



**HAL**  
open science

## Distance Between Mutually Reachable Petri Net Configurations

Jérôme Leroux

► **To cite this version:**

Jérôme Leroux. Distance Between Mutually Reachable Petri Net Configurations. 39th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2019, December 11-13, 2019, Bombay, India, Nov 2019, Bombay, India. <hal-02156549>

**HAL Id: hal-02156549**

**<https://hal.science/hal-02156549v1>**

Submitted on 14 Jun 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# Distance Between Mutually Reachable Petri Net Configurations

Jérôme Leroux

LaBRI, CNRS, Univ. Bordeaux, France

jerome.leroux@labri.fr

## Abstract

Petri nets are a classical model of concurrency widely used and studied in formal verification with many applications in modeling and analyzing hardware and software, data bases, and reactive systems. The reachability problem is central since many other problems reduce to reachability questions. In 2015, we proved that a variant of the reachability problem, called the reversible reachability problem is exponential-space complete. Recently, this problem found several unexpected applications in particular in the theory of population protocols. In this paper we propose to revisit the reversible reachability problem in order to prove that the minimal distance in the reachability graph of two mutually reachable configurations is linear with respect to the Euclidian distance between those two configurations.

**2012 ACM Subject Classification** General and reference → General literature; General and reference

**Keywords and phrases** Dummy keyword

**Digital Object Identifier** 10.4230/LIPIcs.CVIT.2016.23

**Funding** *Jérôme Leroux*: The author is supported by the grant ANR-17-CE40-0028 of the French National Research Agency ANR (project BRAVAS)

## 1 Introduction

Petri nets are a classical model of concurrency widely used and studied in formal verification with many applications in modeling and analyzing hardware and software, data bases, and reactive systems. The reachability problem is central since many other problems reduce to reachability questions. Unfortunately, the reachability problem is difficult for several reasons. In fact, from a complexity point of view, we recently proved that the problem is non-elementary [6] by observing that the worst case complexity in space is at least a tower of exponential with height growing linearly in the dimension of the Petri nets. Moreover, even in practice, the reachability problem is difficult. Nowadays, no tool exists for deciding that problem since the known algorithms are difficult to be implemented and require many enumerations in exponentially large state spaces (see [13] for the state-of-the-art algorithm deciding the reachability problem).

Fortunately, easier natural variants of the reachability problems can be applied in various contexts. For instance, the coverability problem which consists in deciding if a configuration can be covered by a reachable one can be applied in the analysis of concurrent programs [1] (in that context, covered means component-wise smaller than or equal). The coverability problem is known to be exponential-space complete [16, 5], and efficient tools exist [4, 8]. Another variant is the reversible reachability problem. This problem consists in deciding if two configurations are mutually reachable one from the other. This problem was proved to be exponential-space complete in [11] and find unexpected applications in population protocols [7], trace logics [12], universality problems related to structural liveness problems [10], and in solving the home state problem [2].

**Contribution.** The exponential-space complexity lower-bound of the reversible reachability



© Jérôme Leroux;  
licensed under Creative Commons License CC-BY  
42nd Conference on Very Important Topics (CVIT 2016).

Editors: John Q. Open and Joan R. Access; Article No. 23; pp. 23:1–23:15



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

44 bility problem proved in [11] is obtained by observing that if two configurations are mutually  
 45 reachable, then the two configurations belong to a cycle of the (infinite) reachability graph  
 46 with a length at most doubly-exponential with respect to the size in binary of the two  
 47 configurations. In this paper, we focus on the minimal length of such a cycle (called the  
 48 distance in the sequel) with respect to the Euclidian distance between those two configu-  
 49 rations. We prove that the distance is linearly bounded by the Euclidian distance up-to a  
 50 doubly-exponential constant that only depends on the Petri net. As a direct consequence, this  
 51 result generalizes [11] and is shows that the distance between two nearby (for the Euclidian  
 52 distance) mutually reachable configurations is small.

53 **Outline.** In Section 2 we introduce our main problem about the distance between  
 54 mutually reachable Petri net configurations and we motivate the problem. In Section 3 we  
 55 present the Petri net with states basically given as a finite state automaton with transitions  
 56 labeled by Petri net actions. We also introduce the subclass of structurally reversible Petri  
 57 nets. Intuitively a Petri net with states is structurally reversible if the effect of every transition  
 58 can be reverted as soon as we execute that transition from a large enough configuration.  
 59 We provide in that section a sufficient condition to decide the reachability problem for  
 60 structurally reversible Petri nets between large configurations. In Section 4 and Section 5, we  
 61 recall some techniques called *rackoff extraction* to extract from executions or configurations  
 62 components that are “very large” compared to others. Those techniques are applied in  
 63 Section 6 in order to extract from a strongly connected component of the reachability graph  
 64 of a Petri net, a structurally reversible Petri net with states. Intuitively, this Petri net with  
 65 states is obtained by projecting away components that can be large in the considered strongly  
 66 connected component. From that Petri net with states, and thanks to the sufficient condition  
 67 for reachability introduced in Section 3, we proved in Section 7 our main result about the  
 68 distance between mutually reachable configurations.

69 In the paper,  $d$  is a positive natural number denoting the number of components of  
 70 vectors. Given a vector  $\mathbf{x}$  in the set of reals  $\mathbb{R}^d$ , we denote by  $\mathbf{x}(1), \dots, \mathbf{x}(d)$  its components in  
 71 such a way  $\mathbf{x} = (\mathbf{x}(1), \dots, \mathbf{x}(d))$ . Moreover, we introduce the norms  $\|\mathbf{x}\| \stackrel{\text{def}}{=} \sum_{i=1}^d |\mathbf{x}(i)|$  and  
 72  $\|\mathbf{x}\|_\infty \stackrel{\text{def}}{=} \max_{1 \leq i \leq d} |\mathbf{x}(i)|$ . The set of integers and the set of non-negative natural numbers  
 73 are denoted as  $\mathbb{N}$  and  $\mathbb{Z}$  respectively.

## 74 2 Petri Nets

75 A *Petri net*  $A$  (*PN* for short) is a finite set of pairs  $(\mathbf{a}_-, \mathbf{a}_+)$  in  $\mathbb{N}^d \times \mathbb{N}^d$  called *actions*. In  
 76 the literature, vectors  $\mathbf{a}_-$  and  $\mathbf{a}_+$  are respectively usually called the *pre-condition* and the  
 77 *post-condition* of  $a$ . A *configuration* is a vector in  $\mathbb{N}^d$ . We associate to an action  $a = (\mathbf{a}_-, \mathbf{a}_+)$   
 78 the binary relation  $\xrightarrow{a}$  over the configurations defined by  $\mathbf{x} \xrightarrow{a} \mathbf{y}$  if for some configuration  $\mathbf{c}$   
 79 we have  $\mathbf{x} = \mathbf{a}_- + \mathbf{c}$  and  $\mathbf{y} = \mathbf{a}_+ + \mathbf{c}$ . We denote by  $\xrightarrow{A}$  the one-step reachability relation of  
 80  $A$  defined by  $\mathbf{x} \xrightarrow{A} \mathbf{y}$  if there exists an action  $a$  in  $A$  such that  $\mathbf{x} \xrightarrow{a} \mathbf{y}$ . A PN  $A$  defines an  
 81 infinite graph  $(\mathbb{N}^d, \xrightarrow{A})$  called the *reachability graph* of  $A$ .

A  $\sigma$ -*execution* where  $\sigma = a_1 \dots a_k$  is a word of actions is a non-empty word of configura-  
 tions  $e = \mathbf{c}_0 \mathbf{c}_1 \dots \mathbf{c}_k$  such that the following relations hold:

$$\mathbf{c}_0 \xrightarrow{a_1} \mathbf{c}_1 \cdots \xrightarrow{a_k} \mathbf{c}_k$$

82 We denote by  $\text{src}(e)$  and  $\text{tgt}(e)$  the configurations  $\mathbf{c}_0$  and  $\mathbf{c}_k$  respectively. An  $A^*$ -*execution* is  
 83 a  $\sigma$ -execution for some word  $\sigma$  over  $A$ . An  $A^\omega$ -*execution*  $e$  is an infinite word of configurations

84 such that every finite non-empty prefix is an  $A^*$ -execution. We associate to a word  $\sigma$  of actions  
 85 the binary relation  $\xrightarrow{\sigma}$  over the configurations defined by  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$  if there exists a  $\sigma$ -execution  
 86 from  $\mathbf{x}$  to  $\mathbf{y}$ . The *displacement* of a word  $\sigma = a_1 \dots a_k$  is the vector  $\Delta(\sigma) \stackrel{\text{def}}{=} \sum_{j=1}^k \Delta(a_j)$   
 87 where  $\Delta(a) \stackrel{\text{def}}{=} \mathbf{a}_+ - \mathbf{a}_-$  for every action  $a = (\mathbf{a}_-, \mathbf{a}_+)$ . Notice that  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$  implies  $\Delta(\sigma) = \mathbf{y} - \mathbf{x}$   
 88 but the converse is not true in general. We introduce the *reachability relation*  $\xrightarrow{A^*}$  defined as  
 89 the union of the relations  $\xrightarrow{\sigma}$  where  $\sigma \in A^*$ . Notice that this relation is the reflexive and  
 90 transitive closure of  $\xrightarrow{A}$ .

91 The Petri net reachability problem consists in deciding given a PN  $A$  and two configura-  
 92 tions  $\mathbf{x}$  and  $\mathbf{y}$  if  $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$ . In [6], we provided a non-elementary complexity lower-bound for  
 93 the PN reachability problem. Moreover, we prove that for every natural number  $h$ , there  
 94 exists a PN  $A_h$  such that the reachability problem for that PN is at least  $h$ -exponential  
 95 space hard. It means that the minimal length of a word  $\sigma \in A_h^*$  satisfying  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$  is at least  
 96  $(h + 1)$ -exponential with respect to  $\|\mathbf{x}\| + \|\mathbf{y}\|$ . This huge lower bound is no longer valid for  
 97 mutually reachable configurations.

Two configurations  $\mathbf{x}$  and  $\mathbf{y}$  are said to be *mutually reachable* for a PN  $A$  if  $\mathbf{x} \xrightarrow{A^*} \mathbf{y}$  and  
 $\mathbf{y} \xrightarrow{A^*} \mathbf{x}$ . The PN reversible reachability problem consists in deciding given a PN  $A$  and two  
 configurations  $\mathbf{x}$  and  $\mathbf{y}$  if they are mutually reachable for  $A$ . In [11], we proved that the  
 PN reversible reachability problem is decidable in exponential-space by proving that there  
 exists at most doubly-exponential long words  $\sigma$  and  $w$  in  $A^*$  such that  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$  and  $\mathbf{y} \xrightarrow{w} \mathbf{x}$ .  
 This result can be refined by introducing the notion of distance. The *distance* between two  
 mutually reachable configurations  $\mathbf{x}$  and  $\mathbf{y}$  for a PN  $A$  is formally defined as follows:

$$\text{dist}_A(\mathbf{x}, \mathbf{y}) \stackrel{\text{def}}{=} \min_{\sigma, w \in A^*} \{|\sigma w| \mid \mathbf{x} \xrightarrow{\sigma} \mathbf{y} \xrightarrow{w} \mathbf{x}\}$$

98 A simple lower bound on the distance can be obtained by observing that configurations along  
 99 an execution are relatively closed one from each other as shown in the proof of the following  
 100 lemma.

► **Lemma 1.** *Let us consider a PN  $A \subseteq \{0, \dots, m\}^d \times \{0, \dots, m\}^d$  for some positive natural  
 number  $m$ . For every mutually reachable configurations  $\mathbf{x}$  and  $\mathbf{y}$ , we have:*

$$\text{dist}_A(\mathbf{x}, \mathbf{y}) \geq \|\mathbf{y} - \mathbf{x}\| \frac{2}{dm}$$

101 **Proof.** Let  $\sigma$  be a word in  $A^*$  such that  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$  and let us prove that  $\|\mathbf{y} - \mathbf{x}\| \leq m|\sigma|$ . Assume  
 102 that  $\sigma = a_1 \dots a_k$ . Since  $\Delta(a_j) \in \{-m, \dots, m\}^d$ , it follows that  $\Delta(\sigma) \in \{-mk, \dots, mk\}^d$ .  
 103 In particular  $\|\Delta(\sigma)\| \leq dm k$ . As  $\Delta(\sigma) = \mathbf{y} - \mathbf{x}$  and  $k = |\sigma|$ , we deduce the relation  
 104  $\|\mathbf{y} - \mathbf{x}\| \leq md|\sigma|$ . Now, let us consider a word  $w$  in  $A^*$  such that  $\mathbf{y} \xrightarrow{w} \mathbf{x}$  and observe that  
 105 we have  $\|\mathbf{x} - \mathbf{y}\| \leq dm|w|$  by symmetry. It follows that  $|\sigma w| \geq \|\mathbf{y} - \mathbf{x}\| \frac{2}{dm}$  and we have  
 106 proved the lemma. ◀

This paper focus on an upper-bound of the form  $\text{dist}_A(\mathbf{x}, \mathbf{y}) \leq f_A(\|\mathbf{y} - \mathbf{x}\|)$  where  $f_A$  is a  
 function that only depends on the PN  $A$  and not on the two mutually reachable configurations  
 $\mathbf{x}$  and  $\mathbf{y}$ . Such a bound cannot be derived from [11]. In fact, the best upper bound that can  
 be derived from that paper is the following one:

$$\text{dist}_A(\mathbf{x}, \mathbf{y}) \leq 34d^2 n^{15d^{d+2}}$$

107 where  $n = (1 + 2m)(1 + 2 \max\{\|\mathbf{x}\|, \|\mathbf{y}\|\})$ .

108 In this paper we prove that such a function  $f_A$  exists. Moreover a linear one exists as  
 109 shown by the following theorem.

► **Theorem 2.** Let us consider a PN  $A \subseteq \{0, \dots, m\}^d \times \{0, \dots, m\}^d$  for some positive natural number  $m$ . For every mutually reachable configurations  $\mathbf{x}$  and  $\mathbf{y}$ , we have:

$$\text{dist}_A(\mathbf{x}, \mathbf{y}) \leq \|\mathbf{y} - \mathbf{x}\|_{c_{d,m}}$$

where:

$$c_{d,m} \leq (3dm)^{(d+1)^{2d+4}}$$

110 ► **Remark 3.** The previous theorem provides as a corollary a new proof that the reversible  
111 reachability problem is exponential space. It also provides a bound on the minimal elements  
112 defining the *domain of reversibility* (introduced in [11]) of an action  $a$  in  $A$  defined as  
113  $\mathbf{D}_{a,A} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{N}^d \mid \exists \mathbf{y} \mathbf{x} \xrightarrow{a} \mathbf{y} \xrightarrow{A^*} \mathbf{x}\}$ . In fact, this set is *upward closed* and if  $\mathbf{x}$  is a minimal  
114 element for  $\leq$  in  $\mathbf{D}_{a,A}$  then the vector  $\mathbf{y}$  satisfying  $\mathbf{x} \xrightarrow{a} \mathbf{y}$  is such that  $\text{dist}_A(\mathbf{x}, \mathbf{y}) \leq dm c_{d,m}$   
115 since  $\|\mathbf{y} - \mathbf{x}\| = \|\Delta(a)\| \leq dm$ . We deduce that there exists a word  $\sigma$  of actions in  $A$  such  
116 that  $\mathbf{y} \xrightarrow{\sigma} \mathbf{x}$  with a length bounded by  $dm c_{d,m}$ . If a component of  $\mathbf{x}$  is larger than  $m|\sigma|$ , the  
117 vector  $\mathbf{x}$  cannot be minimal since the vector  $\mathbf{x}'$  obtained from  $\mathbf{x}$  by replacing that coordinate  
118 by  $m|\sigma|$  satisfies  $\mathbf{x}' \xrightarrow{a} \mathbf{y}' \xrightarrow{\sigma} \mathbf{x}'$  where  $\mathbf{y}' \stackrel{\text{def}}{=} \mathbf{x}' + \Delta(a)$ . Hence  $\|\mathbf{x}\| \leq dm^2 c_{d,m}$ .

### 119 3 Structurally Reversible Petri Nets With States

120 A *Petri net with states* (PNS for short) is a tuple  $\langle Q, A, T \rangle$  where  $Q$  is a non empty finite  
121 set of elements called *states*,  $A$  is a Petri net, and  $T$  is a set of triples in  $Q \times A \times Q$  called  
122 *transitions*. A *path*  $\pi$  from a state  $p$  to a state  $q$  labeled by a word  $\sigma$  of actions is a word of  
123 transitions of the form  $(q_0, a_1, q_1) \dots (q_{k-1}, a_k, q_k)$  for some states  $q_0, \dots, q_k$  satisfying  $q_0 = p$   
124 and  $q_k = q$ , and for some actions  $a_1, \dots, a_k$  satisfying  $\sigma = a_1 \dots a_k$ . The *displacement* of  
125  $\pi$  is the vector  $\Delta(\pi) \stackrel{\text{def}}{=} \Delta(\sigma)$ . A path is said to be *elementary* if  $q_i = q_j$  implies  $i = j$ . A  
126 path such that  $q_0 = q_k$  is called a *cycle* on  $q_0$ . A cycle is said to be *simple* if  $q_i = q_j$  with  
127  $i < j$  implies  $i = 0$  and  $j = k$ . A pair  $(q, \mathbf{x})$  in  $Q \times \mathbb{N}^d$  is called a *state-configuration* and it  
128 is denoted as  $q(\mathbf{x})$  in the sequel. We associate to a path  $\pi$  the binary relation  $\xrightarrow{\pi}$  over the  
129 state-configurations defined by  $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$  if  $\pi$  is a path from  $p$  to  $q$  labeled by a word  $\sigma$  of  
130 actions such that  $\mathbf{x} \xrightarrow{\sigma} \mathbf{y}$ .

131 A PNS is said to be *structurally reversible* if for every transition  $(p, a, q)$  there exists a  
132 path  $\pi$  from  $q$  to  $p$  such that  $\Delta(a) + \Delta(\pi) = \mathbf{0}$ . Structurally reversible PNSes are such that  
133 the displacement of any cycle can be canceled by the displacement of another cycle as shown  
134 by the following lemma.

135 ► **Lemma 4.**  $-\Delta(\theta)$  is the displacement of a cycle on  $q$  for every cycle  $\theta$  on a state  $q$ .

136 **Proof.** Assume that  $\theta = (q_0, a_1, q_1) \dots (q_{k-1}, a_k, q_k)$  with  $q_0 = q = q_k$ . Since the PNS  
137 is structurally reversible, for every  $j \in \{1, \dots, k\}$ , there exists a path  $\pi_j$  from  $q_j$  to  $q_{j-1}$   
138 such that  $\Delta(a_j) + \Delta(\pi_j) = \mathbf{0}$ . Now, observe that  $\theta' \stackrel{\text{def}}{=} \pi_k \dots \pi_1$  is a cycle on  $q$  such that  
139  $\Delta(\theta') = -\Delta(\theta)$ . ◀

140 A *partial configuration* is a vector  $\mathbf{x} \in \mathbb{N}^I$  where  $I \subseteq \{1, \dots, d\}$ . We associate to a  
141 configuration  $\mathbf{x} \in \mathbb{N}^d$  and a set  $I \subseteq \{1, \dots, d\}$  the partial configuration  $\mathbf{x}|_I$  in  $\mathbb{N}^I$  defined by  
142  $\mathbf{x}|_I(i) = \mathbf{x}(i)$  for every  $i \in I$ . Given an action  $a = (\mathbf{a}_-, \mathbf{a}_+)$  of a Petri net, we extend the binary  
143 relation  $\xrightarrow{a}$  over the partial configurations by  $\mathbf{x} \xrightarrow{a} \mathbf{y}$  if  $\mathbf{x}, \mathbf{y}$  are two partial configurations  
144 in  $\mathbb{N}^I$  such that there exists a partial configuration  $\mathbf{c} \in \mathbb{N}^I$  satisfying  $\mathbf{x} = \mathbf{a}_-|_I + \mathbf{c}$  and  
145  $\mathbf{y} = \mathbf{a}_+|_I + \mathbf{c}$ .

146 A *flow function* is a function  $F : Q \rightarrow \mathbb{N}^I$  for some subset  $I \subseteq \{1, \dots, d\}$  such that  
147  $F(p) \xrightarrow{a} F(q)$  for every transition  $(p, a, q)$  in  $T$ . In this section we prove the following result.

148 ► **Lemma 5.** *Let us consider a structurally reversible PNS with at most  $r$  states and with*  
 149 *actions in  $\{0, \dots, m\}^d \times \{0, \dots, m\}^d$  for some positive natural number  $m$ , let  $p(\mathbf{x})$  and  $q(\mathbf{y})$*   
 150 *be two state-configurations such that the following conditions hold for some flow function*  
 151  $F : Q \rightarrow \mathbb{N}^I$ :

- 152 ■  $\mathbf{x}|_I = F(p)$  and  $\mathbf{y}|_I = F(q)$ ,
- 153 ■  $\mathbf{x}(i), \mathbf{y}(i) \geq mr^3(3drm)^d$  for every  $i \notin I$ , and
- 154 ■  $\mathbf{y} - \mathbf{x}$  is the sum of the displacement of a path from  $p$  to  $q$  and a vector in the subgroup
- 155 of  $(\mathbb{Z}^d, +)$  generated by the displacements of the cycles.

156 Then  $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$  for a path  $\pi$  such that  $|\pi| \leq (\|\mathbf{y} - \mathbf{x}\| + drm)r^3(3drm)^{2d}$ .

157 In this section, we fix such a PNS  $G$ . Since  $G$  is a disjoint union of strongly connected  
 158 components, we can assume without loss of generality that  $G$  is strongly connected. The  
 159 proof of Lemma 5 follows an extended form of the *zigzag-freeness* approach introduced  
 160 in [14]. Intuitively, we fix an elementary path  $\pi_0$  from  $p$  to  $q$ , and we prove that there exist a  
 161 sequence  $\theta_1, \dots, \theta_k$  of short cycles on  $q$  such that for every  $n \in \{0, \dots, k\}$  the displacement  
 162 of  $\Delta(\theta_1 \dots \theta_n)$  is almost the vector  $\frac{n-d}{k}\mathbf{z}$  where  $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{y} - \mathbf{x} - \Delta(\pi_0)$ . This result is based on  
 163 the following lemma.

► **Lemma 6** ([9]). *Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  be a non-empty sequence of vectors in  $\mathbb{R}^d$  such that*  
 *$\|\mathbf{v}_j\|_\infty \leq 1$  for every  $1 \leq j \leq k$  and let  $\mathbf{v} = \sum_{j=1}^k \mathbf{v}_j$ . There exists a permutation  $\sigma$  of*  
 *$\{1, \dots, k\}$  such that for every  $n \in \{d, \dots, k\}$ , we have:*

$$\left\| \sum_{j=1}^n \mathbf{v}_{\sigma(j)} - \frac{n-d}{k} \mathbf{v} \right\|_\infty \leq d$$

164 From the previous lemma we deduce the following two corollaries.

165 ► **Corollary 7.** *Let  $\mathbf{Z}$  be a set of vectors in  $\{-m, \dots, m\}^d$  for some positive natural number*  
 166  *$m$ , and assume that  $\mathbf{z}$  is a finite sum of vectors in  $\mathbf{Z}$ . Then  $\mathbf{z}$  is a finite sum of at most*  
 167  $(\|\mathbf{z}\| + 1)(3dm)^d$  *vectors in  $\mathbf{Z}$ .*

168 **Proof.** By symmetry, we can assume without loss of generality that  $\mathbf{z} \geq \mathbf{0}$ . Let  $k$  be the  
 169 minimal natural number such that there exists a sequence  $\mathbf{z}_1, \dots, \mathbf{z}_k$  of vectors in  $\mathbf{Z}$  such  
 170 that  $\mathbf{z} = \mathbf{z}_1 + \dots + \mathbf{z}_k$ . If  $k = 0$  the lemma is proved, so let us assume that  $k \geq 1$ . Observe  
 171 that there exists a sequence  $\mathbf{e}_1, \dots, \mathbf{e}_k$  of vectors in  $\mathbb{N}^d$  such that  $\mathbf{z} = \sum_{j=1}^k \mathbf{e}_j$  and such  
 172 that  $\mathbf{e}_j(i) \leq \max\{0, \mathbf{z}_j(i)\}$  for every  $1 \leq i \leq d$  and every  $1 \leq j \leq k$ . We introduce the  
 173 sequence  $\mathbf{v}_1, \dots, \mathbf{v}_k$  defined by  $\mathbf{v}_j \stackrel{\text{def}}{=} \mathbf{z}_j - \mathbf{e}_j$ . Notice that  $\|\mathbf{v}_j\|_\infty \leq m$  and  $\sum_{j=1}^k \mathbf{v}_j = \mathbf{0}$ .  
 174 We introduce  $\mathbf{x}_n \stackrel{\text{def}}{=} \sum_{j=1}^n \mathbf{v}_j$ . By applying a permutation, Lemma 6 shows that we can  
 175 assume without loss of generality that  $\mathbf{x}_n \in \mathbf{X}$  for every  $0 \leq n \leq k$  where  $\mathbf{X}$  is the set  
 176 of vectors  $\mathbf{x} \in \mathbb{Z}^d$  such that  $\|\mathbf{x}\|_\infty \leq md$ . Observe that the cardinal of  $\mathbf{X}$  is bounded by  
 177  $(1+2dm)^d \leq (3dm)^d$ . Now, assume by contradiction that there exists  $\ell \in \{0, \dots, k - (3dm)^d\}$   
 178 satisfying  $\mathbf{e}_j = \mathbf{0}$  for every  $j \in \{\ell + 1, \dots, \ell + (3dm)^d\}$ . Notice that there exists  $p < q$  in  
 179  $\{\ell, \dots, \ell + (3dm)^d\}$  such that  $\mathbf{x}_p = \mathbf{x}_q$  since the cardinal of  $\mathbf{X}$  is bounded by  $(3dm)^d$ .  
 180 It follows that  $\sum_{j=p+1}^q \mathbf{v}_j = \mathbf{0}$ . From  $\mathbf{e}_j = \mathbf{0}$  for every  $j \in \{\ell + 1, \dots, \ell + (3dm)^d\}$  it  
 181 follows that  $\mathbf{v}_j = \mathbf{z}_j$  for every  $j \in \{p+1, \dots, q\}$ . In particular  $\sum_{j=p+1}^q \mathbf{z}_j = \mathbf{0}$ . Hence  $k$   
 182 is not minimal since we can remove the vectors  $\mathbf{z}_{p+1}, \dots, \mathbf{z}_q$  from the sequence  $\mathbf{z}_1, \dots, \mathbf{z}_k$ ,  
 183 and we get a contradiction. It follows that for every  $\ell \in \{0, \dots, k - (3dm)^d\}$  there exists  
 184  $j \in \{\ell + 1, \dots, \ell + (3dm)^d\}$  such that  $\mathbf{e}_j \neq \mathbf{0}$ . From  $\|\mathbf{z}\| = \sum_{j=1}^k \|\mathbf{e}_j\|$ , it follows that  
 185  $\|\mathbf{z}\| \geq \frac{k}{(3dm)^d} - 1$ . Hence  $k \leq (\|\mathbf{z}\| + 1)(3dm)^d$ . ◀

► **Corollary 8.** Assume that  $\mathbf{z} = \mathbf{z}_1 + \cdots + \mathbf{z}_k$  where  $\mathbf{z}_1, \dots, \mathbf{z}_k$  are vectors in  $\{-m, \dots, m\}^d$  for some positive natural number  $m \geq 1$ . There exists a permutation  $\sigma$  of  $\{1, \dots, k\}$  such that for every  $n \in \{0, \dots, k\}$ , we have:

$$\sum_{j=1}^n \mathbf{z}_{\sigma(j)}(i) \geq \min\{\mathbf{z}(i), 0\} - md$$

186 **Proof.** If  $k = 0$  the lemma is proved. So, we can assume that  $k \geq 1$ . By applying a  
 187 permutation, Lemma 6 shows that we can assume without loss of generality that for every  $n \in$   
 188  $\{0, \dots, k\}$ , there exists a vector  $\mathbf{e}_n \in \mathbb{R}^d$  such that  $\|\mathbf{e}_n\|_\infty \leq md$  and such that  $\mathbf{x}_n = \frac{n-d}{k}\mathbf{z} + \mathbf{e}_n$   
 189 where  $\mathbf{x}_n \stackrel{\text{def}}{=} \sum_{j=1}^k \mathbf{z}_j$ . Let  $i \in \{1, \dots, d\}$  and let us prove that  $\mathbf{x}_n(i) \geq \min\{\mathbf{z}(i), 0\} - md$ .  
 190 Observe that if  $n \in \{0, \dots, d\}$  then the property is immediate since  $\mathbf{x}_n(i) \geq -md$ . So, let  
 191 us assume that  $n > d$ . If  $\mathbf{z}(i) \geq 0$  then  $\frac{n-d}{k}\mathbf{z}(i) \geq 0$  and we get  $\mathbf{x}_n(i) \geq \mathbf{e}_n(i) \geq -md$ . If  
 192  $\mathbf{z}(i) \leq 0$  then  $\frac{n-d}{k}\mathbf{z}(i) \geq \mathbf{z}(i)$ . In particular  $\mathbf{x}_n(i) \geq \min\{\mathbf{z}(i), 0\} - md$  also in that case. ◀

193 A cycle is said to be *full-state* if every state occurs in the cycle. We first prove that there  
 194 exists a “short” full-state cycle with a zero displacement thanks to the following lemma.

195 ► **Lemma 9.** Every transition occurs on a finite sequence  $\theta_1, \dots, \theta_n$  of simple cycles such  
 196 that  $\Delta(\theta_1) + \cdots + \Delta(\theta_n) = \mathbf{0}$  and such that  $n \leq (3drm)^d$

**Proof.** Let  $t$  be a transition. Since  $G$  is strongly connected, the transition  $t$  occurs in a  
 simple cycle  $\theta_0$ . Lemma 4 shows that  $-\Delta(\theta_0)$  is a finite sum of displacements of simple cycles.  
 In particular  $-\Delta(\theta_0)$  is in the cone generated by the displacements of simple cycles. From  
 Carathéodory theorem,  $-\Delta(\theta_0)$  is in the cone generated by the displacements of  $d$  simple  
 cycles  $\theta_1, \dots, \theta_d$ . It follows that the following linear system over the sequences  $(\beta_j)_{0 \leq j \leq d}$  of  
 natural numbers

$$\sum_{j=0}^d \beta_j \mathbf{v}_j = \mathbf{0}$$

197 admits a solution satisfying  $\beta_0 > 0$  where  $\mathbf{v}_j \stackrel{\text{def}}{=} \Delta(\theta_j)$ .

From [15], it follows that solutions of that system can be decomposed as finite sums  
 of “minimal” solutions  $(\beta_j)_{1 \leq j \leq k}$  of the same system satisfying additionally the following  
 constraint:

$$\sum_{j=0}^d \beta_j \leq (1 + (d + 1)rm)^d$$

198 From  $1 + (d + 1)rm \leq (3drm)$ , we derive  $(1 + (d + 1)rm)^d \leq (3drm)^d$ . Since there exist  
 199 solutions of that system with  $\beta_0 > 0$ , there exists at least a minimal one satisfying the same  
 200 constraint. We have proved the lemma. ◀

201 ► **Lemma 10.** There exists a full-state cycle with a zero displacement with a length bounded  
 202 by  $r^2(r - 1)(3drm)^d$ .

203 **Proof.** Let us consider the set  $H$  of pairs  $(p, q) \in Q \times Q$  such that there exists a transition  
 204 from  $p$  to  $q$  with  $p \neq q$ . For every  $h \in H$  of the form  $(p, q)$ , we select a transition  $t_h \in T$  from  
 205  $p$  to  $q$ . Lemma 9 shows that for every  $h \in H$ , there exists a sequence of at most  $(3drm)^d$   
 206 simple cycles with a zero total displacement. It follows that there exists a sequence of at most  
 207  $|H|(3drm)^d$  simple cycles with a zero total displacement that contains all the transitions  $t_h$   
 208 with  $h \in H$ . Since the set of transitions that occurs in that sequence is strongly connected,

209 Euler's Lemma shows that there exists a cycle  $\theta$  with the same Parikh image as the sum of  
 210 the Parikh images of the cycles occurring in the sequence. It follows that  $|\theta| \leq r|H|(3drm)^d$ .  
 211 Notice that  $\Delta(\theta) = \mathbf{0}$  and  $\theta$  is a full-state cycle. From  $|H| \leq r(r-1)$  we are done.  $\blacktriangleleft$

212 Now, let us prove Lemma 5. Let  $\pi_0$  be an elementary path from  $p$  to  $q$ , and let  
 213  $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{y} - \mathbf{x} - \Delta(\pi_0)$ .

214 Let us first explain why  $\mathbf{z}$  is a finite sum of displacements of simple cycles. By hypothesis,  
 215 there exists a path  $\pi_1$  from  $p$  to  $q$  such that  $\mathbf{y} - \mathbf{x} - \Delta(\pi_1)$  is in the group  
 216 generated by displacements of cycles. Let  $\pi'$  be a path from  $q$  to  $p$  and observe that  
 217  $\mathbf{z} = (\mathbf{y} - \mathbf{x} - \Delta(\pi_1)) + \Delta(\pi'\pi_0) - \Delta(\pi'\pi_1)$ . Since  $\pi'\pi_0$  and  $\pi'\pi_1$  are two cycles, it follows that  
 218  $\mathbf{z}$  is in the group generated by the displacements of the cycles. Lemma 4 shows that  $\mathbf{z}$  is  
 219 finite sum of displacements of simple cycles.

Corollary 7 and Corollary 8 shows that there exists a sequence  $\mathbf{z}_1, \dots, \mathbf{z}_k$  of displacements  
 of simple cycles such that  $\mathbf{z} = \sum_{j=1}^k \mathbf{z}_j$ ,  $k \leq (1 + \|\mathbf{z}\|)(3drm)^d$ , and such that for every  
 $n \in \{0, \dots, k\}$ , we have:

$$\sum_{j=1}^n \mathbf{z}_j(i) \geq \min\{0, \mathbf{z}(i)\} - drm$$

Lemma 10 shows that there exists a full-state cycle  $\theta_0$  with a zero displacement with a length  
 bounded by  $r^2(r-1)(3drm)^d$ . Thanks to a rotation of  $\theta_0$ , we can assume that  $\theta_0$  is a cycle on  
 $q$ . Now, observe that for every  $1 \leq j \leq k$ , there exists a simple cycles  $\theta'_j$  with a displacement  
 equals to  $\mathbf{z}_j$ . By inserting  $\theta'_j$  in the full-state cycle  $\theta_0$ , we get a cycle  $\theta_j$  on  $q$ . Notice that  
 $\Delta(\theta_j) = \mathbf{z}_j$  and  $|\theta_j| \leq r^2(r-1)(3drm)^d + r$ . We introduce the path  $\pi$  defined as follows:

$$\pi \stackrel{\text{def}}{=} \pi_0 \theta_1 \dots \theta_n$$

220 We are going to prove that  $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ . To do so, let  $u$  be a prefix of  $\pi$  and let  
 221  $i \in \{1, \dots, d\}$  and let us prove that  $\mathbf{x}(i) + \Delta(u)(i) \geq 0$ . Observe that if  $u$  is a prefix of  $\pi_0$   
 222 the property is immediate since  $\Delta(u)(i) \geq -m|u| \geq -mr$ . In particular  $\mathbf{x}(i) + \Delta(u)(i) \geq 0$ .  
 223 So, we can assume that there exists  $n \in \{1, \dots, k\}$  and a prefix  $u'$  of  $\theta_n$  such that  $u =$   
 224  $\pi_0 \theta_1 \dots \theta_{n-1} u'$ . It follows that  $\Delta(u) = \Delta(\pi_0) + \Delta(u') + \sum_{j=1}^{n-1} \mathbf{z}_j(i)$ . Moreover, notice that  
 225  $\Delta(u')(i) \geq -m|u'| \geq -mr^2(r-1)(3drm)^d - mr$  for every  $i \in \{1, \dots, d\}$ .

226 We decompose the proof that  $\mathbf{x}(i) + \Delta(u)(i) \geq 0$  in two cases following that  $\mathbf{z}(i) \leq 0$  or  
 227  $\mathbf{z}(i) \geq 0$ .

228 ■ Assume first that  $\mathbf{z}(i) \geq 0$ . In that case  $\sum_{j=1}^{n-1} \mathbf{z}_j(i) \geq -md$ . It follows that  $\Delta(u)(i) \geq$   
 229  $-mr - mr^2(r-1)(3drm)^d - mr \geq -mr^3(3drm)^d$ . Hence  $\mathbf{x}(i) + \Delta(u)(i) \geq 0$ .

230 ■ Now, assume that  $\mathbf{z}(i) \leq 0$ . In that case  $\sum_{j=1}^{n-1} \mathbf{z}_j(i) \geq \mathbf{z}(i) - md$ . It follows that  
 231  $\mathbf{x}(i) + \Delta(u)(i) \geq \mathbf{x}(i) + \Delta(\pi_0) + \mathbf{z}(i) + \Delta(u')(i) - drm = \mathbf{y}(i) - \Delta(u')(i) - drm \geq$   
 232  $\mathbf{y}(i) - mr^2(r-1)(3drm)^d - mr \geq 0$ .

233 We have proved that  $p(\mathbf{x}) \xrightarrow{\pi} q(\mathbf{y})$ . Now, observe that  $|\pi| \leq r + k(r^2(r-1)(3drm)^d + r)$ . From  
 234  $k \leq (1 + \|\mathbf{z}\|)(3drm)^d$  and  $\|\mathbf{z}\| \leq \|\mathbf{y} - \mathbf{x}\| + d(r-1)m$ , we get  $|\pi| \leq (\|\mathbf{y} - \mathbf{x}\| + drm)r^3(3drm)^{2d}$ .

235 Lemma 5 is proved.

## 236 4 Extractors

237 A  $d$ -dimensional extractor  $\lambda$  is a non-decreasing sequence  $(\lambda_0 \leq \dots \leq \lambda_{d+1})$  of positive  
 238 natural numbers. Given a  $d$ -dimensional extractor  $\lambda$  and a set  $I \subseteq \{1, \dots, d\}$ , a  $(\lambda, I)$ -small  
 239 set of a set  $\mathbf{X} \subseteq \mathbb{N}^d$  is a subset  $J \subseteq I$  such that  $\mathbf{x}(j) < \lambda_{|J|}$  for every  $j \in J$  and  $\mathbf{x} \in \mathbf{X}$ . The  
 240 following lemma shows that there exists a unique maximal for the inclusion  $(\lambda, I)$ -small set.  
 241 We denote by  $\text{extract}_{\lambda, \mathbf{X}}(I)$  this set.

242 ► **Lemma 11.** *The class of  $(\lambda, I)$ -small sets of a set  $\mathbf{X} \subseteq \mathbb{N}^d$  is non empty and stable by*  
 243 *union.*

244 **Proof.** We adapt the proof of [11, Section 8]. Since the class contains the empty set, it is  
 245 nonempty. Now, let us prove the stability by union by considering two  $(\lambda, I)$ -small sets  $J_1$   
 246 and  $J_2$  of  $\mathbf{X}$  and let us prove that  $J \stackrel{\text{def}}{=} J_1 \cup J_2$  is a  $(\lambda, I)$ -small set of  $\mathbf{X}$ . Since  $J_1, J_2 \subseteq I$ ,  
 247 we derive  $J \subseteq I$ . Let  $\mathbf{x} \in \mathbf{X}$  and  $j \in J$ . If  $j \in J_1$  then  $\mathbf{x}(j) < \lambda_{|J_1|} \leq \lambda_{|J|}$  since  $|J_1| \leq |J|$ .  
 248 Symmetrically, if  $j \in J_2$  we deduce that  $\mathbf{x}(j) < \lambda_{|J_2|} \leq \lambda_{|J|}$ . We have proved that  $J$  is a  
 249  $(\lambda, I)$ -small set of  $\mathbf{X}$ . ◀

250 The following lemma shows that components that are not in  $\text{extract}_{\lambda, \mathbf{X}}(I)$  are large for  
 251 at least one vector in  $\mathbf{X}$ .

► **Lemma 12.** *For every  $i \in I \setminus \text{extract}_{\lambda, \mathbf{X}}(I)$  there exists  $\mathbf{x} \in \mathbf{X}$  such that:*

$$\mathbf{x}(i) \geq \lambda_{|I|+1}$$

252 **Proof.** Let  $J \stackrel{\text{def}}{=} \text{extract}_{\lambda, \mathbf{X}}(I)$  and assume that for some  $i \in I \setminus J$ , we have  $\mathbf{x}(i) < \lambda_{|J|+1}$  for  
 253 every  $\mathbf{x} \in \mathbf{X}$ . Let  $J' \stackrel{\text{def}}{=} J \cup \{i\}$  and observe that  $J'$  is a  $(\lambda, I)$ -small set of  $\mathbf{X}$ . In fact, for every  
 254  $\mathbf{x} \in \mathbf{X}$  and for every  $j \in J'$ , we have  $\mathbf{x}(j) < \lambda_{|J|} \leq \lambda_{|J'|}$  if  $j \in J$ , and  $\mathbf{x}(j) < \lambda_{|J|+1} = \lambda_{|J'|}$  if  
 255  $j = i$ . We get a contradiction by maximality of  $\text{extract}_{\lambda, \mathbf{X}}(I)$ . We deduce the lemma. ◀

256 Given a set  $I \subseteq \{1, \dots, d\}$  we define  $\text{extract}_{\lambda, e}(I)$  for a finite word  $e$  of configurations by  
 257  $\text{extract}_{\lambda, \varepsilon}(I) \stackrel{\text{def}}{=} I$ , and by  $\text{extract}_{\lambda, e\mathbf{c}}(I) \stackrel{\text{def}}{=} \text{extract}_{\lambda, \{\mathbf{c}\}}(\text{extract}_{\lambda, e}(I))$  for every  $\mathbf{c} \in \mathbb{N}^d$  and  
 258 for every finite word  $e$  of configurations. Given an infinite word  $e$  of configurations, we observe  
 259 that  $(\text{extract}_{\lambda, e_n}(I))_{n \in \mathbb{N}}$  where  $e_n$  is the finite prefix of  $e$  of length  $n$  is a non-increasing  
 260 sequence of sets in  $\{1, \dots, d\}$ . It follows that this sequence is asymptotically constant and  
 261 equals to a set included in  $\{1, \dots, d\}$ . We denote  $\text{extract}_{\lambda, e}(I)$  that set. The following lemma  
 262 shows that extracting along a word of configurations in  $\mathbf{X}$  asymptotically coincides with an  
 263 extraction of  $\mathbf{X}$ .

264 ► **Lemma 13.** *Let us consider a set  $I \subseteq \{1, \dots, d\}$ , an extractor  $\lambda$ , a set  $\mathbf{X}$  of configura-*  
 265 *tions, and an infinite word  $e$  over  $\mathbf{X}$ . We have  $\text{extract}_{\lambda, \mathbf{X}}(I) \subseteq \text{extract}_{\lambda, e}(I)$ . Moreover,*  
 266  *$\text{extract}_{\lambda, \mathbf{X}}(I) = \text{extract}_{\lambda, e}(I)$  if every configuration of  $\mathbf{X}$  occurs infinitely often in  $e$ .*

267 **Proof.** We introduce  $J \stackrel{\text{def}}{=} \text{extract}_{\lambda, \mathbf{X}}(I)$ ,  $J_\infty \stackrel{\text{def}}{=} \text{extract}_{\lambda, e}(I)$ , the prefix  $e_n$  of length  $n$  of  $e$ ,  
 268 and  $J_n \stackrel{\text{def}}{=} \text{extract}_{\lambda, e_n}(I)$ .

269 Let us prove that  $J \subseteq J_n$  for every  $n$ . Since  $J_0 = I$  the property is proved for  $n = 0$ .  
 270 Assume that  $J \subseteq J_{n-1}$  for some  $n \geq 1$  and let us prove that  $J \subseteq J_n$ . There exists  $\mathbf{x} \in \mathbf{X}$   
 271 such that  $e_n = e_{n-1}\mathbf{x}$ . Since  $\mathbf{x} \in \mathbf{X}$ , it follows that  $\mathbf{x}(j) < \lambda_{|J|}$  for every  $j \in J$ . As  $J \subseteq J_{n-1}$ ,  
 272 we deduce that  $J$  is a  $(\lambda, J_{n-1})$ -small set of  $\{\mathbf{x}\}$ . Since  $J_n$  is the maximal set satisfying that  
 273 property, we get  $J \subseteq J_n$  and we have proved the induction. It follows that  $J \subseteq J_n$  for every  
 274  $n \in \mathbb{N}$ . Moreover, since  $J_\infty = \bigcap_{n \in \mathbb{N}} J_n$ , we deduce the inclusion  $J \subseteq J_\infty$ .

275 Now, assume that every  $\mathbf{x} \in \mathbf{X}$  occurs in  $e$  infinitely often. Since  $(J_n)_{n \in \mathbb{N}}$  is a non  
 276 increasing sequence of  $\{1, \dots, d\}$ , there exists  $N$  such that  $J_n = J_\infty$  for every  $n \geq N$ . Let  
 277  $\mathbf{x} \in \mathbf{X}$ . There exists  $n > N$  such that  $e_n = e_{n-1}\mathbf{x}$ . From  $J_n = \text{extract}_{\lambda, \{\mathbf{x}\}}(J_{n-1})$  and  
 278  $J_n = J_{n-1} = J_\infty$ , we derive  $J_\infty = \text{extract}_{\lambda, \{\mathbf{x}\}}(J_\infty)$ . In particular  $\mathbf{x}(j) < \lambda_{|J_\infty|}$  for every  
 279  $j \in J_\infty$ . We have proved that  $\mathbf{x}(j) < \lambda_{|J_\infty|}$  for every  $j \in J_\infty$  and for every  $\mathbf{x} \in \mathbf{X}$ . As  
 280  $J_\infty \subseteq I$ , we deduce that  $J_\infty$  is a  $(\lambda, I)$ -small set of  $\mathbf{X}$ . Since  $J$  is the maximal set satisfying  
 281 that property, we deduce that  $J_\infty \subseteq J$ . It follows that  $J = J_\infty$ . ◀

## 5 Rackoff Extraction

An  $A^*$ -execution  $e$  is said to be  $I$ -cyclic for some  $I \subseteq \{1, \dots, d\}$  if  $\text{src}(e)|_I = \text{tgt}(e)|_I$ . We say that a word  $\sigma = \mathbf{a}_1 \dots \mathbf{a}_k$  of actions in  $A$  is obtained from an  $A^*$ -execution  $e$  by removing  $I$ -cycles where  $I \subseteq \{1, \dots, d\}$ , if there exists a decomposition of  $e$  into a concatenation  $e_0 \dots e_k$  of  $I$ -cyclic  $A^*$ -executions  $e_0, \dots, e_k$  such that  $\text{tgt}(e_{j-1}) \xrightarrow{a_j} \text{src}(e_j)$  for every  $1 \leq j \leq k$ .

An extractor  $\lambda = (\lambda_0 \leq \dots \leq \lambda_{d+1})$  is said to be  $m$ -adapted if for every  $n \in \{0, \dots, d\}$ :

$$\lambda_{n+1} \geq \lambda_n + m\lambda_n^n$$

► **Lemma 14** (slight extension of [16]). *Let  $\lambda$  be an  $m$ -adapted extractor and  $e$  be an  $A^*$ -execution for a PN  $A \subseteq \{0, \dots, m\}^d \times \mathbb{N}^d$ . Let  $I \stackrel{\text{def}}{=} \text{extract}_{\lambda, e}(\{1, \dots, d\})$ . There exists a word  $\sigma$  that can be obtained from  $e$  by removing  $I$ -cycles such that*

$$|\sigma| \leq \sum_{j=1}^d \lambda_j^j$$

and such that  $\text{src}(e) \xrightarrow{\sigma} \mathbf{c}$  for some configuration  $\mathbf{c}$  satisfying  $\mathbf{c}(i) = \text{tgt}(e)(i)$  for every  $i \in I$ , and such that for every  $i \notin I$  we have:

$$\mathbf{c}(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$$

► **Proof.** The proof follows a similar approach than the original one from Rackoff [16]. A detail proof is given in appendix. ◀

## 6 Strongly-Connected Components of Configurations

A *strongly-connected component of configurations* of a PN  $A$  (SCCC for short) is a strongly-connected component of the reachability graph  $(\mathbb{N}^d, \xrightarrow{A})$ .

We associate to an extractor  $\lambda$  and a SCCC  $\mathbf{C}$  of a PN  $A$ , a PNS  $G$  defined as follows. We introduce the set  $I \stackrel{\text{def}}{=} \text{extract}_{\lambda, \mathbf{C}}(\{1, \dots, d\})$ , the set of states  $Q \stackrel{\text{def}}{=} \{\mathbf{c}|_I \mid \mathbf{c} \in \mathbf{C}\}$  and the set of transitions  $T \stackrel{\text{def}}{=} \{(\mathbf{x}|_I, a, \mathbf{y}|_I) \mid (\mathbf{x}, a, \mathbf{y}) \in \mathbf{C} \times A \times \mathbf{C} \wedge \mathbf{x} \xrightarrow{a} \mathbf{y}\}$ . Notice that  $Q$  is finite since it contains at most  $\lambda_{|I|}^{|I|}$  elements. In particular  $T$  is finite as well. The PNS  $G$  is defined as the tuple  $\langle Q, A, T \rangle$ .

► **Lemma 15.** *The PNS  $G$  is structurally reversible.*

► **Proof.** Let  $(p, a, q)$  be a transition in  $T$ . There exist  $\mathbf{x}, \mathbf{y} \in \mathbf{C}$  such that  $\mathbf{x} \xrightarrow{a} \mathbf{y}$  and such that  $p = \mathbf{x}|_I$  and  $q = \mathbf{y}|_I$ . Moreover since  $\mathbf{C}$  is a SCCC, there exists a word  $\sigma$  of actions in  $A$  such that  $\mathbf{y} \xrightarrow{\sigma} \mathbf{x}$ . We deduce that there exists a path in  $G$  from  $q$  to  $p$  labeled by  $\sigma$ . Notice that  $\Delta(a) + \Delta(\sigma) = \mathbf{y} - \mathbf{x} + \mathbf{x} - \mathbf{y} = \mathbf{0}$ . It follows that  $G$  is structurally reversible. ◀

Let us prove the following technical lemma.

► **Lemma 16.** *If  $\mathbf{C}$  is not reduced to a singleton, there exists an  $A^\omega$ -execution  $e$  of configurations in  $\mathbf{C}$  such that every configuration of  $\mathbf{C}$  occurs infinitely often in  $e$ .*

## 23:10 Short Runs

305 **Proof.** Since  $\mathbf{C}$  is countable, there exists an infinite sequence  $(\mathbf{c}_n)_{n \in \mathbb{N}}$  such that  $\mathbf{C} = \{\mathbf{c}_n \mid n \in$   
306  $\mathbb{N}\}$ . Moreover, by replacing that sequence by the sequence  $s_0, s_1, \dots$  where  $s_n \stackrel{\text{def}}{=} \mathbf{c}_0, \dots, \mathbf{c}_n$ ,  
307 we can assume without loss of generality that every configuration of  $\mathbf{C}$  occurs infinitely often  
308 in the sequence  $(\mathbf{c}_n)_{n \in \mathbb{N}}$ . Since  $\mathbf{C}$  is a SCCC, for every positive natural number  $n$ , there exists  
309 an  $A^*$ -execution from  $\mathbf{c}_{n-1}$  to  $\mathbf{c}_n$  of the form  $e_n \mathbf{c}_n$ . Let us introduce the word  $e \stackrel{\text{def}}{=} e_1 e_2 \dots$   
310 Notice that since  $\mathbf{C}$  is not reduced to a singleton, the word  $e$  is infinite. Moreover, notice  
311 that  $e$  is an  $A^\omega$ -execution satisfying the lemma.  $\blacktriangleleft$

312 Now, assume that  $\lambda$  is  $m$ -adapted for some positive natural number  $m$ .

► **Lemma 17.** *If  $A \subseteq \{0, \dots, m\}^d \times \mathbb{N}^d$ , for every  $\mathbf{x} \in \mathbf{C}$ , there exists a cycle in  $G$  on  $\mathbf{x}|_I$  labeled by a word  $u$  such that:*

$$|u| \leq \sum_{j=1}^d \lambda_j^j$$

313 *and a configuration  $\mathbf{x}'$  such that  $\mathbf{x} \xrightarrow{u} \mathbf{x}'$ ,  $\mathbf{x}'|_I = \mathbf{x}|_I$  and such that  $\mathbf{x}'(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$*   
314 *for every  $i \notin I$ .*

**Proof.** Observe that if  $\mathbf{C}$  is reduced to a singleton, the lemma is trivial with  $u \stackrel{\text{def}}{=} \varepsilon$ . So, we can assume that  $\mathbf{C}$  is not a singleton. Lemma 16 shows that there exists an  $A^\omega$ -execution  $e = \mathbf{c}_0 \mathbf{c}_1 \dots$  of configurations in  $\mathbf{C}$  such that every configuration of  $\mathbf{C}$  occurs infinitely often. Without loss of generality, by replacing  $e$  by a suffix of  $e$  we can assume that  $\mathbf{x} = \mathbf{c}_0$ . Lemma 13 shows that  $\text{extract}_{\lambda, e}(\{1, \dots, d\}) = I$ . It follows that there exists  $N \in \mathbb{N}$  such that for every  $n \geq N$  the prefix  $e_n$  of  $e$  of length  $n$  satisfies  $\text{extract}_{\lambda, e_n}(\{1, \dots, d\}) = I$ . Since  $\mathbf{x}$  occurs infinitely often in  $e$ , there exists  $n \geq N$  such that  $\mathbf{x}$  is the last configuration of  $e_n$ . Lemma 14 shows that there exists a word  $u$  that can be obtained from  $e_n$  by removing  $I$ -cycles such that

$$|u| \leq \sum_{j=1}^d \lambda_j^j$$

and such that  $\mathbf{x} \xrightarrow{u} \mathbf{x}'$  for some configuration  $\mathbf{x}'$  satisfying  $\mathbf{x}'|_I = \mathbf{x}|_I$ , and such that for every  $i \notin I$  we have:

$$\mathbf{x}'(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$$

315 Since  $u$  can be obtained from  $e_n$  by removing  $I$ -cycles, it follows that  $u$  is the label of a cycle  
316 on  $\mathbf{x}|_I$  in the PNS  $G$ .  $\blacktriangleleft$

317 Symmetrically, we deduce a similar backward property.

► **Lemma 18.** *If  $A \subseteq \mathbb{N}^d \times \{0, \dots, m\}^d$ , for every  $\mathbf{y} \in \mathbf{C}$ , there exists a cycle in  $G$  on  $\mathbf{y}|_I$  labeled by a word  $v$  such that:*

$$|v| \leq \sum_{j=1}^d \lambda_j^j$$

318 *and a configuration  $\mathbf{y}'$  such that  $\mathbf{y}' \xrightarrow{v} \mathbf{y}$ ,  $\mathbf{y}'|_I = \mathbf{y}|_I$ , and such that for every  $i \notin I$ :*  
319  *$\mathbf{y}'(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$ .*

**Proof.** Let us introduce the PN  $A' \stackrel{\text{def}}{=} \{(\mathbf{a}_+, \mathbf{a}_-) \mid (\mathbf{a}_-, \mathbf{a}_+) \in A\}$ . Observe that  $\mathbf{C}$  is a SCCC of  $A'$ . Let  $G'$  be the PNS associated to the extractor  $\lambda$  and the SCCC  $\mathbf{C}$  of  $A'$ . Lemma 19 shows that there exists a cycle in  $G'$  on  $\mathbf{y}|_I$  labeled by a word  $u$  such that:

$$|u| \leq \sum_{j=1}^d \lambda_j^j$$

320 and a configuration  $\mathbf{y}'$  such that  $\mathbf{y} \xrightarrow{u} \mathbf{y}'$ ,  $\mathbf{y}|_I = \mathbf{y}'|_I$ , and such that  $\mathbf{y}'(i) \geq \lambda_{|I|+1} - m \sum_{j=1}^{|I|} \lambda_j^j$   
 321 for every  $i \notin I$ . Assume that  $u = a'_1 \dots a'_n$  with  $a'_j = (\mathbf{x}_j, \mathbf{y}_j)$  and let  $v \stackrel{\text{def}}{=} a_1 \dots a_n$  with  
 322  $a_j \stackrel{\text{def}}{=} (\mathbf{y}_j, \mathbf{x}_j)$ . Observe that since  $u$  is a cycle on  $\mathbf{y}|_I$  in  $G'$ , then  $v$  is a cycle on  $\mathbf{y}|_I$  in  $G$ .  
 323 Moreover, from  $\mathbf{y} \xrightarrow{u} \mathbf{y}'$  we derive  $\mathbf{y}' \xrightarrow{v} \mathbf{y}$ . We have proved the lemma. ◀

## 7 Mutually Reachable Configurations

324 In this section, we prove Theorem 2. We consider a PN  $A \subseteq \{0, \dots, m\}^d \times \{0, \dots, m\}^d$  for  
 325 some positive natural number  $m$ . We consider two mutually reachable configurations  $\mathbf{x}, \mathbf{y}$   
 326 for  $A$ . Since the theorem is trivial when  $\mathbf{x} = \mathbf{y}$ , we can assume that  $\mathbf{x} \neq \mathbf{y}$ . In particular  
 327  $\|\mathbf{y} - \mathbf{x}\| \geq 1$ .  
 328

We let  $\mathbf{C}$  be the SCCC of  $A$  containing  $\mathbf{x}$  and  $\mathbf{y}$ . We introduce the extractor  $\lambda$  satisfying  $\lambda_0 = 1$ , and for every  $n \in \{0, \dots, d\}$ :

$$\lambda_{n+1} \stackrel{\text{def}}{=} m \sum_{j=1}^n \lambda_j^j + m \lambda_n^{3n} (3d \lambda_n^n m)^d$$

329 Observe that  $\lambda$  is  $m$ -adapted. We introduce  $I \stackrel{\text{def}}{=} \text{extract}_{\lambda, \mathbf{C}}(\{1, \dots, d\})$  and the structurally  
 330 reversible PNS  $G$  associated to  $\mathbf{C}$ ,  $\lambda$  and  $\mathbf{A}$ . Notice that the number of states of  $G$  is bounded  
 331 by  $r \stackrel{\text{def}}{=} \lambda_{|I|}^{|I|}$ . We introduce the states  $p, q$  of  $G$  defined as  $p \stackrel{\text{def}}{=} \mathbf{x}|_I$  and  $q \stackrel{\text{def}}{=} \mathbf{y}|_I$ . Observe  
 332 that  $\mathbf{y} - \mathbf{x}$  is the displacement of a path from  $p$  to  $q$  in  $G$ . We introduce the flow function  
 333  $F : Q \rightarrow \mathbb{N}^I$  defined as the identity.

334 Let us observe that  $\lambda_j \leq \lambda_d$  for every  $j \in \{1, \dots, d\}$ . In particular  $r \leq \lambda_d^d$ .

▶ **Lemma 19.** *The PNS  $G$  admits a cycle on  $p$  labeled by a word  $u$  and a cycle on  $q$  labeled by a word  $v$  such that:*

$$|u|, |v| \leq d \lambda_d^d$$

and such that there exist configurations  $\mathbf{x}', \mathbf{y}'$  such that  $\mathbf{x} \xrightarrow{u} \mathbf{x}'$ ,  $\mathbf{y}' \xrightarrow{v} \mathbf{y}$ , and such that for every  $i \notin I$ , we have:

$$\mathbf{x}'(i), \mathbf{y}'(i) \geq mr^3 (3drm)^d$$

335 **Proof.** This lemma is a direct corollary of Lemma 17 and Lemma 18. ◀

336 From  $\mathbf{y}' - \mathbf{x}' = \mathbf{y} - \mathbf{x} - \Delta(u) - \Delta(v)$ , we deduce from Lemma 5 that there exists a  
 337 word  $\sigma$  of actions in  $\mathbf{A}$  such that  $\mathbf{x}' \xrightarrow{\sigma} \mathbf{y}'$  and such that  $|\sigma| \leq (\|\mathbf{y}' - \mathbf{x}'\| + drm)r^3 (3drm)^{2d}$ .  
 338 Observe that we have:

$$\begin{aligned} \|\mathbf{y}' - \mathbf{x}'\| &\leq \|\mathbf{y} - \mathbf{x}\| + \|\Delta(u)\| + \|\Delta(v)\| \\ &\leq \|\mathbf{y} - \mathbf{x}\| + dm(|u| + |v|) \\ &\leq \|\mathbf{y} - \mathbf{x}\| + 2d^2 m \lambda_d^d \end{aligned}$$

## 23:12 Short Runs

343 Let  $w = u\sigma v$ . Observe that  $\mathbf{x} \xrightarrow{w} \mathbf{y}$ . We derive:

$$\begin{aligned}
 344 \quad |w| &\leq 2d\lambda_d^d + (\|\mathbf{y} - \mathbf{x}\| + 2d\lambda_d^d m + d\lambda_d^d m)\lambda_d^{3d}(3d\lambda_d^d m)^{2d} \\
 345 \quad &\leq \|\mathbf{y} - \mathbf{x}\|8d\lambda_d^d m\lambda_d^{3d}(3d\lambda_d^d m)^{2d} \\
 346 \quad &\leq \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|(3d\lambda_d^d m)^{6d}
 \end{aligned}$$

347

348

From the following Lemma 20 we derive:

$$|w| \leq \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|(3dm)^{(d+1)^{2d+4}}$$

349 We deduce Theorem 2.

► **Lemma 20.** *We have:*

$$(3d\lambda_d^d m)^{6d} \leq (3dm)^{(d+1)^{2d+4}}$$

350 **Proof.** Assume first that  $d = 1$ . In that case, the definition of  $\lambda_{n+1}$  with  $n = 0$  provides  
 351  $\lambda_1 = 3m^2$  and the lemma is immediate. So, let us assume that  $d \geq 2$ . Notice that  $\lambda_j^j \leq \lambda_n^n$   
 352 for every  $j \in \{1, \dots, n\}$  for every  $n \in \{0, \dots, d-1\}$ . It follows that we have:

$$\begin{aligned}
 353 \quad \lambda_{n+1} &\leq 2d\lambda_n^{3n}m(3d\lambda_n^n m)^d \\
 354 \quad &\leq (3d\lambda_n m)^{(d+1)^{2n}-4} \\
 355
 \end{aligned}$$

By induction, we deduce that for every  $n \in \{0, \dots, d\}$ , we have:

$$\lambda_n \leq (3dm)^{n((d+1)^2-4)^n}$$

In particular:

$$3d\lambda_d^d m \leq (3dm)^{d^2(d+1)^{2d}}$$

Hence

$$(3d\lambda_d^d m)^{6d} \leq (3dm)^{6d^3(d+1)^{2d}} \leq (3dm)^{(d+1)^{2d+4}}$$

356 where we use the inequality  $6d^3 \leq (d+1)^4$ . ◀

## 357 **8 Conclusion**

358 In this paper we proved that the distance in the reachability graph between two mutually  
 359 reachable configurations is linear with respect to the Euclidian distance between those two  
 360 configurations. As a future work, we would like to apply that result to provide lower bounds  
 361 on the number of states of population protocols computing some predicates [3].

362 The author thanks Alain Finkel, Igor Khmelnitsky, and Serge Haddad for promoting him  
 363 a Karp and Miller problem that motivated this work, and Matthias Englert and Ranko Lazic  
 364 for pointing out the Steinitz constant lemma [9].

## 365 — References —

- 366 1 Gérard Basler, Michele Mazzucchi, Thomas Wahl, and Daniel Kroening. Symbolic counter  
 367 abstraction for concurrent software. In *CAV*, volume 5643 of *Lecture Notes in Computer*  
 368 *Science*, pages 64–78. Springer, 2009.

- 369 2 Eike Best and Javier Esparza. Existence of home states in petri nets is decidable. *Inf. Process.*  
370 *Lett.*, 116(6):423–427, 2016.
- 371 3 Michael Blondin, Javier Esparza, and Stefan Jaax. Large flocks of small birds: on the  
372 minimal size of population protocols. In Rolf Niedermeier and Brigitte Vallée, editors, *35th*  
373 *Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March*  
374 *3, 2018, Caen, France*, volume 96 of *LIPICs*, pages 16:1–16:14. Schloss Dagstuhl - Leibniz-  
375 Zentrum fuer Informatik, 2018. URL: <https://doi.org/10.4230/LIPICs.STACS.2018.16>,  
376 doi:10.4230/LIPICs.STACS.2018.16.
- 377 4 Michael Blondin, Alain Finkel, Christoph Haase, and Serge Haddad. Approaching the  
378 coverability problem continuously. In Marsha Chechik and Jean-François Raskin, editors, *Tools*  
379 *and Algorithms for the Construction and Analysis of Systems - 22nd International Conference,*  
380 *TACAS 2016, Held as Part of the European Joint Conferences on Theory and Practice of*  
381 *Software, ETAPS 2016, Eindhoven, The Netherlands, April 2-8, 2016, Proceedings*, volume  
382 9636 of *Lecture Notes in Computer Science*, pages 480–496. Springer, 2016.
- 383 5 E. Cardoza, Richard J. Lipton, and Albert R. Meyer. Exponential space complete problems  
384 for Petri nets and commutative semigroups: Preliminary report. In *STOC'76*, pages 50–54.  
385 ACM, 1976. doi:10.1145/800113.803630.
- 386 6 Wojciech Czerwinski, Slawomir Lasota, Ranko Lazic, Jérôme Leroux, and Filip Mazowiecki.  
387 The reachability problem for petri nets is not elementary (extended abstract). In *STOC*. ACM  
388 Computer Society, 2019. to appear.
- 389 7 Javier Esparza, Pierre Ganty, Jérôme Leroux, and Rupak Majumdar. Verification of population  
390 protocols. *Acta Inf.*, 54(2):191–215, 2017.
- 391 8 Thomas Geffroy, Jérôme Leroux, and Grégoire Sutre. Occam’s razor applied to the petri net  
392 coverability problem. *Theor. Comput. Sci.*, 750:38–52, 2018.
- 393 9 V. S. Grinberg and S. V. Sevast’yanov. Value of the steinitz constant. *Functional Analysis*  
394 *and Its Applications*, 14(2):125–126, Apr 1980. URL: <https://doi.org/10.1007/BF01086559>,  
395 doi:10.1007/BF01086559.
- 396 10 Petr Jancar, Jérôme Leroux, and Grégoire Sutre. Co-finiteness and co-emptiness of reachability  
397 sets in vector addition systems with states. In *Petri Nets*, volume 10877 of *Lecture Notes in*  
398 *Computer Science*, pages 184–203. Springer, 2018.
- 399 11 Jérôme Leroux. Vector addition system reversible reachability problem. *Logical Methods in*  
400 *Computer Science*, 9(1), 2013.
- 401 12 Jérôme Leroux, M. Praveen, and Grégoire Sutre. A relational trace logic for vector addition  
402 systems with application to context-freeness. In *CONCUR*, volume 8052 of *Lecture Notes in*  
403 *Computer Science*, pages 137–151. Springer, 2013.
- 404 13 Jérôme Leroux and Sylvain Schmitz. Reachability in vector addition systems is primitive-  
405 recursive in fixed dimension. In *LICS*. IEEE Computer Society, 2019. to appear.
- 406 14 Jérôme Leroux and Grégoire Sutre. On flatness for 2-dimensional vector addition systems  
407 with states. In *CONCUR*, volume 3170 of *Lecture Notes in Computer Science*, pages 402–416.  
408 Springer, 2004.
- 409 15 Loic Pottier. Minimal solutions of linear diophantine systems: Bounds and algorithms. In  
410 R. V. Book, editor, *Proceedings 4th Conference on Rewriting Techniques and Applications,*  
411 *Como (Italy)*, volume 488 of *Lecture Notes in Computer Science*, pages 162–173. Springer,  
412 1991.
- 413 16 C. Rackoff. The covering and boundedness problems for vector addition systems. *Theoretical*  
414 *Computer Science*, 6(2):223–231, 1978.

415 **A Proof of Lemma 14**

416 **Proof.** The proof of this lemma is inspired by [16]. We prove the lemma by induction over  $d$ .  
 417 Naturally, if  $d = 0$  the lemma is immediate. Assume the lemma proved for every dimension  
 418 strictly smaller than  $d \geq 1$  and let us consider an  $m$ -adapted extractor  $\lambda = (\lambda_0 \leq \dots \leq$   
 419  $\lambda_{d+1})$  and an  $A^*$ -execution  $e = \mathbf{c}_0 \dots \mathbf{c}_k$  for a PN  $A \subseteq \{0, \dots, m\}^d \times \mathbb{N}^d$ . We introduce  
 420  $J_n \stackrel{\text{def}}{=} \text{extract}_{\lambda, \mathbf{c}_0 \dots \mathbf{c}_{n-1}}(\{1, \dots, d\})$  for every  $n \in \{0, \dots, k+1\}$ . Since  $J_0 = \{1, \dots, d\}$ , there  
 421 exists a maximal  $h \in \{0, \dots, k+1\}$  such that  $J_h = \{1, \dots, d\}$ . For every  $0 \leq n < h$ ,  
 422 since  $J_n = \{1, \dots, d\}$ , we deduce that  $\mathbf{c}_n \in \{0, \dots, \lambda_d - 1\}^d$ . It follows that the cardinal of  
 423  $\{\mathbf{c}_n \mid 0 \leq n < h\}$  is bounded by  $\lambda_d^d$ . Without loss of generality, by removing cycles from the  
 424  $A^*$ -execution  $e$ , we can assume that  $\mathbf{c}_0, \dots, \mathbf{c}_{h-1}$  are distinct. It follows that  $h \leq \lambda_d^d$ . Notice  
 425 that if  $h = k+1$  we are done. So, we can assume that  $h \leq k$ .

426 Let us introduce  $J \stackrel{\text{def}}{=} J_{h+1}$ . By maximality of  $h$ , it follows that  $J$  is strictly included in  
 427  $\{1, \dots, d\}$ . We introduce  $d' = |J|$ . Thanks to a permutation of the components, we can assume  
 428 without loss of generality that  $J = \{1, \dots, d'\}$ . Lemma 12 shows that  $\mathbf{c}_h(i) \geq \lambda_{d'+1}$  for every  
 429  $i \in \{d'+1, \dots, d\}$ . We let  $f : \mathbb{N}^d \mapsto \mathbb{N}^{d'}$  be the function defined by  $f(\mathbf{z}) = (\mathbf{z}(1), \dots, \mathbf{z}(d'))$   
 430 for every  $\mathbf{z} \in \mathbb{N}^d$ . We also introduce the  $d'$ -dimensional extractor  $\lambda' = (\lambda_0 \leq \dots \leq \lambda_{d'+1})$   
 431 and the PN  $A' = \{(f(\mathbf{a}_-), f(\mathbf{a}_+)) \mid (\mathbf{a}_-, \mathbf{a}_+) \in A\}$ . Let us introduce the  $(A')^*$ -execution  
 432  $e' = \mathbf{c}'_{h+1} \dots \mathbf{c}'_k$  where  $\mathbf{c}'_n \stackrel{\text{def}}{=} f(\mathbf{c}_n)$ , and let us introduce the sequence  $J'_h, \dots, J'_{k+1}$  defined  
 433 by  $J'_n \stackrel{\text{def}}{=} \text{extract}_{\lambda', \mathbf{c}'_h \dots \mathbf{c}'_{n-1}}(\{1, \dots, d'\})$  for every  $n \in \{h+1, \dots, k+1\}$ .

434 Let us first prove that  $J'_n = J_n$  for every  $n \in \{h+1, \dots, k+1\}$ . First of all notice  
 435 that  $J'_{h+1} \subseteq J_{h+1}$ . Moreover, for every  $i \in J_{h+1}$  we have  $\mathbf{c}'_h(i) < \lambda'_{|J_{h+1}|}$ . Hence  $J_{h+1}$  is a  
 436  $(\lambda', J'_{h+1})$ -small set of  $\{\mathbf{c}'_h\}$ . By maximality of  $J'_{h+1}$  we get  $J_{h+1} \subseteq J'_{h+1}$ . Hence  $J'_{h+1} = J_{h+1}$ .  
 437 Assume by induction the property true for some  $n \in \{h+1, \dots, k\}$ . Since  $J'_{n+1}$  is a  $(\lambda', J'_n)$ -  
 438 small set of  $\{\mathbf{c}'_n\}$ , we deduce that  $J'_{n+1} \subseteq J'_n$  and  $\mathbf{c}'_n(j) < \lambda'_{|J'_n|}$  for every  $j \in J'_n$ . As  $J'_n = J_n$ ,  
 439 and  $\mathbf{c}'_n(j) = \mathbf{c}_n(j)$  for every  $j \in \{1, \dots, d'\}$ , we deduce that  $J'_n$  is a  $(\lambda, J_n)$ -small set of  $\mathbf{c}_n$ .  
 440 By maximality of  $J_{n+1}$ , we get  $J'_{n+1} \subseteq J_{n+1}$ . Symmetrically, since  $J_{n+1}$  is a  $(\lambda, J_n)$ -small set  
 441 of  $\mathbf{c}_n$ , we deduce that  $J_{n+1} \subseteq J'_n$  and  $\mathbf{c}_n(j) < \lambda_{|J'_n|}$  for every  $j \in J'_n$ . As  $J'_n = J_n$ , we deduce  
 442 that  $J_n$  is a  $(\lambda', J'_n)$ -small set of  $\mathbf{c}'_n$ . By maximality of  $J'_{n+1}$ , we get  $J_{n+1} \subseteq J'_{n+1}$ . We have  
 443 proved that  $J'_n = J_n$  for every  $n \in \{h+1, \dots, k+1\}$ .

It follows that  $J'_{k+1} = J_{k+1} = I$ . By induction, there exists a word  $\sigma'$  that can be  
 obtained from  $e'$  by removing  $I$ -cycles such that

$$|\sigma'| \leq \sum_{j=1}^{d'} \lambda_j^j$$

and such that  $\mathbf{c}'_h \xrightarrow{\sigma'} \mathbf{c}'$  for some configuration  $\mathbf{c}' \in \mathbb{N}^{d'}$  satisfying  $\mathbf{c}'(i) = \mathbf{c}'_k(i)$  for every  
 $i \in I$  and such that for every  $i \in \{1, \dots, d'\} \setminus I$  we have:

$$\mathbf{c}'(i) \geq \lambda_{|I|+1} - m \sum_{j=0}^{|I|} \lambda_j^j$$

444 Since  $\sigma'$  can be obtained from  $e'$  by removing  $I$ -cycles, it follows that there exists a word  
 445  $w$  that can be obtained from  $\mathbf{c}_h \dots \mathbf{c}_k$  by removing  $I$ -cycles, and such that  $\sigma'$  is the word  
 446 obtained from  $w$  by applying the function  $f$  on each action. Notice that for every prefix  $u$  of

447  $w$  and for every  $i \in \{d' + 1, \dots, d\}$  we have:

$$\begin{aligned}
 448 \quad \mathbf{c}_h(i) + \Delta(u)(i) &\geq \lambda_{d'+1} - m|w| \\
 449 \quad &\geq \lambda_{d'+1} - m \sum_{j=1}^{d'} \lambda_j^j \\
 450 \quad &\geq \lambda_{|I|+1} - m \sum_{j=0}^{|I|} \lambda_j^j \\
 451 \quad &
 \end{aligned}$$

The last inequality is obtained by induction by observing that  $\lambda$  is  $m$ -adapted. We deduce that  $\mathbf{c}_h(i) + \Delta(u)(i) \geq \lambda_0$  with the same kind of induction. In particular the configuration  $\mathbf{c} \in \mathbb{N}^d$  defined by  $\mathbf{c}(i) \stackrel{\text{def}}{=} \mathbf{c}'(i)$  if  $i \in \{1, \dots, d'\}$  and  $\mathbf{c}(i) \stackrel{\text{def}}{=} \mathbf{c}_{h+1}(i) + \Delta(w)(i)$  if  $i \in \{d' + 1, \dots, d\}$  satisfies  $\mathbf{c}_h \xrightarrow{w} \mathbf{c}$ . Notice that  $\mathbf{c}(i) = \mathbf{c}_k(i)$  for every  $i \in I$ , and for every  $i \notin I$ , we have:

$$\mathbf{c}(i) \geq \lambda_{|I|+1} - m \sum_{j=0}^{|I|} \lambda_j^j$$

Let us introduce  $\sigma \stackrel{\text{def}}{=} \mathbf{a}_1 \dots \mathbf{a}_h w$  where  $\mathbf{a}_n \stackrel{\text{def}}{=} \mathbf{c}_n - \mathbf{c}_{n-1}$  for every  $n \in \{1, \dots, h\}$ . Observe that  $\mathbf{c}_0 \xrightarrow{\sigma} \mathbf{c}$  and moreover we have:

$$|\sigma| \leq h + \sum_{j=1}^{d'} \lambda_j^j \leq \sum_{j=1}^d \lambda_j^j$$

452 We have proved the induction. ◀