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# Schrödinger operators with Leray-Hardy potential singular on the boundary

Huyuan Chen\*

Laurent Véron †

## Abstract

We study the kernel function of the operator  $u \mapsto \mathcal{L}_\mu u = -\Delta u + \frac{\mu}{|x|^2} u$  in a bounded smooth domain  $\Omega \subset \mathbb{R}_+^N$  such that  $0 \in \partial\Omega$ , where  $\mu \geq -\frac{N^2}{4}$  is a constant. We show the existence of a Poisson kernel vanishing at 0 and a singular kernel with a singularity at 0. We prove the existence and uniqueness of weak solutions of  $\mathcal{L}_\mu u = 0$  in  $\Omega$  with boundary data  $\nu + k\delta_0$ , where  $\nu$  is a Radon measure on  $\partial\Omega \setminus \{0\}$ ,  $k \in \mathbb{R}$  and show that this boundary data corresponds in a unique way to the boundary trace of positive solution of  $\mathcal{L}_\mu u = 0$  in  $\Omega$ .

**Key Words:** Hardy Potential, Harnack inequality, limit set, Radon Measure.

**MSC2010:** 35B44, 35J75.

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## 1 Introduction

We denote by  $\mathcal{L}_\mu$  the Schrödinger operator defined in a domain  $\Omega \subset \mathbb{R}^N$  by

$$\mathcal{L}_\mu u := -\Delta u + \frac{\mu}{|x|^2} u,$$

where  $\mu$  is a real constant and  $N \geq 2$ . This operator which is associated to the Hardy inequality has been thoroughly studied in the last thirty years. When the singular point 0 belongs to  $\Omega$ , it appears a critical value

$$\mu_0 = -\left(\frac{N-2}{2}\right)^2$$

and the range of the  $\mu$  in which the operator is bounded from below is  $[\mu_0, \infty)$ . This is linked to the Hardy inequality

$$\int_{\Omega} |\nabla \phi|^2 + \mu_0 \int_{\Omega} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Furthermore this inequality is never achieved if  $\Omega$  is bounded, in which case a remainder was shown to exist by Brézis and Vázquez [4]. When  $\lambda$  is a Radon measure in  $\Omega$ , the associated Dirichlet problem

$$\begin{cases} \mathcal{L}_\mu u = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied in its full generality in [8] and [9] thanks to the introduction of a notion of very weak solution associated to some specific weight. Thanks to this new formulation an extensive treatment of the associated semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $g : \mathbb{R} \mapsto \mathbb{R}$  is a continuous nondecreasing function is developed in [9].

In this article we assume that the singular point of the potential lies on the boundary of the domain  $\Omega$ , and we are mainly interested in the two problems:

1- To define a notion of *very weak solution* for the problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\nu$  is a Radon measure on  $\partial\Omega$ , and more generally on  $\partial\Omega \setminus \{0\}$ ;

2- To prove the existence of a boundary trace for any positive  $\mathcal{L}_\mu$ -harmonic function, i.e. solution of  $\mathcal{L}_\mu u = 0$  in  $\Omega$  and to connect it to the problem (1.1).

The model example is  $\Omega = \mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$  although it is not a bounded domain. There exists a critical value

$$\mu \geq \mu_1 := -\frac{N^2}{4}. \quad (1.2)$$

This value is fundamental for the operator  $\mathcal{L}_\mu$  to be bounded from below since there holds,

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}_+^N). \quad (1.3)$$

The analysis of the model case is explicit. Let  $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$  be the spherical coordinates in  $\mathbb{R}_+^N$ , then, if (1.2) is satisfied, there exist two different types of positive  $\mathcal{L}_\mu$ -harmonic functions vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$ ,

$$\gamma_\mu(r, \sigma) = r^{\alpha_+} \psi_1(\sigma) \quad \text{and} \quad \phi_\mu(r, \sigma) = \begin{cases} r^{\alpha_-} \psi_1(\sigma) & \text{if } \mu > \mu_1, \\ r^{-\frac{N-2}{2}} \ln(r^{-1}) \psi_1(\sigma) & \text{if } \mu = \mu_1, \end{cases} \quad (1.4)$$

where  $\psi_1(\sigma) = \frac{x_N}{|x|}$  generates  $\ker(-\Delta' + (N-1)I)$  in  $H_0^1(\mathbb{S}_+^{N-1})$ , and where

$$\alpha_+ := \alpha_+(\mu) = \frac{2-N}{2} + \sqrt{\mu + \frac{N^2}{4}} \quad \text{and} \quad \alpha_- := \alpha_-(\mu) = \frac{2-N}{2} - \sqrt{\mu + \frac{N^2}{4}}. \quad (1.5)$$

Put  $d\gamma_\mu(x) = \gamma_\mu(x)dx$ . We define the  $\gamma_\mu$ -dual operator  $\mathcal{L}_\mu^*$  of  $\mathcal{L}_\mu$  by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu} \langle \nabla \gamma_\mu, \nabla \zeta \rangle \quad \text{for all } \zeta \in C^2(\overline{\mathbb{R}_+^N}), \quad (1.6)$$

and we prove that  $\phi_\mu$  is, in some sense, the fundamental solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \mathbb{R}_+^N, \\ u = \delta_0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

since it satisfies

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in C_c(\overline{\mathbb{R}_+^N}) \cap C^{1,1}(\mathbb{R}_+^N)$$

such that  $\rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N)$ , where  $c_\mu > 0$  is a normalized constant and  $\rho(x) = \text{dist}(x, \partial\Omega)$ . Here  $\rho(x) = x_N$  when  $\Omega = \mathbb{R}_+^N$ .

When  $\mathbb{R}_+^N$  is replaced by a bounded domain  $\Omega$  satisfying the condition

$$(C-1) \quad 0 \in \partial\Omega, \quad \Omega \subset \mathbb{R}_+^N \quad \text{and} \quad \langle x, \mathbf{n} \rangle = O(|x|^2) \quad \text{for all } x \in \partial\Omega,$$

where  $\mathbf{n} = \mathbf{n}_x$  is the outward normal vector at  $x$ , inequality (1.3) holds but it is never achieved in the Hilbert space  $H_0^1(\Omega)$ . Note that the last condition in (C-1) holds if  $\Omega$  is a  $C^2$  domain. It is proved in [5] that there exists a remainder under the following form:

$$\int_\Omega |\nabla \phi|^2 + \mu_1 \int_\Omega \frac{\phi^2}{|x|^2} dx \geq \frac{1}{4} \int_\Omega \frac{\phi^2}{|x|^2 \ln^2(|x|R_\Omega^{-1})} dx \quad \text{for all } \phi \in C_c^\infty(\Omega), \quad (1.7)$$

where  $R_\Omega = \max_{z \in \Omega} |z|$ . Under the assumption (C-1), there holds

$$\ell_\mu^\Omega := \inf \left\{ \int_\Omega \left( |\nabla v|^2 + \frac{\mu}{|x|^2} v^2 \right) dx : v \in C_c^1(\Omega), \int_\Omega v^2 dx = 1 \right\} > 0.$$

This first eigenvalue is achieved in  $H_0^1(\Omega)$  if  $\mu > \mu_1$ , or in the space  $H(\Omega)$  which is the closure of  $C_c^1(\Omega)$  for the norm

$$v \mapsto \|v\|_{H(\Omega)} := \sqrt{\int_{\Omega} \left( |\nabla v|^2 + \frac{\mu_1}{|x|^2} v^2 \right) dx},$$

when  $\mu = \mu_1$ . In the sequel we set

$$H_{\mu}(\Omega) = \begin{cases} H_0^1(\Omega) & \text{if } \mu > \mu_1 \\ H(\Omega) & \text{if } \mu = \mu_1. \end{cases}$$

Moreover, under the assumption (C-1) the imbedding of  $H_{\mu}(\Omega)$  is compact (see e.g. [6]). We denote by  $\gamma_{\mu}^{\Omega}$  the positive eigenfunction, its satisfies

$$\begin{cases} \mathcal{L}_{\mu} \gamma_{\mu}^{\Omega} = \ell_{\mu}^{\Omega} \gamma_{\mu}^{\Omega} & \text{in } \Omega, \\ \gamma_{\mu}^{\Omega} = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \quad (1.8)$$

We prove that there exist  $c_j = c_j(\Omega, \mu) > 0$ ,  $j=1, 2$ , such that

$$\begin{aligned} (i) \quad & \gamma_{\mu}^{\Omega}(x) = c_1 \rho(x) |x|^{\alpha+1} (1 + o(1)) \quad \text{as } x \rightarrow 0, \\ (ii) \quad & |\nabla \gamma_{\mu}^{\Omega}(x)| \leq c_2 \frac{\gamma_{\mu}^{\Omega}(x)}{\rho(x)} \quad \text{for all } x \in \Omega. \end{aligned} \quad (1.9)$$

This function will play the role of a weight function for replacing  $\gamma_{\mu}$ . Next we construct the Poisson kernel  $\mathbb{K}_{\mu}^{\Omega}$  of  $\mathcal{L}_{\mu}$  in  $\Omega \times \partial\Omega$ . When  $\mu \geq 0$  this construction can be made by truncation as in [18], considering for  $\epsilon > 0$  and  $\lambda \in \mathfrak{M}_+(\partial\Omega)$  the solution  $u_{\epsilon}$  of

$$\begin{cases} -\Delta u + \frac{\mu}{\max\{\epsilon^2, |x|^2\}} u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

By a more elaborate method, we also construct the Poisson kernel when  $\mu_1 \leq \mu < 0$ . It is important to notice that when  $\mu > 0$  the kernel has the property that

$$K_{\mu}^{\Omega}(x, 0) = 0 \quad \text{for all } x \in \bar{\Omega} \setminus \{0\} \quad (1.10)$$

by [18, Theorem A.1]. Because of (1.10) it is clear that the Poisson kernel cannot be the tool for describing all the positive  $\mathcal{L}_{\mu}$ -harmonic functions. Our first concern in this article is to clarify the Poisson kernel of  $\mathcal{L}_{\mu}$ .

We first characterize the positive  $\mathcal{L}_{\mu}$ -harmonic functions which are singular at 0.

**Theorem A** *Let  $\Omega$  be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$  and  $\mu \geq \mu_1$ . If  $u$  is a nonnegative  $\mathcal{L}_{\mu}$ -harmonic function vanishing on  $B_{r_0}(0) \cap (\partial\Omega \setminus \{0\})$  for some  $r_0 > 0$ , there exists  $k \geq 0$  such that*

$$\lim_{x \rightarrow 0} \frac{u(x)}{\rho(x) |x|^{\alpha-1}} = k,$$

if  $\mu > \mu_1$  and

$$\lim_{x \rightarrow 0} \frac{|x|^{\frac{N}{2}} u(x)}{\rho(x) \ln |x|} = -k,$$

if  $\mu = \mu_1$ .

Actually the above convergence hold in a stronger way. In order to prove that such solutions truly exist we construct the *kernel function*  $\phi_\mu^\Omega$  (see [13] for the denomination) which is the analogue in a bounded domain of the explicit singular solution  $\phi_\mu$  defined in  $\mathbb{R}_+^N$ .

**Theorem B** *Let  $\Omega$  be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$  satisfying (C-1) and  $\mu \geq \mu_1$ . Then there exists a positive  $\mathcal{L}_\mu$ -harmonic function in  $\Omega$ , which vanishes on  $\partial\Omega \setminus \{0\}$  which satisfies,*

$$\phi_\mu^\Omega(x) = \rho(x)|x|^{\alpha-1}(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.11)$$

if  $\mu > \mu_1$ , and

$$\phi_{\mu_1}^\Omega(x) = \rho(x)|x|^{-\frac{N}{2}}(|\ln|x||+1)(1+o(1)) \quad \text{as } x \rightarrow 0, \quad (1.12)$$

if  $\mu = \mu_1$ .

As in the model case, we define the  $\gamma_\mu^\Omega$ -dual operator of  $\mathcal{L}_\mu$  by

$$\mathcal{L}_\mu^* \zeta = -\Delta \zeta - \frac{2}{\gamma_\mu^\Omega} \langle \nabla \gamma_\mu^\Omega, \nabla \zeta \rangle + \ell_\mu^\Omega \zeta \quad \text{for all } \zeta \in C^{1,1}(\Omega).$$

The following commutation formula holds

$$\mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) = \gamma_\mu^\Omega \mathcal{L}_\mu^* \zeta. \quad (1.13)$$

**Corollary C** *Let  $\Omega$  be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$  satisfying (C-1) and  $\mu \geq \mu_1$ . Then  $\phi_\mu^\Omega$  is the unique function belonging to  $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$  which satisfies*

$$\int_\Omega u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = kc_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \quad (1.14)$$

where and in the sequel the test function space

$$\mathbb{X}_\mu(\Omega) = \left\{ \zeta \in C(\Omega) : \gamma_\mu^\Omega \zeta \in H_\mu(\Omega) \text{ and } \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\Omega) \right\}.$$

Furthermore, if  $u$  is a nonnegative  $\mathcal{L}_\mu$ -harmonic function vanishing on  $\partial\Omega \setminus \{0\}$ , there exists  $k \geq 0$  such that  $u = k\phi_\mu^\Omega$ .

We let  $\sigma_\mu^\Omega \in H_\mu(\Omega)$  be the unique variational solution of

$$\mathcal{L}_\mu u = \frac{\gamma_\mu^\Omega}{\rho} \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.15)$$

Note that  $\sigma_\mu^\Omega \in C^2(\overline{\Omega} \setminus \{0\})$  is a positive classical solution of (1.15) with zero Dirichlet boundary condition on  $\partial\Omega \setminus \{0\}$ , i.e.  $\sigma_\mu^\Omega = 0$  on  $\partial\Omega \setminus \{0\}$ . Moreover,  $\frac{\partial \sigma_\mu^\Omega}{\partial \mathbf{n}} < 0$  on  $\partial\Omega \setminus \{0\}$ . We set

$$\eta = \frac{\sigma_\mu^\Omega}{\gamma_\mu^\Omega} \quad \text{in } \Omega,$$

which satisfies

$$\mathcal{L}_\mu^* \eta = \frac{1}{\rho} \quad \text{in } \Omega, \quad (1.16)$$

play a key role in the sequel. Clearly,  $\eta \in C^2(\overline{\Omega} \setminus \{0\})$  and  $\eta > 0$  in  $\overline{\Omega} \setminus \{0\}$ . We denote by  $\mathfrak{M}(\Omega; \sigma_\mu^\Omega)$  the set of Radon measures  $\nu$  in  $\Omega$  such that

$$\sup \left\{ \int_\Omega \zeta d|\lambda| : \zeta \in C_c(\Omega), 0 \leq \zeta \leq \sigma_\mu^\Omega \right\} := \int_\Omega \sigma_\mu^\Omega d|\nu| < +\infty.$$

If  $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$  the measure  $\sigma_\mu^\Omega \nu$  is a nonnegative bounded measure in  $\Omega$ . Put

$$\beta_\mu^\Omega(x) = -\frac{\partial \gamma_\mu^\Omega(x)}{\partial \mathbf{n}_x}, \quad \forall x \in \partial\Omega. \quad (1.17)$$

The set Radon measures  $\lambda$  in  $\partial\Omega \setminus \{0\}$  such that

$$\sup \left\{ \int_{\partial\Omega \setminus \{0\}} \zeta d|\lambda| : \zeta \in C_c(\partial\Omega \setminus \{0\}), 0 \leq \zeta \leq \eta\beta_\mu^\Omega \right\} := \int_{\partial\Omega \setminus \{0\}} \eta\beta_\mu^\Omega d|\lambda| < +\infty$$

is denoted by  $\mathfrak{M}(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$ . The extension of  $\lambda \in \mathfrak{M}_+(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$  as a measure  $\eta\beta_\mu^\Omega \lambda$  in  $\partial\Omega$  is given by

$$\int_{\partial\Omega} \zeta d(\eta\beta_\mu^\Omega \lambda) = \sup \left\{ \int_{\partial\Omega} v \eta\beta_\mu^\Omega d\lambda : v \in C_c(\partial\Omega \setminus \{0\}), 0 \leq v \leq \zeta \right\} \quad \text{for all } \zeta \in C_c(\partial\Omega), \zeta \geq 0$$

and  $\eta\beta_\mu \lambda = \eta\beta_\mu \lambda_+ - \eta\beta_\mu \lambda_-$  if  $\lambda$  is a signed measure in  $\mathfrak{M}(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$ , and this defines the set  $\mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$  of all such extensions. The Dirac mass at 0 does not belong to  $\mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ , but it is the limit of sequences of measures in this space in the same way as it is a limit of measures in  $\mathfrak{M}_+(\partial\Omega \setminus \{0\}; \eta\beta_\mu^\Omega)$ . In the next result we prove the existence and uniqueness of a solution to

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = \lambda + k\delta_0 & \text{on } \partial\Omega. \end{cases} \quad (1.18)$$

Thanks to (1.7) the Green kernel  $G_\mu^\Omega$  is easily constructible. If  $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$  and  $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$  the following expressions are well defined

$$\mathbb{K}_\mu^\Omega[\lambda](x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{and} \quad \mathbb{G}_\mu^\Omega[\nu](x) = \int_\Omega G_\mu^\Omega(x, y) d\nu(y).$$

Our main existence result is the following.

**Theorem D** *Let  $\Omega$  be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$  satisfying (C-1) and  $\mu \geq \mu_1$ . If  $\nu \in \mathfrak{M}_+(\Omega; \sigma_\mu^\Omega)$ ,  $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$  and  $k \in \mathbb{R}$ , the function*

$$u = \mathbb{G}_\mu^\Omega[\nu] + \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega := \mathbb{H}_\mu^\Omega[(\nu, \lambda, k)] \quad (1.19)$$

is the unique solution of (1.18) in the very weak sense that  $u \in L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$  and

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta d(\gamma_\mu^\Omega \nu) + \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda) + kc_\mu \zeta(0) \quad (1.20)$$

for all  $\zeta \in \mathbb{X}_\mu(\Omega)$ .

In the next result we prove that all the positive  $\mathcal{L}_\mu$ -harmonic functions in  $\Omega$  are described by formula (1.19) (with  $\nu = 0$ ).

**Theorem E** *Let  $\Omega$  be a  $C^2$  bounded domain such that  $0 \in \partial\Omega$  satisfying (C-1),  $\mu \geq \mu_1$  and  $u$  be a nonnegative  $\mathcal{L}_\mu$ -harmonic functions in  $\Omega$ . Then there exist  $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$  and  $k \geq 0$ , such that*

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega = \mathbb{H}_\mu^\Omega[(0, \lambda, k)].$$

The couple  $(\lambda, k\delta_0)$  is called the boundary trace of  $u$ .

The rest of this paper is organized as follows. In section 2, we introduce the distributional identity of  $\mathcal{L}_\mu$  harmonic function  $\phi_\mu$  in  $\mathbb{R}_+^N$ . Section 3 is devoted to build the Kato's type inequalities, to construct Poisson kernel and related properties. Section 4 is addressed to classify the boundary isolated singular  $\mathcal{L}_\mu$  harmonic functions in a bounded domain, i.e. **Theorem A** and to show the existence and related distributional identity in a (C-1) domain: proofs of **Theorem B** and **Corollary C**. We classify the boundary trace for general  $\mathcal{L}_\mu$  harmonic functions and give the existence of  $\mathcal{L}_\mu$  harmonic functions with the boundary trace  $(\lambda, k\delta_0)$ : **Theorem D** and **Theorem E** respectively in Section 5. Finally, we show Estimates (1.9) in Appendix.

In a forthcoming article [10] we study the semilinear problem

$$\begin{cases} \mathcal{L}_\mu u + g(u) = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

## 2 The half-space setting

Let  $\mathbb{R}_+^N := \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$ ,  $(r, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_+^{N-1}$  be the spherical coordinates in  $\mathbb{R}_+^N$  and  $\Delta'$  is the Laplace Beltrami operator on  $\mathbb{S}^{N-1}$ . Then

$$\mathcal{L}_\mu u = -\partial_{rr}u - \frac{N-1}{r}\partial_r u - \frac{1}{r^2}\Delta' u + \frac{\mu}{r^2}u.$$

If  $u(r, \sigma) = r^\alpha \phi(\sigma)$  is a (separable) solution of  $\mathcal{L}_\mu u = 0$  vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$ , then  $\phi$  satisfies

$$\begin{cases} -\Delta' \phi = \lambda_k \phi & \text{in } \mathbb{S}_+^{N-1} := \mathbb{S}^{N-1} \cap \mathbb{R}_+^N, \\ \phi = 0 & \text{in } \partial\mathbb{S}_+^{N-1} \approx \mathbb{S}^{N-2}, \end{cases}$$

where  $\lambda_k$  a constant which necessarily belongs to the spectrum

$$\sigma_{\mathbb{S}_+^{N-1}}(-\Delta') = \{\lambda_k = k(N+k-2) : k \in \mathbb{N}^*\},$$



and  $\alpha = \alpha_{k+}, \alpha_{k-}$  is a root of

$$\alpha^2 + (N-2)\alpha - \lambda_k - \mu = 0. \quad (2.1)$$

The fundamental state corresponds to  $k = 1$ , in which case since  $\lambda_1 = N - 1$ , existence of real roots of (2.1) necessitates  $\mu \geq \mu_1 = -\frac{N^2}{4} = \mu_1$  and we denote  $\alpha_{1+} = \alpha_+$  and  $\alpha_{1-} = \alpha_-$ . Note that this value is connected to the boundary Hardy

$$\int_{\mathbb{R}_+^N} |\nabla \phi|^2 + \mu_1 \int_{\mathbb{R}_+^N} \frac{\phi^2}{|x|^2} dx \geq 0 \quad \text{for all } \phi \in C_0^\infty(S_+^{N-1}).$$

If this condition is fulfilled, the two roots  $\alpha_+$  and  $\alpha_-$  corresponding to  $k = 1$  and  $\lambda_1$  are

$$\alpha_+ = \frac{2-N}{2} + \sqrt{\mu - \mu_1} \quad \text{and} \quad \alpha_- = \frac{2-N}{2} - \sqrt{\mu - \mu_1}. \quad (2.2)$$

The corresponding positive separable solutions  $\gamma_\mu$  and  $\phi_\mu$  of  $\mathcal{L}_\mu u = 0$  vanishing on  $\partial\mathbb{R}_+^N \setminus \{0\}$  are defined by (1.4). We set  $d\gamma_\mu(x) = \gamma_\mu(x)dx$  and define the operator  $\mathcal{L}_\mu^*$  by (1.6).

**Proposition 2.1** *The function  $\phi_\mu$  belongs to  $L_{loc}^1(\mathbb{R}_+^N, \rho^{-1}d\gamma_\mu)$ . It satisfies*

$$\int_{\mathbb{R}_+^N} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) = c_\mu \zeta(0) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N), \quad (2.3)$$

where

$$c_\mu = \begin{cases} 2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1 \end{cases} \quad (2.4)$$

and  $\mathbb{X}_\mu(\mathbb{R}_+^N) = \left\{ \zeta \in C_c(\overline{\mathbb{R}_+^N}) \text{ s.t. } \rho \mathcal{L}_\mu^* \zeta \in L^\infty(\mathbb{R}_+^N) \right\}$ .

**Proof.** Let  $\zeta \in \mathbb{X}_\mu(\mathbb{R}_+^N)$ ,  $\epsilon > 0$  and set  $B_\epsilon^+ = B_\epsilon(0) \cap \mathbb{R}_+^N$ ,  $(B_\epsilon^+)^c = B_\epsilon^c(0) \cap \mathbb{R}_+^N$  and  $\Gamma_\epsilon^+ = \partial B_\epsilon(0) \cap \overline{\mathbb{R}_+^N}$

$$\begin{aligned} 0 &= \int_{(B_\epsilon^+)^c} \zeta \gamma_\mu \mathcal{L}_\mu \phi_\mu dx \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \int_{\Gamma_\epsilon^+} \left( -\frac{\partial \phi_\mu}{\partial \mathbf{n}} \zeta \gamma_\mu + \left( \gamma_\mu \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu}{\partial \mathbf{n}} \right) \phi_\mu \right) dS \\ &= \int_{(B_\epsilon^+)^c} \phi_\mu \mathcal{L}_\mu^* \zeta d\gamma_\mu(x) + \zeta(0) A(\epsilon) \end{aligned}$$

and  $\mathbf{n} = -\epsilon^{-1}x$  on  $\Gamma_\epsilon^+$ , and

$$A(\epsilon) = \begin{cases} -2\sqrt{\mu - \mu_1} \int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS + O(\epsilon) & \text{if } \mu > \mu_1, \\ -\int_{\mathbb{S}_+^{N-1}} \phi_1^2 dS + O(\epsilon) & \text{if } \mu = \mu_1, \end{cases} \quad (2.5)$$

which implies (2.3)-(2.4).  $\square$

### 3 The Poisson kernel

In this section we assume that  $\Omega$  is a bounded  $C^2$  domain included in  $B_1$  (which can always be assumed by scaling) and  $0 \in \partial\Omega$ . We start with the following identity of commutation valid for all  $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$  and  $\zeta \in C^{1,1}(\overline{\Omega})$

$$-\int_{\partial\Omega} \frac{\partial(\zeta\gamma_\mu^\Omega)}{\partial\mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega\lambda) \quad \text{for all } \zeta \in C^{1,1}(\overline{\Omega}), \quad (3.1)$$

where  $\beta_\mu^\Omega$  is defined in (1.17).

The following inequality extends the classical Kato inequality to our framework.

**Lemma 3.1** *Assume  $N \geq 3$  and  $\mu \geq \mu_1$  or  $N = 2$  and  $\mu > \mu_1$ . Then for any  $f \in L^1(\Omega, \sigma_\mu^\Omega dx)$ ,  $h \in L^1(\partial\Omega, \beta_\mu^\Omega dx)$  there exists a unique weak solution  $u$  to*

$$\begin{cases} \mathcal{L}_\mu u = f & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega \end{cases} \quad (3.2)$$

in the sense that

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h \zeta d\beta_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega). \quad (3.3)$$

Furthermore, for any  $\zeta \in \mathbb{X}_\mu(\Omega)$ ,  $\zeta \geq 0$ , there holds

$$\int_{\Omega} |u| \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} |h| \zeta d\beta_\mu^\Omega \quad (3.4)$$

and

$$\int_{\Omega} u^+ \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\Omega} \text{sgn}_+(u) f \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} h^+ \zeta d\beta_\mu^\Omega. \quad (3.5)$$

**Proof.** *Uniqueness.* Assume that  $u$  is a weak solution of (3.2) with  $f = h = 0$ . Then for any  $\zeta \in \mathbb{X}_\mu(\Omega)$  there holds

$$\int_{\Omega} u \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = 0.$$

Let  $\phi \in C_c^\infty(\Omega)$ , and  $v \in H_\mu(\Omega)$  be the variational solution of

$$\mathcal{L}_\mu v = \frac{\gamma_\mu^\Omega}{\rho} \phi \quad \text{and} \quad u \in H_\mu(\Omega).$$

Then  $v \in C^\infty(\Omega)$ ,  $|v| \leq \|\phi\|_{L^\infty} \sigma_\mu^\Omega$ ; the equation is satisfied everywhere and in the sense of distributions in  $\Omega$ . Clearly  $w = (\gamma_\mu^\Omega)^{-1} v$  belongs to  $C^\infty(\Omega)$  and satisfies

$$\mathcal{L}_\mu^* w = \frac{1}{\rho} \phi.$$

Thus,

$$\int_{\Omega} \frac{u}{\rho} \phi d\gamma_\mu^\Omega = 0.$$

Since  $\phi$  is arbitrary, we have that  $u = 0$ .

*Existence and estimates.* We proceed by approximation as in [8, Prop. 2.1]. We assume that  $\{(f_n, h_n)\} \subset C_0^1(\Omega) \times C_0^1(\partial\Omega \setminus \{0\})$  is a sequence which converges to  $(f, h)$  in  $L^1(\Omega, \gamma_\mu^\Omega dx) \times L^1(\partial\Omega, \beta_\mu^\Omega dx)$ . We set  $V(x) = |x|^{-2}$ , denote by  $\mathbb{K}^\Omega$  the Poisson potential of  $-\Delta$  in  $\Omega$  and consider the approximate problem

$$\begin{cases} \mathcal{L}_\mu w_n = f_n - \mu V \mathbb{K}[h_n] & \text{in } \Omega, \\ w_n = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.6)$$

Near 0, we have  $V \mathbb{K}[h_n](x) = O(\frac{\rho(x)}{|x^2|})$ , hence, if  $N \geq 3$ ,  $V \mathbb{K}[h_n] \in L^2(\Omega)$ . If  $N = 2$ , the function  $x \mapsto \frac{\rho(x)}{|x|^2}$  belongs to the Lorentz space  $L^{2,\infty}(\Omega)$  which is the dual of  $L^{2,1}(\Omega)$ . Since  $H(\Omega) \subset L^{2,1}(\Omega)$  by (1.7), it follows that  $H'(\Omega) \subset L^{2,\infty}(\Omega)$ . Hence, by Lax-Milgram theorem there exists a unique  $w_n \in H(\Omega)$  such that (3.6) holds in the variational sense. Then  $u_n = w_n + \mathbb{K}[h_n]$ , which has the same regularity as  $w_n$ , satisfies

$$\begin{cases} \mathcal{L}_\mu u_n = f_n & \text{in } \Omega, \\ u_n = h_n & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

For  $\sigma > 0$ , we set

$$m_\sigma(t) = \begin{cases} |t| - \frac{\sigma}{2} & \text{if } |t| \geq \sigma, \\ \frac{t^2}{2\sigma} & \text{if } |t| < \sigma. \end{cases}$$

The  $m_\sigma$  is convex,  $|m'_\sigma(t)| \leq 1$  and  $m'_\sigma(t) \rightarrow \text{sign}_0(t)$  as  $\sigma \downarrow 0$ . Let  $\zeta \in C^{1,1}(\overline{\Omega})$ ,  $\zeta \geq 0$ . We have that

$$\begin{aligned} & \int_{\Omega} (\langle \nabla w_n, \nabla(\zeta m'_\sigma(u_n) \gamma_\mu^\Omega) \rangle + \mu V w_n \zeta m'_\sigma(u_n) \gamma_\mu^\Omega) dx \\ &= \int_{\Omega} \zeta m'_\sigma(u_n) \gamma_\mu^\Omega f_n dx - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_\sigma(u_n) \gamma_\mu^\Omega dx =: \mathbb{R}(\sigma). \end{aligned}$$

By the fact  $u_n = w_n + \mathbb{K}[h_n]$ , we have that

$$\begin{aligned}
\mathbb{R}(\sigma) &= \int_{\Omega} \langle \nabla u_n, \nabla(\zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega}) \rangle dx - \int_{\Omega} \langle \nabla \mathbb{K}[h_n], \nabla(\zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega}) \rangle dx \\
&\quad + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&= \int_{\Omega} |\nabla u_n|^2 m''_{\sigma}(u_n) \zeta \gamma_{\mu}^{\Omega} dx + \int_{\Omega} \langle \nabla m_{\sigma}(u_n), \nabla(\zeta \gamma_{\mu}^{\Omega}) \rangle dx + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\geq - \int_{\Omega} m_{\sigma}(u_n) \Delta(\zeta \gamma_{\mu}^{\Omega}) dx + \int_{\partial\Omega} m_{\sigma}(h_n) \zeta \frac{\partial \gamma_{\mu}^{\Omega}}{\partial \mathbf{n}} dS + \mu \int_{\Omega} V u_n \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx \\
&\geq \int_{\Omega} m_{\sigma}(u_n) \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} - \int_{\partial\Omega} m_{\sigma}(h_n) d\beta_{\mu}^{\Omega} + \mu \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \gamma_{\mu}^{\Omega} \zeta dx \\
&\quad - \mu \int_{\Omega} V \mathbb{K}[h_n] \zeta m'_{\sigma}(u_n) \gamma_{\mu}^{\Omega} dx,
\end{aligned}$$

thus, we obtain that

$$\begin{aligned}
&\int_{\Omega} m_{\sigma}(u_n) \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} + \mu \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \zeta d\gamma_{\mu}^{\Omega} \\
&\leq \int_{\Omega} \zeta m'_{\sigma}(u_n) f_n d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} m_{\sigma}(h_n) \zeta d\beta_{\mu}^{\Omega}.
\end{aligned} \tag{3.8}$$

Since  $m_{\sigma}$  is convex,  $u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n) \geq 0$ . Hence for  $\mu \geq 0$ , we can let  $\sigma \rightarrow 0$  in (3.8) and obtain (3.4).

For  $\mu \in [\mu_1, 0)$ , we note that

$$0 \leq u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n) \leq \frac{|u_n|^2}{2\sigma^2} \chi_{\{|u_n| \leq \sigma\}}$$

and

$$0 \leq \int_{\Omega} V (u_n m'_{\sigma}(u_n) - m_{\sigma}(u_n)) \zeta d\gamma_{\mu}^{\Omega} \leq \frac{\|\zeta\|_{L^{\infty}}}{2} \int_{\{|u_n| \leq \sigma\}} |x|^{-1 - \frac{N}{2} + \sqrt{\mu - \mu_1}} dx. \tag{3.9}$$

Hence if  $N \geq 3$ , or  $N = 2$  and  $\mu > \mu_1 = -1$ , the right-hand side of (3.9) tends to 0 as  $\sigma \rightarrow 0$  and then we obtain (3.4). The proof of (3.5) is similar.

Applying estimate (3.4) to  $u_n - u_m$ , we obtain for all  $\zeta \in \mathbb{X}_{\mu}(\Omega)$ ,  $\zeta \geq 0$ ,

$$\int_{\Omega} |u_n - u_m| \mathcal{L}_{\mu}^* \zeta d\gamma_{\mu}^{\Omega} \leq \int_{\Omega} |f_n - f_m| \zeta d\gamma_{\mu}^{\Omega} + \int_{\partial\Omega} |h_n - h_m| \zeta d\beta_{\mu}^{\Omega},$$

For test function, we take  $\eta$ , the solution of (1.16), then

$$\int_{\Omega} \frac{|u_n - u_m|}{\rho} d\gamma_{\mu}^{\Omega} \leq \int_{\Omega} |f_n - f_m| d\sigma_{\mu}^{\Omega} + \int_{\partial\Omega} |h_n - h_m| d(\eta \beta_{\mu}^{\Omega}),$$

Therefore  $\{u_n\}$  is a cauchy sequence in  $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$  with limit  $u$ . Since  $u_n$  satisfies (3.7), we let  $n$  go to infity in

$$\int_{\Omega} u_n \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta \sigma_\mu^\Omega f_n dx + \int_{\partial\Omega} \zeta \beta_\mu^\Omega h_n dS$$

and obtain (3.3).  $\square$

**Lemma 3.2** *Assume  $\lambda \in \mathfrak{M}(\partial\Omega; \beta_\mu^\Omega)$  and  $\zeta \in \mathbb{X}_\mu(\Omega)$ , then there holds*

$$\int_{\Omega} \left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}^\Omega[\lambda] d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

**Proof.** By (1.13) we have almost everywhere in  $\Omega$ ,

$$\left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right) \mathbb{K}^\Omega[\lambda] \gamma_\mu^\Omega = \left( \mathcal{L}_\mu(\gamma_\mu^\Omega \zeta) - \frac{\mu}{|x|^2} \gamma_\mu^\Omega \zeta \right) \mathbb{K}^\Omega[\lambda] = -\Delta(\gamma_\mu^\Omega \zeta) \mathbb{K}^\Omega[\lambda].$$

If we assume that  $\lambda$  vanishes in a neighborhood of 0 we derive from (3.1)

$$-\int_{\Omega} \Delta(\gamma_\mu^\Omega \zeta) \mathbb{K}^\Omega[\lambda] dx = -\int_{\partial\Omega} \frac{\partial(\zeta \gamma_\mu^\Omega)}{\partial \mathbf{n}} d\lambda = \int_{\partial\Omega} \zeta d(\beta_\mu^\Omega \lambda).$$

Since  $\gamma_\mu^\Omega \left( \mathcal{L}_\mu^* \zeta - \frac{\mu}{|x|^2} \zeta \right)$  is bounded, we obtain the result first if  $\lambda$  is nonnegative by considering the sequence  $\{\chi_{B_\epsilon} \lambda\}$  and letting  $\epsilon \rightarrow 0$ , and then for any  $\lambda = \lambda^+ - \lambda^-$ .  $\square$

We observe also that the existence of the Green kernel follows from Lax-Milgram theorem which gives the existence of a unique variational solution in  $H(\Omega)$  to

$$\begin{cases} -\Delta u + \frac{\mu}{|x|^2} u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by  $G_\mu^\Omega$  the Green kernel and by  $\mathbb{G}_\mu^\Omega$  the corresponding Green operator.

### 3.1 Construction of the Poisson kernel when $\mu > 0$

For the sake of completeness, we recall the construction in [18]. For  $\epsilon > 0$  we set  $V_\epsilon(x) = \max\{\epsilon^{-2}, |x|^{-2}\}$  and  $V_0(x) = V(x) = |x|^{-2}$ , and if  $\lambda \in \mathfrak{M}(\partial\Omega)$  let  $u_\epsilon$  be the solution of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases}$$

Then

$$u_\epsilon(x) = \int_{\partial\Omega} K_{\mu, \epsilon}^\Omega(x, y) d\lambda(y) = \mathbb{K}_{\mu, \epsilon}^\Omega[\lambda].$$

We obtain by the maximum principle,

$$K_{\mu, \epsilon}^\Omega \leq K_{\mu', \epsilon'}^\Omega \leq K^\Omega \quad \text{for all } \mu \geq \mu' \geq 0 \text{ and } \epsilon' \geq \epsilon > 0,$$

where  $K^\Omega$  is the usual Poisson kernel in  $\Omega$  and there exists

$$K_\mu^\Omega(x, y) = \lim_{\epsilon \rightarrow 0} K_{\mu, \epsilon}^\Omega(x, y) \quad \text{for all } (x, y) \in \Omega \times \partial\Omega.$$

Therefore we infer, firstly by monotone convergence if  $\lambda \geq 0$ , and then for any  $\lambda \in \mathfrak{M}(\partial\Omega)$ , that

$$\lim_{\epsilon \rightarrow 0} u_\epsilon(x) = u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega. \quad (3.10)$$

Since  $V$  is finite in  $B_\epsilon^c \cap \Omega$ , for any  $x \in \Omega$ ,  $K_\mu^\Omega(x, y) > 0$  for all  $y \in \partial\Omega \setminus \{0\}$ . If  $G^\Omega$  is the Green kernel in  $\Omega$ , there holds

$$u_\epsilon(x) + \mu \int_{\Omega} G^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy = \int_{\partial\Omega} K^\Omega(x, y) d\lambda(y)$$

If  $\lambda \geq 0$ , we have by Fatou's lemma,

$$\int_{\Omega} G^\Omega(x, y) V(y) u(y) dy \leq \liminf_{\epsilon \rightarrow 0} \int_{\Omega} G^\Omega(x, y) V_\epsilon(y) u_\epsilon(y) dy. \quad (3.11)$$

Combined with (3.10) it yields

$$u(x) + \mu \int_{\Omega} G^\Omega(x, y) V(y) u(y) dy \leq \int_{\partial\Omega} K^\Omega(x, y) d\lambda(y) \quad \text{for all } x \in \Omega.$$

Since the function  $u + \mu \mathbb{G}[Vu]$  is nonnegative and harmonic in  $\Omega$ , it admits a boundary trace which is a nonnegative Radon measure  $\lambda^*$  and there holds

$$u(x) + \mu \int_{\Omega} G^\Omega(x, y) V(y) u(y) dy = \int_{\partial\Omega} K^\Omega(x, y) d\lambda^*(y) \quad \text{for all } x \in \Omega. \quad (3.12)$$

Because of (3.11)  $0 \leq \lambda^* \leq \lambda$ . The measure  $\lambda^*$  is the reduced measure associated to  $\lambda$ . Since (3.12) is equivalent to

$$u(x) = \int_{\partial\Omega} K_\mu^\Omega(x, y) d\lambda^*(y),$$

there holds

$$\int_{\partial\Omega} K_\mu^\Omega(x, y) d(\lambda - \lambda^*)(y) = 0.$$

This implies that  $\lambda = \lambda^*$  in  $\partial\Omega \setminus \{0\}$ . With the notations of [18], we recall that

$$\begin{aligned} \text{Sing}_V(\Omega) &:= \{y \in \partial\Omega : \exists x_0 \in \Omega \text{ s.t. } K_\mu^\Omega(x_0, y) = 0\} \\ &\subset Z_V := \left\{ y \in \partial\Omega : \int_{\Omega} K_0^\Omega(x, y) V(x) \rho(x) dx = \infty \right\}. \end{aligned}$$

Actually, if  $y \in \text{Sing}_V(\Omega)$ ,  $K_\mu^\Omega(x_0, y) = 0$  for any  $x_0 \in \Omega$  by Harnack inequality. Clearly  $0 \in Z_V$  and if  $y \neq 0$  the integral term in the definition of  $Z_V$  is finite. Hence  $\text{Sing}_V(\Omega) \subset Z_V = \{0\}$ . Since for any truncated cone  $C_{0, \delta} \Subset \Omega$  with vertex 0, there holds

$$\int_{C_{0, \delta}} V(x) \frac{dx}{|x - y|^{N-2}} = \infty,$$

it follows by Ancona's result [18, Theorem A1] that  $0 \in \text{Sing}_V(\Omega)$ . Finally

$$K_\mu^\Omega(x, 0) = 0 \quad \text{for all } x \in \Omega.$$

### 3.2 Construction of the Poisson kernel when $\mu_1 \leq \mu < 0$

For  $\epsilon > 0$  and  $\lambda \in C(\partial\Omega)$ ,  $\lambda \geq 0$  we denote by  $w = w_{\epsilon,\lambda}$  the variational solution in  $H(\Omega)$  of

$$\begin{cases} -\Delta w + \mu V_\epsilon w = -\mu V_\epsilon \mathbb{K}[\lambda] & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Then  $w_{\epsilon,\lambda} \geq 0$  and  $u = u_{\epsilon,\lambda} := w_\epsilon + \mathbb{K}[\lambda]$  satisfies

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega. \end{cases} \quad (3.13)$$

Since  $-\Delta u_{\epsilon,\lambda} + \mu V u_{\epsilon,\lambda} \leq 0$ , there holds from Lemma 3.1

$$\int_{\Omega} u_{\epsilon,\lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda \zeta d\beta_\mu^\Omega \quad \text{for } \zeta \in \mathbb{X}_\mu(\Omega), \zeta \geq 0$$

and in particular

$$\int_{\Omega} \frac{u_{\epsilon,\lambda}}{\rho} d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \lambda d(\eta\beta_\mu^\Omega). \quad (3.14)$$

If  $\epsilon > \epsilon' > 0$  and  $\lambda' > \lambda > 0$  we have

$$-\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} = \mu (V_{\epsilon'} - V_\epsilon) \leq 0.$$

Since

$$\begin{aligned} & \int_{\Omega} \left( -\frac{\Delta u_{\epsilon,\lambda}}{u_{\epsilon,\lambda}} + \frac{\Delta u_{\epsilon',\lambda'}}{u_{\epsilon',\lambda'}} \right) (u_{\epsilon,\lambda}^2 - u_{\epsilon',\lambda'}^2)_+ dx \\ &= \int_{\{u_{\epsilon,\lambda} \geq u_{\epsilon',\lambda'}\}} \left( \left| \nabla u_{\epsilon,\lambda} - \frac{u_{\epsilon,\lambda}}{u_{\epsilon',\lambda'}} \nabla u_{\epsilon',\lambda'} \right|^2 + \left| \nabla u_{\epsilon',\lambda'} - \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}} \nabla u_{\epsilon,\lambda} \right|^2 \right) dx, \end{aligned}$$

we deduce that the function  $x \mapsto \frac{u_{\epsilon',\lambda'}}{u_{\epsilon,\lambda}}(x)$  is constant on the set  $\{x : u_{\epsilon,\lambda}(x) > u_{\epsilon',\lambda'}(x)\}$ . If this set is non-empty we get a contradiction since it is strictly included in  $\Omega$ . Therefore the mapping

$$(\epsilon, \lambda) \mapsto u_{\epsilon,\lambda}$$

is decreasing in  $\epsilon$  and increasing in  $\lambda$ .

Next we can assume that  $\lambda \in \mathfrak{M}_+(\partial\Omega)$  vanishes in  $B_\delta \cap \partial\Omega$  and that  $\{\lambda_n\} \subset C(\partial\Omega)$  is a sequence of functions which converge to  $\lambda$  in the weak sense of measures. We denote by  $u_{\epsilon,\lambda_n}$  the solution of (3.13) with  $\lambda$  replaced by  $\lambda_n$ . Since  $\mu < 0$ ,  $\alpha_+ < 1$ ,  $\rho^{-1}\gamma_\mu^\Omega \sim |x|^{\alpha_+-1} \geq R_\omega^{\alpha_+-1}$ , where  $R_\omega = \max\{|z| : z \in \Omega\}$ . Hence

$$\int_{\Omega} u_{\epsilon,\lambda_n} dx \leq c_8 \int_{\partial\Omega} \lambda_n d(\eta\beta_\mu^\Omega) \leq c_9 \|\lambda\|_{\mathfrak{M}(\partial\Omega)}.$$

Hence  $u_{\epsilon,\lambda_n}$  and  $V_\epsilon u_{\epsilon,\lambda_n}$  are uniformly bounded in  $L^1(\Omega)$ . From standard regularity estimates the sequence  $\{u_{\epsilon,\lambda_n}\}_{n \in \mathbb{N}}$  is bounded in the Lorentz spaces  $L^{\frac{N}{N-1}, \infty}(\Omega)$  and weakly relatively

compact in  $L^1(\Omega)$  (see e.g. [12]). This implies that, up to a subsequence,  $u_{\epsilon, \lambda_n}$  converges in  $L^1(\Omega)$  and a.e. in  $\Omega$  to a weak solution  $u_{\epsilon, \lambda}$  of

$$\begin{cases} -\Delta u + \mu V_\epsilon u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega, \end{cases}$$

that is a function which satisfies

$$\int_{\Omega} (-u_{\epsilon, \lambda} \Delta \zeta + \mu V_\epsilon u_{\epsilon, \lambda} \zeta) dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}). \quad (3.15)$$

Furthermore (3.14) holds (with the same notation). For test function  $\zeta$  in (3.15), we take  $\zeta = \theta_1$  be the solution of

$$\begin{cases} -\Delta \theta_1 = 1 & \text{in } \Omega, \\ \theta_1 = 0 & \text{on } \partial\Omega \end{cases}$$

Then

$$\int_{\Omega} u_{\epsilon, \lambda} dx = -\mu \int_{\Omega} V_\epsilon u_{\epsilon, \lambda} \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

By the monotone convergence theorem we obtain that  $V_\epsilon u_{\epsilon, \lambda} \rightarrow V u_\lambda$  in  $L^1(\Omega, \theta_1 dx)$  by letting  $\epsilon \rightarrow 0$  and

$$\int_{\Omega} u_\lambda dx = -\mu \int_{\Omega} V u_\lambda \theta_1 dx - \int_{\partial\Omega} \frac{\partial \theta_1}{\partial \mathbf{n}} d\lambda.$$

Hence

$$\int_{\Omega} (-\Delta \zeta + \mu V \zeta) u_\lambda dx = - \int_{\partial\Omega} \frac{\partial \zeta}{\partial \mathbf{n}} d\lambda \quad \text{for all } \zeta \in C_0^{1,1}(\overline{\Omega}).$$

We also have

$$\int_{\Omega} u_{\epsilon, \lambda_n} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\Omega} (V - V_\epsilon) u_{\epsilon, \lambda_n} \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta \lambda_n d\beta_\mu^\Omega,$$

for all  $\zeta \in \mathbb{X}_\mu(\Omega)$ . Since  $u_{\epsilon, \lambda_n}$  converges in  $L^1(\Omega)$  we obtain if  $\zeta \geq 0$ ,

$$\int_{\Omega} u_{\epsilon, \lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\Omega} (V - V_\epsilon) u_{\epsilon, \lambda} \zeta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \leq \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega),$$

since  $\mu(V - V_\epsilon) \leq 0$ . When  $\epsilon \rightarrow 0$ ,  $u_{\epsilon, \lambda}$  increases and converges to some  $u_\lambda$  in  $L^1(\Omega, \rho^{-1} d\gamma_\mu^\Omega)$  which satisfies

$$\int_{\Omega} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \leq \int_{\partial\Omega} \zeta d(\lambda \beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega), \zeta \geq 0.$$

For  $\delta > 0$  denote by  $\zeta_\delta$  the solution of

$$\mathcal{L}_\mu^* \zeta_\delta = \chi_{\Omega_\delta} \mathcal{L}_\mu^* \zeta \quad \text{where } \Omega_\delta = \{x \in \Omega : \rho(x) > \delta\}.$$

As  $\zeta \in \mathbb{X}_\mu(\Omega)$ ,  $|\mathcal{L}_\mu^* \zeta| \leq c_{10} \rho$ , hence  $\zeta_\delta \in \mathbb{X}_\mu(\Omega)$ ,  $|\zeta_\delta| \leq c_{10} \eta$  and  $\zeta_\delta \rightarrow \zeta$  when  $\delta \rightarrow 0$ . Furthermore, since  $c_{11}|x|$  is a supersolution for  $c_{11} > 0$  large enough,  $\eta_\delta \leq c_{11}|x|$ . Hence

$$\int_{\Omega_\delta} u_{\epsilon, \lambda} \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \mu \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon, \lambda} \zeta_\delta d\gamma_\mu^\Omega + \int_{\partial\Omega} \zeta_\delta d(\lambda \beta_\mu^\Omega).$$



Because  $|x|^{-2}u_{\epsilon,\lambda}|\zeta_\delta| \leq c_{11}\rho^{-1}u_{\epsilon,\lambda}$  and  $u_{\epsilon,\lambda} \rightarrow u_\lambda$  in  $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$ , we derive that

$$\lim_{\epsilon \rightarrow 0} \int_{\{|x| < \epsilon\}} \frac{1}{|x|^2} u_{\epsilon,\lambda} \zeta_\delta d\gamma_\mu^\Omega = 0$$

which implies

$$\int_{\Omega_\delta} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta_\delta d(\lambda\beta_\mu^\Omega),$$

Letting  $\delta \rightarrow 0$  we obtain by monotonicity

$$\int_{\Omega} u_\lambda \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega). \quad (3.16)$$

Finally, if  $\lambda \in \mathfrak{M}_+(\partial\Omega, \eta\gamma_\mu^\Omega)$  we replace it by  $\lambda_\delta = \chi_{B_\delta^c} \lambda$  and denote by  $u_{\lambda_\delta}$  the weak solution of

$$\begin{cases} -\Delta u + \mu V u = 0 & \text{in } \Omega, \\ u = \lambda_\delta & \text{on } \partial\Omega. \end{cases}$$

The mapping  $\delta \mapsto u_{\lambda_\delta}$  is monotone. Hence, by the monotone convergence theorem  $u_{\lambda_\delta}$  increases and converges to some  $u_\lambda$  in  $L^1(\Omega, \rho^{-1}d\gamma_\mu^\Omega)$  and clearly  $u_\lambda$  satisfies (3.16) for all  $\zeta \in \mathbb{X}_\mu(\Omega)$ .

## 4 The singular kernel

In this section we construct the singular kernel  $\phi_\mu^\Omega$  and prove that it satisfies estimates (1.11)-(1.12) and it is associated to Dirac mass at 0. Up to a rotation we can assume that the inward normal direction to  $\partial\Omega$  at 0 is  $\mathbf{e}_N = (0', 1) \in \mathbb{R}^{N-1} \times \mathbb{R}$ . Hence the tangent hyperplane to  $\partial\Omega$  at 0 is  $\partial\mathbb{R}_+^N = \mathbb{R}^{N-1}$ . For  $R > 0$  set  $B'_R = \{x' \in \mathbb{R}^{N-1} : |x'| < R\}$  and  $D_R = B'_R \times (-R, R)$ . Then there exist  $R > 0$  and a  $C^2$  function  $\theta : B'_R \mapsto \mathbb{R}$  such that  $\partial\Omega \cap D_R = \{x = (x', x_N) : x_N = \theta(x') \text{ for } x' \in B'_R\}$  and  $\Omega \cap D_R = \{x = (x', x_N) : \theta(x') < x_N < R\}$ . Furthermore  $\nabla\theta(0) = 0$ .

### 4.1 Classification of Boundary isolated singularities

We characterize the positive solutions of  $\mathcal{L}_\mu u = 0$  which vanish on  $\partial\Omega \setminus \{0\}$ .

**Lemma 4.1** *Let  $\mu \geq \mu_1$  and  $u \in C^2(\overline{\Omega} \setminus \{0\})$  be a positive solution of  $\mathcal{L}_\mu u = 0$  in  $\Omega$  vanishing on  $\partial\Omega \setminus \{0\}$ . Then there exist  $a > 0$  and  $c_{12} > 0$  such that*

$$u(x) \leq c_{12}|x|^{-a-1}\rho(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}. \quad (4.1)$$

**Proof.** This is a direct consequence of Boundary Harnack inequality [3, Th. 2.7].

**Proposition 4.2** *Assume that  $\mu \geq \mu_1$  and  $u \in C^2(\overline{\Omega} \setminus \{0\})$  is a positive solution of  $\mathcal{L}_\mu u = 0$ , vanishing on  $\partial\Omega \setminus \{0\}$  satisfying (4.1) with  $a \geq -\alpha_-$ . Then the following convergences hold in  $C^1(\mathbb{S}_+^{N-1})$ :*

(i) *If  $\mu > \mu_1$  and  $a = -\alpha_-$ , there exists  $c_{13} \geq 0$  such that*

$$\lim_{r \rightarrow 0} \frac{u(r, \cdot)}{r^{\alpha_-}} = c_{13}\phi \quad \text{as } r \rightarrow 0. \quad (4.2)$$

(ii)  $\mu \geq \mu_1$  and  $a > -\alpha_-$  there exist  $\tau > a + \alpha_-$  depending on  $a$  and  $\mu$ , and  $c_{14} \geq 0$  such that

$$u(x) \leq c_{14}|x|^{-a-1+\tau}\rho(x) \quad \text{for all } x \in \bar{\Omega} \setminus \{0\}.$$

**Proof.** *Step 1. Straightening the boundary.* We define the function  $\Theta = (\Theta_1, \dots, \Theta_N)$  on  $D_R$  by  $y_j = \Theta_j(x) = x_j$  if  $1 \leq j \leq N-1$  and  $y_N = \Theta_N(x) = x_N - \theta(x')$ . Since  $D\Theta(0) = Id$  we can assume that  $\Theta$  is a diffeomorphism from  $D_R$  onto  $\Theta(D_R)$ . We set

$$u(x) = \tilde{u}(y) \quad \text{for all } x \in D_R^+ = B'_R \times [0, R]. \quad (4.3)$$

Then

$$\begin{aligned} u_{x_j x_j} &= \tilde{u}_{y_j y_j} - 2\theta_{x_j} \tilde{u}_{y_j y_N} - \theta_{x_j, x_j} \tilde{u}_{y_N} + \theta_{x_j}^2 \tilde{u}_{y_N y_N} \quad \text{for } 1 \leq j \leq N-1, \\ u_{x_N x_N} &= \tilde{u}_{y_N y_N} \end{aligned}$$

and

$$\Delta \tilde{u} + |\nabla \theta|^2 \tilde{u}_{y_N y_N} - 2\langle \nabla \theta, \nabla \tilde{u}_{y_N} \rangle - \tilde{u}_{y_N} \Delta \theta - \frac{\mu}{|\Theta^{-1}(y)|^2} \tilde{u} = 0.$$

We use here the spherical coordinates  $(r, \sigma)$  in the variable  $y$  and we recall that  $\Delta'$  is the Laplace-Beltrami operator on  $\mathbb{S}^{N-1}$  and  $\nabla'$  is the tangential gradient on  $\mathbb{S}^{N-1}$  identified with the covariant derivative via the isometric imbedding  $\mathbb{S}^{N-1} \hookrightarrow \mathbb{R}^N$  which enables the formula

$$\nabla \tilde{u}(y) = \left( \tilde{u}_r \mathbf{n} + \frac{1}{r} \nabla' \tilde{u} \right) (r, \sigma) \quad \text{with } \mathbf{n} = |y|^{-1} y.$$

After a lengthy computation the details of which can be found in [12, P 298-300] we obtain

$$\begin{aligned} & r^2 \tilde{u}_{rr} [1 - 2\theta_r \langle \mathbf{n}, \mathbf{e}_N \rangle + |\nabla \theta|^2 (\langle \mathbf{n}, \mathbf{e}_N \rangle)^2] \\ & + r \tilde{u}_r [N-1 - r \langle \mathbf{n}, \mathbf{e}_N \rangle \Delta \theta + r |\nabla \theta|^2 (\langle \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle), \mathbf{e}_N \rangle - 2\langle \nabla' \theta, \nabla' (\langle \mathbf{n}, \mathbf{e}_N \rangle) \rangle)] \\ & + \langle \nabla' \tilde{u}, \mathbf{e}_N \rangle [-r \Delta \theta + 2\theta_r - |\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle] + r \langle \nabla' \tilde{u}_r, \mathbf{e}_N \rangle [2\theta_r + 2|\nabla \theta|^2 \langle \mathbf{n}, \mathbf{e}_N \rangle] \\ & - 2\langle \nabla' \tilde{u}, \nabla' \theta \rangle \langle \mathbf{n}, \mathbf{e}_N \rangle + \langle \nabla' (\langle \nabla' \tilde{u}, \mathbf{e}_N \rangle), |\nabla \theta|^2 \mathbf{e}_N - 2r^{-1} \nabla' \theta \rangle + \Delta' \tilde{u} - \frac{\mu}{|\Theta^{-1}(y)|^2} = 0. \end{aligned}$$

Next we set

$$\tilde{u}(r, \sigma) = r^{-a} v(t, \sigma) \quad \text{with } t = \ln r,$$

and we assume that

$$a \neq \frac{N-2}{2}. \quad (4.4)$$

We notice that

$$\begin{aligned} r^2 &= \sum_{j=1}^N y_j^2 = \sum_{j=1}^{N-1} x_j^2 + (x_N - \theta(x'))^2 = |x^2| - 2x_N \theta(x') = |x|^2 (1 + O(r)) \quad \text{as } r \rightarrow 0 \\ &= |x|^2 (1 + O(e^t)) \quad \text{as } t \rightarrow -\infty. \end{aligned}$$

By a straightforward computation we find that  $v$  satisfies the following asymptotically autonomous equation in  $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$

$$(1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 - 2a + \epsilon_2(t, \cdot))v_t + (a(a + 2 - N) - \mu + \epsilon_3(t, \cdot))v + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \quad (4.5)$$

where the  $\epsilon_j$  satisfies

$$|\epsilon_j(t, \cdot)| + |\partial_t \epsilon_j(t, \cdot)| + |\nabla' \epsilon_j(t, \cdot)| \leq c_{15} e^t. \quad (4.6)$$

This is due to the fact that  $|\theta(x')| = O(|x'|^2)$  near 0.

*Step 2. The convergence process.* Since  $v$  is bounded in  $(-\infty, r_0] \times \mathbb{S}_+^{N-1}$  and vanishes on  $(-\infty, r_0] \times \partial \mathbb{S}_+^{N-1}$  and all the coefficients are continuous functions, we obtain that  $v$  is bounded in  $W^{2,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$  independently of  $T \leq r_0 - 2$ , for any  $q < \infty$ . Hence  $v$  is bounded in any  $C^{1,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$  for any  $\tau \in [0, 1)$ . Differentiating the equation and using the standard elliptic equation regularity, we obtain that  $v$  is bounded in  $W^{3,q}([T-1, T+1] \times \mathbb{S}_+^{N-1})$  and in  $C^{2,\tau}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})$ . We consider the negative trajectory of  $v$  in  $C_0^1(\overline{\mathbb{S}_+^{N-1}})$  defined by

$$\mathcal{T}_-(v) = \bigcup_{t \leq r_0 - 1} \{v(t, \cdot)\}.$$

By the previous estimates and the Arzela-Ascoli theorem, it is a relatively compact subset of  $C_0^1(\overline{\mathbb{S}_+^{N-1}})$ , hence its limit set at  $-\infty$  (or alpha-limit set), denoting  $A(\mathcal{T}_-(v))$ , is a non-empty connected compact subset of  $C_0^1(\overline{\mathbb{S}_+^{N-1}})$ . Multiplying (4.5) by  $v_t$  and integrating on  $\mathbb{S}_+^{N-1}$  yields

$$\begin{aligned} & \int_{\mathbb{S}_+^{N-1}} \left( N - 2 - 2a + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right) v_t^2 dS - \frac{1}{2} \int_{\mathbb{S}_+^{N-1}} \partial_t \epsilon_3 v^2 dS \\ &= \frac{d}{dt} \left[ \int_{\mathbb{S}_+^{N-1}} \left( \frac{1}{2} |\nabla v|^2 - \frac{1}{2} [a(a + 2 - N) - \mu + \epsilon_3] v^2 - \frac{1}{2} (1 + \epsilon_1) v_t^2 \right) dS \right] \\ & \quad - \int_{\mathbb{S}_+^{N-1}} (\langle \nabla'v, \epsilon_4 \rangle + \langle \nabla'v_t, \epsilon_5 \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6 \rangle) v_t^2 dS. \end{aligned} \quad (4.7)$$

Next we integrate over  $(-\infty, r_2)$  for some  $r_2$  large enough so that

$$\left| N - 2 - 2a + \epsilon_2 - \frac{1}{2} \partial_t \epsilon_1 \right| \geq \frac{1}{2} |N - 2 - 2a| > 0,$$

here we use the crucial assumption (4.4). Since all the terms on the right-hand side of (4.7) are integrable on  $(-\infty, r_2)$  because of (4.6) and the bounds on  $v$ , we obtain that

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_t^2 dS < \infty. \quad (4.8)$$

Differentiating (4.5) with respect to  $t$  and using the estimates on  $v$  and the  $\epsilon_j$  we obtain (see [12, p. 302] for a similar calculation)

$$\int_{-\infty}^{r_2} \int_{\mathbb{S}_+^{N-1}} v_{tt}^2 dS < \infty. \quad (4.9)$$

Because  $v_t$  and  $v_{tt}$  are uniformly continuous on  $(-\infty, r_1]$ , we infer from (4.8) and (4.9)

$$\lim_{t \rightarrow -\infty} \left( \|v_t(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} + \|v_{tt}(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})} \right) = 0.$$

Therefore the set  $A(\mathcal{T}_-(v))$  is a compact connected subset of the set of nonnegative solutions of

$$\begin{cases} \Delta' \omega + (a(a+2-N) - \mu) \omega = 0 & \text{in } \mathbb{S}_+^{N-1}, \\ \omega = 0 & \text{on } \partial \mathbb{S}_+^{N-1}. \end{cases}$$

*Step 3. The case  $a(a+2-N) - \mu = N-1$ .* The set  $A(\mathcal{T}_-(v))$  is a subset of  $\ker(-\Delta' - (N-1)Id)$  in  $H_0^1(\mathbb{S}_+^{N-1})$  and more precisely  $A(\mathcal{T}_-(v)) = \{m\psi_1 : m \in I^*\}$  where  $I^*$  is a compact interval of  $(0, \infty)$ . We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS.$$

Then  $X$  satisfies

$$X''(t) + (N-2-2a)X'(t) + F(t) = 0, \quad (4.10)$$

where

$$F(t) = \int_{\mathbb{S}_+^{N-1}} \left[ \epsilon_1(t, \cdot) v_{tt} + \epsilon_2(t, \cdot) v_t + \epsilon_3(t, \cdot) v + \langle \nabla' v, \epsilon_4(t, \cdot) \rangle + \langle \nabla' v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla' v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle \right] \psi_1 dS.$$

Then  $|F(t)| \leq c_{16}e^t$ . We consider a sequence  $\{t_n\}$  converging to  $-\infty$  and  $c^* \in I^*$  such that  $X(t_n) \rightarrow c^*$ . Since  $X'(t)$  and  $X''(t)$  converges to 0 as  $t \rightarrow -\infty$ , we integrate (4.10) on  $(t_n, t)$  and let  $n \rightarrow \infty$ . Then we get

$$X'(t) + (N-2-2a)(X(t) - c^*) + O(e^t) = 0.$$

Letting  $t \rightarrow -\infty$  yields  $X(t) \rightarrow c^*$ . Hence we have proved that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c^* \psi_1 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}).$$

*Step 4. The case  $a(a+2-N) - \mu \neq N-1$ .* Clearly  $A(\mathcal{T}_-(v)) = \{0\}$  and

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = 0 \quad \text{in } C^1(\overline{\mathbb{S}_+^{N-1}}). \quad (4.11)$$

Furthermore, since we have assumed  $a \geq -\alpha_-$ , there holds actually  $a > -\alpha_-$ . We recall that  $\lambda_k$  is the  $k$ -th eigenvalue of  $-\Delta'$  in  $H_0^1(\mathbb{S}_+^{N-1})$  and put

$$H_k = \ker(-\Delta' - \lambda_k Id) = \text{span}\langle \phi_{k,1}, \phi_{k,2}, \dots, \phi_{k,j_k} \rangle \quad \text{and} \quad H_0^1(\mathbb{S}_+^{N-1}) = \bigoplus_{k=1}^{\infty} H_k.$$

We denote

$$P_k(x) = x^2 + (N-2)x - \mu - \lambda_k.$$

Then  $P_1(\alpha_-) = 0$  and  $P_k(\alpha_-) = \lambda_1 - \lambda_k < 0$  for  $k \geq 2$ . Since  $a(a+2-N) - \mu \neq N-1$  by assumption, we define a partition of  $\mathbb{N}^*$  by setting

$$N_1 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k \geq 0\}, \quad N_2 := \{k \in \mathbb{N}^* : a(a+2-N) - \mu - \lambda_k < 0\}$$

and

$$W_1 = \bigoplus_{k \in N_1} H_k \quad \text{and} \quad W_2 = \bigoplus_{k \in N_2} H_k.$$

Then

$$-\int_{\mathbb{S}_+^{N-1}} \phi \Delta' \phi dS \geq \gamma \int_{\mathbb{S}_+^{N-1}} \phi^2 dS \quad \text{for all } \phi \in W_2, \quad (4.12)$$

where

$$\gamma = \mu + \lambda_{k_2} - a(a+2-N) > 0 \quad \text{with } k_2 = \inf N_2.$$

We denote by  $P_j$  the orthogonal projector onto  $W_j$  in  $H_0^1(\mathbb{S}_+^{N-1})$  and set  $v = P_1 v + P_2 v = v_1 + v_2$ . Then the projection of (4.5) on to  $W_2$  is

$$(v_2)_{tt} + (N-2-2a)(v_2)_t + (a(a+2-N) - \mu)v_2 + \Delta' v_2 = F_2(t, \cdot),$$

where  $F_2$  satisfies the same estimates (4.6) as  $\epsilon_j$ . Then, using (4.6) and (4.12)

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS + (N-2-2a) \int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS - \gamma \int_{\mathbb{S}_+^{N-1}} v_2^2 dS \geq -c_{17} e^t \left( \int_{\mathbb{S}_+^{N-1}} v_2^2 dS \right)^{\frac{1}{2}}.$$

Put  $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$ , because

$$\int_{\mathbb{S}_+^{N-1}} (v_2)_t v_2 dS = Y'(t)Y(t) \quad \text{and} \quad \int_{\mathbb{S}_+^{N-1}} (v_2)_{tt} v_2 dS \geq Y''(t)Y(t),$$

we obtain the following differential inequality

$$Y'' + (N-2-2a)Y' - \gamma Y \geq -c_{17} e^t \quad \text{in } \mathcal{D}'(-\infty, r_2).$$

The characteristic roots of the equation  $y'' + (N-2-2a)y' - \gamma y = 0$  are

$$\begin{aligned} a_{k_2,-} &= a + \frac{1}{2} \left( 2 - N - \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2} \right) = \alpha_{k_2,-} + a < 0 \\ a_{k_2,+} &= a + \frac{1}{2} \left( 2 - N + \sqrt{4\mu + 4\lambda_{k_2} + (N-2)^2} \right) = \alpha_{k_2,+} + a > 0. \end{aligned} \quad (4.13)$$

where the  $\alpha_{k_2,\pm}$  are the roots of equations (2.1) with  $k = k_2$ . The solutions of

$$z'' + (N-2-2a)z' - \gamma z = -c_{17} e^t \quad \text{in } \mathcal{D}'(-\infty, r_2).$$

endow the form  $z(t) = Ae^{ta_{k_2,-}} + Be^{ta_{k_2,+}} + c_{18}e^t$  if  $a_{k_2,+} \neq 1$  or  $z(t) = Ae^{ta_{k_2,-}} + Be^t + Cte^t$  if  $a_{k_2,+} = 1$ , for some explicit constant  $c_{18}$  depending on  $c_{17}$  and the coefficients in the equation. Since  $Y(t) \rightarrow 0$  when  $t \rightarrow -\infty$  by (4.11), it follows from the maximum principle that

$$Y(t) \leq c_{19} e^{ta_{k_2,+}} + c_{18} e^t \quad \text{if } a_{k_2,+} \neq 1, \quad \text{or } Y(t) \leq c_{20} |t| e^t \quad \text{if } a_{k_2,+} = 1 \quad \text{for } t \leq r_2.$$

Then using standard elliptic equations a priori estimates, initially in  $L^2(\mathbb{S}_+^{N-1})$ , then in  $L^p(\mathbb{S}_+^{N-1})$  and finally in  $C^\tau(\overline{\mathbb{S}_+^{N-1}})$ , we obtain that for  $t \leq r_3$ ,

$$\|v_2(t, \cdot)\|_{C^1(\mathbb{S}_+^{N-1})} \leq \begin{cases} c_{21}e^{ta_{k_2,+}} + c_{22}e^t & \text{if } a_{k_2,+} \neq 1, \\ c_{23}|t|e^t & \text{if } a_{k_2,+} = 1, \end{cases} \quad (4.14)$$

where  $r_3 \leq r_2 - 1$ .

For the components in  $W_1$  we have

$$v_1(t, \cdot) = \sum_{k \in N_1} \sum_{1 \leq j \leq j_k} w_{k,j}(t) \phi_{k,j}(\cdot), \quad (4.15)$$

where the  $\phi_{k,j}$  form an orthonormal basis of  $H_k$ . Then

$$w_{k,j}'' + (N - 2 - 2a)w_{k,j}' + (a(a + 2 - N) - \mu - \lambda_k)w_{k,j} = F_{k,j}(t) \quad (4.16)$$

The characteristic roots of equation  $z'' + (N - 2 - 2a)z' + (a(a + 2 - N) - \mu - \lambda_k)z = 0$  are given in (4.13) with a general  $k$ ,  $a_{k-} = a + \alpha_{k-}$  and  $a_{k+} = a + \alpha_{k+}$  where  $\alpha_{k\pm}$  are the roots of (2.1). They have same sign (including 0) since  $a(a + 2 - N) - \mu - \lambda_k \geq 0$ , furthermore, their sum is positive since  $N - 2 - 2a < 0$ , as a consequence of  $a > -\alpha_-$ . By standard calculation the solution of (4.16) has the form

$$w_{k,j}(t) = m_1 e^{ta_{k+}} + m_2 e^{ta_{k-}} - \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds. \quad (4.17)$$

Since  $|F_{k,j}(s)| \leq c_{24}e^s$  there holds

$$\left| \int_t^0 \frac{e^{(t-s)a_{k+}} - e^{(t-s)a_{k-}}}{a_{k+} - a_{k-}} F_{k,j}(s) ds \right| \leq c_{25} \begin{cases} |t|e^t & \text{if } a_{k-} = 1 \\ \max\{e^t, e^{ta_{k-}}\} & \text{if } a_{k-} \neq 1 \end{cases} \quad (4.18)$$

In particular, if  $k_1 = \max N_1$ , then  $a_{k_1\pm} = \min\{a_{k\pm} : k \in N_1\}$ .

We assume first that  $a_{k_1-} > 0$ . Combining this fact with (4.15) and (4.18) we obtain

$$\|v_1(t, \cdot)\|_{L^\infty(\mathbb{S}_+^{N-1})} \leq c_{26} \begin{cases} |t|e^t & \text{if } a_{k_1-} = 1 \\ \max\{e^t, e^{ta_{k_1-}}\} & \text{if } a_{k_1-} \neq 1 \end{cases} \quad (4.19)$$

Furthermore, because of the explicit formulation and (4.6), the left-hand side of (4.18) can be replaced by  $\|v_1(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})}$ . Combining (4.14) and (4.18) we obtain the result since  $v(t, \cdot) = 0$

on  $(-\infty, r_1) \times \partial\mathbb{S}_+^{N-1}$ .

Next we suppose that  $a_{k_1-} = 0$ . Then for  $k = k_1$ , (4.17) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1+}} + m_2 - \frac{1}{a_{k_1+}} \int_t^0 (e^{(t-s)a_{k_1+}} - 1) F_{k_1,j}(s) ds. \quad (4.20)$$

This implies that

$$w_{k_1,j}(t) \rightarrow m_2 + \frac{1}{a_{k_1+}} \int_{-\infty}^0 F_{k_1,j}(s) ds := A_{k_1,j} \quad \text{as } t \rightarrow \infty.$$

If  $A_{k_1,j} \neq 0$  it would imply that  $\sum_{j=1}^{j_k} A_{k_1,j} \phi_{k_1,j}$  is a nonzero eigenfunction of order  $k_1 > 1$ , hence it changes sign and it would imply that  $v$  changes sign at  $-\infty$  (notice that all the other terms  $w_{k,j}(t)$  tends to 0 exponentially because of (4.17)-(4.18)). Hence  $A_{k_1,j} = 0$  and (4.20) endows the form

$$w_{k_1,j}(t) = m_1 e^{ta_{k_1,+}} - \frac{1}{a_{k,+}} \int_t^0 e^{(t-s)a_{k,+}} F_{k_1,j}(s) ds - \int_{-\infty}^t F_{k_1,j}(s) ds.$$

Because

$$\int_{-\infty}^t F_{k_1,j}(s) ds = O(e^t) \quad \text{as } t \rightarrow \infty,$$

we conclude that for  $k = k_1$ , there holds

$$|w_{k_1,j}(t)| \leq c_{27} \begin{cases} |t|e^t & \text{if } a_{k_1,+} = 1, \\ \max\{e^t, e^{ta_{k_1,+}}\} & \text{if } a_{k_1,+} \neq 1 \end{cases}$$

and finally we infer (4.19), which complete the proof.  $\square$

**Proof of Theorem A.** Assume that  $u \in C^2(\overline{\Omega} \setminus \{0\})$  is a positive solution of  $\mathcal{L}_\mu u = 0$  vanishing on  $\partial\Omega \setminus \{0\}$ .

*Case 1:*  $\mu > \mu_1$ . We claim (4.2) holds for some  $c_{13} \geq 0$ .

By Lemma 4.1, (4.1) holds for some  $a > 0$ . If  $a < -\alpha_-$ , then (4.2) holds with  $c_{13} = 0$ . If  $a = -\alpha_-$ , then (4.2) holds by Proposition 4.2-(i). Hence we are left with the case  $a > -\alpha_-$ . As in the proof of Proposition 4.2 we define  $k_1$  and  $k_2$ . By replacing  $a$  by  $a' = a + \epsilon$ , we can assume that  $a_{k_2,+} \neq 1$  and  $a_{k_1,-} \neq 1$ , to avoid the resonance complication in (4.14) and (4.19), hence

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{27} (e^{ta_{k_2,+}} + e^{ta_{k_1,-}} + e^t).$$

Furthermore  $k_2 = k_1 + 1$  and

$$a_{k_2,+} - a_{k_1,-} = \frac{1}{2} \left( \sqrt{4\mu + 4\lambda_{k_1+1} + (N-2)^2} + \sqrt{4\mu + 4\lambda_{k_1} + (N-2)^2} \right) > 0,$$

which yields

$$\|v(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} \leq c_{28} (e^{ta_{k_1,-}} + e^t).$$

This implies that  $u$  satisfies

$$u(x) \leq c_{29} (|x|^{\alpha_{k_1,-}} + |x|^{1-a}) \rho(x).$$

We iterate this procedure up to obtain

$$u(x) \leq c_{30} |x|^{\alpha_-} \rho(x)$$

and we conclude as in the proof of Proposition 4.2, Step 3.  $\square$

*Case:  $\mu = \mu_1$ .* In this case, the difficulty comes from the fact that there is no dissipation of energy in (4.7) for  $a = -\alpha_- = \frac{N-2}{2}$ . But from the above iterative procedure in *Case:  $\mu > \mu_1$ ,* we could obtain that for some  $\delta \in (0, 1)$ ,

$$u(x) \leq c_{31}|x|^{-\frac{N-2}{2}-\delta}\rho(x).$$

We finally show that there exists  $c_{32} \geq 0$  such that

$$\lim_{r \rightarrow 0} r^{\frac{N-2}{2}} \frac{u(r, \cdot)}{\ln r} = -c_{32}\psi_1(\cdot) \quad (4.21)$$

in  $C^1(\mathbb{S}_+^{N-1})$  and

$$\lim_{r \rightarrow 0} r^{\frac{N}{2}} \frac{u_r(r, \cdot)}{\ln r} = \frac{(N-2)c_{32}}{2}\psi_1(\cdot) \quad (4.22)$$

uniformly in  $\mathbb{S}_+^{N-1}$ .

Note that (4.5) reduces that

$$\begin{aligned} (1 + \epsilon_1(t))v_{tt} + \epsilon_2(t)v_t + (N-1 + \epsilon_3(t))v + \Delta'v \\ + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned}$$

in  $(-\infty, r_0) \times \mathbb{S}_+^{N-1}$ , vanishes on  $(-\infty, r_0) \times \partial\mathbb{S}_+^{N-1}$  and the  $\epsilon_j$  verify (4.6), and

$$v(t, \sigma) \leq c_{33}e^{-\delta t}.$$

Since the operator involved in the equation is uniformly elliptic we have by standard regularity theory

$$\begin{aligned} \|v\|_{C^{2,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_t\|_{C^{1,\delta}([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} + \|v_{tt}\|_{C^\delta([T-1, T+1] \times \overline{\mathbb{S}_+^{N-1}})} \\ \leq c_{34}\|v\|_{L^\infty((T-2, T+2) \times \mathbb{S}_+^{N-1})} \\ \leq c_{35}e^{-\delta T} \end{aligned}$$

for any  $T \leq r_0 + 3$ . We set

$$X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot) \psi_1 dS,$$

then

$$X''(t) + F(t) = 0 \quad (4.23)$$

where

$$F(t) = \int_{\mathbb{S}_+^{N-1}} (\epsilon_1 v_{tt} + \epsilon_2 v_t + \epsilon_3 v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle) \psi_1 dS.$$

Hence

$$|F(t)| \leq c_{36}e^{(1-\delta)t}. \quad (4.24)$$



This implies that  $X'(t)$  admits a limit  $c_{37} \leq 0$  when  $t \rightarrow -\infty$  and

$$\lim_{t \rightarrow -\infty} t^{-1}X(t) = c_{37}.$$

Set

$$W_2 = \bigoplus_{k \geq 2} \ker(\Delta' + \lambda_k Id),$$

and denote by  $v_2$  the orthogonal projection of  $v$  onto  $W_2$ . Then

$$v_2 tt + (N - 1)v_2 + \Delta' v_2 = F_2(t, \cdot), \quad (4.25)$$

where

$$|F_2(t, \cdot)| \leq c_{38}e^{(1-\delta)t}.$$

Since  $\lambda_2 = 2N$ , the function  $Y(t) = \|v_2(t, \cdot)\|_{L^2(\mathbb{S}_+^{N-1})}$  satisfies in  $\mathcal{D}'(-\infty, r_1)$

$$Y'' - (N + 1)Y \geq -c_{38}e^{(1-\delta)t}.$$

Because  $Y(t) = o(e^{-\sqrt{N+1}t})$  when  $t \rightarrow -\infty$ , it follows by the maximum principle that  $Y(t) = O(e^{\sqrt{N+1}t} + e^{(1-\delta)t}) = O(e^{(1-\delta)t})$ . Using again the standard regularity estimates for elliptic equations, we derive

$$\|v_2(t, \cdot)\|_{C^1(\overline{\mathbb{S}_+^{N-1}})} + \|v_2(t, \cdot)\|_{C(\overline{\mathbb{S}_+^{N-1}})} \leq c_{39}e^{(1-\delta)t}. \quad (4.26)$$

Combining (4.23) and (4.25) we derive (4.21). Since  $v(t, \cdot) = X(t)\psi_1 + v_2(t, \cdot)$  it follows from (4.26) that

$$\lim_{t \rightarrow -\infty} v_t(t, \cdot) = c_{37}\psi_1 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

Thus, the identity  $u_r(r, \cdot) = r^{-\frac{N}{2}} \left( \frac{2-N}{2}v(t, \cdot) + v_t(t, \cdot) \right)$  implies (4.21) and (4.22).  $\square$

## 4.2 Existence and uniqueness

**Proof of Theorem B.** We still assume that  $\Omega$  satisfies the condition (C-1) and  $\partial\mathbb{R}_+^N$  is tangent to  $\partial\Omega$  at 0. For  $\epsilon > 0$  let  $u_\epsilon$  be the solution of

$$\begin{cases} \mathcal{L}_\mu u_\epsilon = 0 & \text{in } \Omega_\epsilon := \Omega \setminus \overline{B}_\epsilon, \\ u_\epsilon = 0 & \text{in } \partial\Omega \cap \overline{B}_\epsilon^c, \\ u_\epsilon = \phi_\mu & \text{in } \Omega \cap \partial B_\epsilon. \end{cases} \quad (4.27)$$

Since  $\Omega \subset \mathbb{R}_+^N$ ,  $u_\epsilon \leq \phi_\mu$  in  $\Omega_\epsilon$  and

$$\frac{\partial u_\epsilon}{\partial \mathbf{n}} \Big|_{\Omega \cap \partial B_\epsilon} \leq \frac{\partial \phi_\mu}{\partial \mathbf{n}} \Big|_{\Omega \cap \partial B_\epsilon} < 0, \quad (4.28)$$

where  $\mathbf{n} = \epsilon^{-1}\mathbf{x}$ . Furthermore, if  $0 < \epsilon' < \epsilon$ ,  $u_{\epsilon'}|_{\Omega \cap \partial B_\epsilon} \leq u_\epsilon|_{\Omega \cap \partial B_\epsilon} = \phi_\mu$ , hence  $u_{\epsilon'} \leq u_\epsilon$  in  $\Omega_\epsilon$ . There exists  $u_0 = \lim_{\epsilon \rightarrow 0} u_\epsilon$  and  $u_0$  is a nonnegative solution of  $\mathcal{L}_\mu u = 0$  in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$  and is smaller than  $\phi_\mu$ .

Let  $\zeta \in \mathbb{X}_\mu(\Omega)$ ,  $\zeta > 0$ , then, with  $\mathbf{n}' = -\frac{x}{|x|} = -\mathbf{n}$ ,

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu u_\epsilon dx \\ &= \int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\partial B_\epsilon \cap \Omega} \left( -\frac{\partial u_\epsilon}{\partial \mathbf{n}'} \zeta \gamma_\mu^\Omega + \left( \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}'} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}'} \right) u_\epsilon \right) dS. \end{aligned}$$

Using (4.27) and (4.28) we obtain

$$\int_{\Omega_\epsilon} u_\epsilon \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega \geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega - \left( \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} + \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS.$$

We take  $\zeta = 1$ , hence  $\mathcal{L}_\mu^* \zeta = \ell_\mu^\Omega$  and we get

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case  $\mu > \mu_1$ , and

$$\begin{aligned} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega &\geq \int_{\partial B_\epsilon \cap \Omega} \left( \frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \gamma_\mu^\Omega - \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \phi_\mu^\Omega \right) dS \\ &\geq \left( \frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS - o(1), \end{aligned}$$

in the case  $\mu = \mu_1$ . Since  $u_\epsilon \leq \phi_\mu^\Omega$ ,

$$u_\epsilon \gamma_\mu^\Omega \leq \gamma_\mu^\Omega \phi_\mu^\Omega = r^{2-N} \psi_1^2 \in L^1(\Omega).$$

Therefore, by dominated convergence theorem, we conclude that

$$\ell_\mu^\Omega \int_{\Omega} u_0 d\gamma_\mu^\Omega \geq \begin{cases} 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left( \frac{N}{2} - 1 \right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1. \end{cases} \quad (4.29)$$

We infer that the function  $u_0$  is nonzero. It is a positive solution of  $\mathcal{L}_\mu u_0 = 0$  in  $\Omega$  which vanishes on  $\partial\Omega \setminus \{0\}$ . It follows from **Theorem A** that there exists  $k \geq 0$  such that

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{u(x)}{\rho(x)|x|^{\alpha-1}} &= k & \text{if } \mu > \mu_1, \\ \lim_{x \rightarrow 0} \frac{u(x)}{\rho(x)|x|^{-N/2} \ln|x|} &= k & \text{if } \mu = \mu_1. \end{aligned}$$

Next we next show that  $k = 1$ . In fact, if  $k < 1$ , there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in (0, \epsilon_0)$

$$u_\epsilon \leq \frac{k+1}{2} \phi_\mu^\Omega,$$

and then

$$\lim_{\epsilon \rightarrow 0^+} \ell_\mu^\Omega \int_{\Omega_\epsilon} u_\epsilon d\gamma_\mu^\Omega \leq \frac{k+1}{2} \ell_\mu^\Omega \int_{\Omega_\epsilon} \phi_\mu^\Omega d\gamma_\mu^\Omega < \begin{cases} 2\sqrt{\mu + \mu_1} \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu > \mu_1, \\ \left(\frac{N}{2} - 1\right) \int_{\mathbb{S}_+^{N-1}} \psi_1^2 dS & \text{if } \mu = \mu_1, \end{cases}$$

which contradicts (4.29). Thus, (1.11) and (1.12) hold true.  $\square$

**Proof of Corollary C.** *Identity (1.14).* As a consequence of Proposition 4.2, for any  $\zeta \in \mathbb{X}_\mu(\Omega)$  and  $\epsilon > 0$  we set  $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$ , and there holds

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} \zeta \gamma_\mu^\Omega \mathcal{L}_\mu \phi_\mu^\Omega dx \\ &= \int_{\Omega_\epsilon} \phi_\mu^\Omega \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega + \int_{\Omega \cap \partial B_\epsilon} \left( -\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left( \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS. \end{aligned}$$

Using Proposition ??, we have

$$\int_{\Omega \cap \partial B_\epsilon} \left( -\frac{\partial \phi_\mu^\Omega}{\partial \mathbf{n}} \zeta \gamma_\mu^\Omega + \left( \gamma_\mu^\Omega \frac{\partial \zeta}{\partial \mathbf{n}} + \zeta \frac{\partial \gamma_\mu^\Omega}{\partial \mathbf{n}} \right) \phi_\mu^\Omega \right) dS = -\zeta(0)A(\epsilon)(1 + o(1)),$$

where  $A(\epsilon)$  is defined in (2.5).

The uniqueness follows direct from Kato's inequality (3.4).  $\square$

## 5 The Dirichlet problem

**Proof of Theorem D.** Note that in section §3.2 for  $\lambda \in \mathfrak{M}(\partial\Omega; \eta\beta_\mu^\Omega)$ , problem

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega, \\ u = \lambda & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting  $\mathbb{K}_\mu^\Omega(\lambda)$ , which verifies the identity

$$\int_{\Omega} \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\partial\Omega} \zeta d(\lambda\beta_\mu^\Omega) \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Moreover, problem

$$\begin{cases} \mathcal{L}_\mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution, denoting  $\mathbb{G}_\mu^\Omega(\nu)$ , which verifies the identity

$$\int_{\Omega} \mathbb{K}_\mu^\Omega(\lambda) \mathcal{L}_\mu^* \zeta d\gamma_\mu^\Omega = \int_{\Omega} \zeta d\gamma_\mu^\Omega \quad \text{for all } \zeta \in \mathbb{X}_\mu(\Omega).$$

Together with **Corollary C** and the linearity of operator  $\mathcal{L}_\mu$ , we have that  $\mathbb{K}_\mu^\Omega(\lambda) + \mathbb{G}_\mu^\Omega(\nu) + k\phi_\Omega^\mu$  is a weak solution of (1.18) satisfying (1.20) and the uniqueness follows directly from Kato's inequality (3.4).  $\square$

Our final part is to classify the boundary data for nonnegative  $\mathcal{L}_\mu$ -harmonic function.

**Proof of Theorem E.** Let  $\Omega$  be a bounded  $C^2$  domain and  $u$  be a nonnegative  $\mathcal{L}_\mu$ -harmonic function in  $\Omega$ . We now show that there exists a nonnegative measure  $\lambda$  on  $\partial\Omega \setminus \{0\}$  and  $k \geq 0$  such that

$$u = \mathbb{K}_\mu^\Omega[\lambda] + k\phi_\mu^\Omega. \quad (5.1)$$

For  $\epsilon > 0$  the term  $\mu|x|^{-2}$  is bounded in  $\Omega_\epsilon = \Omega \cap \overline{B}_\epsilon^c$ . Hence there exists a nonnegative Radon measure  $\lambda_\epsilon$  such that  $u$  is the unique solution of

$$\begin{cases} \mathcal{L}_\mu u = 0 & \text{in } \Omega_\epsilon, \\ u = \lambda_\epsilon & \text{on } \partial\Omega_\epsilon. \end{cases}$$

Furthermore  $\lambda_\epsilon$  is the boundary trace is achieved in *dynamical sense*, see [14] and references therein. Hence for any  $\zeta \in C(\overline{\Omega})$  vanishing on  $B_\epsilon$ , there holds

$$\lim_{\delta \rightarrow 0} \int_{\Sigma_\delta} u \zeta dS = \int_{\partial\Omega \cap B_\epsilon^c} \zeta d\lambda_\epsilon,$$

where  $\Sigma_\delta = \{x \in \Omega : \rho(x) = \delta\}$ . If we write

$$\lambda_\epsilon = \lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c} + u|_{\Omega \cap \partial B_\epsilon},$$

it proves that for  $0 < \epsilon' < \epsilon$ ,  $\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c} = \lambda_{\epsilon'}|_{\partial\Omega \cap B_\epsilon^c}$ . This defines in a unique way a nonnegative Radon  $\lambda$  on  $\partial\Omega \setminus \{0\}$  measure such that (5.1) holds for all  $\zeta \in \mathbb{X}_\mu(\Omega)$  vanishing near 0. Furthermore  $\rho u \in L^1(\Omega_\epsilon)$  for any  $\epsilon > 0$ . Denote by  $\mathbb{K}_\mu^{\Omega_\epsilon}$  the Poisson potential of  $\mathcal{L}_\mu$  in  $\Omega_\epsilon$ . Then

$$u|_{\Omega_\epsilon} = \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}] + \mathbb{K}_\mu^{\Omega_\epsilon}[u|_{\Omega \cap \partial B_\epsilon}].$$

For  $0 < \epsilon' < \epsilon$ , one has that  $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega \cap B_{\epsilon'}^c}]|_{\Omega \cap \partial B_\epsilon} \geq 0$ . Therefore  $\mathbb{K}_\mu^{\Omega_{\epsilon'}}[\lambda_{\epsilon'}|_{\partial\Omega \cap B_{\epsilon'}^c}] \geq \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}]$  in  $\Omega_\epsilon$ . Hence

$$\lim_{\epsilon \rightarrow 0} \mathbb{K}_\mu^{\Omega_\epsilon}[\lambda_\epsilon|_{\partial\Omega \cap B_\epsilon^c}] = \mathbb{K}_\mu^\Omega[\lambda] \leq u \quad \text{in } \Omega.$$

Next we aim to characterize the behaviour at 0. By contradiction we assume that

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = \infty.$$

Then for any  $m > 0$  there exists a sequence  $\{\epsilon_{m,k}\} \subset \mathbb{R}_+^*$  tending to 0 and a sequence  $\{\ell_{m,k}\} \subset \mathbb{R}_+^*$  tending to  $\infty$  such that

$$\int_{\Omega \cap \partial B_{\epsilon_k}} \min\{u, \ell_{m,k}\} d\beta_\mu^\Omega = m.$$

Set  $\tau_{m,k} = \min\{u|_{\Omega \cap \partial B_{\epsilon_k}}, \ell_{m,k}\}$  and set  $u_{m,k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\tau_{m,k} \chi_{\Omega \cap \partial B_{\epsilon_k}}]$ . Then

$$u_{m,k} \leq u \quad \text{in } \Omega_{\epsilon_k},$$

and we recall that

$$\int_{\Omega \cap \partial B_{\epsilon_k}} \phi_\mu^\Omega d\beta_\mu^\Omega = c_\mu(1 + o(1)),$$

where  $c_\mu$  is the constant defined in (2.4). Combining the boundary Harnack inequality with the standard Harnack inequality, one infers

$$c_{47} \frac{\rho(x)}{\rho(y)} \leq c_{46} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq \frac{u_{m,k}(x)}{u_{m,k}(y)} \leq c_{44} \frac{\phi_\mu^\Omega(x)}{\phi_\mu^\Omega(y)} \leq c_{45} \frac{\rho(x)}{\rho(y)}, \quad (5.2)$$

for all  $x, y \in \Omega$  such that  $|x| = |y| \geq 2\epsilon_k$ . If we set  $\dot{\phi}_\mu^\Omega(x) = \frac{\phi_\mu^\Omega(x)}{\rho(x)}$  and  $\dot{u}_{m,k}(x) = \frac{u_{m,k}(x)}{\rho(x)}$  then (5.2) becomes

$$c_{47} \leq c_{46} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq \frac{\dot{u}_{m,k}(x)}{\dot{u}_{m,k}(y)} \leq c_{44} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq c_{45}. \quad (5.3)$$

Assume for a while that we have proved that there exists  $\theta > 0$ , independent of  $m$  and  $k$  such that for any

$$\int_{\partial B_{2\epsilon_k} \cap \Omega} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) = \theta m. \quad (5.4)$$

If we assume that for  $\delta \leq 2\epsilon_{k_0}$

$$2c_\mu \geq \int_{\Omega \cap \partial B_\delta} \dot{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega) \geq \frac{c_\mu}{2},$$

one has for  $k \geq k_0$ ,

$$\int_{\Omega \cap \partial B_{2\epsilon_k}} \dot{u}_{m,k} d(\rho\beta_\mu^\Omega) \geq \theta m \geq \frac{\theta m}{2c_\mu} \int_{\Omega \cap \partial B_{2\epsilon_k}} \dot{\phi}_\mu^\Omega d(\rho\beta_\mu^\Omega)$$

Since

$$\dot{\phi}_\mu^\Omega(x) \leq \frac{c_{45}}{c_{46}} \dot{\phi}_\mu^\Omega(y)$$

and

$$\dot{u}_{m,k}(x) \geq c_{47} \dot{u}_{m,k}(y),$$

we derive

$$\frac{1}{c_{47}} \dot{u}_{m,k}(x) \int_{\partial B_{2\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) \geq \frac{\theta m c_{46}}{2c_\mu c_{45}} \dot{\phi}_\mu^\Omega(x) \int_{\Omega \cap \partial B_{2\epsilon_k}} d(\rho\beta_\mu^\Omega).$$

Therefore

$$u_{m,k}(x) \geq c_{48}m\phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega \text{ s.t. } |x| = 2\epsilon_k,$$

and  $c_{48} > 0$  is independent of  $m$  and  $\epsilon_k$ . This implies by the maximum principle and letting  $\epsilon_k \rightarrow 0$

$$u(x) \geq u_{m,k}(x) \geq c_{48}m\phi_\mu^\Omega(x) \quad \text{for all } x \in \Omega. \quad (5.5)$$

Since  $m$  is arbitrary we obtain a contradiction. Hence there holds

$$\limsup_{\epsilon \rightarrow 0} \int_{\Omega \cap \partial B_\epsilon} u d\beta_\mu^\Omega = \lim_{\epsilon_k \rightarrow 0} \int_{\Omega \cap \partial B_{\epsilon_k}} u d\beta_\mu^\Omega = m_u < \infty. \quad (5.6)$$

Then inequality (5.5) holds without truncation with  $m$  replaced by  $m_u$ . We recall that

$$w_\epsilon := \mathbb{K}_\mu^{\Omega_\epsilon} [u|_{\Omega \cap \partial B_\epsilon}] = u|_{\Omega_\epsilon} - \mathbb{K}_\mu^{\Omega_\epsilon} [\lambda_\epsilon|_{\partial \Omega \cap B_\epsilon}] \quad \text{in } \Omega_\epsilon. \quad (5.7)$$

*Case 1:* We first assume that  $m_u > 0$ . Then (5.4) combined with the maximum principle yields

$$\int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \int_{\partial B_{2\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) \geq \theta \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k} d(\rho\beta_\mu^\Omega) = \theta m_u (1 + o(1)),$$

with  $\dot{w}_\epsilon = \rho^{-1}w_\epsilon$ . Inequality (5.3) is replaced by

$$c_{47} \leq c_{46} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq \frac{\dot{w}_\epsilon(x)}{\dot{w}_\epsilon(y)} \leq c_{44} \frac{\dot{\phi}_\mu^\Omega(x)}{\dot{\phi}_\mu^\Omega(y)} \leq c_{45} \quad \text{in } \Omega_{2\epsilon}. \quad (5.8)$$

Therefore, for  $\epsilon_k$  small enough and  $|x| = 2\epsilon_k$ ,

$$\begin{aligned} \dot{w}_\epsilon(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\leq c_{45} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \leq 2c_{45}m_u \\ &\leq \frac{4c_{45}m_u}{c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \dot{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega), \end{aligned}$$

which implies

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.9)$$

Hence

$$w_{\epsilon_k}(x) \leq \frac{4c_{44}c_{45}m_u}{c_\mu c_{47}} \phi_\mu^\Omega(x) := c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}. \quad (5.10)$$

Since  $w_{\epsilon_k}$  and  $\phi_\mu$  are  $\mathcal{L}_\mu$ -harmonic in  $\Omega_{2\epsilon_k}$ , and vanishes on  $\partial \Omega \cap B_{2\epsilon_k}$  it follows that inequality (5.10) also holds for any  $x \in \Omega_{2\epsilon_k}$ . By definition  $w_{\epsilon_k} = \mathbb{K}_\mu^{\Omega_{\epsilon_k}} [u|_{\Omega \cap \partial B_{\epsilon_k}}]$ , hence

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}} [u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \leq c_{49}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.11)$$

Next we obtain the estimate from below. From (5.8), with  $|x| = 2\epsilon_k$ ,

$$\begin{aligned} \dot{w}_{\epsilon_k}(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega) &\geq c_{47} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{w}_{\epsilon_k}(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}m_u}{2} \\ &\geq \frac{c_{47}m_u}{4c_\mu} \int_{\partial B_{\epsilon_k} \cap \Omega} \dot{\phi}_\mu^\Omega(y) d(\rho\beta_\mu^\Omega) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \dot{\phi}_\mu^\Omega(x) \int_{\partial B_{\epsilon_k} \cap \Omega} d(\rho\beta_\mu^\Omega). \end{aligned}$$

Hence

$$w_{\epsilon_k}(x) \geq \frac{c_{47}c_{44}m_u}{4c_\mu c_{45}} \phi_\mu^\Omega(x) := c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

It follows that

$$\mathbb{K}_\mu^{\Omega_{\epsilon_k}}[u|_{\Omega \cap \partial B_{\epsilon_k}}](x) \geq c_{50}m_u \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega_{2\epsilon_k}. \quad (5.12)$$

From (5.7), (5.12) and (5.11) we infer

$$c_{50}m_u \phi_\mu^\Omega \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^c}] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}. \quad (5.13)$$

This implies, by letting  $\epsilon_k \rightarrow 0$ ,

$$c_{50}m_u \phi_\mu^\Omega \leq u - \mathbb{K}_\mu^\Omega[\lambda] \leq c_{48}m_u \phi_\mu^\Omega \quad \text{in } \Omega.$$

Therefore, the function  $u - \mathbb{K}_\mu^\Omega[\lambda]$  is  $\mathcal{L}_\mu$ -harmonic and positive in  $\Omega$  and it vanishes on  $\partial\Omega$ . By Corollary C, it implies that it coincides with  $c\phi_\mu^\Omega$  for some  $c \geq 0$  (and in that case  $c_{50}m_u \leq c \leq c_{49}m_u$ ).

*Case 2:* Assume  $m_u = 0$ . Following the same inequalities as in Case 1, (5.9) is replaced by: for any  $\delta > 0$  there exists  $k_0 > 0$  such that for  $k \geq k_0$ ,

$$w_{\epsilon_k}(x) \leq \delta \phi_\mu^\Omega(x) \quad \text{for } x \in \Omega \cap \partial B_{2\epsilon_k}.$$

Hence (5.13) is transformed into

$$0 \leq u|_{\Omega_{\epsilon_k}} - \mathbb{K}_\mu^{\Omega_{\epsilon_k}}[\lambda_{\epsilon_k}|_{\partial\Omega \cap B_{\epsilon_k}^c}] \leq \delta \phi_\mu^\Omega \quad \text{in } \Omega_{2\epsilon_k}.$$

Letting successively  $\epsilon_k \rightarrow 0$  and  $\delta \rightarrow 0$  yields  $u - \mathbb{K}_\mu^\Omega[\lambda] = 0$  in  $\Omega$ , which ends the proof.  $\square$

## Appendix: Estimates (1.9)

**Proposition A.1** *Assume  $\Omega$  is a bounded  $C^2$  domain such that  $0 \in \partial\Omega$  satisfying condition (C-1) and let  $\gamma_\mu^\Omega$  be defined by (1.8) and normalized by  $\|\gamma_\mu^\Omega\|_{L^2(\Omega)} = 1$ . Then*

$$\lim_{r \rightarrow 0} r^{\frac{N}{2}-1} \gamma_\mu^\Omega(r, \cdot) = c_1 \psi_1 \quad \text{in } C_{loc}^1(\mathbb{S}_+^{N-1})$$

and

$$\lim_{r \rightarrow 0} r^{\frac{N}{2}} \gamma_{\mu r}^\Omega(r, \cdot) = c_1 \left(1 - \frac{N}{2}\right) \psi_1 \quad \text{locally uniformly in } \mathbb{S}_+^{N-1}.$$

**Proof.** Since  $\alpha_+ + (N - 2)\alpha_+ - \mu + 1 - N = 0$ , the function  $x \mapsto w(x) := |x|^{\alpha_+}$  satisfies

$$\tilde{\mathcal{L}}_\mu w(x) := \mathcal{L}_\mu w - \ell_\mu^\Omega w = |x|^{\alpha_+ - 2} (N - 1 - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Furthermore,  $\nabla w \in L_{loc}^2(\mathbb{R}^N)$ . Let  $R_0 > 0$  such that  $N - 1 \geq \ell_\mu^\Omega R_0^2$  and  $m > 0$  such that  $mw \geq \gamma_\mu^\Omega$  on  $\Omega \cap B_{R_0}^c$ . Then the function  $(\gamma_\mu^\Omega - mw)_+$  belongs to  $H_\mu(\Omega)$  and satisfies  $\tilde{\mathcal{L}}_\mu(\gamma_\mu^\Omega - mw)_+ \leq 0$  in the dual of  $H_\mu(\Omega)$ . Hence

$$\int_\Omega \left( |\nabla(\gamma_\mu^\Omega - mw)_+|^2 + \left( \frac{\mu}{|x|^2} - \ell_\mu^\Omega \right) (\gamma_\mu^\Omega - mw)_+^2 \right) dx \leq 0.$$

Therefore  $(\gamma_\mu^\Omega - mw)_+ \leq 0$ , which implies that

$$0 < \gamma_\mu^\Omega(x) \leq m|x|^{\alpha_+} \quad \text{for all } x \in \Omega.$$

Then we proceed as in the proof of Proposition 4.2. We flatten the boundary near 0 and set

$$v(t, \sigma) = r^{-\alpha_+} \tilde{\gamma}_\mu^\Omega(r, \sigma) \quad \text{with } t = \ln r,$$

where the function  $\tilde{\gamma}_\mu^\Omega$  is defined similarly as  $\tilde{u}$  in (4.3). Then  $v$  is bounded in  $(-\infty, T_0] \times \mathbb{S}_+^{N-1}$  where it satisfies

$$\begin{aligned} (1 + \epsilon_1(t, \cdot))v_{tt} + (N - 2 + 2\alpha_+ + \epsilon_2(t, \cdot))v_t + (\alpha_+(\alpha_+ + N - 2) - \mu + \epsilon_3(t, \cdot) + e^{2t}\ell_\mu^\Omega)v \\ + \Delta'v + \langle \nabla'v, \epsilon_4(t, \cdot) \rangle + \langle \nabla'v_t, \epsilon_5(t, \cdot) \rangle + \langle \nabla'(\langle \nabla'v, \mathbf{e}_N \rangle), \epsilon_6(t, \cdot) \rangle = 0, \end{aligned}$$

instead of (4.5). It vanishes on  $(-\infty, T_0] \times \partial\mathbb{S}_+^{N-1}$  and the  $\epsilon_j$  satisfy again (4.6).

*Case 1:*  $\mu > \mu_1$ . The energy method used in proof of Proposition 4.2 applies with no modification and we infer that there exists  $c_{51} \geq 0$  such that

$$v(t, \cdot) \rightarrow c_{51}\psi_1 \quad \text{as } t \rightarrow -\infty$$

in  $C^1(\mathbb{S}_+^{N-1})$  and  $v_t(t, \cdot) \rightarrow 0$  uniformly in  $\mathbb{S}_+^{N-1}$ . If  $c_{51} = 0$ , we can prove, as in Proposition 4.2-(ii) that there exists  $\tau > 0$  such that

$$\gamma_\mu^\Omega(x) \leq c_{52}|x|^{\alpha_+ + \tau} \quad \text{for all } x \in \Omega. \quad (5.14)$$

Iterating this process, we infer that (5.14) holds for any  $\tau > 0$ . For  $k > 1$ , let  $\alpha_{k,+}$  be the positive root of (2.1) and put  $w_k(x) = |x|^{\alpha_{k,+}}$ . Then

$$\tilde{\mathcal{L}}_\mu w_k(x) = |x|^{\alpha_{k,+} - 2} (\lambda_k - \ell_\mu^\Omega |x|^2) \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Since  $\lambda_k \rightarrow \infty$ , as  $k \rightarrow \infty$ , we choose  $k$  such that  $\lambda_k > \ell_\mu^\Omega (\text{diam}(\Omega))^2$ . Hence  $w_k$  is a supersolution of  $\tilde{\mathcal{L}}_\mu$ . Because  $\gamma_\mu^\Omega(x) = o(w_k(x))$  near  $x = 0$ , it follows that  $\gamma_\mu^\Omega(x) \leq \epsilon o(w_k(x))$  in  $\Omega$  for any  $x \in \Omega$ . Hence  $\gamma_\mu^\Omega = 0$ , which is a contradiction. Finally it implies that  $c_{51} > 0$ , which yields (1.9)-(i). Because the convergence holds in  $C^1(\mathbb{S}_+^{N-1})$  and  $v_t(t, \cdot) \rightarrow 0$ , we infer

$$\lim_{r \rightarrow 0} r^{1 - \alpha_+} \nabla \tilde{\gamma}_\mu^\Omega(r, \cdot) = c_{51} (\alpha_+ \psi_1 \mathbf{e} + \nabla' \psi_1)$$



where  $\mathbf{e} = \frac{x}{|x|}$ . This implies the claim.

*Case 2:*  $\mu = \mu_1$ . Set  $v(t, \cdot) = r^{\frac{N}{2}-1}u(r, \cdot)$  with  $t = \ln r$  and  $X(t) = \int_{\mathbb{S}_+^{N-1}} v(t, \cdot)\psi_1 dS$  and obtain again (4.23), where  $F(t, \cdot)$  satisfies (4.24). Since  $X'(t) \rightarrow 0$  and  $X$  is bounded, it follows that  $X(t)$  admits a limit  $c_{52} \geq 0$  when  $t \rightarrow -\infty$ . As in the proof of **Theorem A**, we infer that

$$\lim_{t \rightarrow -\infty} v(t, \cdot) = c_{52}\psi_1 \quad \text{in } C^1(\mathbb{S}_+^{N-1}) \quad \text{and} \quad \lim_{t \rightarrow -\infty} v_t(t, \cdot) = 0 \quad \text{uniformly in } \mathbb{S}_+^{N-1}.$$

If  $c_{52} = 0$  we derive a contradiction as in the first case. □

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## References

- [1] P. Bauman. Positive solutions of elliptic equations in nondivergence form and their adjoints, *Ark. Mat.* **22** (1984), 153-173.
- [2] G. Barbatis , S. Filippas, A. Tertikas. Sharp Hardy and Hardy-Sobolev inequalities with point singularities on the boundary. *J. Math. Pures Appl.* **117** (2018), 146-184.
- [3] M.F. Bidaut-Véron, R. Borghol, L. Véron. Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations, *Calc. Var. Part. Diff. Eq.* **27(2)** (2006), 159-177.
- [4] H. Brezis, J. L. Vazquez. Blow-up solutions of some nonlinear elliptic equations, *Rev. Mat. Complut.* **10** (1997) 443-469.
- [5] C. Cazacu. On Hardy inequalities with singularities on the boundary, *C. R. Math. Acad. Sci. Paris* **349** (2011) 273-277.
- [6] C. Cazacu. Hardy inequality and Pohozaev identity for operators with boundary singularities: Some applications. *C. R. Math. Acad. Sci. Paris* **349** (2011)1167-1172.
- [7] C. Cazacu. *Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results*, *J. Funct. Anal.* **263** (2012), 3741-3782.
- [8] H. Chen, A. Quaas, F. Zhou. On nonhomogeneous elliptic equations with the Hardy- Leray potentials, arXiv:1705.08047.
- [9] H. Chen, L. Véron. Weak solutions of semilinear elliptic equations with Leray-Hardy potential and measure data, *Mathematics in Engineering* **1** (2019), 391-418.
- [10] H. Chen, L. Véron. Semilinear elliptic equations with Leray-Hardy potential singular on the boundary and measure data. *In preparation*.

- [11] Filippas S., Tertikas A., Tidblom J. On the structure of Hardy-Sobolev-Mazya inequalities. *J. Eur. Math. Soc.* **11** (2009), 1165-1185.
- [12] A. Gmira, L. Véron. Boundary singularities of solutions of nonlinear elliptic equations, *Duke Math. J.* **64** (1991), 271-324.
- [13] R. A. Hunt and R. L. Wheeden. Positive harmonic functions on Lipschitz domains, *Trans. Amer. Math. Soc.* **147** (1970), 507-527.
- [14] M. Marcus, L. Véron, *Nonlinear Second Order Elliptic Equations Involving Measures*, De Gruyter Series in Nonlinear Analysis and Applications **21** (2013), xiii+pp. 1-262.
- [15] M. Marcus, L. Véron, Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains: the subcritical case, *Ann. Scu. Norm. Sup. Pisa Ser. V* **Vol. X** (2011), 913-984.
- [16] M. Marcus, L. Véron, Boundary trace of positive solutions of supercritical elliptic equations in dihedral domains, *Ann. Scu. Norm. Sup. Pisa Ser. V*, **Vol. XV** (2016), 501-542.
- [17] L. Véron. Elliptic Equations Involving Measures, in *Stationary Partial differential equations* Vol. I, 593-712. *Handb. Differ. Equ.*, North-Holland, Amsterdam (2004).
- [18] L. Véron, C. Yarur. Boundary value problems with measures for elliptic equations with singular potentials, *J. Funct. Anal.* **262** (2012), 733-772.
- [19] L. Véron. Existence and Stability of Solutions of General Semilinear Elliptic Equations with Measure Data, *Advanced Nonlinear Studies* **13** (2013), 447-460.