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Improper vs finitely additive distributions as limits of countably additive probabilities

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Abstract:

In Bayesian statistics, improper distributions and finitely additive probabilities (FAPs) are the two main alternatives to proper distributions, i.e. countably additive probabilities. Both of them can be seen as limits of proper distribution sequences w.r.t. to some specific convergence modes. Therefore, some authors attempt to link these two notions by this means, partly using heuristic arguments. The aim of the paper is to compare these two kinds of limits. We show that improper distributions and FAPs represent two distinct characteristics of a sequence of proper distributions and therefore, surprisingly, cannot be connected by the mean of proper distribution sequences. More specifically, for a sequence of proper distribution which converge to both an improper distribution and a set of FAPs, we show that another sequence of proper distributions can be constructed having the same FAP limits and converging to any given improper distribution. This result can be mainly explained by the fact that improper distributions describe the behavior of the sequence inside the domain after rescaling, whereas FAP limits describe how the mass concentrates on the boundary of the domain. We illustrate our results with several examples and we show the difficulty to define properly a uniform FAP distribution on the natural numbers as an equivalent of the improper flat prior.

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diffuse distribution, remote probability, bayesian statistics.

1. Introduction

Improper priors and finitely additive probabilities (FAP) are the two main alternatives to the standard Bayesian Paradigm using proper priors, i.e. countably additive probabilities in the Kolmogorov axiomatic. Both alternatives are involved in paradoxical phenomena such as non-conglomerability, marginalization paradox, etc. As an heuristic argument, some authors such as [Stone \(1982\)](#) or [Kadane et al. \(1986, p.218\)](#), consider proper prior sequences, for example sequence of uniform prior, and derive under two different topologies two kind of limits: FAPs limits and improper limits. The choice between FAP distribution and improper distribution has been largely debated in the Bayesian literature (see [Hartigan, 1983, p.15](#)).

The aim of FAP limits is to preserve the total mass equal to 1, while sacrificing the countable additivity. This point of view has been mainly defended by [de Finetti \(1972\)](#). On the other hand, improper distributions aim to preserve the countable additivity, while sacrificing a total mass equal to 1. Improper distribution appears naturally in the framework of conditional probability, see [Rényi \(1955\)](#) and more recently [Taraldsen and Lindqvist \(2010, 2016\)](#) and [Lindqvist and Taraldsen \(2018\)](#). Conditional probability spaces are also related to projective spaces of measures ([Rényi, 1970](#)) which have a natural quotient space topology and a natural convergence mode, named q -vague convergence by [Bioche and Druilhet \(2016\)](#).

In Bayesian inference some paradoxes such as non-conglomerability ([Stone, 1976, 1982](#)) or the marginalization paradox ([Dawid et al., 1973](#)) occur with improper or diffuse FAP priors ([Dubins, 1975](#)), but not with proper priors. This lead some authors to include the likelihood in the definition of a convergence mode for the priors, by for instance considering the convergence of the posterior distribution w.r.t. to an entropy criterion ([Akaike, 1980](#)) or an integrated version of this criterion ([Berger et al., 2009](#)) when the posterior is proper. Bayesian inference with improper posterior is justified by [Taraldsen et al. \(2019\)](#) from a theoretical point of view. [Bord et al. \(2018\)](#) consider the convergence of proper to an improper posterior for Bayesian estimation of abundance by removal sampling. [Tufto et al. \(2018\)](#) propose to adapt MCMC for the estimation of improper

posterior.

In this paper, we only consider convergence of distributions regardless to any statistical model. The implication of our result in Bayesian inference, especially the construction of specific sequences of distribution used in the proof of Theorem 3 is left for future works. In Section 2, we define several concept of convergence in the setting of improper and FAP distributions. In Section 3 we revisit the notion of uniform distribution on integer. In Section 4, we illustrate the fundamental difference between convergence to an improper prior and to an FAP.

2. Convergence of probability sequences

We denote by \mathcal{C}_b the set of continuous real-valued bounded functions, by \mathcal{C}_K the set of continuous real-valued functions with compact support. For a σ -finite measure π , we denote $\pi(f) = \int f(\theta) d\pi(\theta)$. Consider a sequence of proper priors $\{\pi_n\}_{n \in \mathbb{N}}$. The usual converge mode of $\{\pi_n\}_{n \in \mathbb{N}}$ to a proper prior π is the narrow convergence, also called weak convergence or convergence in law, defined by:

$$\pi_n \xrightarrow[n \rightarrow +\infty]{\text{narrowly}} \pi \iff \pi_n(f) \xrightarrow[n \rightarrow +\infty]{} \pi(f) \quad \forall f \in \mathcal{C}_b \quad (1)$$

When it exists, the narrow limit of $\{\pi_n\}_n$ is necessarily unique. In this section, we consider two alternative convergence modes when there is no narrow limit, and especially when the total mass tends to concentrate around the boundary on the domain, more precisely when $\lim_n \pi_n(f) = 0$ for all f in \mathcal{C}_K . The idea is to consider a proper prior as a special case of FAP or as a special case of a Radon measure, and for each case to define in a formalized way, a convergence mode. In both cases the limit is not unique in general but, as a requirement, must coincide with the narrow convergence when Eq. (1) holds.

In the following, we consider that Θ is a metric, locally compact, second countable Hausdorff space Θ . This is the case, for example, for usual topological finite-dimensional vector spaces or denumerable sets with the discrete topology. In the latter case, any function is continuous and a compact set is a finite set.

2.1. Convergence to an improper distribution

To extend the notion of the narrow limit, we consider here proper distributions within the set of projective space of positive Radon measures as follows: we denote by \mathcal{R} the set of non-null Radon measures, that is regular countably additive measures with finite mass on each compact set. Note that, in the discrete case any σ -finite measure is a Radon measure.

We define an improper distribution as an unbounded Radon measure which appear in parametric Bayesian statistics (see, e.g. [Jeffreys, 1970](#)). The *projective space* $\overline{\mathcal{R}}$ associated to \mathcal{R} is the quotient space for the equivalence relation \sim defined by $\pi_1 \sim \pi_2$ iff $\pi_2 = \alpha \pi_1$ for some positive scalar factor α . To a Radon measure π , it can be associated a unique equivalence class $\overline{\pi} = \{\pi' = \alpha \pi; \alpha > 0\}$. Therefore, a projective space is a space where objects are defined up to a positive scalar factor. It is natural in Bayesian statistics to consider such projective space since two equivalent priors give the same posterior. The projective space \mathcal{R} is also naturally linked with conditional probability spaces ([Rényi, 1955](#)). All the results presented below on the convergence mode w.r.t. to the projective space $\overline{\mathcal{R}}$ can be found in [Bioche and Druilhet \(2016\)](#). The usual topology on \mathcal{R} is the vague topology defined by

$$\pi_n \xrightarrow[n \rightarrow +\infty]{\text{vaguely}} \pi \iff \pi_n(f) \xrightarrow[n \rightarrow +\infty]{} \pi(f) \quad \forall f \in \mathcal{C}_K \quad (2)$$

where \mathcal{C}_K is the set of all real-valued functions on Θ with compact support.

From the related quotient topology, we can derive a convergence mode, called *q-vague convergence*: a sequence $\{\pi_n\}_n$ in \mathcal{R} converge *q-vaguely* to a (non-null) improper distribution π in \mathcal{R} if $\overline{\pi}_n$ converges to $\overline{\pi}$ w.r.t. the quotient topology where $\overline{\pi}_n = \{\alpha \pi_n; \alpha > 0\}$ is the equivalence class associated to π_n and similarly for $\overline{\pi}$. The limit $\overline{\pi}$ is unique whereas π is unique only up to a positive scalar factor. It is not always tractable to check a convergence in the quotient space. However, there is an equivalent definition in the initial space \mathcal{R} : $\{\pi_n\}_n$ converges *q-vaguely* to π if there exists some scalar factors α_n such that $\{\alpha_n \pi_n\}_n$ converges vaguely to π :

$$\pi_n \xrightarrow[n \rightarrow +\infty]{q\text{-vaguely}} \pi \iff a_n \pi_n \xrightarrow[n \rightarrow +\infty]{\text{vaguely}} \pi \quad \text{for some } a_1, a_2, \dots > 0 \quad (3)$$

The *q-vague convergence* can be considered as an extension of the narrow convergence in the sense that if $\{\pi_n\}_n$ and π are proper distributions and $\{\pi_n\}_n$

converge narrowly to π then $\{\pi_n\}_n$ converge q-vaguely to π . Note that the converse part holds if and only if $\{\pi_n\}_n$ is tight (see [Bioche and Druilhet, 2016](#), Proposition 2.8).

When a sequence $\{\pi_n\}_n$ of proper distributions converges q-vaguely to an improper distribution, then $\lim_n \pi_n(K) = 0$ for any compact K ([Bioche and Druilhet, 2016](#), Prop. 2.11). The following lemma gives an apparently stronger, but in fact equivalent, result. It will be useful to establish the main result and to construct examples in Section 4.3.

Lemma 1. *Let $\{\pi_n\}_n$ be a sequence of proper distributions such that $\lim_n \pi_n(K) = 0$ for any compact K . Then there exists a non-decreasing sequence of compact sets K_n such that $\cup_n K_n = \Theta$ and $\lim_n \pi_n(K_n) = 0$. Moreover, K_n may be chosen such that, for any compact K , there exists an integer N such that $K \subset K_N$.*

Proof. Let $\tilde{K}_m, m \geq 1$, be an increasing sequence of compact sets with $\cup_m \tilde{K}_m = \Theta$. For each m , $\lim_n \pi_n(\tilde{K}_m) = 0$, so there exists an integer N_m such that $N_m > N_{m-1}$ and $\pi_n(\tilde{K}_m) \leq 1/m$ for $n > N_m$. Consider now such a sequence of integers $N_m, m \geq 1$. For any n there exists a unique integer m such that $N_m \leq n < N_{m+1}$. We define K_n by $K_n = \tilde{K}_m$. So, $\pi_n(K_n) = \pi_n(\tilde{K}_m) \leq 1/m$. Since m increases with n , $\lim_n \pi_n(K_n) = 0$. Furthermore, the sequence \tilde{K}_m can be chosen such that, for any compact K , K is a subset of all but finitely many \tilde{K}_m , (see e.g. [Bauer, 2001](#), Lemma 29.8). By construction, the same property holds for the sequence K_n . \square

Note that, $\lim_n \pi_n(K) = 0$ for any compact set K does not imply that $\{\pi_n\}$ converge q-vaguely. See Section 4 for some examples of such sequences.

2.2. Convergence to a FAP

In this section, we consider a proper distribution as special cases of FAPs. When a sequence $\{\pi_n\}$ of proper distribution does not converge narrowly to a proper prior, we need to define a weaker convergence mode so that the sequence can converge to some FAPs. This convergence mode will be named FAP convergence, and the corresponding limits, the FAP limits.

Denote by \mathcal{F}_b the set of bounded real-valued measurable functions on Θ . A FAP π can be defined as a linear functional on \mathcal{F}_b which is positive, i.e. $\pi(f) \geq 0$

if $f \geq 0$ and which satisfies $\pi(1) = 1$. Therefore, the set of FAPs belongs to the topological dual of \mathcal{F}_b equipped with the sup-norm. For any measurable set $E \in \Theta$, we define $\pi(E) = \pi(\mathbb{1}_E)$, where $\mathbb{1}_E(x) = 1$ if $x \in E$ and 0 otherwise. We also denote $\int f(\theta) d\pi(\theta) = \pi(f)$. Since proper distributions are required to be special cases of FAPs, we do impose here measurability conditions in the definition of \mathcal{F}_b , contrary to many authors (see, e.g. [Heath and Sudderth, 1978](#)). In the case where Θ is a denumerable set equipped with the usual discrete topology, any functions or sets are measurable and therefore this distinction is not relevant.

Consider a sequence $\{\pi_n\}_n$ of FAPs. The minimal requirement for π to be a FAP limit is that $\lim_n \pi_n(f) = \pi(f)$ for any $f \in \mathcal{S}_c$, where \mathcal{S}_c is the set of $f \in \mathcal{F}_b$ such that $\lim_n \pi(f_n)$ exists. The existence of a limit π that satisfies this requirement is guaranteed by the Hahn-Banach theorem (see [Rudin, 1991](#); [Huisman, 2016](#)) as follows: define the linear function Φ on \mathcal{S}_c by $\Phi(f) = \lim_n \pi_n(f)$ and the sublinear functional $p(f) = \limsup_n \pi_n(f)$. Then, there exists a linear functional π on \mathcal{F}_b that coincides with Φ on \mathcal{S}_c and that satisfies $\pi(f) \leq p(f)$ on \mathcal{F}_b . The functional π will be called a *FAP limit*. The condition $\pi(f) \leq p(f)$ implies that π is a FAP. Conversely, a FAP necessarily satisfies $\pi(f) \leq p(f)$. Replacing f by $-f$ gives $\pi(f) \geq \liminf_n \pi_n(f)$. Therefore, a FAP limit can be characterized by the following lemma:

Lemma 2. *A FAP π is a FAP-limit of the sequence $\{\pi_n\}_n$ if and only if for any $f \in \mathcal{F}_b$*

$$\liminf_n \pi_n(f) \leq \pi(f) \leq \limsup_n \pi_n(f) \quad (4)$$

or equivalently if and only if for any measurable set E

$$\liminf_n \pi_n(E) \leq \pi(E) \leq \limsup_n \pi_n(E). \quad (5)$$

When $\mathcal{S}_c = \mathcal{F}_b$, the FAP limit is unique and corresponds to the convergence associated to the weak*-topology. In the general case, the FAP limit is not unique and its existence relies on the axiom of choice. Unlike the q-vague convergence, the FAP convergence cannot be considered as an extension of the narrow convergence. For example, consider the proper distributions $\pi_n = \delta_{\sqrt{2}/n}$, where δ is the Dirac measure. The sequence $\{\pi_n\}_n$ converges narrowly to $\pi = \delta_0$ but π is not a FAP limit of $\{\pi_n\}_n$. To show this, consider $f(\theta) = \mathbb{1}_{\mathbb{Q}}(\theta) \in \mathcal{F}_b$,

with \mathbb{Q} the set rational numbers, we have $\lim_n \pi_n(f) = 0 \neq \pi(f) = 1$. However, in the special case where Θ is a denumerable set, any real-valued function on Θ is continuous and therefore if a sequence of proper distribution $\{\pi\}_n$ converges narrowly to a proper distribution π , then π is a FAP limit.

Another way to define a FAP limit in a formalized way has been proposed by Stone (1982) for denumerable sets but can be extended to more general sets considered in this paper. The existence of a limit relies on the Banach-Alaoglu-Bourbaki theorem (see, e.g. Rudin, 1991), since a FAP belongs to the unit ball in the dual of \mathcal{F}_b which is compact for the weak*-topology. Hence, for any sequence $\{\pi_n\}_n$ of FAPs, there exists at least one accumulation point π which is defined as a FAP limit. We recall that π is an accumulation point of $\{\pi_n\}_n$ for the weak*-topology if and only if for any integer p , any f_1, \dots, f_p in \mathcal{F}_b and any $\varepsilon > 0$, there exists an infinite number of n such that $|\pi_n(f_i) - \pi(f_i)| \leq \varepsilon$, $i = 1, \dots, p$. Note that, since \mathcal{F}_b is not in general first-countable, there does not necessarily exist a subsequence $\{\pi_{n_k}\}_k$ that converges to π . We can only say that, for any $f \in \mathcal{F}_b$, there exists a subsequence $\{\pi_{n_k}(f)\}_k$ such that $\pi_{n_k}(f)$ converges to $\pi(f)$.

Therefore, the set of FAP limits of $\{\pi_n\}_n$ obtained by Stone's approach is included in the set of FAP limits obtained by using the Hahn-Banach theorem as above and (4) or (5) still hold but are not sufficient conditions. The converse inclusion is false in general. It is easy to see that the closed convex hull of the set of FAP limits defined by Stone is included in the set of FAP limits defined in this paper. We conjecture that, conversely, the set of FAP limits defined by (4) is the convex hull for the limits defined by Stone. Consider for example $\pi_{2n} = \delta_0$ and $\pi_{2n+1} = \delta_1$. There are only two FAP limits δ_0 and δ_1 with Stone's construction, whereas any $\pi = \alpha\delta_0 + (1 - \alpha)\delta_1$, $0 \leq \alpha \leq 1$ is a FAP limit with our construction. In Section 4.1, we illustrate the difference between the two convergence modes with another example.

Even if the two definitions of FAP limits are not equivalent, the main results, especially Theorem 3, Corollary 4, Proposition 6, Lemma 7 and 8 hold for both of them. In the following, we consider only the first definition of FAP limits.

2.3. FAP limits vs q-vague convergence

The fact that a sequence of proper distributions have both improper and FAP limits may suggest a connection between the two notions as proposed heuristically by many authors. The following results show that this is not the case. Roughly speaking, it is shown that any FAP which is a FAP limit of some proper distribution sequence can be connected to any improper prior by this mean.

Theorem 3. *Let $\{\pi_n\}_n$ be a sequence of proper distributions such that $\lim_n \pi_n(K) = 0$ for any compact set K . Then, for any improper distribution π , it can be constructed a sequence $\{\tilde{\pi}_n\}_n$ which converges q-vaguely to π and which has the same set of FAP limits as $\{\pi_n\}_n$.*

Proof. For any FAP or any proper or improper distribution μ we define the distribution $(\mathbb{1}_A \mu)$ by $(\mathbb{1}_A \mu)(f) = \mu(\mathbb{1}_A f)$. From Lemma 1, it can be constructed an exhaustive increasing sequence K_n of compact sets such that $\lim_n \pi_n(K_n) = 0$. Put $\gamma_n = \pi_n(K_n)$ and define the sequence of proper distributions $\tilde{\pi}_n = \gamma_n \frac{1}{\pi(K_n)} \mathbb{1}_{K_n} \pi + (1 - \gamma_n) \frac{1}{\pi_n(K_n^c)} \mathbb{1}_{K_n^c} \pi_n$, with K^c the complement of K . By Lemma 8 and 7 in Appendix A, $\tilde{\pi}_n$ has the same FAP limits as $\{\pi_n\}$. By Lemma 9, $\tilde{\pi}_n$ converges q-vaguely to π . □

Corollary 4. *Let $\{\pi_n\}_n$ be a sequence of proper distributions that converges q-vaguely to an improper distribution $\pi^{(1)}$. Then, for any other improper distribution $\pi^{(2)}$, it can be constructed a sequence $\{\tilde{\pi}_n\}_n$ that converges q-vaguely to $\pi^{(2)}$ and that has the same FAP limits as $\{\pi_n\}_n$.*

The only link that can be established between improper q-vague limits and FAP limits of the same proper distribution sequence is that the FAP limits are necessarily *diffuse*, i.e. they assign a probability 0 to any compact set.

3. Uniform distribution on integers

In this section, we compare different notions of uniform distributions on the set of integers \mathbb{N} . by using several considerations such as limit of proper uniform distributions.

We illustrate the fact that FAP uniform distributions are not well defined objects (de Finetti, 1972, pp.122,224). Contrary to uniform improper distribu-

tions, FAP limits of uniform distributions on an exhaustive sequence of compact sets are highly dependent on the choice of that sequence.

3.1. Uniform improper distribution

There are several equivalent ways to define a uniform improper prior on integers. These definitions lead to a unique, up to a scalar factor, distribution. The uniform distribution can be defined directly as a flat distribution, i.e. $\pi(k) \propto 1$ for k integer. It is also the unique (up to a scalar factor) measure that is shift invariant, i.e. such that $\pi(k + A) = \pi(A)$ for any integer k and any set of integers A . The uniform distribution is also the q-vague limit of the sequence of uniform proper distributions on $K_n = \{0, 1, \dots, n\}$. More generally and equivalently, the uniform distribution is the q-vague limit of any sequence of proper uniform priors on an exhaustive increasing sequence $\{K_n\}_n$ of finite subsets of integers.

3.2. Finitely additive uniform distribution

The notion of uniform finitely additive probabilities is more complex. Contrary to the improper case, there is no explicit definition since $\pi(k) = 0$ for any integer k . We present here several non equivalent approaches to define a uniform FAP. The first two ones can be found in [Kadane and O'Hagan \(1995\)](#) and [Schirokauer and Kadane \(2007\)](#).

3.2.1. Shift invariant (SI) uniform distribution

As for the improper case, a uniform FAP distribution π can be defined as been any shift invariant FAP, i.e. FAPs satisfying $\pi(A) = \pi(A + k)$ for any subset of integers A and any integer k . Such a distribution will be called SI-uniform. In that case, one necessarily has : $\pi(k_1 + k_2 \times \mathbb{N}) = k_2^{-1}$, for any $(k_1, k_2) \in \mathbb{N} \times \mathbb{N}^*$. [Kadane and O'Hagan \(1995\)](#) also investigate the properties of FAPs satisfying only $\pi(k_1 + k_2 \times \mathbb{N}) = k_2^{-1}$, where $k_1 + k_2 \times \mathbb{N}$ are called *residue classes*.

3.2.2. Limiting relative frequency (LRF) uniform distributions.

Kadane and O'Hagan (1995) consider a stronger condition to define uniformity. For a subset A , define its limiting relative frequency $\text{LRF}(A)$ by

$$\text{LRF}(A) = \lim_{N \rightarrow \infty} \frac{\#\{k \leq N, \text{ s.t. } k \in A\}}{N + 1},$$

when this limit exists. A FAP π on \mathbb{N} is said to be *LRF-uniform* if $\pi(A) = p$ when $\text{LRF}(A) = p$.

Let π_n be the uniform proper distribution on $K_n = \{0, 1, \dots, n\}$, then $\text{LRF}(A) = \lim_{n \rightarrow \infty} \pi_n(A)$. Therefore a FAP π is LRF uniform if and only if it is a FAP limit of $\{\pi_n\}_n$. It is worth noting that, unlike the improper case, the FAP limits are highly dependent on the choice of the increasing exhaustive sequence of finite sets K_n . Changing the sequence $\{K_n\}_n$ changes the notion of uniformity. For example, if $\tilde{\pi}_n$ is the uniform distribution on $K_n = \{2k; 0 \leq k \leq n^2\} \cup \{2k+1; 0 \leq k \leq n\}$, then $\lim_n \tilde{\pi}_n(2\mathbb{N}) = 1$, whereas $\lim_n \pi_n(2\mathbb{N}) = 1/2$.

3.2.3. Bernoulli Scheme (BS) uniform distribution

We propose here another notion of uniformity that is not dependent of the choice a particular increasing sequence of K_n as for the LRF uniformity. Consider a Bernoulli Scheme, that is a sequence $\{X_k\}_{k \in \mathbb{N}}$ of i.i.d. Bernoulli distributed random variables with mean $p \in [0; 1]$. Define the random set $A(X) = \{k \in \mathbb{N}, \text{ s.t. } X_k = 1\}$. A FAP π is said to be *BS-uniform* if, for any $p \in [0; 1]$, $\pi(A(X)) = p$, almost surely. Note that the strong law of large numbers, $\text{LRF}(A(X)) = p$, almost surely.

Proposition 5. *Let $\{K_n\}$ be an increasing sequence of finite subsets of \mathbb{N} , with $\cup_{n \in \mathbb{N}} K_n$ being infinite. Then any FAP-limit of the sequence π_n of uniform distributions on K_n is BS-uniform.*

When $\cup_{n \in \mathbb{N}} K_n = \mathbb{N}$, this proposition shows that any FAP limit of uniform distribution is BS-uniform. In particular, a LRF uniform FAP is also BS uniform. However, if for example K_n is the set of even numbers less or equal to n , then any FAP-limit of the sequence of uniform distributions on K_n will be BS-uniform, although it is intuitively, certainly not uniform on \mathbb{N} but on $2\mathbb{N}$. Therefore, BS uniformity looks much more like a necessary condition for a FAP to be uniform, than like a complete definition.

4. Comparison of convergence modes on examples

We consider here some examples that illustrate the difference between convergence of proper distributions to an improper distribution or a to FAP.

4.1. FAP limits on \mathbb{N} .

For a sequence $\{\pi_n\}$ of proper distribution on \mathbb{N} , it is known that there does not necessarily exist a q -vague limit, but if it exists, it is unique in the projective space of Radon measures, i.e. unique up to a scalar factor. At the opposite, we have seen that a FAP limit always exists but is not necessarily unique.

We illustrate the non-uniqueness of FAP limits with an extreme case. Consider the sequence of proper distributions $\pi_n = \delta_n$, where δ_n is the Dirac measure on n . This sequence has no q -vague limit since $\pi_n(k) = 0$ for $n > k$. Moreover, for any subset A of \mathbb{N} so that A and A^c are both infinite:

$$0 = \liminf_{n \rightarrow \infty} \pi_n(A) \leq \pi(A) \leq \limsup_{n \rightarrow \infty} \pi_n(A) = 1,$$

whereas, for any finite set A , $\lim_n \pi_n(A) = 0 = \pi(A)$ and for any cofinite set A , $\lim_n \pi_n(A) = 1 = \pi(A)$. Therefore, from (5), the set of FAP limits of π_n is the set of all diffuse FAPs on \mathbb{N} . This shows that all the diffuse FAPs are connected through the same sequence π_n .

Let's examine the set of FAP limits of $\pi_n = \delta_n$ obtained with Stone's definition of FAP convergence (see Section 2.2). For any subset A , there exists a subsequence $\{\pi_{n_k}\}$ such that $\pi_{n_k}(A)$ converges to $\pi(A)$. So, $\pi(A) \in \{0, 1\}$. Therefore the FAP limits of π_n in Stone's sense are all *remote* FAPs, that is diffuse FAPs π such that $\pi(A) \in \{0, 1\}$, as defined by Dubins (1975, p.92). This also proves the existence of remote FAPs. Note that a remote distribution is neither BS uniform nor SI and therefore cannot be LRF uniform.

4.2. Convergence of sequence of Poisson distributions

We consider the sequence $\{\pi_n\}_n$ of Poisson distributions with mean n . For any finite set K , we have $\lim_n \pi_n(K) = 0$. However this sequence of proper priors does not converge q -vaguely to any improper distribution (Bioche and Druilhet, 2016, §5.2). As a remark, let $\tilde{\pi}_n$ be shifted measures on \mathbb{Z} , defined by

$\pi_n(B) = \pi_n(B+n)$, where π_n can be seen as a measure on the set of all integers \mathbb{Z} with $\tilde{\pi}_n(k) = 0$ for $k < 0$. Then, using the approximation of the Poisson distribution by a the normal distribution, it can be shown that the sequence $\tilde{\pi}_n$ converges q -vaguely to the improper uniform measure on \mathbb{Z} .

We consider now the FAP limits of the sequence $\{\pi_n\}_n$. The next result shows that the limit have some properties of uniformity described in Section 3 but not all of them. The proof is given in Appendix B.

Proposition 6. *Any FAP limit π of the sequence $\{\pi_n\}_n$ of Poisson distribution with mean n is shift invariant and BS-uniform but not necessarily LRF uniform.*

Therefore, the FAP limits of the Poisson distribution sequence are examples of SI- and BS-uniform distributions that are not LRF uniform. Kadane and Jin (2014) give another example of SI but not LRF uniform FAPs using paths of random walks. Even if they consider FAP on a subset of bounded function, it can be extended to \mathcal{F}_b using Hahn-Banach theorem similarly to Section 2.2.

4.3. FAP vs q -vague convergence of uniform proper distributions

To illustrate the fact that any FAP limit can be related with any improper distribution, consider again the sequence $\{\pi_n\}_n$ of Poisson distributions with mean n and any given improper distribution π^0 on the integers. Since $\lim_n \pi_n(K) = 0$ for any finite set, the proof of Lemma 1 shows how to construct an exhaustive sequence of finite set K_n such that $\lim_n \pi_n(K_n) = 0$. For example, choose $K_n = \{k \in \mathbb{N}, k \leq n/2\}$ and define the sequence of proper distributions $\tilde{\pi}_n$ by :

$$\tilde{\pi}_n(A) = \pi_n(K_n) \frac{\pi^0(A \cap K_n)}{\pi^0(K_n)} + (1 - \pi_n(K_n)) \frac{\pi_n(A \cap K_n^c)}{\pi_n(K_n^c)} \quad (6)$$

for any set A . From Theorem 3, $\tilde{\pi}_n$ converge q -vaguely to π^0 and has the same FAP limit as the Poisson distribution sequence.

As another example, consider the sequence $\{\pi_n\}_n$ of uniform distribution on $\{0, 1, \dots, n\}$ and choose $K_n = \{k \in \mathbb{N}, k \leq \sqrt{n}\}$. We have $\lim_n \pi_n(K_n) = 0$. Therefore, for any improper distribution π^0 on the set of integers, the sequence constructed as in (6) have the same FAP limits as that of sequence of uniform distributions $\{\pi_n\}_n$ and converge q -vaguely to π^0 . This shows again the difficulty to connect improper and FAP uniform distributions by limits of proper distribution.

4.4. Convergence of beta distributions

In this section, we consider the limit of the sequence of beta distribution $\pi_{a_n, b_n} = \text{Beta}(a_n, b_n)$ defined on $\Theta =]0, 1[$ when a_n and b_n go to 0. We see that the FAP limits depend on the way a_n and b_n goes to 0, which is not the case for the q-vague improper limit. This illustrates the difference between FAP limits and q-vague limits of proper distribution sequences.

This illustrates the way FAP limits capture how the mass concentrates on the boundary and q-vague limits describe the behavior of the sequence inside the domain.

The density of a beta distribution $\text{Beta}(a, b)$ is given by

$$\pi_{a,b}(x) = \frac{1}{\beta(a,b)} x^{a-1} (1-x)^{b-1} \text{ for } x \in]0, 1[$$

where $\beta(a, b)$ is the beta function.

From [Bioche and Druilhet \(2016\)](#), the unique (up to a scalar factor) q-vague limit of $\text{Beta}(a_n, b_n)$ when a_n and b_n go to 0 is the Haldane improper distribution:

$$\pi_H(x) = \frac{1}{x(1-x)} \text{ for } x \in]0, 1[.$$

The q-vague limit gives no information on the relative concentration of the mass around 0 and 1: for $0 < u < v < 1$, $\pi_H(]0, u]) = \pi_H(]v, 1[) = +\infty$. To explore this concentration, we temporarily replace the space Θ by $\tilde{\Theta} = [0, 1]$. This has no consequence on the Beta distribution but will change radically the q-vague limit. Put $c_n = a_n/b_n$ and assume that $\{c_n\}_n$ converges to some $c \in [0, 1]$. The sequence $\{\pi_{a_n, b_n}\}_n$ converges narrowly, and hence q-vaguely, to the proper distribution $\tilde{\pi} = \frac{1}{1+c}\delta_0 + \frac{c}{1+c}\delta_1$. Contrary to the Haldane prior, $\tilde{\pi}$ shows how the mass concentrate on the boundary of the domain, but gives no information on the behavior of the sequence inside the domain. Note that the Haldane distribution is not a Radon measure on $\tilde{\Theta}$ since $\pi_H([0, 1]) = +\infty$ where $[0, 1]$ is a compact set. Therefore π_H cannot be a candidate for the q-vague limit on $\tilde{\Theta}$.

We now consider the FAP limits on $\Theta =]0, 1[$ of π_{a_n, b_n} and we show that they give an information similar to that given by $\tilde{\pi}$ on the way the mass concentrate on the boundary of the domain. Again, we assume that $c_n = a_n/b_n$

converge to some $c \in [0, 1]$. Easy calculations show that for any $0 < \varepsilon < 1$ $\lim_n \pi_{a_n, b_n}([0, \varepsilon]) = \frac{1}{1+c}$ and $\lim_n \pi_{a_n, b_n}([1 - \varepsilon, 1]) = \frac{c}{1+c}$. Therefore, for any FAP-limit π and for any $\varepsilon \in]0, 1[$, we have $\pi([0, \varepsilon]) = \frac{1}{1+c}$ and $\pi([1 - \varepsilon, 1]) = \frac{c}{1+c}$, whereas $\pi([u, v]) = 0$ for $0 < u < v < 1$.

5. Conclusion and perspectives

In this paper, we have shown that improper distributions and FAPs limits give quite different information on the behaviour of the sequence of proper distribution and are therefore complementary.

In a Bayesian context, under some mild regularity condition on the likelihood $f(x|\theta)$, these results can be applied to the joint distribution $f_n(x, \theta) = f(x|\theta)\pi_n(\theta)$ and to the marginal distribution $p_n(x) = \int f(x|\theta)\pi_n(d\theta)$ when it is well defined, both converging to an improper prior if the sequence of prior converge to an improper distribution, similarly for FAP limits. As limits of proper priors, FAP limits and improper prior limits represent two different features of the sequence : the first one describes the behavior of the sequence inside the domain, whereas the second one describes how the mass concentrate on the boundary of the domain. We hope to use these differences in future works to understand some paradoxical phenomenon that occurs in Bayesian statistics.

Appendix A

We establish some lemmas useful to prove Theorem 3. The first one is straightforward.

Lemma 7. *Let $\{\pi_n^{(1)}\}_n$ and $\{\pi_n^{(2)}\}_n$ be two sequences of proper distributions and $0 \leq \gamma_n \leq 1$ be a sequence of scalars that converges to 0. Then, the sequence defined by $\tilde{\pi}_n = \gamma_n \pi_n^{(1)} + (1 - \gamma_n) \pi_n^{(2)}$ has the same FAP limits as $\{\pi_n^{(2)}\}_n$.*

Proof. For any $f \in \mathcal{F}_b$, $\limsup_n \tilde{\pi}_n(f) = \limsup_n \pi_n^{(2)}(f)$ and $\liminf_n \tilde{\pi}_n(f) = \liminf_n \pi_n^{(2)}(f)$. The result follows from Lemma 2. \square

Lemma 8. *Let $\{\pi_n\}$ be a sequence of proper priors and K_n be a non-decreasing sequence of compact sets such that $\lim_n \pi_n(K_n) = 0$, then the sequence defined by $\tilde{\pi}_n = \frac{1}{\pi_n(K_n^c)} \mathbb{1}_{K_n^c} \pi_n$ has the same FAP limits than $\{\pi_n\}_n$.*

Proof. First, note that $\{\pi_n\}_n$ is not defined when $\pi_n(K_n) = 1$, but this cannot occur more than a finite number of times. For any $f \in \mathcal{F}_b$, $\pi_n(f) = \mathbb{1}_{K_n} \pi_n(f) + \mathbb{1}_{K_n^c} \pi_n(f) = \pi_n(\mathbb{1}_{K_n} f) + \pi_n(K_n^c) \tilde{\pi}_n(f)$. Since f is bounded, $\lim_n \pi_n(\mathbb{1}_{K_n} f) = 0$. Moreover, $\lim_n \pi_n(K_n^c) = 1$. Therefore, $\pi_n(f)$ and $\tilde{\pi}_n(f)$ have the same limit-sup and limit-inf and thus they have the same FAP limits by Lemma 2. \square

At the opposite of Lemma 8, the following lemma shows that if we consider the restriction of a sequence $\{\pi_n\}_n$ of a proper or improper distribution on a exhaustive increasing sequence $\{K_n\}_n$ of compact sets, we preserve the q-vague limits.

Lemma 9. *Let K_n be a non-decreasing sequence of compact sets such that $\cup_n K_n = \Theta$ and such that, for any compact K , there exists N such that $K \subset K_N$. A sequence $\{\pi_n\}$ of Radon measures converges q-vaguely to the Radon measure π if and only if $\tilde{\pi}_n = \frac{1}{\pi_n(K_n)} \mathbb{1}_{K_n} \pi_n$ converges q-vaguely to π .*

Proof. Assume that π_n converges q-vaguely to π , then there exists some positive scalars $\{a_n\}_n$ such that for any f in \mathcal{C}_K , $\lim_n a_n \pi_n(f) = \pi(f)$. Put $\tilde{a}_n = a_n \pi_n(K_n)$ and denote by K_f a compact set that includes the support of f . Then, there exists an integer N such that $K_f \subset K_n$ for $n > N$. Therefore, for $n > N$, $\tilde{a}_n \tilde{\pi}_n(f) = a_n \pi_n(f)$. The result and its reciprocal follow. \square

Appendix B

We prove here Proposition 6 of section 4.2.

In order to show that π is SI, we consider π_n as distribution on \mathbb{Z} , extending them by 0 on the non-positive integers. Define the $\pi_n^{(k)}$ by $\pi_n^{(k)}(A) = \pi_n(A+k)$, for any subset of \mathbb{Z} . It is known that that $\|\pi_n^{(k)} - \pi_n\|_{TV} \leq \frac{k}{\sqrt{2\pi n}}$, where $\|\cdot\|_{TV}$ is the total variation norm. Therefore, for any subset of \mathbb{N} , $\lim_{n \rightarrow \infty} |\pi_n(A+k) - \pi_n(A)| = 0$. Letting n go to infinity, we deduce that, for any FAP limit π of π_n , and any integer k : $\pi(A+k) = \pi(A)$.

The fact that π is uniform in BS sense comes from an easy adaptation of the Hoeffding inequality in that context. Let $(X_k)_{k \in \mathbb{N}}$ be a Bernoulli scheme, of parameter p , and denote by \mathbb{P} being the associated probability. Hoeffding inequality gives, that, for any n :

$$\mathbb{P} \left\{ \left| \sum_{k=0}^{\infty} e^{-n} \frac{k^n}{n!} (X_k(\omega) - p) \right| \geq t \right\} \leq 2e^{-2c\sqrt{2\pi n} t^2},$$

for some positive constant c . The expected conclusion is then obtained thanks to the Borel-Cantelli lemma.

The fact that some of the FAP limits π are not LRF uniform is a direct consequence of the following lemma.

Lemma 10. *For any $0 \leq p, p' \leq 1$, there exists a set A and some FAP limits π of $\{\pi\}_n$ such that $LRF(A) = p$ and $\pi(A) = p'$.*

Proof. First note that, for any set A' , $LRF(A') = p$ if, and only if, $\#\{k \leq n, k \in A'\} = pn + o(n)$. Therefore, for any set A with $LRF(A) = p$ and for any set B such that $\#\{k \leq n, k \in B\} = o(N)$, one has both $LRF(A \cup B) = p$ and $LRF(A \setminus B) = p$. Take now for set B the following :

$$B = \bigcup_{k \in \mathbb{N}} \{u \in \mathbb{N} : 4^k - 2^k k \leq u \leq 4^k + 2^k k\}.$$

For that B , one has :

$$\limsup_{n \rightarrow \infty} \frac{\#\{k \leq n, k \in B\}}{n+1} = \lim_{k \rightarrow \infty} \frac{\sum_{i=0}^k 2^{i+1} i}{4^k + 2^k k} \leq \lim_{k \rightarrow \infty} \frac{(k+1)2^{k+2}}{4^k} = 0,$$

and thus $LRF(B) = 0$. However, $\pi_{4^k}(B)$ converges to 1. Indeed, if U_k is some random variable with law π_{4^k} , one has :

$$\pi_{4^k}(\{u \in \mathbb{N} : 4^k - 2^k k \leq u \leq 4^k + 2^k k\}) = \mathbb{P}\left(\frac{U_k - 4^k}{\sqrt{4^k}} \in [-k; k]\right).$$

The righthand side term above converges to 1 thanks to the central limit theorem. Hence $LRF(A \cup B) = LRF(A \setminus B) = p$ while $\pi_{4^k}(A \cup B)$ converges to 1, and $\pi_{4^k}(A \setminus B)$ converges to 0. Now, for any $p' \in [0; 1]$, choose two numbers $a < b$, so that $p' = \int_a^b \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$. Take the set B' to be :

$$B' = \bigcup_{k \in \mathbb{N}} \{u \in \mathbb{N} : 4^k + 2^k \max(-k, a) \leq u \leq 4^k + 2^k \min(k, b)\},$$

then $LRF(B) = 0$ again and $\pi_{4^k}(B')$ converges to p' , still thanks to the central limit theorem. Let $A = (A' \setminus B) \cup B'$. Then $LRF(A) = p$ and $\lim_{k \rightarrow \infty} \pi_{4^k}(A) = p'$. Now, any FAP-limit π of subsequence π_{4^k} is also a FAP-limit of π_k : π is therefore shift uniform and BS uniform, but one has $\pi(A) = p'$. \square

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