# Schrödinger operators with Leray-Hardy potential singular on the boundary

Huyuan Chen, Laurent Veron

# Schrödinger operators with Leray-Hardy potential singular on the boundary

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**Abstract**

We study the kernel function of the operator $u \mapsto \mathcal{L}_\mu u = -\Delta u + \frac{\mu}{|x|^2}u$ in a bounded smooth domain $\Omega \subset \mathbb{R}^N_+$ such that $0 \in \partial\Omega$, where $\mu \geq -\frac{N^2}{4}$ is a constant. We show the existence of a Poisson kernel vanishing at 0 and a singular kernel with a singularity at 0. We prove the existence and uniqueness of weak solutions of $\mathcal{L}_\mu u = 0$ in $\Omega$ with boundary data $\nu + k\delta_0$, where $\nu$ is a Radon measure on $\partial\Omega \setminus \{0\}$, $k \in \mathbb{R}$ and show that this boundary data corresponds in a unique way to the boundary trace of positive solution of $\mathcal{L}_\mu u = 0$ in $\Omega$.

**Key Words**: Hardy Potential, Harnack inequality, limit set, Radon Measure.

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## Contents



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# 1 Introduction

We denote by $\mathcal{L}_\mu$ the Schrödinger operator defined in a domain $\Omega \subset \mathbb{R}^N$ by

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2}u,$$

where $\mu$ is a real constant and $N \geq 2$. This operator which is associated to the Hardy inequality has been thoroughly studied in the last thirty years. When the singular point 0 belongs to $\Omega$, it appears a critical value

$$\mu_0 = -\left(\frac{N-2}{2}\right)^2$$

and the range of the $\mu$ in which the operator is bounded from below is $[\mu_0, \infty)$. This is linked to the Hardy inequality

$$\int_\Omega |\nabla \phi|^2 + \mu_0 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Furthermore this inequality is never achieved if $\Omega$ is bounded, in which case a remainder was shown to exist by Brézis and Vázquez [4]. When $\lambda$ is a Radon measure in $\Omega$, the associated Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied in its full generality in [8] and [9] thanks to the introduction of a notion of very weak solution associated to some specific weight. Thanks to this new formulation an extensive treatment of the associated semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g : \mathbb{R} \mapsto \mathbb{R}$ is a continuous nondecreasing function is developed in [9].

In this article we assume that the singular point of the potential lies on the boundary of the domain $\Omega$, and we are mainly interested in the two problems:
1- To define a notion of *very weak solution* for the problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\nu$ is a Radon measure on $\partial\Omega$, and more generaly on $\partial\Omega \setminus \{0\}$;
2- To prove the existence of a boundary trace for any positive $\mathcal{L}_\mu$-*harmonic function*, i.e. solution of $\mathcal{L}_\mu u = 0$ in $\Omega$ and to connect it to the problem (1.1).

The model example is $\Omega = \mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \ x_N > 0\}$ although it is not a bounded domain. There exists a critical value

$$\mu \geq \mu_1 := -\frac{N^2}{4}. \tag{1.2}$$

This value is fundamental for the operator $\mathcal{L}_\mu$ to be bounded from below since there holds,

$$\int_{\mathbb{R}^N_+} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}^N_+} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N_+). \tag{1.3}$$

The analysis of the model case is explicit. Let $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}_+$ be the spherical coordinates in $\mathbb{R}^N_+$, then, if (1.2) is satisfied, there exist two different types of positive $\mathcal{L}_\mu$-harmonic functions vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$),

$$\gamma_\mu(r,\sigma) = r^{\alpha_+}\psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r,\sigma) = \begin{cases} r^{\alpha_-}\psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1})\psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \tag{1.4}$$

where $\psi_1(\sigma) = \frac{x_N}{|x|}$ generates $\ker(-\Delta' + (N-1)I)$ in $H^1_0(\mathbb{S}^{N-1}_+)$, and where

$$\alpha_+ := \alpha_+(\mu) = \frac{2-N}{2} + \sqrt{\mu + \frac{N^2}{4}} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2-N}{2} - \sqrt{\mu + \frac{N^2}{4}}. \tag{1.5}$$

Put $d\gamma_\mu(x) = \gamma_\mu(x)dx$. We define the $\gamma_\mu$-dual operator $\mathcal{L}^*_\mu$ of $\mathcal{L}_\mu$ by

$$\mathcal{L}^*_\mu \zeta = -\Delta\zeta - \frac{2}{\gamma_\mu}\langle \nabla\gamma_\mu, \nabla\zeta\rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}}^N_+), \tag{1.6}$$

and we prove that $\phi_\mu$ is, in some sense, the fundamental solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \mathbb{R}^N_+, \\ u = \delta_0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

since it satisfies

$$\int_{\mathbb{R}^N_+} \phi_\mu \mathcal{L}^*_\mu \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}^N_+}) \cap C^{1,1}(\mathbb{R}^N_+)$$

such that $\rho\mathcal{L}^*_\mu\zeta \in L^\infty(\mathbb{R}^N_+)$, where $c_\mu > 0$ is a normalized constant and $\rho(x) = \mathrm{dist}(x, \partial\Omega)$. Here $\rho(x) = x_N$ when $\Omega = \mathbb{R}^N_+$.

When $\mathbb{R}^N_+$ is replaced by a bounded domain $\Omega$ satisfying the condition

($\mathcal{C}$-1) $$0 \in \partial\Omega\ ,\ \Omega \subset \mathbb{R}^N_+ \text{ and } \langle x, \mathbf{n}\rangle = O(|x|^2) \text{ for all } x \in \partial\Omega,$$

where $\mathbf{n} = \mathbf{n}_x$ is the outward normal vector at $x$, inequality (1.3) holds but it is never achieved in the Hilbert space $H^1_0(\Omega)$. Note that the last condition in ($\mathcal{C}$-1) holds if $\Omega$ is a $C^2$ domain. It is proved in [5] that there exists a remainder under the following form:

$$\int_\Omega |\nabla\phi|^2 + \mu_1 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq \frac{1}{4}\int_\Omega \frac{\phi^2}{|x|^2 \ln^2(|x|R_\Omega^{-1})} dx \qquad \text{for all } \phi \in C_c^\infty(\Omega), \tag{1.7}$$

where $R_\Omega = \max_{z\in\Omega} |z|$. Under the assumption ($\mathcal{C}$-1), there holds

$$\ell^\Omega_\mu := \inf\left\{ \int_\Omega \left( |\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx : v \in C^1_c(\Omega), \int_\Omega v^2 dx = 1 \right\} > 0.$$

This first eigenvalue is achieved in $H_0^1(\Omega)$ if $\mu > \mu_1$, or in the space $H(\Omega)$ which is the closure of $C_c^1(\Omega)$ for the norm

$$v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_\Omega \left(|\nabla v|^2 + \frac{\mu_1}{|x|^2} v^2\right) dx},$$

when $\mu = \mu_1$. In the sequel we set

$$H_\mu(\Omega) = \begin{cases} H_0^1(\Omega) & \text{if } \mu > \mu_1 \\ H(\Omega) & \text{if } \mu = \mu_1. \end{cases}$$

Moreover, under the assumption ($\mathcal{C}$-1) the imbedding of $H_\mu(\Omega)$ is compact (see e.g. [6]). We denote by $\gamma_\mu^\Omega$ the positive eigenfunction, its satisfies

$$\begin{cases} \mathcal{L}_\mu \gamma_\mu^\Omega = \ell_\mu^\Omega \gamma_\mu^\Omega & \text{in } \Omega, \\ \gamma_\mu^\Omega = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{1.8}$$

We prove that there exist $c_j = c_j(\Omega, \mu) > 0$, j=1, 2, such that

$$\begin{aligned} &(i) && \gamma_\mu^\Omega(x) = c_1 \rho(x) |x|^{\alpha_+ - 1}(1 + o(1)) \quad \text{as } x \to 0, \\ &(ii) && |\nabla \gamma_\mu^\Omega(x)| \le c_2 \frac{\gamma_\mu^\Omega(x)}{\rho(x)} \quad \text{for all } x \in \Omega. \end{aligned} \tag{1.9}$$

This function will play the role of a weight function for replacing $\gamma_\mu$. Next we construct the Poisson kernel $\mathbb{K}_\mu^\Omega$ of $\mathcal{L}_\mu$ in $\Omega \times \partial\Omega$. When $\mu \ge 0$ this construction can be made by truncation as in [18], considering for $\epsilon > 0$ and $\lambda \in \mathfrak{M}_+(\partial\Omega)$ the solution $u_\epsilon$ of

$$\begin{cases} -\Delta u + \dfrac{\mu}{\max\{\epsilon^2, |x|^2\}} u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

By a more elaborate method, we also construct the Poisson kernel when $\mu_1 \le \mu < 0$. It is important to notice that when $\mu > 0$ the kernel has the property that

$$K_\mu^\Omega(x, 0) = 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\} \tag{1.10}$$

by [18, Theorem A.1]. Because of (1.10) it is clear that the Poisson kernel cannot be the tool for describing all the positive $\mathcal{L}_\mu$-harmonic functions. Our first concern in this article is to clarify the Poisson kernel of $\mathcal{L}_\mu$.

We first characterize the positive $\mathcal{L}_\mu$-harmonic functions which are singular at 0.

**Theorem A** *Let $\Omega$ be a $C^2$ bounded domain such that $0 \in \partial\Omega$ and $\mu \ge \mu_1$. If $u$ is a nonnegative $\mathcal{L}_\mu$-harmonic function vanishing on $B_{r_0}(0) \cap (\partial\Omega \setminus \{0\})$ for some $r_0 > 0$, there exists $k \ge 0$ such that*

$$\lim_{x \to 0} \frac{u(x)}{\rho(x)|x|^{\alpha_- - 1}} = k,$$

*if* $\mu > \mu_1$ *and*

$$\lim_{x\to 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x)\ln|x|} = -k,$$

*if* $\mu = \mu_1$.

Actually the above convergence hold in a stronger way. In order to prove that such solutions truly exist we construct the *kernel function* $\phi_\mu^\Omega$ (see [13] for the denomination) which is the analogue in a bounded domain of the explicit singular solution $\phi_\mu$ defined in $\mathbb{R}^N_+$.

**Theorem B** *Let* $\Omega$ *be a* $C^2$ *bounded domain such that* $0 \in \partial\Omega$ *satisfying* ($\mathcal{C}$-1) *and* $\mu \geq \mu_1$. *Then there exists a positive* $\mathcal{L}_\mu$*-harmonic function in* $\Omega$, *which vanishes on* $\partial\Omega \setminus \{0\}$ *which satisfies,*

$$\phi_\mu^\Omega(x) = \rho(x)|x|^{\alpha_- - 1}(1 + o(1)) \quad as \;\; x \to 0, \tag{1.11}$$

*if* $\mu > \mu_1$, *and*

$$\phi_{\mu_1}^\Omega(x) = \rho(x)|x|^{-\frac{N}{2}}(|\ln|x|| + 1)(1 + o(1)) \quad as \;\; x \to 0, \tag{1.12}$$

*if* $\mu = \mu_1$.

As in the model case, we define the $\gamma_\mu^\Omega$-dual operator of $\mathcal{L}_\mu$ by

$$\mathcal{L}_\mu^* \zeta = -\Delta\zeta - \frac{2}{\gamma_\mu^\Omega}\langle \nabla\gamma_\mu^\Omega, \nabla\zeta\rangle + \ell_\mu^\Omega \zeta \quad \text{for all } \zeta \in C^{1,1}(\Omega).$$

The following commutation formula holds

$$\mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) = \gamma_\mu^\Omega \mathcal{L}_\mu^* \zeta. \tag{1.13}$$

**Corollary C** *Let* $\Omega$ *be a* $C^2$ *bounded domain such that* $0 \in \partial\Omega$ *satisfying* ($\mathcal{C}$-1) *and* $\mu \geq \mu_1$. *Then* $\phi_\mu^\Omega$ *is the unique function belonging to* $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ *which satisfies*

$$\int_\Omega u\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = kc_\mu\zeta(0) \quad for\ all\ \zeta \in \mathbb{X}_\mu(\Omega), \tag{1.14}$$

*where and in the sequel the test function space*

$$\mathbb{X}_\mu(\Omega) = \left\{\zeta \in C(\Omega) : \gamma_\mu^\Omega\zeta \in H_\mu(\Omega) \text{ and } \rho\mathcal{L}_\mu^*\zeta \in L^\infty(\Omega)\right\}.$$

*Furthermore, if* $u$ *is a nonnegative* $\mathcal{L}_\mu$*-harmonic function vanishing on* $\partial\Omega \setminus \{0\}$, *there exists* $k \geq 0$ *such that* $u = k\phi_\mu^\Omega$.

We let $\sigma_\mu^\Omega \in H_\mu(\Omega)$ be the unique variational solution of

$$\mathcal{L}_\mu u = \frac{\gamma_\mu^\Omega}{\rho} \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

Note that $\sigma_\mu^\Omega \in C^2(\overline{\Omega}\setminus\{0\})$ is a positive classical solution of (1.15) with zero Dirichlet boundary condition on $\partial\Omega \setminus \{0\}$, i.e. $\sigma_\mu^\Omega = 0$ on $\partial\Omega \setminus \{0\}$. Moreover, $\dfrac{\partial\sigma_\mu^\Omega}{\partial\mathbf{n}} < 0$ on $\partial\Omega \setminus \{0\}$. We set

$$\eta = \frac{\sigma_\mu^\Omega}{\gamma_\mu^\Omega} \quad \text{in } \Omega,$$

which satisfies

$$\mathcal{L}_\mu^*\eta = \frac{1}{\rho} \quad \text{in } \Omega, \tag{1.16}$$

play a key role in the sequel. Clearly, $\eta \in C^2(\overline{\Omega} \setminus \{0\})$ and $\eta > 0$ in $\overline{\Omega} \setminus \{0\}$. We denote by $\mathfrak{M}(\Omega; \sigma_\mu^\Omega)$ the set of Radon measures $\nu$ in $\Omega$ such that

$$\sup\left\{\int_\Omega \zeta d|\lambda| : \zeta \in C_c(\Omega),\, 0 \le \zeta \le \sigma_\mu^\Omega\right\} := \int_\Omega \sigma_\mu^\Omega d|\nu| < +\infty.$$

If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$ the measure $\sigma_\mu^\Omega \nu$ is a nonnegative bounded measure in $\Omega$. Put

$$\beta_\mu^\Omega(x) = -\frac{\partial \gamma_\mu^\Omega(x)}{\partial \mathbf{n_x}}, \quad \forall\, x \in \partial\Omega. \tag{1.17}$$

The set Radon measures $\lambda$ in $\partial\Omega \setminus \{0\}$ such that

$$\sup\left\{\int_{\partial\Omega\setminus\{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial\Omega \setminus \{0\}),\, 0 \le \zeta \le \eta\beta_\mu^\Omega\right\} := \int_{\partial\Omega\setminus\{0\}} \eta\beta_\mu d|\lambda| < +\infty$$

is denoted by $\mathfrak{M}(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$. The extension of $\lambda \in \mathfrak{M}_+(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$ as a measure $\eta\beta_\mu^\Omega\lambda$ in $\partial\Omega$ is given by

$$\int_{\partial\Omega} \zeta d(\eta\beta_\mu^\Omega\lambda) = \sup\left\{\int_{\partial\Omega} \upsilon\eta\beta_\mu^\Omega\, d\lambda : \upsilon \in C_c(\partial\Omega \setminus \{0\}),\, 0 \le \upsilon \le \zeta\right\} \quad \text{for all } \zeta \in C_c(\partial\Omega)\,,\ \zeta \ge 0$$

and $\eta\beta_\mu\lambda = \eta\beta_\mu\lambda_+ - \eta\beta_\mu\lambda_-$ if $\lambda$ is a signed measure in $\mathfrak{M}(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$, and this defines the set $\mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ of all such extensions. The Dirac mass at 0 does not belong to $\mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$, but it is the limit of sequences of measures in this space in the same way as it is a limit of measures in $\mathfrak{M}_+(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$. In the next result we prove the existence and uniqueness of a solution to

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ \quad u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

Thanks to (1.7) the Green kernel $G_\mu^\Omega$ is easily constructible. If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$ and $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ the following expressions are well defined

$$\mathbb{K}_\mu^\Omega[\lambda](x) = \int_{\partial\Omega} K_\mu^\Omega(x,y) d\lambda(y) \quad \text{and} \quad \mathbb{G}_\mu^\Omega[\nu](x) = \int_\Omega G_\mu^\Omega(x,y) d\nu(y).$$

Our main existence result is the following.

**Theorem D** *Let $\Omega$ be a $C^2$ bounded domain such that $0 \in \partial\Omega$ satisfying ($\mathcal{C}$-1) and $\mu \ge \mu_1$. If $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$, $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ and $k \in \mathbb{R}$, the function*

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda, k)] \tag{1.19}$$

*is the unique solution of (1.18) in the very weak sense that* $u \in L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ *and*

$$\int_\Omega u\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + kc_\mu\zeta(0) \tag{1.20}$$

*for all* $\zeta \in \mathbb{X}_\mu(\Omega)$.

In the next result we prove that all the positive $\mathcal{L}_\mu$-harmonic functions in $\Omega$ are described by formula (1.19) (with $\nu = 0$).

**Theorem E** *Let* $\Omega$ *be a* $C^2$ *bounded domain such that* $0 \in \partial\Omega$ *satisfying* ($\mathcal{C}$-1), $\mu \geq \mu_1$ *and* $u$ *be a nonnegative* $\mathcal{L}_\mu$*-harmonic functions in* $\Omega$. *Then there exist* $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ *and* $k \geq 0$, *such that*

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = \mathbb{H}_\mu^\Omega[(0, \lambda, k)].$$

*The couple* $(\lambda, k\delta_0)$ *is called the boundary trace of* $u$.

The rest of this paper is organized as follows. In section 2, we introduce the distributional identity of $\mathcal{L}_\mu$ harmonic function $\phi_\mu$ in $\mathbb{R}^N_+$. Section 3 is devoted to build the Kato's type inequalities, to construct Poisson kernel and related properties. Section 4 is addressed to classify the boundary isolated singular $\mathcal{L}_\mu$ harmonic functions in a bounded domain, i.e. **Theorem A** and to show the existence and related distributional identity in a ($\mathcal{C}$-1) domain: proofs of **Theorem B** and **Corollary C**. We classify the boundary trace for general $\mathcal{L}_\mu$ harmonic functions and give the existence of $\mathcal{L}_\mu$ harmonic functions with the boundary trace $(\lambda, k\delta_0)$: **Theorem D** and **Theorem E** respectively in Section 5. Finally, we show Estimates (1.9) in Appendix.

In a forthcomming article [10] we study the semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

## 2 The half-space setting

Let $\mathbb{R}^N_+ := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$, $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}_+$ be the spherical coordinates in $\mathbb{R}^N_+$ and $\Delta'$ is the Laplace Beltrami operator on $\mathbb{S}^{N-1}$. Then

$$\mathcal{L}_\mu u = -\partial_{rr}u - \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\Delta' u + \frac{\mu}{r^2}u.$$

If $u(r, \sigma) = r^\alpha\phi(\sigma)$ is a (separable) solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$, then $\phi$ satisfies

$$\begin{cases} -\Delta'\phi = \lambda_k\phi & \text{in } \mathbb{S}^{N-1}_+ := \mathbb{S}^{N-1} \cap \mathbb{R}^N_+, \\ \phi = 0 & \text{in } \partial\mathbb{S}^{N-1}_+ \approx \mathbb{S}^{N-2}, \end{cases}$$

where $\lambda_k$ a constant which necessarily belongs to the spectrum

$$\sigma_{\mathbb{S}^{N-1}_+}(-\Delta') = \{\lambda_k = k(N + k - 2) : k \in \mathbb{N}^*\},$$

and $\alpha = \alpha_{k\,+}, \alpha_{k\,-}$ is a root of

$$\alpha^2 + (N-2)\alpha - \lambda_k - \mu = 0. \tag{2.1}$$

The fundamental state corresponds to $k = 1$, in which case since $\lambda_1 = N-1$, existence of real roots of (2.1) necessitates $\mu \geq \mu_1 = -\frac{N^2}{4} = \mu_1$ and we denote $\alpha_{1\,+} = \alpha_+$ and $\alpha_{1\,-} = \alpha_-$. Note that this value is connected to the boundary Hardy

$$\int_{\mathbb{R}^N_+} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}^N_+} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(S^{N-1}_+).$$

If this condition is fulfilled, the two roots $\alpha_+$ and $\alpha_-$ corresponding to $k = 1$ and $\lambda_1$ are

$$\alpha_+ = \frac{2-N}{2} + \sqrt{\mu - \mu_1} \quad \text{and } \alpha_- = \frac{2-N}{2} - \sqrt{\mu - \mu_1}. \tag{2.2}$$

The corresponding positive separable solutions $\gamma_\mu$ and $\phi_\mu$ of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\mathbb{R}^N_+ \setminus \{0\}$ are defined by (1.4). We set $d\gamma_\mu(x) = \gamma_\mu(x)dx$ and define the operator $\mathcal{L}^*_\mu$ by (1.6).

**Proposition 2.1** *The function $\phi_\mu$ belongs to $L^1_{loc}(\mathbb{R}^N_+, \rho^{-1}d\gamma_\mu)$. It satisfies*

$$\int_{\mathbb{R}^N_+} \phi_\mu \mathcal{L}^*_\mu \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \textit{for all } \zeta \in \mathbb{X}_\mu(\mathbb{R}^N_+), \tag{2.3}$$

*where*

$$c_\mu = \begin{cases} 2\sqrt{\mu - \mu_1} \displaystyle\int_{\mathbb{S}^{N-1}_+} \psi_1^2 dS & \textit{if } \ \mu > \mu_1, \\ \displaystyle\int_{\mathbb{S}^{N-1}_+} \psi_1^2 dS & \textit{if } \ \mu = \mu_1 \end{cases} \tag{2.4}$$

*and* $\mathbb{X}_\mu(\mathbb{R}^N_+) = \left\{\zeta \in C_c(\overline{\mathbb{R}^N_+}) \ \textit{s.t. } \rho\mathcal{L}^*_\mu\zeta \in L^\infty(\mathbb{R}^N_+)\right\}$.

**Proof.** Let $\zeta \in \mathbb{X}_\mu(\mathbb{R}^N_+)$, $\epsilon > 0$ and set $B^+_\epsilon = B_\epsilon(0) \cap \mathbb{R}^N_+$, $(B^+_\epsilon)^c = B^c_\epsilon(0) \cap \mathbb{R}^N_+$ and $\Gamma^+_\epsilon = \partial B_\epsilon(0) \cap \overline{\mathbb{R}}^N_+$

$$\begin{aligned} 0 &= \int_{(B^+_\epsilon)^c} \zeta \gamma_\mu \mathcal{L}_\mu \phi_\mu dx \\ &= \int_{(B^+_\epsilon)^c} \phi_\mu \mathcal{L}^*_\mu \zeta d\gamma_\mu(x) + \int_{\Gamma^+_\epsilon} \left( -\frac{\partial \phi_\mu}{\partial \mathbf{n}} \zeta \gamma_\mu + \left( \gamma_\mu \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu}{\partial \mathbf{n}} \right) \phi_\mu \right) dS \\ &= \int_{(B^+_\epsilon)^c} \phi_\mu \mathcal{L}^*_\mu \zeta d\gamma_\mu(x) + \zeta(0) A(\epsilon) \end{aligned}$$

and $\mathbf{n} = -\epsilon^{-1}x$ on $\Gamma^+_\epsilon$, and

$$A(\epsilon) = \begin{cases} -2\sqrt{\mu - \mu_1} \displaystyle\int_{\mathbb{S}^{N-1}_+} \phi_1^2 dS + O(\epsilon) & \text{if } \ \mu > \mu_1, \\ -\displaystyle\int_{\mathbb{S}^{N-1}_+} \phi_1^2 dS + O(\epsilon) & \text{if } \ \mu = \mu_1, \end{cases} \tag{2.5}$$

which implies (2.3)-(2.4). □

## 3 The Poisson kernel

In this section we assume that $\Omega$ is a bounded $C^2$ domain included in $B_1$ (which can always be assumed by scaling) and $0 \in \partial\Omega$. We start with the following identity of commutation valid for all $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $\zeta \in C^{1,1}(\overline{\Omega})$

$$-\int_{\partial\Omega} \frac{\partial(\zeta\gamma_\mu^\Omega)}{\partial \mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) \quad \text{for all } \zeta \in C^{1,1}(\overline{\Omega}), \tag{3.1}$$

where $\beta_\mu^\Omega$ is defined in (1.17).

The following inequality extends the classical Kato inequality to our framework.

**Lemma 3.1** *Assume $N \geq 3$ and $\mu \geq \mu_1$ or $N = 2$ and $\mu > \mu_1$. Then for any $f \in L^1(\Omega, \sigma_\mu^\Omega dx)$, $h \in L^1(\partial\Omega, \beta_\mu^\Omega dx)$ there exists a unique weak solution $u$ to*

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

*in the sense that*

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_\Omega f\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h\zeta d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \tag{3.3}$$

*Furthermore, for any $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta \geq 0$, there holds*

$$\int_\Omega |u| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_\Omega sgn(u) f\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h|\zeta d\beta_\mu^\Omega \tag{3.4}$$

*and*

$$\int_\Omega u^+ \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_\Omega sgn_+(u) f\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h^+\zeta d\beta_\mu^\Omega. \tag{3.5}$$

**Proof.** *Uniqueness.* Assume that $u$ is a weak solution of (3.2) with $f = h = 0$. Then for any $\zeta \in \mathbb{X}_\mu(\Omega)$ there holds

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = 0.$$

Let $\phi \in C_c^\infty(\Omega)$, and $\upsilon \in H_\mu(\Omega)$ be the variational solution of

$$\mathcal{L}_\mu \upsilon = \frac{\gamma_\mu^\Omega}{\rho}\phi \quad \text{and} \quad u \in H_\mu(\Omega).$$

Then $\upsilon \in C^\infty(\Omega)$, $|\upsilon| \leq \|\phi\|_{L^\infty}\sigma_\mu^\Omega$; the equation is satisfied everywhere and in the sense of distributions in $\Omega$. Clearly $w = (\gamma_\mu^\Omega)^{-1}\upsilon$ belongs to $C^\infty(\Omega)$ and satisfies

$$\mathcal{L}_\mu^* w = \frac{1}{\rho}\phi.$$

Thus,

$$\int_\Omega \frac{u}{\rho}\phi d\gamma_\mu^\Omega = 0.$$

Since $\phi$ is arbitrary, we have that $u = 0$.

*Existence and estimates.* We proceed by approximation as in [8, Prop. 2.1]. We assume that $\{(f_n, h_n)\} \subset C_0^1(\Omega) \times C_0^1(\partial\Omega \setminus \{0\})$ is a sequence which converges to $(f, h)$ in $L^1(\Omega, \gamma_\mu^\Omega dx) \times L^1(\partial\Omega, \beta_\mu^\Omega dx)$. We set $V(x) = |x|^{-2}$, denote by $\mathbb{K}^\Omega$ the Poisson potential of $-\Delta$ in $\Omega$ and consider the approximate problem

$$\begin{cases} \mathcal{L}_\mu w_n = f_n - \mu V \mathbb{K}[h_n] & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Near 0, we have $V\mathbb{K}[h_n](x) = O(\frac{\rho(x)}{|x^2|})$, hence, if $N \geq 3$, $V\mathbb{K}[h_n] \in L^2(\Omega)$. If $N = 2$, the function $x \mapsto \frac{\rho(x)}{|x|^2}$ belongs to the Lorentz space $L^{2,\infty}(\Omega)$ which is the dual of $L^{2,1}(\Omega)$. Since $H(\Omega) \subset L^{2,1}(\Omega)$ by (1.7), it follows that $H'(\Omega) \subset L^{2,\infty}(\Omega)$. Hence, by Lax-Milgram theorem there exists a unique $w_n \in H(\Omega)$ such that (3.6) holds in the variational sense. Then $u_n = w_n + \mathbb{K}[h_n]$, which has the same regularity as $w_n$, satisfies

$$\begin{cases} \mathcal{L}_\mu u_n = f_n & \text{in } \Omega, \\ u_n = h_n & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

For $\sigma > 0$, we set

$$m_\sigma(t) = \begin{cases} |t| - \frac{\sigma}{2} & \text{if } |t| \geq \sigma, \\ \frac{t^2}{2\sigma} & \text{if } |t| < \sigma. \end{cases}$$

The $m_\sigma$ is convex, $|m'_\sigma(t)| \leq 1$ and $m'_\sigma(t) \to \mathrm{sign}_0(t)$ as $\sigma \downarrow 0$. Let $\zeta \in C^{1,1}(\overline{\Omega})$, $\zeta \geq 0$. We have that

$$\int_\Omega \left( \langle \nabla w_n, \nabla(\zeta m'_\sigma(u_n)\gamma_\mu^\Omega) \rangle + \mu V w_n \zeta m'_\sigma(u_n)\gamma_\mu^\Omega \right) dx$$
$$= \int_\Omega \zeta m'_\sigma(u_n)\gamma_\mu^\Omega f_n dx - \mu \int_\Omega V\mathbb{K}[h_n]\zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx =: \mathbb{R}(\sigma).$$

By the fact $u_n = w_n + \mathbb{K}[h_n]$, we have that

$$\begin{aligned}
\mathbb{R}(\sigma) &= \int_\Omega \langle \nabla u_n, \nabla(\zeta m'_\sigma(u_n)\gamma_\mu^\Omega)\rangle dx - \int_\Omega \langle \nabla \mathbb{K}[h_n], \nabla(\zeta m'_\sigma(u_n)\gamma_\mu^\Omega)\rangle dx \\
&\quad + \mu \int_\Omega V u_n \zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx - \mu \int_\Omega V \mathbb{K}[h_n]\zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx \\
&= \int_\Omega |\nabla u_n|^2 m''_\sigma(u_n)\zeta\gamma_\mu^\Omega dx + \int_\Omega \langle \nabla m_\sigma(u_n), \nabla(\zeta\gamma_\mu^\Omega)\rangle dx + \mu \int_\Omega V u_n \zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx \\
&\quad - \mu \int_\Omega V \mathbb{K}[h_n]\zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx \\
&\geq -\int_\Omega m_\sigma(u_n)\Delta(\zeta\gamma_\mu^\Omega)dx + \int_{\partial\Omega} m_\sigma(h_n)\zeta\frac{\partial\gamma_\mu^\Omega}{\partial \mathbf{n}} dS + \mu \int_\Omega V u_n \zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx \\
&\quad - \mu \int_\Omega V \mathbb{K}[h_n]\zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx \\
&\geq \int_\Omega m_\sigma(u_n)\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega - \int_{\partial\Omega} m_\sigma(h_n) d\beta_\mu^\Omega + \mu \int_\Omega V\left(u_n m'_\sigma(u_n) - m_\sigma(u_n)\right)\gamma_\mu^\Omega \zeta dx \\
&\quad - \mu \int_\Omega V \mathbb{K}[h_n]\zeta m'_\sigma(u_n)\gamma_\mu^\Omega dx,
\end{aligned}$$

thus, we obtain that

$$\begin{aligned}
&\int_\Omega m_\sigma(u_n)\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega + \mu\int_\Omega V\left(u_n m'_\sigma(u_n) - m_\sigma(u_n)\right)\zeta d\gamma_\mu^\Omega \\
&\qquad\qquad \leq \int_\Omega \zeta m'_\sigma(u_n) f_n d\gamma_\mu^\Omega + \int_{\partial\Omega} m_\sigma(h_n)\zeta d\beta_\mu^\Omega.
\end{aligned} \tag{3.8}$$

Since $m_\sigma$ is convex, $u_n m'_\sigma(u_n) - m_\sigma(u_n) \geq 0$. Hence for $\mu \geq 0$, we can let $\sigma \to 0$ in (3.8) and obtain (3.4).

For $\mu \in [\mu_1, 0)$, we note that

$$0 \leq u_n m'_\sigma(u_n) - m_\sigma(u_n) \leq \frac{|u_n|^2}{2\sigma^2}\chi_{\{|u_n|\leq\sigma\}}$$

and

$$0 \leq \int_\Omega V\left(u_n m'_\sigma(u_n) - m_\sigma(u_n)\right)\zeta d\gamma_\mu^\Omega \leq \frac{\|\zeta\|_{L^\infty}}{2}\int_{\{|u_n|\leq\sigma\}} |x|^{-1-\frac{N}{2}+\sqrt{\mu-\mu_1}} dx. \tag{3.9}$$

Hence if $N \geq 3$, or $N = 2$ and $\mu > \mu_1 = -1$, the right-hand side of (3.9) tends to 0 as $\sigma \to 0$ and then we obtain (3.4). The proof of (3.5) is similar.

Applying estimate (3.4) to $u_n - u_m$, we obtain for all $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta \geq 0$,

$$\int_\Omega |u_n - u_m|\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega \leq \int_\Omega |f_n - f_m|\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h_n - h_m|\zeta d\beta_\mu^\Omega,$$

For test function, we take $\eta$, the solution of (1.16), then

$$\int_\Omega \frac{|u_n - u_m|}{\rho} d\gamma_\mu^\Omega \leq \int_\Omega |f_n - f_m| d\sigma_\mu^\Omega + \int_{\partial\Omega} |h_n - h_m| d(\eta\beta_\mu^\Omega),$$

Therefore $\{u_n\}$ is a cauchy sequence in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ with limit $u$. Since $u_n$ satisfies (3.7), we let $n$ go to infty in

$$\int_\Omega u_n \mathcal{L}_\mu^* \zeta \, d\gamma_\mu^\Omega = \int_\Omega \zeta \sigma_\mu^\Omega f_n dx + \int_{\partial\Omega} \zeta \beta_\mu^\Omega h_n dS$$

and obtain (3.3). □

**Lemma 3.2** *Assume $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$ and $\zeta \in \mathbb{X}_\mu(\Omega)$, then there holds*

$$\int_\Omega \left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2}\zeta \right) \mathbb{K}^\Omega[\lambda] d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

**Proof.** By (1.13) we have almost everywhere in $\Omega$,

$$\left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2}\zeta \right) \mathbb{K}^\Omega[\lambda] \gamma_\mu^\Omega = \left( \mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) - \frac{\mu}{|x|^2}\gamma_\mu^\Omega \zeta \right) \mathbb{K}^\Omega[\lambda] = -\Delta(\gamma_\mu^\Omega \zeta)\mathbb{K}^\Omega[\lambda].$$

If we assume that $\lambda$ vanishes in a neighborood of 0 we derive from (3.1)

$$-\int_\Omega \Delta(\gamma_\mu^\Omega \zeta)\mathbb{K}^\Omega[\lambda] dx = -\int_{\partial\Omega} \frac{\partial(\zeta\gamma_\mu^\Omega)}{\partial \mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

Since $\gamma_\mu^\Omega \left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2}\zeta \right)$ is bounded, we obtain the result first if $\lambda$ is nonnegative by considering the sequence $\{\chi_{B_\epsilon^c}\lambda\}$ and letting $\epsilon \to 0$, and then for any $\lambda = \lambda^+ - \lambda^-$. □

We observe also that the existence of the Green kernel follows from Lax-Milgram theorem which gives the existence of a unique variational solution in $H(\Omega)$ to

$$\begin{cases} -\Delta u + \dfrac{\mu}{|x|^2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by $G_\mu^\Omega$ the Green kernel and by $\mathbb{G}_\mu^\Omega$ the corresponding Green operator.

### 3.1 Construction of the Poisson kernel when $\mu > 0$

For the sake of completeness, we recall the construction in [18]. For $\epsilon > 0$ we set $V_\epsilon(x) = \max\{\epsilon^{-2}, |x|^{-2}\}$ and $V_0(x) = V(x) = |x|^{-2}$, and if $\lambda \in \mathfrak{M}(\partial\Omega)$ let $u_\epsilon$ be the solution of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

Then

$$u_\epsilon(x) = \int_{\partial\Omega} K_{\mu,\epsilon}^\Omega(x,y) d\lambda(y) = \mathbb{K}_{\mu,\epsilon}^\Omega[\lambda].$$

We obtain by the maximum principle,

$$K_{\mu,\epsilon}^\Omega \le K_{\mu',\epsilon'}^\Omega \le K^\Omega \quad \text{for all } \ \mu \ge \mu' \ge 0 \text{ and } \ \epsilon' \ge \epsilon > 0,$$

where $K^\Omega$ is the usual Poisson kernel in $\Omega$ and there exists

$$K^\Omega_\mu(x,y) = \lim_{\epsilon\to 0} K^\Omega_{\mu,\epsilon}(x,y) \qquad \text{for all } (x,y) \in \Omega\times\partial\Omega.$$

Therefore we infer, firstly by monotone convergence if $\lambda \geq 0$, and then for any $\lambda \in \mathfrak{M}(\partial\Omega)$, that

$$\lim_{\epsilon\to 0} u_\epsilon(x) = u(x) = \int_{\partial\Omega} K^\Omega_\mu(x,y)d\lambda(y) \qquad \text{for all } x\in\Omega. \tag{3.10}$$

Since $V$ is finite in $B^c_\epsilon \cap \Omega$, for any $x\in\Omega$, $K^\Omega_\mu(x,y) > 0$ for all $y \in \partial\Omega\setminus\{0\}$. If $G^\Omega$ is the Green kernel in $\Omega$, there holds

$$u_\epsilon(x) + \mu\int_\Omega G^\Omega(x,y)V_\epsilon(y)u_\epsilon(y)dy = \int_{\partial\Omega} K^\Omega(x,y)d\lambda(y)$$

If $\lambda\geq 0$, we have by Fatou's lemma,

$$\int_\Omega G^\Omega(x,y)V(y)u(y)dy \leq \liminf_{\epsilon\to 0}\int_\Omega G^\Omega(x,y)V_\epsilon(y)u_\epsilon(y)dy. \tag{3.11}$$

Combined with (3.10) it yields

$$u(x) + \mu\int_\Omega G^\Omega(x,y)V(y)u(y)dy \leq \int_{\partial\Omega} K^\Omega(x,y)d\lambda(y) \qquad \text{for all } x\in\Omega.$$

Since the function $u+\mu\mathbb{G}[Vu]$ is nonnegative and harmonic in $\Omega$, it admits a boundary trace which is a nonnegative Radon measure $\lambda^*$ and there holds

$$u(x) + \mu\int_\Omega G^\Omega(x,y)V(y)u(y)dy = \int_{\partial\Omega} K^\Omega(x,y)d\lambda^*(y) \qquad \text{for all } x\in\Omega. \tag{3.12}$$

Because of (3.11) $0\leq\lambda^*\leq\lambda$. The measure $\lambda^*$ is the reduced measure associated to $\lambda$. Since (3.12) is equivalent to

$$u(x) = \int_{\partial\Omega} K^\Omega_\mu(x,y)d\lambda^*(y),$$

there holds

$$\int_{\partial\Omega} K^\Omega_\mu(x,y)d(\lambda-\lambda^*)(y) = 0.$$

This implies that $\lambda = \lambda^*$ in $\partial\Omega\setminus\{0\}$. With the notations of [18], we recall that

$$\mathcal{S}ing_{_V}(\Omega) := \left\{y\in\partial\Omega : \exists x_0\in\Omega \text{ s.t. } K^\Omega_\mu(x_0,y) = 0\right\}$$

$$\subset Z_{_V} := \left\{y\in\partial\Omega : \int_\Omega K^\Omega_0(x,y)V(x)\rho(x)dx = \infty\right\}.$$

Actually, if $y\in\mathcal{S}ing_{_V}(\Omega)$, $K^\Omega_\mu(x_0,y) = 0$ for any $x_0\in\Omega$ by Harnack inequality. Clearly $0\in Z_V$ and if $y\neq 0$ the integral term in the definition of $Z_V$ is finite. Hence $\mathcal{S}ing_{_V}(\Omega)\subset Z_{_V} = \{0\}$. Since for any truncated cone $C_{0,\delta}\Subset\Omega$ with vertex 0, there holds

$$\int_{C_{0,\delta}} V(x)\frac{dx}{|x-y|^{N-2}} = \infty,$$

it follows by Ancona's result [18, Theorem A1] that $0\in\mathcal{S}ing_{_V}(\Omega)$. Finally

$$K^\Omega_\mu(x,0) = 0 \quad \text{for all } x\in\Omega.$$

### 3.2 Construction of the Poisson kernel when $\mu_1 \leq \mu < 0$

For $\epsilon > 0$ and $\lambda \in C(\partial\Omega)$, $\lambda \geq 0$ we denote by $w = w_{\epsilon,\lambda}$ the variational solution in $H(\Omega)$ of

$$\begin{cases} -\Delta w + \mu V_\epsilon w = -\mu V_\epsilon \mathbb{K}[\lambda] & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $w_{\epsilon,\lambda} \geq 0$ and $u = u_{\epsilon,\lambda} := w_\epsilon + \mathbb{K}[\lambda]$ satisfies

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Since $-\Delta u_{\epsilon,\lambda} + \mu V u_{\epsilon,\lambda} \leq 0$, there holds from Lemma 3.1

$$\int_\Omega u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda \zeta d\beta_\mu^\Omega \quad \text{for } \zeta \in \mathbb{X}_\mu(\Omega),\ \zeta \geq 0$$

and in particular

$$\int_\Omega \frac{u_{\epsilon,\lambda}}{\rho} d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda d(\eta\beta_\mu^\Omega). \tag{3.14}$$

If $\epsilon > \epsilon' > 0$ and $\lambda' > \lambda > 0$ we have

$$-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} = \mu\left(V_{\epsilon'} - V_\epsilon\right) \leq 0.$$

Since

$$\begin{aligned} &\int_\Omega \left(-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}}\right)(u_{\epsilon,\lambda}^2 - u_{\epsilon',\lambda'}^2)_+ dx \\ &\quad = \int_{\{u_{\epsilon,\lambda} \geq u_{\epsilon',\lambda'}\}} \left(\left|\nabla u_{\epsilon,\lambda} - \frac{u_{\epsilon,\lambda}}{u_{\epsilon',\lambda'}} \nabla u_{\epsilon',\lambda'}\right|^2 + \left|\nabla u_{\epsilon',\lambda'} - \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}} \nabla u_{\epsilon,\lambda}\right|^2\right) dx, \end{aligned}$$

we deduce that the function $x \mapsto \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}}(x)$ is constant on the set $\{x : u_{\epsilon,\lambda}(x) > u_{\epsilon',\lambda'}(x)\}$. If this set is non-empty we get a contradiction since it is strictly included in $\Omega$. Therefore the mapping

$$(\epsilon, \lambda) \mapsto u_{\epsilon,\lambda}$$

is decreasing in $\epsilon$ and increasing in $\lambda$.

Next we can assume that $\lambda \in \mathfrak{M}_+(\partial\Omega)$ vanishes in $B_\delta \cap \partial\Omega$ and that $\{\lambda_n\} \subset C(\partial\Omega)$ is a sequence of functions which converge to $\lambda$ in the weak sense of measures. We denote by $u_{\epsilon,\lambda_n}$ the solution of (3.13) with $\lambda$ replaced by $\lambda_n$. Since $\mu < 0$, $\alpha_+ < 1$, $\rho^{-1}\gamma_\mu^\Omega \sim |x|^{\alpha_+ - 1} \geq R_\omega^{\alpha_+ - 1}$, where $R_\omega = \max\{|z| : z \in \Omega\}$. Hence

$$\int_\Omega u_{\epsilon,\lambda_n} dx \leq c_8 \int_{\partial\Omega} \lambda_n d(\eta\beta_\mu^\Omega) \leq c_9 \|\lambda\|_{\mathfrak{M}(\partial\Omega)}.$$

Hence $u_{\epsilon,\lambda_n}$ and $V_\epsilon u_{\epsilon,\lambda_n}$ are uniformly bounded in $L^1(\Omega)$. From standard regularity estimates the sequence $\{u_{\epsilon,\lambda_n}\}_{n\in\mathbb{N}}$ is bounded in the Lorentz spaces $L^{\frac{N}{N-1},\infty}(\Omega)$ and weakly relatively

compact in $L^1(\Omega)$ (see e.g. [12]). This implies that, up to a subsequence, $u_{\epsilon,\lambda_n}$ converges in $L^1(\Omega)$ and a.e. in $\Omega$ to a weak solution $u_{\epsilon,\lambda}$ of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

that is a function which satisfies

$$\int_\Omega (-u_{\epsilon,\lambda}\Delta\zeta + \mu V_\epsilon u_{\epsilon,\lambda}\zeta)\, dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}). \tag{3.15}$$

Furthermore (3.14) holds (with the same notation). For test function $\zeta$ in (3.15), we take $\zeta = \theta_1$ be the solution of

$$\begin{cases} -\Delta\theta_1 = 1 & \text{in } \Omega, \\ \theta_1 = 0 & \text{on } \partial\Omega \end{cases}$$

Then

$$\int_\Omega u_{\epsilon,\lambda} dx = -\mu \int_\Omega V_\epsilon u_{\epsilon,\lambda}\theta_1 dx - \int_{\partial\Omega} \frac{\partial\theta_1}{\partial\mathbf{n}} d\lambda.$$

By the monotone convergence theorem we obtain that $V_\epsilon u_{\epsilon,\lambda} \to V u_\lambda$ in $L^1(\Omega, \theta_1 dx)$ by letting $\epsilon \to 0$ and

$$\int_\Omega u_\lambda dx = -\mu \int_\Omega V u_\lambda \theta_1 dx - \int_{\partial\Omega} \frac{\partial\theta_1}{\partial\mathbf{n}} d\lambda.$$

Hence

$$\int_\Omega (-\Delta\zeta + \mu V\zeta)\, u_\lambda dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}).$$

We also have

$$\int_\Omega u_{\epsilon,\lambda_n} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_\Omega (V - V_\epsilon) u_{\epsilon,\lambda_n}\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta\lambda_n d\beta_\mu^\Omega,$$

for all $\zeta \in \mathbb{X}_\mu(\Omega)$. Since $u_{\epsilon,\lambda_n}$ converges in $L^1(\Omega)$ we obtain if $\zeta \geq 0$,

$$\int_\Omega u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_\Omega (V - V_\epsilon) u_{\epsilon,\lambda}\zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega) \leq \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega),$$

since $\mu(V - V_\epsilon) \leq 0$. When $\epsilon \to 0$, $u_{\epsilon,\lambda}$ increases and converges to some $u_\lambda$ in $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$ which satisfies

$$\int_\Omega u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega),\ \zeta \geq 0.$$

For $\delta > 0$ denote by $\zeta_\delta$ the solution of

$$\mathcal{L}_\mu^* \zeta_\delta = \chi_{\Omega_\delta} \mathcal{L}_\mu^* \zeta \quad \text{where } \Omega_\delta = \{x \in \Omega : \rho(x) > \delta\}.$$

As $\zeta \in \mathbb{X}_\mu(\Omega)$, $|\mathcal{L}_\mu^*\zeta| \leq c_{10}\rho$, hence $\zeta_\delta \in \mathbb{X}_\mu(\Omega)$, $|\zeta_\delta| \leq c_{10}\eta$ and $\zeta_\delta \to \zeta$ when $\delta \to 0$. Furthermore, since $c_{11}|x|$ is a supersolution for $c_{11} > 0$ large enough, $\eta_\delta \leq c_{11}|x|$. Hence

$$\int_{\Omega_\delta} u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\{|x|<\epsilon\}} \frac{1}{|x|^2} u_{\epsilon,\lambda}\zeta_\delta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta_\delta d(\lambda\beta_\mu^\Omega).$$

Because $|x|^{-2}u_{\epsilon,\lambda}|\zeta_\delta| \le c_{11}\rho^{-1}u_{\epsilon,\lambda}$ and $u_{\epsilon,\lambda} \to u_\lambda$ in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$, we derive that

$$\lim_{\epsilon\to 0}\int_{\{|x|<\epsilon\}} \frac{1}{|x|^2}u_{\epsilon,\lambda}\zeta_\delta d\gamma_\mu^\Omega = 0$$

which implies

$$\int_{\Omega_\delta} u_\lambda \mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = \int_{\partial\Omega}\zeta_\delta d(\lambda\beta_\mu^\Omega),$$

Letting $\delta \to 0$ we obtain by monotonicity

$$\int_{\Omega} u_\lambda \mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = \int_{\partial\Omega}\zeta d(\lambda\beta_\mu^\Omega). \tag{3.16}$$

Finally, if $\lambda \in \mathfrak{M}_+(\partial\Omega,\, \eta\gamma_\mu^\Omega)$ we replace it by $\lambda_\delta = \chi_{_{B_\delta^c}}\lambda$ and denote by $u_{\lambda_\delta}$ the weak solution of

$$\begin{cases} -\Delta u + \mu V u = 0 & \text{in } \Omega, \\ \qquad\qquad u = \lambda_\delta & \text{on } \partial\Omega. \end{cases}$$

The mapping $\delta \mapsto u_{\lambda_\delta}$ is monotone. Hence, by the monotone convergence theorem $u_{\lambda_\delta}$ increases and converges to some $u_\lambda$ in $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ and clearly $u_\lambda$ satisfies (3.16) for all $\zeta \in \mathbb{X}_\mu(\Omega)$.

# 4 The singular kernel

In this section we construct the singular kernel $\phi_\mu^\Omega$ and prove that it satisfies estimates (1.11)-(1.12) and it is associated to Dirac mass at 0. Up to a rotation we can assume that the inward normal direction to $\partial\Omega$ at 0 is $\mathbf{e}_N = (0', 1) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Hence the tangent hyperplane to $\partial\Omega$ at 0 is $\partial\mathbb{R}^N_+ = \mathbb{R}^{N-1}$. For $R > 0$ set $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$ and $D_R = B'_R \times (-R, R)$. Then there exist $R > 0$ and a $C^2$ function $\theta : B'_R \mapsto \mathbb{R}$ such that $\partial\Omega \cap D_R = \{x = (x', x_N) :\ x_N = \theta(x')$ for $x' \in B'_R\}$ and $\Omega \cap D_R = \{x = (x', x_N) : \theta(x') < x_N < R\}$. Furthermore $\nabla\theta(0) = 0$.

## 4.1 Classification of Boundary isolated singularities

We characterize the positive solutions of $\mathcal{L}_\mu u = 0$ which vanish on $\partial\Omega \setminus \{0\}$.

**Lemma 4.1** *Let $\mu \ge \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ be a positive solution of $\mathcal{L}_\mu u = 0$ in $\Omega$ vanishing on $\partial\Omega \setminus \{0\}$. Then there exist $a > 0$ and $c_{12} > 0$ such that*

$$u(x) \le c_{12}|x|^{-a-1}\rho(x) \qquad \text{for all } \ x \in \overline{\Omega} \setminus \{0\}. \tag{4.1}$$

**Proof.** This is a direct consequence of Boundary Harnack inequality [3, Th. 2.7].

**Proposition 4.2** *Assume that $\mu \ge \mu_1$ and $u \in C^2(\overline{\Omega} \setminus \{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$, vanishing on $\partial\Omega \setminus \{0\}$ satisfying (4.1) with $a \ge -\alpha_-$. Then the following convergences hold in $C^1(\mathbb{S}^{N-1}_+)$:*
*(i) If $\mu > \mu_1$ and $a = -\alpha_-$, there exists $c_{13} \ge 0$ such that*

$$\lim_{r\to 0}\frac{u(r,\cdot)}{r^{\alpha_-}} = c_{13}\phi \qquad \text{as } \ r \to 0. \tag{4.2}$$

*(ii)* $\mu \geq \mu_1$ *and* $a > -\alpha_-$ *there exist* $\tau > a + \alpha_-$ *depending on* $a$ *and* $\mu$*, and* $c_{14} \geq 0$ *such that*

$$u(x) \leq c_{14}|x|^{-a-1+\tau}\rho(x) \quad \textit{for all } x \in \bar{\Omega} \setminus \{0\}.$$

**Proof.** *Step 1. Straightening the boundary.* We define the function $\Theta = (\Theta_1, ..., \Theta_N)$ on $D_R$ by $y_j = \Theta_j(x) = x_j$ if $1 \leq j \leq N-1$ and $y_N = \Theta_N(x) = x_N - \theta(x')$. Since $D\Theta(0) = Id$ we can assume that $\Theta$ is a diffeomorphism from $D_R$ onto $\Theta(D_R)$. We set

$$u(x) = \tilde{u}(y) \qquad \text{for all } x \in D_R^+ = B'_R \times [0, R). \tag{4.3}$$

Then

$$\begin{array}{ll} u_{x_j x_j} & = \tilde{u}_{y_j y_j} - 2\theta_{x_j}\tilde{u}_{y_j y_N} - \theta_{x_j,x_j}\tilde{u}_{y_N} + \theta_{x_j}^2\tilde{u}_{y_N y_N} \quad \text{for } 1 \leq j \leq N-1, \\ u_{x_N x_N} & = \tilde{u}_{y_N y_N} \end{array}$$

and

$$\Delta\tilde{u} + |\nabla\theta|^2\tilde{u}_{y_N y_N} - 2\langle\nabla\theta, \nabla\tilde{u}_{y_N}\rangle - \tilde{u}_{y_N}\Delta\theta - \frac{\mu}{|\Theta^{-1}(y)|^2}\tilde{u} = 0.$$

We use here the spherical coordinates $(r, \sigma)$ in the variable $y$ and we recall that $\Delta'$ is the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$ and $\nabla'$ is the tangential gradient on $\mathbb{S}^{N-1}$ identified with the covariant derivative via the isometric imbedding $\mathbb{S}^{N-1} \subset\!\!> \mathbb{R}^N$ which enables the formula

$$\nabla\tilde{u}(y) = \left(\tilde{u}_r\mathbf{n} + \frac{1}{r}\nabla'\tilde{u}\right)(r, \sigma) \quad \text{with } \mathbf{n} = |y|^{-1}y.$$

After a lengthy computation the details of which can be found in [12, P 298-300] we obtain

$$\begin{aligned} & r^2\tilde{u}_{rr}\left[1 - 2\theta_r\langle\mathbf{n}, \mathbf{e}_N\rangle + |\nabla\theta|^2(\langle\mathbf{n}, \mathbf{e}_N\rangle)^2\right] \\ & + r\tilde{u}_r\left[N - 1 - r\langle\mathbf{n}, \mathbf{e}_N\rangle\Delta\theta + r|\nabla\theta|^2\left(\langle\nabla'(\langle\mathbf{n}, \mathbf{e}_N\rangle), \mathbf{e}_N\rangle - 2\langle\nabla'\theta, \nabla'(\langle\mathbf{n}, \mathbf{e}_N\rangle)\rangle\right)\right] \\ & + \langle\nabla'\tilde{u}, \mathbf{e}_N\rangle\left[-r\Delta\theta + 2\theta_r - |\nabla\theta|^2\langle\mathbf{n}, \mathbf{e}_N\rangle\right] + r\langle\nabla'\tilde{u}_r, \mathbf{e}_N\rangle\left[2\theta_r + 2|\nabla\theta|^2\langle\mathbf{n}, \mathbf{e}_N\rangle\right] \\ & - 2\langle\nabla'\tilde{u}, \nabla'\theta\rangle\langle\mathbf{n}, \mathbf{e}_N\rangle + \left\langle\nabla'(\langle\nabla'\tilde{u}, \mathbf{e}_N\rangle), |\nabla\theta|^2\mathbf{e}_N - 2r^{-1}\nabla'\theta\right\rangle + \Delta'\tilde{u} - \frac{\mu}{|\Theta^{-1}(y)|^2} = 0. \end{aligned}$$

Next we set

$$\tilde{u}(r, \sigma) = r^{-a}v(t, \sigma) \quad \text{with } t = \ln r,$$

and we assume that

$$a \neq \frac{N-2}{2}. \tag{4.4}$$

We notice that

$$\begin{aligned} r^2 = \sum_{j=1}^N y_j^2 = \sum_{j=1}^{N-1} x_j^2 + (x_N - \theta(x'))^2 = |x^2| - 2x_N\theta(x') = |x|^2(1 + O(r)) \ \text{ as } \ r \to 0 \\ = |x|^2(1 + O(e^t)) \ \text{ as } \ t \to -\infty. \end{aligned}$$

By a straightforward computation we find that $v$ satisfies the following asymptotically autonomous equation in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$

$$\begin{aligned}(1+\epsilon_1(t,\cdot))v_{tt} + (N-2-2a+\epsilon_2(t,\cdot))\, v_t + (a(a+2-N)-\mu+\epsilon_3(t,\cdot))\, v \\ + \Delta' v + \langle \nabla' v, \epsilon_4(t,\cdot)\rangle + \langle \nabla' v_t, \epsilon_5(t,\cdot)\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6(t,\cdot)\rangle = 0,\end{aligned} \tag{4.5}$$

where the $\epsilon_j$ satisfies

$$|\epsilon_j(t,\cdot)| + |\partial_t \epsilon_j(t,\cdot)| + |\nabla' \epsilon_j(t,\cdot)| \leq c_{15} e^t. \tag{4.6}$$

This is due to the fact that $|\theta(x')| = O(|x'|^2)$ near 0.

*Step 2. The convergence process.* Since $v$ is bounded in $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$ and vanishes on $(-\infty, r_0] \times \partial\mathbb{S}_+^{N-1}$ and all the coefficients are continuous functions, we obtain that $v$ is bounded in $W^{2,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ independently of $T \leq r_0 - 2$, for any $q < \infty$. Hence $v$ is bounded in any $C^{1,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$ for any $\tau \in [0, 1)$. Differentiating the equation and using the standard elliptic equation regularity, we obtain that $v$ is bounded in $W^{3,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$ and in $C^{2,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$. We consider the negative trajectory of $v$ in $C_0^1(\overline{\mathbb{S}_+^{N-1}})$ defined by

$$\mathcal{T}_-(v) = \bigcup_{t \leq r_0 - 1} \{v(t,.)\}.$$

By the previous estimates and the Arzela-Ascoli theorem, it is a relatively compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$, hence its limit set at $-\infty$ (or alpha-limit set), denoting $A(\mathcal{T}_-(v))$, is a non-empty connected compact subset of $C_0^1(\overline{\mathbb{S}_+^{N-1}})$. Multiplying (4.5) by $v_t$ and integrating on $\mathbb{S}_+^{N-1}$ yields

$$\begin{aligned}&\int_{\mathbb{S}_+^{N-1}} \left(N-2-2a+\epsilon_2 - \frac{1}{2}\partial_t \epsilon_1\right) v_t^2 dS - \frac{1}{2}\int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\ &= \frac{d}{dt}\left[\int_{\mathbb{S}_+^{N-1}} \left(\frac{1}{2}|\nabla v|^2 - \frac{1}{2}\left[a(a+2-N)-\mu+\epsilon_3\right] v^2 - \frac{1}{2}(1+\epsilon_1) v_t^2\right) dS\right] \\ &\quad - \int_{\mathbb{S}_+^{N-1}} \left(\langle \nabla' v, \epsilon_4\rangle + \langle \nabla' v_t, \epsilon_5\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6\rangle\right) v_t^2 dS.\end{aligned} \tag{4.7}$$

Next we integrate over $(-\infty, r_2)$ for some $r_2$ large enough so that

$$\left|N-2-2a+\epsilon_2 - \frac{1}{2}\partial_t \epsilon_1\right| \geq \frac{1}{2}\left|N-2-2a\right| > 0,$$

here we use the crucial assumption (4.4). Since all the terms on the right-hand side of (4.7) are integrable on $(-\infty, r_2)$ because of (4.6) and the bounds on $v$, we obtain that

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_t^2 dS < \infty. \tag{4.8}$$

Differentiating (4.5) with respect to t and using the estimates on $v$ and the $\epsilon_j$ we obtain (see [12, p. 302] for a similar calculation)

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_{tt}^2 dS < \infty. \tag{4.9}$$

Because $v_t$ and $v_{tt}$ are uniformly continuous on $(-\infty, r_1]$, we infer from (4.8) and (4.9)

$$\lim_{t\to-\infty}\left(\|v_t(t,.)\|_{L^2(\mathbb{S}^{N-1}_+)}+\|v_{tt}(t,.)\|_{L^2(\mathbb{S}^{N-1}_+)}\right)=0.$$

Therefore the set $A(\mathcal{T}_-(v))$ is a compact connected subset of the set of nonnegative solutions of

$$\begin{cases}\Delta'\omega+(a(a+2-N)-\mu)\,\omega &= 0 \qquad \text{in } \mathbb{S}^{N-1}_+,\\ \omega &= 0 \qquad \text{on } \partial\mathbb{S}^{N-1}_+.\end{cases}$$

*Step 3. The case* $a(a+2-N)-\mu=N-1$. The set $A(\mathcal{T}_-(v))$ is a subset of $\ker(-\Delta'-(N-1))I_d$ in $H^1_0(\mathbb{S}^{N-1}_+)$ and more precisely $A(\mathcal{T}_-(v))=\{m\psi_1 : m\in I^*\}$ where $I^*$ is a compact interval of $[0,\infty)$. We set

$$X(t)=\int_{\mathbb{S}^{N-1}_+} v(t,.)\psi_1 dS.$$

Then $X$ satisfies

$$X''(t)+(N-2-2a)X'(t)+F(t)=0, \tag{4.10}$$

where

$$F(t)=\int_{\mathbb{S}^{N-1}_+}\big[\epsilon_1(t,.)v_{tt}+\epsilon_2(t,.)v_t+\epsilon_3(t,.)v+\langle\nabla' v,\epsilon_4(t,.)\rangle \\ +\langle\nabla' v_t,\epsilon_5(t,.)\rangle+\langle\nabla'(\langle\nabla' v,\mathbf{e}_N\rangle),\epsilon_6(t,.)\rangle\big]\,\psi_1 dS.$$

Then $|F(t)|\le c_{16}e^t$. We consider a sequence $\{t_n\}$ converging to $-\infty$ and $c^*\in I^*$ such that $X(t_n)\to c^*$. Since $X'(t)$ and $X''(t)$ converges to 0 as $t\to-\infty$, we integrate (4.10) on $(t_n,t)$ and let $n\to\infty$. Then we get

$$X'(t)+(N-2-2a)(X(t)-c^*)+O(e^t)=0.$$

Letting $t\to-\infty$ yields $X(t)\to c^*$. Hence we have proved that

$$\lim_{t\to-\infty} v(t,.)=c^*\psi_1 \quad \text{in } C^1(\overline{\mathbb{S}^{N-1}_+}).$$

*Step 4. The case* $a(a+2-N)-\mu\neq N-1$. Clearly $A(\mathcal{T}_-(v))=\{0\}$ and

$$\lim_{t\to-\infty} v(t,.)=0 \qquad \text{in } C^1(\overline{\mathbb{S}^{N-1}_+}). \tag{4.11}$$

Furthermore, since we have assumed $a\ge-\alpha_-$, there holds actually $a>-\alpha_-$. We recall that $\lambda_k$ is the k-th eigenvalue of $-\Delta'$ in $H^1_0(\mathbb{S}^{N-1}_+)$ and put

$$H_k=\ker(-\Delta'-\lambda_k Id)=\text{span}\langle\phi_{k,1},\phi_{k,2},...,\phi_{k,j_k}\rangle \text{ and } H^1_0(\mathbb{S}^{N-1}_+)=\overset{\infty}{\underset{k=1}{\oplus}} H_k.$$

We denote

$$P_k(x)=x^2+(N-2)x-\mu-\lambda_k.$$

Then $P_1(\alpha_-) = 0$ and $P_k(\alpha_-) = \lambda_1 - \lambda_k < 0$ for $k \geq 2$. Since $a(a+2-N) - \mu \neq N-1$ by assumption, we define a partition of $\mathbb{N}^*$ by setting

$$N_1 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k \geq 0\}, \quad N_2 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k < 0\}$$

and

$$W_1 = \underset{k\in N_1}{\oplus} H_k \text{ and } W_2 = \underset{k\in N_2}{\oplus} H_k.$$

Then

$$-\int_{\mathbb{S}^{N-1}_+} \phi \Delta' \phi dS \geq \gamma \int_{\mathbb{S}^{N-1}_+} \phi^2 dS \quad \text{for all } \ \phi \in W_2, \tag{4.12}$$

where

$$\gamma = \mu + \lambda_{k_2} - a(a+2-N) > 0 \ \text{ with } \ k_2 = \inf N_2.$$

We denote by $P_j$ the orthognal projector onto $W_j$ in $H^1_0(\mathbb{S}^{N-1}_+)$ and set $v = P_1 v + P_2 v = v_1 + v_2$. Then the projection of (4.5) on to $W_2$ is

$$(v_2)_{tt} + (N-2-2a)\,(v_2)_t + (a(a+2-N) - \mu)\, v_2 + \Delta' v_2 = F_2(t,.),$$

where $F_2$ satisfies the same estimates (4.6) as $\epsilon_j$. Then, using (4.6) and (4.12)

$$\int_{\mathbb{S}^{N-1}_+} (v_2)_{tt} v_2 dS + (N-2-2a) \int_{\mathbb{S}^{N-1}_+} (v_2)_t v_2 dS - \gamma \int_{\mathbb{S}^{N-1}_+} v_2^2 dS \geq -c_{17} e^t \left( \int_{\mathbb{S}^{N-1}_+} v_2^2 dS \right)^{\frac{1}{2}}.$$

Put $Y(t) = \|v_2(t,.)\|_{L^2(\mathbb{S}^{N-1}_+)}$, because

$$\int_{\mathbb{S}^{N-1}_+} (v_2)_t v_2 dS = Y'(t) Y(t) \ \text{ and } \ \int_{\mathbb{S}^{N-1}_+} (v_2)_{tt} v_2 dS \geq Y''(t) Y(t),$$

we obtain the following differential inequality

$$Y'' + (N-2-2a) Y' - \gamma Y \geq -c_{17} e^t \ \text{ in } \ \mathcal{D}'(-\infty, r_2).$$

The characteristic roots of the equation $y'' + (N-2-2a)y' - \gamma y = 0$ are

$$\begin{aligned} a_{k_2,-} &= a + \frac{1}{2}\left(2 - N - \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2}\right) = \alpha_{k_2,-} + a < 0 \\ a_{k_2,+} &= a + \frac{1}{2}\left(2 - N + \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2}\right) = \alpha_{k_2,+} + a > 0. \end{aligned} \tag{4.13}$$

where the $\alpha_{k_2,\pm}$ are the roots of equations (2.1) with $k = k_2$. The solutions of

$$z'' + (N-2-2a) z' - \gamma z = -c_{17} e^t \ \text{ in } \ \mathcal{D}'(-\infty, r_2).$$

endow the form $z(t) = Ae^{t a_{k_2,-}} + Be^{t a_{k_2,+}} + c_{18} e^t$ if $a_{k_2,+} \neq 1$ or $z(t) = Ae^{t a_{k_2,-}} + Be^t + Cte^t$ if $a_{k_2,+} = 1$, for some explicit constant $c_{18}$ depending on $c_{17}$ and the coefficients in the equation. Since $Y(t) \to 0$ when $t \to -\infty$ by (4.11), it follows from the maximum principle that

$$Y(t) \leq c_{19} e^{t a_{k_2+}} + c_{18} e^t \ \text{ if } \ a_{k_2+} \neq 1, \ \text{ or } \ Y(t) \leq c_{20} |t| e^t \ \text{ if } \ a_{k_2+} = 1 \quad \text{for } \ t \leq r_2.$$

Then using standard elliptic equations a priori estimates, initialy in $L^2(\mathbb{S}^{N-1}_+)$, then in $L^p(\mathbb{S}^{N-1}_+)$ and finally in $C^\tau(\overline{\mathbb{S}^{N-1}_+})$, we obtain that for $t \le r_3$,

$$\|v_2(t,.)\|_{C^1(\mathbb{S}^{N-1}_+)} \le \begin{cases} c_{21}e^{ta_{k_2,+}} + c_{22}e^t & \text{if } \ a_{k_2,+} \neq 1, \\ c_{23}|t|e^t & \text{if } \ a_{k_2,+} = 1, \end{cases} \tag{4.14}$$

where $r_3 \le r_2 - 1$.
For the components in $W_1$ we have

$$v_1(t,\cdot) = \sum_{k\in N_1} \sum_{1\le j\le j_k} w_{k,j}(t)\phi_{k,j}(\cdot), \tag{4.15}$$

where the $\phi_{k,j}$ form an orthoromal basis of $H_k$. Then

$$w''_{k,j} + (N-2-2a)w'_{k,j} + (a(a+2-N) - \mu - \lambda_k)\, w_{k,j} = F_{k,j}(t) \tag{4.16}$$

The characteristic roots of equation $z'' + (N-2-2a)z' + (a(a+2-N) - \mu - \lambda_k)\, z = 0$ are given in (4.13) with a general $k$, $a_{k-} = a + \alpha_{k-}$ and $a_{k+} = a + \alpha_{k+}$ where $\alpha_{k\pm}$ are the roots of (2.1). They have same sign (including 0) since $a(a+2-N) - \mu - \lambda_k \ge 0$, furthermore, their sum is positive since $N - 2 - 2a < 0$, as a consequence of $a > -\alpha_-$. By standard calculation the solution of (4.16) has the form

$$w_{k,j}(t) = m_1 e^{ta_{k+}} + m_2 e^{ta_{k-}} - \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s)ds. \tag{4.17}$$

Since $|F_{k,j}(s)| \le c_{24}e^s$ there holds

$$\left| \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s)ds \right| \le c_{25} \begin{cases} |t|e^t & \text{if } a_{k-} = 1 \\ \max\{e^t, e^{ta_{k-}}\} & \text{if } a_{k-} \neq 1 \end{cases} \tag{4.18}$$

In particular, if $k_1 = \max N_1$, then $a_{k_1\pm} = \min\{a_{k\pm} : k \in N_1\}$.
We assume first that $a_{k_1-} > 0$. Combining this fact with (4.15) and (4.18) we obtain

$$\|v_1(t,.)\|_{L^\infty(\mathbb{S}^{N-1}_+)} \le c_{26} \begin{cases} |t|e^t & \text{if } a_{k_1-} = 1 \\ \max\{e^t, e^{ta_{k_1-}}\} & \text{if } a_{k_1-} \neq 1 \end{cases} \tag{4.19}$$

Furthermore, because of the explicit formulation and (4.6), the left-hand side of (4.18) can be replaced by $\|v_1(t,.)\|_{C^1(\overline{\mathbb{S}^{N-1}_+})}$. Combining (4.14) and (4.18) we obtain the result since $v(t,.) = 0$ on $(-\infty, r_1) \times \partial\mathbb{S}^{N-1}_+$.
Next we suppose that $a_{k_1-} = 0$. Then for $k = k_1$, (4.17) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1+}} + m_2 - \frac{1}{a_{k_1+}} \int_t^0 (e^{(t-s)a_{k_1+}} - 1)F_{k_1,j}(s)ds. \tag{4.20}$$

This implies that

$$w_{k_1,j}(t) \to m_2 + \frac{1}{a_{k_1+}} \int_{-\infty}^0 F_{k_1,j}(s)ds := A_{k_1,j} \quad \text{as } \ t \to \infty.$$

If $A_{k_1,j} \neq 0$ it would imply that $\sum_{j=1}^{j_k} A_{k_1,j}\phi_{k_1,j}$ is a nonzero eigenfunction of order $k_1 > 1$, hence it changes sign and it would imply that $v$ changes sign at $-\infty$ (notice that all the other terms $w_{k,j}(t)$ tends to 0 exponentially because of (4.17)-(4.18)). Hence $A_{k_1,j} = 0$ and (4.20) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1,+}} - \frac{1}{a_{k+}}\int_t^0 e^{(t-s)a_{k,+}} F_{k_1,j}(s)ds - \int_{-\infty}^t F_{k_1,j}(s)ds.$$

Because

$$\int_{-\infty}^t F_{k_1,j}(s)ds = O(e^t) \quad \text{as} \ \ t \to \infty,$$

we conclude that for $k = k_1$, there holds

$$|w_{k_1,j}(t)| \leq c_{27} \begin{cases} |t|e^t & \text{if } \ a_{k_1,+} = 1, \\ \max\{e^t, e^{ta_{k_1,+}}\} & \text{if } \ a_{k_1,+} \neq 1 \end{cases}$$

and finally we infer (4.19), which complete the proof. □

**Proof of Theorem A.** Assume that $u \in C^2(\overline{\Omega}\setminus\{0\})$ is a positive solution of $\mathcal{L}_\mu u = 0$ vanishing on $\partial\Omega \setminus \{0\}$.

*Case 1:* $\mu > \mu_1$. We claim (4.2) holds for some $c_{13} \geq 0$.

By Lemma 4.1, (4.1) holds for some $a > 0$. If $a < -\alpha_-$, then (4.2) holds with $c_{13} = 0$. If $a = -\alpha_-$, then (4.2) holds by Proposition 4.2-(i). Hence we are left with the case $a > -\alpha_-$. As in the proof of Proposition 4.2 we define $k_1$ and $k_2$. By replacing $a$ by $a' = a + \epsilon$, we can assume that $a_{k_2,+} \neq 1$ and $a_{k_1,-} \neq 1$, to avoid the resonance complication in (4.14) and (4.19), hence

$$\|v(t,.)\|_{C^1(\overline{\mathbb{S}^{N-1}_+})} \leq c_{27}\left(e^{ta_{k_2,+}} + e^{ta_{k_1,-}} + e^t\right).$$

Furthermore $k_2 = k_1 + 1$ and

$$a_{k_2+} - a_{k_1-} = \frac{1}{2}\left(\sqrt{4\mu + 4\lambda_{k_1+1} + (N-2)^2} + \sqrt{4\mu + 4\lambda_{k_1} + (N-2)^2}\right) > 0,$$

which yields

$$\|v(t,.)\|_{C^1(\overline{\mathbb{S}^{N-1}_+})} \leq c_{28}\left(e^{ta_{k_1,-}} + e^t\right).$$

This implies that $u$ satisfies

$$u(x) \leq c_{29}\left(|x|^{\alpha_{k_1,-}} + |x|^{1-a}\right)\rho(x).$$

We iterate this procedure up to obtain

$$u(x) \leq c_{30}|x|^{\alpha_-}\rho(x)$$

and we conclude as in the proof of Proposition 4.2, Step 3. □

*Case:* $\mu = \mu_1$. In this case, the difficulty comes from the fact that there is no dissipation of energy in (4.7) for $a = -\alpha_- = \frac{N-2}{2}$. But from the above iterative procedure in *Case:* $\mu > \mu_1$, we could obtain could obtain that for some $\delta \in (0,\, 1)$,

$$u(x) \leq c_{31}|x|^{-\frac{N-2}{2}-\delta}\rho(x).$$

*We finally show that there exists* $c_{32} \geq 0$ *such that*

$$\lim_{r\to 0} r^{\frac{N-2}{2}} \frac{u(r,.)}{\ln r} = -c_{32}\psi_1(.) \tag{4.21}$$

*in* $C^1(\mathbb{S}^{N-1}_+)$ *and*

$$\lim_{r\to 0} r^{\frac{N}{2}} \frac{u_r(r,.)}{\ln r} = \frac{(N-2)c_{32}}{2}\psi_1(.) \tag{4.22}$$

*uniformly in* $\mathbb{S}^{N-1}_+$.

Note that (4.5) reduces that

$$\begin{aligned}(1+\epsilon_1(t))v_{tt} + \epsilon_2(t)v_t + (N-1+\epsilon_3(t))v + \Delta' v \\ + \langle \nabla' v, \epsilon_4(t,.)\rangle + \langle \nabla' v_t, \epsilon_5(t,.)\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6(t,.)\rangle = 0,\end{aligned}$$

in $(-\infty, r_0)\times \mathbb{S}^{N-1}_+$, vanishes on $(-\infty, r_0)\times \partial\mathbb{S}^{N-1}_+$ and the $\epsilon_j$ verify (4.6), and

$$v(t,\sigma) \leq c_{33}e^{-\delta t}.$$

Since the operator involved in the equation is uniformly elliptic we have by standard regularity theory

$$\begin{aligned}\|v\|_{C^{2,\delta}([T-1,T+1]\times\overline{\mathbb{S}^{N-1}_+})} + \|v_t\|_{C^{1,\delta}([T-1,T+1]\times\overline{\mathbb{S}^{N-1}_+})} + \|v_{tt}\|_{C^{\delta}([T-1,T+1]\times\overline{\mathbb{S}^{N-1}_+})} \\ \leq c_{34}\|v\|_{L^\infty((T-2,T+2)\times\mathbb{S}^{N-1}_+} \\ \leq c_{35}e^{-\delta T}\end{aligned}$$

for any $T \leq r_0 + 3$. We set

$$X(t) = \int_{\mathbb{S}^{N-1}_+} v(t,.)\psi_1 dS,$$

then

$$X''(t) + F(t) = 0 \tag{4.23}$$

where

$$F(t) = \int_{\mathbb{S}^{N-1}_+} \big(\epsilon_1 v_{tt} + \epsilon_2 v_t + \epsilon_3 v + \langle \nabla' v, \epsilon_4(t,.)\rangle + \langle \nabla' v_t, \epsilon_5(t,.)\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6(t,.)\rangle\big)\,\psi_1 dS.$$

Hence

$$|F(t)| \leq c_{36}e^{(1-\delta)t}. \tag{4.24}$$

This implies that $X'(t)$ admits a limit $c_{37} \leq 0$ when $t \to -\infty$ and

$$\lim_{t\to-\infty} t^{-1}X(t) = c_{37}.$$

Set

$$W_2 = \bigoplus_{k\geq 2} \ker(\Delta' + \lambda_k Id),$$

and denote by $v_2$ the orthogonal projection of $v$ onto $W_2$. Then

$$v_{2\,tt} + (N-1)v_2 + \Delta' v_2 = F_2(t,.), \tag{4.25}$$

where

$$|F_2(t,.)| \leq c_{38}e^{(1-\delta)t}.$$

Since $\lambda_2 = 2N$, the function $Y(t) = \|v_2(t,.)\|_{L^2(\mathbb{S}^{N-1}_+)}$ satisfies in $\mathcal{D}'(-\infty, r_1)$

$$Y'' - (N+1)Y \geq -c_{38}e^{(1-\delta)t}.$$

Because $Y(t) = o(e^{-\sqrt{N+1}t})$ when $t \to -\infty$, it follows by the maximum principle that $Y(t) = O(e^{\sqrt{N+1}t} + e^{(1-\delta)t}) = O(e^{(1-\delta)t})$. Using again the standard regularity estimates for elliptic equations, we derive

$$\|v_2(t,.)\|_{C^1(\overline{\mathbb{S}^{N-1}_+})} + \|v_{2\,t}(t,.)\|_{C(\overline{\mathbb{S}^{N-1}_+})} \leq c_{39}e^{(1-\delta)t}. \tag{4.26}$$

Combining (4.23) and (4.25) we derive (4.21). Since $v(t,.) = X(t)\psi_1 + v_2(t,.)$ it follows from (4.26) that

$$\lim_{t\to-\infty} v_t(t,.) = c_{37}\psi_1 \quad \text{uniformly in } \mathbb{S}^{N-1}_+.$$

Thus, the indentity $u_r(r,\cdot) = r^{-\frac{N}{2}}\left(\frac{2-N}{2}v(t,\cdot) + v_t(t,\cdot)\right)$ implies (4.21) and (4.22). □

## 4.2 Existence and uniqueness

**Proof of Theorem B.** We still assume that $\Omega$ satisfies the condition ($\mathcal{C}$-1) and $\partial\mathbb{R}^N_+$ is tangent to $\partial\Omega$ at 0. For $\epsilon > 0$ let $u_\epsilon$ be the solution of

$$\begin{cases} \mathcal{L}_\mu u_\epsilon = 0 & \text{in } \Omega_\epsilon := \Omega \setminus \overline{B}_\epsilon, \\ u_\epsilon = 0 & \text{in } \partial\Omega \cap \overline{B}^c_\epsilon, \\ u_\epsilon = \phi_\mu & \text{in } \Omega \cap \partial B_\epsilon. \end{cases} \tag{4.27}$$

Since $\Omega \subset \mathbb{R}^N_+$, $u_\epsilon \leq \phi_\mu$ in $\Omega_\epsilon$ and

$$\frac{\partial u_\epsilon}{\partial \mathbf{n}}\Big|_{\Omega\cap\partial B_\epsilon} \leq \frac{\partial \phi_\mu}{\partial \mathbf{n}}\Big|_{\Omega\cap\partial B_\epsilon} < 0, \tag{4.28}$$

where $\mathbf{n} = \epsilon^{-1}\mathbf{x}$. Furthermore, if $0 < \epsilon' < \epsilon$, $u_{\epsilon'}\lfloor_{\Omega\cap\partial B_\epsilon} \leq u_\epsilon\lfloor_{\Omega\cap\partial B_\epsilon} = \phi_\mu$, hence $u_{\epsilon'} \leq u_\epsilon$ in $\Omega_\epsilon$. There exists $u_0 = \lim_{\epsilon\to 0} u_\epsilon$ and $u_0$ is a nonnegative solution of $\mathcal{L}_\mu u = 0$ in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$ and is smaller than $\phi_\mu$.

Let $\zeta \in \mathbb{X}_\mu(\Omega)$, $\zeta > 0$, then, with $\mathbf{n}' = -\frac{x}{|x|} = -\mathbf{n}$,

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu u_\epsilon dx \\ &= \int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\partial B_\epsilon \cap \Omega} \left( -\frac{\partial u_\epsilon}{\partial \mathbf{n}'} \zeta \gamma_\mu^\Omega + \left( \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}'} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}'} \right) u_\epsilon \right) dS. \end{aligned}$$

Using (4.27) and (4.28) we obtain

$$\int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega - \left( \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS.$$

We take $\zeta = 1$, hence $\mathcal{L}_\mu^* \zeta = \ell_\mu^\Omega$ and we get

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case $\mu > \mu_1$, and

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq \left( \frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case $\mu = \mu_1$. Since $u_\epsilon \leq \phi_\mu^\Omega$,

$$u_\epsilon \gamma_\mu^\Omega \leq \gamma_\mu^\Omega \phi_\mu^\Omega = r^{2-N} \psi_1^2 \in L^1(\Omega).$$

Therefore, by dominated convergence theorem, we conclude that

$$\ell_\mu^\Omega \int_\Omega u_0 d\gamma_\mu^\Omega \geq \begin{cases} 2\sqrt{\mu + \mu_1} \displaystyle\int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \ \mu > \mu_1, \\ \left( \dfrac{N}{2} - 1 \right) \displaystyle\int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \ \mu = \mu_1. \end{cases} \tag{4.29}$$

We infer that the function $u_0$ is nonzero. It is a positive solution of $\mathcal{L}_\mu u_0 = 0$ in $\Omega$ which vanishes on $\partial\Omega \setminus \{0\}$. It follows from **Theorem A** that there exists $k \geq 0$ such that

$$\begin{aligned} &\lim_{x \to 0} \frac{u(x)}{\rho(x)|x|^{\alpha_- - 1}} = k && \text{if } \ \mu > \mu_1, \\ &\lim_{x \to 0} \frac{u(x)}{\rho(x)|x|^{-N/2} \ln |x|} = k && \text{if } \ \mu = \mu_1. \end{aligned}$$

Next we next show that $k=1$. In fact, if $k<1$, there exists $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$

$$u_\epsilon\le\frac{k+1}{2}\phi_\mu^\Omega,$$

and then

$$\lim_{\epsilon\to0^+}\ell_\mu^\Omega\int_{\Omega_\epsilon}u_\epsilon d\gamma_\mu^\Omega\le\frac{k+1}{2}\ell_\mu^\Omega\int_{\Omega_\epsilon}\phi_\mu^\Omega d\gamma_\mu^\Omega<\begin{cases}2\sqrt{\mu+\mu_1}\displaystyle\int_{\mathbb{S}_+^{N-1}}\psi_1^2dS & \text{if } \mu>\mu_1,\\ \left(\dfrac{N}{2}-1\right)\displaystyle\int_{\mathbb{S}_+^{N-1}}\psi_1^2dS & \text{if } \mu=\mu_1,\end{cases}$$

which contradicts (4.29). Thus, (1.11) and (1.12) hold true. □

**Proof of Corollary C.** *Identity (1.14).* As a consequence of Proposition 4.2, for any $\zeta\in\mathbb{X}_\mu(\Omega)$ and $\epsilon>0$ we set $\Omega_\epsilon=\Omega\cap\overline{B}_\epsilon^c$, and there holds

$$\begin{aligned}0&=\int_{\Omega^\epsilon}\zeta\gamma_\mu^\Omega\mathcal{L}_\mu\phi_\mu^\Omega dx\\&=\int_{\Omega^\epsilon}\phi_\mu^\Omega\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega+\int_{\Omega\cap\partial B_\epsilon}\left(-\frac{\partial\phi_\mu^\Omega}{\partial\mathbf{n}}\zeta\gamma_\mu^\Omega+\left(\gamma_\mu^\Omega\frac{\partial\zeta}{\partial\mathbf{n}}+\zeta\frac{\partial\gamma_\mu^\Omega}{\partial\mathbf{n}}\right)\phi_\mu^\Omega\right)dS.\end{aligned}$$

Using Proposition **??**, we have

$$\int_{\Omega\cap\partial B_\epsilon}\left(-\frac{\partial\phi_\mu^\Omega}{\partial\mathbf{n}}\zeta\gamma_\mu^\Omega+\left(\gamma_\mu^\Omega\frac{\partial\zeta}{\partial\mathbf{n}}+\zeta\frac{\partial\gamma_\mu^\Omega}{\partial\mathbf{n}}\right)\phi_\mu^\Omega\right)dS=-\zeta(0)A(\epsilon)(1+o(1)),$$

where $A(\epsilon)$ is defined in (2.5).

The uniqueness follows direct from Kato's inequality (3.4). □

# 5 The Dirichlet problem

**Proof of Theorem D.** Note that in section §3.2 for $\lambda\in\mathfrak{M}(\partial\Omega;\eta\beta_\mu^\Omega)$, problem

$$\begin{cases}\mathcal{L}_\mu u=0 & \text{in } \Omega,\\ u=\lambda & \text{on } \partial\Omega\end{cases}$$

has a unique solution, denoting $\mathbb{K}_\mu^\Omega(\lambda)$, which verifies the indentity

$$\int_\Omega\mathbb{K}_\mu^\Omega(\lambda)\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega=\int_{\partial\Omega}\zeta d(\lambda\beta_\mu^\Omega)\quad\text{for all }\zeta\in\mathbb{X}_\mu(\Omega).$$

Moreover, problem

$$\begin{cases}\mathcal{L}_\mu u=\nu & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega\end{cases}$$

has a unique solution, denoting $\mathbb{G}_\mu^\Omega(\nu)$, which verifies the indentity

$$\int_\Omega \mathbb{K}_\mu^\Omega(\lambda)\mathcal{L}_\mu^*\zeta d\gamma_\mu^\Omega = \int_\Omega \zeta d\gamma_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Together with **Corollary C** and the linearity of operator $\mathcal{L}_\mu$, we have that $\mathbb{K}_\mu^\Omega(\lambda)+\mathbb{G}_\mu^\Omega(\nu)+k\phi_\Omega^\mu$ is a weak solution of (1.18) satisfying (1.20) and the uniqueness follows directly from Kato's inequality (3.4). □

Our final part is to classify the boundary data for nonnegative $\mathcal{L}_\mu$-harmonic function.

**Proof of Theorem E.** Let $\Omega$ be a bounded $C^2$ domain and $u$ be a nonnegative $\mathcal{L}_\mu$-harmonic function in $\Omega$. We now show that there exists a nonnegative measure $\lambda$ on $\partial\Omega \setminus \{0\}$ and $k \geq 0$ such that

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega. \tag{5.1}$$

For $\epsilon > 0$ the term $\mu|x|^{-2}$ is bounded in $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$. Hence the exists a nonnegative Radon measure $\lambda_\epsilon$ such that $u$ is the unique solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega_\epsilon, \\ u = \lambda_\epsilon & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Furthermore $\lambda_\epsilon$ is the boundary trace is achieved in *dynamical sense*, see [14] and references therein. Hence for any $\zeta \in C(\overline{\Omega})$ vanishing on $B_\epsilon$, there holds

$$\lim_{\delta\to 0}\int_{\Sigma_\delta} u\zeta dS = \int_{\partial\Omega\cap B_\epsilon^c} \zeta d\lambda_\epsilon,$$

where $\Sigma_\delta = \{x \in \Omega : \rho(x) = \delta\}$. If we write

$$\lambda_\epsilon = \lambda_\epsilon\lfloor_{\partial\Omega\cap B_\epsilon^c} + u\lfloor_{\Omega\cap\partial B_\epsilon},$$

it proves that for $0 < \epsilon' < \epsilon$, $\lambda_\epsilon\lfloor_{\partial\Omega\cap B_\epsilon^c} = \lambda_{\epsilon'}\lfloor_{\partial\Omega\cap B_\epsilon^c}$. This defines in a unique way a nonnegative Radon $\lambda$ on $\partial\Omega \setminus \{0\}$ measure such that (5.1) holds for all $\zeta \in \mathbb{X}_\mu(\Omega)$ vanishing near 0. Furthermore $\rho u \in L^1(\Omega_\epsilon)$ for any $\epsilon > 0$. Denote by $\mathbb{K}_\mu^{\Omega_\epsilon}$ the Poisson potential of $\mathcal{L}_\mu$ in $\Omega_\epsilon$. Then

$$u\lfloor_{\Omega_\epsilon} = \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon\lfloor_{\partial\Omega\cap B_\epsilon^c}] + \mathbb{K}_\mu^{\Omega_\epsilon}[u\lfloor_{\Omega\cap\partial B_\epsilon}].$$

For $0 < \epsilon' < \epsilon$, one has that $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}\lfloor_{\partial\Omega\cap B_{\epsilon'}^c}]\lfloor_{\Omega\cap\partial B_\epsilon} \geq 0$. Therefore $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}\lfloor_{\partial\Omega\cap B_{\epsilon'}^c}] \geq \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon\lfloor_{\partial\Omega\cap B_\epsilon^c}]$ in $\Omega_\epsilon$. Hence

$$\lim_{\epsilon\to 0}\mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon\lfloor_{\partial\Omega\cap B_\epsilon^c}] = \mathbb{K}_\mu^\Omega[\lambda] \leq u \quad \text{in } \Omega.$$

Next we aim to characterize the behaviour at 0. By contradiction we assume that

$$\limsup_{\epsilon\to 0}\int_{\Omega\cap\partial B_\epsilon} ud\beta_\mu^\Omega = \lim_{\epsilon_k\to 0}\int_{\Omega\cap\partial B_{\epsilon_k}} ud\beta_\mu^\Omega = \infty.$$

Then for any $m>0$ there exists a sequence $\{\epsilon_{m,k}\}\subset\mathbb{R}_+^*$ tending to 0 and a sequence $\{\ell_{m,k}\}\subset\mathbb{R}_+^*$ tending to $\infty$ such that

$$\int_{\Omega\cap\partial B_{\epsilon_k}}\min\{u,\ell_{m,k}\}d\beta_\mu^\Omega=m.$$

Set $\tau_{m,k}=\min\{u\lfloor_{\Omega\cap\partial B_{\epsilon_k}},\ell_{m,k}\}$ and set $u_{m,k}=\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\tau_{m,k}\chi_{\Omega\cap\partial B_{\epsilon_k}}]$. Then

$$u_{m,k}\le u\qquad\text{in }\ \Omega_{\epsilon_k},$$

and we recall that

$$\int_{\Omega\cap\partial B_{\epsilon_k}}\phi_\mu^\Omega d\beta_\mu^\Omega=c_\mu(1+\circ(1)),$$

where $c_\mu$ is the constant defined in (2.4). Combining the boundary Harnack inequality with the standard Harnack inequality, one infers

$$c_{47}\frac{\rho(x)}{\rho(y)}\le c_{46}\frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)}\le\frac{u_{m,k}(x)}{u_{m,k}(y)}\le c_{44}\frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)}\le c_{45}\frac{\rho(x)}{\rho(y)},\tag{5.2}$$

for all $x,y\in\Omega$ such that $|x|=|y|\ge2\epsilon_k$. If we set $\dot\phi_\mu^\Omega(x)=\dfrac{\phi_\mu^\Omega(x)}{\rho(x)}$ and $\dot u_{m,k}(x)=\dfrac{u_{m,k}(x)}{\rho(x)}$ then (5.2) becomes

$$c_{47}\le c_{46}\frac{\dot\phi_\mu^\Omega(x)}{\dot\phi_\mu^\Omega(y)}\le\frac{\dot u_{m,k}(x)}{\dot u_{m,k}(y)}\le c_{44}\frac{\dot\phi_\mu^\Omega(x)}{\dot\phi_\mu^\Omega(y)}\le c_{45}.\tag{5.3}$$

Assume for a while that we have proved that there exists $\theta>0$, independent of $m$ and $k$ such that for for any

$$\int_{\partial B_{2\epsilon_k}\cap\Omega}\dot u_{m,k}d\left(\rho\beta_\mu^\Omega\right)\ge\theta\int_{\partial B_{\epsilon_k}\cap\Omega}\dot u_{m,k}d\left(\rho\beta_\mu^\Omega\right)=\theta m.\tag{5.4}$$

If we assume that for $\delta\le2\epsilon_{k_0}$

$$2c_\mu\ge\int_{\Omega\cap\partial B_\delta}\dot\phi_\mu^\Omega d\left(\rho\beta_\mu^\Omega\right)\ge\frac{c_\mu}{2},$$

one has for $k\ge k_0$,

$$\int_{\Omega\cap\partial B_{2\epsilon_k}}\dot u_{m,k}d\left(\rho\beta_\mu^\Omega\right)\ge\theta m\ge\frac{\theta m}{2c_\mu}\int_{\Omega\cap\partial B_{2\epsilon_k}}\dot\phi_\mu^\Omega d\left(\rho\beta_\mu^\Omega\right)$$

Since

$$\dot\phi_\mu^\Omega(x)\le\frac{c_{45}}{c_{46}}\dot\phi_\mu^\Omega(y)$$

and

$$\dot u_{m,k}(x)\ge c_{47}\dot u_{m,k}(y),$$

we derive

$$\frac{1}{c_{47}}\dot u_{m,k}(x)\int_{\partial B_{2\epsilon_k}\cap\Omega}d\left(\rho\beta_\mu^\Omega\right)\ge\frac{\theta mc_{46}}{2c_\mu c_{45}}\dot\phi_\mu^\Omega(x)\int_{\Omega\cap\partial B_{2\epsilon_k}}d\left(\rho\beta_\mu^\Omega\right).$$

Therefore

$$u_{m,k}(x) \geq c_{48} m \phi_\mu^\Omega(x) \quad \text{for all } \ x \in \Omega \ \text{ s.t. } \ | \, x \, | = 2\epsilon_k,$$

and $c_{48} > 0$ is independent of $m$ and $\epsilon_k$. This implies by the maximum principle and letting $\epsilon_k \to 0$

$$u(x) \geq u_{m,k}(x) \geq c_{48} m \phi_\mu^\Omega(x) \quad \text{for all } \ x \in \Omega. \tag{5.5}$$

Since $m$ is arbitrary we obtain a contradiction. Hence there holds

$$\limsup_{\epsilon \to 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \to 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = m_u < \infty. \tag{5.6}$$

Then inequality (5.5) holds without truncation with $m$ replaced by $m_u$. We recall that

$$w_\epsilon := \mathbb{K}_\mu^{\Omega_\epsilon}[u \lfloor_{\Omega \cap \partial B_\epsilon}] = u \lfloor_{\Omega_\epsilon} - \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon \lfloor_{\partial\Omega \cap B_\epsilon^c}] \quad \text{in } \ \Omega_\epsilon. \tag{5.7}$$

*Case 1: We first assume that* $m_u > 0$. Then (5.4) combined with the maximum principle yields

$$\int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d\left(\rho \beta_\mu^\Omega\right) \geq \int_{\partial B_{2\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d\left(\rho \beta_\mu^\Omega\right) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d\left(\rho \beta_\mu^\Omega\right) = \theta m_u (1 + o(1)),$$

with $\dot{w}_\epsilon = \rho^{-1} w_\epsilon$. Inequality (5.3) is replaced by

$$c_{47} \leq c_{46} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq \frac{\dot{w}_\epsilon(x)}{\dot{w}_\epsilon(y)} \leq c_{44} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq c_{45} \quad \text{in } \ \Omega_{2\epsilon}. \tag{5.8}$$

Therefore, for $\epsilon_k$ small enough and $|x| = 2\epsilon_k$,

$$\begin{aligned} \dot{w}_\epsilon(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d\left(\rho \beta_\mu^\Omega\right) &\leq c_{45} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k}(y) d\left(\rho \beta_\mu^\Omega\right) \leq 2 c_{45} m_u \\ &\leq \frac{4 c_{45} m_u}{c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{\phi}_\mu^\Omega(y) d\left(\rho \beta_\mu^\Omega\right) \leq \frac{4 c_{44} c_{45} m_u}{c_\mu c_{47}} \dot{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d\left(\rho \beta_\mu^\Omega\right), \end{aligned}$$

which implies

$$w_{\epsilon_k}(x) \leq \frac{4 c_{44} c_{45} m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49} m_u \phi_\mu^\Omega(x) \quad \text{for } \ x \in \Omega \cap \partial B_{2\epsilon_k}. \tag{5.9}$$

Hence

$$w_{\epsilon_k}(x) \leq \frac{4 c_{44} c_{45} m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49} m_u \phi_\mu^\Omega(x) \quad \text{for } \ x \in \Omega \cap \partial B_{2\epsilon_k}. \tag{5.10}$$

Since $w_{\epsilon_k}$ and $\phi_\mu$ are $\mathcal{L}_\mu$-harmonic in $\Omega_{2\epsilon_k}$, and vanishes on $\partial\Omega \cap B_{2\epsilon_k}$ it follows that inequality (5.10) also holds for any $x \in \Omega_{2\epsilon_k}$. By definition $w_{\epsilon_k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u \lfloor_{\Omega \cap \partial B_{\epsilon_k}}]$, hence

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u \lfloor_{\Omega \cap \partial B_{\epsilon_k}}](x) \leq c_{49} m_u \phi_\mu^\Omega(x) \quad \text{for } \ x \in \Omega_{2\epsilon_k}. \tag{5.11}$$

Next we obtain the estimate from below. From (5.8), with $|x| = 2\epsilon_k$,

$$\dot{w}_{\epsilon_k}(x)\int_{\partial B_{\epsilon_k}\cap\Omega} d\left(\rho\beta_\mu^\Omega\right) \geq\ c_{47}\int_{\partial B_{\epsilon_k}\cap\Omega}\dot{w}_{\epsilon_k}(y)d\left(\rho\beta_\mu^\Omega\right)\ \geq \frac{c_{47}m_u}{2}$$
$$\geq \frac{c_{47}m_u}{4c_\mu}\int_{\partial B_{\epsilon_k}\cap\Omega}\dot{\phi}_\mu^\Omega(y)d\left(\rho\beta_\mu^\Omega\right) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}}\dot{\phi}_\mu^\Omega(x)\int_{\partial B_{\epsilon_k}\cap\Omega} d\left(\rho\beta_\mu^\Omega\right).$$

Hence

$$w_{\epsilon_k}(x) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}}\phi_\mu^\Omega(x) := c_{50}m_u\phi_\mu^\Omega(x) \quad \text{for}\ \ x\in\Omega\cap\partial B_{2\epsilon_k}.$$

It follows that

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u\lfloor_{\Omega\cap\partial B_{\epsilon_k}}](x) \geq c_{50}m_u\phi_\mu^\Omega(x) \quad \text{for}\ \ x\in\Omega_{2\epsilon_k}. \tag{5.12}$$

From (5.7), (5.12) and (5.11) we infer

$$c_{50}m_u\phi_\mu^\Omega \leq u\lfloor_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}\lfloor_{\partial\Omega\cap B^c_{\epsilon_k}}] \leq c_{48}m_u\phi_\mu^\Omega \quad \text{in}\ \ \Omega_{2\epsilon_k}. \tag{5.13}$$

This implies, by letting $\epsilon_k\to 0$,

$$c_{50}m_u\phi_\mu^\Omega \leq u - \mathbb{K}_\mu^\Omega[\lambda] \leq c_{48}m_u\phi_\mu^\Omega \quad \text{in}\ \ \Omega.$$

Therefore, the function $u-\mathbb{K}_\mu^\Omega[\lambda]$ is $\mathcal{L}_\mu$-harmonic and positive in $\Omega$ and it vanishes on $\partial\Omega$. By Corollary C, it implies that it coincides with $c\phi_\mu^\Omega$ for some $c\geq 0$ (and in that case $c_{50}m_u\leq c\leq c_{49}m_u$).

*Case 2: Assume $m_u = 0$.* Following the same inequalities as in Case 1, (5.9) is replaced by: for any $\delta>0$ there exists $k_0>0$ such that for $k\geq k_0$,

$$w_{\epsilon_k}(x) \leq \delta\phi_\mu^\Omega(x) \quad \text{for}\ \ x\in\Omega\cap\partial B_{2\epsilon_k}.$$

Hence (5.13) is transformed into

$$0 \leq u\lfloor_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}\lfloor_{\partial\Omega\cap B^c_{\epsilon_k}}] \leq \delta\phi_\mu^\Omega \quad \text{in}\ \ \Omega_{2\epsilon_k}.$$

Letting successively $\epsilon_k\to 0$ and $\delta\to 0$ yields $u-\mathbb{K}_\mu^\Omega[\lambda]=0$ in $\Omega$, which ends the proof. □

## Appendx: Estimates (1.9)

**Proposition A.1** *Assume $\Omega$ is a bounded $C^2$ domain such that $0\in\partial\Omega$ satisfying condition (C-1) and let $\gamma_\mu^\Omega$ be defined by (1.8) and normalized by $\|\gamma_\mu^\Omega\|_{L^2(\Omega)}=1$. Then*

$$\lim_{r\to 0} r^{\frac{N}{2}-1}\gamma_\mu^\Omega(r,.) = c_1\psi_1 \quad \textit{in}\ \ C^1_{loc}(\mathbb{S}^{N-1}_+)$$

*and*

$$\lim_{r\to 0} r^{\frac{N}{2}}\gamma_{\mu\,r}^\Omega(r,.) = c_1\left(1-\frac{N}{2}\right)\psi_1 \quad \textit{locally uniformly in}\ \mathbb{S}^{N-1}_+.$$

**Proof.** Since $\alpha_+ + (N-2)\alpha_+ - \mu + 1 - N = 0$, the function $x \mapsto w(x) := |x|^{\alpha_+}$ satisfies

$$\tilde{\mathcal{L}}_\mu w(x) := \mathcal{L}_\mu w - \ell_\mu^\Omega w = |x|^{\alpha_+ - 2}\left(N - 1 - \ell_\mu^\Omega |x|^2\right) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore, $\nabla w \in L^2_{loc}(\mathbb{R}^N)$. Let $R_0 > 0$ such that $N - 1 \geq \ell_\mu^\Omega R_0^2$ and $m > 0$ such that $mw \geq \gamma_\mu^\Omega$ on $\Omega \cap B^c_{R_0}$. Then the function $(\gamma_\mu^\Omega - mw)_+$ belongs to $H_\mu(\Omega)$ and satisfies $\tilde{\mathcal{L}}_\mu(\gamma_\mu^\Omega - mw)_+ \leq 0$ in the dual of $H_\mu(\Omega)$. Hence

$$\int_\Omega \left( |\nabla(\gamma_\mu^\Omega - mw)_+|^2 + \left(\frac{\mu}{|x^2|} - \ell_\mu^\Omega\right)(\gamma_\mu^\Omega - mw)_+^2 \right) dx \leq 0.$$

Therefore $(\gamma_\mu^\Omega - mw)_+ \leq 0$, which implies that

$$0 < \gamma_\mu^\Omega(x) \leq m|x|^{\alpha_+} \quad \text{for all } \ x \in \Omega.$$

Then we proceed as in the proof of Proposition 4.2. We flatten the boundary near 0 and set

$$v(t,\sigma) = r^{-\alpha_+}\tilde{\gamma}_\mu^\Omega(r,\sigma) \quad \text{with } \ t = \ln r,$$

where the function $\tilde{\gamma}_\mu^\Omega$ is defined similarly as $\tilde{u}$ in (4.3). Then $v$ is bounded in $(-\infty, T_0] \times \mathbb{S}^{N-1}_+$ where it satisfies

$$\begin{aligned}(1 + \epsilon_1(t,.))v_{tt} + (N - 2 + 2\alpha_+ + \epsilon_2(t,.))\, v_t + \left(\alpha_+(\alpha_+ + N - 2) - \mu + \epsilon_3(t,.) + e^{2t}\ell_\mu^\Omega\right) v \\ + \Delta' v + \langle \nabla' v, \epsilon_4(t,.)\rangle + \langle \nabla' v_t, \epsilon_5(t,.)\rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N\rangle), \epsilon_6(t,.)\rangle = 0,\end{aligned}$$

instead of (4.5). It vanishes on $(-\infty, T_0] \times \partial\mathbb{S}^{N-1}_+$ and the $\epsilon_j$ satisfy again (4.6).

*Case 1:* $\mu > \mu_1$. The energy method used in proof of Proposition 4.2 applies with no modification and we infer that there exists $c_{51} \geq 0$ such that

$$v(t,.) \to c_{51}\psi_1 \quad \text{as } \ t \to -\infty$$

in $C^1(\mathbb{S}^{N-1}_+)$ and $v_t(t,.) \to 0$ uniformly in $\mathbb{S}^{N-1}_+$. If $c_{51} = 0$, we can prove, as in Proposition 4.2-(ii) that there exists $\tau > 0$ such that

$$\gamma_\mu^\Omega(x) \leq c_{52}|x|^{\alpha_+ + \tau} \quad \text{for all } \ x \in \Omega. \tag{5.14}$$

Iterating this process, we infer that (5.14) holds for any $\tau > 0$. For $k > 1$, let $\alpha_{k,+}$ be the positive root of (2.1) and put $w_k(x) = |x|^{\alpha_{k,+}}$. Then

$$\tilde{\mathcal{L}}_\mu w_k(x) = |x|^{\alpha_{k,+} - 2}\left(\lambda_k - \ell_\mu^\Omega |x|^2\right) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since $\lambda_k \to \infty$, as $k \to \infty$, we choose $k$ such that $\lambda_k > \ell_\mu^\Omega(\mathrm{diam}(\Omega))^2$. Hence $w_k$ is a supersolution of $\tilde{\mathcal{L}}_\mu$. Because $\gamma_\mu^\Omega(x) = o(w_k(x))$ near $x = 0$, it follows that $\gamma_\mu^\Omega(x) \leq \epsilon o(w_k(x))$ in $\Omega$ for any $x \in \Omega$. Hence $\gamma_\mu^\Omega = 0$, which is a contradiction. Finally it implies that $c_{51} > 0$, which yields (1.9)-(i). Because the convergence holds in $C^1(S^{N-1}_+)$ and $v_t(t,.) \to 0$, we infer

$$\lim_{r \to 0} r^{1-\alpha_+}\nabla\tilde{\gamma}_\mu^\Omega(r,.) = c_{51}\left(\alpha_+\psi_1\mathbf{e} + \nabla'\psi_1\right)$$

where $\mathbf{e} = \frac{x}{|x|}$. This implies the claim.

*Case 2:* $\mu = \mu_1$. Set $v(t,.) = r^{\frac{N}{2}-1}u(r,.)$ with $t = \ln r$ and $X(t) = \displaystyle\int_{\mathbb{S}_+^{N-1}} v(t,.)\psi_1 dS$ and obtain again (4.23), where $F(t,.)$ satisfies (4.24). Since $X't) \to 0$ and $X$ is bounded, it follows that $X(t)$ admits a limit $c_{52} \geq 0$ when $t \to -\infty$. As in the proof of **Theorem A**, we infer that

$$\lim_{t\to-\infty} v(t,.) = c_{52}\psi_1 \quad \text{in } C^1(\mathbb{S}_+^{N-1}) \quad \text{and} \quad \lim_{t\to-\infty} v_t(t,.) = 0 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

If $c_{52} = 0$ we derive a contradiction as in the first case. □

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