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TYPICAL PROPERTIES OF INTERVAL MAPS PRESERVING THE LEBESGUE MEASURE

JOZEF BOBOK AND SERGE TROUBETZKOY

ABSTRACT. Let us denote λ the Lebesgue measure on $[0, 1]$, put

$$C(\lambda) = \{f \in C([0, 1]); \forall A \subset [0, 1], A \text{ Borel} : \lambda(A) = \lambda(f^{-1}(A))\}.$$

We endow the set $C(\lambda)$ by the uniform metric ρ and investigate dynamical properties of typical maps in the complete metric space $(C(\lambda), \rho)$.

1. INTRODUCTION AND SUMMARY OF RESULTS

This article is about typical properties of continuous maps of the interval which preserve the Lebesgue measure. Throughout the article the word typical will mean that with respect to the uniform topology there is a dense G_δ of maps having this property. Such results are in the domain of approximation theory. The use of approximation techniques in dynamical systems was apparently started in 1941 by Oxtoby and Ulam who considered a simplicial polyhedron with a non-atomic measure which is positive on open sets, in this setting they showed that the set of ergodic measure-preserving homeomorphisms is typical in the strong topology [19]. In 1944 Halmos introduced approximation techniques to a purely metric situation, the study of invertible mode 0 maps of the interval $[0, 1]$ which preserve the Lebesgue measure, he showed that the typical invertible map is weakly mixing, i.e., has continuous spectrum [8],[9],[10]. In 1967 Yuzvinskii improved this last result showing the typicality of simple continuous spectrum [21]. In 1948 Rohlin showed that the set of (strongly) mixing measure preserving invertible maps is of first category [16]. In 1967 Katok and Stepin [13] introduced the notation of speed of approximation, one of the notable applications of their method is the typicality of ergodicity and weak mixing for certain classes of interval exchange transformations. Katok has shown that interval exchange transformations are never mixing [12].

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The study of typical properties of homeomorphism of compact manifolds which preserve regular measures was continued and generalized by Katok and Stepin in 1970 [14] who showed the typicality of ergodicity, of simple continuous spectrum, and the absence of mixing. More recently Alpern and Prasad have unified the studies of homeomorphisms and of measure preserving transformations and shown that if a property is typical for measure preserving transformations then it is typical for homeomorphisms [1].

All of these results are about invertible maps. In this article we consider the set $C(\lambda)$ of continuous non-invertible maps of the unit interval $[0, 1]$ which preserve the Lebesgue measure λ . Every such map has a dense set of periodic points. Furthermore, except for the two exceptional maps id and $1 - id$, every such map has positive metric entropy. The $C(\lambda)$ -typical function (all the properties will be defined later in the article):

- is weakly mixing with respect to λ ,
- is leo,
- maps a set of the Lebesgue measure zero onto $[0, 1]$,
- has a knot point at λ almost every point [6],
- has Hausdorff dimension = lower Box dimension = $1 <$ upper Box dimension = 2 [18],
- has infinite topological entropy.

Furthermore, in analogy to Rohlin's result, we show that the set of mixing maps in $C(\lambda)$ is of first category. We also show that for any $c > 0$ as well as for $c = \infty$ the set of maps having metric entropy c is dense in $C(\lambda)$.

2. MAPS IN $C(\lambda)$

Let λ denote the Lebesgue measure on $[0, 1]$ and \mathcal{B} the Borel sets in $[0, 1]$. Let $C(\lambda)$ consist of all continuous λ -preserving functions from $[0, 1]$ onto $[0, 1]$, i.e.,

$$C(\lambda) = \{f: [0, 1] \rightarrow [0, 1]: \forall A \in \mathcal{B}, \lambda(A) = \lambda(f^{-1}(A))\}.$$

We consider the uniform metric ρ on $C(\lambda)$: $\rho(f, g) := \sup_{x \in [0, 1]} |f(x) - g(x)|$.

Proposition 1. $(C(\lambda), \rho)$ is a complete metric space.

We leave the standard proof of this result to the reader.

Definition 2. We say that continuous maps $f, g: [a, b] \subset [0, 1] \rightarrow [0, 1]$ are λ -equivalent if for each Borel set $A \in \mathcal{B}$,

$$\lambda(f^{-1}(A)) = \lambda(g^{-1}(A)).$$

For $f \in C(\lambda)$ and $[a, b] \subset [0, 1]$ we denote by $C(f; [a, b])$ the set of all continuous maps λ -equivalent to $f \upharpoonright [a, b]$ and

$$C_*(f; [a, b]) = \{h \in C(f; [a, b]): h(a) = f(a), h(b) = f(b)\}.$$

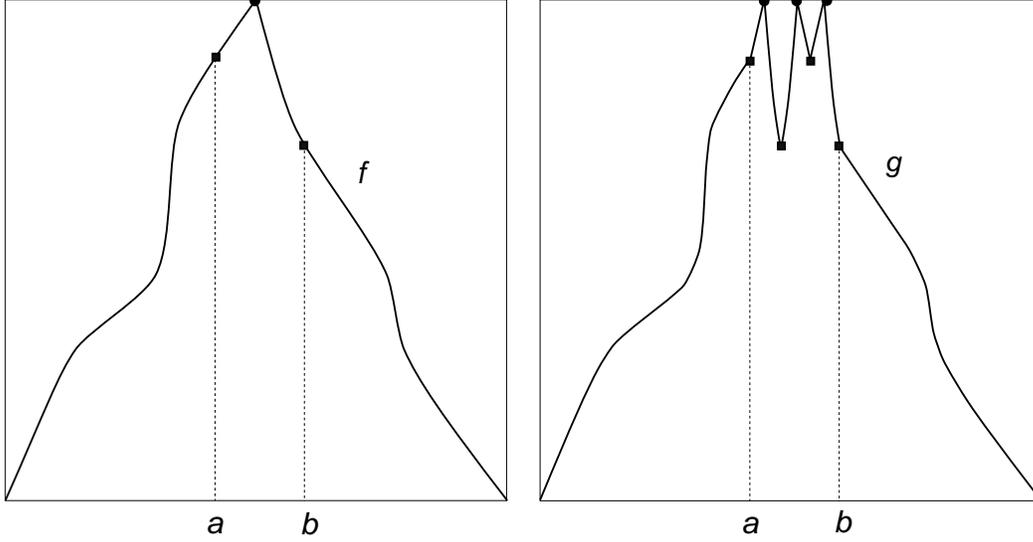


FIGURE 1. $f \in C(\lambda)$, $m = 3$, $h = h\langle f; [a, b], 3 \rangle \in C_*(f; [a, b])$.
 g is a regular 3-fold window perturbation of f by h .

Example 1. Let f be from $C(\lambda)$ and $[a, b] \subset [0, 1]$. For any fixed $m \in \mathbb{N}$, let us define the map $h = h\langle f; [a, b], m \rangle: [a, b] \rightarrow [0, 1]$ by $j \in \{0, \dots, m-1\}$ and

$$(1) \quad h(a+x) = \begin{cases} f\left(a + m\left(x - \frac{j(b-a)}{m}\right)\right) & \text{if } x \in \left[\frac{j(b-a)}{m}, \frac{(j+1)(b-a)}{m}\right], \quad j \text{ even,} \\ f\left(a + m\left(\frac{(j+1)(b-a)}{m} - x\right)\right) & \text{if } x \in \left[\frac{j(b-a)}{m}, \frac{(j+1)(b-a)}{m}\right], \quad j \text{ odd.} \end{cases}$$

Then $h\langle f; [a, b], m \rangle \in C(f; [a, b])$ for each m and $h\langle f; [a, b], m \rangle \in C_*(f; [a, b])$ for each m odd (See Figure 1).

For a fix $h \in C_*(f; [a, b])$, the map $g = g\langle f, h \rangle \in C(\lambda)$ defined by

$$(2) \quad g(x) = \begin{cases} f(x) & \text{if } x \notin [a, b], \\ h(x) & \text{if } x \in [a, b] \end{cases}$$

will be called the *window perturbation* of f (by h on $[a, b]$). In particular, if $h = h\langle f; [a, b], m \rangle$, m odd, resp. h is piecewise affine, we will speak of *regular m -fold*, resp. *piecewise affine window perturbation g of f* (on $[a, b]$) - see Figure 1.

The following useful observation will be repeatedly used in our text. We will omit its proof since it is a straightforward consequence of the uniform continuity of f .

Lemma 3. Let f be from $C(\lambda)$. For each $\varepsilon > 0$ there is $\delta > 0$ such that

$$(3) \quad \forall [a, b] \subset [0, 1], \quad b - a < \delta \quad \forall h \in C_*(f; [a, b]): \rho(f, g\langle f, h \rangle) < \varepsilon.$$

In particular, for each $\varepsilon > 0$ there is a positive integer $n_0 > 0$ such that for each $n > n_0$, if $I_j = [\frac{j}{n}, \frac{(j+1)}{n}]$ and

$$g \upharpoonright I_j = h\langle f; I_j, m(j) \rangle$$

with odd numbers $m(j)$ for every $j \in \{0, \dots, n-1\}$, then $\rho(f, g) < \varepsilon$ independently of the numbers $m(j)$.

Below we introduce three classical types of mixing in a topological dynamics. We consider them in the context of $C(\lambda)$.

A map $f \in C(\lambda)$ is called

- *transitive* if for each pair of nonempty open sets U, V , there is $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$,
- *topological mixing* if for each pair of nonempty open sets U, V , there is $n_0 \geq 0$ such that $f^n(U) \cap V \neq \emptyset$ for every $n \geq n_0$,
- *leo (locally eventually onto)* if for every nonempty open set U there is an $n \in \mathbb{N}$ such that $f^n(U) = [0, 1]$.

For each map $f \in C(\lambda)$ the set of periodic points is dense in $[0, 1]$, this is a consequence of the Poincaré Recurrence Theorem and the fact that in dynamical system given by an interval map the closures of recurrent points and periodic points coincide [7]. Thus Proposition 4 and Lemma 5, stated below, apply to elements of $C(\lambda)$.

For two intervals J_1, J_2 with pairwise disjoint non-empty interiors we write $J_1 < J_2$ if $x_1 < x_2$ for some points $x_1 \in J_1$ and $x_2 \in J_2$.

Proposition 4. [3] *Suppose f has a dense set of periodic points. The following assertions hold (Figure 2).*

- (i) *There is a collection (perhaps finite or empty) $\mathcal{J} = \{J_1, J_2, \dots\}$ of closed subintervals of $[0, 1]$ with mutually disjoint interiors, such that for each i , $f^2(J_i) = J_i$, and there is a point $x_i \in J_i$ such that $\{f^{4n}(x_i) : n \geq 0\}$ is dense in J_i .*
- (ii) *If $\#\mathcal{J} \geq 2$ then either*
 - (4) $\forall J_1, J_2 \in \mathcal{J} : J_1 < J_2 \implies f(J_1) < f(J_2)$
 - or
 - (5) $\forall J_1, J_2 \in \mathcal{J} : J_1 < J_2 \implies f(J_1) > f(J_2)$.
- (iii) *For each $J \in \mathcal{J}$, $f(J) \in \mathcal{J}$ and $f^{-1}(f(J)) = J$.*
- (iv) *If (4) is true then $f(J) = J$ for each $J \in \mathcal{J}$.*
- (v) *If (5) holds true $f(J) = J$ if and only if $J^\circ \cap \text{Fix}(f) \neq \emptyset$ and there is at most one such interval.*
- (vi) *If $x \in (0, 1) \setminus \bigcup_{i \geq 1} J_i^\circ$, then $f^2(x) = x$.*

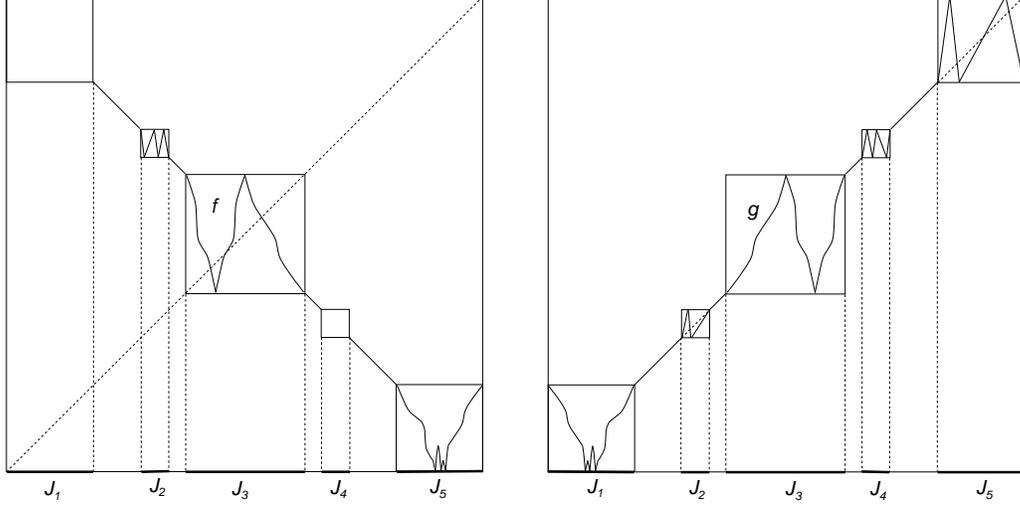


FIGURE 2. $f, g \in C(\lambda)$; Prop. 4(5): $\mathcal{J}(f) = \{J_i\}_{i=1}^5$, $f(J_1) = J_5$, $f(J_2) = J_4$, $f(J_3) = J_3$, $f(J_4) = J_2$, $f(J_5) = J_1$; Prop. 4(4): $\mathcal{J}(g) = \{J_i\}_{i=1}^5$, $g(J_i) = J_i$ for each i .

- (vii) For each $J \in \mathcal{J}$, the map $f^2 \upharpoonright J$ is topologically mixing.
- (viii) For each $J \in \mathcal{J}$, if $f(J) = J$ then the map $f \upharpoonright J$ is topologically mixing.
- (ix) The map f is surjective.

Proof. Properties (i)-(vi) had been proved in [3]. The other ones easily follow. (vii) It is well known that an interval map $g: J \rightarrow J$ is topologically mixing if and only if g^2 is transitive [4, Theorem 46]. By (i), the set $\{f^{4n}(x) : n \geq 0\}$ is dense in J for some $x \in J$, hence $f^4 \upharpoonright J$ is transitive, i.e., $f^2 \upharpoonright J$ is topologically mixing.

(viii) From (vii) we know that $f^2 \upharpoonright J$ is topologically mixing hence also transitive and as in (vii) we can deduce that $f \upharpoonright J: J \rightarrow J$ is topologically mixing.

(ix) Since f is continuous the image $f([0, 1])$ is a closed interval and by our assumption it contains a dense subset of $[0, 1]$, so $f([0, 1]) = [0, 1]$. \square

Lemma 5. *Suppose f has a dense set of periodic points.*

- (i) The map f is transitive but not topologically mixing if and only if $\mathcal{J} = \{[0, b], [b, 1]\}$ and $f([0, b]) = [b, 1]$.
- (ii) The map f is topologically mixing if and only if $\mathcal{J} = \{[0, 1]\}$.
- (iii) The map f is leo if and only if $\mathcal{J} = \{[0, 1]\}$ and both of the sets $f^{-2}(0) \cap (0, 1)$ and $f^{-2}(1) \cap (0, 1)$ are non-empty.

Proof. Parts (i) and (ii) follow immediately from Proposition 4, thus we begin by the only if direction of (iii).

By (ii), if $\mathcal{J} \neq \{[0, 1]\}$ then f is not topologically mixing, and thus not leo, thus we suppose $\mathcal{J} = \{[0, 1]\}$. Suppose first that $f^{-2}(0) \cap (0, 1) = \emptyset$. The leo map f is continuous and surjective hence every point in $(0, 1)$ must have at least one preimage in $(0, 1)$. This fact and our assumption $f^{-2}(0) \cap (0, 1) = \emptyset$ imply $f^{-1}(0) \subset \{0, 1\}$.

If $f(1) \neq 0$ then $f^{-1}(0) = \{0\}$ and $0 \notin f^n((0, 1])$ for every n positive. If $f(1) = 0$ then by our assumption, $f^{-1}(1) \cap (0, 1) = \emptyset$ hence $f(0) = 1$. It implies that $f^2(0) = 0$, $f^2(1) = 1$ and $0 \notin f^{2n}((0, 1])$, what contradicts the leo property of f . The case when $f^{-2}(1) \cap (0, 1) = \emptyset$ can be proven analogously.

We turn to the if direction. We assume that $\mathcal{J} = \{[0, 1]\}$ and that $f^{-2}(0) \cap (0, 1) \neq \emptyset \neq f^{-2}(1) \cap (0, 1) = \emptyset$; since by (ii) f is topologically mixing, for every nonempty open $L \subset [0, 1]$ there has to exist a positive n for which $f^n(L) \cap f^{-2}(0) \neq \emptyset \neq f^n(L) \cap f^{-2}(1)$ hence $f^{n+2}(L) = [0, 1]$. \square

For any set $X \subset C([0, 1])$ we denote by $X_{property}$ the set of all maps in X having a *property* (in lower index abbreviated) in question. We denote by $PA(\lambda)$ the set of all piecewise affine maps from $C(\lambda)$.

Proposition 6. *The set $PA(\lambda)_{leo}$ is dense in $C(\lambda)$.*

Proof. Fix an $f \in C(\lambda)$ and $\varepsilon > 0$. It had been shown in [6] that there exists a map $d^* \in PA(\lambda)$ such that $\rho(d^*, f) < \varepsilon$.

Let us show with the help of Lemma 3 that there exists a map $d^{**} \in PA(\lambda)_{leo}$ for which $\rho(d^{**}, d^*) < \varepsilon$. First we prove

Claim. $PA(\lambda)_{leo} = PA(\lambda)_{tmix}$.

Proof of Claim. Any leo map is topological mixing. So let $f \in PA(\lambda)_{tmix}$ and show that $f \in PA(\lambda)_{leo}$. If $f^{-1}(0) \cap (0, 1) = f^{-1}(1) \cap (0, 1) = \emptyset$, then since f is surjective either $f^{-1}(0) = \{0\}$, $f^{-1}(1) = \{1\}$ or $f^{-1}(0) = \{1\}$ and $f^{-1}(1) = \{0\}$. But $f \in C(\lambda)$, so $f' \equiv 1$ on some neighborhood of $\{0, 1\}$ in the first case or $f' \equiv -1$ on some neighborhood of $\{0, 1\}$ in the latter case - a contradiction with topological mixing of f . So assume that $f^{-1}(0) \cap (0, 1) \neq \emptyset$. As in the proof of Lemma 5, since f is continuous and surjective, $f^{-2}(0) \cap (0, 1) \neq \emptyset$ and by Lemma 5(iii), it is sufficient to show that $f^{-2}(1) \cap (0, 1) \neq \emptyset$, resp. $f^{-1}(1) \cap (0, 1) \neq \emptyset$. Let $f^{-1}(1) \cap (0, 1) = \emptyset$. We are done if $\{0\} \subset f^{-1}(1)$, since then $f^{-2}(1) \cap (0, 1) \cap f^{-1}(0) \neq \emptyset$. It remains to comment the case $f^{-1}(1) = \{1\}$. Then since $f \in C(\lambda)$, $f' \equiv 1$ on some neighborhood of 1 - a contradiction with topological mixing of f . The case when $f^{-1}(1) \cap (0, 1) \neq \emptyset$ can be captured analogously. This finishes the proof of the claim.

By our claim we are done if $d^* \in PA(\lambda)_{mix}$, so assume that this is not the case. Using Proposition 4 and a finite number of regular m -fold, $m \geq 3$, piecewise affine window perturbations on sufficiently small intervals as described in Lemma 3 - see Figure 3(Left), without loss of generality we can assume that $\mathcal{J}(d^*) = \{J_i: i = 1, \dots, k\}$ with $2 \leq k < \infty$ and $[0, 1] \setminus \bigcup_{1 \leq i \leq k} J_i$ is a finite set. Let $J = [a, b]$ and $J' = [b, c]$ be two adjacent element of $\mathcal{J}(d^*)$ - see Figure 3(Right).

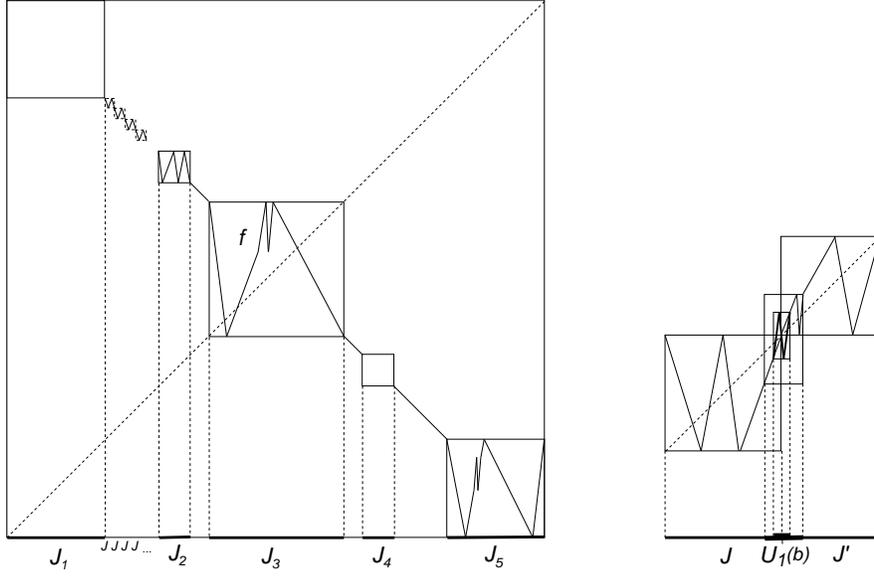


FIGURE 3. Left: $f \in C(\lambda)$, $\bigcup \mathcal{J}(f) = \bigcup \{J_i\}_{i=1}^5$ is not dense, regular 3-fold perturbations of f on new J 's; Right: Perturbation on $U_1(b)$ and $U_2(b) \subset U_1(b)$ from the proof of Prop. 6.

- Using a piecewise affine window perturbation (not necessarily regular) on some neighborhood of b if necessary, w.l.o.g. we can assume that d^* is strictly monotone with a constant slope on neighborhood $U_1(b)$ of b .
- Choosing sufficiently small $\varepsilon_1 < \varepsilon$ we can consider the regular 3-fold window perturbation of d^* on $U_2(b) \subset U_1(b)$ resulting to the map $d_1^* \in PA(\lambda)$ satisfying $\rho(d^*, d_1^*) < \varepsilon_1$. Moreover, by Proposition 4 and Lemma 5 either
 - $\#\mathcal{J}(d_1^*) = k - 1 < \#\mathcal{J}(d^*) = k$ in the case of Equation (4), resp. Equation (5) with $b \in \text{Fix}(f)$
 - or
 - $\#\mathcal{J}(d_1^*) = k - 2 < \#\mathcal{J}(d^*) = k$ for \hat{f} Equation (5) with $b \notin \text{Fix}(f)$.

Finitely many modifications of d^* with $\varepsilon_1, \dots, \varepsilon_\ell$, $\ell \leq k - 1$, satisfying

$$\varepsilon_1 + \dots + \varepsilon_\ell < \varepsilon$$

result to maps $d_1^*, \dots, d_\ell^*, d_\ell^* \in C(\lambda)$, $\#\mathcal{J}(d_\ell^*) = 1$, the union $\bigcup \mathcal{J}(d_\ell^*)$ dense in $[0, 1]$ and

$$\rho(d^*, d_\ell^*) < \varepsilon_1 + \dots + \varepsilon_\ell < \varepsilon.$$

Summarizing, from Lemma 5(iii) we obtain that $d^{**} = d_\ell^*$ is topologically mixing hence also from $PA(\lambda)_{leo}$ and $\rho(d^{**}, f) < 2\varepsilon$. \square

We denote by $PAM(\lambda)$ the set of all piecewise affine Markov maps in $PA(\lambda)$, i.e., maps for which all points of discontinuity of the derivative and also both endpoints $0, 1$ are eventually periodic.

Proposition 7. *The set $PAM(\lambda)_{leo}$ is dense in $C(\lambda)$.*

Proof. Let $f \in PA(\lambda)_{leo}$, fix $\varepsilon > 0$. Denote $R(f)$ the set containing $\{0, 1\}$ and all points of discontinuity of the derivative of f , let $S(f) \subset R(f)$ be eventually periodic points from $R(f)$ and $T(f) = R(f) \setminus S(f)$.

Clearly the set $R(f)$ is finite. Fix $t \in T = T(f)$. By Proposition 4(ix) and Lemma 3 we can consider a periodic orbit $P = \{p_1 < \dots < p_k\}$ of f such that for some three consecutive points $p_{i-1} < p_i < p_{i+1}$

- $f \upharpoonright [p_{i-1}, p_{i+1}]$ is affine,
- $\text{orb}(S(f), f) \cap [p_{i-1}, p_{i+1}] = \emptyset$,
- every window perturbation of f by $h \in C(f; [p_{i-1}, p_{i+1}])$ on $[p_{i-1}, p_{i+1}]$ is ε/m -close to f , where $m = \#T$,
- every piecewise affine window perturbation of f on $[p_{i-1}, p_{i+1}]$ belongs to $PA(\lambda)_{leo}$,
- $\text{orb}(t, f) \cap (p_{i-1}, p_{i+1}) \neq \emptyset$.

Let $f^\ell(t)$ be the first iterate of t in (p_{i-1}, p_{i+1}) . By Lemma 3 there exists a 5-fold piecewise affine window perturbation (not necessarily regular) g_1 of f by h on $[p_{i-1}, p_{i+1}]$ satisfying

$$g_1(f^\ell(t)) = g_1(p_i) = f(p_i).$$

Then $\#R(g_1) = \#R(f) + 6$ and $\#S(g_1) \geq \#S(f) + 7$ hence

$$\begin{aligned} \#T(f) - 1 = m - 1 &= \#R(f) + 6 - (\#S(f) + 7) \geq \\ &\geq \#R(g_1) - \#S(g_1) = \#T(g_1). \end{aligned}$$

Repeating the above procedure maximally $m = \#T$ -times, we obtain the required Markov map $g \in PAM(\lambda)_{leo}$. \square

Theorem 8. *The $C(\lambda)$ -typical function is leo.*

Proof. By Proposition 6 we can fix a countable dense collection $\{f_n\}_n$ from $PA(\lambda)_{leo}$. Using a 2-fold piecewise affine window perturbation of f_n on $[0, \varepsilon]$,

resp. $[1 - \varepsilon, 1]$ if necessary - see Example 1 and Lemma 3 - without loss of generality we can assume that for each $n \in \mathbb{N}$ we have $f_n(0) \in (0, 1)$ and $f_n(1) \in (0, 1)$.

Let $B(g, \varepsilon) := \{f \in C(\lambda) : \rho(f, g) < \varepsilon\}$. For a given sequence $\{\varepsilon_n : \varepsilon_n > 0\}_n$ which we will choose later, we consider the dense G_δ set

$$G := \bigcap_{N \geq 1} \bigcup_{n \geq N} B(f_n, \varepsilon_n).$$

We claim that we can choose ε_n in such a way that any $f \in G$ is leo.

Consider a sequence $(J_m)_m$ of all open rational subintervals of $(0, 1)$. For each n, m there is a $j(n, m)$ such that $f_n^{j(n, m)}(J_m) = [0, 1]$. Choose $\varepsilon_n > 0$ so small so that for all $f \in B(f_n, \varepsilon_n)$ we have $f^{j(n, m)}(J_m) \supset (1/n, 1 - 1/n)$ for $m = 1, 2, \dots, n$. Additionally we assume that $\varepsilon_n > 0$ is so small that $f(0) \in (0, 1)$ and $f(1) \in (0, 1)$.

Now consider an $f \in G$. Then there exists an infinite sequence n_k so that $f \in B(f_{n_k}, \varepsilon_{n_k})$. By Proposition 4(ix) the map f is surjective. Thus there are points $a, b \in [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$. By the choice of ε_n we have such points $a, b \in (0, 1)$.

Fix an open interval $J \subset [0, 1]$. Choose an m so that $J_m \subset J$. Suppose n_k satisfies the following conditions (i) $n_k \geq m$ and (ii) $a, b \in (1/n_k, 1 - 1/n_k)$. Assume $a < b$, the other case being similar. By construction of G and the above two assumptions we have

$$f^{j(n_k, m)}(J) \supset f^{j(n_k, m)}(J_m) \supset (1/n_k, 1 - 1/n_k) \supset [a, b].$$

Thus $f^{j(n_k, m)+1}(J) \supset f[a, b] = [0, 1]$. □

3. MIXING PROPERTIES IN $C(\lambda)$

We start by introducing three classical types of mixing in a measure-theoretical dynamics [20]. We state them in the context of $C(\lambda)$.

Definition 9. A map $f \in C(\lambda)$ is called

(i) *ergodic*, if for every $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \lambda(f^{-j}(A) \cap B) = \lambda(A)\lambda(B).$$

(ii) *weakly mixing*, if for every $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\lambda(f^{-j}(A) \cap B) - \lambda(A)\lambda(B)| = 0.$$

(iii) *strongly mixing, if for every $A, B \in \mathcal{B}$,*

$$\lim_{n \rightarrow \infty} \lambda(f^{-j}(A) \cap B) = \lambda(A)\lambda(B).$$

Analogously as before, for a subset $X \subset C(\lambda)$ we denote by $X_{slope>1}$ the set of all maps f from X for which $|f'(x)| > 1$ for all $x \in [0, 1]$ at which derivative of f exists.

Let f be from $PAM(\lambda)_{slope>1}$ with a Markov partition

$$\mathcal{A} = \{A_0 = [x_0, x_1] \leq \dots \leq A_{N-1} = [x_{N-1}, x_N]\},$$

where the set $P_f = \{0 = x_0 < \dots < x_N = 1\}$ contains all orbits of points of discontinuity of derivative of f and of the endpoints. To each point $x \in [0, 1]$ we associate its itinerary $\Phi(x) = (\phi_i(x))_{i \geq 0}$ with respect to \mathcal{A} , i.e., $\phi_i(x) \in \{0, 1, \dots, N-1\}$ and $f^i(x) \in A_{\phi_i(x)}$ for each $i \geq 0$ (in this settings Φ is a one-to-finite multivalued map). Since f is continuous, the system $(\Phi([0, 1]), \sigma)$ is a subshift of the full shift $(\{0, 1, \dots, N-1\}^{\mathbb{N}_0}, \sigma)$ on the symbols $\{0, 1, \dots, N-1\}$ [20].

Any map from $PAM(\lambda)_{leo}$ satisfies the hypothesis of [2, Theorem 3.2]. So any such map is in fact exact, i.e., for every $A \in \bigcap_{n \geq 0} T^{-n}(\mathcal{B})$, $\lambda(A)\lambda(A^c) = 0$. It is known that every exact map has one-sided countable Lebesgue spectrum and hence is strongly mixing [20, p. 115]. For our purpose it will be convenient to prove explicitly the following.

Lemma 10. *Let f be from $PAM(\lambda)_{slope>1}$, consider \mathcal{A} and Φ as above. The system $([0, 1], \mathcal{B}, \lambda, f)$ is isomorphic to the one-sided Markov shift $(\Phi([0, 1]), \mathcal{B}', \mu, \sigma)$, where the measure μ on the Borel σ -algebra \mathcal{B}' is given by the probability vector*

$$(6) \quad p = (\lambda(A_0), \dots, \lambda(A_{N-1}))$$

and the stochastic matrix $P = (p_{ij})_{i,j=0}^{N-1}$, where

$$(7) \quad p_{ij} = \begin{cases} \frac{\lambda(A_j)}{\lambda(f(A_i))}, & f(A_i) \supset A_j \\ 0, & \text{otherwise} \end{cases}.$$

In particular every map from $PAM(\lambda)_{leo}$ is strongly mixing.

Proof. For the definition of isomorphic measure theoretic systems see [20, Definition 2.4]. Clearly, the vector p is a probability vector and, since $f \in C(\lambda)$ with constant derivative on each A_i , the matrix P is stochastic and $pP = p$. So the measure μ is defined well on the Borel σ -algebra \mathcal{B}' generated by the cylinders in $\Phi([0, 1])$. Since $f \in PAM(\lambda)_{slope>1}$, Φ is injective. Let N be the set of those points x from $[0, 1]$ for which the set $\Phi(x)$ consists of more itineraries. Then N is countable and, since Φ is a one-to-finite multivalued

map, the set $\Phi(N)$ is also countable. Hence $\Phi: [0, 1] \setminus N \rightarrow \Phi([0, 1] \setminus N)$ is a bijection and $\lambda([0, 1] \setminus N) = \mu(\Phi([0, 1] \setminus N)) = 1$. Obviously,

$$\Phi \circ f = \sigma \circ \Phi \text{ on } [0, 1] \setminus N.$$

To finish the proof we need to show that

$$\lambda(\Phi^{-1}(A)) = \mu(A) \text{ for each } A \in \mathcal{B}'.$$

Obviously it is sufficient to verify the last equality for cylinders, i.e., the sets

$$C_{\phi_0, \dots, \phi_{k-1}} = \{(\phi_i(x))_{i \geq 0} \in \phi([0, 1]) : \phi_0(x) = \phi_0, \dots, \phi_{k-1}(x) = \phi_{k-1}\},$$

where $k \in \mathbb{N}$ and $\phi_0, \dots, \phi_{k-1} \in \{0, \dots, N-1\}$. By the definition of the Markov shift

$$\mu(C_{\phi_0, \dots, \phi_{k-1}}) = \lambda(A_{\phi_0}) \prod_{j=1}^{k-1} \frac{\lambda(A_{\phi_j})}{\lambda(f(A_{\phi_{j-1}}))} = \clubsuit,$$

where the second factor equals to one if $k = 1$. Since f has a constant derivative on each A_i ,

$$\clubsuit = \lambda(\Phi^{-1}(C_{\phi_0, \dots, \phi_k})).$$

If $f \in PAM(\lambda)_{leo} \cap PAM(\lambda)_{slope > 1}$, the matrix P is irreducible and aperiodic, hence

$$(\Phi([0, 1]), \mathcal{B}', \mu, \sigma)$$

is strongly mixing [20, Theorem 1.31]. By the previous, it is also true for isomorphic $([0, 1], \mathcal{B}, \lambda, f)$. \square

Corollary 11. *The set $C(\lambda)_{smix}$ of strongly mixing maps is dense in $C(\lambda)$.*

Proof. It is a consequence of Proposition 7 and Lemma 10. \square

Theorem 12. *$C(\lambda)$ -typical function is weakly mixing.*

Proof. By Proposition 7 and Lemma 10 we can consider a countable dense set $\{f_n\}_n$ of weakly mixing maps. Suppose ε_n are strictly positive. Let

$$\mathcal{G} := \bigcap_{N \geq 1} \bigcup_{n \geq N} B(f_n, \varepsilon_n).$$

Clearly \mathcal{G} is a dense G_δ . We will show that the ε_i can be chosen in such a way that all the configurations in \mathcal{G} are weakly mixing.

Let $\{h_j\}_{j \geq 1}$ be a countable, dense collection of continuous functions in $L^1(X \times X)$. For any $f \in C(\lambda)$ and $\ell \geq 1$, let

$$S_\ell^f h_j(x, y) := \frac{1}{\ell} \sum_{k=0}^{\ell-1} h_j((f \times f)^k(x, y)).$$

The map f is weakly mixing if and only if the map $f \times f$ is ergodic, and by the Birkhoff ergodic theorem, the map $f \times f$ is ergodic if and only if we have

$$\lim_{\ell \rightarrow \infty} S_\ell^f h_j(x) = \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t))$$

for all $j \geq 1$.

For each n since f_n is weakly mixing, there exists a set $B_n \subset X \times X$ and a positive integer ℓ_n such that $\lambda(B_n) > 1 - \frac{1}{i}$ and

$$\left| S_{\ell_n}^{f_n} h_j(x, y) - \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t)) \right| < \frac{1}{i}$$

for all $(x, y) \in B_n$, $1 \leq j \leq n$. We can assume that $\lim_{n \rightarrow \infty} \ell_n = \infty$.

Now we would like to extend these estimates to the neighborhood $B(f_n, \varepsilon_n)$ for a sufficiently small strictly positive ε_n . By the triangular inequality we have:

$$\begin{aligned} & \left| S_{\ell_n}^g h_j(x, y) - \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t)) \right| \leq \\ & \left| S_{\ell_n}^g h_j(x, y) - S_{\ell_n}^{f_n} h_j(x, y) \right| + \left| S_{\ell_n}^{f_n} h_j(x, y) - \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t)) \right|. \end{aligned}$$

For any point (x, y) , and any $\ell \geq 1$ the point $g^\ell(x, y)$ varies continuously with g in a small neighborhood of f_n ; thus we can find $\varepsilon_n > 0$, and a set $\hat{B}_n \subset B_n$ of measure larger than $1 - \frac{2}{i}$ so that that if $g \in B(f_n, \varepsilon_n)$, then

$$\left| S_{\ell_n}^g h_j(x, y) - \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t)) \right| < \frac{2}{i}$$

for all $(x, y) \in \hat{B}_n$, $1 \leq j \leq i$.

For each $g \in \mathcal{G}$ there is an infinite sequence n_k such that $g \in B(f_{n_k}, \varepsilon_{n_k})$. Consider $\mathcal{B}(g) = \bigcap_{M=1}^{\infty} \bigcup_{i=M}^{\infty} \hat{B}_{n_k}$. Since $\lambda(\hat{B}_{n_k}) > 1 - \frac{1}{n_k}$, it follows that $\lambda(\mathcal{B}(g)) = 1$.

We can thus conclude that for λ -a.e. (x, y) , for all $j \geq 1$,

$$(8) \quad \lim_{k \rightarrow \infty} S_{\ell_n}^g h_j(x, y) = \int_{X \times X} h_j(s, t) d(\lambda(s) \times \lambda(t)),$$

and thus g is weakly mixing. \square

Definition 13. We say a piecewise monotone map $f: [0, 1] \rightarrow [0, 1]$ is expanding if there is a constant $c > 1$ such that $|f(x) - f(y)| > c|x - y|$ whenever x and y lie in the same monotone piece. If f is expanding Markov and a finite set $P_f = \{x_0 < \dots < x_N\}$ contains orbits of all points of discontinuity of the derivative and also of the endpoints $0, 1$, we let $P^* = \{0, \dots, N\}$ and define $f^*: P^* \rightarrow P^*$ by $f^*(i) = j$ if $f(x_i) = x_j$.

Remark 14. *The set P from Definition 13 is not uniquely determined. Any set $P' = \bigcup_{k=0}^n f^{-k}(P)$, $n \in \mathbb{N}$, is also a finite set that contains orbits of all points of discontinuity of the derivative and also of the endpoints $0, 1$.*

Theorem 15. [5, Theorem 2.1] *Expanding Markov maps f and g are topologically conjugate via an increasing homeomorphism h if and only if $f^* = g^*$. In this case $h(P_f) = P_g$, where $g = h \circ f \circ h^{-1}$.*

The next part of this paragraph will be devoted to the strong mixing maps in $C(\lambda)$. We start with one useful lemma.

Lemma 16. *Let f be from $PAM(\lambda)_{leo}$. For each $\varepsilon > 0$ there exists a strongly mixing measure $\mu \neq \lambda$ preserved by the map f and a homeomorphism $h: [0, 1] \rightarrow [0, 1]$ such that for $\nu = (\mu + \lambda)/2$*

$$\lambda = h^*\nu, \quad \text{i.e., } g = h \circ f \circ h^{-1} \in C(\lambda) \text{ and } \|f - g\| < \varepsilon.$$

Proof. Consider the Markov partition

$$\mathcal{A} = \{A_0 = [x_0, x_1] \leq \dots \leq A_{N-1} = [x_{N-1}, x_N]\}$$

for f , where the set $P_f = \{0 = x_0 < \dots < x_N = 1\}$ contains all orbits of points of discontinuity of derivative of f and of the endpoints. Using Definition 13 and Remark 14 we can assume that for some $\{x_{i-1} < x_i < x_{i+1}\} \subset P_f$ there are points $\{x_{\ell-1} < x_\ell < x_{\ell+1} \leq x_{r-1} < x_r < x_{r+1}\} \subset P_f$ such that

$$f(\{x_{\ell-1}, x_{\ell+1}\}) = f(\{x_{r-1}, x_{r+1}\}) = \{x_{i-1}, x_{i+1}\}, \quad f(x_\ell) = f(x_r) = x_i$$

and

$$(9) \quad f^{-1}(x_\ell) \cap P_f = \emptyset = f^{-1}(x_r) \cap P_f.$$

The last conditions in (9) imply that for every $A_j \in \mathcal{A}$ and $s \in \{\ell, r\}$

$$(10) \quad \emptyset \neq f(A_j) \cap (A_{s-1} \cup A_s)^\circ \implies f(A_j) \supset A_{s-1} \cup A_s.$$

In what follows we introduce a map α from $PAM(\lambda)_{leo}$ such that P_α differs from P_f only in the points x_ℓ, x_r . Fix $\delta > 0$. We can consider $\delta_1 \in (0, \delta)$ and

$$P_\alpha = \{y_0 < \dots < y_{\ell-1} < y_\ell < y_{\ell+1} \leq y_{r-1} < y_r < y_{r+1} < \dots < y_N\}$$

satisfying

- $x_i = y_i$ for $i \notin \{\ell, r\}$ and $0 < |x_\ell - y_\ell| < \delta_1$, $0 < |x_r - y_r| < \delta_1$,
- $\tilde{\alpha}(y_i) = y_j$ if and only if $f(x_i) = x_j$
- the connect-the-dots map α extending $\tilde{\alpha}$ from the set P_α to the whole interval $[0, 1]$ satisfies $\alpha \in PAM(\lambda)_{leo}$.

Since both maps f and α are expanding, by (10) also Markov and $f^* = \alpha^*$, from Theorem 15 we obtain that $\alpha = h_1 \circ f \circ h_1^{-1}$ with $h_1(P_f) = P_\alpha$. By Remark 14 we can consider the set P_f δ -dense in $[0, 1]$ hence the homeomorphism h_1 fulfils $0 < \|h_1 - \text{id}\| < 2\delta$.

By Lemma 10 the map α with respect to λ is measure isomorphic to a one-sided Markov shift given by the probability vector $q = (\lambda([y_0, y_1]), \dots, \lambda([y_{N-1}, y_N]))$ and the stochastic matrix $Q = (q_{ij})_{i,j=0}^{N-1}$, where

$$(11) \quad q_{ij} = \begin{cases} \frac{\lambda([y_{j-1}, y_j])}{\lambda(\alpha([y_{i-1}, y_i]))}, & \alpha([y_{i-1}, y_i]) \supset [y_{j-1}, y_j] \\ 0, & \text{otherwise} \end{cases}.$$

Since $\alpha = h_1 \circ f \circ h_1^{-1}$, the measure $\mu \neq \lambda$ (for $h_1(y_\ell) \neq x_\ell$) given by $\lambda = h_1^* \mu$ is a strongly mixing measure preserved by the map f . It follows that the measure $\nu = \frac{\mu + \lambda}{2}$, as a convex combination of two strongly mixing measures, is a nonergodic measure with $\text{supp } \nu = [0, 1]$ and preserved by the map f . Let us consider a homeomorphism $h: [0, 1] \rightarrow [0, 1]$ defined by

$$\lambda = h^* \nu.$$

Then from $|h_1(x) - x| < 2\delta$ fulfilling for each $x \in [0, 1]$ we obtain

$$x - \delta < \nu([0, x]) = h(x) = \frac{\mu([0, x]) + x}{2} = \frac{h_1(x) + x}{2} < x + \delta,$$

i.e., $\|h - \text{id}\| < \delta$. Now, taking δ sufficiently small we obtain

$$g = h \circ f \circ h^{-1} \in C(\lambda) \text{ and } \|f - g\| < \varepsilon.$$

□

Theorem 17. *The set of all strongly mixing maps in $C(\lambda)$ is of the first category.*

Proof. As before we denote $C(\lambda)_{\text{smix}}$ the set of all strongly mixing maps in $C(\lambda)$. Then

$$(12) \quad C(\lambda)_{\text{smix}} = \bigcap_{\varepsilon > 0} \bigcap_{A, B \in \mathcal{B}} \bigcup_{n \geq 1} \bigcap_{k \geq n} F_{\varepsilon, A, B, k}$$

where

$$F_{\varepsilon, A, B, k} = \{f \in C(\lambda) : |\lambda(f^{-k}(A) \cap B) - \lambda(A)\lambda(B)| \leq \varepsilon\}.$$

is a closed set for each pair $A, B \in \mathcal{B}$. Using Propositions 7 and Corollary 11 we can consider a dense sequence $\{f_j\}_j$ of piecewise affine leo, strongly mixing maps in $C(\lambda)$.

For a positive sequence $\{\varepsilon_m\}_m$ converging to 0 and a map f_j let us denote $\mu_{j,m}$, $h_{j,m}$, $\nu_{j,m} = (\mu_{j,m} + \lambda)/2$ all objects guaranteed in Lemma 16(ii) for $f = f_j$ and $\varepsilon = \varepsilon_m$. Since each $\mu_{j,m}$ is orthogonal to λ , there is a Borel set $A_{j,m}$ satisfying

$\lambda(A_{j,m}) = 1$ and $\mu_{j,m}(A_{j,m}) = 0$. Put $A = \bigcap_{j,m} A_{j,m}$. Then $\lambda(A) = 1$ and we can write for the map $g_{j,m} = h_{j,m} \circ f_j \circ h_{j,m}^{-1} \in C(\lambda)$ and each $k \in \mathbb{N}$

$$(13) \quad |\lambda(g_{j,m}^{-k}(h_{j,m}(A)) \cap h_{j,m}(A)) - \lambda(h_{j,m}(A))\lambda(h_{j,m}(A))| = \\ = |\nu_{j,m}(f_j^{-k}(A) \cap A) - \nu_{j,m}(A)\nu_{j,m}(A)| = \left| \frac{1}{2}\lambda(A) - \frac{1}{2}\lambda(A)\frac{1}{2}\lambda(A) \right| = \frac{1}{4}.$$

It shows that the closed set $F_{1/5,A,A,k}$ is nowhere dense for each k hence by (12) the set $C(\lambda)_{smix}$ is of the first category in $C(\lambda)$. \square

The following theorem states a general result analogous to one of V. Jarník [11]. Recall that by a *knot point* of function f we mean a point x where $D^+f(x) = D^-f(x) = \infty$ and $D_+f(x) = D_-f(x) = -\infty$.

Theorem 18. [6] *$C(\lambda)$ -typical function has a knot point at λ -almost every point.*

Corollary 19. *$C(\lambda)$ -typical function maps at least one λ -null set onto $[0, 1]$.*

Proof. Let K be the set of knot points of f . Each level set contains its maximum, it cannot be a knot point. Thus $f(K^c) = [0, 1]$. \square

4. METRIC ENTROPY IN $C(\lambda)$

We start this section by an easy application of the Rohlin entropy formula (see for example Theorem 1.9.7 in [15]).

Lemma 20. *Let f be from $PA(\lambda)$. Then*

$$h_\lambda(f) = \int_0^1 \log |f'(x)| \, d\lambda(x).$$

It follows from [20, Corollary 4.14.3] that if $f \in C(\lambda) \setminus \{\text{id}, 1 - \text{id}\}$ then $h_\lambda(f) > 0$. Analogously as before, for $c \in (0, \infty]$ and $X \subset C(\lambda)$ we denote by $X_{entr < c}$, resp. $X_{entr = c}$ the set of all maps f from X for which $h_\lambda(f) < c$, resp. $h_\lambda(f) = c$.

Proposition 21. *For every $c \in (0, \infty)$ the set $PAM(\lambda)_{entr = c}$ is dense in $C(\lambda)$.*

Proof. We claim that for each $\varepsilon > 0$

$$(14) \quad \forall f \in PA(\lambda)_{slope > 1} \exists \delta > 0: B(f; \delta) \cap PA(\lambda)_{entr < \varepsilon} \neq \emptyset.$$

In order to verify (14) we will proceed in several steps.

I. Let $F: [0, 1] \rightarrow [0, 1]$ be a continuous piecewise affine map with $m > 1$ full laps, i.e., for which there are points $0 = x_0 < x_1 < \dots < x_m = 1$ such that $F \upharpoonright [x_i, x_{i+1}]$, $i = 0, \dots, m-1$, is affine and $F([x_i, x_{i+1}]) = [0, 1]$ for each i .

Clearly $F \in PAM(\lambda)_{slope>1}$ and $|F'(x)| = 1/\alpha_i$ for $\alpha_i = \lambda([x_i, x_{i+1}])$ and each $x \in (x_i, x_{i+1})$. In fact the map F is uniquely determined by the $(m+1)$ -tuple $(\pm, \alpha_0, \dots, \alpha_{m-1})$ (we write $F \sim (+, \alpha_0, \dots, \alpha_{m-1})$) satisfying

$$(15) \quad \sum_{i=0}^{m-1} \alpha_i = 1,$$

$\alpha_i > 0$ for each i and in which the first coordinate indicates if F increases (+), resp. decreases (-) on the interval $[0, \alpha_0]$. Let us assume that $\alpha_i = p_i/q \in \mathbb{Q}$ for each i and for $\eta > 0$ and integer $M > 2$ put

$$r(\eta, M) = \frac{\eta}{M(m-1)} \text{ and } s(\eta, M) = \frac{1-\eta}{M}.$$

For $\eta \in (0, 1)$, an M divisible by q where $\alpha_i = p_i/q$, for $i \in \{0, \dots, m-1\}$ define a continuous map $h_i = h_i[\eta, M]: [0, Mq^{-1}\gamma_i] \rightarrow \mathbb{R}$, where $r = r(\eta, M)$, $s = s(\eta, M)$, $\gamma_i = \gamma_i(\eta, M) = p_i s + (q - p_i)r$ and

- h_i is affine with slope $\frac{1}{1-\eta}$ on $[q_i r, q_i r + p_i s]$, where $q_i = \sum_{j \leq i-1} p_j$
- h_i is affine with slope $\frac{m-1}{\eta}$ on $[0, q_i r]$ and $[q_i r + p_i s, \gamma_i]$
- $h_i(x) = h_i(x - (\ell-1)\gamma_i) + \frac{(\ell-1)q}{M}$ for $x \in [(\ell-1)\gamma_i, \ell\gamma_i]$, $1 \leq \ell \leq Mq^{-1}$
- $h_i(0) = 0$.

We leave the straightforward verification of the following properties to the reader (see Figure 4).

- (i) $h_i(0) = 0$, $h_i(Mq^{-1}\gamma_i) = 1$
- (ii) h_i is strictly increasing
- (iii) h_i is a piecewise affine map with two slopes and $\frac{m-1}{\eta}$ and $\frac{1}{1-\eta}$, the latter one on Mq^{-1} pairwise disjoint closed intervals
- (iv) $\lim_{\eta \rightarrow 0+} Mq^{-1}\gamma_i(\eta, M) = \alpha_i$ and

$$\forall \iota, \kappa > 0 \exists \eta', M' \forall \eta < \eta', M > M': \max_{x \in [x_i + \iota, x_{i+1} - \iota]} |F(x) - h_i[\eta, M](x)| < \kappa.$$

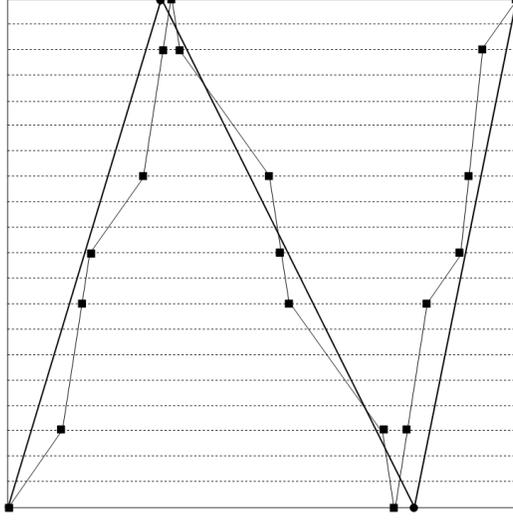
- (v) $Mq^{-1} \sum_{i=0}^{m-1} \gamma_i(\eta, M) = 1$ for each pair η, M

Let $H = H[\eta, M]: [0, 1] \rightarrow [0, 1]$ be defined by (we put $\beta_i = Mq^{-1}\gamma_i(\eta, M)$)

$$H(x) := \begin{cases} h_i(x - \sum_{j \leq i-1} \beta_j), & \text{for } x \in [\sum_{j \leq i-1} \beta_j, \sum_{j \leq i} \beta_j], \text{ } i \text{ even} \\ h_i(\sum_{j \leq i} \beta_j - x), & \text{for } x \in [\sum_{j \leq i-1} \beta_j, \sum_{j \leq i} \beta_j], \text{ } i \text{ odd.} \end{cases}$$

Clearly $H \in PA(\lambda)_{slope>1}$ and

$$\rho(f, H[\eta, M]) \rightarrow 0 \text{ for } \eta \rightarrow 0+, M \rightarrow \infty.$$


 FIGURE 4. $F \sim (+, 3/10, 1/2, 1/5)$, $\eta = 3/20$, $q = 10$, $M = 20$.

Then

$$\begin{aligned}
 \int_0^1 \log |H'[\eta, M](x)| \, d\lambda(x) &= \sum_{i=0}^{m-1} \int_0^{Mq^{-1}\gamma_i} \log |h'_i| \, d\lambda = Mq^{-1} \sum_{i=0}^{m-1} \int_0^{\gamma_i} \log |h'_i| \, d\lambda \\
 &= Mq^{-1} \sum_{i=0}^{m-1} \left(p_i s \log \frac{1}{1-\eta} + (\gamma_i - p_i s) \log \frac{m-1}{\eta} \right) \\
 (16) \qquad \qquad \qquad &= \eta \log \frac{1}{\eta} + (1-\eta) \log \frac{m-1}{1-\eta}
 \end{aligned}$$

where the last equality follows from (v), Equation (15) and the easily verifiable fact that $Mq^{-1}p_i s = \alpha_i(1-\eta)$.

Thus, for each M , for any $c \in (0, \log(m-1))$ there is an η such that the entropy of $H(\eta, M)$ equals c .

II. Fix $f \in PA(\lambda)_{slope > 1}$, let $0 = y_0 < \dots < y_n = 1$ be such that (y_j, y_{j+1}) , $j = 0, \dots, n-1$, are the maximal open intervals on which the map

$$\text{card} f^{-1}: [0, 1] \rightarrow \mathbb{N}, \quad \text{card} f^{-1}(y) = m_j, \quad y \in (y_j, y_{j+1})$$

is constant. Let us denote α_i^j , $i = 0, \dots, m_j - 1$, the Lebesgue measure of the i th (from the left) connected components of $f^{-1}((y_j, y_{j+1}))$; Fix the vector of m_j 's and n as above, and consider the system of equations ($z_0 = 0$, $z_n = 1$, all other variables free)

$$\sum_{i=0}^{m_j-1} \beta_i^j = z_{j+1} - z_j \quad \text{for each } j \in \{0, \dots, n-1\}.$$

A solution of this equation is a $n - 1 + \sum_{j=0}^{n-1} m_j$ -tuple. Since $f \in C(\lambda)$, $\sum_{i=0}^{m_j-1} \alpha_i^j = y_{j+1} - y_j$ for each $j \in \{0, \dots, n - 1\}$, thus it is a solution for this system. But this system has a open set of solutions, so it has a solutions (arbitrarily close to the fixed solution) such that all the numbers α_i^j and the y_j are rational for all i, j . Each such solution corresponds to to a map in $C(\lambda)$, thus wlog we can replace f by a close “rational” map.

Now we essentially repeat step I for f restricted so that its image is (y_j, y_{j+1}) . More precisely for each j we can consider the map $F^{(j)} \sim (*, \alpha_0^j, \dots, \alpha_{m_j-1}^j)$ with $\sum_{i=0}^{m_j-1} \alpha_i^j < 1$, i.e.,

$$F^{(j)}: [0, \sum_{i=0}^{m_j-1} \alpha_i^j] \rightarrow [0, \sum_{i=0}^{m_j-1} \alpha_i^j],$$

where $* = +$, resp. $* = -$ if f increases, resp. decreases on the leftmost connected components of $f^{-1}([y_j, y_{j+1}])$. Let $\alpha_i^j = p_i^j/q$. Using part I with $\eta \in (0, 1)$ and M divisible by q , for each j there is a map $H^{(j)}[\eta, M]$ that approximates (for small η and large M) the map $F^{(j)}$. Moreover, each $H^{(j)}$ is composed from $h_i^{(j)}$, $i = 0, \dots, m_j - 1$ and we can use those maps to approximate the map f by the uniquely determined map $H = H[\eta, M]: [0, 1] \rightarrow [0, 1]$ in the following way.

Let C_i^j denote the i th connected component of $f^{-1}([y_j, y_{j+1}])$. Remember from step I that the graph of $H^{(j)}$ is produced by gluing various horizontally shifted copies of $h_i^{(j)}$. The graph of H is produced by gluing various the same pieces, but with new vertical (with respect to j) and horizontal (with respect to C_i^j) shifts. More precisely the copies of $H_i^{(j)} := h_i^{(j)} + y_j$ are glued in the same combinatorial order of the interval C_i^j , i.e., the rightmost value (either y_j or y_{j+1}) of preceding $H_i^{(j)}$ coincides with the leftmost value of the following $H_{i'}^{(j')}$.

Note that $H[\eta, M] \in PA(\lambda)_{slope>1}$, Equality (16) holds and

$$\rho(f, H[\eta, M]) \rightarrow 0 \text{ for } \eta \rightarrow 0_+ \text{ and } M \rightarrow \infty.$$

As in part I, for each M , for any $c \in (0, \log(m - 1))$ there is an η such that the entropy of $H(\eta, M)$ equals c . In particular, the proof of (14) is finished.

III. Fix a map $f \in PA(\lambda)_{slope>1}$ and ε and δ positive. Using part II, for a sufficiently small η and large M , the map $H = H[\eta, M] \in PA(\lambda)_{slope>1}$ satisfies

$$\rho(f, H) < \delta/2, \int_0^1 \log |H'(x)| \, d\lambda(x) = \eta \log \frac{1}{\eta} + (1 - \eta) \log \frac{m - 1}{1 - \eta} < \varepsilon/2.$$

If H is not Markov, we can use sufficiently small window perturbations of the piecewise affine map H analogous to the ones from the proofs of Propositions

6 and 7 to obtain $\tilde{H} \in PAM(\lambda)_{slope>1}$ still satisfying

$$\rho(f, \tilde{H}) < \delta, \quad h_\lambda(\tilde{H}) = \int_0^1 \log |\tilde{H}'(x)| \, d\lambda(x) < \varepsilon.$$

Let $S = S(\tilde{H})$ be the set consisting of all orbits of points of discontinuity of the derivative of \tilde{H} and also both endpoints 0, 1. Since \tilde{H} is Markov, S is finite and there exists a periodic orbit P and its two consecutive points $p, p' \in P$ such that $[p, p'] \cap S = \emptyset$, (i.e., $\tilde{H} \upharpoonright [p, p']$ is affine) and, p and p' are so close that using Lemma 3, for every $m \geq 3$ any m -fold piecewise affine perturbation (not necessarily regular) of \tilde{H} on $[p, p']$ is still from $B(f; \delta)$. Notice that each such perturbation \hat{H} is again from $PAM(\lambda)_{slope>1}$ hence by Lemma 20 the entropy is given by the integral formula and

- $h_\lambda(\hat{H}) \in \left(h_\lambda(\tilde{H}), h_\lambda(\tilde{H}) + (\log m)(p' - p) \right]$,
- Lemma 20 implies that the entropy $h_\lambda(\hat{H})$ is a continuous function of the slopes of the affine pieces of $\hat{H} \upharpoonright [p, p']$ and that each value from $\left(h_\lambda(\tilde{H}), h_\lambda(\tilde{H}) + (\log m)(p' - p) \right]$ is the entropy of some piecewise affine m -fold perturbation \hat{H} of \tilde{H} on $[p, p']$

To see that for every $c \in (0, \infty)$ the set $PAM(\lambda)_{entr=c}$ is dense in $C(\lambda)$, we proceed as follows. As mentioned in the beginning of the proof of Proposition 6 $PA(\lambda)$ is dense in $C(\lambda)$. For each $f \in PA(\lambda)$ the slope of each affine piece is at least one, using a window perturbation we can make an arbitrarily small perturbation replacing these affine pieces with ones whose slopes are strictly greater than one, obtaining that $PA(\lambda)_{slope>1}$ is dense in $C(\lambda)$.

Fix c and an $g \in C(\lambda)$, choose $f \in PA(\lambda)_{slope>1}$ arbitrarily close to g . By Equation (14) we can find a $H \in PA(\lambda)_{slope>1}$ arbitrarily close of g with entropy strictly less than c . The above construction yields Markov map \tilde{H} with small entropy, then for large enough m it yields a Markov map \hat{H} with entropy exactly c . \square

Proposition 22. *The set $C(\lambda)_{entr=+\infty}$ is dense in $C(\lambda)$.*

Proof. We proceed like in the proof of the previous lemma. We fix $g \in C(\lambda)$, and we repeat steps I and II, and then in step III we realize a sequence of window perturbations: sequences $(H_n)_{n \geq 1}$, $([p_n, p'_n])_{n \geq 1}$ and $(m_n)_{n \geq 1}$ such that for each n ,

- $[p_n, p'_n] \supset [p_{n+1}, p'_{n+1}]$,
- H_{n+1} is a m_n -fold window perturbation of H_n on $[p_n, p'_n]$,
- $H_n \in PAM(\lambda)_{slope>1}$,
- $h_\lambda(H_n) = h_\lambda(H_n, \mathcal{A}_n) > n$, where \mathcal{A}_n is a Markov partition for H_n ,

- for some $H_\infty \in B(g; \delta)$, $\rho(H_n, H_\infty) \rightarrow 0$ for $n \rightarrow \infty$,
- $h_\lambda(H_\infty, \mathcal{A}_n) \geq h_\lambda(H_n, \mathcal{A}_n) > n$ hence $h_\lambda(H_\infty) = \infty$.

□

For completeness we prove the following fact, which is well known in many situations.

Proposition 23. *The set $C(\lambda)_{h_{top}=\infty}$ is a dense G_δ subset of $C(\lambda)$.*

Proof. Every map $f \in C(\lambda) \setminus \{id\}$, has a fixed point b where the graph of f is transverse to the diagonal at b . Using an $(n+2)$ -fold window perturbation on a neighborhood of b , we can create a map $g \in C(\lambda)$ arbitrarily close to f with a horseshoe with entropy $\log n$ in the window. Since horseshoes are stable under perturbations, there is an open ball $B(g, \delta)$ such that each h in this ball has topological entropy at least $\log n$. □

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