, problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting $\mathbb{K}_\mu^\Omega(\lambda)$, which verifies the identity

$$\int_{\Omega} \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Moreover, problem

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting $\mathbb{G}_\mu^\Omega(\nu)$, which verifies the identity

$$\int_{\Omega} \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta d\gamma_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Together with **Corollary C** and the linearity of operator \mathcal{L}_μ , we have that $\mathbb{K}_\mu^\Omega(\lambda) + \mathbb{G}_\mu^\Omega(\nu) + k\phi_\Omega^\mu$ is a weak solution of (1.21) satisfying (1.23) and the uniqueness follows directly from Kato's inequality (3.4). \square

Our final part is to classify the boundary data for nonnegative \mathcal{L}_μ -harmonic function.

Proof of Theorem E. Let Ω be a bounded C^2 domain and u be a nonnegative \mathcal{L}_μ -harmonic function in Ω . We now show that there exists a nonnegative measure λ on $\partial\Omega \setminus \{0\}$ and $k \geq 0$ such that

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega. \quad (5.1)$$

For $\epsilon > 0$ the term $\mu|x|^{-2}$ is bounded in $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$. Hence there exists a nonnegative Radon measure λ_ϵ such that u is the unique solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega_\epsilon, \\ u = \lambda_\epsilon & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Furthermore λ_ϵ is the boundary trace is achieved in *dynamical sense*, see [14] and references therein. Hence for any $\zeta \in C(\overline{\Omega})$ vanishing on B_ϵ , there holds

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \zeta dS = \int_{\partial\Omega \cap B_\epsilon^c} \zeta d\lambda_\epsilon,$$

where $\Sigma_\delta = \{x \in \Omega : \rho(x) = \delta\}$. If we write

$$\lambda_\epsilon = \lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c} + u|_{\Omega \cap \partial B_\epsilon},$$

it proves that for $0 < \epsilon' < \epsilon$, $\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c} = \lambda_{\epsilon'}|_{\partial\Omega \cap B_\epsilon^c}$. This defines in a unique way a nonnegative Radon λ on $\partial\Omega \setminus \{0\}$ measure such that (5.1) holds for all $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing near 0. Furthermore $\rho u \in L^1(\Omega_\epsilon)$ for any $\epsilon > 0$. Denote by $\mathbb{K}_\mu^{\Omega_\epsilon}$ the Poisson potential of \mathcal{L}_μ in Ω_ϵ . Then

$$u|_{\Omega_\epsilon} = \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}] + \mathbb{K}_\mu^{\Omega_\epsilon}[u|_{\Omega \cap \partial B_\epsilon}].$$

For $0 < \epsilon' < \epsilon$, one has that $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega \cap B_{\epsilon'}^c}]|_{\Omega \cap \partial B_\epsilon} \geq 0$. Therefore $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega \cap B_{\epsilon'}^c}] \geq \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}]$ in Ω_ϵ . Hence

$$\lim_{\epsilon \rightarrow 0} \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}] = \mathbb{K}_\mu^\Omega[\lambda] \leq u \quad \text{in } \Omega.$$

Next we aim to characterize the behaviour at 0. By contradiction we assume that

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = \infty.$$

Then for any $m > 0$ there exists a sequence $\{\epsilon_{m,k}\} \subset \mathbb{R}_+^*$ tending to 0 and a sequence $\{\ell_{m,k}\} \subset \mathbb{R}_+^*$ tending to ∞ such that

$$\int_{\Omega \cap \partial B_{\epsilon_k}} \min\{u, \ell_{m,k}\} d\beta_\mu^\Omega = m.$$

Set $\tau_{m,k} = \min\{u|_{\Omega \cap \partial B_{\epsilon_k}}, \ell_{m,k}\}$ and set $u_{m,k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\tau_{m,k} \chi_{\Omega \cap \partial B_{\epsilon_k}}]$. Then

$$u_{m,k} \leq u \quad \text{in } \Omega_{\epsilon_k},$$

and we recall that

$$\int_{\Omega \cap \partial B_{\epsilon_k}} \phi_\mu^\Omega d\beta_\mu^\Omega = c_\mu(1 + o(1)),$$

where c_μ is the constant defined in (2.4). Combining the boundary Harnack inequality with the standard Harnack inequality, one infers

$$c_{47} \frac{\rho(x)}{\rho(y)} \leq c_{46} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq \frac{u_{m,k}(x)}{u_{m,k}(y)} \leq c_{44} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq c_{45} \frac{\rho(x)}{\rho(y)} \quad (5.2)$$

for all $x, y \in \Omega$ such that $|x| = |y| \geq 2\epsilon_k$. If we set $\dot{\phi}_\mu^\Omega(x) = \frac{\phi_\mu^\Omega(x)}{\rho(x)}$ and $\dot{u}_{m,k}(x) = \frac{u_{m,k}(x)}{\rho(x)}$, then (5.2) becomes

$$c_{47} \leq c_{46} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq \frac{\dot{u}_{m,k}(x)}{\dot{u}_{m,k}(y)} \leq c_{44} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq c_{45}. \quad (5.3)$$

Assume for a while that we have proved that there exists $\theta > 0$, independent of m and k such that for any

$$\int_{\partial B_{2\epsilon_k} \cap \Omega} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) = \theta m. \quad (5.4)$$

If we assume that for $\delta \leq 2\epsilon_{k_0}$

$$2c_\mu \geq \int_{\Omega \cap \partial B_\delta} \dot{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega) \geq \frac{c_\mu}{2},$$

one has for $k \geq k_0$,

$$\int_{\Omega \cap \partial B_{2\epsilon_k}} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta m \geq \frac{\theta m}{2c_\mu} \int_{\Omega \cap \partial B_{2\epsilon_k}} \dot{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega)$$

Since

$$\dot{\phi}_\mu^\Omega(x) \leq \frac{c_{45}}{c_{46}} \dot{\phi}_\mu^\Omega(y)$$

and

$$\dot{u}_{m,k}(x) \geq c_{47} \dot{u}_{m,k}(y),$$

we derive

$$\frac{1}{c_{47}} \dot{u}_{m,k}(x) \int_{\partial B_{2\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) \geq \frac{\theta m c_{46}}{2c_\mu c_{45}} \dot{\phi}_\mu^\Omega(x) \int_{\Omega \cap \partial B_{2\epsilon_k}} d(\rho\beta_\mu^\Omega).$$

Therefore

$$u_{m,k}(x) \geq c_{48}m\phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega \text{ s.t. } |x| = 2\epsilon_k,$$

and $c_{48} > 0$ is independent of m and ϵ_k . This implies by the maximum principle and letting $\epsilon_k \rightarrow 0$

$$u(x) \geq u_{m,k}(x) \geq c_{48}m\phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega. \quad (5.5)$$

Since m is arbitrary we obtain a contradiction. Hence there holds

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = m_u < \infty. \quad (5.6)$$

Then inequality (5.5) holds without truncation with m replaced by m_u . We recall that

$$w_\epsilon := \mathbb{K}_\mu^{\Omega_\epsilon} [u|_{\Omega \cap \partial B_\epsilon}] = u|_{\Omega_\epsilon} - \mathbb{K}_\mu^{\Omega_\epsilon} [\lambda_\epsilon|_{\partial \Omega \cap B_\epsilon}] \quad \text{in } \Omega_\epsilon. \quad (5.7)$$

Case 1: We first assume that $m_u > 0$. Then (5.4) combined with the maximum principle yields

$$\int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \int_{\partial B_{2\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) = \theta m_u (1 + o(1))$$

with $\dot{w}_\epsilon = \rho^{-1}w_\epsilon$. Inequality (5.3) is replaced by

$$c_{47} \leq c_{46} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq \frac{\dot{w}_\epsilon(x)}{\dot{w}_\epsilon(y)} \leq c_{44} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq c_{45} \quad \text{in } \Omega_{2\epsilon}. \quad (5.8)$$

Therefore, for ϵ_k small enough and $|x| = 2\epsilon_k$,

$$\begin{aligned} \dot{w}_\epsilon(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\leq c_{45} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \leq 2c_{45}m_u \\ &\leq \frac{4c_{45}m_u}{c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \dot{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega), \end{aligned}$$

which implies

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.9)$$

Hence

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.10)$$

Since w_{ϵ_k} and ϕ_μ are \mathcal{L}_μ -harmonic in $\Omega_{2\epsilon_k}$, and vanishes on $\partial \Omega \cap B_{2\epsilon_k}$ it follows that inequality (5.10) also holds for any $x \in \Omega_{2\epsilon_k}$. By definition $w_{\epsilon_k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}} [u|_{\Omega \cap \partial B_{\epsilon_k}}]$, hence

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}} [u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \leq c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.11)$$

Next we obtain the estimate from below. From (5.8), with $|x| = 2\epsilon_k$,

$$\begin{aligned} \dot{w}_{\epsilon_k}(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\geq c_{47} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}m_u}{2} \\ &\geq \frac{c_{47}m_u}{4c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \dot{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega). \end{aligned}$$

Hence

$$w_{\epsilon_k}(x) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \phi_\mu^\Omega(x) := c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

It follows that

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \geq c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.12)$$

From (5.7), (5.12) and (5.11) we infer

$$c_{50}m_u \phi_\mu^\Omega \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^\epsilon}] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}. \quad (5.13)$$

This implies, by letting $\epsilon_k \rightarrow 0$,

$$c_{50}m_u \phi_\mu^\Omega \leq u - \mathbb{K}_\mu^\Omega[\lambda] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega.$$

Therefore, the function $u - \mathbb{K}_\mu^\Omega[\lambda]$ is \mathcal{L}_μ -harmonic and positive in Ω and it vanishes on $\partial\Omega$. By Corollary C, it implies that it coincides with $c\phi_\mu^\Omega$ for some $c \geq 0$ (and in that case $c_{50}m_u \leq c \leq c_{49}m_u$).

Case 2: Assume $m_u = 0$. Following the same inequalities as in Case 1, (5.9) is replaced by: for any $\delta > 0$ there exists $k_0 > 0$ such that for $k \geq k_0$,

$$w_{\epsilon_k}(x) \leq \delta \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

Hence (5.13) is transformed into

$$0 \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^\epsilon}] \leq \delta \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}.$$

Letting successively $\epsilon_k \rightarrow 0$ and $\delta \rightarrow 0$ yields $u - \mathbb{K}_\mu^\Omega[\lambda] = 0$ in Ω , which ends the proof. \square

Appendix: Estimates (1.9) and (1.16)

Proposition A.1 *Assume Ω is a bounded C^2 domain such that $0 \in \partial\Omega$ satisfying condition (C-1) and let γ_μ^Ω be defined by (1.8) and normalized by $\|\gamma_\mu^\Omega\|_{L^2(\Omega)} = 1$. Then*

$$\lim_{r \rightarrow 0} r^{1-\alpha+\gamma_\mu^\Omega}(r, \cdot) = c_1 \psi_1 \quad \text{in } C_{loc}^1(\mathbb{S}_+^{N-1})$$

and

$$\lim_{r \rightarrow 0} r^{-\alpha+\gamma_\mu^\Omega}(r, \cdot) = c_1 \left(1 - \frac{N}{2}\right) \psi_1 \quad \text{locally uniformly in } \mathbb{S}_+^{N-1}.$$

Proof. Since $\alpha_+ + (N - 2)\alpha_+ - \mu + 1 - N = 0$, the function $x \mapsto w(x) := |x|^{\alpha_+}$ satisfies

$$\tilde{\mathcal{L}}_\mu w(x) := \mathcal{L}_\mu w - \ell_\mu^\Omega w = |x|^{\alpha_+ - 2} (N - 1 - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore, $\nabla w \in L_{loc}^2(\mathbb{R}^N)$. Let $R_0 > 0$ such that $N - 1 \geq \ell_\mu^\Omega R_0^2$ and $m > 0$ such that $mw \geq \gamma_\mu^\Omega$ on $\Omega \cap B_{R_0}^c$. Then the function $(\gamma_\mu^\Omega - mw)_+$ belongs to $H_\mu(\Omega)$ and satisfies $\tilde{\mathcal{L}}_\mu(\gamma_\mu^\Omega - mw)_+ \leq 0$ in the dual of $H_\mu(\Omega)$. Hence

$$\int_\Omega \left(|\nabla(\gamma_\mu^\Omega - mw)_+|^2 + \left(\frac{\mu}{|x|^2} - \ell_\mu^\Omega \right) (\gamma_\mu^\Omega - mw)_+^2 \right) dx \leq 0.$$

Therefore $(\gamma_\mu^\Omega - mw)_+ \leq 0$, which implies that

$$0 < \gamma_\mu^\Omega(x) \leq m|x|^{\alpha_+} \quad \text{for all } x \in \Omega.$$

Then we proceed as in the proof of Proposition 4.2. We flatten the boundary near 0 and set

$$v(t, \sigma) = r^{-\alpha_+} \tilde{\gamma}_\mu^\Omega(r, \sigma) \quad \text{with } t = \ln r,$$

where the function $\tilde{\gamma}_\mu^\Omega$ is defined similarly as \tilde{u} in (4.3). Then v is bounded in $(-\infty, T_0] \times \mathbb{S}_+^{N-1}$ where it satisfies

$$\begin{aligned} (1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 + 2\alpha_+ + \epsilon_2(t, \cdot))v_t + (\alpha_+(\alpha_+ + N - 2) - \mu + \epsilon_3(t, \cdot) + e^{2t}\ell_\mu^\Omega)v \\ + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned}$$

instead of (4.7). It vanishes on $(-\infty, T_0] \times \partial\mathbb{S}_+^{N-1}$ and the ϵ_j satisfy again (4.8).

Case 1: $\mu > \mu_1$. The energy method used in proof of Proposition 4.2 applies with no modification and we infer that there exists $c_{51} \geq 0$ such that

$$v(t, \cdot) \rightarrow c_{51}\psi_1 \quad \text{as } t \rightarrow -\infty$$

in $C^1(\mathbb{S}_+^{N-1})$ and $v_t(t, \cdot) \rightarrow 0$ uniformly in \mathbb{S}_+^{N-1} . If $c_{51} = 0$, we can prove, as in Proposition 4.2-(ii) that there exists $\tau > 0$ such that

$$\gamma_\mu^\Omega(x) \leq c_{52}|x|^{\alpha_+ + \tau} \quad \text{for all } x \in \Omega. \quad (5.14)$$

Iterating this process, we infer that (5.14) holds for any $\tau > 0$. For $k > 1$, let $\alpha_{k,+}$ be the positive root of (2.1) and put $w_k(x) = |x|^{\alpha_{k,+}}$. Then

$$\tilde{\mathcal{L}}_\mu w_k(x) = |x|^{\alpha_{k,+} - 2} (\lambda_k - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since $\lambda_k \rightarrow \infty$, as $k \rightarrow \infty$, we choose k such that $\lambda_k > \ell_\mu^\Omega (\text{diam}(\Omega))^2$. Hence w_k is a supersolution of $\tilde{\mathcal{L}}_\mu$. Because $\gamma_\mu^\Omega(x) = o(w_k(x))$ near $x = 0$, it follows that $\gamma_\mu^\Omega(x) \leq \epsilon o(w_k(x))$ in Ω for any $x \in \Omega$. Hence $\gamma_\mu^\Omega = 0$, which is a contradiction. Finally it implies that $c_{51} > 0$, which yields (1.9)-(i). Because the convergence holds in $C^1(\mathbb{S}_+^{N-1})$ and $v_t(t, \cdot) \rightarrow 0$, we infer

$$\lim_{r \rightarrow 0} r^{1 - \alpha_+} \nabla \tilde{\gamma}_\mu^\Omega(r, \cdot) = c_{51} (\alpha_+ \psi_1 \mathbf{e} + \nabla' \psi_1)$$

where $\mathbf{e} = \frac{x}{|x|}$. This implies the claim.

Case 2: $\mu = \mu_1$. Set $v(t, \cdot) = r^{\frac{N}{2}-1}u(r, \cdot)$ with $t = \ln r$ and $X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS$ and obtain again (4.25), where $F(t, \cdot)$ satisfies (4.26). Since $X'(t) \rightarrow 0$ and X is bounded, it follows that $X(t)$ admits a limit $c_{52} \geq 0$ when $t \rightarrow -\infty$. As in the proof of **Theorem A**, we infer that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c_{52} \psi_1 \quad \text{in } C^1(\mathbb{S}_+^{N-1}) \quad \text{and} \quad \lim_{t \rightarrow -\infty} v_t(t, \cdot) = 0 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

If $c_{52} = 0$ we derive a contradiction as in the first case. \square

Proof of (1.16). Since $\rho^* \leq \frac{1}{i_\mu^\Omega}$, then comparison principle implies that $\sigma_\mu^\Omega \geq \gamma_\mu^\Omega$ in Ω . Next we show $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$ in Ω . In fact, we only have to show this inequality holds in a neighborhood of the origin.

Case 1: the boundary is flat at the origin. We first prove above inequality when Ω is flat in a neighborhood of the origin, i.e. $B'_R \times [0, R) \subset \Omega \subset \mathbb{R}_+^N$ for some $R > 0$.

For $\tau \in \mathbb{R}$, denote

$$w_\tau(x) = |x|^\tau x_N \quad \text{and} \quad \tilde{w}_\tau(x) = |x|^\tau x_N^2 \quad \text{in } \mathbb{R}_+^N,$$

and direct calculation shows that

$$\mathcal{L}_\mu w_\tau(x) = [\mu - \tau(\tau + N)]|x|^{\tau-2} x_N, \quad \mathcal{L}_\mu \tilde{w}_\tau(x) = [\mu - \tau(\tau + N + 2)]|x|^{\tau-2} x_N^2 - 2|x|^\tau \quad \text{for } x \in \mathbb{R}_+^N.$$

Let

$$\bar{u}(x) = \begin{cases} w_{\alpha_+ - 1} - \frac{1}{2} \tilde{w}_{\alpha_+ - 1} & \text{if } \alpha_+ \geq 0, \\ w_{\alpha_+ - 1} - \frac{N-2}{2(N+2)} w_{\alpha_+} - \frac{2}{N+2} \tilde{w}_{\alpha_+ - 1} & \text{if } \alpha_+ < 0, \end{cases}$$

by resetting $R \in (0, 1]$ such that $\bar{u} > 0$ in $\mathbb{R}^{N-1} \times (0, R]$.

When $\alpha_+ \geq 0$, we have that $\mu - (\alpha_+ - 1)(\alpha_+ - 1 + N) = 0$ and

$$\mathcal{L}_\mu \bar{u}(x) = \alpha_+ |x|^{\alpha_+ - 3} x_N^2 + |x|^{\alpha_+ - 1} \geq |x|^{\alpha_+ - 1},$$

thus there exists $t_1 > 0$ such that $t_1 \bar{u} \geq \sigma_\mu^\Omega$ on $\Omega \cap (\mathbb{R}^{N-1} \times \{R\})$ and

$$\mathcal{L}_\mu(t_1 \bar{u})(x) \geq t_1 |x|^{\alpha_+ - 1} \geq \mathcal{L}_\mu \sigma_\mu^\Omega(x) \quad \text{in } \Omega \cap (\mathbb{R}^{N-1} \times (0, R))$$

By comparison principle, we have that

$$\sigma_\mu^\Omega \leq t_1 \bar{u}$$

which, together with the inequality $\bar{u} \leq 2t_1 w_{\alpha_+ - 1}$, implies that $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$.

When $\alpha_+ \in [\frac{2-N}{2}, 0)$ if $N \geq 3$,

$$\begin{aligned} \mathcal{L}_\mu \bar{u}(x) &= \frac{N-2}{2(N+2)} (2\alpha_+ + 2 + N) |x|^{\alpha_+ - 2} x_N + \frac{4}{N+2} \alpha_+ |x|^{\alpha_+ - 3} x_N^2 + \frac{4}{N+2} |x|^{\alpha_+ - 1} \\ &\geq \left(\frac{N-2}{2(N+2)} (2\alpha_+ + 2 + N) - \frac{4\alpha_+}{N+2} \right) |x|^{\alpha_+ - 2} x_N + \frac{4}{N+2} |x|^{\alpha_+ - 1} \\ &\geq \frac{4}{N+2} |x|^{\alpha_+ - 1}. \end{aligned}$$

The remaining of the proof is similar to the previous one and we omit it.

Case 2: the boundary is not flat at origin.

We define the function $\Theta = (\Theta_1, \dots, \Theta_N)$ on D_R by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$ we can assume that Θ is a diffeomorphism from D_R onto $\Theta(D_R)$. We set

$$u_1(x) = \bar{u}(y) \quad \text{for all } y \in D_R^+ = B'_R \times [0, R).$$

Then by (4.4) and (4.5), we have that

$$(-\Delta u_1(x) + \frac{\mu}{|x|^2})u_1(x) = (-\Delta \bar{u}(y) + \frac{\mu}{|y|^2}\bar{u}(y)) + O(|y|) \left(|D^2 \bar{u}(y)| + \frac{\mu}{|y|^2} \bar{u}(y) \right)$$

Then by resetting $R > 0$ small and the calculation in *Case 1*, we have that

$$\mathcal{L}_\mu u_1(x) \geq c_{53}|x|^{\alpha_+-1}, \quad \forall x \in \Theta^{-1}(D_R^+).$$

By Hopf's Lemma, there exists $t_2 > 0$ such that $t_2 u_1 \geq \sigma_\mu^\Omega$ on $\Theta^{-1}(\partial B'_R \times [0, R))$ and by compactness of $\Theta^{-1}(\bar{B}'_R \times \{R\})$, there exists $t_3 > 0$ such that $t_3 u_1 \geq \sigma_\mu^\Omega$ on $\Theta^{-1}(\bar{B}'_R \times \{R\})$. Applying comparison principle, for some $t_4 \geq \max\{t_2, t_3\}$, we have that

$$\sigma_\mu^\Omega \leq t_4 \bar{u} \quad \text{in } \Theta^{-1}(D_R^+)$$

and we have $\sigma_\mu^\Omega \leq c_2 \gamma_\mu^\Omega$ near the origin. □

Proposition A.2 *Under the assumption of Proposition A.1 there exists $c_{53} > 0$ such that*

$$\lim_{r \rightarrow 0} r^{1-\alpha_+} \sigma_\mu^\Omega(r, \cdot) = c_{53} \psi_1 \quad \text{in } C_{loc}^1(\mathbb{S}_+^{N-1}) \quad (5.15)$$

and

$$\lim_{r \rightarrow 0} r^{-\alpha_+} \sigma_{\mu r}^\Omega(r, \cdot) = c_{53} \left(1 - \frac{N}{2}\right) \psi_1 \quad \text{locally uniformly in } \mathbb{S}_+^{N-1}. \quad (5.16)$$

Proof. We follow the proof of Proposition A.1, flattening the boundary near 0 and defining a new function $\tilde{\sigma}_\mu^\Omega$ as previously. By (1.16) the function

$$v(t, \sigma) = r^{-\alpha_+} \tilde{\sigma}_\mu^\Omega(r, \sigma) \quad \text{with } t = \ln r,$$

is bounded and it satisfies

$$\begin{aligned} (1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 + 2\alpha_+ + \epsilon_2(t, \cdot))v_t + (\alpha_+(\alpha_+ + N - 2) - \mu + \epsilon_3(t, \cdot))v \\ + \Delta' v + \langle \nabla' v, \epsilon_4(t, \cdot) \rangle + \langle \nabla' v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = e^t m(t, \cdot), \end{aligned}$$

instead of (4.7), where the function m is bounded as well as its gradient. Then v satisfies the same bounds as the ones in the proof of Proposition 4.2. The only difference is that the energy

estimate (4.9) is replaced by

$$\begin{aligned}
& \int_{\mathbb{S}_+^{N-1}} \left(N - 2 - 2\alpha_+ + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right) v_t^2 dS - \frac{1}{2} \int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\
&= \frac{d}{dt} \left[\int_{\mathbb{S}_+^{N-1}} \left(\frac{1}{2} |\nabla v|^2 - \frac{1}{2} [\alpha_+ (\alpha_+ + 2 - N) - \mu + \epsilon_3] v^2 - \frac{1}{2} (1 + \epsilon_1) v_t^2 \right) dS \right] \\
& - \int_{\mathbb{S}_+^{N-1}} (\langle \nabla' v, \epsilon_4 \rangle + \langle \nabla' v_t, \epsilon_5 \rangle + \langle \nabla' (\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6 \rangle) v_t^2 dS + e^t \int_{\mathbb{S}_+^{N-1}} m(t, \cdot) v(t, \cdot) dS.
\end{aligned} \tag{5.17}$$

Therefore, if $2\alpha_+ \neq N - 2$, we conclude that (4.10) holds, and (4.11) follows. We infer (5.15) and (5.16) as in the proof of Proposition 4.2. When $2\alpha_+ = N - 2$ the proof of (4.10) and (4.10) in the case $2\alpha_+ = N - 2$ is carried out as in the proof of Proposition 4.2-Step 3. \square

Acknowledgements H. Chen is supported by NSF of China, No: 11726614, 11661045, by the Jiangxi Provincial Natural Science Foundation, No: 20161ACB20007, and by the Alexander von Humboldt Foundation.

References

- [1] P. Bauman. Positive solutions of elliptic equations in nondivergence form and their adjoints, *Ark. Mat.* **22** (1984), 153-173.
- [2] G. Barbatis, S. Filippas, A. Tertikas. Sharp Hardy and Hardy-Sobolev inequalities with point singularities on the boundary, *J. Math. Pures Appl.* **117** (2018), 146-184.
- [3] M.F. Bidaut-Véron, R. Borghol, L. Véron. Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations, *Calc. Var. Part. Diff. Eq.* **27(2)** (2006), 159-177.
- [4] H. Brezis, J. L. Vazquez. Blow-up solutions of some nonlinear elliptic equations, *Rev. Mat. Complut.* **10** (1997) 443-469.
- [5] C. Cazacu. On Hardy inequalities with singularities on the boundary, *C. R. Math. Acad. Sci. Paris* **349** (2011) 273-277.
- [6] C. Cazacu. Hardy inequality and Pohozaev identity for operators with boundary singularities: Some applications. *C. R. Math. Acad. Sci. Paris* **349** (2011) 1167-1172.
- [7] C. Cazacu. *Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results*, *J. Funct. Anal.* **263** (2012), 3741-3782.
- [8] H. Chen, A. Quaas, F. Zhou. On nonhomogeneous elliptic equations with the Hardy- Leray potentials, arXiv:1705.08047.
- [9] H. Chen, L. Véron. Weak solutions of semilinear elliptic equations with Leray-Hardy potential and measure data, *Mathematics in Engineering* **1** (2019), 391-418.

- [10] H. Chen, L. Véron. Semilinear elliptic equations with Leray-Hardy potential singular on the boundary and measure data. *In preparation*.
- [11] S. Filippas, A. Tertikas and J. Tidblom. On the structure of Hardy-Sobolev-Mazya inequalities. *J. Eur. Math. Soc.* **11** (2009), 1165-1185.
- [12] A. Gmira, L. Véron. Boundary singularities of solutions of nonlinear elliptic equations, *Duke Math. J.* **64** (1991), 271-324.
- [13] R. A. Hunt and R. L. Wheeden. Positive harmonic functions on Lipschitz domains, *Trans. Amer. Math. Soc.* **147** (1970), 507-527.
- [14] M. Marcus, L. Véron, *Nonlinear Second Order Elliptic Equations Involving Measures*, De Gruyter Series in Nonlinear Analysis and Applications **21** (2013), xiii+pp. 1-262.
- [15] M. Marcus, L. Véron, Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains: the subcritical case, *Ann. Scu. Norm. Sup. Pisa Ser. V* **Vol. X** (2011), 913-984.
- [16] M. Marcus, L. Véron, Boundary trace of positive solutions of supercritical elliptic equations in dihedral domains, *Ann. Scu. Norm. Sup. Pisa Ser. V*, **Vol. XV** (2016), 501-542.
- [17] L. Véron. Elliptic Equations Involving Measures, in *Stationary Partial differential equations* Vol. I, 593-712. *Handb. Differ. Equ.*, North-Holland, Amsterdam (2004).
- [18] L. Véron, C. Yarur. Boundary value problems with measures for elliptic equations with singular potentials, *J. Funct. Anal.* **262** (2012), 733-772.
- [19] L. Véron. Existence and Stability of Solutions of General Semilinear Elliptic Equations with Measure Data, *Advanced Nonlinear Studies* **13** (2013), 447-460.