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Abstract

We analyse the Landau damping mechanism for variants of Vlasov equations, with a time dependent linear force term and a self-consistent potential that involves an additional memory effect. This question is directly motivated by a model describing the interaction of particles with their environment, through momentum and energy exchanges with a vibrating field. We establish the stability of homogeneous states. We bring out how the coupling influences the stability criterion, in comparison to the standard Vlasov case.

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1 Introduction

In this work, we go back to the analysis of Landau damping mechanisms in kinetic equations. This effect has been brought out for the Vlasov equation of plasma physics in the pioneering work of L. Landau [23], and extended to gravitational models in astrophysics [26, 27], where it is thought to play a key role in the stability of galaxies. It can be interpreted as a stability statement about steady solutions, leading to a decay of the self-consistent force. A complete mathematical analysis of the Landau damping for non linear Vlasov equations has been performed in [28], and revisited later on in [6, 7] (see also [21]). Similar behaviors have been revealed for the 2D Euler system [5]. The phenomena are surprising since they describe damping mechanisms, counter-intuitive for *reversible* equations which apparently do not present any dissipative process.

The starting point of this contribution comes from an original model introduced by L. Bruneau and S. De Bièvre [8] describing the motion of a *single* classical particle interacting with its environment. The particle is described by its position $t \mapsto q(t) \in \mathbb{R}^d$, while the behavior of the environment is embodied into a scalar field $(t,x,z) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^n \mapsto \psi(t,x,z)$. The dynamic is modeled by the following set of differential equations

$$\begin{cases} \ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\iint_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-y)\ \sigma_2(z)\ \nabla_x \Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,z) - c^2\Delta_z\Psi(t,x,z) = -\sigma_2(z)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ z\in\mathbb{R}^n. \end{cases} \tag{1}$$

It corresponds to the intuition of a particle moving through an infinite set of n-dimensional elastic membranes, one for each position $x \in \mathbb{R}^d$. The physical properties of the membranes are characterized by the wave speed $c > 0$. The coupling between the particles and the environment is governed by two form functions σ_1, σ_2, which are both non negative, smooth and radially symmetric functions; they can be seen as determining the influence domain of the particle in each direction, the direction of particle's motion and the direction of wave propagation, respectively. It is therefore relevant to assume both form functions have a compact support. The particle exchanges its kinetic energy with the vibrations of the membranes. These mechanisms eventually act like a friction force since particle's energy is evacuated in the membranes, and, depending on the shape of the external potential $x \mapsto V(x)$, they determine the large time behavior of the particle. We refer the reader to [1, 11, 12, 13, 22, 32] for further studies of the system (1), that include numerical experiments and interpretation by means of random walks.

The system (1) can be generalized by considering a set of N particles going through the membranes. The mean field regime $N\to\infty$ leads to the following PDE system

$$\partial_t F + v\cdot\nabla_x F - \nabla_x(V+\Phi[\Psi])\cdot\nabla_v F = 0, \qquad t\geq 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \tag{2a}$$

$$(\partial^2_{tt}\Psi - c^2\Delta_z\Psi)(t,x,z) = -\sigma_2(z)\ \int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y,\ t\geq 0,\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n, \tag{2b}$$

$$\rho(t,x) = \int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v, \tag{2c}$$

$$\Phi[\Psi](t,x) = \iint_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_1(x-y)\sigma_2(z)\Psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y, \qquad t\geq 0,\ x\in\mathbb{R}^d, \tag{2d}$$

where now $(t, x, v) \mapsto F(t, x, v)$ is interpreted as the particles distribution function in phase space, $x \in \mathbb{R}^d$ being the position variable, $v \in \mathbb{R}^d$ being the velocity variable. The system (2a)–(2d) is completed by initial conditions

$$F\big|_{t=0} = F_0, \qquad (\Psi, \partial_t \Psi)\big|_{t=0} = (\Psi_0, \Psi_1). \tag{3}$$

We refer the reader to [17, 33] for the derivation of the N-particles system and the analysis of the mean field regime that leads to (2a)–(2d). The existence of solutions of (2a)–(2d) is investigated in [9]. Furthermore, asymptotic issues are also discussed that reveal an unexpected connection with the *gravitational* Vlasov-Poisson equation. This relation with another model of statistical physics can guide the intuition to analyze further mathematical properties of (2a)–(2d). In this spirit, the existence of equilibrium states and their stability is discussed in [2], adding in the kinetic model a dissipative effect with the Fokker–Planck operator, and in [10] where a variational approach is adopted for the collisionless model, following [19, 20, 35].

We wish to continue this analysis, adopting a different viewpoint. In [2, 10] the effect of a confining potential $x \mapsto V(x)$ is considered, which governs the shape of the equilibrium states. Here, we change the geometry of the problem, replacing the confining assumption on the external potential, by the assumption that particles' motion holds in the $d-$dimensional torus $\mathbb{T}^d$. In such a framework, like for the usual Vlasov-Poisson system, we can find space–homogeneous stationary solutions, and we wish to investigate their stability. This question is directly reminiscient to the well-known phenomena of damping brought out in plasma physics by L. Landau [23]: for the electrostatic Vlasov-Poisson system, it can be shown that the electric field of the linearized system decays exponentially fast. For gravitational interactions a similar discussion dates back to D. Lynden–Bell [26, 27]. In fact, Landau's analysis [23] was concerned with the linearized equation only. Of course the linearization procedure is questionable and the non linear dynamics might significantly depart form the linear behavior, as pointed out in [3]. A stunning analysis of the non linear problem in the analytic framework has been recently performed by C. Mouhot & C. Villani [28, 34]. A simplified analysis of the Landau damping has been proposed in [6]; we also refer the reader to [16] for results based on Sobolev regularity (with a definition of the force which involves only a finite number of Fourier modes, though) and [21] for an alternative approach that uses integration along phase-space characteristics. The Landau damping around homogeneous solutions has also been investigated in the whole space $\mathbb{R}^d$ [7], thus dealing with a set of particles having an infinite mass. We wish to address these issues for the system (2a)–(2d), still when $V = 0$. The analysis of the non-linear equations is quite involved; it requires a complex functional framework and fine estimates in order to control the non linear effects, the so–called "plasma echoes", that can break the damping mechanisms observed on the linearized model. By the way, it has been recently shown that insufficient regularity of the perturbation can annihilate the damping mechanisms, and the proof (which, though, is very specific to the coupling with the Poisson equation; it is not clear that the argument applies for more regular convolution kernels) precisely uses the role of the plasma echoes against damping [4]. Nevertheless it turns out that identifying stability conditions for the linearized problem plays a central role in the analysis of the non linear stability, see [28, Condition **(L)**]. Beyond their interest for the specific model (2a)–(2d) of particles interacting with their environment, the results we are going to discuss can be thought of with some generality. Indeed, as we shall detail below, the equation for the particle distribution function can be recast as follows

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \Phi_I \cdot \nabla_v F - \nabla_x \Phi_S \cdot \nabla_v F = 0,$$

where the potential splits into two parts, that both induce new issues compared to the case of the "standard" Vlasov system (hereafter simply refered to as the "Vlasov equation"):

- $\Phi_I(t,x)$ does not depend on F: this is a *linear* contribution in the equation. The damping then relies on suitable time-decay properties, here related to the dispersion properties of the free wave equation.
- the self-consistent potential $\Phi_S(t,x)$ is defined by a convolution with respect to space, combined with a half-convolution with respect to time

  $$\Phi_S(t,x) = -\int_0^t \int \Sigma(x-y) p_c(t-s) \rho(s,y) \, \mathrm{d}y \, \mathrm{d}s.$$

  Then the Landau damping relies on properties of the kernel Σ, which is quite similar to the analysis of the Vlasov case, but also on decay properties of the kernel p_c.

The discussion is organized as follows. We start by checking that we can find homogeneous solutions in Section 2. We also introduce different, but complementary, ways to think of the equations. We complete this preliminary section with a series of comments explaining how the problem differs from the usual Vlasov system. In Section 3, which is the heart of this work, we turn to the linearized problem and we discuss the stability criterion. At least, it turns out that stability can be verified when c, the speed of wave propagation, is large enough. Next, we fully detail the proof of the Landau damping for the free space problem, for which the functional framework is less intricate, in Section 4. We present how the main arguments should be adapted for the torus in Section 5. This content is completed by several Appendices which have their own interest. Appendix A details the analysis of the Volterra equation associated to the linearized problem, offering a unified description of the derivation of the stability criterion for both the Vlasov and the Vlasov-Wave equation. Appendix B discusses in further details the stability criterion, in the spirit of the Penrose criterion. Quite surprisingly, we are led to an intricate expression, much more complicated than for the Vlasov model, which, nevertheless, allows us to establish some conclusions close to the gravitational Vlasov case. We also propose several interpretations of criteria that lead to (un)stable solutions. Finally, Appendix C briefly goes back to the Cauchy theory for analytic solutions of the system.

2 Preliminaries

In what follows, $\mathbb{X}^d$ stands indifferently for $\mathbb{T}^d$ or $\mathbb{R}^d$, and for given functions $\phi : x \in \mathbb{X}^d \mapsto \phi(x)$ and $g : v \in \mathbb{R}^d \mapsto g(v)$, we denote

$$\langle \varphi \rangle_{\mathbb{X}^d} = \int_{\mathbb{X}^d} \varphi(x)\,\mathrm{d}x, \qquad \langle g \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} g(v)\,\mathrm{d}v,$$

where $\mathrm{d}x$ is either the usual Lebesgue measure on $\mathbb{X}^d = \mathbb{R}^d$ or the normalized Lebesgue measure on $\mathbb{X}^d = \mathbb{T}^d$. We shall also use indifferently the notation $\widehat{\cdot}$ for the Fourier coefficients of a $\mathbb{T}^d-$periodic function

$$\varphi : \mathbb{T}^d \to \mathbb{R}, \qquad \widehat{\varphi}(k) = \int_{\mathbb{T}^d} e^{-ik\cdot x}\varphi(x)\,\mathrm{d}x \text{ for } k \in \mathbb{Z}^d,$$

or the Fourier transform over $\mathbb{R}^m$ (with $m = d$ or $m = n$)

$$\varphi : \mathbb{R}^m \to \mathbb{R}, \qquad \widehat{\varphi}(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi}\varphi(x)\,\mathrm{d}x \text{ for } \xi \in \mathbb{R}^m.$$

We equally use the same notation for a function ϕ depending on $x \in \mathbb{X}^d$ and $v \in \mathbb{R}^d$

$$\widehat{\varphi}(k,\xi) = \iint_{\mathbb{X}^d\times\mathbb{R}^m} e^{-ik\cdot x}e^{-i\xi\cdot v}\varphi(x,v)\,\mathrm{d}v\ \mathrm{d}x,$$

for $\xi \in \mathbb{R}^m$ and either $k \in \mathbb{Z}^d$ (case $\mathbb{X}^d = \mathbb{T}^d$) or $k \in \mathbb{R}^d$ (case $\mathbb{X}^d = \mathbb{R}^d$). In the sequel, we shall use the shorthand notation $k \in \mathbb{X}^{\star d}$ to encompass these two situations.

2.1 Rewriting the equations

Due to the linearity of the wave equation, the solution of (2b) can be split into a contribution that depends only on the initial condition (Ψ_0, Ψ_1) and a contribution that depends only on ρ, see [9, Eq. (6)–(8)]. Accordingly, we split the potential into

$$\Phi = \Phi_I + \Phi_S,$$

where Φ_I depends only on (Ψ_0, Ψ_1) as follows

$$\Phi_I(t,x) = \frac{1}{(2\pi)^n}\iint_{\mathbb{R}^n\times\mathbb{X}^d} \sigma_1(x-y)\left(\widehat{\Psi_0}(y,\zeta)\cos(c|\zeta|t) + \widehat{\Psi_1}(y,\zeta)\frac{\sin(c|\zeta|t)}{c|\zeta|}\right)\ \widehat{\sigma_2}(\zeta)\ \mathrm{d}y\,\mathrm{d}\zeta \tag{4}$$

and the coupling term reads

$$\begin{aligned} &\Phi_S(t,x) = -\int_0^t p_c(t-s)\Sigma \star \rho(s,x)\,\mathrm{d}s,\\ &\Sigma = \sigma_1 \star \sigma_1,\\ &p_c(t) = \int_{\mathbb{R}^n} \frac{\sin(c|\zeta|t)}{c|\zeta|}\ |\widehat{\sigma}_2(\zeta)|^2 \frac{\mathrm{d}\zeta}{(2\pi)^n}. \end{aligned} \tag{5}$$

The properties of the function $t \mapsto p_c(t)$ play a crucial role in the asymptotic analysis of (2a)–(2d).

In what follows, we shall use the following general assumptions

(H1) $n \geq 3$ is odd,

(H2) $\sigma_2 \in C^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\sigma_2) \subset B(0, R_2)$.

With $n \geq 3$, according to [9, Lemma 4.4], we know that $p_c \in L^1((0,\infty))$ with

$$\int_0^\infty p_c(t)\,\mathrm{d}t = \frac{\kappa}{c^2}, \qquad \kappa = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma}_2(\zeta)|^2}{|\zeta|^2}\,\mathrm{d}\zeta.$$

In particular, the condition $n \geq 3$ guarantees that the integral that defines κ makes sense. (Note that [8] makes the case $n = 3$ the most relevant.) Finite speed of propagation and energy conservation for the wave equation can be used to deduce fundamental estimates on the function p_c: the following simple observation strengthens [9, Lemma 4.4] by taking full advantage of **(H1)**–**(H2)**.

Lemma 2.1 *Assume **(H1)**–**(H2)**. Then the function $t \mapsto p_c(t)$ has a compact support, included in $[0, \frac{2R_2}{c}]$ and it satisfies*

$$|p_c(t)| \leq C_S \frac{\|\sigma_2\|_{L^{2n/(n+2)}} \|\sigma_2\|_{L^2}}{c},$$

for a certain constant $C_S > 0$.

Proof. The kernel $p_c(t)$ can be rewritten as

$$p_c(t) = \int_{\mathbb{R}^n} \sigma_2(z)\Upsilon(t,z)\,\mathrm{d}z$$

where Υ is the solution of the wave equation with initial impulsion σ_2:

$$\begin{aligned}&(\partial_{tt}^2 - c^2\Delta_z)\Upsilon(t,z) = 0,\\ &(\Upsilon, \partial_t\Upsilon)\big|_{t-0} = (0, \sigma_2).\end{aligned}$$

With **(H2)**, Huygens' principle implies that

$$\text{if } ct \geq R_2 + |z| \text{ then } \Upsilon(t,z) = 0.$$

Therefore, see Fig. 1, when $t \geq \frac{2R_2}{c}$, the product $\sigma_2(z)\Upsilon(t,z)$ vanishes, and $p_c(t) = 0$.

Next, we start with the Hölder inequality, bearing in mind $n \geq 3$

$$|p_c(t)| \leq \|\sigma_2\|_{L^{2n/(n+2)}} \|\Upsilon(t,\cdot)\|_{L^{2n/(n-2)}}.$$

We dominate the right hand side by making use of the Sobolev embedding, see e. g. [25, Lemma 8.3], $\|\Upsilon(t,\cdot)\|_{L^{2n/(n-2)}} \leq C_S \|\nabla_z \Upsilon(t,\cdot)\|_{L^2}$, while energy conservation for the wave equation tells us that

$$\|\nabla_z\Upsilon(t,\cdot)\|_{L^2}^2 \leq \frac{1}{c^2}\Big(\|\partial_t\Upsilon(t,\cdot)\|_{L^2}^2 + c^2\|\nabla_z\Upsilon(t,\cdot)\|_{L^2}^2\Big) \leq \frac{1}{c^2}\|\sigma_2\|_{L^2}^2$$

holds. ∎

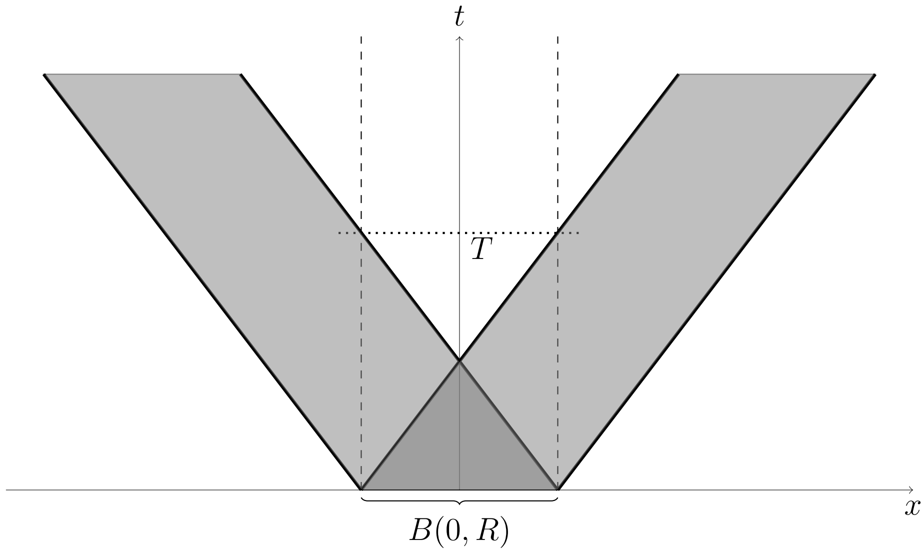

Figure 1: Propagation cone: the signal emanating from the ball $B(0,R)$ cannot be felt in this ball after time T

Remark 2.2 *Integrability of p_c and Huygens' principle play a central role in our analysis and, more generally, in the qualitative properties of the model introduced in [8]: they imply a strong dissipation mechanism of energy through the vibration of the medium. For instance, in dimension $n = 1$, a direct computation by means of D'Alembert formula shows that*

$$p_c(t) = \frac{1}{2c}\int_{-\infty}^{+\infty} \sigma_2(z) \left(\int_{z-ct}^{z+ct} \sigma_2(s)\,\mathrm{d}s \right) \mathrm{d}z \xrightarrow[t\to\infty]{} \frac{1}{2c}\|\sigma_2\|^2_{L^1_z} > 0.$$

Hence, in this case $p_c \notin L^1(0,\infty)$, there is no loss of memory at all, and numerical simulations [18] indeed confirm that there is no damping phenomena. Similarly, working in the torus $\mathbb{T}^n$ for the wave equation leads to

$$p_c(t) = \sum_{\ell\neq 0} \frac{|\widehat{\sigma}_2(\ell)|^2}{c|\ell|} \sin(c|\ell|t) + |\widehat{\sigma}_2(0)|^2 t.$$

It prevents p_c for being integrable over $(0,\infty)$ and shows that there is no possible energy dispersion mechanism in this geometry.

The case of $\mathbb{R}^n$ with an even dimension is more subtle. It seems that the analysis performed on the torus uses crucially the compactness of the support of p_c and this case cannot be handled. For the free space problem it is less clear whether or not the dispersion mechanisms of the wave equation in even dimensions are enough. The estimates we are using are not fine enough to handle this situation. However, the alternative proof of [21], which is less demanding in terms of regularity, could be adapted in order to extend the result in this direction.

2.2 Homogeneous solutions

Let $\rho_0 > 0$ and let $v \mapsto M(v)$ be a given function such that $\int_{\mathbb{R}^d} M(v)\,\mathrm{d}v = 1$. We claim that

$$\mathscr{M} : (x,v) \in \mathbb{X}^d \times \mathbb{R}^d \longmapsto \mathscr{M}(x,v) = \rho_0 M(v)$$

is a stationary solution of (2a)–(2d), associated to a spatially homogeneous potential Φ, when starting from spatially homogeneous data for the wave equation. On the torus, since M and $\mathrm{d}x$ are normalized, ρ_0 is the mass of the solution $\mathscr{M}$. With $F = \mathscr{M}$, the right hand side of the wave equation (2b) becomes

$$-\sigma_2(z) \iint_{\mathbb{X}^d \times \mathbb{R}^d} \sigma_1(x-y)\mathscr{M}(y,v)\,\mathrm{d}v \;\; \mathrm{d}y = -\sigma_2(z) \;\; \rho_0 \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle M \rangle_{\mathbb{R}^d},$$

which depends only on the variable $z \in \mathbb{R}^n$. Therefore, considering space-homogeneous initial data $(x,z) \mapsto (\Psi_0^H(z), \Psi_1^H(z))$, the solution of the wave equation

$$\partial^2_{tt} \Psi^H - c^2 \Delta_z \Psi^H = -\sigma_2(z) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d}$$

is given by the inverse Fourier transform of

$$\widehat{\Psi}^H(t,\xi) = \widehat{\Psi^0_H}(\xi) \cos(c|\xi|t) + \widehat{\Psi^1_H}(\xi) \frac{\sin(c|\xi|t)}{c|\xi|} - \frac{1-\cos(c|\xi|t)}{c^2|\xi|^2} \; \widehat{\sigma}_2(\xi) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d},$$

and it does not depend on the space variable x. Accordingly, the associated potential

$$\Phi[\Psi^H](t,x) = \langle \sigma_1 \rangle_{\mathbb{X}^d} \iint_{\mathbb{R}^n} \sigma_2(z) \Psi^H(t,z)\,\mathrm{d}z$$

does not depend on x. We obtain

$$(\partial_t + v \cdot \nabla_x)\mathscr{M} = 0 = \nabla_x \Phi[\Psi^H] \cdot \nabla_v \mathscr{M},$$

and finally $(\mathscr{M}, \Psi^H)$ is a homogeneous solution of (2a)–(2d). We bring the attention of the reader to the fact that, in the case $\mathbb{X}^d = \mathbb{R}^d$, the homogeneous solutions have infinite mass and infinite energy.

Remark 2.3 (Stationary solutions) *A specific case of interest corresponds to* stationary *solutions. Let us associate to $\mathscr{M}$, the function*

$$\Psi_{\mathrm{eq}}(z) = \frac{1}{c^2} \Gamma(z) \langle \sigma_1 \rangle_{\mathbb{X}^d} \langle \mathscr{M} \rangle_{\mathbb{R}^d},$$

where Γ is the solution of $\Delta_z \Gamma(z) = \sigma_2(z)$. It defines a stationary solution Ψ_{eq} for the wave equation (2c) *(with initial data $\Psi_0^H = \Psi_{\mathrm{eq}}$ and $\Psi_1^H = 0$). The associated potential thus reads*

$$\iint_{\mathbb{X}^d \times \mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\Psi_{\mathrm{eq}}(z) \;\; \mathrm{d}x\,\mathrm{d}z = \langle \sigma_1 \rangle_{\mathbb{X}^d} \int_{\mathbb{R}^n} \sigma_2(z) \Psi_{\mathrm{eq}}(z)\,\mathrm{d}z,$$

which does not depend on the space variable $x \in \mathbb{X}^d$, nor on the time variable t.

2.3 Equations for the fluctuations

Given a space-homogeneous solution $(\mathscr{M}, \Psi^H)$, we expand the solution as

$$F(t,x,v) = \mathscr{M}(v) + f(t,x,v), \qquad \Psi(t,x,z) = \Psi^H(t,z) + \psi(t,x,z). \tag{6}$$

The fluctuations (f, ψ) satisfy

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi[\psi] \cdot \nabla_v(\mathscr{M} + f) = 0, \tag{7a}$$

$$\Phi[\psi](t,x) = \iint_{X^d \times \mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, \tag{7b}$$

$$\partial^2_{tt}\psi - c^2\Delta_z\psi = -\sigma_2(z)\int_{\mathbb{R}^d}\sigma_1(x-y)\varrho(t,y)\,\mathrm{d}y, \tag{7c}$$

$$\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \tag{7d}$$

completed by the initial conditions

$$f(0,x,v) = f_0(x,v), \qquad (\psi(0,x,z), \partial_t\psi(0,x,z)) = (\psi_0(x,z), \psi_1(x,z)). \tag{8}$$

As said above, it can be convenient to set $\psi(t,x,z) = \psi_I(t,x,z) + \psi_S(t,x,z)$, with the contribution from the initial data

$$\widehat{\psi}_I(t,x,\xi) = \widehat{\psi_0}(x,\xi)\cos(c|\xi|t) + \widehat{\psi_1}(x,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}$$

and the self-consistent contribution

$$\widehat{\psi}_S(t,x,\xi) = -\int_0^t \frac{\sin(c|\xi|[t-\tau])}{c|\xi|}\widehat{\sigma}_2(\xi)\sigma_1 \star \varrho(\tau,x)\,\mathrm{d}\tau.$$

Plugging this into the expression of the potential, we get

$$\Phi[\psi](t,x) = \sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))(x),$$

where we have set

$$\mathscr{F}_I(t,x) = \int_{\mathbb{R}^n} \sigma_2(z)\psi_I(t,x,z)\,\mathrm{d}z$$

and

$$\mathscr{G}_\varrho(t,x) = \int_0^t p_c(t-\tau)\varrho(\tau,x)\,\mathrm{d}\tau.$$

Hence, the evolution equation for the fluctuation f can be recast as

$$\partial_t f + v \cdot \nabla_x f - \nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho) \cdot \nabla_v(\mathscr{M} + f) = 0. \tag{9}$$

Finally, let us introduce

$$g(t,x,v) = f(t, x+tv, v),$$

which allows us to get rid of the advection operator. We remark that

$$\partial_t g(t,x,v) = (\partial_t + v \cdot \nabla_x) f(t, x+tv, v)$$

and

$$(\nabla_v f)(t,x+tv,v) = \nabla_v\Big[f(t,x+tv,v)\Big] - t\nabla_x f(t,x+tv,v) = (\nabla_v - t\nabla_x)g(t,x,v).$$

Thus, (9) becomes

$$\partial_t g(t,x,v) = \nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)(t,x+tv)\cdot(\nabla_v - t\nabla_x)(\mathscr{M}+g)(t,x,v), \tag{10a}$$

$$g(0,x,v) = f_0(x,v). \tag{10b}$$

The following rough statement gives the flavor of the result we wish to justify.

Theorem *We assume that the data $\sigma_1, \sigma_2, \psi_0, \psi_1, f_0$ are smooth enough. We assume, furthermore, that the analog of the* **(L)***-condition for the Vlasov-Wave equation holds. If, initially, the fluctuation is small enough, then, we can find an asymptotic profile g^∞ so that $g(t) - g^\infty$ and the applied force $\nabla\sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)$ tend to 0 as $t \to \infty$.*

The precise statements are given in Theorem 4.8 (case $\mathbb{X}^d = \mathbb{R}^d$) and Theorem 5.8 (case $\mathbb{X}^d = \mathbb{T}^d$) Let us make a few comments to announce the forthcoming analysis.

- The stability condition (**L**) (see Section 3.1 and the comments in Appendix B), like for the usual Vlasov equation, imposes that a certain symbol cannot reach the value 1. In particular, the stability condition holds provided the wave speed c is large enough, see Proposition 3.4.
- The functional framework is a bit intricate. Roughly speaking, we distinguish two types of results, depending whether we work with analytic functions and regularity measured by means of Gevrey spaces (for the torus, the result applies only in this framework), or with functions having enough Sobolev regularity (the result on $\mathbb{R}^d$ applies in this context, and we can also establish the damping for the *linearized* problems in both cases $\mathbb{X}^d = \mathbb{R}^d$ and $\mathbb{X}^d = \mathbb{T}^d$).
- Typically the smallness assumption is imposed on a certain space X (of Gevrey or Sobolev type), but the damping holds in slightly “less regular” spaces Y, with $X \subset Y$.
- The rate of convergence depends on the functional framework (Gevrey vs. Sobolev) and how far Y is from X.
- For the problem on $\mathbb{R}^d$, we shall need to assume $d \geq 3$; the method breaks down in smaller dimensions, for reasons that already appeared for the Vlasov-Poisson system [7].

For the usual Vlasov equation, the main ingredients to justify the Landau damping can be recapped as follows:

- the transport operator induces a phase mixing phenomena, which is a source of decay for the macroscopic density ϱ;

- when linearizing the system around the homogeneous solution, we observe that the Fourier modes of ϱ decouple, leading to a Volterra equation for the Fourier transform of the density. It permits us to identify a stability criterion, that depends on the homogeneous solution and on the potential so that the linear dynamics induced by the force term does not annihilate the effects of the phase mixing;
- it remains to control the non linear effects, with the plasma echoes that tend to contribute against the phase mixing.

Technically, in order to address this program, one needs a Cauchy theory (in analytic regularity for the problem on $\mathbb{T}^d$) such that a control on a "weak" norm is enough to assert that the solution can be extended. Moreover, assuming the smallness of the data, the norms used in this Cauchy theory should permit us to justify uniform boundedness with respect to time, and, eventually, the Landau damping. In particular, the echoes should be controlled by means of these norms. Rewriting the potential with (4)–(5), we realize that the system (2a)–(2d) substantially differs from the usual Vlasov system dealt with in [28] and [6, 7] in the following aspects:

- there is an additional term
$$\nabla_x \Phi_I \cdot \nabla_v F,$$
with a force *independent* on the particles density. This linear perturbation could drive the solution far from the homogeneous state $\mathscr{M}$;
- the self-consistent potential Φ_S involves a half-convolution with respect to the time variable, inducing a sort of memory effect. In particular, the function p_c dramatically influences the expression of the stability criterion.

As we shall see, the analysis of the linearized problem, and the stability criterion, sensibly differ from the Vlasov case. Nevertheless, this linearized analysis remains at the heart of the proof of the Landau damping: once the Landau damping established for the linearized equation, the arguments of [28] and [6, 7] can be adapted to handle the nonlinear problem. Furthermore, we will also bring out the analogies with the gravitational Vlasov-Poisson problem, in terms of conditions of the equilibrium profile. We address both the confined case $\mathbb{X}^d = \mathbb{T}^d$ and the free space problem $\mathbb{X}^d = \mathbb{R}^d$, underlying the differences needed depending on the technical framework.

3 Analysis of the linearized Landau damping

3.1 The linearized system

In the expansion (6), let us assume that the fluctuations f and ψ remain small, so that we neglect the quadratic term (with respect to the perturbations) $\nabla_x \Phi[\psi] \cdot \nabla_v f$ in the evolution equations (note in particular that this assumes the smallness of the initial fluctuations (ψ_0, ψ_1)). We are thus led to the following linearized system

$$\partial_t f + v \cdot \nabla_x f = \rho_0 \nabla_x \phi \cdot \nabla_v M, \qquad t \geq 0,\ x \in \mathbb{X}^d,\ v \in \mathbb{R}^d, \tag{11a}$$

$$(\partial_{tt}^2 \psi - c^2 \Delta_z \psi)(t,x,z) = -\sigma_2(z) \int_{\mathbb{X}^d} \sigma_1(x-y)\varrho(t,y)\ \mathrm{d}y,\ t \geq 0,\ x \in \mathbb{X}^d,\ z \in \mathbb{R}^n, \tag{11b}$$

$$\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \tag{11c}$$

$$\phi(t,x) = \iint_{\mathbb{X}^d \times \mathbb{R}^n} \sigma_1(x-y)\psi(t,y,z)\sigma_2(z)\,\mathrm{d}z\ \mathrm{d}y, \qquad t \geq 0,\ x \in \mathbb{X}^d. \tag{11d}$$

The system is completed by initial conditions

$$f\big|_{t=0} = f_0, \qquad (\psi, \partial_t \psi)\big|_{t=0} = (\psi_0, \psi_1). \tag{12}$$

The expected result can be explained as follows: let us assume that the fluctuation does not provide additional mass: $\iint f(0,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0$, and, to fix ideas, $\psi_0 = 0$ and $\psi_1 = 0$. In such a case, linearized Landau damping asserts that ϱ converges strongly to 0, while f converges weakly to 0, as $t \to \infty$. Moreover, the potential ϕ also vanishes for large times. We are going to establish that such a behavior holds for the system (11a)–(12).

We start by applying the Fourier transform, with respect to x and v to (11a). It yields

$$(\partial_t - k \cdot \nabla_\xi)\widehat{f}(t,k,\xi) = -\rho_0\ k \cdot \xi\ \widehat{\phi}(t,k)\ \widehat{M}(\xi).$$

The equation can be integrated along characteristics, which leads to the following Duhamel formula

$$\widehat{f}(t,k,\xi) = \widehat{f}_0(k, \xi + kt) - \rho_0 \int_0^t (\xi + k(t-\tau)) \cdot k\ \widehat{\phi}(\tau,k)\ \widehat{M}(\xi + k(t-\tau))\,\mathrm{d}\tau. \tag{13}$$

We turn to the expression of the Fourier coefficients of the potential. We remind the reader that we can split the potential into

$$\phi = \phi_I + \phi_S,$$

where ϕ_I depends only on (ψ_0, ψ_1) as follows

$$\phi_I(t,x) = \frac{1}{(2\pi)^n} \iint_{\mathbb{R}^n \times \mathbb{X}^d} \sigma_1(x-y) \underbrace{\left(\widehat{\psi_0}(y,\zeta)\cos(c|\zeta|t) + \widehat{\psi_1}(y,\zeta)\frac{\sin(c|\zeta|t)}{c|\zeta|}\right)}_{=\widehat{\psi}_I(t,y,\zeta)} \widehat{\sigma_2}(\zeta)\ \mathrm{d}y\,\mathrm{d}\zeta \tag{14}$$

and the coupling term reads

$$\phi_S(t,x) = -\int_0^t p_c(t-\tau)\Sigma \star \varrho(\tau,x)\,\mathrm{d}\tau.$$

Plugging the expression of $\phi = \phi_I + \phi_S$ into (13), we obtain

$$
\begin{aligned}
\widehat{f}(t,k,\xi) &= \widehat{f}_0(k,\xi+kt) - \rho_0 \int_0^t (\xi + k(t-\tau))\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{M}(\xi+k(t-\tau))\,\mathrm{d}\tau \\
&\quad +\rho_0|\widehat{\sigma_1}(k)|^2 \int_0^t (\xi+k(t-\tau))\cdot k\ \left(\int_0^\tau p_c(\tau-\varsigma)\widehat{\varrho}(\varsigma,k)\,\mathrm{d}\varsigma\right)\ \widehat{M}(\xi+k(t-\tau))\,\mathrm{d}\tau \\
&= \widehat{f}_0(k,\xi+kt) - \rho_0 \int_0^t (\xi + k(t-\tau))\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{M}(\xi+k(t-\tau))\,\mathrm{d}\tau \\
&\quad +\rho_0|\widehat{\sigma_1}(k)|^2 \int_0^t \widehat{\varrho}(\varsigma,k)\left(\int_\varsigma^t \widehat{M}(\xi+k(t-\tau))(\xi+k(t-\tau))\cdot k\ p_c(\tau-\varsigma)\,\mathrm{d}\tau\right)\mathrm{d}\varsigma \\
&= \widehat{f}_0(k,\xi+kt) - \rho_0 \int_0^t (\xi + k(t-\tau))\cdot k\ \widehat{\phi}_I(\tau,k)\ \widehat{M}(\xi+k(t-\tau))\,\mathrm{d}\tau \\
&\quad +\rho_0|\widehat{\sigma_1}(k)|^2 \int_0^t \widehat{\varrho}(\varsigma,k)\left(\int_0^{t-\varsigma} \widehat{M}(\xi+k(t-\varsigma-\tau))(\xi+k(t-\varsigma-\tau))\cdot k\ p_c(\tau)\,\mathrm{d}\tau\right)\mathrm{d}\varsigma.
\end{aligned}
$$

We are led to an integral equation for the (Fourier coefficients of) the macroscopic density by considering this relation for $\xi = 0$. Let us set

$$
a(t,k) = \widehat{f}_0(k,tk) - \rho_0|k|^2 \int_0^t \widehat{\phi}_I(\tau,k)\ (t-\tau)\widehat{M}(k(t-\tau))\,\mathrm{d}\tau \tag{15}
$$

and

$$
\mathscr{K}(t,k) = \rho_0|k|^2\ |\widehat{\sigma_1}(k)|^2 \int_0^t (t-\tau)\widehat{M}(k(t-\tau))p_c(\tau)\,\mathrm{d}\tau. \tag{16}
$$

Then, we obtain an integral equation for the fluctuation of the macroscopic density

$$
\widehat{\varrho}(t,k) = a(t,k) + \int_0^t \mathscr{K}(t-\varsigma,k)\widehat{\varrho}(\varsigma,k)\,\mathrm{d}\varsigma. \tag{17}
$$

The analysis of this relation makes use of the Laplace transform

$$
\varphi : (0,\infty)\to\mathbb{C}, \qquad \mathscr{L}\varphi(\omega) = \int_0^\infty e^{-\omega t}\varphi(t)\,\mathrm{d}t \text{ for } \omega\in\mathbb{C},
$$

which is well defined for $\mathrm{Re}(\omega)$ large enough. We wish to apply directly the following claim [34, Lemma 3.5], see also [28, Lemma 3.6].

Lemma 3.1 *Let $a, \mathscr{K} : (0,\infty)\to\mathbb{C}$. We suppose that*

i) there exists $\alpha,\lambda>0$ such that, for any $t\geq 0$, $|a(t)|\leq \alpha e^{-\lambda t}$;

ii) there exists C_0,λ_0 such that, for any $t\geq 0$, $|\mathscr{K}(t)|\leq C_0 e^{-\lambda_0 t}$;

iii) there exists $\Lambda>0$ such that $\mathscr{L}\mathscr{K}(\omega)\neq 1$ for any $\omega\in\mathbb{C}$ verifying $\mathrm{Re}(\omega)\geq-\Lambda$.

Let φ satisfy

$$
\varphi(t) = a(t) + \int_0^t \mathscr{K}(t-\tau)\varphi(\tau)\,\mathrm{d}\tau.
$$

Then, for any $\lambda' < \min(\lambda,\lambda_0,\Lambda)$, there exists $C'>0$ such that, for any $t\in(0,\infty)$, we have

$$
|\varphi(t)|\leq C' e^{-\lambda' t}.
$$

Condition iii) gives rise to a stability criterion on the stationary profile $\mathscr{M}$. Since the operator $\mathscr{K}$ involves the kernel p_c the detailed condition substantially differs from the usual Vlasov case. In Eq. (17), the Fourier index k appears as a parameter. For applying Lemma 3.1 in order to establish the exponential decay of the potential, the time variable will be replaced by $|k|t$ and estimates i-iii) should be satisfied uniformly with respect to k, see [28, Theorem 3.1 & Lemma 3.6] (and the constant C' in the final estimate might depend on k). This requires appropriate regularity and decay assumptions on the equilibrium function, on the initial data (12) and on the coefficients.

According to [28] and [6], it is convenient to work in the *analytic* setting, which amounts to introduce the following assumptions on the equilibrium M, the initial data f_0 and the form function σ_1. Compared to the standard Vlasov equation, the model involves an additional term associated to the initial perturbation of the wave equation; for the linearized problem it appears as a new contribution in the term $a(t,k)$ of the Volterra equation (17). We thus also need to specify the assumption of ψ_0, ψ_1. The requirements on the data state as follows:

(H3) we have $\mathrm{supp}(\psi_0, \psi_1) \subset \mathbb{X}^d \times B(0, R_I)$, for some $0 < R_I < \infty$, and

$$\sup_{k\in\mathbb{X}^{\star d}} \left\{ \int_{\mathbb{R}^n} \left(|\widehat{\psi}_1(k,z)|^2 + c^2|\nabla_z \widehat{\psi}_0(k,z)|^2\right) \mathrm{d}z \right\} = \mathscr{E}_I < \infty,$$

(**R1**) there exists $C_0, \lambda_0 > 0$ such that for any $\xi \in \mathbb{R}^d$, $k \in \mathbb{X}^{\star d}$ we have

$$|\widehat{M}(\xi)| \leq C_0 e^{-\lambda_0|\xi|}, \qquad |\widehat{f}_0(k,\xi)| \leq C_0 e^{-\lambda_0|\xi|},$$

(**R2**) the function $\sigma_1 : \mathbb{X}^d \to (0,\infty)$ is radially symmetric and real analytic, and in particular (see [34, Proposition 3.16]) there exists $C_1, \lambda_1 > 0$ such that, for any $k \in \mathbb{X}^{\star d}$, $|\widehat{\sigma}_1(k)| \leq C_1 e^{-\lambda_1|k|}$.

Namely, we assume analytic regularity on the data with (**R1**) and (**R2**). Note that (**R2**) is not a strong restriction in the present context, contrarily to what it could be for the Vlasov case, since for this model σ_1 is naturally smooth. In fact, physically the form function σ_1 would naturally be compactly supported (the support being interpreted as the "domain of influence" of the particle), which does not make sense in the analytic framework. Thus, we should here think σ_1 as a peaked bump function. We also bear in mind the fact that σ_1 is radially symmetric: its Fourier coefficients are real and we have $\widehat{\sigma_1 \star \sigma_1}(k) = |\widehat{\sigma_1}(k)|^2 \geq 0$. These assumptions, together with the finite speed of propagation for the wave equation, allow us to control the "initial data" contribution in (15) and the kernel (16). Let us explain the role of **(H3)** for the associated contribution to (14) in (15). In (14), ψ_I is the solution of the wave equation on $\mathbb{R}^n$, starting form initial data (ψ_0, ψ_1). The space variable $x \in \mathbb{X}^d$ appears only as a parameter in this equation. Assumption **(H3)** means that the Fourier transform (with respect to the parameter), of the initial data has finite and uniformly bounded energy. When $\mathbb{X}^d = \mathbb{T}^d$, **(H3)** holds under the condition

$$\iint_{\mathbb{X}^d\times\mathbb{R}^n} \left(|\psi_1(x,z)|^2 + c^2|\nabla_z \psi_0(x,z)|^2\right) \mathrm{d}z \ \mathrm{d}x = \mathscr{E}_I < \infty,$$

which implies that the Fourier coefficients of the energy lies in $\ell^2(\mathbb{Z}^d)$, and thus in $\ell^\infty(\mathbb{Z}^d)$. This assumption is quite natural since this quantity is involved in the global energy balance for (2a)–(2d), see [9, 10, 33]. Working in $\mathbb{R}^d$, this has to be replaced by condition **(H3)**.

Lemma 3.2 *Assume **(H1)**–**(H3)** and* (**R1**)–(**R2**). *Let $a(t,k)$ be defined by* (15). *Then, there exists $\alpha, \lambda > 0$ such that $|a(t,k)| \leq \alpha e^{-\lambda|k|t}$ holds for any $t \geq 0$, $k \in \mathbb{X}^{\star d}$.*

Proof. Assumption (**R1**) implies that

$$|\widehat{f}_0(k,tk)| \leq C_0 e^{-\lambda_0 t|k|}.$$

Relation (14) can be recast as

$$\phi_I(t,x) = \int_{\mathbb{X}^d} \sigma_1(x-y) \left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_I(t,x,z)\,\mathrm{d}z \right) \mathrm{d}y$$

with ψ_I the solution of the free wave equation

$$\begin{array}{l} (\partial^2_{tt} - c^2\Delta_z)\psi_I = 0, \\ (\psi_I, \partial_t\psi_I)\big|_{t=0} = (\psi_0, \psi_1). \end{array}$$

Assumption **(H1)** & **(H2)** allow us to make use of Huygens' principle which tells us that

$$\mathrm{supp}(\psi_I(t,x,\cdot)) \subset \{z \in \mathbb{R}^n,\ ct - R_I \leq |z| \leq ct + R_I\}.$$

This can be read directly on the representation formula, see e. g. [15, Section 2.4, Theorem 2]

$$\begin{aligned} \psi_I(t,x,z) \quad = \quad & \frac{1}{\gamma_n}\partial_t \left(\frac{1}{t}\partial_t\right)^{(n-3)/2} \left(t^{n-2} \fint_{|z-z'|=ct} \psi_0(x,z')\,\mathrm{d}S(z') \right) \\ & + \frac{1}{\gamma_n} \left(\frac{1}{t}\partial_t\right)^{(n-3)/2} \left(t^{n-2} \fint_{|z-z'|=ct} \psi_1(x,z')\,\mathrm{d}S(z') \right) \end{aligned}$$

where $\gamma_n = 1\times 3 \times \ldots \times (n-2)$. Therefore, by virtue of **(H2)**, the product $\sigma_2(z)\psi_I(t,x,z)$ vanishes when $t \geq \frac{R_I+R_2}{c} = S_0$, see Fig. 1, for any $x \in \mathbb{X}^d$, $z \in \mathbb{R}^n$. Hence, ϕ_I is supported in $[0,S_0] \times \mathbb{X}^d$ and we can write, for $t \geq S_0$,

$$\begin{aligned} \left| \rho_0|k|^2 \int_0^t \widehat{\phi}_I(s,k)\ (t-s)\widehat{M}\big(k(t-s)\big)\,\mathrm{d}s \right| &= \left| \rho_0 \int_0^{S_0} k\widehat{\phi}_I(s,k)\cdot k(t-s)\widehat{M}\big(k(t-s)\big)\,\mathrm{d}s \right| \\ &\leq \rho_0 \int_0^{S_0} |k\widehat{\phi}_I(s,k)|\ \big|k(t-s)\widehat{M}\big(k(t-s)\big)\big|\,\mathrm{d}s. \end{aligned}$$

Assuming (**R1**), for any $0 < \lambda < \lambda_0$, we obtain

$$\begin{aligned} \int_0^{S_0} |k|(t-s)|\widehat{M}(k(t-s))|\,\mathrm{d}s \quad &\leq \quad C_0 \int_0^{S_0} |k|(t-s)e^{-\lambda_0|k|(t-s)}\,\mathrm{d}s \\ &\leq \quad C_0 \int_0^{S_0} |k|(t-s)e^{-(\lambda_0-\lambda)|k|(t-s)}e^{-\lambda|k|(t-s)}\,\mathrm{d}s \\ &\leq \quad \frac{e^{\lambda|k|S_0}}{(\lambda_0-\lambda)e}\ \frac{e^{-\lambda|k|t}}{\lambda}, \end{aligned}$$

where we have used the elementary inequality $ue^{-\lambda u} \le \frac{1}{\lambda e}$, which holds for any $\lambda > 0$, $u \ge 0$. Next, we observe that

$$\widehat{\phi}_I(s,k) = \widehat{\sigma_1}(k) \times \int_{\mathbb{R}^n} \sigma_2(z)\widehat{\psi}_I(s,k,z)\,\mathrm{d}z,$$

where $\widehat{\psi}_I(s,k,z)$ satisfies

$$\begin{array}{l}(\partial^2_{tt} - c^2\Delta_z)\widehat{\psi}_I = 0,\\ (\widehat{\psi}_I, \partial_t\widehat{\psi}_I)(0,k,z) = (\widehat{\psi_0}(k,z), \widehat{\psi_1}(k,z)).\end{array}$$

Standard energy conservation for the wave equation yields, for any $t \ge 0$,

$$\int_{\mathbb{R}^n} \Big(|\partial_t\widehat{\psi}_I|^2 + c^2|\nabla_z\widehat{\psi}_I|^2\Big)(t,k,z)\,\mathrm{d}z = \int_{\mathbb{R}^n} \Big(|\widehat{\psi_1}|^2 + c^2|\nabla_z\widehat{\psi_0}|^2\Big)(k,z)\,\mathrm{d}z \le \mathscr{E},$$

by using **(H3)**. It follows that (mind the conditions **(H1)**–**(H3)** which allow us to make use of Sobolev's embedding, see [2, Lemma 4.4] for similar reasoning)

$$\big|k\widehat{\phi}_I(s,k)\big| = |k|\Big|\widehat{\sigma_1}(k)\int_{\mathbb{R}^n} \sigma_2(z)\widehat{\psi}_I(s,k,z)\,\mathrm{d}z\Big| \le C_1|k|e^{-\lambda_1|k|}\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}},$$

where we have used (**R2**). With $\lambda_1 > \lambda' > 0$ we get

$$\big|k\widehat{\phi}_I(s,k)\big| \le C_1|k|e^{-(\lambda_1-\lambda')|k|}e^{-\lambda'|k|}\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}} \le \frac{C_1}{e(\lambda_1-\lambda')}\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}}e^{-\lambda'|k|}.$$

Gathering these estimates together, we arrive at

$$\begin{aligned}&\left|\rho_0|k|^2\int_0^t \widehat{\phi}_I(s,k)\ (t-s)\widehat{M}\big(k(t-s)\big)\,\mathrm{d}s\right|\\ &\qquad\le C_1\rho_0\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}} \times \frac{e^{(\lambda S_0-\lambda')|k|}}{(\lambda_1-\lambda')\lambda(\lambda_0-\lambda)e^2}e^{-\lambda|k|t}.\end{aligned}$$

We use this relation with $\lambda S_0 < \lambda' < \lambda_1$. We conclude that $a(t,k)$ is dominated by $\mathscr{O}(e^{-\lambda|k|t})$, uniformly with respect to k, for $0 < \lambda < \min(\lambda_0, \lambda_1/S_0)$. (Note that S_0 behaves like $1/c$; as c becomes large, only λ_0 is relevant in this condition.) ∎

Next, with (**R1**), (**R2**) and Lemma 2.1, we can estimate as follows

$$\begin{aligned}|\mathscr{K}(t,k)| &= \rho_0|k\widehat{\sigma_1}(k)|^2\Big|\int_0^{2R_2/c}(t-\tau)\widehat{M}\big(k(t-\tau)\big)p_c(\tau)\,\mathrm{d}\tau\Big|\\ &\le \rho_0|k\widehat{\sigma_1}(k)|^2\int_0^{2R_2/c}\frac{\tau}{(2\pi)^n}\|\widehat{\sigma}_2\|^2_{L^2}(t-\tau)C_0e^{-\lambda_0(t-\tau)|k|}\,\mathrm{d}\tau\\ &\le \frac{2\rho_0R_2}{c}\|\sigma_2\|^2_{L^2}\ C_0C_1^2|k|e^{2(R_2\lambda/c-\lambda_1)|k|}\ \frac{e^{-\lambda|k|t}}{\lambda(\lambda_0-\lambda)e}\end{aligned}$$

for $0 < \lambda < \lambda_0$. We conclude that $\mathscr{K}(t,k)$ is dominated by $\mathscr{O}(e^{-\lambda|k|t})$, uniformly with respect to k, provided $0 < \lambda < \min\big(\lambda_0, \frac{c\lambda_1}{R_2}\big)$.

Lemma 3.3 *Assume **(H1)**–**(H3)** and* (**R1**)–(**R2**). *Let $\mathscr{K}(t,k)$ be defined by* (16). *Then, there exists $C, \lambda > 0$ such that $|\mathscr{K}(t,k)| \leq Ce^{-\lambda|k|t}$ holds for any $t \geq 0$, $k \in \mathbb{X}^{\star d}$.*

We have justified that properties i) and ii) in Lemma 3.1 hold. We turn to investigate the Laplace transform of $\mathscr{K}$. It reads

$$\mathscr{L}\mathscr{K}(\omega,k) = \rho_0|\widehat{\sigma_1}(k)|^2\ \mathscr{L}p_c(\omega)\mathscr{L}(|k|^2 t\widehat{M}(kt))(\omega).$$

A detailed expression will be discussed in Section B below. The stability condition in Lemma 3.1-iii) should take into account the dependence with respect to the frequency: in view of the estimates in Lemma 3.2 and Lemma 3.3, we expect the decay of all modes $\widehat{\rho}(t,k)$, $k \neq 0$, with an exponential rate proportional to $|k|$. To this end, it remains to check the "(**L**)-condition" in [28]: it amounts to find $\kappa, \Lambda > 0$, such that

$$\inf_{k\in\mathbb{X}^{\star d}\setminus\{0\}} \big|1 - \mathscr{L}\mathscr{K}(\omega|k|,k)\big| \geq \kappa \quad \text{for } 0 \geq \mathrm{Re}(\omega) \geq -\Lambda. \tag{L}$$

In fact, for the Vlasov equation, such a property holds under a smallness assumption, see [28, Condition (a) in Proposition 2.1]. Here, this condition can be rephrased by means of a condition on the wave speed $c \gg 1$. The latter confirms the intuition that the damping is related to the ability to evacuate the particles energy through the membranes, see [8]. (It also raises the issue to determine whether or not there exist stable equilibrium for $c \ll 1$; we shall go back to this issue in Proposition B.11 and Remark B.13.) A similar smallness condition on $1/c$ appears in the asymptotic statements for a single particle [8, Theorem 2, 3 & 4], for the analysis of the relaxation to equilibrium for the Vlasov-Wave-Fokker-Planck model [2, Theorem 2.3], and the stability analysis in [10]. Moreover, as mentioned in the Introduction, up to a suitable c-dependent rescaling of the coupling, the regime $c \to \infty$ leads to the usual Vlasov system [8]; we check accordingly that the stability criterion for large c's is consistent to the condition exhibited for the Vlasov equation, see Remark B.10. A forthcoming work investigates on numerical grounds the role of the wave speed c on the damping phenomenom [18].

Proposition 3.4 (Stability criterion for large c's) *Assume **(H1)**–**(H2)** and* (**R2**). *There exists $c_0 > 0$ such that if $c > c_0$ then condition* (**L**) *is fulfilled.*

Proof. Let $\Lambda \in (0, \lambda_0)$. Let $\omega = \alpha + i\beta$, with $-\lambda_0 < -\Lambda \leq \alpha \leq 0$, and $\beta \in \mathbb{R}$. On the one hand, we have, for $k \neq 0$,

$$\begin{aligned}
|\mathscr{L}(|k|^2 t\widehat{M}(kt))(\omega|k|)| &= \Big|\int_0^\infty s\widehat{M}\Big(\frac{k}{|k|}s\Big)e^{-\omega s}\,\mathrm{d}s\Big| \\
&\leq C_0\int_0^\infty se^{-\lambda_0 s}e^{-\alpha s}\,\mathrm{d}s \leq C_0\int_0^\infty se^{-(\lambda_0-\Lambda)s}\,\mathrm{d}s \\
&\leq \frac{C_0}{(\lambda_0-\Lambda)^2}.
\end{aligned}$$

On the other hand, Lemma 2.1 allows us to estimate as follows

$$\big|\mathscr{L}p_c(\omega|k|)\big| \leq \|p_c\|_{L^\infty}\int_0^{2R_2/c} e^{-\alpha|k|s}\,\mathrm{d}s \leq C_S\frac{\|\sigma_2\|_{L^{2n/(n+2)}}\|\sigma_2\|_{L^2}}{\Lambda}\ \frac{e^{2\Lambda|k|R_2/c}}{c}.$$

Owing to (**R2**), we obtain

$$|\widehat{\sigma_1}(k)|^2|\mathscr{L}p_c(\omega|k|)| \leq C_1^2 C_S \frac{\|\sigma_2\|_{L^{2n/(n+2)}}\|\sigma_2\|_{L^2}}{\Lambda}\ \frac{e^{-2(\lambda_1-\Lambda R_2/c)|k|}}{c}.$$

We observe that the right hand side tends to 0 as $c\to\infty$. Therefore, for any $\kappa \in (0,1)$, provided c is large enough, we have

$$\sup_{k\neq 0}|\mathscr{L}\mathscr{K}(\omega|k|,k)| \leq 1-\kappa$$

for any $\omega \in \mathbb{C}$ with $-\lambda_0 < -\Lambda \leq \mathrm{Re}(\omega) \leq 0$, which implies $\inf_{k\neq 0}|\mathscr{L}\mathscr{K}(\omega|k|,k)-1| \geq \kappa > 0$. This is exactly condition (**L**) in [28]. It allows us to apply the reasoning as in [28, Theorem 3.1], which will thus imply the Landau damping for c large enough. ■

3.2 Linearized Landau damping: main statements

Let us collect here various statements that will be discussed for the linearized Landau damping, depending on whether $\mathbb{X}^d = \mathbb{T}^d$ (confined case) or $\mathbb{X}^d = \mathbb{R}^d$ (dispersive case), and on the decay/regularity assumptions made on the data.

Proposition 3.5 (Linearized Landau damping on $\mathbb{T}^d$ with analytic regularity) *Let $\mathbb{X}^d = \mathbb{T}^d$. Let us assume **(H1)**–**(H3)**,* (**R1**)–(**R2**) *and* (**L**)*. Then as $t\to+\infty$, the solution of the linearized problem* (11a)–(11d) *with data* (12) *converges weakly to the mean value $f_\infty(v) = \langle f_0(\cdot,v)\rangle_{\mathbb{T}^d}$ while $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$ converges strongly to $\rho_\infty = \iint_{\mathbb{T}^d\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x$. To be more specific, we can find $0<\mu<\min(\lambda_0,\lambda,\Lambda)$ such that*

for any $(k,\xi)\in\mathbb{Z}^d\times\mathbb{R}^d$, there exists $C>0$ (independent of k,ξ) verifying

$$|\widehat{f}(t,k,\xi)-\widehat{f}_\infty(k,\xi)| \leq Ce^{-\mu|\xi+kt|},$$

for any $r\in\mathbb{N}$, there exists $M_r>0$ verifying $\|\varrho(t,\cdot)-\rho_\infty\|_{C^r} \leq M_r\ e^{-\mu t}$.

By virtue of Proposition 3.4 the damping holds provided c is large enough. Note that f_∞ depends on the velocity variable only. Therefore, we have $\widehat{f}_\infty(k,\xi)=0$ for any $k\neq 0$. It can be natural to assume that the initial perturbation f_0 does not provide additional mass to the system; in this case $\rho_\infty = 0$ and the macroscopic mass fluctuation $\varrho(t,\cdot)$ tends to 0 exponentially fast. This statement also implies that the applied force tends to 0 as t goes to infinity, which is the essence of Landau damping. Indeed, we have seen that ϕ_I is compactly supported with respect to the time variable while for large times, ϕ_S casts as

$$\phi_S(t,x) = -\int_{t-2R_2/c}^{t} p_c(t-s)\Sigma\star\varrho(s)\,\mathrm{d}s.$$

The bounds on p_c and on $\varrho(t,x)$ allow us to conclude. The corresponding force

$$\begin{aligned}
\nabla_x\phi_S(t,x) &= -\int_{\mathbb{T}^d}\nabla\Sigma(x-y)\left(\int_0^t p_c(\tau)\varrho(t-\tau,y)\,\mathrm{d}\tau\right)\mathrm{d}y\\
&= -\int_{\mathbb{T}^d}\nabla\Sigma(x-y)\left(\int_0^t p_c(\tau)\left[\varrho(t-\tau,y)-\rho_\infty\right]\mathrm{d}\tau\right)\mathrm{d}y
\end{aligned}$$

then satisfies

$$\begin{aligned}\|\nabla_x \phi_S(t)\|_{C^r} &\leq \int_{\mathbb{T}^d} |\nabla\Sigma(x-y)| \left(\int_0^t |p_c(\tau)|\, \|\varrho(t-\tau)-\rho_\infty\|_{C^r}\, \mathrm{d}\tau\right) \mathrm{d}y \\ &\leq \|\nabla\Sigma\|_{L^1} \int_0^t |p_c(\tau)| M_r e^{-\mu(t-\tau)}\, \mathrm{d}\tau \\ &\leq M_r \|\nabla\Sigma\|_{L^1} \left(\int_0^{2R_2/c} |p_c(\tau)| e^{+\mu\tau}\, \mathrm{d}\tau\right) e^{-\mu t}.\end{aligned}$$

We can equally state that the shifted distribution $g(t,x,v) = f(t,x+tv,v)$ converges strongly while $f(t,x,v)$ converges only weakly.

Proposition 3.6 (Linearized Landau damping on $\mathbb{R}^d$ with analytic regularity) *Let $\mathbb{X}^d = \mathbb{R}^d$. Let us assume **(H1)–(H3)**, (**R1**)–(**R2**) and (**L**). Then as $t \to +\infty$, the Fourier transform of the solution f of the linearized problem (11a)–(11d) with data (12) converges almost everywhere to 0, while $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$ converges strongly to 0. To be more specific, we can find $0 < \mu < \min(\lambda_0, \lambda, \Lambda)$ such that*

for any $(k,\xi) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$, there exists $C > 0$ (independent of k,ξ) verifying
$$|\widehat{f}(t,k,\xi)| \leq C e^{-\mu|\xi+kt|},$$
for any $r \in \mathbb{N}$, there exists $M_r > 0$ verifying $\|\varrho(t,\cdot)\|_{C^r} \leq \frac{M_r}{1+t^{d/2}}$.

The decay of the macroscopic density to 0 holds, even for an initial fluctuation that brings some mass in the system; this is a dispersion mechanism which also governs the decay rate.

We can modify the assumptions on the data. In particular, the analyticity condition can be relaxed into a polynomial decay, up to a suitable adaptation of Lemma 3.1. Namely, we can replace (**R1**) by

(**R1′**) there exists $C_0 > 0$ such that for any $\xi \in \mathbb{R}^d$, $k \in \mathbb{X}^{\star d}$ we have, for some $p > 2$,

$$|\widehat{M}(\xi)| \leq \frac{C_0}{(1+|\xi|^2)^{p/2}}, \qquad |\widehat{f}_0(k,\xi)| \leq \frac{C_0}{(1+|\xi|^2)^{p/2}}.$$

It is also possible to relax the condition on σ_1 which does not need to be an analytic function in this framework (for instance it can be assumed to be Schwartz' class; further relaxation can be especially interesting when $(\psi_0, \psi_1) = 0$). In this framework, we slightly modify the stability condition; which now states as follows

$$\inf_{k \in \mathbb{X}^{\star d}\setminus\{0\}} |\mathscr{L}\mathscr{K}(\omega,k) - 1| \geq \kappa > 0 \text{ for any } \omega \in i\mathbb{R}. \qquad (\mathbf{L'})$$

Note that with (**R1′**), $\mathscr{L}\mathscr{K}(\omega,k)$ is well-defined for $\mathrm{Re}(\omega) \geq 0$, but it does not make sense *a priori* for $\mathrm{Re}(\omega) < 0$ contrarily to what happened in the analytic framework. In

Appendix A, we will unify conditions ($\mathbf{L}$) and ($\mathbf{L'}$) and explain that the statements for analytic data applies with ($\mathbf{L'}$) replacing ($\mathbf{L}$). We warn the reader that the following result in finite regularity on $\mathbb{T}^d$ applies only to the *linearized* problem. The non linear Landau damping on $\mathbb{T}^d$ requires to work within the analytic framework, due to the echoes phenomena that cannot be controlled by the dispersive effect of the transport operator, see [4] for further hints in this direction.

Proposition 3.7 (Linearized Landau damping on $\mathbb{T}^d$ with finite regularity) *Let $\mathbb{X}^d = \mathbb{T}^d$. Let us assume **(H1)–(H3)**,* ($\mathbf{R1'}$)–($\mathbf{R2}$) *and* ($\mathbf{L'}$). *Then as $t \to +\infty$, the solution of the linearized problem* (11a)–(11d) *with data* (12) *converges weakly to the mean value $f_\infty(v) = \langle f_0(\cdot, v)\rangle_{\mathbb{T}^d}$ while $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$ converges strongly to $\rho_\infty = \iint_{\mathbb{T}^d\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x$. To be more specific,*

for any $(k,\xi) \in \mathbb{T}^d \times \mathbb{R}^d$, there exists $C > 0$ verifying

$$|\widehat{f}(t,k,\xi) - \widehat{f}_\infty(k,\xi)| \le C(1 + |\xi|^2 + t^2|k|^2)^{-(p-3)/2},$$

for any $r \in [0, p-4-d/2]$, there exists $M_r > 0$ verifying

$$\|\varrho(t,\cdot) - \rho_\infty\|_{H^r} \le M_r(1+t^2)^{-(p-3)/2+r/2+(d+1)/4}.$$

Proposition 3.8 (Linearized Landau damping on $\mathbb{R}^d$ with finite regularity) *Let $\mathbb{X}^d = \mathbb{R}^d$. Let us assume **(H1)–(H3)**,* ($\mathbf{R1'}$)–($\mathbf{R2}$) *and* ($\mathbf{L'}$). *Then as $t \to +\infty$, the solution of the linearized problem* (11a)–(11d) *with data* (12) *converges weakly to* 0 *while $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$ converges strongly to* 0. *To be more specific,*

for any $(k,\xi) \in \mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d$, there exists $C > 0$ verifying

$$|\widehat{f}(t,k,\xi)| \le C(1 + |\xi|^2 + t^2|k|^2)^{-(p-3)/2},$$

for any $r \in [0, p-4-d/2]$, there exists $M_r > 0$ verifying

$$\|\varrho(t,\cdot)\|_{H^r} \le M_r(1+t^2)^{-d/4}.$$

As said above, it is possible to significantly relax the regularity assumption ($\mathbf{R2}$) on σ_1, for instance just assuming that σ_1 is Schwartz' class or with a sufficiently large Sobolev regularity (see Appendix A). We shall see that the analysis of the *non linear* equation in $\mathbb{R}^d$ requires a restriction on the space dimension; namely the non linear Landau damping occurs when $d \ge 3$. As far as we are concerned with the *linearized* problem, there is no such restriction on d. The statements on finite regularity can be completed by the strong convergence to 0 of the force field and the strong convergence of the shifted distribution $g(t,x,v) = f(t, x+tv, v)$. We shall state in the forthcoming Section a different formulation of the linearized damping in finite regularity, in a fashion similar to [7, Proposition 2.2], which will be convenient to study the non linear problem, see Proposition 4.14.

We collect in Appendix A the detailed analysis of the Volterra equation (17), paying attention to bring out the differences with the standard Vlasov case where the potential is defined by a mere space-convolution. We also discuss in Appendix B a Penrose-like stability criterion.

4 Analysis of the Landau damping on $\mathbb{R}^d$

We shall see that the damping in $\mathbb{R}^d$ occurs with a restriction on the space dimension: we should assume $d \geq 3$. As in [7], the analysis in the whole space relies on dispersive phenomena attached to the free transport operator; these effects are indeed strong enough to dominate the plasma echoes when $d \geq 3$.

4.1 Functional framework

We shall make use of Sobolev-type spaces. To this end, let us introduce a few notation. For $x \in \mathbb{R}^m$, $m \in \mathbb{N} \setminus \{0\}$, we denote

$$\langle x \rangle = (1 + |x|^2)^{1/2},$$

which is the weight involved in the definition of Sobolev spaces:

$$H^s(\mathbb{R}^m) = \Big\{ u : \mathbb{R}^m \to \mathbb{R},\ \int_{\mathbb{R}^m} \langle x \rangle^{2s} |\widehat{u}(x)|^2 \, \mathrm{d}x \Big\}.$$

Given x and y in $\mathbb{R}^d$, x, y stands for the vector in $\mathbb{R}^{2d}$ that results from the concatenation of x and y. Consequently, we can set

$$\langle x, y \rangle = (1 + |x|^2 + |y|^2)^{1/2}.$$

With $\alpha = (\alpha_1, \dots \alpha_d) \in \mathbb{N}^d$, we introduce the differential operator

$$\mathrm{D}^\alpha_\xi = (-i\partial^{\alpha_1}_{\xi_1}) \cdots (-i\partial^{\alpha_d}_{\xi_d}).$$

For $s \geq 0$, H^s stands for the standard Sobolev space. We shall make use of the norms introduced in [7]. We deal with functions $f : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, and for $P \in \mathbb{N}$, $s \geq 0$, we denote

$$\|f(t)\|^2_{H^s_P} = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq P}} \|(x, v) \mapsto v^\alpha f(t, x, v)\|^2_{H^s} = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq P}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle k, \xi \rangle^{2s} \left| \mathrm{D}^\alpha_\xi \widehat{f}(t, k, \xi) \right|^2 \, \mathrm{d}k \, \mathrm{d}\xi. \tag{18}$$

It is also convenient to consider

$$\begin{aligned} \|\langle t\nabla_x, \nabla_v \rangle f(t)\|^2_{H^s_P} &= \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq P}} \|(x, v) \mapsto \langle t\nabla_x, \nabla_v \rangle v^\alpha f(t, x, v)\|^2_{H^s} \\ &= \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \leq P}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle tk, \xi \rangle^2 \langle k, \xi \rangle^{2s} \left| \mathrm{D}^\alpha_\xi \widehat{f}(t, k, \xi) \right|^2 \, \mathrm{d}k \, \mathrm{d}\xi \end{aligned}$$

(there is a slight abuse of notation here since the right hand side is actually *equivalent* to the definition of $\|\langle t\nabla_x,\nabla_v\rangle f(t)\|^2_{H^s_P}$ based on (18)) and

$$\begin{aligned}\Big\| |\nabla_x|^\delta f(t)\Big\|^2_{H^s_P} &= \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \|(x,v)\mapsto |\nabla_x|^\delta v^\alpha f(t,x,v)\|^2_{H^s}\\ &= \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} |k|^{2\delta}\langle k,\xi\rangle^{2s}\left|D^\alpha_\xi \widehat{f}(t,k,\xi)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi.\end{aligned}$$

We shall also use L^∞-type estimate on Fourier transforms; we set

$$\left\|\widehat{\langle\nabla_{x,v}\rangle^s f}\right\|_{L^\infty_{(t)}L^\infty_{(k,\xi)}} = \sup_{t\in[0,T]}\left(\sup_{k,\xi\in\mathbb{R}^d}\left\{\langle k,\xi\rangle^s\left|\widehat{f}(t,k,\xi)\right|\right\}\right).$$

For a function $(t,x)\in(0,\infty)\times\mathbb{R}^d\mapsto\varrho(t,x)\in\mathbb{R}$ we introduce the modified Sobolev norm

$$\int_{\mathbb{R}^d}|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\,\mathrm{d}k = \|A_s(t)\widehat{\varrho}(t)\|_{L^2_{(k)}},$$

where we have set

$$A_s(t,k)=|k|^{1/2}\langle k,tk\rangle^s,$$

and we shall also use

$$\|A_s\widehat{\varrho}\|_{L^2_{(k,t)}} = \int_0^T\int_{\mathbb{R}^d}|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\,\mathrm{d}k\,\mathrm{d}t,$$

and

$$\|A_s\widehat{\varrho}\|_{L^\infty_{(k)}L^2_{(t)}} = \sup_{k\in\mathbb{R}^d}\left(\int_0^T|k|\langle k,tk\rangle^{2s}|\widehat{\varrho}(t,k)|^2\right)^{1/2}.$$

The norms defined on the macroscopic density ϱ equally apply to the kinetic quantity g, replacing $\widehat{\varrho}(t,k)$ by $\widehat{g}(t,k,tk)$.

In what follows, we shall use the notation $A\lesssim B$, meaning that we can find a constant $C>0$ such that $A\le CB$. Here, A, B are in general functions of time, space, velocity, or their associated Fourier variables; it is thus understood that C is uniform over these variables.

We go back to the formulation (9). Compared to the usual Vlasov equation, the expression of the potential $\Phi[\psi]$ now involves the contribution of the initial data $\mathscr{F}_I$, and the self-consistent part $\mathscr{G}_\varrho$ presents a memory effect, through the kernel p_c. It is convenient to think of the problem with some generality on these quantities. Thus, let us collect the hypothesis on the data of the problem: $\mathscr{F}_I$, p_c and σ_1. It is not obvious to translate these assumptions on the original data σ_2, ψ_0, ψ_1... Nevertheless, it can be checked that these assumptions are satisfied in the specific cases where **(H1)**–**(H3)** and (**R1**)–(**R2**) hold. (For instance we remind the reader that **(H1)**–**(H3)** imply that ϕ_I, and thus $\mathscr{F}_I$, has a compact support with respect to the time variable, by virtue of Huygens' principle.)

(D1) $(t,x)\mapsto \mathscr{F}_I(t,x)$ decays faster than polynomially with respect to the time variable, in norms $L^2(\mathbb{R}_+;L^1(\mathbb{R}^d_x))$ and $L^\infty(\mathbb{R}_+;L^1(\mathbb{R}^d_x))$: for any $\alpha\geq 0$, we have

$$\int_0^{+\infty}\langle t\rangle^\alpha\|\mathscr{F}_I(t)\|^2_{L^1(\,\mathrm{d}x)}\,\mathrm{d}t<+\infty \;\text{ and }\; \sup_{t\in\mathbb{R}_+}\langle t\rangle^\alpha\|\mathscr{F}_I(t)\|_{L^1(\,\mathrm{d}x)}<+\infty.$$

(D2) $t\mapsto p_c(t)$ decays faster than polynomially: for any $\alpha\geq 0$, we have

$$\int_0^{+\infty}\langle t\rangle^\alpha|p_c(t)|\,\mathrm{d}t<+\infty.$$

(H4) $\sigma_1\in\mathscr{S}(\mathbb{R}^d)$: for any $\alpha\geq 0$ we have

$$\lim_{|k|\to+\infty}\langle k\rangle^\alpha|\widehat{\sigma}_1(k)|=0.$$

Remark 4.1 *Some results are strengthened by replacing **(D1)** by the stronger assumption*

(D1') *$\mathscr{F}_I$ is compactly supported with respect to time: there exists $S_0>0$ such that for any $|t|\geq S_0$ and $x\in\mathbb{R}^d$, we have $\mathscr{F}_I(t,x)=0$.*

*In particular, as observed in the proof of Lemma 3.2, **(D1')** holds when **(H1)**–**(H3)** are fulfilled.*

This formulation of the hypothesis has the advantage of pushing the generality of the result, both on the "linear" perturbation due to the data through $\mathscr{F}_I$ and on the memory effects in the self-consistent potential through p_c. The following claim is crucial for our purposes: roughly speaking, it explains why the situation is not very different from the Vlasov case, once the role of p_c well understood, and it justifies that the approach of [7] is robust enough to be adapted. Note that **(D1)** is the assumption that makes the constants $C_1(\mathscr{F}_I)$ and $C_2(\mathscr{F}_I)$ below meaningful.

Proposition 4.2 *Let **(D1)**, **(D2)** and **(H4)** be fulfilled. Then for any $0<T<\infty$ and any $s\geq 0$ the following three estimates hold*

$$\left\|A_s\widehat{\sigma}_1\left(\widehat{\mathscr{F}_I}-\widehat{\sigma}_1\widehat{\mathscr{G}_\varrho}\right)\right\|^2_{L^2_{(t)}L^2_{(k)}}\lesssim C_1(\mathscr{F}_I)+\|A_s\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}}, \tag{19a}$$

$$\left\|A_s\widehat{\sigma}_1\left(\widehat{\mathscr{F}_I}-\widehat{\sigma}_1\widehat{\mathscr{G}_\varrho}\right)\right\|^2_{L^\infty_{(k)}L^2_{(t)}}\lesssim C_1(\mathscr{F}_I)+\|A_s\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}}, \tag{19b}$$

$$\begin{aligned}\sup_{t\in[0,T]}\sup_{k\in\mathbb{R}^d}\langle k,tk\rangle^s|\widehat{\sigma}_1(k)|\left|\widehat{\mathscr{F}_I}(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}_\varrho}(t,k)\right|&\\ \lesssim C_2(\mathscr{F}_I)+\sup_{t\in[0,T]}\sup_{k\in\mathbb{R}^d}\langle k,tk\rangle^s\left|\widehat{\varrho}(t,k)\right|,&\end{aligned} \tag{19c}$$

with

$$C_1(\mathscr{F}_I)=\int_0^{+\infty}\langle t\rangle^{2s}\|\mathscr{F}_I(t)\|^2_{L^1(\,\mathrm{d}x)}\,\mathrm{d}t\;\text{ and }\;C_2(\mathscr{F}_I)=\sup_{t\in\mathbb{R}_+}\langle t\rangle^s\|\mathscr{F}_I(t)\|_{L^1(\,\mathrm{d}x)}.$$

Remark 4.3 *We shall use the following variant of the statement : for any polynomial $k \mapsto P(k)$, we have*

$$\left\| P A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^2_{(t)} L^2_{(k)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^2_{(t)} L^2_k} \tag{20}$$

$$\left\| P A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^\infty_{(k)} L^2_{(t)}} \lesssim C_1(\mathscr{F}_I) + \|A_s \widehat{\varrho}\|^2_{L^\infty_k L^2_{(t)}} \tag{21}$$

$$\begin{aligned} \sup_{t \in [0,T]} \sup_{k \in \mathbb{R}^d} \langle k, tk \rangle^s P(k) |\widehat{\sigma}_1(k)| \left| \widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k) \widehat{\mathscr{G}_\varrho}(t,k) \right| \\ \lesssim C_2(\mathscr{F}_I) + \sup_{t \in [0,T]} \sup_{k \in \mathbb{R}^d} \langle t \rangle^{-\alpha} \langle k, tk \rangle^s \left| \widehat{\varrho}(t,k) \right|. \end{aligned} \tag{22}$$

These estimates can be justified since σ_1 lies in the Schwartz class and thus $P(k)\widehat{\sigma}_1(k)$ remains a function with fast decay.

Proof. In order to prove (19a), we analyse separately the contribution from $\widehat{\mathscr{F}_I}$ and $\widehat{\mathscr{G}_\varrho}$ as follows

$$\begin{aligned} &\left\| A_s \widehat{\sigma}_1 \left(\widehat{\mathscr{F}_I} - \widehat{\sigma}_1 \widehat{\mathscr{G}_\varrho} \right) \right\|^2_{L^2_{(t)} L^2_k} \\ &\lesssim \underbrace{\int_0^T \int_{\mathbb{R}^d_k} |k| \langle k, tk \rangle^{2s} |\widehat{\sigma}_1(k)|^2 |\widehat{\mathscr{F}_I}(t,k)|^2 \,\mathrm{d}k \,\mathrm{d}t}_{=\mathrm{I}} + \underbrace{\int_0^T \int_{\mathbb{R}^d_k} |k| \langle k, tk \rangle^{2s} |\widehat{\sigma}_1(k)|^4 |\widehat{\mathscr{G}_\varrho}(t,k)|^2 \,\mathrm{d}k \,\mathrm{d}t}_{=\mathrm{II}}. \end{aligned}$$

For I, by using $\langle k, tk \rangle^2 \le \langle k \rangle^2 \langle t \rangle^2$, we readily obtain

$$\mathrm{I} \le \left(\int_{\mathbb{R}^d_k} |k| \langle k \rangle^{2s} |\widehat{\sigma}_1(k)|^2 \,\mathrm{d}k \right) \left(\int_0^{+\infty} \langle t \rangle^{2s} \|\mathscr{F}_I(t)\|^2_{L^1(\,\mathrm{d}x)} \,\mathrm{d}t \right).$$

For II we start by applying Cauchy-Schwarz' inequality

$$|\widehat{\mathscr{G}_\varrho}(t,k)|^2 = \left| \int_0^t p_c(t-\tau) \varrho(\tau,k) \,\mathrm{d}\tau \right|^2 \le \left(\int_0^t |p_c(t-\tau)| \,\mathrm{d}\tau \right) \left(\int_0^t |p_c(t-\tau)| |\widehat{\varrho}(\tau,k)|^2 \,\mathrm{d}\tau \right).$$

Going back to II, we are led to

$$\mathrm{II} \le \|p_c\|_{L^1} \int_0^T \int_0^t |p_c(t-\tau)| \left(\int_{\mathbb{R}^d_k} |k| \langle k, \tau k \rangle^{2s} \frac{\langle k, tk \rangle^{2s}}{\langle k, \tau k \rangle^{2s}} |\widehat{\sigma}_1(k)|^4 |\widehat{\varrho}(t,k)|^2 \,\mathrm{d}k \right) \mathrm{d}\tau \ \mathrm{d}t.$$

A simple study of function shows that (for $t \ge \tau$)

$$\sup_{k \in \mathbb{R}^d} \frac{\langle k, tk \rangle^{2s}}{\langle k, \tau k \rangle^{2s}} \le \frac{\langle t \rangle^{2s}}{\langle \tau \rangle^{2s}}.$$

Since $|\widehat{\sigma}_1(k)| \le \|\sigma_1\|_{L^1} \lesssim 1$, and using Fubini's theorem, we obtain

$$\begin{aligned} \mathrm{II} &\lesssim \|p_c\|_{L^1} \int_0^T \left(\int_\tau^T |p_c(t-\tau)| \frac{\langle t \rangle^{2s}}{\langle \tau \rangle^{2s}} \|A_s \widehat{\varrho}(\tau)\|^2_{L^2_{(k)}} \,\mathrm{d}t \right) \mathrm{d}\tau \\ &\lesssim \|p_c\|_{L^1} \int_0^T \|A_s \widehat{\varrho}(\tau)\|^2_{L^2_{(k)}} \left(\int_0^{T-\tau} |p_c(u)| \frac{\langle u+\tau \rangle^{2s}}{\langle \tau \rangle^{2s}} \,\mathrm{d}u \right) \mathrm{d}\tau. \end{aligned}$$

Since $\langle u+\tau\rangle^{2s} \lesssim \langle u\rangle^{2s}\langle\tau\rangle^{2s}$, we arrive at

$$\mathrm{II} \lesssim \|p_c\|_{L^1}\left(\int_0^{+\infty}\langle u\rangle^{2s}|p_c(u)|\,\mathrm{d}u\right)\|A_s\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}}.$$

It ends the proof of (19a).

Estimate (19b) follows the same strategy: for $k\in\mathbb{R}^d$, we split as follows

$$\int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^2\left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2\,\mathrm{d}t$$

$$\underbrace{\leq \int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^2|\widehat{\mathscr{F}}_I(t,k)|^2\,\mathrm{d}t}_{=\mathrm{J}} + \underbrace{\int_0^T |k|\langle k,tk\rangle^{2s}|\widehat{\sigma}_1(k)|^4|\widehat{\mathscr{G}}_\varrho(t,k)|^2\,\mathrm{d}t}_{=\mathrm{JJ}}.$$

Proceeding as above, we obtain

$$\mathrm{J}\leq\left(\sup_{k\in\mathbb{R}^d}|k|\langle k\rangle^{2s}|\widehat{\sigma}_1(k)|^2\right)\left(\int_0^{+\infty}\langle t\rangle^{2s}\|F_I(t)\|^2_{L^1(\,\mathrm{d}x)}\,\mathrm{d}t\right)$$

and

$$\begin{aligned}\mathrm{JJ} &\lesssim \|p_c\|_{L^1}\int_0^T\left(\int_\tau^T|p_c(t-\tau)|\frac{\langle t\rangle^{2s}}{\langle\tau\rangle^{2s}}|k|\langle k,\tau k\rangle^{2s}|\widehat{\varrho}(\tau,k)|^2\,\mathrm{d}t\right)\,\mathrm{d}\tau\\ &\lesssim \|p_c\|_{L^1}\left(\int_0^{+\infty}\langle u\rangle^{2s}|p_c(u)|\,\mathrm{d}u\right)\left(\int_0^T|k|\langle k,\tau k\rangle^{2s}|\widehat{\varrho}(\tau,k)|^2\,\mathrm{d}\tau\right).\end{aligned}$$

We proceed with a slightly different approach for (19c) when dealing with the contribution involving $\widehat{\mathscr{G}}_\varrho$. For any $t\in[0,T]$ and $k\in\mathbb{R}^d$, we write

$$\begin{aligned}&\langle k,tk\rangle^s|\widehat{\sigma}_1(k)|\left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|\\ &\qquad\lesssim\left(\sup_{k\in\mathbb{R}^d}\langle k\rangle^s|\widehat{\sigma}_1(k)|\right)\left(\sup_{t\in[0,T]}\langle t\rangle^s\|\mathscr{F}_I(t)\|_{L^1(\,\mathrm{d}x)}\right)+\langle k,tk\rangle^s|\widehat{\mathscr{G}}_\varrho(t,k)|.\end{aligned}$$

Since

$$\begin{aligned}\langle k,tk\rangle^s|\widehat{\mathscr{G}}_\varrho(t,k)| &\leq \int_0^t|p_c(t-\tau)|\frac{\langle k,tk\rangle^s}{\langle k,\tau k\rangle^s}\langle k,\tau k\rangle^s|\widehat{\varrho}(\tau,k)|\,\mathrm{d}\tau\\ &\lesssim \left(\int_0^t|p_c(t-\tau)|\frac{\langle t\rangle^s}{\langle\tau\rangle^s}\,\mathrm{d}\tau\right)\left(\sup_{\tau\in[0,T]}\sup_{k\in\mathbb{R}^d}\langle k,\tau k\rangle^s|\widehat{\varrho}(\tau,k)|\right),\end{aligned}$$

we are left with the task of showing the finiteness of the integral that involves p_c:

$$\begin{aligned}\int_0^t |p_c(t-\tau)| \frac{\langle t\rangle^s}{\langle \tau\rangle^s}\,\mathrm{d}\tau &= \langle t\rangle^s \int_0^t |p_c(u)| \frac{1}{\langle t-u\rangle^s}\,\mathrm{d}u \\ &= \langle t\rangle^s \int_0^{t/2} |p_c(u)| \frac{1}{\langle t-u\rangle^s}\,\mathrm{d}u + \langle t\rangle^s \int_{t/2}^t |p_c(u)| \frac{1}{\langle t-u\rangle^s}\,\mathrm{d}u \\ &\lesssim \langle t\rangle^s \int_0^{t/2} |p_c(u)| \frac{1}{\langle t/2\rangle^s}\,\mathrm{d}u + \frac{\langle t\rangle^s}{\langle t/2\rangle^s} \int_{t/2}^t \langle t/2\rangle^s |p_c(u)|\,\mathrm{d}u \\ &\lesssim \int_0^{t/2} |p_c(u)|\,\mathrm{d}u + \int_{t/2}^t \langle u\rangle^s |p_c(u)|\,\mathrm{d}u \\ &\lesssim \|p_c\|_{L^1} + \int_0^{+\infty} \langle u\rangle^s |p_c(u)|\,\mathrm{d}u \lesssim 1,\end{aligned}$$

by virtue of **(D2)**. ■

Let us now collect a few technical results, more or less extracted from [7], which will be useful for the proof of the Landau damping.

Lemma 4.4 (Trace Lemma) *Let $f \in H^s(\mathbb{R}^d)$ with $s > \frac{d-1}{2}$. Let $\mathscr{C} \subset \mathbb{R}^d$ be a submanifold with dimension larger or equal to 1. We have*

$$\|f\|_{L^2(\mathscr{C})} \lesssim \|f\|_{H^s}.$$

This claim, which will be further used in the sequel, allows us to obtain the following estimates.

Lemma 4.5 *Let f_0 be in H^s_P with $P > d/2$. Then,*

1. *we have*

$$\int_0^T \int_{\mathbb{R}^d} |A_s \widehat{f}_0(k, tk)|^2\,\mathrm{d}k\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} |k| \langle k, tk\rangle^{2s} |\widehat{f}_0(k, tk)|^2\,\mathrm{d}k\,\mathrm{d}t \lesssim \|f_0\|^2_{H^s_P}. \tag{23}$$

2. *if, moreover, $(x, v) \mapsto x^\alpha f_0(x, v) \in H^s_P$, for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \le P$, we have*

$$\sup_{k,\xi} \Big(\langle k, \xi\rangle^s |\widehat{f}(k, \xi)|\Big) \lesssim \sum_{|\alpha|\le P} \|x^\alpha f_0(x, v)\|_{H^s_P}. \tag{24}$$

3. *if, moreover $(x, v) \mapsto x^\alpha f_0(x, v) \in H^{s+1}_P$ for any $\alpha \in \mathbb{N}^d$ with $|\alpha| \le P$, we have*

$$\sup_{k\in\mathbb{R}^d} \int_0^T |k| \langle k, tk\rangle^{2s} |\widehat{f}_0(k, tk)|^2\,\mathrm{d}t \lesssim \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le P}} \|x^\alpha f_0(x, v)\|_{H^{s+1}_P}. \tag{25}$$

Proof. Since $f_0 \in H^s_P$, we have

$$(k,\xi) \longmapsto \langle k,\xi\rangle^s \widehat{f}_0(k,\xi) \in L^2_{(k)} H^P_{(\xi)}.$$

Indeed,

$$\begin{aligned}
\left|\mathrm{D}^\alpha_\xi \left(\xi \mapsto \langle k,\xi\rangle^s \widehat{f}_0(k,\xi)\right)\right| &= \Big|\sum_{\substack{j\in\mathbb{N}^d\\ j\le\alpha}} \binom{\alpha}{j} \mathrm{D}^{\alpha-j}_\xi \left(\xi \mapsto \langle k,\xi\rangle^s\right) \mathrm{D}^j_\xi \widehat{f}_0(k,\xi)\Big| \\
&\lesssim \sum_{\substack{j\in\mathbb{N}^d\\ j\le\alpha}} \langle k,\xi\rangle^s \left|\mathrm{D}^j_\xi \widehat{f}_0(k,\xi)\right|
\end{aligned}$$

yields

$$\underbrace{\left\|(k,\xi)\mapsto\langle k,\xi\rangle^s\widehat{f}_0(k,\xi)\right\|^2_{L^2_{(k)}H^P_{(\xi)}}}_{=\int_{\mathbb{R}^d_k}\|\xi\mapsto\langle k,\xi\rangle^s\widehat{f}_0(k,\xi)\|^2_{H^P_{(\xi)}}\,\mathrm{d}k} \lesssim \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le P}} \int_{\mathbb{R}^d_k\times\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2s} \left|\mathrm{D}^\alpha_\xi \widehat{f}_0(k,\xi)\right|^2 \mathrm{d}k\,\mathrm{d}\xi = \|f_0\|^2_{H^s_P}. \tag{26}$$

Next, we observe that

$$\begin{aligned}
\int_0^T \|A_s\widehat{f}_0(\cdot,t\cdot)\|^2_{L^2_{(k)}}\,\mathrm{d}t &= \int_{\mathbb{R}^d_k}\left(\int_0^T \langle k,tk\rangle^{2s}|\widehat{f}_0(k,tk)|^2|k|\,\mathrm{d}t\right)\mathrm{d}k \\
&= \int_{\mathbb{R}^d_k}\left(\int_0^{|k|T} \langle k,uk/|k|\rangle^{2s}|\widehat{f}_0(k,uk/|k|)|^2\,\mathrm{d}u\right)\mathrm{d}k \\
&\le \int_{\mathbb{R}^d_k}\left(\sup_{\omega\in\mathbb{S}^{d-1}}\int_{-\infty}^{+\infty} \langle k,u\omega\rangle^{2s}|f_0(k,u\omega)|^2\,\mathrm{d}u\right)\mathrm{d}k.
\end{aligned}$$

Therefore coming back to (26), with $P > d/2$, we deduce that

$$\|\xi\mapsto\langle k,\xi\rangle^s\widehat{f}_0(k,\xi)\|^2_{H^P_{(\xi)}}$$

is finite for almost every $k\in\mathbb{R}^d$. We can apply the Trace Lemma 4.4 for almost every $k\in\mathbb{R}^d$, which leads to

$$\int_{-\infty}^{+\infty}\langle k,u\omega\rangle^{2s}|\widehat{f}_0(k,u\omega)|^2\,\mathrm{d}u \lesssim \|\xi\mapsto\langle k,\xi\rangle^s\widehat{f}_0(k,\xi)\|^2_{H^P_{(\xi)}}.$$

(Note that the constant in the estimate of the Trace Lemma 4.4 only depends on the submanifold $\mathscr{C}$, and the estimate does not involve the parameter k.) Integrating over k we conclude that

$$\int_0^T \|A_s\widehat{f}_0(\cdot,t\cdot)\|^2_{L^2_{(k)}}\,\mathrm{d}t \lesssim \|f_0\|^2_{H^s_P}.$$

For the second estimate, we remark that $(x,v)\mapsto x^\alpha f_0(x,v)\in H^s_P$ implies that $\langle k,\xi\rangle^{s+1}\widehat{f_0}(k,\xi)$ lies in $H^P_{(k)}H^P_{(\xi)}$, which embeds into the space of continuous functions; the third estimate then follows immediately, see [7, Lemma 2.6]. ■

The following statement will be repeatedly used for proving Proposition 4.10, see [7, Lemma 2.9].

Lemma 4.6 *Let g_1 et g_2 be in $L^2(\mathbb{R}^d_k \times \mathbb{R}^d_\xi)$ and let $r \in L^1(\mathbb{R}^d_n)$. Then, we have*

$$\left| \int_{\mathbb{R}^{3d}_{k,\xi,n}} g_1(k,\xi) r(n) g_2(k-n, \xi - tn) \,\mathrm{d}n \,\mathrm{d}k \,\mathrm{d}\xi \right| \lesssim \|g_1\|_{L^2_{(k,\xi)}} \|g_2\|_{L^2_{(k,\xi)}} \|r\|_{L^1_{(n)}}. \tag{27}$$

Let $g_1 \in L^2(\mathbb{R}^d_k \times \mathbb{R}^d_\xi)$, $g_2 \in L^1(\mathbb{R}^d_k; L^2(\mathbb{R}^d_\xi))$ and $r \in L^2(\mathbb{R}^d_n)$. Then, we have

$$\left| \int_{\mathbb{R}^{3d}_{k,\xi,n}} g_1(k,\xi) r(n) g_2(k-n, \xi - tn) \,\mathrm{d}n \,\mathrm{d}k \,\mathrm{d}\xi \right| \lesssim \|g_1\|_{L^2_{(k,\xi)}} \|g_2\|_{L^1(\mathbb{R}^d_k; L^2(\mathbb{R}^d_\xi))} \|r\|_{L^2_{(n)}}. \tag{28}$$

We now state an existence-uniqueness result for the Cauchy problem (10a)–(10b), in the functional spaces of interest. Again we refer the reader to [7] for a similar result for the screened Vlasov equation.

Proposition 4.7 *Let $P > d/2$ be an integer. Let $f_0 \in H^s_P$ with $s > d/2+1$. Then, there exists $T^\star > 0$ such that, for any $0 < T < T^\star$, the problem (10a)–(10b) admits a unique solution $g \in C^0([0,T]; H^s_P)$ on $[0,T]$. Moreover, if for some $T \le T^\star$ there exists $s \ge s'$ such that*

$$\limsup_{t \to T} \|g(t)\|_{H^{s'}_P} < +\infty,$$

then, actually, $T < T^\star$.

The analysis of the Landau Damping, as it is already clear for the linearized problem, relies heavily on the formulation of the problem by means of the Fourier variables. Let us collect the useful formula from which the reasoning starts. Integrating (10a)–(10b) over $[0,t]$, we get

$$g(t,x,v) = f_0(x,v) + \int_0^t \nabla_x \sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)(\tau, x + \tau v) \cdot (\nabla_v - \tau \nabla_x)(\mathscr{M}(v) + g(\tau, x, v)) \,\mathrm{d}\tau.$$

We check that

$$\int_{\mathbb{R}^{2d}} u(x+\tau v, v) e^{-ik\cdot x} e^{-i\xi\cdot v} \,\mathrm{d}v \,\mathrm{d}x = \int_{\mathbb{R}^{2d}} u(y,v) e^{-ik\cdot y} e^{-i(\xi - \tau k)\cdot v} \,\mathrm{d}v \,\mathrm{d}x = \widehat{u}(k, \xi - \tau k).$$

We also bear in mind that $\widehat{\mathbf{1}(v)}(\xi) = \delta(\xi = 0)$ and $\widehat{\mathbf{1}(x)}(k) = \delta(k = 0)$. We thus obtain

$$\begin{aligned}
&\widehat{g}(t,k,\xi) \\
&= \widehat{f}_0(k,\xi) \\
&\quad - \int_0^t \int_{\mathbb{R}^{2d}} n \widehat{\sigma_1}(n) (\widehat{\mathscr{F}_I} - \widehat{\sigma_1} \widehat{\mathscr{G}_\varrho})(\tau, n) \delta(\zeta = \tau n) \cdot (\xi - \zeta) \widehat{\mathscr{M}}(\xi - \zeta) \delta(n = k) \,\mathrm{d}n \,\mathrm{d}\zeta \,\mathrm{d}\tau \\
&\quad - \int_0^t \int_{\mathbb{R}^{2d}} n \widehat{\sigma_1}(n) (\widehat{\mathscr{F}_I} - \widehat{\sigma_1} \widehat{\mathscr{G}_\varrho})(\tau, n) \delta(\zeta = \tau n) \cdot (\xi - \zeta - \tau(k-n)) \widehat{g}(\tau, k-n, \xi - \zeta) \,\mathrm{d}n \,\mathrm{d}\zeta \,\mathrm{d}\tau \\
&= \widehat{f}_0(k,\xi) - \int_0^t k \widehat{\sigma_1}(k) (\widehat{\mathscr{F}_I} - \widehat{\sigma_1} \widehat{\mathscr{G}_\varrho})(\tau, \tau k) \cdot (\xi - \tau k) \widehat{\mathscr{M}}(\xi - \tau k) \,\mathrm{d}\tau \\
&\qquad - \int_0^t \int_{\mathbb{R}^d} n \widehat{\sigma_1}(n) (\widehat{\mathscr{F}_I} - \widehat{\sigma_1} \widehat{\mathscr{G}_\varrho})(\tau, n) \cdot (\xi - \tau k) \widehat{g}(\tau, k-n, \xi - \tau n) \,\mathrm{d}n \,\mathrm{d}\tau.
\end{aligned} \tag{29}$$

Eventually, the macroscopic density is evaluated by

$$\begin{aligned}\widehat{\varrho}(t,k) &= \int_{\mathbb{R}^{2d}} f(t,x,v)e^{-ik\cdot x}\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^{2d}} g(t,x-tv,v)e^{-ik\cdot x}\,\mathrm{d}v\,\mathrm{d}x\\ &= \int_{\mathbb{R}^{2d}} g(t,y,v)e^{-ik\cdot y}e^{-itk\cdot v}\,\mathrm{d}v\,\mathrm{d}y = \widehat{g}(t,k,tk).\end{aligned}$$

Going back to (29) with $\xi = tk$, we arrive at

$$\begin{aligned}\widehat{\varrho}(t,k) &= \widehat{f}_0(k,tk)\\ &= -\int_0^t k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,\tau k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau\\ &\quad -\int_0^t\int_{\mathbb{R}^d} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\cdot((t-\tau)k)\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}n\,\mathrm{d}\tau.\end{aligned} \tag{30}$$

4.2 Main result

We are ready now to state the main result about the non linear Landau damping. As said above, the proof makes the constraint $d \geq 3$ on the space dimension appear.

Theorem 4.8 (Landau damping in $\mathbb{R}^d$) *Let $d \geq 3$. Suppose **(D1)**, **(D2)**, **(H4)**. There exists universal constants ε_0, $R_0 > 0$ and $r \in (0, R_0)$ such that if $s > R_0$,*

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \|x^\alpha f_0\|^2_{H^s_P} \leq \varepsilon_0^2 \qquad \int_0^{+\infty} \langle t\rangle^{2s}\|\mathscr{F}_I(t)\|^2_{L^1(\mathrm{d}x)}\,\mathrm{d}t \leq \varepsilon_0^2, \quad \sup_{t\in\mathbb{R}_+}\langle t\rangle^s\|\mathscr{F}_I(t)\|_{L^1(\mathrm{d}x)} \leq \varepsilon_0,$$

and $\mathscr{M} \in H^{\tilde{s}}_P(\mathbb{R}^d_v)$ with $P > d/2$ and $\tilde{s} \geq s + 2d$ satisfies $(\mathbf{L})$, then, the unique solution g of (10a)–(10b) is globally defined. Moreover, there exists $g^\infty \in H^r_P$ such that

$$\begin{aligned}\|g(t) - g^\infty\|^2_{H^\sigma_P} &\lesssim \varepsilon_0^2\langle t\rangle^{-d-1+\eta_0} \quad \text{for } 0 \leq \sigma \leq r, &(31)\\ |\widehat{g}(t,k,tk)| &\lesssim \varepsilon_0\langle k,tk\rangle^{-(r+d+2)}\langle t\rangle^{\eta_0} &(32)\\ \|\langle\nabla_x\rangle^\sigma\nabla\sigma_1\star(\mathscr{F}_I(t) - \sigma_1\star\mathscr{G}_g(t))\|_{L^\infty(\mathrm{d}x)} &\lesssim \varepsilon_0\langle t\rangle^{-d-1+\eta_0} \quad \text{for } \sigma \geq 0 &(33)\end{aligned}$$

holds where $\eta_0 > 0$ stands for a arbitrarily small positive number (but the constants might blow up as $\eta_0 \to 0$).

Remark 4.9 *With **(D1')** the statement holds with $\eta_0 = 0$. Estimate (33) holds because σ_1 is assumed to be in the Schwartz class; this assumption can be relaxed at the price of introducing constraints on the regularity exponent σ.*

The proof of the Landau Damping in fact relies on a bootstrap estimate, see [7, Proposition 2.5], which states as follows.

Proposition 4.10 (Bootstrap) *Let the hypothesis of Theorem 4.8 be fulfilled. Let $0 < \eta < 1$ and $0 < \delta < 1/2$. There exists real numbers $2(d+1)+1 < s_1 < s_2 < s_3 <$*

$s_4 < s$ and $K_1, ..., K_5 \geq 1$ such that, for any $g \in C^0([0,T], H^s_P)$ solution of (10a)–(10b) *on the time interval $[0,T]$ verifying*

$$\begin{aligned}
\|\langle t\nabla_x, \nabla_v\rangle g(t)\|^2_{H^{s_4}_P} &\leq 4K_1\varepsilon^2\langle t\rangle^5, &(34)\\
\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}} &\leq 4K_2\varepsilon^2, &(35)\\
\||\nabla_x|^\delta g(t)\|^2_{H^{s_3}_P} &\leq 4K_3\varepsilon^2, &(36)\\
\|A_{s_2}\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}} &\leq 4K_4\varepsilon^2\langle T\rangle^\eta, &(37)\\
\|\widehat{\langle\nabla_{x,v}\rangle^{s_1} g}(t)\|_{L^\infty_{(k,\xi)}} &\leq 4K_5\varepsilon\langle t\rangle^\eta, &(38)
\end{aligned}$$

for $0 < \varepsilon \leq \varepsilon_0$ small enough, the following estimates hold on $[0,T]$

$$\begin{aligned}
\|\langle t\nabla_x, \nabla_v\rangle g(t)\|^2_{H^{s_4}_P} &\leq 2K_1\varepsilon^2\langle t\rangle^5, &(39)\\
\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(t)}L^2_{(k)}} &\leq 2K_2\varepsilon^2, &(40)\\
\||\nabla_x|^\delta g(t)\|^2_{H^{s_3}_P} &\leq 2K_3\varepsilon^2, &(41)\\
\|A_{s_2}\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}} &\leq 2K_4\varepsilon^2\langle T\rangle^\eta, &(42)\\
\|\widehat{\langle\nabla_{x,v}\rangle^{s_1} g}(t)\|_{L^\infty_{(k,\xi)}} &\leq 2K_5\varepsilon\langle t\rangle^\eta. &(43)
\end{aligned}$$

Remark 4.11 *We shall see within the proof how the s_i's are chosen, according to some compatibility conditions. This choice determines the possible value for R_0 that arises in Theorem 4.8 as a threshold for the Sobolev regularity in which the damping is evaluated. To be specific, Proposition 4.10 holds for $s > s_4 + 2d$ and $s_i > s_{i-1} + 2d$ and in Theorem 4.8, we can set*

$$R_0 = s_4 + 2d, \qquad r = s_1 - d - 2.$$

The condition on ε_0 imposes a smallness constraint on the initial perturbation.

Remark 4.12 *The parameter $\eta > 0$ does not arise in the analysis of the Vlasov system [7]. In fact, we can prove a logarithmic growth on the solution, but the proof of the Landau damping is simpler by using the algebraic decay as stated here. Looking at the details of the proof, K_4 and K_5 blow up as η goes to 0; ε should be chosen small enough, depending on all the K_j's, and it thus shrinks as η becomes smaller. When* ***(D1')*** *holds the statement applies with $\eta = 0$.*

Remark 4.13 *It might be surprising that the half-convolution with respect to time plays a relatively weak role in this statement, compared to the Vlasov case. At first sight, we would suspect that the memory effect changes a lot the control of the force terms, or that it imposes further restrictions. In fact, the heart of the proof relies on the estimates in Proposition 4.2, and the main impact of the memory term is rather on the stability condition, where it completely modifies, in a quite intricate way, the expression of the symbol $\mathscr{L}\mathscr{K}$. This can be seen as a confirmation of the robustness of the approach designed in [28, 6, 7].*

We now explain how the Landau damping can be justified, having at hand the bootstrap statement. The arguments follow closely the analysis performed in [7]. However, we think valuable to make the discussion as self-contained as possible and not to hide any difficulty, explaining in full details how we proceed to obtain the estimates.

Proof of Landau damping. Proposition 4.7 justifies the local existence of a solution to (10a)–(10b); Proposition 4.10 tells us that the solution is in fact *globally* defined and it satisfies (39)–(43) over $[0, \infty)$. We are going to use these estimates to analyse the Landau Damping.

From this, (43) implies

$$|\widehat{\varrho}(t,k)| \lesssim \varepsilon \langle k, tk\rangle^{-s_1} \langle t\rangle^{\eta}.$$

For the force term, we shall use the general estimate, for $\sigma \geq 0$,

$$\|\langle \nabla_x\rangle^{\sigma} F(t,\cdot)\|_{L^\infty(\,\mathrm{d}x)} \leq \int_{\mathbb{R}^d} \langle k\rangle^{\sigma} |\widehat{F}(t,k)| \,\mathrm{d}k.$$

Next, we apply successively (19c) and (43); we obtain

$$\begin{aligned}
\|\langle \nabla_x\rangle^{\sigma} \nabla_x \Phi[\psi](t,\cdot)\|_{L^\infty(\,\mathrm{d}x)} &\leq \int_{\mathbb{R}^d} \langle k\rangle^{\sigma} |k| |\widehat{\sigma}_1(k)| \left| \widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k) \widehat{\mathscr{G}_\varrho}(t,k) \right| \mathrm{d}k \\
&\lesssim \int_{\mathbb{R}^d} \langle k, tk\rangle^{-s_1} |k| \varepsilon \langle t\rangle^{\eta} \,\mathrm{d}k \\
&\lesssim \varepsilon \langle t\rangle^{\eta-1} \int_{\mathbb{R}^d} \langle k, tk\rangle^{1-s_1} \,\mathrm{d}k \lesssim \varepsilon \langle t\rangle^{-d-1+\eta}
\end{aligned}$$

where we used **(H4)** to incorporate $\langle k\rangle^{\sigma}$ with $|\widehat{\sigma}_1(k)|$ and the elementary inequality $|k|\langle t\rangle \leq \langle k, tk\rangle$.

It remains to show that the behavior of $g(t,x,v)$ is driven by free transport. To this end, we are going to define g^∞ as the solution of

$$\begin{aligned}
&g^\infty(x,v) = f_0(x,v) \\
&+ \int_0^{+\infty} \nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\,(x+tv) \cdot (\nabla_v \mathscr{M}(v) + (\nabla_v - t\nabla_x) g(t,x,v)) \,\mathrm{d}t,
\end{aligned}$$

which, indeed, lies in some H^r_P. From this, we can establish the convergence of g to g^∞ in H^σ_P-norm, with $0 \leq \sigma \leq r = s_1 - d - 2$. To this aim, we go back to (29) and we get

$$\begin{aligned}
&\langle k,\xi\rangle^{\sigma} \mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi) = \langle k,\xi\rangle^{\sigma} \mathrm{D}^\alpha_\xi \widehat{f}_0(k,\xi) \\
&- \int_0^t \underbrace{\langle k,\xi\rangle^{\sigma} k \widehat{\sigma}_1(k) \left(\widehat{\mathscr{F}_I}(\tau,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}_\varrho}(\tau,k)\right) \cdot \mathrm{D}^\alpha_\xi (\xi - \tau k) \widehat{\mathscr{M}}(\xi - \tau k)}_{\mathrm{L}(\tau,k,\xi)} \,\mathrm{d}\tau \\
&- \int_0^t \underbrace{\int_{\mathbb{R}^d_n} \langle k,\xi\rangle^{\sigma} n \widehat{\sigma}_1(n) \left(\widehat{\mathscr{F}_I}(\tau,n) - \widehat{\sigma}_n \widehat{\mathscr{G}_\varrho}(\tau,n)\right) \cdot \mathrm{D}^\alpha_\xi (\xi - tk) \widehat{g}(\tau, k-n, \xi - \tau n) \,\mathrm{d}n}_{\mathrm{NL}(\tau,k,\xi)} \,\mathrm{d}\tau.
\end{aligned}$$

For the linear term, we combine (19c), (43), together with the elementary inequalities $\langle k,\xi\rangle^{2\sigma} \lesssim \langle k,\tau k\rangle^{2\sigma}\langle \xi-\tau k\rangle^{2\sigma}$ and $|k|\langle\tau\rangle \le \langle k,\tau k\rangle$; we are led to

$$\begin{aligned}
\|\mathrm{L}(\tau)\|^2_{L^2_{(k,\xi)}} \;\lesssim\;& \int_{\mathbb{R}^{2d}_{k,\xi}} \langle k,\tau k\rangle^{2\sigma}|k|^2|\widehat{\sigma}_1(k)|^2\left|\widehat{\mathscr{F}}_I(\tau,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(\tau,k)\right|^2\\
&\qquad\times\langle\xi-\tau k\rangle^{2\sigma}\left|\mathrm{D}^\alpha_\xi(\xi-\tau k)\widehat{\mathscr{M}}(\xi-\tau k)\right|^2\,\mathrm{d}k\,\mathrm{d}\xi\\
\lesssim\;& \left(\int_{\mathbb{R}^d_k}\langle k,\tau k\rangle^{2\sigma}|k|^2\langle k,\tau k\rangle^{-2s_1}\varepsilon^2\langle\tau\rangle^{2\eta}\,\mathrm{d}k\right)\left(\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^{2\sigma}\left|\widehat{\nabla_v\mathscr{M}}(\xi)\right|^2\,\mathrm{d}\xi\right)\\
\lesssim\;& \varepsilon^2\langle\tau\rangle^{-2+2\eta}\int_{\mathbb{R}^d_k}\langle k,\tau k\rangle^{2\sigma+2-2s_1}\,\mathrm{d}k \lesssim \varepsilon^2\langle\tau\rangle^{-d-2+2\eta},
\end{aligned}$$

where we used the assumption $\mathscr{M}\in H^{\tilde{s}}_P$ with $\tilde{s}>\sigma$; the last estimate holds provided $2\sigma+2-2s_1<-d$, that is $\sigma<s_1-d/2-1$.

For the non linear term, the Cauchy-Schwarz inequality, with $\langle k,\xi\rangle \le \langle n,\tau n\rangle\langle k-n,\xi-\tau n\rangle$, yields

$$\begin{aligned}
&\int_{\mathbb{R}^d_n}\langle k,\xi\rangle^\sigma|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|\left|\mathrm{D}^\alpha_\xi\widehat{\nabla_v g}(\tau,k-n,\xi-\tau n)\right|\,\mathrm{d}n\\
&\le\left(\int_{\mathbb{R}^d_n}\frac{\langle n,\tau n\rangle^\sigma}{|k-n|^{2\delta}}|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|\,\mathrm{d}n\right)^{1/2}\\
&\qquad\times\Big(\int_{\mathbb{R}^d_n}\langle n,\tau n\rangle^\sigma|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|\\
&\qquad\times|k-n|^{2\delta}\langle k-n,\xi-\tau n\rangle^{2\sigma}\left|\mathrm{D}^\alpha_\xi\widehat{\nabla_v g}(\tau,k-n,\xi-\tau n)\right|^2\,\mathrm{d}n\Big)^{1/2}.
\end{aligned}$$

Next, combining (19c), (43), (41) and $|n|\langle\tau\rangle \le \langle n,\tau n\rangle$, leads to

$$
\begin{aligned}
&\|\mathrm{NL}(\tau)\|^2_{L^2_{(k,\xi)}}\\
&\lesssim \int_{\mathbb{R}^{2d}_{k,\xi}} \left(\int_{\mathbb{R}^d_n} \frac{\langle n,\tau n\rangle^\sigma}{|k-n|^{2\delta}} |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\right)\\
&\qquad \times\Big(\int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^\sigma |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|\\
&\qquad \times |k-n|^{2\delta}\langle k-n,\xi-\tau n\rangle^{2\sigma} \left|\mathrm{D}^\alpha_\xi \widehat{\nabla_v g(t)}(k-n,\xi-\tau n)\right|^2 \,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}\xi\\
&\lesssim \left(\sup_{k\in\mathbb{R}^d} \int_{\mathbb{R}^d_n} \frac{\langle n,\tau n\rangle^\sigma}{|k-n|^{2\delta}} |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\right)\\
&\qquad \times \left(\int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^\sigma |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\right) \left\| |\nabla_x|^\delta \nabla_v g(\tau)\right\|^2_{H^\sigma_P}\\
&\lesssim \left(\sup_{k\in\mathbb{R}^d} \int_{\mathbb{R}^d_n} \frac{\langle n,\tau n\rangle^\sigma}{|k-n|^{2\delta}} |n|\langle n,\tau n\rangle^{-s_1}\varepsilon\langle\tau\rangle^\eta \,\mathrm{d}n\right)\\
&\qquad \times \left(\int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^\sigma |n|\langle n,\tau n\rangle^{-s_1}\varepsilon\langle\tau\rangle^\eta \,\mathrm{d}n\right) \left\| |\nabla_x|^\delta g(\tau)\right\|^2_{H^{s_3}_P}\\
&\lesssim \varepsilon^4\langle\tau\rangle^{2\eta-2} \left(\sup_{k\in\mathbb{R}^d} \int_{\mathbb{R}^d_n} \frac{\langle n,\tau n\rangle^{\sigma+1-s_1}}{|k-n|^{2\delta}} \,\mathrm{d}n\right) \left(\int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^{\sigma+1-s_1} \,\mathrm{d}n\right)
\end{aligned}
$$

where we have used the condition $s_3 \ge \sigma+1$. Remarking that $\langle n,\tau n\rangle^2 = 1+\langle\tau\rangle^2|n|^2 = \langle\langle\tau\rangle n\rangle^2$, a simple change of variable yields

$$
\int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^{\sigma+1-s_1}\,\mathrm{d}n = \langle\tau\rangle^{-d}\int_{\mathbb{R}^d_n} \langle n\rangle^{\sigma+1-s_1}\,\mathrm{d}n \lesssim \langle\tau\rangle^{-d}
$$

provided $\sigma+1-s_1 < -d$, that is $\sigma < s_1-d-1$. Proceeding with the same change of variable, we obtain, for any $k\in\mathbb{R}^d$,

$$
\begin{aligned}
\int_{\mathbb{R}^d_n} \frac{\langle n,\tau n\rangle^{\sigma+1-s_1}}{|k-n|^{2\delta}}\,\mathrm{d}n &= \langle\tau\rangle^{-d+2\delta}\int_{\mathbb{R}^d_n} \frac{\langle n\rangle^{\sigma+1-s_1}}{|\langle\tau\rangle k-n|^{2\delta}}\,\mathrm{d}n\\
&= \langle\tau\rangle^{-d+2\delta}\left(\int_{B(\langle\tau\rangle k,1)} + \int_{\complement B(\langle\tau\rangle k,1)} \frac{\langle n\rangle^{\sigma+1-s_1}}{|\langle\tau\rangle k-n|^{2\delta}}\,\mathrm{d}n\right)\\
&\le \langle\tau\rangle^{-d+2\delta}\left(\int_{B(0,1)} \frac{1}{|n|^{2\delta}}\,\mathrm{d}n + \int_{\mathbb{R}^d_n} \langle n,\tau n\rangle^{\sigma+1-s_1}\,\mathrm{d}n\right)\\
&\lesssim \langle\tau\rangle^{-d+2\delta},
\end{aligned}
$$

(since $\delta < d$). This is indeed bounded uniformly with respect to k. Eventually, we arrive at

$$\|\mathrm{NL}(\tau)\|^2_{L^2_{(k,\xi)}} \lesssim \varepsilon^4 \langle \tau\rangle^{-2d+2\eta-2+2\delta}.$$

The conclusion is two-fold: on the one hand, the definition of g^∞ is meaningful, and it gives an element of H^σ_P for any $0 \le \sigma \le r = s_1 - d - 2$; on the other hand, for any $\sigma \in [0, s_1 - d - 1)$, we have

$$\begin{aligned} \|g(t) - g^\infty\|^2_{H^\sigma_P} &\lesssim \varepsilon^2 \int_t^{+\infty} \langle\tau\rangle^{-d-2+2\eta}\,\mathrm{d}\tau + \varepsilon^4 \int_t^{+\infty} \langle\tau\rangle^{-2d+2\eta-2+2\delta}\,\mathrm{d}\tau \\ &\lesssim \varepsilon^2\langle t\rangle^{-d-1+2\eta+} + \varepsilon^4\langle t\rangle^{-2d-1+2\eta+2\delta+}. \end{aligned}$$

This ends the proof. ∎

The proof of the bootstrap property relies on fine estimates for the linearized problem. Let us state the linearized damping property in the functional framework adapted to our purposes, see [7, Proposition 2.2]. In Appendix A we clarify the connection between this statement and the Propositions given in Section 3.2.

Proposition 4.14 (Linearized damping on $\mathbb{R}^d$) *Let the assumptions of Theorem 4.8 be fulfilled. We consider a family of functions $\{t \in [0,T] \mapsto a(t,k),\ k \in \mathbb{R}^d\}$. We suppose that, for any $k \in \mathbb{R}^d$,*

$$\int_0^T |k|\langle k, tk\rangle^{2s}|a(t,k)|^2\,\mathrm{d}t < +\infty,$$

holds. Then, we can find a constant C_{LD} (which does not depend on k and T) such that any solution $(t,k) \mapsto \phi(t,k)$ of the system

$$\begin{aligned} \phi(t,k) &= a(t,k) + \int_0^t \mathscr{K}(t-\tau,k)\phi(\tau,k)\,\mathrm{d}\tau \\ &= a(t,k) + \int_0^t |\widehat{\sigma}_1(k)|^2|k|^2(t-\tau)\widehat{\mathscr{M}}([t-\tau]k)\left(\int_0^\tau p_c(\tau-\sigma)\phi(\sigma,k)\,\mathrm{d}\sigma\right)\mathrm{d}\tau, \end{aligned}$$

on $[0,T]$ satisfies the following estimate: for any $k \in \mathbb{R}^d$

$$\int_0^T |k|\langle k, tk\rangle^{2s}|\phi(t,k)|^2\,\mathrm{d}t \le C_{LD}\int_0^T |k|\langle k, tk\rangle^{2s}|a(t,k)|^2\,\mathrm{d}t.$$

4.3 Bootstrap analysis: proof of Proposition 4.10

As in [7], we introduce the time-response kernel

$$\bar{K}(t,\tau,k,n) = \frac{|k|^{1/2}|n|^{1/2}|k(t-\tau)|}{\langle n\rangle^2}\left|\widehat{g}(\tau, k-n, tk-\tau n)\right|.$$

The following statement is crucial to the analysis of the echo phenomena. It involves the constraint on s_1 involved in Proposition 4.10. Technically, this statement is substantially different when $\mathbb{X}^d = \mathbb{T}^d$ or when $\mathbb{X}^d = \mathbb{R}^d$. In the torus, the proof needs analytic regularity but is free of constraint on the space dimension d. For the free space problem, the argument relies on dispersion mechanisms of the transport operator which are strong enough only when $d \geq 2$; in this situation it is thus possible to work in finite regularity.

Lemma 4.15 *Let $0 < T < \infty$. Let $s_1 > 2(d+1)+1$. The following two estimates hold*

$$\sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}n\,\mathrm{d}\tau \lesssim \sup_{\tau\in[0,T]} \sup_{k,\xi\in\mathbb{R}^d} \frac{\langle k,\xi\rangle^{s_1}}{\langle \tau\rangle^{\eta}} |\widehat{g}(\tau,k,\xi)|$$

and

$$\sup_{\tau\in[0,T]} \sup_{n\in\mathbb{R}^d} \int_\tau^T \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}k\,\mathrm{d}t \lesssim \sup_{\tau\in[0,T]} \sup_{k,\xi\in\mathbb{R}^d} \frac{\langle k,\xi\rangle^{s_1}}{\langle \tau\rangle^{\eta}} |\widehat{g}(\tau,k,\xi)|\,.$$

We refer the reader to [7, Section 3] for a proof of this claim dealing with the Vlasov equation.

Remark 4.16 *The factor $1/\langle n\rangle^2$ in the kernel $\bar{K}$ comes from the convolution kernel used in [7]. Here, since σ_1 is Schwartz class, this factor can be replaced by $1/\langle n\rangle^m$ with $m \in \mathbb{N}$ as large as we wish.*

We follow closely the arguments of [7], up to the perturbation due to $\mathscr{F}_I$; as pointed out above, the half convolution with respect to time in $\mathscr{G}_\varrho$ does not substantially modify the analysis, owing to Proposition 4.2.

4.4 Estimates on ϱ

We start from the expression of $\widehat{\varrho}(t,k)$ in (30) and we apply Proposition 4.14 in order to estimate the $L^2_{(t)}$ norm of $A_{s_i}\widehat{\varrho}$ (with $i \in \{2,4\}$). We get

$$\begin{aligned}
&\|A_{s_i}\widehat{\varrho}(\cdot,k)\|^2_{L^2_{(t)}} \\
&\quad\lesssim \int_0^T |k|\langle k,tk\rangle^{2s_i}|\widehat{f}_0(k,tk)|^2\,\mathrm{d}t \\
&\qquad + \int_0^T \left|\int_0^t |k|^{1/2}\langle k,tk\rangle^{s_4} k\widehat{\sigma}_1(k)\widehat{\mathscr{F}_I}(\tau,k)\cdot[t-\tau]k\widehat{\mathscr{M}}([t-\tau]k)\,\mathrm{d}\tau\right|^2 \mathrm{d}t \\
&\qquad + \int_0^T \Big|\int_0^t \int_{\mathbb{R}^d_n} |k|^{1/2}\langle k,tk\rangle^{s_4} n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}_I}(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right) \\
&\qquad\qquad \cdot[t-\tau]k\,\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}\tau\,\mathrm{d}n\Big|^2\,\mathrm{d}t.
\end{aligned} \tag{44}$$

4.4.1 Estimate of the $L^2_{(k)}L^2_{(t)}$ norm of $A_{s_4}\widehat{\varrho}$.

Integrating (44) with respect to k yields

$$
\begin{aligned}
&\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(k)}L^2_{(t)}} \\
&\quad\lesssim \int_{\mathbb{R}^d}\int_0^T |k|\langle k,tk\rangle^{2s_4}\left|\widehat{f_0}(k,tk)\right|^2 \,\mathrm{d}k\,\mathrm{d}t \\
&\qquad + \int_{\mathbb{R}^d}\int_0^t \left|\int_0^t |k|^{1/2}\langle k,tk\rangle^{s_4} k\widehat{\sigma}_1(k)\widehat{\mathscr{F}_I}(\tau,k)\cdot(t-\tau)k\widehat{f}^0([t-\tau]k)\,\mathrm{d}\tau\right|^2 \mathrm{d}k\,\mathrm{d}t \\
&\qquad + \int_{\mathbb{R}^d}\int_0^T \Big|\int_0^t\int_{\mathbb{R}^d} |k|^{1/2}\langle k,tk\rangle^{s_4} n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}_I}(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(\tau,n)\right) \\
&\qquad\qquad \cdot(t-\tau)k\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}\tau\,\mathrm{d}n\Big|^2 \mathrm{d}k\,\mathrm{d}t.
\end{aligned}
$$

We denote the three terms in the right hand side as CT1, CT2 and NLT, respectively (for "constant term 1 and 2, non linear term"). In what follows, we are going to split the discussion according to the estimate NLT $\lesssim$ NLTT + NLTR, where NLTT (for transport) and NLTR (for reaction) stand for the contributions that arise from the following decomposition

$$
\langle k,tk\rangle^{s_4} \lesssim \langle k-n,tk-\tau n\rangle^{s_4} + \langle n,\tau n\rangle^{s_4}.
$$

Estimate on CT1**.** Owing to the fact that

$$
\sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \|(x,v)\mapsto x^\alpha f_0(x,v)\|^2_{H^s_P} \leq \varepsilon^2,
$$

Lemma 4.5 ensures

$$
\mathrm{CT1} \lesssim \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \|(x,v)\mapsto x^\alpha f_0(x,v)\|^2_{H^s_P} \leq \varepsilon^2
$$

as well.

Estimate of CT2**.** This term induces new difficulties since it does not appear in the analysis of the Vlasov equation. It is far from clear whether or not this perturbation annihilates the Landau Damping mechanisms. With the strengthened assumption **(D1')** we shall see that we can obtain the necessary estimates on both $A_{s_4}\widehat{\varrho}$ in norm $L^2_{(k)}L^2_{(t)}$ and $A_{s_2}\widehat{\varrho}$ in norm $L^\infty_{(k)}L^2_{(t)}$. With **(D1)** only, we will be able to control the $L^2_{(k)}L^2_{(t)}$ norms, provided $d\geq 3$, but a singularity remains for the $L^\infty_{(k)}L^2_{(t)}$ norm, which will thus require a specific analysis. The former estimate holds uniformly with respect to T, but the singularity in the latter yields the weight with T^η.

Let us write

$$
\mathrm{CT2} = \int_0^T\int_{\mathbb{R}^d} |I(t,k)|^2\,\mathrm{d}k\,\mathrm{d}t.
$$

Since $\mathscr{M} \in H^{\tilde{s}}_P$, we have $\xi \mapsto \langle \xi \rangle^{\tilde{s}} \widehat{\mathscr{M}}(\xi) \in H^P_{(\xi)}$, where $P > d/2$, and Sobolev's embedding yields $|\widehat{\mathscr{M}}(\xi)| \lesssim \|\widehat{\mathscr{M}}\|_{H^P} \langle \xi \rangle^{-\tilde{s}}$. By using this together with the relations $|k|\langle t \rangle \le \langle k, tk \rangle$, $\langle k, tk \rangle \lesssim \langle k, \tau k \rangle \langle [t-\tau]k \rangle$ and $\langle k, \tau k \rangle \lesssim \langle k \rangle \langle \tau \rangle$, we obtain

$$\begin{aligned}
|I(t,k)| &\le \int_0^t |k|^{1/2} \langle k, tk \rangle^{s_4} |k| |\widehat{\sigma}_1(k)| |\widehat{\mathscr{F}}_I(\tau,k)| \left| \widehat{\nabla_v \mathscr{M}}([t-\tau]k) \right| \, \mathrm{d}\tau \\
&\lesssim \langle t \rangle^{-3/2} \int_0^t \langle k, \tau k \rangle^{s_4+3/2} |\widehat{\sigma}_1(k)| |\widehat{\mathscr{F}}_I(\tau,k)| \langle [t-\tau]k \rangle^{s_4-\tilde{s}} \, \mathrm{d}\tau \\
&\lesssim \left(\sup_{\tau \ge 0} \langle \tau \rangle^{s_4+3/2} \|\mathscr{F}_I(\tau)\|_{L^1(\mathrm{d}x)} \right) \langle k \rangle^{s_4+3/2} |\widehat{\sigma}_1(k)| \langle t \rangle^{-3/2} \\
&\qquad\qquad \times \int_0^t \langle [t-\tau]k \rangle^{s_4-\tilde{s}} \, \mathrm{d}\tau \\
&\lesssim \varepsilon \frac{\langle k \rangle^{s_4+3/2}}{|k|} |\widehat{\sigma}_1(k)| \langle t \rangle^{-3/2} \int_0^{+\infty} \langle u \rangle^{s_4-\tilde{s}} \, \mathrm{d}u \\
&\lesssim \varepsilon \frac{\langle k \rangle^{s_4+3/2}}{|k|} |\widehat{\sigma}_1(k)| \langle t \rangle^{-3/2}
\end{aligned}$$

where we use $s_4 - \tilde{s} < -1$. When, $d \ge 3$, $k \mapsto \frac{1}{|k|^2}$ is locally integrable. Therefore, the singularity with $1/|k|$ does not raise any difficulty as far as we are interested in the integral of the square of I with respect to k. To be more specific, we have

$$\mathrm{CT2} = \int_0^T \int_{\mathbb{R}^d} |I(t,k)|^2 \, \mathrm{d}k \, \mathrm{d}t \lesssim \varepsilon^2 \left(\int_{\mathbb{R}^d} \frac{\langle k \rangle^{2s_4+3}}{|k|^2} |\widehat{\sigma}_1(k)|^2 \, \mathrm{d}k \right) \left(\int_0^T \langle t \rangle^{-3} \, \mathrm{d}t \right) \lesssim \varepsilon^2.$$

Of course, the quantity $I(t,k)$ enters in the derivation of the estimate of $A_{s_2} \widehat{\varrho}$ in $L^\infty_{(k)} L^2_{(t)}$ norm in (42), just changing s_4 into s_2. In contrast, the singularity then becomes an obstacle to obtain such a $L^\infty_{(k)} L^2_{(t)}$ estimate. To treat the difficulty, we can modify (from the passage from the first to the second estimate) the previous inequality into

$$|I(t,k)| \lesssim \varepsilon \langle k \rangle^{s_2+1/2} |\widehat{\sigma}_1(k)| \langle t \rangle^{-1/2},$$

which is indeed bounded on $\mathbb{R} \times \mathbb{R}^d$, but which is not square-integrable with respect to the time variable. This difficulty disappears when **(D1')** is assumed. Indeed, for $t \le S_0$, we have already seen that

$$|I(t,k)| \lesssim \varepsilon.$$

For $t > S_0$, we proceed as follows

$$\begin{aligned}
|I(t,k)| &\leq \int_0^{S_0} |k|^{1/2}\langle k, tk\rangle^{s_2}|k||\widehat{\sigma}_1(k)||\widehat{\mathscr{F}}_I(\tau,k)|\left|\widehat{\nabla_v\mathscr{M}}([t-\tau]k)\right| \,\mathrm{d}\tau \\
&\lesssim \langle t\rangle^{-3/2}\int_0^{S_0} \langle k, \tau k\rangle^{s_2+3/2}|\widehat{\sigma}_1(k)||\widehat{\mathscr{F}}_I(\tau,k)|\langle [t-\tau]k\rangle^{s_2-\tilde{s}} \,\mathrm{d}\tau \\
&\lesssim \langle t\rangle^{-3/2}\int_0^{S_0} \langle k\rangle^{s_2+3/2}\langle \tau\rangle^{s_2+3/2}|\widehat{\sigma}_1(k)|\|\widehat{\mathscr{F}}_I\|_{L^\infty_{\tau,k}} \,\mathrm{d}\tau \\
&\lesssim \langle t\rangle^{-3/2}\left(\sup_{k\in\mathbb{R}^d}\langle k\rangle^{s_2+3/2}|\widehat{\sigma}_1(k)|\right)\langle S_0\rangle^{s_2+5/2}\|\mathscr{F}_I\|_{L^\infty_{(t)}L^1_x} \lesssim \varepsilon\langle t\rangle^{-3/2}.
\end{aligned}$$

This estimate tells us that I is square integrable with respect to the time variable, and uniformly bounded with respect to k, when **(D1')** holds. We shall go back to the $L^\infty_{(k)}L^2_{(t)}$ later on.

Estimate on NLTT. As said above, having Proposition 4.2 at hand permits us to readily adapt the arguments of [7]. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathrm{NLTT} \leq &\int_{\mathbb{R}^d}\int_0^T \left(\int_0^t\int_{\mathbb{R}^d}\langle\tau\rangle^{5/2}|n||\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)|\,\mathrm{d}\tau\,\mathrm{d}n\right) \\
&\times\Big(\int_0^t\int_{\mathbb{R}^d}\langle\tau\rangle^{-5/2}|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \\
&\times|k|\langle k-n, tk-\tau n\rangle^{2s_4}|[t-\tau]k|^2|\widehat{g}(\tau,k-n,tk-\tau n)|^2\,\mathrm{d}\tau\,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}t.
\end{aligned}$$

Now, (19c) and (38) ensure that

$$\langle n,\tau n\rangle^{s_1}|\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)| \lesssim (1+K_5)\varepsilon\langle\tau\rangle^\eta.$$

Since $|n|\langle\tau\rangle \leq \langle n,\tau n\rangle$, we get

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}^d}\langle\tau\rangle^{5/2}|n||\widehat{\sigma}_1(n)||\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)|\,\mathrm{d}\tau\,\mathrm{d}n \\
&\lesssim \left(\int_0^t\langle\tau\rangle^{5/2+\eta}\int_{\mathbb{R}^d_n}|n|\langle n,\tau n\rangle^{-s_1}\,\mathrm{d}n\ \mathrm{d}\tau\right)(1+K_5)\varepsilon \\
&\lesssim \left(\int_0^{+\infty}\langle\tau\rangle^{5/2+\eta-d-1}\,\mathrm{d}\tau\right)(1+K_5)\varepsilon \lesssim (1+K_5)\varepsilon
\end{aligned}$$

where the last estimate assumes the condition $5/2+\eta-d-1 < -1$, that is $d > 5/2+\eta$. This is one of the constraints on the space dimension d which imply that the analysis applies only when $d \geq 3$. Furthermore, when $d = 3$, we see that $\eta < 1/2$ is necessary.

Going back to NLTT we are led to (by using $(|t-\tau)k|\leq \langle \tau(k-n), tk-\tau n\rangle)$

$$\begin{aligned}
\mathrm{NLTT} &\lesssim (1+K_5)\varepsilon \int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \\
&\quad \times \langle\tau\rangle^{-5}|k|\langle k-n, tk-\tau n\rangle^{2s_4}\langle \tau(k-n), tk-\tau n\rangle^2 |\widehat{g}(\tau,k-n,tk-\tau n)|^2 \,\mathrm{d}\tau\,\mathrm{d}n\Big)\,\mathrm{d}k\,\mathrm{d}t \\
&\lesssim (1+K_5)\varepsilon \int_{\mathbb{R}^d}\int_0^T \left(\int_\tau^T \int_{\mathbb{R}^d} \langle\tau\rangle^{-5}|k|\langle k-n, tk-\tau n\rangle^{2s_4}\right. \\
&\quad \times \langle \tau(k-n), tk-\tau n\rangle^2 |\widehat{g}(\tau,k-n,tk-\tau n)|^2 \,\mathrm{d}t\,\mathrm{d}k\Big) \\
&\quad \times \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\,\mathrm{d}\tau \\
&\lesssim (1+K_5)\varepsilon \Big(\sup_{0\leq\tau\leq T}\sup_{n\in\mathbb{R}^d}\langle\tau\rangle^{-5}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty}\langle k-n, tk-\tau n\rangle^{2s_4}\langle \tau(k-n), tk-\tau n\rangle^2 \\
&\qquad \times |\widehat{g}(\tau,k-n,tk-\tau n)|^2 |k|\,\mathrm{d}t\ \mathrm{d}k\Big) \\
&\quad \times \left(\int_{\mathbb{R}^d}\int_0^T \langle\tau\rangle^{+5/2}|n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \,\mathrm{d}n\,\mathrm{d}\tau\right) \\
&\lesssim (1+K_5)^2\varepsilon^2 \left(\sup_{0\leq\tau\leq T}\sup_{n\in\mathbb{R}^d}\langle\tau\rangle^{-5}\int_{\mathbb{R}^d}|k|\right. \\
&\quad \left.\int_{-\infty}^{+\infty} |\langle \tau(k-n), tk-\tau n\rangle\langle k-n, tk-\tau n\rangle^{s_4}\widehat{g}(\tau,k-n,tk-\tau n)|^2\ \mathrm{d}t\,\mathrm{d}k\right).
\end{aligned}$$

With two changes of variables and by applying the Trace Lemma 4.4 as in the proof of Proposition 4.5, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}|k|\int_{-\infty}^{+\infty} |\langle \tau(k-n), tk-\tau n\rangle\langle k-n, tk-\tau n\rangle^{s_4}\widehat{g}(\tau,k-n,tk-\tau n)|^2\ \mathrm{d}t\,\mathrm{d}k \\
&= \int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} \left|\langle \tau(k-n), t\frac{k}{|k|}-\tau n\rangle\langle k-n, t\frac{k}{|k|}-\tau n\rangle^{s_4}\widehat{g}(\tau,k-n,tk-\tau n)\right|^2\ \mathrm{d}t\,\mathrm{d}k \\
&\leq \sup_{\omega\in\mathbb{S}^{d-1}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} |\langle \tau(k-n), t\omega+x\rangle\langle k-n, t\omega+x\rangle^{s_4}\widehat{g}(\tau,k-n,t\omega+x)|^2\ \mathrm{d}t\,\mathrm{d}k \\
&\leq \sup_{\omega\in\mathbb{S}^{d-1}}\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{-\infty}^{+\infty} |\langle \tau k, t\omega+x\rangle\langle k, t\omega+x\rangle^{s_4}\widehat{g}(\tau,k-n,t\omega+x)|^2\ \mathrm{d}t\,\mathrm{d}k \\
&\lesssim \|\langle\tau\nabla_x,\nabla_v\rangle g(\tau)\|^2_{H_P^{s_4}}\,.
\end{aligned}$$

Finally, combining this with (34) we obtain

$$\mathrm{NLTT} \lesssim (1+K_5)^2 K_1 \varepsilon^4.$$

Estimate on NLTR**.** We make the time-response kernel $\bar{K}$ appear; Cauchy-Schwarz' inequality and Fubini's theorem allow us to obtain

$$\begin{aligned}
\mathrm{NLTR} = \int_{\mathbb{R}^d}\int_0^T \Bigg(\int_0^t \int_{\mathbb{R}^d} & \bar{K}(t,\tau,k,n) \langle n, \tau n\rangle^{s_4} |n|^{1/2} \langle n\rangle^2 |\widehat{\sigma}_1(n)| \\
& \times \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right| \mathrm{d}\tau\,\mathrm{d}n \Bigg)^2 \mathrm{d}k\,\mathrm{d}t \\
\lesssim \int_{\mathbb{R}^d}\int_0^T & \left(\int_0^t \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n\right) \\
\times \Bigg(\int_0^t \int_{\mathbb{R}^d} & \bar{K}(t,\tau,k,n)\langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n \Bigg) \mathrm{d}k\,\mathrm{d}t \\
\lesssim \Bigg(\sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} & \int_0^t \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n \Bigg) \\
\times \int_0^T \int_{\mathbb{R}^d} & \left(\int_\tau^T \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}t\,\mathrm{d}k\right) \langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \\
& \times \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n \\
\lesssim \Bigg(\sup_{t\in[0,T]} \sup_{k\in\mathbb{R}^d} & \int_0^t \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\,\mathrm{d}n \Bigg) \left(\sup_{\tau\in[0,T]} \sup_{n\in\mathbb{R}^d} \int_\tau^T \int_{\mathbb{R}^d} \bar{K}(t,\tau,k,n)\,\mathrm{d}t\,\mathrm{d}k \right) \\
& \times \int_0^T \int_{\mathbb{R}^d} \langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n.
\end{aligned}$$

By using (19a) and (35), we obtain

$$\int_0^T \int_{\mathbb{R}^d} \langle n,\tau n\rangle^{2s_4}|n|\langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n \lesssim (1+K_2)\varepsilon^2.$$

Gathering this with Lemma 4.15 and (38), we are led to

$$\mathrm{NLTR} \lesssim (1+K_2) K_5^2 \varepsilon^4.$$

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying (34)–(38) on $[0,T]$, then

$$\|A_{s_4}\widehat{\varrho}\|^2_{L^2_{(k)}L^2_{(t)}} \lesssim \left(1 + (1+K_5)^2 K_1 \varepsilon^2 + (1+K_2)K_5^2\varepsilon^2\right)\varepsilon^2.$$

Let us denote C_1 the constant hidden in the symbol $\lesssim$ of this estimate. Choosing $K_2 \geq C_1$ and $\varepsilon \ll 1$ so that

$$(1+K_5)^2 K_1 \varepsilon^2 + (1+K_2) K_5^2 \varepsilon^2 \leq 1$$

allows us to conclude that (40) holds.

4.4.2 Estimate of the $L^\infty_{(k)} L^2_{(t)}$ norm of $A_{s_2}\widehat{\varrho}$

We start from (44) which allows us to write

$$\|A_{s_2}\varrho(\cdot,k)\|^2_{L^2_{(t)}} \lesssim \mathrm{CT1} + \mathrm{CT2} + \mathrm{NLT}.$$

We split again the non linear term as NLT = NLTR + NLTT based on

$$\langle k, tk\rangle^{s_2} \lesssim \langle n, \tau n\rangle^{s_2} + \langle k-n, tk-\tau n\rangle^{s_2}.$$

Estimate on CT1**.** Owing to the assumptions on f_0 and Lemma 4.5, we have

$$\mathrm{CT1} \lesssim \varepsilon^2.$$

Estimate on CT2**.** We have already shown in the previous Section that **(D1')** implies

$$\mathrm{CT2} \lesssim \varepsilon^2.$$

Therefore, when **(D1')** holds, the control of CT2 allows us to reproduce the same arguments as in [7], using as far it is necessary the estimates (19a)–(19c). This is why under **(D1')**, the Landau Damping still holds with $\eta = 0$.

When assuming only **(D1)**, we obtain

$$\mathrm{CT2} \lesssim \varepsilon^2 \int_0^T \langle t\rangle^{-1}\,\mathrm{d}t \lesssim \varepsilon^2 \langle T\rangle^\eta$$

with η as small as we wish. Therefore, new difficulties arise: the force term associated to $\mathscr{F}_I$ can push the solution g far from the equilibrium which might lead to a loss of control of the norms (39)–(43). In what follows, we should keep track carefully of the associated contributions in order to justify that the control of the norms remains possible: we are going to establish (39)–(43) from (34)–(38). This will imply the damping of the force term. We should pay attention to the compatibility between the constraints that appear from the estimates in order to verify such a control. Indeed, we have already seen that $\eta < 1/2$ when $d = 3$. Therefore, we should check carefully that the other constraints keep η in the range $(0, 1/2)$.

Estimate on NLTR. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\mathrm{NLTR} &= \int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} |k|^{1/2} \langle n, \tau n\rangle^{s_2} |n| |\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(\tau,n)\right| \\
&\qquad \times |(t-\tau)k| |\widehat{g}(\tau, k-n, tk-\tau n)| \,\mathrm{d}\tau\,\mathrm{d}n \Big)^2 \mathrm{d}t \\
&\leq \int_0^T \left(\int_0^t \int_{\mathbb{R}^d} |n| \langle n, \tau n\rangle^{2s_4} \langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n \right) \\
&\qquad \times \left(\int_0^t \int_{\mathbb{R}^d} \frac{|k|^3 |t-\tau|^2 |n|}{\langle n\rangle^4 \langle n, \tau n\rangle^{2s_4 - 2s_2}} |\widehat{g}(\tau, k-n, tk-\tau n)|^2 \,\mathrm{d}\tau\,\mathrm{d}n \right) \mathrm{d}t.
\end{aligned}$$

We combine (19a) with (35) and we obtain

$$\int_0^t \int_{\mathbb{R}^d} |n| \langle n, \tau n\rangle^{2s_4} \langle n\rangle^4 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2 \mathrm{d}\tau\,\mathrm{d}n \lesssim (1+K_2)\varepsilon^2$$

while (38) implies

$$\langle k-n, tk-\tau n\rangle^{s_1} |\widehat{g}(\tau, k-n, tk-\tau n)| \lesssim K_5 \varepsilon \langle \tau\rangle^\eta.$$

Hence, we get

$$\mathrm{NLTR} \lesssim (1+K_2) K_5^2 \varepsilon^4 \int_0^T \int_0^t \int_{\mathbb{R}^d} \frac{|k|^3 |t-\tau|^2 |n| \langle \tau\rangle^{2\eta}}{\langle n\rangle^4 \langle n, \tau n\rangle^{2s_4 - 2s_2} \langle k-n, tk-\tau n\rangle^{2s_1}} \,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}n.$$

We are left with the task of proving

$$\sup_{T\geq 0} \sup_{k\in\mathbb{R}^d} \int_0^T \int_0^t \int_{\mathbb{R}^d} \frac{|k|^3 |t-\tau|^2 |n| \langle \tau\rangle^{2\eta}}{\langle n\rangle^4 \langle n, \tau n\rangle^{2s_4 - 2s_2} \langle k-n, tk-\tau n\rangle^{2s_1}} \,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}n \lesssim 1.$$

We postpone the proof of this estimate to Section 4.6.

Estimate on NLTT. By virtue of (19c) and (38), we obtain

$$|\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(\tau,n)\right| \lesssim \frac{\langle \tau\rangle^\eta}{\langle n\rangle^2 \langle n, \tau n\rangle^{s_1}} (1+K_5)\varepsilon.$$

Since

$$\langle k-n, tk-\tau n\rangle^{s_2} |(t-\tau)k| \leq \frac{\langle \tau\rangle}{\langle k-n, tk-\tau n\rangle^{s_3 - s_2 - 1}} \langle k-n, tk-\tau n\rangle^{s_3},$$

the Cauchy-Schwarz inequality allow us to obtain

$$
\begin{aligned}
\mathrm{NLTT} = \int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} |k|^{1/2} \langle k-n, tk-\tau n\rangle^{s_2} |n| |\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}}_I(\tau,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(\tau,n) \right| \\
\times |(t-\tau)k| |\widehat{g}(\tau, k-n, tk-\tau n)| \,\mathrm{d}\tau\,\mathrm{d}n \Big)^2 \,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\lesssim\ & \int_0^T \Big(\int_0^t \int_{\mathbb{R}^d} |k|^{1/2} \frac{|n| \langle \tau\rangle^{\eta}}{\langle n\rangle^2 \langle n, \tau n\rangle^{s_1}} \frac{\langle \tau\rangle}{\langle k-n, tk-\tau n\rangle^{s_3-s_2-1}} \frac{1}{|k-n|^{\delta}} \\
& \times |k-n|^{\delta} \langle k-n, tk-\tau n\rangle^{s_3} \left| \widehat{g}(\tau, k-n, tk-\tau n) \right| \,\mathrm{d}\tau\,\mathrm{d}n \Big)^2 \,\mathrm{d}t \\
\lesssim\ & (1+K_5)^2 \varepsilon^2 \int_0^T \left(\int_0^t \int_{\mathbb{R}^d} \frac{|n|^2 \langle\tau\rangle^{2\eta+2}}{\langle n\rangle^2 \langle n, \tau n\rangle^{2s_1}} \frac{|k|}{\langle k-n, tk-\tau n\rangle^{2s_3-2s_2-2}} \frac{1}{|k-n|^{2\delta}} \,\mathrm{d}\tau\,\mathrm{d}n \right) \\
& \times \left(\int_0^t \int_{\mathbb{R}^d} |k-n|^{2\delta} \langle k-n, tk-\tau n\rangle^{2s_3} \left| \widehat{g}(\tau, k-n, tk-\tau n) \right|^2 \,\mathrm{d}\tau\,\mathrm{d}n \right) \mathrm{d}t.
\end{aligned}
$$

Then, by the Trace Lemma (see the proof of Lemma 4.5 for more details) and (36), we have (for $k \neq 0$)

$$
\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} |k-n|^{2\delta} \langle k-n, tk-\tau n\rangle^{2s_3} \left| \widehat{g}(\tau, k-n, tk-\tau n) \right|^2 \,\mathrm{d}\tau\,\mathrm{d}n \\
=\ & \int_0^t \int_{\mathbb{R}^d} |n|^{2\delta} \langle n, (t-\tau)k-\tau n\rangle^{2s_3} \left| \widehat{g}(\tau, n, (t-\tau)k-\tau n) \right|^2 \,\mathrm{d}\tau\,\mathrm{d}n \\
\leq\ & \sup_{s\in[0,T]} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{-\infty}^{+\infty} |n|^{2\delta} \langle n, \eta+\tau k\rangle \left| \widehat{g}(s, n, \eta+\tau k) \right|^2 \,\mathrm{d}n\,\mathrm{d}\tau \\
\lesssim\ & \sup_{s\in[0,T]} \left\| |\nabla_x|^{\delta} g(s) \right\|_{H^{s_3}}^2 \lesssim K_3 \varepsilon^2
\end{aligned}
$$

Going back to NLTT we are finally led to

$$
\begin{aligned}
\mathrm{NLTT}\ \lesssim\ & (1+K_5)^2 K_3 \varepsilon^4 \\
& \times \int_0^T \int_0^t \int_{\mathbb{R}^d} \frac{|n|^2 \langle\tau\rangle^{2\eta+2}}{\langle n\rangle^2 \langle n, \tau n\rangle^{2s_1}} \frac{|k|}{\langle k-n, tk-\tau n\rangle^{2s_3-2s_2-2}} \frac{1}{|k-n|^{2\delta}} \,\mathrm{d}t\,\mathrm{d}\tau\,\mathrm{d}n
\end{aligned}
$$

and it remains to check that the integral is uniformly bounded with respect to both k and T. We postpone this integral estimate to Section 4.6.

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying (34)–(38) on $[0,T]$, then

$$\begin{aligned}\|A_{s_2}\widehat{\varrho}\|^2_{L^\infty_{(k)}L^2_{(t)}} &\lesssim \Big(1+\langle T\rangle^\eta + (1+K_2)K_5^2\varepsilon^2 + (1+K_5)^2K_3\varepsilon^2\Big)\,\varepsilon^2 \\ &\lesssim \Big(1+(1+K_2)K_5^2\varepsilon^2 + (1+K_5)^2K_3\varepsilon^2\Big)^{1/2}\,\varepsilon^2\langle T\rangle^\eta.\end{aligned}$$

Let us denote C_2 the constant hidden in the $\lesssim$ symbol of this estimate. Choosing $K_4 \geq C_2$ and $\varepsilon \ll 1$ so that

$$(1+K_2)K_5^2\varepsilon^2 + (1+K_5)^2K_3\varepsilon^2 \leq 1$$

allows us to obtain (42).

4.5 Estimates on g.

We cannot apply directly the estimates coming from the linearized problem. Nevertheless, we are going to justify the estimates (39), (41) and (43) from (34)–(38). To this end, we should play with the constants K_1, K_3 and K_5 that depend themselves on K_2 and K_4. What is crucial is to check the compatibility of the choices of these constants, and the consistency of the smallness assumption on ε.

We begin with the equality, obtained by derivating (29),

$$\begin{aligned}\partial_t\widehat{g}(t,k,\xi) &= \widehat{\nabla\sigma}_1(k)\left(\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right)\cdot\widehat{\nabla_v\mathscr{M}}(\xi - tk) \qquad (45)\\ &+ \int_{\mathbb{R}^d_n}\widehat{\nabla\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right)\cdot(\nabla_v \widehat{- t\nabla_x})g(t)(k-n,\xi - tk)\,\mathrm{d}n.\end{aligned}$$

We remark that

$$\|\langle t\nabla_x,\nabla_v\rangle g(t)\|^2_{H^{s_4}_P} \leq \langle t\rangle^2\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} + \|\langle\nabla_v\rangle g(t)\|^2_{H^{s_4}_P} \leq 2\|\langle t\nabla_x,\nabla_v\rangle g(t)\|^2_{H^{s_4}_P}.$$

The first inequality tells us that it suffices to estimate independently $\|\langle\nabla_v\rangle g(t)\|_{H^{s_4}_P}$ and $\|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}$ to get a control of $\|\langle t\nabla_x,\nabla_v\rangle g(t)\|_{H^{s_4}_P}$. We combine the second inequality with (34), so that

$$\|\langle\nabla_v\rangle g(t)\|^2_{H^{s_4}_P} \leq 8K_1\varepsilon^2\langle t\rangle^5$$

and, moreover,

$$\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} \leq 8K_1\varepsilon^2\langle t\rangle^3. \qquad (46)$$

Hence, we are going to handle separately the $H^{s_4}_P$ norm of $\langle\nabla_v\rangle g(t)$ and $\langle\nabla_x\rangle g(t)$.

4.5.1 Estimate of the $H^{s_4}_P$ norm of $\langle\nabla_v\rangle g(t)$

Let $\alpha \in \mathbb{N}^d$, $|\alpha| \le M$ be given; we are going to estimate

$$\|(x,v) \mapsto \langle\nabla_v\rangle v^\alpha g(t,x,v)\|^2_{H^{s_4}}.$$

We postpone as far as possible the summation over α. We work on the Fourier transform, and applying (45) leads to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(x,v) \mapsto \langle\nabla_v\rangle v^\alpha g(t,x,v)\|^2_{H^{s_4}}\\
&= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi \overline{\widehat{g}(t,k,\xi)}\langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi \partial_t\widehat{g}(t,k,\xi)\,\mathrm{d}k\,\mathrm{d}\xi\\
&= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi \overline{\widehat{g}(t,k,\xi)}\langle\xi\rangle\langle k,\xi\rangle^{s_4}\widehat{\nabla\sigma}_1(k)\\
&\qquad\qquad \times\left(\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right)\mathrm{D}^\alpha_\xi\widehat{\nabla_v\mathscr{M}}(\xi-tk)\,\mathrm{d}k\,\mathrm{d}\xi\\
&+ \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi \overline{\widehat{g}(t,k,\xi)}\langle\xi\rangle\langle k,\xi\rangle^{s_4}\\
&\times\mathrm{D}^\alpha_\xi\left(\xi\mapsto\int_{\mathbb{R}^d}\widehat{\nabla\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right)(\nabla_v \widehat{- t\nabla_x})g(t)(k-n,\xi-tn)\,\mathrm{d}n\right)\mathrm{d}k\,\mathrm{d}\xi\\
&= \mathrm{LT}+\mathrm{NLT}.
\end{aligned}$$

We split the non linear term into two parts NLT = NLT1+NLT2: in NLT1 the operator D^α_ξ acts on $\widehat{g}$ only while in NLT2 it acts on both $\widehat{g}$ and $\xi - tk$,

$$\begin{aligned}
&\mathrm{D}^\alpha_\xi\left[\xi\mapsto(\xi-tk)\widehat{g}(t,k-n,\xi-tn)\right]\\
&= (\xi-tk)\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn) + \sum_{\substack{j\in\mathbb{N}^d\\ |j|=1\ j\le\alpha}}\binom{\alpha}{j} j\mathrm{D}^{\alpha-j}_\xi\widehat{g}(t,k-n,\xi-tn).
\end{aligned}$$

The linear term LT. By using

$$\langle\xi\rangle\langle k,\xi\rangle^{s_4} \lesssim \langle t\rangle\langle k\rangle\langle k,tk\rangle^{s_4}\langle\xi-tk\rangle^{s_4+1},$$

and Cauchy-Schwarz' inequality, we get

$$|\mathrm{LT}| \lesssim \langle t\rangle \int_{\mathbb{R}^{2d}_{k,\xi}} \langle \xi\rangle \langle k,\xi\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle k\rangle \langle k,tk\rangle^{s_4} |k||\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|$$

$$\times \langle \xi - tk\rangle^{s_4+1} \left|\mathrm{D}^\alpha_\xi \widehat{\nabla_v \mathscr{M}}(\xi - tk)\right| \,\mathrm{d}k\,\mathrm{d}\xi$$

$$\lesssim \langle t\rangle \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle \xi\rangle^2 \langle k,\xi\rangle^{2s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle k\rangle^2 \langle k,tk\rangle^{2s_4} |k|^2 |\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \langle \xi - tk\rangle^{2s_4+2} \left|\mathrm{D}^\alpha_\xi \widehat{\nabla_v \mathscr{M}}(\xi - tk)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2}$$

$$\lesssim \langle t\rangle \|\langle \nabla_v\rangle g(t)\|_{H^{s_4}_P} \left(\int_{\mathbb{R}^d_k} \langle k\rangle^2 \langle k,tk\rangle^{2s_4} |k|^2 |\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \,\mathrm{d}k\right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^d_\xi} \langle \xi\rangle^{2s_4+2} \left|\mathrm{D}^\alpha_\xi \widehat{\nabla_v \mathscr{M}}(\xi)\right|^2 \,\mathrm{d}\xi\right)^{1/2}.$$

Let us set

$$B(t) = \left(\int_{\mathbb{R}^d_k} \langle k\rangle^2 \langle k,tk\rangle^{2s_4} |k|^2 |\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \,\mathrm{d}k\right)^{1/2}. \tag{47}$$

We observe that (19a) and (35) lead to

$$\int_0^T B(t)^2 \,\mathrm{d}t \lesssim (1+K_2)\varepsilon^2.$$

From now on, we adopt the convention that B denotes a function which satisfies such an estimate. Moreover $\mathscr{M} \in H^{\tilde{s}}_P$ implies (for $\tilde{s}$ large enough)

$$\int_{\mathbb{R}^d_\xi} \langle \xi\rangle^{2s_4+2} \left|\mathrm{D}^\alpha_\xi \widehat{\nabla_v \mathscr{M}}(\xi)\right|^2 \,\mathrm{d}\xi \lesssim 1,$$

and we are led to (owing to (34))

$$|\mathrm{LT}| \lesssim \sqrt{K_1}\varepsilon \langle t\rangle^{5/2+1} B(t).$$

Remark 4.17 *This estimate is quite rough and it involves a Sobolev regularity $\tilde{s}$ higher than s_4 on $\nabla_v \mathscr{M}$. For the non linear term a finer approach will be necessary since we cannot use a Sobolev regularity beyond s_4 on $\langle \nabla_v\rangle g(t)$; a gain of one derivative with respect to v will be necessary.*

We should pay attention not to have contradiction in the definition of the constant K_1. To this end, we introduce $\delta' > 0$ that can be selected as small as necessary, and we use the following estimate

$$|\mathrm{LT}| \lesssim \frac{\sqrt{\delta'}}{\sqrt{\langle t\rangle}}\sqrt{K_1}\varepsilon\langle t\rangle^{5/2} \times \frac{\sqrt{\langle t\rangle}^3}{\sqrt{\delta'}}B(t) \lesssim \delta' K_1\varepsilon^2\langle t\rangle^4 + \frac{B(t)^2}{\delta'}\langle t\rangle^3.$$

Using this way Young's inequality, we make the square of $B(t)$ appear, which is the quantity that we are able to estimate.

The non linear term NLT1**.** We start by studying the operator

$$\mathcal{L}_t[\varrho] : f \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d) \longmapsto \Big[(x,v) \mapsto \nabla\sigma_1\star(\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\,(x+tv)\cdot(\nabla_v - t\nabla_x)f(x,v)\Big].$$

A simple integration by parts shows that

$$\big(f, \mathcal{L}_t[\varrho]f\big)_{L^2(\,\mathrm{d}x\otimes\mathrm{d}v)} = 0. \tag{48}$$

holds for any $f \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. The operator $\mathcal{L}_t[\varrho]$, as well as the relation (48), can be extended to $f \in H^1(\mathbb{R}^d \times \mathbb{R}^d)$. If

$$f = \mathcal{F}^{-1}\Big((k,\xi) \mapsto \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}_\xi^\alpha\widehat{g}(t,k,\xi)\Big),$$

then, by Fourier-transforming and owing to Plancherel's theorem, (48) tells us

$$\begin{aligned}
0 &= \int_{\mathbb{R}^{2d}_{k,\xi}} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}_\xi^\alpha\overline{\widehat{g}(t,k,\xi)}\,\widehat{\mathcal{L}_{(t)}[\varrho]f}(t,k,\xi)\,\mathrm{d}k\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}_\xi^\alpha\overline{\widehat{g}(t,k,\xi)}\,n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\Big) \\
&\qquad \times\langle\xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4}(\xi - tk)\mathrm{D}_\xi^\alpha\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n.
\end{aligned}$$

Therefore NLT1 can be cast as

$$\begin{aligned}
\mathrm{NLT1} = -&\int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}_\xi^\alpha\overline{\widehat{g}(t,k,\xi)}\,\left[\langle\xi\rangle\langle k,\xi\rangle^{s_4} - \langle\xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4}\right] \\
&\times n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\Big)(\xi - tk)\mathrm{D}_\xi^\alpha\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n.
\end{aligned}$$

We split depending on the leading frequencies

$$\begin{aligned}
\mathrm{NLT1} = -&\int_{\mathbb{R}^{3d}_{k,\xi,n}} \Big(\mathbf{1}_{|n,tn|\geq|k-n,\xi-tn|} + \mathbf{1}_{|n,tn|\leq|k-n,\xi-tn|}\Big)\langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}_\xi^\alpha\overline{\widehat{g}(t,k,\xi)} \\
&\times\left[\langle\xi\rangle\langle k,\xi\rangle^{s_4} - \langle\xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4}\right]\,n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\Big) \\
&\times(\xi - tk)\mathrm{D}_\xi^\alpha\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
&= \mathrm{NLT1R} + \mathrm{NLT1T}.
\end{aligned}$$

We are now going to study the two terms of this splitting.

Estimate on NLT1R**.** We remark that

$$|\xi - tk| \leq \langle t\rangle\langle k-n, \xi - tn\rangle.$$

and when $|n, tn| \geq |k-n, \xi - tn|$, we have

$$|\langle\xi\rangle\langle k,\xi\rangle^{s_4} - \langle\xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4}| \lesssim \langle\xi - tn\rangle\langle t\rangle\langle n\rangle\langle n, tn\rangle^{s_4}.$$

Remark 4.18 *This relation allows us to overcome the difficulty mentioned during the study of the linear term. In the regime $|n,tn| \geq |k-n,\xi-tn|$ we have been able, at the price of an extra factor $\langle t\rangle$, to distribute the weights $\langle n, tn\rangle$ and $\langle k-n,\xi-tn\rangle$ on $\varrho(t)$ and $\langle\nabla_v\rangle g(t)$ so that their estimate does not involve Sobolev exponent larger than s_4. This answers for* NLT1R *the regularity issue risen in Remark 4.17.*

We apply these inequalities to NLT1R, and next we make use of Lemma C.9; we obtain

$$|\mathrm{NLT1R}| \lesssim \langle t\rangle^2 \int_{\mathbb{R}^{3d}_{k,\xi,n}} \mathbf{1}_{|n,tn|\geq|k-n,\xi-tn|}\langle\xi\rangle\langle k,\xi\rangle^{s_4}\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|\langle\xi-tn\rangle\langle n\rangle\langle n,tn\rangle^{s_4}$$

$$\times|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|\langle k-n,\xi-tn\rangle\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|\,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n$$

$$\lesssim \langle t\rangle^2\left(\int_{\mathbb{R}^{2d}_{k,\xi}}\langle\xi\rangle^2\langle k,\xi\rangle^{2s_4}\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2\,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2}$$

$$\times\left(\int_{\mathbb{R}^d_n}\langle n\rangle^2\langle n,tn\rangle^{2s_4}|n|^2|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2\,\mathrm{d}n\right)^{1/2}$$

$$\times\int_{\mathbb{R}^d_k}\left(\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^2\langle k,\xi\rangle^2\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2}\mathrm{d}k$$

$$\lesssim \langle t\rangle^2\|\langle\nabla_v\rangle g(t)\|_{H^{s_4}_P}\,B(t)\int_{\mathbb{R}^d_k}\left(\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^2\langle k,\xi\rangle^2\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2}\mathrm{d}k.$$

where we use again the generic notation $B(t)$ as in (47). Let us consider in details the third term: as far as $\delta < d/2$ (which holds since $\delta < 1$) and s_3 is large enough ($s_3 > d/2 + 2$ is sufficient), the Cauchy-Schwartz inequality yields

$$\int_{\mathbb{R}^d_k}\frac{|k|^\delta\langle k\rangle^{s_3-2}}{|k|^\delta\langle k\rangle^{s_3-2}}\left(\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^2\langle k,\xi\rangle^2\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2}\mathrm{d}k$$

$$\leq\left(\int_{\mathbb{R}^d_k}\frac{1}{|k|^{2\delta}\langle k\rangle^{2s_3-4}}\,\mathrm{d}k\right)^{1/2}\left(\int_{\mathbb{R}^{2d}_{k,\xi}}|k|^{2\delta}\langle k\rangle^{2s_3-4}\langle\xi\rangle^2\langle k,\xi\rangle^2\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2\,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2}$$

$$\lesssim \|\,|\nabla_x|^\delta g(t)\|_{H^{s_3}_P}.$$

Next, with (34) and (36) we get

$$|\mathrm{NLT1R}| \lesssim \sqrt{K_1 K_3}\varepsilon^2 \langle t\rangle^{5/2+2} B(t).$$

In order to make the square of $B(t)$ appear, we decompose the inequality as follows

$$|\mathrm{NLT1R}| \lesssim \frac{1}{\sqrt{\langle t\rangle}}\sqrt{K_1 K_3}\varepsilon^{3/2}\langle t\rangle^{5/2} \times \sqrt{\langle t\rangle}\varepsilon^{1/2}\langle t\rangle^2 B(t) \lesssim K_1 K_3 \varepsilon^3 \langle t\rangle^4 + \varepsilon\langle t\rangle^5 B(t)^2.$$

Estimate on NLT1T. For $|n, tn| \leq |k-n, \xi - tk|$ we have, see [7, Section 5.1.1]

$$\langle \xi\rangle\langle k,\xi\rangle^{s_4} - \langle \xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4} \lesssim \langle n, tn\rangle^2 \left(\langle \xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4-1} + \langle k-n,\xi-tn\rangle^{s_4}\right).$$

Remark 4.19 *Note that we gain one order of Sobolev regularity on g, see Remark 4.17 and 4.18. Like when dealing with* NLT1R*, the idea is to distribute the weights $\langle n, tn\rangle$ and $\langle k-n, \xi-tn\rangle$ on $\varrho(t)$ and $\langle\nabla_v\rangle g(t)$ in order to make estimates with Sobolev exponents smaller or equal to s_4 appear (see the regularity issue explained in Remark 4.17). In the regime $|k-n,\xi-tn| \geq |n,tn|$ we can not use an estimate as rough as for* TNL1R *since all the Sobolev exponents already apply to $\langle k-n,\xi-tn\rangle$. We should take advantage of cancellations between $\langle\xi\rangle\langle k,\xi\rangle^{s_4}$ and $\langle\xi-tn\rangle\langle k-n,\xi-tn\rangle^{s_4}$. This motivates the introduction of the operator $\mathcal{L}_t[\varrho]$, see [6] and [16] where this operator already appeared for similar reasons.*

We use this inequality for estimating NLT1T that we split according to the two terms above. We are led to

$$\begin{aligned}
&|\mathrm{NLT1T}| \lesssim \int_{\mathbb{R}^{3d}_{k,\xi,n}} \mathbf{1}_{|n,tn|\leq|k-n,\xi-tn|}\langle\xi\rangle\langle k,\xi\rangle^{s_4}\left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \\
&\times\langle n,tn\rangle^2\left(\langle\xi-tn\rangle\langle k-n,\xi-tn\rangle^{s_4-1} + \langle k-n,\xi-tn\rangle^{s_4}\right) \\
&\times|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right||\xi-tk|\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|\,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
&= \mathrm{NLT1T1} + \mathrm{NLT1T2}.
\end{aligned}$$

We treat NLT1T1 by applying Lemma C.9 (and $|\xi - tk| \le \langle t\rangle\langle k-n, \xi - tk\rangle$); we get

$$\begin{aligned}
\mathrm{NLT1T1} \lesssim \langle t\rangle \int_{\mathbb{R}^{3d}_{k,\xi,n}} &\langle \xi\rangle \langle k,\xi\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n, tn\rangle^2 |n| |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \\
&\times \langle \xi - tn\rangle \langle k-n, \xi - tn\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
\lesssim \langle t\rangle \|\langle\nabla_v\rangle g(t)\|_{H^{s_4}_P} &\left(\int_{\mathbb{R}^d_n} \langle n, tn\rangle^2 |n| |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \,\mathrm{d}n\right) \\
&\times \left(\int_{\mathbb{R}^{2d}_{k,n}} \langle \xi\rangle^2 \langle k,\xi\rangle^{2s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2} \\
\lesssim \langle t\rangle \|\langle\nabla_v\rangle g(t)\|^2_{H^{s_4}_P} &\left(\int_{\mathbb{R}^d_n} \langle n, tn\rangle^2 |n| |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \,\mathrm{d}n\right).
\end{aligned}$$

However, (19c) and (38) lead to

$$|\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \lesssim \frac{\langle t\rangle^\eta}{\langle n, tn\rangle^{s_1}} (1+K_5)\varepsilon,$$

so that (by using $|n|\langle t\rangle \le \langle n, tn\rangle$)

$$\begin{aligned}
&\int_{\mathbb{R}^d_n} \langle n, tn\rangle^2 |n| |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \,\mathrm{d}n \\
&\lesssim \langle t\rangle^{-1} \left(\int_{\mathbb{R}^d_n} \langle n, tn\rangle^{3-s_1} \,\mathrm{d}n\right) (1+K_5)\varepsilon \langle t\rangle^\eta \lesssim \varepsilon \langle t\rangle^{\eta - d - 1}.
\end{aligned}$$

We gather these estimates with (34), and we arrive at

$$\mathrm{NLT1T1} \lesssim K_1^2 (1+K_5)\varepsilon^3 \langle t\rangle^{5+\eta-d}.$$

For NLT1T2 we proceed similarly by using Lemma C.9 (and remarking that $|\xi - tk| \leq \langle t(k-n), \xi - tn\rangle$ holds); we are led to

$$\begin{aligned}
\mathrm{NLT1T2} \lesssim \int_{\mathbb{R}^{3d}_{k,\xi,n}} &\langle\xi\rangle\langle k,\xi\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle^2 |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\times \langle t(k-n)\xi - tn\rangle\langle k-n,\xi-tn\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
\lesssim \|\langle\nabla_v\rangle g(t)\|_{H^{s_4}_P} &\left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle tk,\xi\rangle^2\langle k,\xi\rangle^{2s_4}\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2} \\
&\times \int_{\mathbb{R}^d} \langle n,tn\rangle^2 \times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n \\
\lesssim \|\langle\nabla_v\rangle g(t)\|_{H^{s_4}_P} &\|\langle t\nabla_x\nabla_v\rangle g(t)\|_{H^{s_4}_P} \\
&\times \int_{\mathbb{R}^d} \langle t\rangle\langle n,tn\rangle^2 \times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n,
\end{aligned}$$

and we deduce that

$$\mathrm{NLT1T2} \lesssim K_1^2(1+K_5)\varepsilon^3\langle t\rangle^{5+\eta-d-1}$$

holds.

Estimate on NLT2**.** Compared to what we just did, we are concerned with a term having less regularity (we do not have the factor $\xi - tk$ which has been derivated). The regularity issue presented in Remark 4.17 does not hold for NLT2 and there is no need to make use of (48). We turn to the second step, by decomposing between low and high frequencies

$$\begin{aligned}
\mathrm{NLT2} = \int_{\mathbb{R}^{3d}_{k,\xi,n}} &\langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}\langle\xi\rangle\langle k,\xi\rangle^{s_4} n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right) \\
&\times \left(\sum_{\substack{j\in\mathbb{N}^d \\ |j|=1\ j\leq\alpha}} \binom{\alpha}{j} j\mathrm{D}^{\alpha-j}_\xi\widehat{g}(t,k-n,\xi-tn)\right) \mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
= \int_{\mathbb{R}^{3d}_{k,\xi,n}} &\left(\mathbf{1}_{|n,tn|\geq|k-n,\xi-tn|} + \mathbf{1}_{|n,tn|\leq|k-n,\xi-tn|}\right)\langle\xi\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)} \\
&\times\langle\xi\rangle\langle k,\xi\rangle^{s_4} n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right) \\
&\times \left(\sum_{\substack{j\in\mathbb{N}^d \\ |j|=1\ j\leq\alpha}} \binom{\alpha}{j} j\mathrm{D}^{\alpha-j}_\xi\widehat{g}(t,k-n,\xi-tn)\right) \mathrm{d}k\,\mathrm{d}\xi\,\mathrm{d}n \\
= \mathrm{NLT2R} &+ \mathrm{NLT2T}.
\end{aligned}$$

On the integration domain of the reaction term, we have

$$\langle \xi \rangle \langle k, \xi \rangle^{s_4} \lesssim \langle \xi - tn \rangle \langle t \rangle \langle n \rangle \langle n, tn \rangle^{s_4}.$$

We apply Lemma C.9 to obtain

$$|\mathrm{NLT2R}| \lesssim \langle t \rangle \sum_{\substack{j \in \mathbb{N}^d \\ |j|=1\ j \leq \alpha}} \int_{\mathbb{R}^{3d}_{k,\xi,n}} \mathbf{1}_{|n,tn| \geq |k-n, \xi - tn|} \langle \xi \rangle \langle k, \xi \rangle^{s_4} \left| \mathrm{D}^{\alpha}_{\xi} \widehat{g}(t,k,\xi) \right|$$

$$\times \langle n \rangle \langle n, tn \rangle^{s_4} |n| |\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}_\varrho}(t,n) \right| \langle \xi - tn \rangle \left| \mathrm{D}^{\alpha-j}_{\xi} \widehat{g}(t, k-n, \xi - tn) \right| \mathrm{d}k \, \mathrm{d}\xi \, \mathrm{d}n$$

$$\lesssim \langle t \rangle \sum_{\substack{j \in \mathbb{N}^d \\ |j|=1\ j \leq \alpha}} \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle \xi \rangle^2 \langle k, \xi \rangle^{2s_4} \left| \mathrm{D}^{\alpha}_{\xi} \widehat{g}(t,k,\xi) \right|^2 \mathrm{d}k \, \mathrm{d}\xi \right)^{1/2}$$

$$\times \left(\int_{\mathbb{R}^d_n} \langle n \rangle^2 \langle n, tn \rangle^{2s_4} |n|^2 |\widehat{\sigma}_1(n)|^2 \left| \widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}_\varrho}(t,n) \right|^2 \mathrm{d}n \right)^{1/2}$$

$$\times \int_{\mathbb{R}^d_k} \left(\int_{\mathbb{R}^d_\xi} \langle \xi \rangle^2 \left| \mathrm{D}^{\alpha-j}_{\xi} \widehat{g}(t,k,\xi) \right|^2 \mathrm{d}\xi \right)^{1/2} \mathrm{d}k.$$

Hence it behaves like the reaction term NLT1R, up to a factor $\langle t \rangle$ which does not appear here; we can dominate the product and we get

$$|\mathrm{NLT2R}| \lesssim \sqrt{K_1 K_3} \varepsilon^2 \langle t \rangle^{5/2+1} B(t) \lesssim K_1 K_3 \varepsilon^3 \langle t \rangle^4 + \varepsilon \langle t \rangle^3 B(t)^2.$$

For the transport term, on the integration domain

$$\langle \xi \rangle \langle k, \xi \rangle^{s_4} \lesssim \langle t \rangle \langle n \rangle \langle \xi - tn \rangle \langle k - n, \xi - tn \rangle^{s_4}$$

holds and applying Lemme C.9 yields

$$|\mathrm{NLT2T}| \lesssim \langle t \rangle \sum_{\substack{j \in \mathbb{N}^d \\ |j|=1\ j \leq \alpha}} \int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle \xi \rangle \langle k, \xi \rangle^{s_4} \left| \mathrm{D}^{\alpha}_{\xi} \widehat{g}(t,k,\xi) \right| \langle n \rangle |n| |\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}_\varrho}(t,n) \right|$$

$$\times \langle \xi - tn \rangle \langle k - n, \xi - tn \rangle^{s_4} \left| \mathrm{D}^{\alpha-j}_{\xi} \widehat{g}(t, k-n, \xi - tn) \right| \mathrm{d}k \, \mathrm{d}\xi \, \mathrm{d}n$$

$$\lesssim \langle t \rangle \sum_{\substack{j \in \mathbb{N}^d \\ |j|=1\ j \leq \alpha}} \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle \xi \rangle^2 \langle k, \xi \rangle^{2s_4} \left| \mathrm{D}^{\alpha}_{\xi} \overline{\widehat{g}(t,k,\xi)} \right|^2 \mathrm{d}k \, \mathrm{d}\xi \right)^{1/2}$$

$$\times \int_{\mathbb{R}^d_n} \langle n \rangle |n| |\widehat{\sigma}_1(n)| \left| \widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}_\varrho}(t,n) \right| \mathrm{d}n$$

$$\times \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle \xi \rangle^2 \langle k, \xi \rangle^{2s_4} \left| \mathrm{D}^{\alpha-j}_{\xi} \widehat{g}(t,k,\xi) \right|^2 \mathrm{d}k \, \mathrm{d}\xi \right)^{1/2}.$$

We finally get
$$|\mathrm{NLT2T}| \lesssim K_1^2(1+K_5)\varepsilon^3\langle t\rangle^{5+\eta-d}.$$

Remark 4.20 *As said above, the regularity issue described in Remarks 4.17 and 4.18 does not hold with* NLT2*. Thus, there is no need to introduce the operator $\mathcal{L}_t[\varrho]$ and we derive a better estimate for* NLT2 *than for* NLT1*. In fact, we will not use this improved estimate. We can also observe that it would be possible to use the obvious estimate $1 \leq \langle \xi - tk\rangle$, which yields*

$$\begin{aligned}
&\left|\mathrm{D}_\xi^\alpha[\xi \mapsto (\xi - tk)\widehat{g}(t,k-n,\xi-tn)]\right| \\
&\leq \left|(\xi-tk)\mathrm{D}_\xi^\alpha\widehat{g}(t,k-n,\xi-tn)\right| + \Big|\sum_{\substack{j\in\mathbb{N}^d\\ |j|=1, j\leq\alpha}} \binom{\alpha}{j} j\mathrm{D}_\xi^{\alpha-j}\widehat{g}(t,k-n,\xi-tn)\Big| \\
&\leq \left|(\xi-tk)\mathrm{D}_\xi^\alpha\widehat{g}(t,k-n,\xi-tn)\right| + \sum_{\substack{j\in\mathbb{N}^d\\ |j|=1, j\leq\alpha}} \binom{\alpha}{j} \langle\xi - tk\rangle\left|\mathrm{D}_\xi^{\alpha-j}\widehat{g}(t,k-n,\xi-tn)\right|.
\end{aligned}$$

From this, NLT2 *can be treated exactly like* NLT1*. In what follows, in similar situations we will only focus the discussion on the most regularity demanding terms.*

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying moreover (34)–(38) on $[0,T]$, then, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(t \longmapsto \|(x,v)\mapsto \langle\nabla_v\rangle v^\alpha g(t,x,v)\|_{H^{s_4}}^2\right) \\
&\lesssim \delta' K_1\varepsilon^2\langle t\rangle^4 + \frac{\langle t\rangle^3}{\delta'}B(t)^2 + K_1K_3\varepsilon^3\langle t\rangle^4 + \varepsilon\langle t\rangle^5 B(t)^2 \\
&+K_1^2(1+K_5)\varepsilon^3\langle t\rangle^{5+\eta-d}.
\end{aligned}$$

(note that we have used the rough estimates that consists in dominating NLT2R like NLT1R, NLT1T2 like NLT2T and NLT2T like NLT1T1). Let C_3 be the constant hidden in the $\lesssim$ symbol; integrating over $[0,T]$ and summing over α, we obtain (with the generic notation (47) for $B(t)$)

$$\begin{aligned}
\|\langle\nabla_v\rangle g(T)\|_{H_P^{s_4}}^2 &\leq \|\langle\nabla_v\rangle g(0)\|_{H_P^{s_4}}^2 + \left(C_3\delta' K_1\langle T\rangle^5 + C_3\frac{\langle T\rangle^3}{\delta'}(1+K_2)\right)\varepsilon^2 \\
&+\left(C_3K_1K_3\varepsilon\langle T\rangle^5 + C_3(1+K_2)\varepsilon\langle T\rangle^5 + C_3K_1^2(1+K_5)\varepsilon\langle T\rangle^{6+\eta-d}\right)\varepsilon^2.
\end{aligned}$$

Since $g(0,x,v) = f_0(x,v)$ and $f_0 \in H_P^s$ with $s > s_4$, we observe that
$$\|\langle\nabla_v\rangle g(0)\|_{H_P^{s_4}}^2 \leq \varepsilon^2.$$

Let $\delta' \ll 1$ so that $C_3\delta' < 1/4$. Once δ' is fixed that way, we choose $K_1 \gg 1$ so that

$$\|\langle\nabla_v\rangle g(0)\|^2_{H^{s_4}_P} + C_3\frac{\langle T\rangle^3}{\delta'}(1+K_2)\varepsilon^2 \le \frac{K_1}{4}\varepsilon^2\langle T\rangle^5$$

holds. Therefore K_1 depends on K_2 and δ'. We are left with the task of determining $\varepsilon \ll 1$ in order to obtain

$$\Big(C_3K_1K_3\varepsilon\langle T\rangle^5 + C_3(1+K_2)\varepsilon\langle T\rangle^5 + C_3K_1^2(1+K_5)\varepsilon\langle T\rangle^{6+\eta-d}\Big)\,\varepsilon^2 \le K_1\varepsilon^2\langle T\rangle^5,$$

which eventually leads to

$$\|\langle\nabla_v\rangle g(T)\|^2_{H^{s_4}_P} \le K_1\varepsilon^2\langle T\rangle^5.$$

4.5.2 Estimate of the $H^{s_4}_P$ norm of $\langle\nabla_x\rangle g(t)$

We proceed like in the previous section: we evaluate the time derivative of $\|\langle\nabla_x\rangle\langle\nabla_x,\nabla_v\rangle^{s_4}v^\alpha g(t)\|^2_{L^2}$ by means of the Fourier variables, and we express $\partial_t\widehat{g}$ with (45). We obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla_x\rangle\langle\nabla_x,\nabla_v\rangle^{s_4}v^\alpha g(t)\|^2_{L^2}\\
&= -\int_{\mathbb{R}^{2d}_{k,\xi}}\langle k\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}\langle k\rangle\langle k,\xi\rangle^{s_4}k\widehat{\sigma}_1(k)\left(\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right)\\
&\qquad\qquad\times\mathrm{D}^\alpha_\xi\left((\xi-tk)\widehat{\mathscr{M}}(\xi-tk)\right)\,\mathrm{d}k\,\mathrm{d}\xi\\
&-\int_{\mathbb{R}^{3d}_{k,\xi,n}}\langle k\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}\langle k\rangle\langle k,\xi\rangle^{s_4}n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right)(\xi-tk)\\
&\qquad\qquad\times\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi\\
&-\sum_{\substack{j\in\mathbb{N}^d\\|j|=1;j\le\alpha}}\binom{\alpha}{j}\int_{\mathbb{R}^{3d}_{k,\xi,n}}\langle k\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}\langle k\rangle\langle k,\xi\rangle^{s_4}n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right)\\
&\qquad\qquad\cdot j\mathrm{D}^{\alpha-j}_\xi\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi\\
&= \mathrm{LT}+\mathrm{NLT1}+\mathrm{NLT2}.
\end{aligned}$$

The analysis of the the first non linear term also covers the second term, see Remark 4.20. Thus we do not detail how to handle NLT2. Note however that similar manipulations as above can lead to a refined estimate on NLT2, but this is not necessary for our purpose.

Estimate on the linear term LT We apply the Cauchy-Schwarz inequality, up to the observation

$$\langle k\rangle\langle k,\xi\rangle^{s_4} \le \langle k\rangle\langle k,tk\rangle^{s_4}\langle\xi-tk\rangle^{s_4};$$

we arrive at

$$
\begin{aligned}
|\mathrm{LT}| &\lesssim \|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}\Big(\int_{\mathbb{R}^{2d}_{k,\xi}}\langle k\rangle^2\langle k,tk\rangle^{2s_4}|k|^2|\widehat{\sigma}_1(k)|^2\left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \\
&\qquad\times\langle\xi-tk\rangle^{2s_4}\left|\mathrm{D}^\alpha_\xi\widehat{\nabla_v\mathscr{M}}(\xi-tk)\right|^2\,\mathrm{d}k\,\mathrm{d}\xi\Big)^{1/2} \\
&\lesssim \|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}\left(\int_{\mathbb{R}^d_k}\langle k\rangle^2\langle k,tk\rangle^{2s_4}|k|^2|\widehat{\sigma}_1(k)|^2\left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2\,\mathrm{d}k\right)^{1/2} \\
&\qquad\times\left(\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^{2s_4}\left|\mathrm{D}^\alpha_\xi\widehat{\nabla_v\mathscr{M}}(\xi)\right|^2\,\mathrm{d}\xi\right)^{1/2} \\
&\lesssim \|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}B(t).
\end{aligned}
$$

(where the assumption $\mathscr{M}\in H^{\tilde{s}}_P$ with $\tilde{s}>s_4+1$ has permitted us to obtain

$$
\int_{\mathbb{R}^d_\xi}\langle\xi\rangle^{2s_4}\left|\mathrm{D}^\alpha_\xi\widehat{\nabla_v\mathscr{M}}(\xi)\right|^2\,\mathrm{d}\xi\lesssim 1,
$$

and we have used the notation (47) for $B(t)$). Again, we introduce a positive number δ'', as small as we wish, and we split the product into two parts so that the constant K_1 is isolated and we make the square of $B(t)$ appear. Namely, we have

$$
|\mathrm{LT}|\lesssim\frac{\delta''}{\langle t\rangle}\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P}+\langle t\rangle\frac{B^2(t)}{\delta''}\lesssim\delta'' K_1\varepsilon^2\langle t\rangle^2+\langle t\rangle\frac{B^2(t)}{\delta''}.
$$

where we have also made use of (46).

Remark 4.21 *Here, in contrast to the previous estimate of $\langle\nabla_v\rangle g(t)$ in norm $H^{s_4}_P$, we make the Sobolev estimate of $\nabla_v\mathscr{M}$ appear with* exactly *the exponent s_4. Nevertheless we are facing a similar regularity difficulty since now we wish to estimate $\langle\nabla_x\rangle g(t)$ in norm $H^{s_4}_P$ (instead of $\langle\nabla_v\rangle g(t)$). Hence, again, we need to gain one derivative. To this end we shall adapt the strategy designed for* NLT1*.*

Estimate on NLT1**.** We use (48) with

$$
f=\mathscr{F}^{-1}\Big((k,\xi)\mapsto\langle k\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\Big).
$$

We split between the contributions of low and high frequencies, so that

$$
\begin{aligned}
\mathrm{NLT1} &= \int_{\mathbb{R}^{3d}_{k,\xi,n}}\Big(\mathbf{1}_{|n,tn|\geq|k-n,\xi-tn|}+\mathbf{1}_{|n,tn|\leq|\xi-tn|}\Big)\langle k\rangle\langle k,\xi\rangle^{s_4}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)} \\
&(\langle k\rangle\langle k,\xi\rangle^{s_4}-\langle k-n\rangle\langle k-n,\xi-tn\rangle^{s_4})\,n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\Big) \\
&\cdot(\xi-tk)\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&= \mathrm{NLT1R}+\mathrm{NLT1T}.
\end{aligned}
$$

Estimate on NLT1R**.** On the integration domain, we have

$$|\langle k\rangle\langle k,\xi\rangle^{s_4} - \langle k-n\rangle\langle k-n,\xi-tn\rangle^{s_4}| \lesssim \langle k-n\rangle\langle n\rangle\langle n,tn\rangle^{s_4}.$$

Going back to Lemma C.9 (and owing to $|\xi - tk| \le \langle t\rangle\langle k-n,\xi-tn\rangle$), we obtain

$$\begin{aligned}
|\mathrm{NLT1R}| &\lesssim \langle t\rangle \int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle k\rangle\langle k,\xi\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n|\langle n\rangle\langle n,tn\rangle^{s_4}|\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\qquad \times\langle k-n\rangle\langle k-n,\xi-tn\rangle \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \langle t\rangle \|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}\, B(t) \left(\int_{\mathbb{R}^d_k}\left(\int_{\mathbb{R}^d_\xi} \langle k\rangle^2\langle k,\xi\rangle^2 \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \mathrm{d}k\right)
\end{aligned}$$

When estimating $\langle\nabla_v\rangle g(t)$ in norm $H^{s_4}_P$ we have seen that (cf. NLT1R)

$$\int_{\mathbb{R}^d_k}\left(\int_{\mathbb{R}^d_\xi} \langle k\rangle^2\langle k,\xi\rangle^2 \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \mathrm{d}k \lesssim \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P}.$$

Then, (46) and (36) ensure that

$$|\mathrm{NLT1R}| \lesssim \sqrt{K_1K_3}\varepsilon^2\langle t\rangle^{3/2+1}B(t).$$

With the Young inequality we make the square of $B(t)$ appear; we conclude that

$$|\mathrm{NLT1R}| \lesssim K_1K_3\varepsilon^3\langle t\rangle^2 + \varepsilon\langle t\rangle^3 B(t)^2.$$

Estimate on NLT1T**.** Again we split NLT1T = NLT1T1 + NLT1T2 by using the fact that, on the integration domain, we have (see [7, Section 5.1.2])

$$\begin{aligned}
&|\langle k\rangle\langle k,\xi\rangle^{s_4} - \langle k-n\rangle\langle k-n,\xi-tn\rangle^{s_4}| \\
&\qquad\qquad \lesssim \langle n,tn\rangle^2\left(\langle k-n\rangle\langle k-n,\xi-tn\rangle^{s_4-1} + \langle k-n,\xi-tn\rangle^{s_4}\right).
\end{aligned}$$

Thus, NLT1T1 stands for the term with the exponent s_4-1. We use Lemma C.9 and $|\xi-tk| \le \langle t\rangle\langle k-n,\xi-tn\rangle$ and we obtain

$$\begin{aligned}
|\mathrm{NLT1T1}| &\lesssim \langle t\rangle \int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle k\rangle\langle k,\xi\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n||\widehat{\sigma}_1(n)|\langle n,tn\rangle^2 \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\qquad \times\langle k-n\rangle\langle k-n,\xi-tn\rangle^{s_4} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \langle t\rangle \|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} \int_{\mathbb{R}^d_n} |n||\widehat{\sigma}_1(n)|\langle n,tn\rangle^2 \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \mathrm{d}n
\end{aligned}$$

Since (19c) and (38) imply

$$|\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \lesssim \frac{\langle t\rangle^\eta}{\langle n,tn\rangle^{s_1}}(1+K_5)\varepsilon,$$

we get (by using addionnally $|n|\langle t\rangle \leq \langle n, tn\rangle$)

$$\int_{\mathbb{R}^d_n} |n|\langle t\rangle|\widehat{\sigma}_1(n)|\langle n, tn\rangle^2 \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \mathrm{d}n$$

$$\lesssim \langle t\rangle^{\eta-1}\left(\int_{\mathbb{R}^d_n} \langle n, tn\rangle^{3-s_1}\,\mathrm{d}n\right)(1+K_5)\varepsilon \lesssim (1+K_5)\varepsilon\langle t\rangle^{\eta-d-1}.$$

Using also (46), we thus show that

$$|\mathrm{NLT1T1}| \lesssim K_1(1+K_5)\varepsilon^3\langle t\rangle^{3+\eta-d}.$$

For NLT1T2, we proceed similarly, by coming back to Lemma C.9, but now we use $|\xi - tk| \leq \langle t(k-n), \xi - tn\rangle$; we obtain

$$\begin{aligned}
|\mathrm{NLT1T2}| \quad &\lesssim \quad \int_{\mathbb{R}^{3d}_{k,\xi,n}} \langle k\rangle\langle k,\xi\rangle^{s_4}\left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n||\widehat{\sigma}_1(n)|\langle n, tn\rangle^2 \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \\
&\qquad \times\langle t(k-n), \xi - tn\rangle\langle k-n, \xi-tn\rangle^{s_4}\left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \quad \|\langle\nabla_x\rangle g(t)\|_{H^{s_4}_P}\|\langle t\nabla_x, \nabla_v\rangle g(t)\|_{H^{s_4}_P} \\
&\qquad \times\int_{\mathbb{R}^d_n} |n||\widehat{\sigma}_1(n)|\langle n, tn\rangle^2 \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \mathrm{d}n.
\end{aligned}$$

Gathering (19c), (38), (34) and (46), this leads to

$$|\mathrm{NLT1T2}| \lesssim K_1(1+K_5)\varepsilon^3\langle t\rangle^{3+\eta-d}.$$

Recap. We have shown that, if g is a solution of (10a)–(10b) which satisfies (34)–(38) on $[0,T]$, then we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} \quad &\lesssim \quad \delta'' K_1\varepsilon^2\langle t\rangle^2 + \langle t\rangle\frac{B(t)^2}{\delta''} \\
&\qquad +K_1K_3\varepsilon^3\langle t\rangle^2 + \varepsilon\langle t\rangle^3 B(t)^2 + K_1(1+K_5)\varepsilon^3\langle t\rangle^{3+\eta-d}.
\end{aligned}$$

Let us denote C_4 the constant hidden in the $\lesssim$ symbol. Integrating over $[0,T]$ yields

$$\begin{aligned}
\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} \quad &\leq \quad \|\langle\nabla_x\rangle g(0)\|^2_{H^{s_4}_P} + C_4\delta'' K_1\varepsilon^2\langle T\rangle^3 + C_4\tfrac{1+K_2}{\delta''}\varepsilon^2\langle T\rangle \\
&\qquad +C_4K_1K_3\varepsilon^3\langle T\rangle^3 + C_4(1+K_2)\varepsilon^3\langle T\rangle^3 + C_4K_1(1+K_5)\varepsilon^3\langle T\rangle^{4+\eta-d}.
\end{aligned}$$

We remind the reader that K_1 and δ' have already been fixed at the previous step. Possibly at the price of making δ' smaller, we can assume that $\delta'' = \delta'$ and $\delta' C_4 < 1/4$. Next, choosing K_1 larger if necessary, we can equally suppose that

$$\|\langle\nabla_x\rangle g(0)\|^2_{H^{s_4}_P} + C_4\frac{1+K_2}{\delta''}\varepsilon^2\langle T\rangle \leq \frac{K_1}{4}\langle T\rangle^3\varepsilon^2$$

holds. Eventually, when $\varepsilon \ll 1$, we have

$$C_4 K_1 K_3 \varepsilon^3 \langle T\rangle^3 + C_4(1+K_2)\varepsilon^3\langle T\rangle^3 + C_4K_1(1+K_5)\varepsilon^3\langle T\rangle^{4+\eta-d} \le \frac{K_1}{2}\varepsilon^2\langle T\rangle^3,$$

and we have shown that

$$\|\langle\nabla_x\rangle g(t)\|^2_{H^{s_4}_P} \le K_1\varepsilon^2\langle T\rangle^3$$

is satisfied.

4.5.3 Estimates of the $H^{s_3}_P$ norm of $|\nabla_x|^\delta g(t)$.

Since $s_4 > s_3$, we can naively think that this term can be dominated by using the estimates on $g(t)$ and $\varrho(t)$ with norms based on $H^{s_4}_P$. However, here we wish to establish estimates *uniform* with respect to t, while the $H^{s_4}_P$ estimates were involving a polynomial weight $\langle t\rangle^5$. Therefore, we shall need refined estimates in order to make use as less as possible of the $H^{s_4}_P$ norm of $\langle t\nabla_x, \nabla_v\rangle g(t)$.

We compute the time derivative of $\big\||k|^\delta\langle k,\xi\rangle^{s_3}\mathrm{D}^\alpha_\xi\widehat{g}(t,k,\xi)\big\|^2_{L^2_{(k,\xi)}}$, using the expression of $\partial_t\widehat{g}$ in (45):

$$\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big\|\,|\nabla_x|^\delta\langle\nabla_{x,v}\rangle^{s_3}v^\alpha g(t)\Big\|^2_{L^2}\\
&=\int_{\mathbb{R}^{2d}_{k,\xi}}|k|^\delta\langle k,\xi\rangle^{s_3}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}|k|^\delta\langle k,\xi\rangle^{s_3}\widehat{\nabla\sigma}_1(k)\Big(\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}(k)\widehat{\mathscr{G}}_\varrho(t,k)\Big)\\
&\qquad\qquad\qquad\mathrm{D}^\alpha_\xi\widehat{\nabla_v\mathscr{M}}(\xi-tk)\,\mathrm{d}k\,\mathrm{d}\xi\\
&\quad-\int_{\mathbb{R}^{3d}_{k,\xi,n}}|k|^\delta\langle k,\xi\rangle^{s_3}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}|k|^\delta\langle k,\xi\rangle^{s_3}n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\Big)\\
&\qquad\qquad\qquad(\xi-tk)\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tk)\,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi\\
&\quad-\sum_{\substack{j\in N^d\\|j|=1\,;\,j\le\alpha}}\int_{\mathbb{R}^{3d}_{k,\xi,n}}|k|^\delta\langle k,\xi\rangle^{s_3}\mathrm{D}^\alpha_\xi\overline{\widehat{g}(t,k,\xi)}|k|^\delta\langle k,\xi\rangle^{s_3}n\widehat{\sigma}_1(n)\Big(\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\Big)\\
&\qquad\qquad\qquad j\mathrm{D}^{\alpha-j}_\xi\widehat{g}(t,k-n,\xi-tk)\,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi\\
&=\mathrm{LT}+\mathrm{NLT1}+\mathrm{NLT2}.
\end{aligned}$$

We shall only detail how to handle NLT1; similar estimates apply for NLT2, see Remark 4.20.

Estimate of LT. Since

$$\langle k,\xi\rangle^{s_3} \lesssim \langle k,tk\rangle^{s_3}\langle\xi-tk\rangle^{s_3} \text{ and } \langle t\rangle^{1/2+\delta}|k|^{1/2+\delta} \le \langle k,tk\rangle^{1/2+\delta},$$

by using the Cauchy-Schwarz inequality and $s_4 - s_3 - 1 - \delta/2 > 0$, we get

$$
\begin{aligned}
|\mathrm{LT}| &\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \int_{k,\xi} |k|^{\delta} \langle k,\xi\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right| \frac{|k|^{1/2+\delta}\langle t\rangle^{1/2+\delta}}{\langle k,tk\rangle^{s_4-s_3}} \\
&\quad \times \langle k,tk\rangle^{s_4} |k|^{1/2} |\widehat{\sigma}_1(k)| \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}(k)\widehat{\mathscr{G}}_\varrho(t,k)\right| \langle \xi - tk\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{\nabla_v \mathscr{M}}(\xi - tk)\right| \,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left(\int_{\mathbb{R}^{2d}_{k,\xi}} |k|^{2\delta} \langle k,\xi\rangle^{2s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle k,tk\rangle^{2s_4} |k| |\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \langle \xi - tn\rangle^{2s_3} \left|\mathrm{D}_\xi^\alpha \widehat{\nabla_v \mathscr{M}}(\xi - tk)\right|^2 \,\mathrm{d}k\,\mathrm{d}\xi\right)^{1/2} \\
&\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\nabla_x|^{\delta} g(t)\right\|_{H^{s_3}_P} \left(\int_{\mathbb{R}^d_k} \langle k,tk\rangle^{2s_4} |k| |\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \,\mathrm{d}k\right)^{1/2} \\
&\quad \times \left(\int_{\mathbb{R}^d_\xi} \langle \xi\rangle^{2s_3} \left|\mathrm{D}_\xi^\alpha \widehat{\nabla_v \mathscr{M}}(\xi)\right|^2 \,\mathrm{d}\xi\right)^{1/2} \\
&\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\nabla_x|^{\delta} g(t)\right\|_{H^{s_3}_P} B(t) \left\|\nabla_v \mathscr{M}\right\|_{H^{s_3}_P} \lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\nabla_x|^{\delta} g(t)\right\|_{H^{s_3}_P} B(t)
\end{aligned}
$$

The Young inequality then yields

$$
|\mathrm{LT}| \lesssim \frac{\tilde{\delta}}{\langle t\rangle^{1+2\delta}} \left\| |\nabla_x|^{\delta} g(t)\right\|^2_{H^{s_3}_P} + \frac{B(t)^2}{\tilde{\delta}} \lesssim \tilde{\delta} K_3 \varepsilon^2 \langle t\rangle^{-1-2\delta} + \frac{B(t)^2}{\tilde{\delta}}
$$

where we have used (36) for the second inequality.

Estimate of NLT1**.** Again, we can use (48), where we set

$$
f = \mathcal{F}^{-1}\left((k,\xi) \mapsto |k|^{\delta} \langle k,\xi\rangle^{s_3} \mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right),
$$

and we split the contributions of low and high frequencies

$$
\begin{aligned}
|\mathrm{NLT1}| &\leq \int_{\mathbb{R}^{3d}_{k,\xi,n}} \left(\mathbf{1}_{|n,tn| \geq |k-n,\xi-tn|} + \mathbf{1}_{|n,tn| \leq |k-n,\xi-tn|}\right) |k|^{\delta} \langle k,\xi\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right| \\
&\quad \times \left| |k|^{\delta} \langle k,\xi\rangle^{s_3} - |k-n|^{\delta} \langle k-n,\xi-tn\rangle^{s_3}\right| |n| |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\quad \times |\xi - tk| \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k-n,\xi-tk)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&= \mathrm{NLT1R} + \mathrm{NLT1T}.
\end{aligned}
$$

Estimate of NLT1R**.** We make 4 terms appear, remarking that $|n, tn| \geq |k-n, \xi - tn|$ and $\delta < 1$ allow us to write

$$\Big||k|^\delta \langle k,\xi\rangle^{s_3} - |k-n|^\delta \langle k-n, \xi - tn\rangle^{s_3}\Big| \lesssim (|n|^\delta + |k-n|^\delta)\langle n, tn\rangle^{s_3}$$

while $|\xi - tk| \leq |\xi - tn| + t|k-n|$. We get

$$\begin{aligned}
\mathrm{NLT1R} \quad &\lesssim \quad \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n|^{1+\delta} |\widehat{\sigma}_1(n)| \langle n, tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\quad \times (|\xi - tn| + t|k-n|) \left|\mathrm{D}^\alpha_\xi \widehat{g}(t, k-n, \xi - tk)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\quad + \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n| |\widehat{\sigma}_1(n)| \langle n, tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\quad \times |k-n|^\delta (|\xi - tn| + t|k-n|) \left|\mathrm{D}^\alpha_\xi \widehat{g}(t, k-n, \xi - tk)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&= \quad R_{1,V} + R_{1,Z} + R_{2,V} + R_{2,Z}
\end{aligned}$$

where $R_{i,V}$ is the term with $|\xi - tn|$ and $R_{i,Z}$ the term with $t|k-n|$.

For $R_{1,V}$ we apply Lemma C.9

$$\begin{aligned}
R_{1,V} \quad &= \quad \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n|^{1+\delta} |\widehat{\sigma}_1(n)| \langle n, tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\quad \times |\xi - tn| \left|\mathrm{D}^\alpha_\xi \widehat{g}(t, k-n, \xi - tk)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \quad \left(\int_{\mathbb{R}^{2d}_{k,\xi}} |k|^{2\delta} \langle k,\xi\rangle^{2s_3} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}k\,\mathrm{d}\xi\right)^{1/2} \\
&\quad \times \frac{1}{\langle t\rangle^{1/2+\delta}} \left(\int_{\mathbb{R}^d_n} \frac{|n|^{1+2\delta}\langle t\rangle^{1+2\delta}}{\langle n, tn\rangle^{2s_4 - 2s_3}} |\widehat{\sigma}_1(n)| \langle n, tn\rangle^{2s_4} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2 \mathrm{d}n\right)^{1/2} \\
&\quad \times \int_{\mathbb{R}^d_k} \left(\int_{\mathbb{R}^d_\xi} |\xi|^2 \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \mathrm{d}k \\
&\lesssim \quad \frac{1}{\langle t\rangle^{1/2+\delta}} \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P} B(t) \int_{\mathbb{R}^d_k} \left(\int_\xi |\xi|^2 \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \mathrm{d}k
\end{aligned}$$

where we have used the relations $|n|\langle t\rangle \leq \langle n, tn\rangle$ and $2s_4 - 2s_3 - 1 - 2\delta > 0$). We have already seen (see the estimate of NLT1R when dealing with the norm $H^{s_4}_P$ of $\langle \nabla_v\rangle g(t)$) that

$$\int_{\mathbb{R}^d_k} \left(\int_{\mathbb{R}^d_\xi} |\xi|^2 \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi\right)^{1/2} \mathrm{d}k \lesssim \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P}.$$

Using (36) and the Young inequality, we obtain

$$R_{1,V} \lesssim K_3^2 \varepsilon^3 \langle t\rangle^{-1-2\delta} + \varepsilon B(t)^2.$$

For $R_{1,Z}$ we apply the second inequality in Lemma C.9 and we get

$$\begin{aligned} R_{1,Z} &= t \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right| |n|^{1+\delta} |\widehat{\sigma}_1(n)| \langle n, tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\ &\quad \times |k-n| \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k-n,\xi-tk)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\ &\lesssim \langle t\rangle \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P} \left(\int_{\mathbb{R}^d_n} |n|^{1+\delta}|\widehat{\sigma}_1(n)|\langle n,tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \mathrm{d}n\right) \\ &\quad \times \left(\int_{\mathbb{R}^d_{k,\xi}} |k-n|^2 \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}k\,\mathrm{d}\xi\right)^{1/2}. \end{aligned}$$

Cauchy-Schwarz's inequality yields

$$\begin{aligned} &\int_{\mathbb{R}^d_n} |n|^{1+\delta}|\widehat{\sigma}_1(n)|\langle n,tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \mathrm{d}n \\ &\lesssim \frac{1}{\langle t\rangle^{1/2+\delta}} \left(\int_{\mathbb{R}^d_n} \frac{|n|^{1+2\delta}\langle t\rangle^{1+2\delta}}{\langle n,tn\rangle^{2s_4-2s_3}} \,\mathrm{d}n\right)^{1/2} \\ &\qquad \times \left(\int_{\mathbb{R}^d_n} |n|\langle n,tn\rangle^{2s_4} |\widehat{\sigma}_1(n)|^2 \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2 \mathrm{d}n\right)^{1/2}. \end{aligned}$$

Since

$$\int_{\mathbb{R}^d_n} \frac{|n|^{1+2\delta}\langle t\rangle^{1+2\delta}}{\langle n,tn\rangle^{2s_4-2s_3}} \,\mathrm{d}n \le \int_{\mathbb{R}^d_n} \frac{1}{\langle n,tn\rangle^{2s_4-2s_3-1-2\delta}} \,\mathrm{d}n \lesssim \frac{1}{\langle t\rangle^d},$$

we deduce that

$$\int_{\mathbb{R}^d_n} |n|^{1+\delta}|\widehat{\sigma}_1(n)|\langle n,tn\rangle^{s_3} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \mathrm{d}n \lesssim \frac{1}{\langle t\rangle^{(d+1)/2+\delta}} B(t).$$

Besides, we can dominate

$$\left(\int_{\mathbb{R}^d_{k,\xi}} |k-n|^2 \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right|^2\right)^{1/2} \lesssim \left\||\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P},$$

since, assuming s_3 large enough, with $\delta < 1$, we have

$$|k|^2 = |k|^{2\delta}|k|^{2-2\delta} \le |k|^{2\delta}\langle k,\xi\rangle^{2-2\delta} \le |k|^{2\delta}\langle k,\xi\rangle^{2s_3}.$$

By applying (36) and the Young inequality, we end up with

$$R_{1,Z} \lesssim \frac{1}{\varepsilon\langle t\rangle^{d-1+2\delta}} \left\||\nabla_x|^\delta g(t)\right\|^4_{H^{s_3}_P} + \varepsilon B^2(t) \lesssim K_3^2\varepsilon^3\langle t\rangle^{1-d-2\delta} + \varepsilon B(t)^2.$$

The expressions of $R_{2,V}$ and $R_{2,Z}$ already involve $|k-n|^\delta$ with $\mathrm{D}_\xi^\alpha \widehat{g}(t,k-n,\xi-tn)$, and we can reproduce similar arguments as for $R_{1,Z}$; we obtain

$$R_{2,V} \lesssim K_3^2 \varepsilon^3 \langle t\rangle^{-1-d} + \varepsilon B(t)^2.$$

and

$$R_{2,Z} \lesssim K_3^2 \varepsilon^3 \langle t\rangle^{1-d} + \varepsilon B(t)^2.$$

Observe that among $R_{1,Z}$, $R_{2,V}$ and $R_{2,Z}$, the worst domination is for $R_{2,Z}$. Thus it will guide the determination of the constants in the final estimate.

Estimate of NLT1T**.** We split as NLT1T = NLT1T1 + NLT1T2 noting that, on the integration domain, see [7, Section 5.2]

$$\begin{aligned}
&\Big||k|^\delta\langle k,\xi\rangle^{s_3} - |k-n|^\delta \langle k-n,\xi-tn\rangle^{s_3}\Big| \\
&\lesssim |k-n|^\delta |n,tn| \langle k-n,\xi-tn\rangle^{s_3-1} + \Big||k|^\delta - |k-n|^\delta\Big| \langle k-n,\xi-tn\rangle^{s_3}.
\end{aligned}$$

Here, NLT1T1 stands for the term that involves the exponent s_3-1). We use Lemma C.9 and $|\xi - tk| \leq \langle t\rangle\langle k-n,\xi-tn\rangle$) so that

$$\begin{aligned}
|\mathrm{NLT1T1}| \quad &\lesssim \quad \langle t\rangle \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right| |n||n,tn||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\qquad \times |k-n|^\delta \langle k-n,\xi-tn\rangle^{s_3} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \quad \langle t\rangle \left\| |\nabla_x|^\delta g(t)\right\|^2_{H^{s_3}_P} \int_{\mathbb{R}^d_n} |n||n,tn||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n.
\end{aligned}$$

By virtue of (19c) and (38), we have

$$|\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \lesssim \frac{\langle t\rangle^\eta}{\langle n,tn\rangle^{s_1}}(1+K_5)\varepsilon,$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d_n} |n||n,tn||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n \\
&\qquad \lesssim \frac{1}{\langle t\rangle}\left(\int_{\mathbb{R}^d_n} |n|\langle t\rangle|n,tn| \frac{\langle t\rangle^\eta}{\langle n,tn\rangle^{s_1}} \,\mathrm{d}n\right)(1+K_5)\varepsilon \\
&\qquad \lesssim (1+K_5)\varepsilon\langle t\rangle^{\eta-1} \int_{\mathbb{R}^d_n} \langle n,tn\rangle^{s_1-2}\,\mathrm{d}n \lesssim (1+K_5)\varepsilon\langle t\rangle^{\eta-d-1}.
\end{aligned}$$

We combine this to (36) and we arrive at

$$|\mathrm{NLT1T1}| \lesssim (1+K_5)K_3\varepsilon^3\langle t\rangle^{\eta-d}.$$

We proceed similarly for NLT1T2, applying Lemma C.9, and remarking that $\left| |k|^\delta - |k-n|^\delta \right| \le |n|^\delta$ and $|\xi - tk| \le \langle t(k-n), \xi - tn\rangle$. We get

$$
\begin{aligned}
|\mathrm{NLT1T2}| &\lesssim \int_{\mathbb{R}^{3d}_{k,\xi,n}} |k|^\delta \langle k,\xi\rangle^{s_3} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| |n|^{1+\delta} |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \\
&\quad \times \langle t(k-n), \xi - tn\rangle^{s_3+1} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \,\mathrm{d}n\,\mathrm{d}k\,\mathrm{d}\xi \\
&\lesssim \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P} \left(\int_{\mathbb{R}^d_n} |n|^{1+\delta} |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n \right) \\
&\quad \times \left(\int_{\mathbb{R}^{2d}_{k,\xi}} \langle tk,\xi\rangle^2 \langle k,\xi\rangle^{2s_3} \left|\mathrm{D}^\alpha_\xi g(t,k,\xi)\right|^2 \right)^{1/2} \\
&\lesssim \left\| |\nabla_x|^\delta g(t)\right\|_{H^{s_3}_P} \|\langle t\nabla_x, \nabla_v\rangle g(t)\|_{H^{s_4}_P} \left(\int_{\mathbb{R}^d_n} |n|^{1+\delta} |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n \right).
\end{aligned}
$$

With (19c) and (38), we show that

$$
\int_{\mathbb{R}^d_n} |n|^{1+\delta} |\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \,\mathrm{d}n \lesssim (1+K_5)\varepsilon \langle t\rangle^{\eta-d-1-\delta},
$$

which eventually leads to

$$
|\mathrm{NLT1T2}| \lesssim \sqrt{K_1K_3}(1+K_5)\varepsilon^3 \langle t\rangle^{5/2+\eta-d-1-\delta}.
$$

Recap. We have shown that, if g is a solution of (10a)–(10b) which satisfies (34)–(38) on $[0,T]$, then we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left\| |\nabla_x|^\delta g(t)\right\|^2_{H^{s_3}_P} &\lesssim \tilde{\delta} K_3 \varepsilon^2 \langle t\rangle^{-1-2\delta} + \frac{B(t)^2}{\tilde{\delta}} \\
&+ K_3^2 \varepsilon^3 \langle t\rangle^{-1-2\delta} + \varepsilon B(t)^2 + K_3^2\varepsilon^3 \langle t\rangle^{1-d} \\
&+ K_3(1+K_5)\varepsilon^3 \langle t\rangle^{\eta-d} + \sqrt{K_1K_3}(1+K_5)\varepsilon^3 \langle t\rangle^{5/2+\eta-d-1-\delta}.
\end{aligned}
$$

Let C_5 be the constant associated to the $\lesssim$ symbol. We integrate over $[0,T]$and we bear in mind that all the exponents of $\langle t\rangle$ are strictly less than 1. We get

$$
\begin{aligned}
\left\| |\nabla_x|^\delta g(T)\right\|^2_{H^{s_3}_P} &\le \left\| |\nabla_x|^\delta g(0)\right\|^2_{H^{s_3}_P} + C_5 \tilde{\delta} K_3 \varepsilon^2 + C_5 \frac{1+K_2}{\tilde{\delta}} \varepsilon^2 \\
&+ C_5 K_3^2 \varepsilon^3 + C_5(1+K_2)\varepsilon^3 + C_5 K_3^2 \varepsilon^3 \\
&+ C_5 K_3 (1+K_5)\varepsilon^3 + C_5 \sqrt{K_1K_3}(1+K_5)\varepsilon^3.
\end{aligned}
$$

First, let $\tilde{\delta} \ll 1$ such that $\tilde{\delta} C_5 < 1/2$. Second, pick $K_3 \gg 1$ so that

$$\left\| |\nabla_x|^{\delta} g(0) \right\|^2_{H^{s_3}_P} + C_5 \frac{1+K_2}{\tilde{\delta}} \varepsilon^2 \leq \frac{K_3}{2} \varepsilon^2.$$

Finally, choose $\varepsilon \ll 1$ such that

$$\begin{aligned} &+C_5 K_3^2 \varepsilon^3 + C_5(1+K_2)\varepsilon^3 + C_5 K_3^2 \varepsilon^3 \\ &\quad + C_5 K_3 (1+K_5)\varepsilon^3 + C_5 \sqrt{K_1 K_3}(1+K_5)\varepsilon^3 \leq K_3 \varepsilon^2. \end{aligned}$$

We conclude that

$$\left\| |\nabla_x|^{\delta} g(T) \right\|^2_{H^{s_3}_P} \leq 2K_3 \varepsilon^2$$

holds.

4.5.4 Estimate of the $L^\infty_{(k,\xi)}$ norm of $\widehat{\langle \nabla_{x,v} \rangle^{s_1} g}(t)$

We go back to (29) and we write

$$\begin{aligned} &\langle k,\xi\rangle^{s_1} |\widehat{g}(T,k,\xi)| \leq \langle k,\xi\rangle^{s_1} |\widehat{f}_0(k,\xi)| \\ &+ \int_0^T \left| k\widehat{\sigma}_1(k) \left(\widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k) \widehat{\mathscr{G}}_\varrho(t,k) \right) \cdot (\xi - tk) \widehat{\mathscr{M}}(\xi - tk) \right| \,\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^d} \left| n\widehat{\sigma}_1(n) \left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n) \widehat{\mathscr{G}}_\varrho(t,n) \right) (\xi - tk) \widehat{g}(\tau, k-n, \xi - tn) \right| \,\mathrm{d}t \,\mathrm{d}n \\ &= \mathrm{CT} + \mathrm{LT} + \mathrm{NLT}. \end{aligned}$$

We also split the non linear term $\mathrm{NLT} = \mathrm{NLT1} + \mathrm{NLT2}$ according to

$$\langle k,\xi\rangle^{s_1} \lesssim \langle n,tn\rangle^{s_1} + \langle k-n,\xi-tn\rangle^{s_1}.$$

Estimate of CT. Since $(x,v) \mapsto x^\alpha f_0(x,v)$ lies in H^s_P, with $|\alpha| \leq P$, it satisfies $|\widehat{f}_0(k,\xi)| \lesssim \langle k,\xi\rangle^{-s}$, see (24). Hence, assuming $s \geq s_1$, we get

$$\mathrm{CT} \lesssim \varepsilon.$$

Estimate of LT. We use

$$\langle k,\xi\rangle^{s_1} \lesssim \langle k,tk\rangle^{s_1} + \langle \xi - tk\rangle^{s_1} \leq \langle k,tk\rangle^{s_2} \langle \xi - tk\rangle^{s_1}.$$

The Cauchy-Schwarz inequality then leads to

$$\begin{aligned} \mathrm{LT} \lesssim &\left(\int_0^T |k| \langle k,tk\rangle^{2s_2} ||k|\widehat{\sigma}_1(k)|^2 \left| \widehat{\mathscr{F}}_I(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k) \right|^2 \,\mathrm{d}t \right)^{1/2} \\ &\times \left(\int_0^T \langle \xi - tk\rangle^{2s_1} \left| \widehat{\nabla_v \mathscr{M}}(\xi - tk) \right|^2 \,\mathrm{d}t \right)^{1/2}. \end{aligned}$$

For the first term, (19b) and (37) allow us to get

$$\left(\int_0^T |k|\langle k,tk\rangle^{2s_2}||k|\widehat{\sigma}_1(k)|^2 \left|\widehat{\mathscr{F}}_I(t,k)-\widehat{\sigma}_1(k)\widehat{\mathscr{G}}_\varrho(t,k)\right|^2 \,\mathrm{d}t\right)^{1/2} \lesssim \sqrt{1+K_4}\;\varepsilon\langle T\rangle^{\eta/2}.$$

For the second term, since $\nabla_v\mathscr{M}\in H_P^{\tilde{s}}$, we can write

$$(\xi\mapsto\langle\xi\rangle^{s_1}\widehat{\nabla_v\mathscr{M}}(\xi))\in H^P_{(\xi)}.$$

Finally, the Trace Lemma 4.4 yields

$$\int_0^T \langle\xi-tk\rangle^{2s_1}\left|\widehat{\nabla_v\mathscr{M}}(\xi-tk)\right|^2 \,\mathrm{d}t \lesssim \|\xi\mapsto\langle\xi\rangle^{s_1}\widehat{\nabla_v\mathscr{M}}(\xi)\|^2_{H^P_{(\xi)}} \lesssim \|\nabla_v\mathscr{M}\|^2_{H_P^{s_1}} \lesssim 1.$$

We have thus shown

$$\mathrm{LT}\lesssim\sqrt{1+K_4}\;\varepsilon\langle T\rangle^{\eta/2}.$$

Estimate of NLT1. The Cauchy-Schwarz inequality yields

$$\mathrm{NLT1}=\int_0^T\int_{\mathbb{R}^d_n}\langle n,tn\rangle^{s_1}|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right||\xi-tk||\widehat{g}(t,k-n,\xi-tk)|\,\mathrm{d}t\,\mathrm{d}n$$

$$\leq\int_{\mathbb{R}^d_n}\left(\int_0^T\langle n\rangle^4\langle n,tn\rangle^{2s_2}|n||\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2\,\mathrm{d}t\right)^{1/2}$$

$$\times\left(\int_0^T\frac{|n||\xi-tk|^2}{\langle n\rangle^4\langle n,tn\rangle^{2s_2-2s_1}}\frac{\langle k-n,\xi-tn\rangle^{2s_1}}{\langle k-n,\xi-tn\rangle^{2s_1}}|\widehat{g}(t,k-n,\xi-tn)|^2\,\mathrm{d}t\right)^{1/2}\,\mathrm{d}n.$$

Next (19b) and (37) lead to

$$\left(\int_0^T\langle n\rangle^4\langle n,tn\rangle^{2s_2}|n||\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2\,\mathrm{d}t\right)^{1/2}\lesssim\sqrt{1+K_4}\;\varepsilon\langle T\rangle^{\eta/2},$$

and (38) ensures that

$$\langle k-n,\xi-tn\rangle^{s_1}|\widehat{g}(t,k-n,\xi-tn)|\lesssim K_5\varepsilon\langle t\rangle^{\eta}.$$

Therefore, we get

$$\mathrm{NLT1}\lesssim\sqrt{1+K_4}\,K_5\varepsilon^2\langle T\rangle^{\eta/2}\left[\int_{\mathbb{R}^d_n}\left(\int_0^T\frac{|n||\xi-tk|^2}{\langle n\rangle^4\langle n,tn\rangle^{2s_2-2s_1}}\frac{\langle t\rangle^{2\eta}}{\langle k-n,\xi-tn\rangle^{2s_1}}\right)^{1/2}\,\mathrm{d}n\right].$$

We are left with the task of justifying that the last integrals bounded uniformly with respect to k, ξ and T; this will be detailed in Section 4.6 below.

Estimate of NLT2. We combine (19c) and (38) so that

$$|\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|\lesssim\frac{\langle t\rangle^\eta}{\langle n\rangle^2\langle n,tn\rangle^{s_1}}(1+K_5)\varepsilon.$$

Applying the Cauchy-Schwarz inequality (and $|\xi-tk|=|\xi-tn+t(n-k)|\le\langle t\rangle\langle k-n,\xi-tn\rangle$) we obtain

$$\begin{aligned}
\text{NLT2} &= \int_0^T\int_{\mathbb{R}^d_n}|n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n))\right| \\
&\qquad\times\langle k-n,\xi-tn\rangle^{s_1}|\xi-tk||\widehat{g}(t,k-n,\xi-tn)|\,\mathrm{d}t\,\mathrm{d}n \\
&\lesssim(1+K_5)\varepsilon\int_0^T\int_{\mathbb{R}^d_n}\frac{|n|\langle t\rangle^{1+\eta}}{\langle n\rangle^2\langle n,tn\rangle^{s_1}}\langle k-n,\xi-tn\rangle^{s_1+1}|\widehat{g}(t,k-n,\xi-tn)|\,\mathrm{d}t\,\mathrm{d}n \\
&\lesssim(1+K_5)\varepsilon\left(\int_0^T\int_{\mathbb{R}^d_n}\frac{|n|^2\langle t\rangle^{2+2\eta}}{\langle n\rangle^4\langle n,tn\rangle^{2s_1}}\frac{1}{|k-n|^{2\delta}}\,\mathrm{d}t\,\mathrm{d}n\right)^{1/2} \\
&\qquad\times\left(\int_0^T\int_{\mathbb{R}^d_n}|k-n|^{2\delta}\langle k-n,\xi-tn\rangle^{2s_1+2}|\widehat{g}(t,k-n,\xi-tn)|^2\,\mathrm{d}t\,\mathrm{d}n\right)^{1/2}.
\end{aligned}$$

Then, by Trace Lemma and (36), we have (for $k\neq0$)

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}|k-n|^{2\delta}\langle k-n,tk-\tau n\rangle^{2s_3}\left|\widehat{g}(\tau,k-n,tk-\tau n)\right|^2\,\mathrm{d}\tau\,\mathrm{d}n \\
&\lesssim\sup_{s\in[0,T]}\left\||\nabla_x|^\delta g(s)\right\|^2_{H^{s_3}}\lesssim K_3\varepsilon^2
\end{aligned}$$

Going back to NLT2 we are finally led to

$$\text{NLT2}\lesssim(1+K_5)\sqrt{K_3}\varepsilon^2\times\left(\int_0^T\int_{\mathbb{R}^d}\frac{|n|^2\langle t\rangle^{2+2\eta}}{\langle n\rangle^2\langle n,tn\rangle^{2s_1}}\frac{1}{|k-n|^{2\delta}}\,\mathrm{d}t\,\mathrm{d}n\right)^{1/2}$$

and it remains to check that the integral is uniformly bounded with respect to both k and T. Again, we postpone this estimate to Section 4.6 below.

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying (34)–(38) on $[0,T]$, then, we have

$$\begin{aligned}
\|\widehat{\langle\nabla_{x,v}\rangle g}(T)\|_{L^\infty_{(k,\xi)}} &\lesssim (1+\sqrt{1+K_4}\langle T\rangle^{\eta/2}+\sqrt{1+K_4}K_5\varepsilon\langle T\rangle^{\eta/2}+(1+K_5)\sqrt{K_3}\varepsilon)\varepsilon \\
&\lesssim (1+\sqrt{K_4}+(1+\sqrt{K_4})K_5\varepsilon+(1+K_5)\sqrt{K_3}\varepsilon)\varepsilon\langle T\rangle^\eta.
\end{aligned}$$

Let C_6 be the constant involved in $\lesssim$. We set $K_5\gg1$ such that $C_6(1+\sqrt{K_4})\le K_5$ and, next, we pick $\varepsilon\ll1$ so that $C_6[(1+\sqrt{K_4})K_5\varepsilon+(1+K_5)\sqrt{K_3}\varepsilon]\le K_5$. We are thus led to

$$\|\widehat{\langle\nabla_{x,v}\rangle g}(T)\|_{L^\infty_{(k,\xi)}}\le2K_5\langle T\rangle^\eta\varepsilon.$$

We have checked at all steps of the proof that the choices of the constants K_i and of the parameter ε are compatible.

4.6 Integral estimates

We collect here the estimates of the four integrals that we need to finish the proof of the bootstrap property. Namely, we wish to control, uniformly with respect to k, ξ and T the following four quantities (in the same order as they appeared within the previous discussion)

$$\begin{aligned}
\mathrm{I1} &= \int_0^T \int_0^t \int_{\mathbb{R}^d} \frac{|k|^3 |t-\tau|^2 |n| \langle \tau \rangle^{2\eta}}{\langle n \rangle^4 \langle n, \tau n \rangle^{2s_4 - 2s_2} \langle k-n, tk-\tau n \rangle^{2s_1}} \, \mathrm{d}t \, \mathrm{d}\tau \, \mathrm{d}n, \\
\mathrm{I2} &= \int_0^T \int_0^t \int_{\mathbb{R}^d} \frac{|n|^2 \langle \tau \rangle^{2\eta+2}}{\langle n \rangle^2 \langle n, \tau n \rangle^{2s_1}} \frac{|k|}{\langle k-n, tk-\tau n} \frac{1}{|k-n|^{2\delta}} \, \mathrm{d}t \, \mathrm{d}\tau \, \mathrm{d}n, \\
\mathrm{I3} &= \int_{\mathbb{R}^d} \left(\int_0^T \frac{|n| |\xi - tk|^2}{\langle n \rangle^4 \langle n, tn \rangle^{2s_2 - 2s_1}} \frac{\langle t \rangle^{2\eta}}{\langle k-n, \xi - tn \rangle^{2s_1}} \, \mathrm{d}t \right)^{1/2} \mathrm{d}n, \\
\mathrm{I4} &= \int_0^T \int_{\mathbb{R}^d} \frac{|n|^2 \langle t \rangle^{2+2\eta}}{\langle n \rangle^2 \langle n, tn \rangle^{2s_1}} \frac{1}{|k-n|^{2\delta}} \, \mathrm{d}t \, \mathrm{d}n.
\end{aligned}$$

Let us start with I4 which satisfies

$$\mathrm{I4} \leq \int_0^T \langle t \rangle^{2\eta} \left(\int_{\mathbb{R}^d} \frac{\langle n, tn \rangle^{-2s_1+2}}{|k-n|^{2\delta}} \, \mathrm{d}n \right) \mathrm{d}t.$$

Given $t \geq 0$, we have seen during the proof of Theorem 4.8 that

$$\int_{\mathbb{R}^d} \frac{\langle n, tn \rangle^{-2s_1+2}}{|k-n|^{2\delta}} \, \mathrm{d}n \lesssim \langle t \rangle^{-d+2\delta}$$

holds. It follows that

$$\mathrm{I4} \lesssim \int_0^T \langle t \rangle^{-2+2\eta+2\delta} \, \mathrm{d}t \lesssim 1.$$

Next, for estimating I3, we observe that $|\xi - tn| \leq \langle t \rangle \langle k-n, \xi - tn \rangle$, so that

$$\begin{aligned}
\mathrm{I3} \quad &\leq \quad \int_{\mathbb{R}^d} \frac{1}{\langle n \rangle^2} \left(\int_0^T \frac{|n| \langle t \rangle^{2+2\eta}}{\langle n, tn \rangle^{2s_2 - 2s_1}} \frac{1}{\langle k-n, \xi - tn \rangle^{2s_1 - 2}} \, \mathrm{d}t \right)^{1/2} \mathrm{d}n \\
&\lesssim \quad \int_{\mathbb{R}^d} \frac{1}{\langle n \rangle^2} \left(\frac{1}{|n|^{1+2\eta}} \right)^{1/2} \left(\int_0^T \frac{\langle n, tn \rangle^{-2s_2+2s_1+2+2\eta}}{\langle k-n \rangle^{2s_1-2}} \, \mathrm{d}t \right)^{1/2} \mathrm{d}n \\
&\lesssim \quad \int_{\mathbb{R}^d} \frac{1}{\langle n \rangle^2} \frac{1}{|n|^{1/2+\eta}} \frac{1}{\langle k-n \rangle^{s_1-1}} \left(\int_0^T \langle n, tn \rangle^{-2s_2+2s_1+2+2\eta} \, \mathrm{d}t \right)^{1/2} \mathrm{d}n.
\end{aligned}$$

For any $n \neq 0$ fixed, we get (with s_2 sufficiently larger than s_1)

$$\int_0^T (1+|n|^2+|n|^2 t^2)^{-s_2+s_1+1+\eta} \, \mathrm{d}t \leq \frac{1}{|n|} \int_0^{|n|T} (1+u^2)^{-s_2+s_1+1+\eta} \, \mathrm{d}t \lesssim \frac{1}{|n|}.$$

Hence, we obtain

$$\mathrm{I3} \lesssim \int_{\mathbb{R}^d} \frac{1}{\langle n\rangle^2} \frac{1}{\langle k-n\rangle^{s_1-1}} \frac{1}{|n|^{1+\eta}} \,\mathrm{d}n \lesssim \int_{\mathbb{R}^d} \frac{1}{\langle n\rangle^{s_1-1}} \frac{1}{|k-n|^{1+\eta}} \,\mathrm{d}n \lesssim 1.$$

We estimate I2 by coming back to I4; indeed, I2 can be recast as

$$\mathrm{I2} = \int_0^T \int_{\mathbb{R}^d} \left(\int_\tau^T \frac{|k|}{\langle k-n, tk-\tau n\rangle^{2s_3-2s_2-2}} \,\mathrm{d}t \right) \frac{|n|^2 \langle \tau\rangle^{2\eta+2}}{\langle n\rangle^2 \langle n, \tau n\rangle^{2s_1}} \frac{1}{|k-n|^{2\delta}} \,\mathrm{d}\tau \,\mathrm{d}n.$$

It thus remains to show that

$$\int_{-\infty}^{+\infty} \frac{|k|}{\langle k-n, tk-\tau n\rangle^{2s_3-2s_2-2}} \,\mathrm{d}t$$

is bounded uniformly with respect to k. To this end, let us set

$$n_\parallel = \frac{k\cdot n}{|k|^2} k, \qquad n_\perp = n - n_\parallel.$$

For $k \neq 0$, we are led to

$$\begin{aligned} \langle k-n, tk-\tau n\rangle^2 &= 1 + |k-n_\parallel|^2 + |n_\perp|^2 + |tk-\tau n_\parallel|^2 + |\tau n_\perp|^2 \\ &\leq 1 + |tk-\tau n_\parallel|^2 = 1 + \left| t|k| - \tau \frac{k\cdot n}{|k|} \right|^2 = \left\langle t|k| - \tau \frac{k\cdot n}{|k|} \right\rangle^2, \end{aligned}$$

It yields

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{|k|}{\langle k-n, tk-\tau n\rangle^{2s_3-2s_2-2}} \,\mathrm{d}t &\leq \int_{-\infty}^{+\infty} \frac{|k|}{\left\langle t|k| - \tau \frac{k\cdot n}{|k|} \right\rangle^{2s_2-2s_3-2}} \,\mathrm{d}t \\ &\leq \int_{-\infty}^{+\infty} \frac{1}{\langle u\rangle^{2s_3-2s_2-2}} \,\mathrm{d}u \lesssim 1. \end{aligned}$$

We finally treat I1 like I2.

5 Analysis of the Landau damping on $\mathbb{T}^d$

The dispersive effect which has been used for proving the Landau damping on $\mathbb{R}^d$ does not exist on the torus. For this reason, in order to control the echoes, we shall work in the analytic framework, following [6]. For the Vlasov-Poisson problem, the analysis of [4] is a hint that this regularity could be necessary. As a counterpart of this regularity, there is no restriction on the space dimension d.

The proof still relies on a bootstrap argument, see [6]. There are two main arguments, like on $\mathbb{R}^d$: firstly, the force term $\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))$ can be controlled, in suitable norms, by the macroscopic density $\varrho(t)$, and, secondly, the contribution associated to the initial data $\int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau v)\cdot\nabla_v \mathscr{M}(v)\,\mathrm{d}\tau$ does not perturb too much the bootstrap property (here, we refer the reader to the remarks made when analyzing the whole space problem).

5.1 Functional framework

We start by introducing several Gevrey norms. Let $g : (0,\infty)_t \times \mathbb{T}^d_x \times \mathbb{R}^d_v \to \mathbb{R}$. The Gevrey norm $\|\cdot\|_{\mathcal{G}^{\lambda,\sigma;s}}$ is defined by

$$\|g(t)\|^2_{\mathcal{G}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda\langle k,\xi\rangle^s} \left|\widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi$$

and we also need the Gevrey norm $\|\cdot\|_{\mathcal{F}^{\lambda,\sigma;s}}$ given by

$$\|g(t)\|^2_{\mathcal{F}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \langle k,tk\rangle^{2\sigma} e^{2\lambda\langle k,tk\rangle^s} \left|\widehat{g}(t,k,tk)\right|^2.$$

Let $\varrho : \mathbb{R}_t \times \mathbb{T}^d_x \to \mathbb{R}$. The Gevrey norm $\|\cdot\|_{\mathcal{F}^{\lambda,\sigma;s}}$ reads

$$\|\varrho(t)\|^2_{\mathcal{F}^{\lambda,\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \langle k,tk\rangle^{2\sigma} e^{2\lambda\langle k,tk\rangle^s} \left|\widehat{\varrho}(t,k)\right|^2.$$

In what follows, we always assume $\lambda,\sigma \geq 0$ and $0 < s \leq 1$.

As a warm-up, we observe that, with $g(t,x,v) = f(t,x+tv,v)$ and $\varrho(t,x) = \int f(t,x,v)\,\mathrm{d}v$, we have

$$\|\varrho(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} = \|g(t)\|_{\mathcal{F}^{\lambda,\sigma;s}}.$$

Moreover, assuming $\sigma > d/2$ we have a $\sigma-$ring property: with $h(t,x,v) = \varrho(t,x+tv)g(t,x,v)$, we have

$$\|h(t)\|_{\mathcal{G}^{\lambda,\sigma;s}} \lesssim \|\varrho(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} \|g(t)\|_{\mathcal{G}^{\lambda,\sigma;s}}.$$

Finally, we shall also need the following Gevrey norm: for $P \in \mathbb{N}$, we define the norm $\|\cdot\|_{\mathcal{G}^{\lambda,\sigma;s}_P}$ of a function $(t,x,v) \mapsto g(t,x,v)$ by

$$\|g(t)\|^2_{\mathcal{G}^{\lambda,\sigma;s}_P} = \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \|(x,v)\mapsto v^\alpha g(t,x,v)\|^2_{\mathcal{G}^{\lambda,\sigma;s}} = \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\leq P}} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda\langle k,\xi\rangle^s} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right|^2 \mathrm{d}\xi.$$

The $\sigma-$ring estimate equally applies to this norm. Note that the weight in the exponential is $\langle k,\xi\rangle$, instead of $|k,\xi|$; this is useful to establish the following embedding property.

Proposition 5.1 *Let $\lambda > 0$, $0 < s \leq 1$ and $P \in \mathbb{N}$.*

i) (σ-ring estimate) Let $\sigma > d/2$, and set $h(t,x,v) = \varrho(t,x+tv)g(t,x,v)$. Then, we have

$$\|h(t)\|_{\mathcal{G}^{\lambda,\sigma;s}_P} \lesssim \|\varrho(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} \|g(t)\|_{\mathcal{G}^{\lambda,\sigma;s}_P}. \tag{49}$$

ii) (embedding) Let $\sigma \geq 0$, and suppose $P > d/2$. Then, there exists $C > 0$ such that for any $(t,x,v) \mapsto g(t,x,v) \in \mathcal{G}^{\lambda,\sigma;s}_P$, we have

$$\|g(t)\|_{\mathcal{F}^{\lambda,\sigma;s}} \leq C \|g(t)\|_{\mathcal{G}^{\lambda,\sigma;s}_P} \tag{50}$$

Proof. Let $\alpha \in \mathbb{N}^d$. We remark that

$$\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle} \lesssim (\langle n,tn\rangle^\sigma + \langle k-n,\xi-tn\rangle^\sigma) e^{\lambda\langle n,tn\rangle^s} e^{\lambda\langle k-n,\xi-tn\rangle^s}.$$

Denoting

$$N(t) = \|(t,x,v)\mapsto\ v^\alpha\,\varrho(t,x+tv)g(t,x,v)\|^2_{\mathcal{G}_P^{\lambda,\sigma;s}},$$

by using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
N(t) &= \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left|\sum_{n\in\mathbb{Z}^d}\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\,\widehat{\varrho}(t,n)\,\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|^2 \mathrm{d}\xi\\
&\lesssim \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left|\sum_{n\in\mathbb{Z}^d}\langle n,tn\rangle^\sigma e^{\lambda\langle n,tn\rangle^s}\,\widehat{\varrho}(t,n)e^{\lambda\langle k-n,\xi-tn\rangle^s}\,\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|^2 \mathrm{d}\xi\\
&\quad+\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left|\sum_{n\in\mathbb{Z}^d} e^{\lambda\langle n,tn\rangle^s}\,\widehat{\varrho}(t,n)\langle k-n,\xi-tn\rangle^\sigma e^{\lambda\langle k-n,\xi-tn\rangle^s}\,\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|^2 \mathrm{d}\xi\\
&\lesssim \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left(\sum_{n\in\mathbb{Z}^d}\langle k-n\rangle^{-2\sigma}\langle n,tn\rangle^{2\sigma}e^{2\lambda\langle n,tn\rangle^s}|\widehat{\varrho}(t,n)|^2\right)\\
&\qquad\times\left(\sum_{n\in\mathbb{Z}^d}\langle k-n,\xi-tn\rangle^{2\sigma}e^{2\lambda\langle k-n,\xi-tn\rangle^s}\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|^2\right)\mathrm{d}\xi\\
&\quad+\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left(\sum_{n\in\mathbb{Z}^d}\langle n,tn\rangle^{2\sigma}e^{2\lambda\langle n,tn\rangle^s}|\widehat{\varrho}(t,n)|^2\right)\\
&\qquad\times\left(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{-2\sigma}\langle k-n,\xi-tn\rangle^{2\sigma}e^{2\lambda\langle k-n,\xi-tn\rangle^s}\left|\mathrm{D}^\alpha_\xi\widehat{g}(t,k-n,\xi-tn)\right|^2\right)\mathrm{d}\xi.
\end{aligned}$$

We conclude that i) holds since the condition $\sigma > d/2$ implies that the series $\sum_k\langle k-n\rangle^{-2\sigma}$ and $\sum_n\langle n\rangle^{-2\sigma}$ are finite.

We turn to the proof of ii). For $0 < s \leq 1$, we get

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\leq P}}\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\left|\mathrm{D}^\alpha_\xi(\xi\mapsto\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi))\right|^2\mathrm{d}\xi \lesssim \|g(t)\|^2_{\mathcal{G}_P^{\lambda,\sigma;s}}. \tag{51}$$

Indeed, since $|\partial_{\xi_i}\langle k,\xi\rangle| = |\xi_i/\langle k,\xi\rangle| \leq 1$, we have

$$\left|\partial_{\xi_i}[\xi\mapsto\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi)]\right| \lesssim \langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}|\widehat{g}(t,k,\xi)| + \langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}|\partial_{\xi_i}\widehat{g}(t,k,\xi)|.$$

Repeating the argument, we establish that, for any multi-index α,

$$\left|\mathrm{D}^\alpha_\xi[\xi\mapsto\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi)]\right| \lesssim \sum_{j\leq\alpha}\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}|\mathrm{D}^j_\xi\widehat{g}(t,k,\xi)|.$$

Going back to (51) shows that, $g(t)$ being an element of $\mathcal{G}_P^{\lambda,\sigma;s}$, for any $k \in \mathbb{Z}^d$, we have

$$\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \int_{\mathbb{R}^d_\xi} \left|\mathrm{D}^\alpha_\xi(\xi \mapsto \langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi))\right|^2 \,\mathrm{d}\xi < +\infty.$$

In other words, $\xi \mapsto \langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi)$ belongs to $H^P(\mathbb{R}^d_\xi)$. Since $P > d/2$, Sobolev's embedding applies: this function is continuous, and, for any $k \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^d$, we get

$$\left|\langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi)\right| \lesssim \left(\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \int_{\mathbb{R}^d_\zeta} \left|\mathrm{D}^\alpha_\zeta[\zeta \mapsto \langle k,\zeta\rangle^\sigma e^{\lambda\langle k,\zeta\rangle^s}\widehat{g}(t,k,\zeta)]\right|^2 \,\mathrm{d}\zeta\right)^{1/2}.$$

It follows that

$$\begin{aligned}\|g(t)\|^2_{\mathcal{F}^{\lambda,\sigma;s}} &= \sum_{k\in\mathbb{Z}^d}\left|\langle k,tk\rangle^\sigma e^{\lambda\langle k,tk\rangle^s}\widehat{g}(t,k,tk)\right|^2\\ &\lesssim \sum_{k\in\mathbb{Z}^d}\sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \int_{\mathbb{R}^d_\xi} \left|\mathrm{D}^\alpha_\xi(\xi \mapsto \langle k,\xi\rangle^\sigma e^{\lambda\langle k,\xi\rangle^s}\widehat{g}(t,k,\xi))\right|^2 \,\mathrm{d}\xi \lesssim \|g(t)\|^2_{\mathcal{G}_P^{\lambda,\sigma;s}}.\end{aligned}$$

■

From now on, we assume that

$$\sigma > d/2, \qquad P > d/2, \qquad 0 < s \le 1.$$

We shall consider the parameter λ as a function of the time variable $\lambda : t \mapsto \lambda(t) \in (0,\infty)$, continuous and decreasing. The estimates (49) and (50) adapt to this context.

In contrast to what we did for the problem on $\mathbb{R}^d$, we do not express general conditions on $\mathscr{F}_I$ and p_c. Instead, we shall use the same assumptions as in the case of the linearized Landau damping. For the sake of convenience, let us recall them here.

(H1) $n \ge 3$ is odd,

(H2) $\sigma_2 \in C^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\sigma_2) \subset B(0,R_2)$.

(H3) $\mathrm{supp}(\psi_i) \subset \mathbb{T}^d \times B(0,R_I)$, $i = 1,2$ and

$$\mathscr{E}_I = \iint_{\mathbb{T}^d\times\mathbb{R}^n} \left(|\psi_1(x,z)|^2 + c^2|\nabla_z\psi_0(x,z)|\right) \mathrm{d}x\,\mathrm{d}z < +\infty.$$

(R2) $\sigma_1 : \mathbb{T}^d \to \mathbb{R}_+$ is radially symmetry and analytic; in particular there exist $C_1, \lambda_1 > 0$ such that $|\widehat{\sigma}_1(k)| \le C_1 \exp(-\lambda_1|k|)$ holds for any $k \in \mathbb{Z}^d$.

Note theat assumption (**R1**) on $\mathscr{M}$ and f_0 will be replaced by $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$.

As a consequence of **(H1)** and **(H2)** the kernel p_c has a compact support: $\mathrm{supp}(p_c) \subset [0, 2R_2/c]$, see Lemma 2.1. By virtue of **(H2)** and **(H3)**, $\mathscr{F}_I$ is compactly supported too: $\mathrm{supp}(\mathscr{F}_I) \subset [0,(R_I+R_c)/2]$, as pointed out in the proof of Lemme 3.2. In what follows, the following parameters will play an important role

$$2R_2/c, \qquad S_0 = (R_I+R_c)/2.$$

The following statement, analog for the torus of Proposition 4.2, is a crucial ingredient to justify the boostrap property.

Proposition 5.2 *Let **(H1)**–**(H3)** and **(R2)** be fulfilled. Let $t \mapsto \lambda(t) > 0$ be a continuous and decreasing function. For any $\sigma \geq 0$ and $0 < s \leq 1$, we get*

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \mathscr{E}_I \mathbf{1}_{0\leq t\leq S_0} + \int_0^t |p_c(t-\tau)|\, \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau, \tag{52}$$

Consequently, the following estimates hold

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \mathscr{E}_I + \int_0^t \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau, \tag{53a}$$

$$\sup_{\tau\in[0,t]} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \lesssim \mathscr{E}_I + \sup_{\tau\in[0,t]} \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}, \tag{53b}$$

$$\int_0^t \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau \lesssim \mathscr{E}_I + \int_0^t \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau. \tag{53c}$$

Remark 5.3 *The following observations will be useful:*

i) In the specific case $s = 1$ we shall need a further assumption on $\lambda(0)$: for this situation, we assume $\lambda(0) < C(\lambda_1, 2R_2/c, S_0) = \min(\lambda_1/\langle S_0\rangle, 2\lambda_1/\langle 2R_2/c\rangle)$.

ii) In contrast to the analysis of the Vlasov-Poisson problem, a control of $\int \|\varrho\|\,\mathrm{d}\tau$ ensures a pointwise control of the force term. This fact, which can be seen as a kind of regularizing effect of the half-time-convolution, simplifies the proof of the bootstrap property.

iii) Like for the whole space problem, the exponential decay of $\widehat{\sigma}_1(k)$ can be used to absorb any polynomial with respect to k that arises in the estimates, see Remark 4.3.

Proof. We estimate separately the contributions from $\mathscr{F}_I$ and $\mathscr{G}_\varrho$:

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \|\nabla\sigma_1 \star \mathscr{F}_I(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} + \|\nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}.$$

For the former, we use $\mathrm{supp}(\mathscr{F}_I) \subset [0,S_0]\times\mathbb{T}^d$ and the estimate (see the proof of Lemma 3.2)

$$|k|\,|\widehat{\sigma}_1(k)|\,|\widehat{\mathscr{F}}_I(t,k)| \leq C_1|k|e^{-\lambda_1|k|}\|\sigma_2\|_{L^{2n/(n+2)}}\sqrt{\mathscr{E}_I}\mathbf{1}_{0\leq t\leq S_0}. \tag{54}$$

We obtain

$$\begin{aligned}
&\|\nabla\sigma_1 \star \mathscr{F}_I(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s}|k|^2 e^{-2\lambda_1|k|^2}\right)\mathscr{E}_I\mathbf{1}_{0\leq t\leq S_0}\\
&\lesssim \left(\sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma}\langle S_0\rangle^{2\sigma} e^{2\lambda(0)\langle k\rangle^s\langle S_0\rangle^s}|k|^2 e^{-2\lambda_1|k|^2}\right)\mathscr{E}_I\mathbf{1}_{0\leq t\leq S_0}.
\end{aligned}$$

When $0 < s < 1$ the sum is finite; when $s = 1$ we should impose the additional condition $\lambda_1 > \lambda(0)\langle S_0\rangle$.

For the latter, we apply the Cauchy-Schwarz inequality, so that

$$
\begin{aligned}
&\| \nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \\
&= \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} |k|^2 |\widehat{\sigma}_1(k)|^4 \left| \int_0^t p_c(t-\tau)\widehat{\varrho}(\tau,k)\,\mathrm{d}\tau \right|^2 \\
&\le \left(\int_0^t |p_c(t-\tau)|\,\mathrm{d}\tau \right) \int_0^t |p_c(t-\tau)| \left(\sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} |\widehat{\varrho}(\tau,k)|^2 \right) \mathrm{d}\tau \\
&= \left(\int_0^t |p_c(t-\tau)|\,\mathrm{d}\tau \right) \int_0^t |p_c(t-\tau)| \left(\sum_{k\in\mathbb{Z}^d} I_k(t,\tau) \langle k, \tau k\rangle^{2\sigma} e^{2\lambda(t)\langle k,\tau k\rangle^s} |\widehat{\varrho}(\tau,k)|^2 \right) \mathrm{d}\tau.
\end{aligned}
$$

It follows that

$$
I_k(t,\tau) = |k|^2 |\widehat{\sigma}_1(k)|^4 \frac{\langle k,tk\rangle^{2\sigma}}{\langle k,\tau k\rangle^{2\sigma}} e^{2(\lambda(t)-\lambda(\tau)\langle k,tk\rangle^s} e^{\lambda(\tau)(\langle k,tk\rangle^s - \langle k,\tau k\rangle^s)}.
$$

Therefore if $I_k(t,\tau)$ is bounded uniformly with respect to k, t and τ, then we get

$$
\| \nabla\Sigma \star \mathscr{G}_\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \lesssim \int_0^t |p_c(t-\tau)|\, \|\varrho(\tau)\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau.
$$

We are left with the task of justify a uniform bound on $I_k(t,\tau)$. To this end, we remember that p_c has a compact support: we can restrict the time integration to $0 \le t-\tau \le 2R_2/c$. For $t \ge \tau$, a simple analysis of function shows that

$$
\sup_{k\in\mathbb{Z}^d} \frac{\langle k,tk\rangle^{2\sigma}}{\langle k,\tau k\rangle^{2\sigma}} \le \frac{\langle t\rangle^{2\sigma}}{\langle \tau\rangle^{2\sigma}} \le \langle t-\tau\rangle^{2\sigma} \le \langle 2R_2/c\rangle^{2\sigma}.
$$

Since $t \mapsto \lambda(t)$ is decreasing, we have $\exp(2(\lambda(t)-\lambda(\tau))\langle k,tk\rangle^s) \le 1$. Finally, with $0 < s \le 1$, we have

$$
|\langle x\rangle^s - \langle y\rangle^s| \le \langle x-y\rangle^s,
$$

so that $\langle k,tk\rangle^s - \langle k,\tau k\rangle^s \le \langle (t-\tau)k\rangle^s \le \langle \frac{2R_2}{c}k\rangle^s$ and $\exp(2\lambda(\tau)\,(\langle k,tk\rangle^s - \langle k,\tau k\rangle^s)) \le \exp(2\lambda(0)\langle \frac{2R_2}{c}\rangle^s\langle k\rangle^s)$. We conclude with

$$
I_k(t,\tau) \le C_1^4 |k|^2 e^{-4\lambda_1|k|} \langle 2R_2/c\rangle^{2\sigma} e^{2\lambda(0)\langle \mathcal{T}_M\rangle^s\langle k\rangle^s},
$$

when $0 < s < 1$, while for $s = 1$ we further assume $4\lambda_1 > 2\lambda(0)2R_2/c$.

Note that we have used in an essential way the fact that p_c is compactly supported. In Proposition 4.2, the polynomial decay **(D2)** was enough. This is due to the different weight that arise in the definition of the norms used for the analysis. ■

We turn to the estimate of the force term $\int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau v)\cdot\nabla_v\mathscr{M}(v)\,\mathrm{d}\tau$ by means of the norms involved in the bootstrap.

Proposition 5.4 *Let **(H1)**–**(H3)** and **(R2)** be fulfilled. Assume that $\mathscr{M} \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$ for some integer $P > d/2$. Let $t \mapsto \lambda(t) > 0$ be continuous, decreasing, and such that*

$\lambda(0) < \widetilde{\lambda_0}$. *Then for any $\sigma \geq 0$ and $0 < s \leq 1$, we have*

$$\int_0^T \left\| \int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x + \tau\, v) \cdot \nabla_v \mathscr{M}(v)\, \mathrm{d}\tau \right\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \mathrm{d}t \lesssim \mathscr{E}_I. \tag{55}$$

Remark 5.5 *Again, when $s = 1$ a constraint on $\lambda(0)$ like $\lambda(0) < C'(\lambda_1, S_0) = \lambda_1/\langle S_0\rangle$ should be imposed.*

Proof. We start with

$$\int_0^T \left\| \int_0^t \nabla\sigma_1 \star \mathscr{F}_I(\tau, x + \tau\, v) \cdot \nabla_v \mathscr{M}(v)\, \mathrm{d}\tau \right\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \mathrm{d}t$$

$$= \int_0^T \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} \left| \int_0^t k\widehat{\sigma}_1(k)\widehat{\mathscr{F}}_I(\tau,k) \cdot [t-\tau]k\widehat{\mathscr{M}}([t-\tau]k)\, \mathrm{d}\tau \right|^2 \mathrm{d}t$$

$$\leq \int_0^T \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \underbrace{\left(\int_0^t \langle k, tk\rangle^\sigma e^{\lambda(t)} \langle k, tk\rangle^s |k|\, |\widehat{\sigma}_1(k)|\, \left|\widehat{\mathscr{F}}_I(\tau,k)\right|\, |t-\tau|\, |k|\, \left|\widehat{\mathscr{M}}([t-\tau]k)\right| \mathrm{d}\tau \right)^2}_{=I(t,k)^2} \mathrm{d}t$$

and we are going to estimate $I(t,k)$. For any $k \neq 0$, we have $\langle t\rangle \leq \langle k, tk\rangle$, and since λ is decreasing, we obtain

$$\begin{aligned} I(t,k) \leq \langle t\rangle^{-1} \int_0^t &\langle k, \tau k\rangle^{\sigma+1} e^{\lambda(\tau)\langle k,\tau k\rangle^s} |k|\, |\widehat{\sigma}_1(k)|\, \left|\widehat{\mathscr{F}}_I(\tau,k)\right| \\ &\times \langle [t-\tau]k\rangle^{\sigma+1} e^{\lambda(\tau)\langle [t-\tau]k\rangle^s} |t-\tau|\, |k|\, \left|\widehat{\mathscr{M}}([t-\tau]k)\right| \mathrm{d}\tau. \end{aligned}$$

By using (54) and remarking that $\mathscr{M} \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$, for $P > d/2$, we are led to

$$|\widehat{\mathscr{M}}(\xi)| \lesssim e^{-\widetilde{\lambda_0}\langle\xi\rangle^s}$$

(since $\|\xi \mapsto \exp(\widetilde{\lambda_0}\langle\xi\rangle^s)\widehat{\mathscr{M}}(\xi)\|_{H^P} \lesssim \|\mathscr{M}\|_{\mathcal{G}_P^{\widetilde{\lambda_0},0;s}}$, and $P > d/2$ allows us to make use of the Sobolev embedding $H^P \hookrightarrow C^0$; we refer the reader to the proof of (50) for further details). We arrive at

$$\begin{aligned} I(t,k) \lesssim \langle t\rangle^{-1}\langle k\rangle^{\sigma+1}\langle S_0\rangle^{\sigma+1} &e^{\lambda(0)\langle k\rangle^s\langle S_0\rangle^s} |k| e^{-\lambda_1|k|} \\ &\times \left(\int_0^t \langle [t-\tau]k\rangle^{\sigma+1} e^{\lambda(0)\langle [t-\tau]k\rangle^s} |t-\tau|\, |k| e^{-\widetilde{\lambda_0}\langle [t-\tau]k\rangle^s} \mathrm{d}\tau \right) \sqrt{\mathscr{E}_I}. \end{aligned}$$

Since $\lambda(0) < \widetilde{\lambda_0}$ we have

$$\int_0^t \langle [t-\tau]k\rangle^{\sigma+1} e^{\lambda(0)\langle [t-\tau]k\rangle^s} |t-\tau|\, |k| e^{-\widetilde{\lambda_0}\langle [t-\tau]k\rangle^s} \mathrm{d}\tau \leq \int_{\mathbb{R}} \langle u\rangle^{\sigma+2} e^{-(\widetilde{\lambda_0}-\lambda(0))\langle u\rangle^s} \mathrm{d}u \lesssim 1.$$

Therefore, when $0 < s < 1$ we obtain $\int_0^T \sum_k I(t,k)^2\, \mathrm{d}t \lesssim \mathscr{E}_I$ and for $s = 1$ we conclude similarly at the price of a constraint like $\lambda_1 > \lambda(0)\langle S_0\rangle$. ∎

We now state an existence-uniqueness result for the Cauchy problem (10a)–(10b), in the functional spaces of interest. We will give a complete proof of this theorem in Appendix C.

Proposition 5.6 *Let $P > d/2$ be an integer and $\sigma > d/2$ be a real number. Let $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;1}$ with $\widetilde{\lambda_0} > 0$. Then, there exists $T^\star > 0$ and a continuous decreasing function $0 < \widetilde{\lambda}(t) < \min(\widetilde{\lambda_0}, \lambda_1/\langle S_0\rangle, 2\lambda_1\langle 2R_2/c\rangle)$ such that the problem* (10a)–(10b) *admits a unique solution $g \in C^0([0,T^\star); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1})$ on $[0,T^\star)$. Moreover, if for some $T \le T^\star$, we have*

$$\limsup_{t\nearrow T} \|g(t)\|_{H_P^\sigma} < +\infty$$

then, actually, $T < T^\star$.

Remark 5.7 *The constraint $\lambda(0) < \min(\widetilde{\lambda_0}, \lambda_1/\langle S_0\rangle, 2\lambda_1\langle 2R_2/c\rangle)$ comes from the fact that the proof uses Proposition 52.*

The analysis of the Landau Damping, as it is already clear for the linearized problem, relies heavily on the formulation of the problem by means of the Fourier variables. Let us collect the useful formula from which the reasoning starts. Integrating (10a)–(10b) over $[0,t]$, we get

$$g(t,x,v) = f_0(x,v) + \int_0^t \nabla_x \sigma_1 \star (\mathscr{F}_I - \sigma_1 \star \mathscr{G}_\varrho)(\tau, x+\tau v)\cdot(\nabla_v - \tau\nabla_x)(\mathscr{M}(v) + g(\tau,x,v))\,\mathrm{d}\tau.$$

Thus, we obtain

$$\widehat{g}(t,k,\xi) = \widehat{f_0}(k,\xi) - \int_0^t k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,k)\cdot(\xi - \tau k)\widehat{\mathscr{M}}(\xi - \tau k)\,\mathrm{d}\tau$$

$$- \sum_{n\in\mathbb{Z}^d} \int_0^t n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\cdot(\xi - \tau k)\widehat{g}(\tau, k-n, \xi - \tau n)\,\mathrm{d}\tau$$

and

$$\widehat{\varrho}(t,k) = \widehat{f_0}(k,tk) - \int_0^t k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau$$

$$- \sum_{n\in\mathbb{Z}^d} \int_0^t n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\cdot(t-\tau)k\,\widehat{g}(\tau, k-n, tk - \tau n)\,\mathrm{d}\tau.$$

5.2 Main result

That the Landau damping holds on the torus can be formulated as follows.

Theorem 5.8 (Landau damping in $\mathbb{T}^d$) *Suppose **(H1)**–**(H3)** and **(R2)**. Let $P > d/2$ be an integer, $0 < s \le 1$ be a real number and $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;s}$ with $\widetilde{\lambda_0} > 0$. There exist a universal constant ε_0, such that if*

$$\|f_0\|_{\mathcal{G}_P^{\widetilde{\lambda_0},\sigma;s}} \le \varepsilon_0 \ \ ; \ \ \mathscr{E}_I \le \varepsilon_0^2$$

and $\mathscr{M}$ satisfies $(\mathbf{L})$*, then, the unique solution g of* (10a)–(10b) *is globally defined. To be more specific, for any $0 < \lambda' < \widetilde{\lambda_0}$, we have $g \in C^0(\mathbb{R}_+; \mathcal{G}^{\lambda',0;s})$ and there exists an asymptotic density $g^\infty \in \mathcal{G}^{\lambda',0;s}$, the space average of which vanishes, such that*

$$\|g(t) - g^\infty\|_{\mathcal{G}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}, \tag{56a}$$

$$\|\varrho(t)\|_{\mathcal{F}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}, \tag{56b}$$

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|_{\mathcal{F}^{\lambda',0;s}} \lesssim \varepsilon_0 e^{-\frac{1}{2}(\widetilde{\lambda_0}-\lambda')\langle t\rangle^s}. \tag{56c}$$

Remark 5.9 *When $s = 1$ the constraint on λ' becomes $\lambda' < \min(\widetilde{\lambda_0}, \lambda_1/\langle S_0\rangle, 2\lambda_1\langle 2R_2/c\rangle)$.*

Like for the problem set on $\mathbb{R}^d$, the proof relies on a bootstrap argument, which, in this context, states as follows.

Proposition 5.10 (Bootstrap) *Let the assumptions of Theorem 5.8 be fulfilled. Let $\alpha_0 = (\widetilde{\lambda_0} + \lambda')/2$ and $\sigma > d/2 + 6$. There exists a function $\lambda : \mathbb{R}_+ \to (\alpha_0, \widetilde{\lambda_0})$, continuous and decreasing, a real $\beta > 2$ and constants $K_1, K_2, K_3, K_4 > 0$ such that if g is a solution of* (10a)–(10b) *on the time interval $[0, T]$ verifying*

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq 4K_1\langle t\rangle^7\varepsilon^2 \tag{57a}$$

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq 4K_2\varepsilon^2 \tag{57b}$$

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \leq 4K_3\varepsilon^2 \tag{57c}$$

for $0 < \varepsilon \leq \varepsilon_0$ small enough, then g also satisfies, on $[0, T]$, the estimates

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq 2K_1\langle t\rangle^7\varepsilon^2 \tag{58a}$$

$$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq 2K_2\varepsilon^2 \tag{58b}$$

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \leq 2K_3\varepsilon^2 \tag{58c}$$

$$\|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \leq 2K_4\langle t\rangle\varepsilon^2 \tag{58d}$$

Remark 5.11 *The role of (58d) is a bit different from its analog for the Vlasov-Poisson problem. Indeed, the interest of this estimate is to provide a pointwise control on the force term. However, here, as said above, such a control can be obtained by estimating $\int \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t$. Consequently (58c) is enough to finish the proof, without using (58d) and the proof slightly simplifies. Nevertheless, we keep (58d) in the statement since it is useful to justify (56b).*

We now explain how the Landau damping can be justified, having at hand the bootstrap statement.

Proof of Landau damping. We only detail the case $0 < s < 1$ and $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;1}$, and we refer the reader to Remark 5.12 for further information.

Step 1 : Global well-posedness. Since $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;1}$, Proposition 5.6 ensures that we can find $T^\star > 0$ and a continuously decreasing function $0 < \widetilde{\lambda}(t) < \min(\widetilde{\lambda_0}, \lambda_1/\langle S_0\rangle, 2\lambda_1\langle 2R_2/c\rangle)$ such that (10a)–(10b) has a unique solution $g \in C^0([0,T^\star); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1;1})$ on $[0,T^\star)$. Moreover, since $0 < s < 1$, this solution equally lies in $C^0([0,T^\star); \mathcal{G}_P^{\lambda(t),\sigma+1;s})$, where now $\lambda(t)$ stands for the function arising from Proposition 5.10. It is still possible to fix the constants so that the estimates (58a)–(58c) hold at $T = 0$, and g is continuous for the corresponding norms. Therefore, we already know that we can find $T > 0$ such that (57a)–(57c) hold on $[0,T]$. Proposition 5.10 together with a reasoning by connectivity ensures that (58a)–(58d) hold on $[0,T^\star)$. Finally, (58a) tells us that

$$\limsup_{t\nearrow T^\star} \|g(t)\|_{H_P^{\sigma+1}} \le \limsup_{t\nearrow T^\star} \|g(t)\|_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \le 2K_1\langle T^\star\rangle^7\varepsilon^2$$

holds, and thus we can go back to the extension argument in Proposition 5.6, and we conclude that $T^\star = +\infty$.

Step 2 : Convergence to 0 of ϱ. Since the space average of $g(t)$ vanishes: $\widehat{\varrho}(t,0) = \widehat{g}(t,0,0) = 0$, we get

$$\|\varrho(t)\|^2_{\mathcal{F}^{\lambda',0;s}} \le \|\varrho\|^2_{\mathcal{F}^{\alpha_0,0;s}} e^{-2(\alpha_0-\lambda')\langle t\rangle}.$$

Next (58d) (with $\sigma > 1/2$) ensures that

$$\begin{aligned}
\|\varrho(t)\|^2_{\mathcal{F}^{\alpha_0,0;s}} &= \sum_{k\in\mathbb{Z}^d\setminus\{0\}} e^{2\alpha_0\langle k,tk\rangle^s}|\widehat{\varrho}(t,k)|^2 \le \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{\langle t\rangle^{2\sigma}}{\langle t\rangle} e^{2\lambda(t)\langle k,tk\rangle^s}|\widehat{\varrho}(t,k)|^2 \\
&\le \frac{1}{\langle t\rangle}\|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \le K_4\varepsilon^2.
\end{aligned}$$

Since $\alpha_0 = (\widetilde{\lambda_0} + \lambda')/2$, we have proved

$$\|\varrho(t)\|_{\mathcal{F}^{\lambda',0;s}} \le \sqrt{K_4}\varepsilon e^{-\frac{1}{2}(\lambda_0-\lambda')\langle t\rangle^s}.$$

Step 3 : Convergence to 0 of the force. This result follows similar arguments. Since the average of the force term vanishes, we have

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\lambda',0;s}} \le \|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\alpha_0,0;s}} e^{-2(\alpha_0-\lambda')\langle t\rangle}.$$

By using (53a) and (58c), we get

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|^2_{\mathcal{F}^{\alpha_0,0;s}} \lesssim \mathscr{E}_I \mathbf{1}_{0\le t\le S_0} + \int_0^t \|\varrho\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}\tau \lesssim \varepsilon^2.$$

we conclude by using $\alpha_0 = (\widetilde{\lambda_0} + \lambda')/2$, again.

Step 4 : Existence of the asymptotic profile. We wish to define the quantity

$$g^\infty : (x,v) \longmapsto f_0(x,v) + \int_0^{+\infty} \mathcal{N}(g)(\tau)\mathrm{d}\tau.$$

Let us check that this makes sense as an element of $\mathcal{G}^{\lambda',0:s}$. Next, we will show that $g(t)$ converges to g^∞ for large times. We start by estimating $\int_0^t \|\mathcal{N}(g)(\tau)\|_{\mathcal{F}^{\lambda',0;s}}\,\mathrm{d}\tau$.

With (49), we get

$$\int_0^t \|\mathcal{N}(g)(\tau)\|_{\mathcal{G}^{\lambda',0;s}}\mathrm{d}\tau \le \int_0^t \|\mathcal{N}(g)(\tau)\|_{\mathcal{G}^{\lambda',d/2+1;s}}\mathrm{d}\tau$$

$$\lesssim \int_0^t \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda',d/2+1;s}} \|(\nabla_v - \tau\nabla_x)(\mathscr{M} + g(\tau))\|_{\mathcal{G}_P^{\lambda',d/2+1;s}}\mathrm{d}\tau.$$

Since $\sigma > d/2+6$, we have

$$\|(\nabla_v - \tau\nabla_x)(\mathscr{M} + g(\tau))\|_{\mathcal{G}_P^{\lambda',d/2+1;s}} \lesssim \langle\tau\rangle\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda',d/2+2;s}} \le \langle\tau\rangle\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda(\tau),\sigma+1;s}}.$$

Moreover, the average of the force term vanishes so that

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda',d/2+1;s}}$$

$$\le \langle\tau\rangle^{-\sigma+d/2+1} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda',\sigma;s}}$$

$$\le \langle\tau\rangle^{-\sigma+d/2+1} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda(\tau),\sigma;s}},$$

and applying (58a) with the Cauchy-Schwarz inequality yields

$$\int_0^t \|\mathcal{N}(g)(\tau)\|_{\mathcal{G}^{\lambda',0;s}}\mathrm{d}\tau$$

$$\lesssim \int_0^t \langle\tau\rangle^{-\sigma+d/2+2} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \left(\|\mathscr{M}\|_{\mathcal{G}_P^{\widetilde{\lambda_0},0;s}} + \sqrt{K_1}\langle\tau\rangle^{7/2}\varepsilon\right)\mathrm{d}\tau$$

$$\lesssim \left(\int_0^t \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau\right)^{1/2} \times \left(\int_0^t \langle\tau\rangle^{-2\sigma+d+11}\left(\|\mathscr{M}\|^2_{\mathcal{G}_P^{\widetilde{\lambda_0},0;s}} + K_1\varepsilon^2\right)\mathrm{d}\tau\right)^{1/2}.$$

By using (53a) and (58c) we see that the left hand side is bounded uniformly with respect to t while the condition $\sigma > d/2+6$ implies that the right hand side is also bounded uniformly with respect to t. Thus g^∞ is well defined in $\mathcal{G}^{\lambda',0;s}$. To be more specific, we have shown that

$$\|g^\infty - f_0\|^2_{\mathcal{G}^{\lambda',0;s}} \lesssim (\mathscr{E}_I + K_3\varepsilon^2)(1 + K_1\varepsilon^2).$$

Since $\mathscr{E}_I \le \varepsilon^2$ it says that g^∞ is at a distance at most ε from f_0.

The convergence of $g(t)$ towards g^∞ relies on the same manipulations. The noticeable difference is in Step 3; using again the fact that the space average of the force term vanishes, we get

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda',d/2+1;s}}$$

$$\le \langle\tau\rangle^{-\sigma+d/2+1}e^{-(\alpha_0-\lambda')\langle\tau\rangle} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\alpha_0,\sigma;s}}$$

$$\le \langle\tau\rangle^{-\sigma+d/2+1}e^{-(\alpha_0-\lambda')\langle\tau\rangle} \|\nabla\sigma_1 \star (\mathscr{F}_I(\tau) - \sigma_1 \star \mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda(\tau),\sigma;s}},$$

It follows that

$$\|g(t)-g^\infty\|_{\mathcal{G}^{\lambda',0;s}} \leq \int_t^{+\infty} \|\mathscr{N}(g)(\tau)\|_{\mathcal{G}^{\lambda',0;s}}\,\mathrm{d}\tau$$

$$\lesssim \int_t^{+\infty} e^{-(\alpha_0-\lambda')\langle\tau\rangle^s}\langle\tau\rangle^{-\sigma+d/2+2}\,\|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda(\tau),\sigma+1;s}}\,\mathrm{d}\tau$$

$$\lesssim e^{-(\alpha_0-\lambda')\langle t\rangle^s}\left(\int_t^{+\infty}\langle\tau\rangle^{-\sigma+d/2+2}\,\|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda(\tau),\sigma+1;s}}\,\mathrm{d}\tau\right)$$

$$\lesssim \varepsilon e^{-(\alpha_0-\lambda')\langle t\rangle^s}.$$

We conclude by using $\alpha_0=(\widetilde{\lambda_0}+\lambda')/2$. ■

Remark 5.12 *We conclude the proof with a couple of remarks.*

- *When the data $\mathscr{M}, f_0$ belong to $\mathcal{G}^{\widetilde{\lambda_0},0;s}$, with $0<s<1$, Step 1 is critical since it relies on Proposition 5.6 which applies for analytic data only. We use a regularization argument: we introduce a sequence $\left(\mathscr{M}^\eta, f_0^\eta\right)_{\eta>0}$ of data that belong to $\mathcal{G}^{\widetilde{\lambda_0},0;1}$ and that converge to $(\mathscr{M}, f_0)$ in $\mathcal{G}^{\widetilde{\lambda_0},0;s}$ as $\eta\to 0$. For any $\eta>0$, the associated solution g^η is globally defined and it satisfies* (58a)–(58d) *on* $[0,+\infty)$ *(remarking that the criterion **(L)** is stable by such a regularisation). We can also check that the constants $K_1,\dots,K_4$ can be defined independently of η and that g^η converges in $C^0([0,+\infty);L^1(\mathbb{R}^d\times\mathbb{R}^d))$ to a certain function g, which is still a solution of* (10a)–(10b), *see [33] and [9, Theorem 4 & Lemma 8]. Moreover, for any $t\geq 0$, we have*

  $$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq \liminf_{\eta\to 0^+}\|g^\eta(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}}$$

  and

  $$\|g(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq \liminf_{\eta\to 0^+}\|g^\eta(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}}.$$

  Indeed, for any fixed t, the sequence $\left(g^\eta(t)\right)_{\eta>0}$ is bounded in $\mathcal{G}_P^{\lambda(t),\sigma+1;s}$ and $\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}$ (owing to (58a) *to* (58b)*); thus, extracting a subsequence (which might depend on t, but this is not an issue here), there exists $\bar{g}_t$ and $\tilde{g}_t$ such $g^\eta(t)$ converges weakly to $\bar{g}_t$ in $\mathcal{G}_P^{\lambda(t),\sigma+1;s}$ (resp. to $\tilde{g}_t$ in $\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}$). By lower-semi-continuity of the norm for the weak topology, we get*

  $$\|\bar{g}_t\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}} \leq \liminf_{\eta\to 0^+}\|g^\eta(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma+1;s}}$$

  and

  $$\|\tilde{g}_t\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}} \leq \liminf_{\eta\to 0^+}\|g^\eta(t)\|^2_{\mathcal{G}_P^{\lambda(t),\sigma-\beta;s}}.$$

  Since $g(t)=\bar{g}_t=\tilde{g}_t$ (by uniqueness of the limit in L^1) almost everywhere, (58a) *and* (58b) *still apply for g. In order to justify that* (58c) *and* (58d) *apply to g, we* *use the fact that, for any t, k, ξ*

$$\widehat{g^\eta}(t,k,\xi) \underset{\eta\to 0^+}{\longrightarrow} \widehat{g}(t,k,\xi).$$

Fatou's lemma then yields

$$\|g(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} = \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} \liminf_{\eta\to 0^+} |\widehat{g^\eta}(t,k,tk)|^2 \le \liminf_{\eta\to 0^+} \|g^\eta(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}.$$

- *When $s = 1$ this is still Step 1 that contains some difficulty. We can apply Proposition 5.6, but we should check the interaction between the function λ given by the bootstrap statement and the function $\widetilde{\lambda}$ arising from Proposition 5.6. Indeed, it is not* a priori *excluded that $\widetilde{\lambda}(t) < \lambda(t)$ at a certain time $t > 0$, which would prevent us from extending the solution in $\mathcal{G}_P^{\lambda(t),\sigma+1;1}$, see [6].*

Like for the problem on $\mathbb{R}^d$, the proof of the bootstrap property relies on fine estimates for the linearized problem. We are therefore going to use the following analog to Proposition 4.14, see [6, Lemma 4.1] and further comments in Appendix A.

Proposition 5.13 (Linearized damping on $\mathbb{T}^d$) *Let the assumptions of Theorem 5.8 and Proposition 5.10 be fulfilled. We consider a family of functions $\{t \in [0,T] \mapsto a(t,k),\ k \in \mathbb{Z}^d\}$. We suppose that*

$$\sum_{k\in\mathbb{Z}^d} \int_0^T \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} |a(t,k)|^2 \,\mathrm{d}t < +\infty,$$

holds. Then, we can find a constant C_{LD} (which does not depend on k and T) such that any solution $(t,k) \mapsto \phi(t,k)$ of the system

$$\begin{aligned}
\phi(t,k) &= a(t,k) + \int_0^t \mathscr{K}(t-\tau,k)\phi(\tau,k)\,\mathrm{d}\tau \\
&= a(t,k) + \int_0^t |\widehat{\sigma}_1(k)|^2 |k|^2 (t-\tau) \widehat{\mathscr{M}}([t-\tau]k) \left(\int_0^\tau p_c(\tau-\sigma)\phi(\sigma,k)\,\mathrm{d}\sigma \right) \mathrm{d}\tau,
\end{aligned}$$

on $[0,T]$ satisfies the following estimate: for any $k \in \mathbb{Z}^d$

$$\int_0^T \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} |\phi(t,k)|^2 \,\mathrm{d}t \le C_{LD} \int_0^T \langle k, tk\rangle^{2\sigma} e^{2\lambda(t)\langle k,tk\rangle^s} |a(t,k)|^2 \,\mathrm{d}t.$$

5.3 Bootstrap analysis: sketch of proof of Proposition 5.10

To start with, let us make a few observations:

- Like for the problem in $\mathbb{R}^d$, the main difficulty relies on the treatment of the echoes. In $\mathbb{R}^d$, the dispersive effect of the transport operator allows us to obtain a control by means of Sobolev norms, at the price of restrictions on the space dimension d, though: in finite regularity we need to assume $d \ge 2$ (the case $d = 2$ being critical for a different reason). On the torus, the dispersive effect does not hold, which motivates the analytic framework. As a consequence of working in such a high regularity, we get rid of the restriction on d.

- In order to adapt the arguments of [6], when we estimate expressions that involve the force term, we make use of Proposition 5.2. It allows us to control the force term by the macroscopic density ϱ, up to a constant term, like for the Vlasov-Poisson system. The constant term is of order ε^2, so that it does not induce new difficulties (see the proof of Proposition 4.2). Finally, when applying Proposition 5.13 in order to estimate $\int_0^T \|\varrho\|^2 \,\mathrm{d}t$, we should pay attention to the force term $\int_0^t \|\nabla\sigma_1 \star \mathscr{F}_I(\tau, x+\tau\, v)\cdot\nabla_v \mathscr{M}(v)\| \,\mathrm{d}\tau$. Proposition 5.4 provides the necessary estimates.

For the sake of brevity, let us just sketch how it is possible to obtain the estimate (58c) from (57a)–(57c), having, on the one hand, the estimates of [6] and, on the other hand, the estimates from Propositions 5.2 and 5.4. As in the free space case, we introduce the time response kernel which contains all the difficulties concerning the control of echos terms: let

$$\bar{K}(t,\tau,k,n) = \frac{1}{\langle n\rangle^{\gamma}} e^{(\lambda(t)-\lambda(\tau))\langle k,tk\rangle^s} e^{c\lambda(\tau)\langle k-n,tk-\tau n\rangle^s}\, |(t-\tau)k\,\widehat{g}(\tau,k-n,tk-\tau n)|\,\mathbf{1}_{n\neq 0}$$

where $c = c(s) \in (0,1)$ is determined by the proof.

Remark 5.14 *i) Since in our case the kernel σ_1 is analytic we can choose γ as large as we wish. In practice, since we use the arguments of [6], for proving a result in Gevrey regularity class $s \in (0,1)$, we should take γ such that $s > 1/(2+\gamma)$ (so the smaller s, the larger γ).*
ii) Note also that the analyticity of σ_1 allows us to replace the term $\langle n\rangle^{-\gamma}$ in the time response kernel by $\exp(-\gamma\langle n\rangle)$. According to [28, Section 7.1.1], this permits us to obtain better estimates on $\bar{K}$, but it is not obvious that these improvements lead to a Landau damping effect in finite regularity on the torus. Since in our context the regularity of σ_1 is also needed to obtain the crucial estimates of Propositions 5.2 and 5.4, and since replacing $\langle n\rangle^{-\gamma}$ by $\exp(-\gamma\langle n\rangle)$ does not improve the result, we chose the definition of the time response kernel with the $\langle n\rangle^{-\gamma}$ factor.

For this time response kernel we will use the followings estimates (see [6, Section 6], which are the analog in the torus of Lemma 4.15.

Lemma 5.15 *Under the assumptions of Proposition 5.10 the following two estimates hold*

$$\sup_{t\in[0,T]}\ \sup_{k\in\mathbb{Z}^d\setminus\{0\}} \int_0^t \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \bar{K}(t,\tau,k,n)\,\mathrm{d}\tau \lesssim \sqrt{K_2}\varepsilon$$

and

$$\sup_{\tau\in[0,T]}\ \sup_{n\in\mathbb{Z}^d\setminus\{0\}} \int_\tau^T \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \bar{K}(t,\tau,k,n)\,\mathrm{d}t \lesssim \sqrt{K_2}\varepsilon.$$

We follow closely the arguments of [6]. We start from

$$\begin{aligned}
\widehat{\varrho}(t,k) &= \widehat{f}_0(k,tk) - \int_0^t k\widehat{\sigma_1}(k)\widehat{\mathscr{F}_I}(\tau,k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau \\
&\quad + \int_0^t k|\widehat{\sigma_1}(k)|^2\widehat{\mathscr{G}_\varrho}(\tau,k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau \\
&\quad - \sum_{n\in\mathbb{Z}^d}\int_0^t n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I} - \widehat{\sigma_1}\widehat{\mathscr{G}_\varrho})(\tau,n)\cdot(t-\tau)k\,\widehat{g}(\tau,k-n,tk-\tau n)\,\mathrm{d}\tau \\
&= \mathrm{CT1}(t,k) + \mathrm{CT2}(t,k) \\
&\quad + \int_0^t k|\widehat{\sigma_1}(k)|^2\widehat{\mathscr{G}_\varrho}(\tau,k)\cdot(t-\tau)k\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau + \mathrm{NLT}(t,k).
\end{aligned}$$

As in the free space problem (see Section 4.4.1), for estimating the non linear term NLT we start by splitting it into several parts. Here this decomposition is slightly more precise than in Section 4.4.1 but the main idea is the same: we consider separately contributions from high and low frequencies coming from ϱ and g: $\mathrm{NLT} = \mathrm{T} + \mathrm{R} + \mathcal{R}$. The transport term T contains ϱ's low frequency terms and g's high frequency terms; the reaction term R contains ϱ's high frequency terms and g's low frequency terms and the remainder term $\mathcal{R}$ contains the other terms, those where ϱ and g have almost the same frequency. The precise decomposition needs the introduction of the Littlewood-Paley decomposition and the paradifferential formalism. We prefer not to detail this aspect here. Then, we apply Proposition 5.13 to obtain (by summing over $k\in\mathbb{Z}^d\setminus\{0\}$)

$$\begin{aligned}
\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t &\lesssim \int_0^T \|\mathrm{CT1}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t + \int_0^T \|\mathrm{CT2}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \\
&\quad + \int_0^T \|\mathrm{T}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t + \int_0^T \|\mathrm{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t + \int_0^T \|\mathcal{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t.
\end{aligned}$$

Constant terms. We estimate the first constant term CT1 as in [6] and we obtain

$$\int_0^T \|\mathrm{CT1}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \lesssim \varepsilon^2.$$

For the second constant term CT2 we use the Proposition 5.4 to obtain

$$\int_0^T \|\mathrm{CT2}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t \lesssim \mathscr{E}_I.$$

Reaction term. Following closely the argument from [6, Section 5.1.1], we are led to the following estimate on R:

$$\begin{aligned}
\int_0^T \|\mathrm{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t &\lesssim \left(\sup_{t\in[0,T]}\sup_{k\in\mathbb{Z}^d\setminus\{0\}}\int_0^t \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \bar{K}(t,\tau,k,n)\,\mathrm{d}\tau\right) \\
&\quad \times \left(\sup_{\tau\in[0,T]}\sup_{n\in\mathbb{Z}^d\setminus\{0\}}\int_\tau^T \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \bar{K}(t,\tau,k,n)\,\mathrm{d}t\right) \\
&\quad \times \left(\int_0^T \|\nabla\sigma_1\star(\mathscr{F}_I(\tau) - \sigma_1\star\mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau\right).
\end{aligned}$$

In order to make the kernel $\bar{K}$ appear, we have to multiply and divide by $\langle n\rangle^{\gamma}$. Hence, we can obtain the same estimate but replacing

$$\begin{aligned}\|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}} \\ = \sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n,\tau n\rangle^{2\sigma}e^{2\lambda(\tau)\langle n,\tau n\rangle^s}|n|^2|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2\end{aligned}$$

by

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\langle n,\tau n\rangle^{2\sigma}e^{2\lambda(\tau)\langle n,\tau n\rangle^s}\,\langle n\rangle^{2\gamma}\,|n|^2|\widehat{\sigma}_1(n)|^2\left|\widehat{\mathscr{F}}_I(\tau,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(\tau,n)\right|^2.$$

Since σ_1 is analytic we can always use, without any bad consequences, a small part of the exponential decay of its Fourier transform to absorb the $\langle k\rangle^{\gamma}$-term (we already dealt with this difficulty in the free space problem, see Remark 4.3). From now on, we always omit this minor detail in the estimates. Then, applying Lemma 5.15 and Proposition 5.2 with (57a), we get

$$\int_0^T\|\mathrm{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t\lesssim K_2\varepsilon^2\left(\mathscr{E}_I+K_3\varepsilon^2\right).$$

Transport term. We follow line by line the estimate of [6, Section 5.1.2], and we are led to

$$\begin{aligned}\int_0^T\|\mathrm{T}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t\lesssim&\left(\int_0^T\|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau\right)\\ &\times\Bigg(\sup_{\tau\geq0}e^{(c-1)\alpha_0\langle\tau\rangle^s}\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\sup_{\omega\in\mathbb{Z}^d\setminus\{0\}}\sup_{x\in\mathbb{R}^d}\int_{-\infty}^{+\infty}\left\langle k,\frac{\omega}{|\omega|}\zeta-x\right\rangle^{2\sigma+2}\\ &\qquad\times e^{2\lambda(\tau)\langle k,\frac{\omega}{|\omega|}\zeta-x\rangle^s}\left|\widehat{g}\left(\tau,k,\frac{\omega}{|\omega|}\zeta-x\right)\right|^2\,\mathrm{d}\zeta\Bigg)\end{aligned}$$

where $c=c(s)\in(0,1)$. Then, applying Proposition 5.2 with (57c) and the Trace Lemma 4.4 with (57a) (see in Section 4.4.1 the paragraph **Estimate on** NLTT for a similar reasoning) yields

$$\int_0^T\|\mathrm{T}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t\lesssim(\mathscr{E}_I+K_3\varepsilon^2)K_1\varepsilon^2.$$

Remainder term. The arguments of [6, Section 5.1.3] allow us to obtain the estimate

$$\begin{aligned}\int_0^T\|\mathcal{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}}\,\mathrm{d}t\lesssim K_1\varepsilon^2&\left(\int_0^T\|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|^2_{\mathcal{F}^{\lambda(\tau),\sigma;s}}\,\mathrm{d}\tau\right)\\ &\times\left(\int_0^T\sum_{n\in\mathbb{Z}^d\setminus\{0\}}e^{2(c'-1)\lambda(\tau)\langle n,\tau n\rangle^s}\langle\tau\rangle^7\,\mathrm{d}\tau\right)\end{aligned}$$

where $c' \in (0,1)$. We conclude by applying Proposition 5.2 with (57c) to obtain

$$\int_0^T \|\mathcal{R}(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \,\mathrm{d}t \lesssim K_1\varepsilon^2(\mathscr{E}_I + K_3\varepsilon^2).$$

Recap. We have shown that, if g is a solution of (10a)–(10b) satisfying (57a)–(57c) on $[0,T]$, then

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \,\mathrm{d}t \lesssim \varepsilon^2 + \mathscr{E}_I + K_2\varepsilon^2 \left(\mathscr{E}_I + K_3\varepsilon^2\right) + (\mathscr{E}_I + K_3\varepsilon^2)K_1\varepsilon^2 + K_1\varepsilon^2(\mathscr{E}_I + K_3\varepsilon^2).$$

Since in Theorem 5.8 the smallness assumption on the fluctuation of the media is $\mathscr{E}_I \leq \varepsilon^2$, this estimate can be rewritten as

$$\int_0^T \|\varrho(t)\|^2_{\mathcal{F}^{\lambda(t),\sigma;s}} \,\mathrm{d}t \lesssim \left(1 + K_2(1+K_3)\varepsilon^2 + K_1(1+K_3)\varepsilon^2\right)\varepsilon^2.$$

Let us denote C_1 the constant hidden in the symbol $\lesssim$ of this estimate. Choosing $K_3 \geq C_1$ and $\varepsilon \ll 1$ so that

$$(K_1 + K_2)(1+K_3)\varepsilon^2 \leq 1$$

allows us to conclude that (58c) holds.

The general idea. Since the structure of the Vlasov-Wave equation is close to the structure of the Vlasov-Poisson equation, we can perform the same estimates than in [6]. The price to be paid is to replace terms of the form $\|\varrho(t)\|_{\mathcal{F}}$ by

$$\|\nabla\sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t))\|_{\mathcal{F}}. \tag{59}$$

Then all the difficulty consists in controlling (59) by means of $\|\varrho(t)\|_{\mathcal{F}}$. Since Proposition 5.2 allows us to perform this kind of estimate, we have a complete proof of the bootstrap statement Proposition 5.10 by applying this strategy. We refer the reader to the detailed analysis performed for the free space problem. The justification of the necessary estimates relies on the understanding of the kernel p_c.

A Analysis of the Volterra equation

This Section is concerned with the analysis of the system of integral equations

$$\varphi_k(t) = a_k(t) + \int_0^t K_k(t-\tau)\varphi_k(\tau)\,\mathrm{d}\tau, \tag{60}$$

parametrized by $k \in \mathbb{X}^{\star d} \setminus \{0\}$ (but note that the equations are uncoupled). The unknowns are the functions $t \mapsto \varphi_k(t)$, while the source a_k and the kernel K_k are given. We wish to establish fine estimates on the solutions, depending on decay assumptions on the data, in the spirit of Lemma 3.1, and keeping track on the dependence with respect to k. The statements on the linearized problem in Section 3.2 are consequences of the discussion below, by virtue of the Volterra equation (17) satisfied by the fluctuation of the macroscopic density ϱ. We discuss precisely the differences between the usual Vlasov equation where the potential is defined by a mere space-convolution $\Phi = W \star \varrho$ and the Vlasov-Wave model under consideration, and in particular we bring out the role of the time kernel p_c. In order to have such a unified presentation, some arguments slightly differ from [28, 6, 7], and we justify in full details that it suffices to satisfy the stability criterion on the imaginary axis.

A.1 Volterra system in analytic regularity

A.1.1 The Vlasov case

For the analysis of the standard Vlasov system, one is led to the following assumptions on the data a and the kernel K:

(A:H1) $|a_k(t)| \leq \alpha e^{-\lambda |k| t}$,

(A:H2) $K_k(t) = -\widehat{W}(k)|k|^2 t \widehat{\mathscr{M}}(tk) \mathbf{1}_{t\geq 0}$ and $|\widehat{\mathscr{M}}(\eta)| \leq C_0 e^{-\lambda_0 |\eta|}$,

(A:LV) There exists $\kappa > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and any $\tilde{\omega} \in \mathbb{R}$, we have

$$|\mathscr{F}(K_k)(|k|\tilde{\omega}) - 1| \geq \kappa$$

Remark A.1 *Hypothesis **(A:H2)** ensures that the following (rescaled) Laplace transform of K_k makes sense for any $\omega \in \mathbb{C}$ such that* $\mathrm{Re}(\omega) > -\lambda_0$ *:*

$$\mathscr{L}K(\omega, k) = \int_0^{+\infty} e^{-|k|\omega t} K_k(t)\, \mathrm{d}t.$$

*The condition **(A:LV)** is expressed by means of the Fourier transform of K_k, which amounts to impose the behavior of the Laplace transform on the imaginary axis.*

Theorem A.2 *Assume **(A:H1)**–**(A:LV)**. We can find $C', \lambda' > 0$ such that, for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and $t \geq 0$, we have*

$$|\varphi_k(t)| \leq C' e^{-\lambda' |k| t}.$$

Let us start with two preliminary statements.

Lemma A.3 *Assume **(A:H1)**–**(A:LV)**. We can find $\Lambda > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and $\omega \in \mathbb{C}$, we have*

$$\textit{if } -\Lambda \leq \mathrm{Re}(\omega) \leq \Lambda \textit{ then } |\mathscr{L}K(\omega, k) - 1| \geq \frac{\kappa}{2}.$$

Proof. The Laplace transform of K_k can be cast as

$$\begin{aligned}
\mathscr{L}K(s + i\tilde{\omega}, k) &= \int_0^{+\infty} e^{-s|k|t} e^{-i\tilde{\omega}|k|t} \left(-\widehat{W}(k)|k|^2 t \widehat{\mathscr{M}}(tk)\right) \mathrm{d}t \\
&= -\widehat{W}(k) \int_0^{+\infty} e^{-su} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}\left(\frac{k}{|k|} u\right) u \, \mathrm{d}u
\end{aligned}$$

for real $s, \tilde\omega$. It follows that

$$\begin{aligned}
|\mathscr{L}K(0+i\tilde\omega,k)-\mathscr{L}K(s+i\tilde\omega,k)| &\le \|W\|_{L^1}\left|\int_0^{+\infty}\left(1-e^{-su}\right)e^{i\tilde\omega u}\widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right)\,\mathrm{d}u\right| \\
&\le C_0\|W\|_{L^1}\int_0^{+\infty}\left|1-e^{-su}\right|e^{-\lambda_0 u}u\,\mathrm{d}u \underset{s\to 0}{\longrightarrow} 0.
\end{aligned}$$

The convergence holds by virtue of the Lebesgue theorem, uniformly with respect to k and $\tilde\omega$: for any $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that for any $s\in\mathbb{R}$,

$$\text{if } |s|\le\delta_\varepsilon \text{ then } C_0\|W\|_{L^1}\int_0^{+\infty}\left|1-e^{su}\right|e^{-\lambda_0 u}u\,\mathrm{d}u\le\varepsilon.$$

Choosing $\varepsilon=\kappa/2$, **(A:LV)** ensures that $\Lambda=\delta_{\kappa/2}$ is suitable. ■

Lemma A.4 *Assume **(A:H1)**–**(A:LV)**. For any $k\in\mathbb{X}^{\star d}\setminus\{0\}$, the open set*

$$\Omega=\{\omega\in\mathbb{C},\ \Lambda<\mathrm{Re}(\omega)\}$$

contains at most a countable set of zeroes of the function $\omega\mapsto\mathscr{L}K(\omega,k)-1$.

Proof. By holomorphy under the integral, the function $\omega\mapsto\mathscr{L}K(\omega,k)$ is holomorphic on the open set

$$U=\{\omega\in\mathbb{C},\ \mathrm{Re}(\omega)>-\lambda_0\}.$$

Then the uniqueness theorem for analytic functions tells us that the zeroes of $\omega\mapsto\mathscr{L}K(\omega,k)-1$ are isolated. ■

We turn to the proof of Theorem A.2

Proof. Let $k\in\mathbb{X}^{\star d}\setminus\{0\}$, We introduce

$$\phi_k(t)=\varphi_k(t)e^{\lambda'|k|t}\mathbf{1}_{t\ge 0},\qquad A_k(t)=a_k(t)e^{\lambda'|k|t}\qquad K_k^0(t)=K_k(t)e^{\lambda'|k|t}\mathbf{1}_{t\ge 0}$$

where we choose λ' such that $0<\lambda'<\min(\lambda,\lambda_0,\Lambda)$, with Λ defined as in Lemma A.3. For any $t\ge 0$, we get

$$\phi_k(t)=A_k(t)+K_k^0\star\phi_k(t).$$

Step 1. We show that

$$\|\phi_k\|_{L^2(\,\mathrm{d}t)}\le\frac{2}{\kappa}\|A_k\|_{L^2(\,\mathrm{d}t)}.$$

Indeed, Grönwall allows us to find $C(k)>0$ such that

$$|\phi_k(t)|\lesssim_k e^{C(k)t}.$$

(The constant hidden in the $\lesssim$ symbol depends on k.) For $\mu\in\mathbb{R}$, we introduce the functions

$$\phi_{k,\mu}(t)=e^{\mu|k|t}\phi_k(t)\qquad A_{k,\mu}(t)=e^{\mu|k|t}a_k(t)\qquad K_{k,\mu}^0(t)=e^{\mu|k|t}K_k(t).$$

We get

$$\phi_{k,\mu}(t)=A_{k,\mu}(t)+K_{k,\mu}^0\star\phi_{k,\mu}(t).$$

Let us Fourier-transform this relation, with $\mu < -C(k)/|k|$; for any $\tilde{\omega} \in \mathbb{R}$, we obtain

$$\left[1 - \mathscr{F}(K^0_{k,\mu})(\tilde{\omega})\right] \mathscr{F}(\phi_{k,\mu})(\tilde{\omega}) = \mathscr{F}(A_{k,\mu})(\tilde{\omega}).$$

Observe that

$$\mathscr{F}(K^0_{k,\mu})(\tilde{\omega}) = \mathscr{L}K(-\lambda' - \mu + i\tilde{\omega}/|k|, k).$$

Let us set

$$N_k = \{\tilde{\omega} \in \mathbb{R} \text{ such that there exists } s > \Lambda \text{ verifying } \mathscr{L}K(s + i\tilde{\omega}/|k|, k) = 1\}.$$

We deduce that, for $\mu < -C(k)/|k|$ and $\tilde{\omega} \in \mathbb{R} \setminus N_k$,

$$\mathscr{F}(\phi_{k,\mu})(\tilde{\omega}) = \frac{\mathscr{F}(A_{k,\mu})(\tilde{\omega})}{1 - \mathscr{L}K(-\lambda' - \mu + i\tilde{\omega}/|k|, k)}.$$

Let $\gamma_\delta(t) = \exp(-\delta t^2/2)$, wit $\delta > 0$. We write

$$\mathscr{F}(\phi_{k,\mu}\gamma_\delta)(\tilde{\omega}) = \mathscr{F}(\phi_{k,\mu}) \star \mathscr{F}(\gamma_\delta)(\tilde{\omega}) = \frac{\mathscr{F}(A_{k,\mu})(\cdot)}{1 - \mathscr{L}K(-\lambda' - \mu + i\cdot/|k|, k)} \star \mathscr{F}(\gamma_\delta)(\tilde{\omega}).$$

The left hand side makes sense for any $\mu \in \mathbb{R}$ and it is analytic with respect to μ. The third term in the equality makes sense provided $\mu < \min(\lambda_0 - \lambda', \lambda - \lambda')$ and $\tilde{\omega} \in \mathbb{R} \setminus N_k$, and it is analytic with respect to μ on an open set that contains the half-line $\{x < \min(\lambda_0 - \lambda', \lambda - \lambda')\}$. The second term is defined for $\mu < -C(k)/|k|$ and the equalities hold when this constraint on μ is fulfilled. The uniqueness theorem for analytic functions tells us that the equality still holds for $\mu < \min(\lambda_0 - \lambda', \lambda - \lambda')$. In particular, with $\mu = 0$, we obtain, for any $\tilde{\omega} \in \mathbb{R} \setminus N_k$

$$\mathscr{F}(\phi_k\gamma_\delta)(\tilde{\omega}) = \frac{\mathscr{F}(A_k)(\cdot)}{1 - \mathscr{L}K(-\lambda' + i\cdot/|k|, k)} \star \mathscr{F}(\gamma_\delta)(\tilde{\omega}).$$

By Lemma A.4 we know that N_k is negligible. Thus taking the L^2 norm of the equality leads to

$$\|\phi_k\gamma_\delta\|_{L^2(\,\mathrm{d}t)} \le \frac{\|A_k\|_{L^2(\,\mathrm{d}t)}}{\kappa/2} \|\mathscr{F}(\gamma_\delta)\|_{L^1(\,\mathrm{d}t)} = \frac{2}{\kappa}\|A_k\|_{L^2(\,\mathrm{d}t)},$$

where the first inequality relies on Lemma A.3 with $\lambda' < \Lambda$. Finally, since $\phi_k\gamma_\delta$ converges monotonically to ϕ_k as $\delta \to 0$, Beppo-Lévi's theorem leads to

$$\|\phi_k\|_{L^2(\,\mathrm{d}t)} \le \frac{2}{\kappa}\|A_k\|_{L^2(\,\mathrm{d}t)}.$$

Step 2. We go back to the equation satisfied by ϕ_k and we estimate the sup-norm:

$$\begin{aligned} \|\phi_k\|_{L^\infty} &\le \|A_k\|_{L^\infty} + \|K^0_k\|_{L^2}\|\phi_k\|_{L^2} \\ &\le \|A_k\|_{L^\infty} + \frac{2}{\kappa}\|K^0_k\|_{L^2}\|A_k\|_{L^2}. \end{aligned}$$

By **(A:H1)** the sup-norm of A_k is bounded, uniformly with respect to k, while the L^2-norm of A_k behaves like $1/|k|^{1/2}$. The expression of K_k tells us that its L^2-norm behaves like $|k|^{1/2}$. Therefore, the product of the L^2-norm of A_k and K^0_k is bounded uniformly with respect to k. (This estimate is particularly crucial for the case $\mathbb{X}^{\star d} = \mathbb{R}^d$ since an estimate of the order of $1/|k|$ would compromise the proof.) ∎

Remark A.5 *For the analysis of the* linearized *Landau damping, this L^∞ estimate on ϕ_k is a crucial ingredient. Note that it is obtained as a consequence of an intermediate estimate with the L^2-norm, that can be recast as*

$$\int_0^{+\infty} e^{2\lambda'|k|t}|\varphi_k(t)|^2\,\mathrm{d}t \le \frac{1}{\kappa^2}\int_0^{+\infty} e^{2\lambda'|k|t}|a_k(t)|^2\,\mathrm{d}t.$$

When studying the non-linear *problem, this L^2 estimate becomes the key argument. Changing $a_k(t)$ into $a_k(t)\mathbf{1}_{0\le t\le T}$, we can equally obtain*

$$\int_0^{T} e^{2\lambda'|k|t}|\varphi_k(t)|^2\,\mathrm{d}t \le \frac{1}{\kappa^2}\int_0^{T} e^{2\lambda'|k|t}|a_k(t)|^2\,\mathrm{d}t.$$

Similarly, with $\langle k\rangle = (1+k^2)^{1/2}$, replacing $\phi_k(t)$ by $e^{\lambda'\langle k\rangle}\phi_k(t)$ and $A_k(t)$ by $e^{\lambda'\langle k\rangle}A_k(t)$, leads to

$$\int_0^{T} e^{2\lambda'(\langle k\rangle+|k|t)}|\varphi_k(t)|^2\,\mathrm{d}t \le \frac{1}{\kappa^2}\int_0^{T} e^{2\lambda'(\langle k\rangle+|k|t)}|a_k(t)|^2\,\mathrm{d}t.$$

Since $\langle k, tk\rangle \lesssim \langle k\rangle + |k|t \lesssim \langle k, tk\rangle$, it becomes

$$\int_0^{T} e^{2\lambda'\langle k,tk\rangle}|\varphi_k(t)|^2\,\mathrm{d}t \le C^2\int_0^{T} e^{2\lambda'\langle k,tk\rangle}|a_k(t)|^2\,\mathrm{d}t.$$

When discussing the Landau damping in finite regularity, we shall see that a polynomial weight (that means a Sobolev correction) can be incorporated in the estimate

$$\int_0^{T} \langle k,tk\rangle^{2\sigma} e^{2\lambda'\langle k,tk\rangle}|\varphi_k(t)|^2\,\mathrm{d}t \le C^2\int_0^{T} \langle k,tk\rangle^{2\sigma} e^{2\lambda'\langle k,tk\rangle}|a_k(t)|^2\,\mathrm{d}t.$$

Eventually, in order to obtain results in Gevrey-norms, it is relevant to consider fractional exponential weights as well:

$$\int_0^{T} \langle k,tk\rangle^{2\sigma} e^{2\lambda'\langle k,tk\rangle^s}|\varphi_k(t)|^2\,\mathrm{d}t \le C^2\int_0^{T} \langle k,tk\rangle^{2\sigma} e^{2\lambda'\langle k,tk\rangle^s}|a_k(t)|^2\,\mathrm{d}t$$

with $0 < s \le 1$. The proof is quite technical, and we refer the reader to [6] for further details on this framework.

A.1.2 The Vlasov-Wave case

Now we investigate (60) with following assumptions on the data a and the kernel K :

(A:H1b) $|a_k(t)| \le \alpha e^{-\lambda|k|t}$,

(A:H2b) $K_k(t) = |\widehat{\sigma}_1(k)|^2|k|^2\int_0^t p_c(t-\tau)\tau\widehat{\mathscr{M}}(\tau k)\,\mathrm{d}\tau\mathbf{1}_{t\ge 0}$ and $|\widehat{\mathscr{M}}(\eta)| \le C_0 e^{-\lambda_0|\eta|}$,

(A:LW) There exists $\kappa > 0$ such that for any $k \in \mathbb{X}^{\star d}\setminus\{0\}$ and any $\tilde\omega \in \mathbb{R}$

$$|\mathscr{F}(K_k)(|k|\tilde\omega) - 1| \ge \kappa,$$

(A:a) $\mathrm{supp}(p_c) \subset [0, R_c]$ and $\|p_c\|_{L^\infty} < +\infty$,

(A:b) $\quad |\widehat{\sigma}_1(k)| \leq C_1 e^{-\lambda_1|k|}$

Remark A.6 *For the Vlasov-Wave problem, with p_c defined as in* (5), ***(A:a)*** *holds under assumption* ***(H1)–(H2)***, *see Lemma 2.1. With* ***(A:H2b)*** *and* ***(A:a)*** *we infer the following estimate*

$$|K_k(t)| \leq C_0 R_c \|p_c\|_{L^\infty} |\widehat{\sigma}_1(k)|^2 |k|^2 e^{\lambda_0|k|R_c} |t| e^{-\lambda_0|k|t}$$

and ***(A:b)*** *ensures that provided λ_1 is large enough, $|\widehat{\sigma}_1(k)|^2|k|^2 e^{\lambda_0|k|R_c}$ is uniformly bounded with respect to k, which has to be compared to* ***(A:H1)***. *In particular, we can again introduce the (rescaled) Laplace transform of K_k, for $\omega \in \mathbb{C}$ such that* $\mathrm{Re}(\omega) > -\lambda_0$*:*

$$\mathscr{L}K(\omega, k) = \int_0^{+\infty} e^{-|k|\omega t} K_k(t)\,\mathrm{d}t.$$

We have

$$\mathscr{L}K(\omega, k) = \mathscr{L}p_c(\omega, k)\mathscr{L}\tilde{K}(\omega, k),$$

where $\tilde{K}$ relies on the space-convolution only and has the same properties as the kernel of the Vlasov case.

Remark A.7 *Note that the rescaling of the Laplace transform still appears through the equilibrium $\mathscr{M}$ but the kernel K_k also involves p_c, which does not have such a homogeneity property. It induces some difficulties for the analysis.*

Theorem A.8 *Assume* ***(A:H1b)–(A:b)***. *Then, there exists $C', \lambda' > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and any $t \geq 0$, we have*

$$|\varphi_k(t)| \leq C' e^{-\lambda'|k|t}.$$

We start by discussing the zeroes of $\omega \mapsto \mathscr{L}K(\omega, k) - 1$.

Lemma A.9 *Assume* ***(A:H1b)–(A:LW)***. *We can find $\Lambda > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and $\omega \in \mathbb{C}$, we have*

$$\textit{if } -\Lambda \leq \mathrm{Re}(\omega) \leq \Lambda \textit{ then } |\mathscr{L}K(\omega, k) - 1| \geq \frac{\kappa}{2}.$$

Proof. With $s, \tilde{\omega} \in \mathbb{R}$ and $\omega = s + i\tilde{\omega}$, the Laplace transform of K_k can be cast as

$$\begin{aligned}
\mathscr{L}K(s + i\tilde{\omega}, k) &= \left(\int_0^{+\infty} e^{-\omega|k|t} p_c(t)\,\mathrm{d}t\right) \\
&\quad \times \left(\int_0^{+\infty} e^{-s|k|t} e^{-i\tilde{\omega}|k|t}\ |\widehat{\sigma}_1(k)|^2 |k|^2 t \widehat{\mathscr{M}}(tk)\,\mathrm{d}t\right) \\
&= \left(\int_0^{+\infty} e^{-\omega|k|t} p_c(t)\,\mathrm{d}t\right) \\
&\quad \times \left(|\widehat{\sigma}_1(k)|^2 \int_0^{+\infty} e^{-su} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right) u\,\mathrm{d}u\right).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&|\mathscr{L}K(0+i\tilde\omega,k)-\mathscr{L}K(s+i\tilde\omega,k)| \\
&\qquad \le C_0\|\sigma_1\|_{L^1}^2\left(\int_0^{+\infty} ue^{-\lambda_0 u}\,\mathrm{d}u\right)\left(\int_0^{+\infty}\left|1-e^{-s|k|t}\right|\,|p_c(t)|\,\mathrm{d}t\right) \\
&\qquad\qquad +C_0\|p_c\|_{L^1}\|\sigma_1\|_{L^1}^2\left(\int_0^{+\infty}|1-e^{-su}|\,ue^{-\lambda_0 u}\,\mathrm{d}u\right)\underset{s\to 0}{\longrightarrow}0
\end{aligned}
\tag{61}
$$

by virtue of the Lebesgue theorem. Notice that the second term converges to 0 uniformly with respect to $\tilde\omega$ and k, but for the first term, it is not clear that the convergence remains uniform with respect to k (it is uniform with respect to $\tilde\omega$). In order to treat this difficulty, we observe that for any $\omega\in\mathbb{C}$, $\mathrm{Re}(\omega)>-\lambda'>-\lambda_0$, we have

$$
\begin{aligned}
|\mathscr{L}K(\omega,k)| &\le |\widehat\sigma_1(k)|^2\left(\int_0^{+\infty}e^{\lambda'|k|t}|p_c(t)|\,\mathrm{d}t\right)\left(\int_0^{+\infty}ue^{-(\lambda_0-\lambda')u}\,\mathrm{d}u\right) \\
&\le |\widehat\sigma_1(k)|^2e^{\lambda'|k|R_c}\|p_c\|_{L^1}\left(\int_0^{+\infty}ue^{-(\lambda_0-\lambda')u}\,\mathrm{d}u\right)\underset{|k|\to+\infty}{\longrightarrow}0.
\end{aligned}
$$

when $\lambda'<\lambda_0$. The convergence holds uniformly with respect to ω. Thus, it suffices to consider (61) for k in a bounded subset of $\mathbb{X}^{\star d}\setminus\{0\}$. When $\mathbb{X}^{\star d}\setminus\{0\}=\mathbb{Z}^d\setminus\{0\}$, such a subset contains a finite number of elements, and the convergence (61) is therefore uniform with respect to k. The case $\mathbb{X}^{\star d}\setminus\{0\}=\mathbb{R}^d\setminus\{0\}$ is more delicate. Let us introduce the function

$$
g:(s,\tau)\in\mathbb{R}\times\mathbb{R}_+\longmapsto\int_0^{+\infty}\left|1-e^{-s\tau t}\right|\,|p_c(t)|\,\mathrm{d}t.
$$

By virtue of the Lebesgue theorem this function is continuous, and thus uniformly continuous over compact sets in $\mathbb{R}\times\mathbb{R}_+$. Hence, for any $|k|\le A<\infty$, we have

$$
g(s,|k|)\underset{s\to 0}{\longrightarrow}0
$$

uniformly with respect to k (but the convergence depends on A). This ends the proof. ■

Lemma A.10 *For any $k\in\mathbb{X}^{\star d}\setminus\{0\}$, the open set*

$$
\Omega=\{\omega\in\mathbb{C}\ \textit{such that}\ \Lambda<\mathrm{Re}(\omega)\}
$$

contains at most a countable set of zeroes of $\omega\mapsto\mathscr{L}K(\omega,k)-1$.

Proof. By holomorphy under the integral, which uses Remark A.6, the function $\omega\mapsto\mathscr{L}K(\omega,k)$ is holomorphic on the open set

$$
\{\omega\in\mathbb{C}\ \text{such that}\ \mathrm{Re}(\omega)>-\lambda_0\}\,.
$$

The uniqueness theorem for analytic functions then tells us that the zeroes of $\omega\mapsto\mathscr{L}K(\omega,k)-1$ are isolated. ■

Proof of Theorem A.8. Lemma A.9 and A.10, together with Remark A.6 allow us to reproduce the arguments of Section A.1.1. In particular the behavior of the kernel observed in Remark A.6 permits us to establish in the second step of the proof that the constant C' can be defined independently of k. ■

A.2 Volterra system in finite regularity

A.2.1 The Vlasov case

The assumptions on a and K become

(A:H3) $|a_k(t)| \leq \alpha \langle |k|t \rangle^{-m}$,

(A:H4) $K_k(t) = -\widehat{W}(k)|k|^2 t \widehat{\mathscr{M}}(tk) \mathbf{1}_{t \geq 0}$ et $|\widehat{\mathscr{M}}(\eta)| \leq C_0 \langle \eta \rangle^{-m_0}$,

(A:Lv) There exists $\kappa > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and any $\tilde{\omega} \in \mathbb{R}$

$$|\mathscr{F}(K_k)(|k|\tilde{\omega}) - 1| \geq \kappa.$$

We remind the reader that $\langle k \rangle$ is a shorthand notation for $\sqrt{1+k^2}$, k being a scalar or a vector.

Remark A.11 *By **(A:H4)** the (rescaled) Laplace transform K_k*

$$\mathscr{L}K(\omega, k) = \int_0^{+\infty} e^{-|k|\omega t} K_k(t) \, \mathrm{d}t$$

is well defined for any $\omega \in \mathbb{C}$ such that $\mathrm{Re}(\omega) \geq 0$.

Theorem A.12 *Let $m_\star = \min(m-1, m_0 - 3)$. There exists $C > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and $t \geq 0$, we have*

$$|\varphi_k(t)| \leq C \langle |k|t \rangle^{-m_\star}.$$

Like in the analytic framework, we need to discuss the location of the zeroes of the function $\omega \mapsto \mathscr{L}K(\omega, k) - 1$.

Lemma A.13 *Assume **(A:H3)**–**(A:Lv)**. We can find $\Lambda > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and any $\omega \in \mathbb{C}$,*

$$\textit{if } 0 \leq \mathrm{Re}(\omega) \leq \Lambda, \textit{ then } |\mathscr{L}K(\omega, k) - 1| \geq \frac{\kappa}{2}.$$

Lemma A.14 *For any $k \in \mathbb{X}^{\star d} \setminus \{0\}$, the open set*

$$\Omega = \{\omega \in \mathbb{C} \textit{ such that } \Lambda < \mathrm{Re}(\omega)\}$$

contains at most a countable set of zeroes of $\omega \mapsto \mathscr{L}K(\omega, k) - 1$.

The proof is completely similar to the analytic case. However, we now need an additional claim.

Lemma A.15 *The following properties hold:*

(i) *If $\beta \leq m$, then the function $t \mapsto (t|k|)^\beta a_k(t)$ bounded uniformly with respect to k and t;*

(ii) *If $\beta \leq m-1$, then the function $t \mapsto (t|k|)^\beta a_k(t)$ is square integrable and*

$$\left\| t \mapsto (t|k|)^\beta a_k(t) \right\|_{L^2} \lesssim \frac{1}{\sqrt{|k|}};$$

(ii) *If $\beta \leq m_0 - 2$, then the function $t \mapsto (t|k|)^\beta K_k(t)$ is square integrable and*

$$\left\| t \mapsto (t|k|)^\beta K_k(t) \right\|_{L^2} \lesssim \sqrt{|k|}.$$

(ii) *For $\mu \leq 0$, let $K^0_{k,\mu}(t) = K_k(t) e^{\mu|k|t} \mathbf{1}_{t \geq 0}$. If $\beta \leq m_0 - 3$, then the function $\tilde{\omega} \in \mathbb{R} \mapsto |k|^\beta \partial^\beta_{\tilde{\omega}} \mathscr{F}(K^0_{k,\mu})(\tilde{\omega})$ is bounded uniformly with respect to k and μ. To be more specific we have*

$$\left| |k|^\beta \partial^\beta_{\tilde{\omega}} \mathscr{F}(K^0_{k,\mu})(\tilde{\omega}) \right| \leq C_0 \|W\|_{L^1} \int_0^{+\infty} u^{\beta+1} \langle u \rangle^{-m_0} \, \mathrm{d}u < +\infty.$$

Proof. The results follow by direct computation. ∎

We turn to the proof of Theorem A.12.

Proof of Theorem A.12. Pick $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and let

$$\phi_k(t) = \varphi_k(t) \mathbf{1}_{t \geq 0}, \qquad K^0_k(t) = K_k(t) \mathbf{1}_{t \geq 0}.$$

For any $t \geq 0$, we have

$$\phi_k(t) = a_k(t) + K^0_k \star \phi_k(t).$$

Step 1. We show that, for any $\beta \in [0, \min(m-1, m_0-3)] \cap \mathbb{N}$ (we start by dealing with integer regularity exponents, the extension to real exponents follows by standard interpolation arguments), we have

$$\left\| t \mapsto |tk|^\beta \phi_k(t) \right\|_{L^2(\,\mathrm{d}t)} \leq C(\beta, \kappa) \left\| t \mapsto \langle tk \rangle^\beta a_k(t) \right\|_{L^2(\,\mathrm{d}t)}.$$

By Grönwall lemma, we can find $C(k) > 0$ such that

$$|\phi_k(t)| \lesssim_k e^{C(k)t}$$

(with an evaluation constant depending on k). For $\mu \in \mathbb{R}$, let

$$\phi_{k,\mu}(t) = e^{\mu|k|t} \phi_k(t), \qquad a_{k,\mu}(t) = e^{\mu|k|t} a_k(t), \qquad K^0_{k,\mu}(t) = e^{\mu|k|t} K_k(t).$$

We get

$$\phi_{k,\mu}(t) = a_{k,\mu}(t) + K^0_{k,\mu} \star \phi_{k,\mu}(t)$$

With $\mu < -C(k)/|k|$, we take the time-Fourier transform and we obtain, for any $\tilde{\omega} \in \mathbb{R}$,

$$\mathscr{F}(\phi_{k,\mu})(\tilde{\omega}) = \mathscr{F}(a_{k,\mu})(\tilde{\omega}) + \mathscr{F}(K^0_{k,\mu})(\tilde{\omega}) \mathscr{F}(\phi_{k,\mu})(\tilde{\omega}).$$

Moreover (still assuming $\mu < -C(k)/|k|$) $\phi_{k,\mu}$, $a_{k,\mu}$ et $K^0_{k,\mu}$ all decay exponentially fast, and thus $\mathscr{F}(\phi_{k,\mu})$, $\mathscr{F}(a_{k,\mu})$ et $\mathscr{F}(K^0_{k,\mu})$ have the C^∞ regularity. Besides **(A:H3)** and **(A:H4)** imply that a_k and K_k decay polynomially fast, and thus $\mathscr{F}(a_{k,\mu})$ and $\mathscr{F}(K^0_{k,\mu})$ are of class C^∞ for $\mu < 0$. We deduce that, for any $\beta \in \mathbb{N}$ and $\mu < -C(k)/|k|$

$$\partial^\beta_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega) = \partial^\beta_{\tilde\omega}\mathscr{F}(a_{k,\mu})(\tilde\omega) + \sum_{j=0}^{\beta}\binom{\beta}{j}\partial^{\beta-j}_{\tilde\omega}\mathscr{F}(K^0_{k,\mu})(\tilde\omega)\partial^j_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega),$$

which can be recast as

$$\left[1-\mathscr{F}(K^0_{k,\mu})(\tilde\omega)\right]\partial^\beta_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega) = \partial^\beta_{\tilde\omega}\mathscr{F}(a_{k,\mu})(\tilde\omega) + \sum_{j=0}^{\beta-1}\binom{\beta}{j}\partial^{\beta-j}_{\tilde\omega}\mathscr{F}(K^0_{k,\mu})(\tilde\omega)\partial^j_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega).$$

We remark that

$$\mathscr{F}(K^0_{k,\mu})(\tilde\omega) = \mathscr{L}K(-\mu + i\tilde\omega/|k|, k).$$

Let

$$N_k = \{\tilde\omega \in \mathbb{R} \text{ such that there exists } s > \Lambda \text{ verifying } \mathscr{L}K(s + i\tilde\omega/|k|, k) = 1\}.$$

We conclude that, for any $\mu < -C(k)/|k|$, $\tilde\omega \in \mathbb{R}\setminus N_k$ and $\beta \in \mathbb{N}$,

$$\partial^\beta_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega) = \frac{\partial^\beta_{\tilde\omega}\mathscr{F}(a_{k,\mu})(\tilde\omega) + \sum_{j=0}^{\beta-1}\binom{\beta}{j}\partial^{\beta-j}_{\tilde\omega}\mathscr{F}(K^0_{k,\mu})(\omega)\partial^j_{\tilde\omega}\mathscr{F}(\phi_{k,\mu})(\tilde\omega)}{1-\mathscr{L}K(-\mu + i\tilde\omega/|k|, k)}$$

We proceed by recursion over β to justify that for any $\beta \in [\![0, \min(m-1, m_0-3)]\!]$, we can find $C = C(\beta,\kappa)$ that satisfies

$$\left\| t \mapsto |tk|^\beta \phi_k(t)\right\|_{L^2(\,\mathrm{d}t)} \le C(\beta,\kappa)\left\| t\mapsto \langle tk\rangle^\beta a_k(t)\right\|_{L^2(\,\mathrm{d}t)}.$$

Initialisation. For any $\tilde\omega \in \mathbb{R}\setminus\{N_k\}$, $\mu < -C(k)/|k|$ and with $\beta = 0$, we get

$$\mathscr{F}(\phi_{k,\mu})(\tilde\omega) = \frac{\mathscr{F}(a_{k,\mu})(\tilde\omega)}{1-\mathscr{L}K(-\mu + i\tilde\omega/|k|, k)}.$$

Let $\gamma_\delta(t) = \exp(-\delta t^2/2)$. We write

$$\mathscr{F}(\phi_{k,\mu}\gamma_\delta) = \mathscr{F}(\phi_{k,\mu}) \star \mathscr{F}(\gamma_\delta)(\tilde\omega) = \frac{\mathscr{F}(a_{k,\mu})(\cdot)}{1-\mathscr{L}K(-\mu + i\cdot/|k|, k)} \star \mathscr{F}(\gamma_\delta)(\tilde\omega).$$

The left hand side is well defined for any $\mu \in \mathbb{R}$ and is analytic with respect to μ. The right hand side is defined for $\mu \le 0$ and any $\tilde\omega \in \mathbb{R}\setminus N_k$; it is analytic with respect to μ on an open set that contains the half real line $\{x < 0\}$. Finally the mid-term makes sense for $\mu < -C(k)/|k|$ and the equality holds when this constraint on μ is fulfilled. The uniqueness theorem for analytic functions implies that the left-hand-side and the right-hand-side coincide for $\mu < 0$:

$$\mathscr{F}(\phi_{k,\mu}\gamma_\delta)(\tilde\omega) = \frac{\mathscr{F}(a_{k,\mu})(\cdot)}{1-\mathscr{L}K(-\mu + i\cdot/|k|, k)} \star \mathscr{F}(\gamma_\delta)(\tilde\omega)$$

Owing to Lemma A.22, we know that N_k is a negligible set. Hence we can take the L^2-norm and we get, for $-\Lambda < \mu < 0$,

$$\|\phi_{k,\mu}\gamma_\delta\|_{L^2(\,\mathrm{d}t)} \le \frac{\|a_{k,\mu}\|_{L^2(\,\mathrm{d}t)}}{\kappa/2}\|\mathscr{F}(\gamma_\delta)\|_{L^1(\,\mathrm{d}t)} = \frac{2}{\kappa}\|a_{k,\mu}\|_{L^2(\,\mathrm{d}t)},$$

the first inequality being a consequence of Lemma A.13. We let μ go to 0: the Lebesgue theorem justifies that the inequality still holds for $\mu = 0$:

$$\|\phi_k \gamma_\delta\|_{L^2(\,\mathrm{d}t)} \leq \frac{2}{\kappa}\|a_k\|_{L^2(\,\mathrm{d}t)}.$$

Since $\phi_k\gamma_\delta$ converges monotonically to ϕ_k as $\delta \to 0$, the Beppo-Lévi theorem implies

$$\|\phi_k\|_{L^2(\,\mathrm{d}t)} \leq \frac{2}{\kappa}\|a_k\|_{L^2(\,\mathrm{d}t)}.$$

Recursion. Suppose $\beta \leq \min(m-1, m_0-3)$ and that for any $m \in [\![0, \beta-1]\!]$ there exists $C(m,\kappa)$ such that

$$\|t \mapsto |tk|^m \phi_k(t)\|_{L^2(\,\mathrm{d}t)} \leq C(m,\kappa)\,\|t \mapsto \langle tk\rangle^m a_k(t)\|_{L^2(\,\mathrm{d}t)}\,.$$

Since

$$\left(\partial_\omega^\beta \mathscr{F}(\phi_{k,\mu})\right) \star \mathscr{F}(\gamma_\delta)(\omega) = \partial_\omega^\beta \left(\mathscr{F}(\phi_{k,\mu}) \star \mathscr{F}(\gamma_\delta)\right)(\omega) = \partial_\omega^\beta \mathscr{F}(\phi_{k,\mu}\gamma_\delta)(\omega)$$

we get

$$\begin{aligned}\partial_\omega^\beta \mathscr{F}(\phi_{k,\mu}\gamma_\delta)(\tilde\omega) &= \left(\partial_{\tilde\omega}^\beta \mathscr{F}(\phi_{k,\mu})\right) \star \mathscr{F}(\gamma_\delta)(\tilde\omega) \\ &= \frac{\partial_{\tilde\omega}^\beta \mathscr{F}(a_{k,\mu})(\cdot) + \sum_{j=0}^{\beta-1} \binom{\beta}{j} \partial_\omega^{\beta-j}\mathscr{F}(K^0_{k,\mu})(\cdot)\partial_\omega^j \mathscr{F}(\phi_{k,\mu})(\cdot)}{1 - \mathscr{L}K(-\mu + i\cdot/|k|, k)} \star \mathscr{F}(\gamma_\delta)(\tilde\omega).\end{aligned}$$

The left hand side is well-defined for any $\mu \in \mathbb{R}$ and it is analytic with respect to μ. The right hand side is defined for $\mu \geq 0$ and any $\tilde\omega \in \mathbb{R} \setminus N_k$; it is analytic with respect to μon an open set that contains the half-line $\{x < 0\}$. The full justification of this assertion uses the recursion assumption : we know that $t \mapsto |t|^m \phi_k(t)$ lies in L^2 for $\mu < 0$, thus $t \mapsto |t|^m \phi_{k,\mu}(t)$ belongs to L^1 and $\partial_{\tilde\omega}^m \mathscr{F}(\phi_{k,\mu})$ is defined everywhere and depends analytically on μ. However, for $\mu = 0$ this quantity is defined almost everywhere only. Finally, the mid-term makes sense for $\mu < -C(k)/|k|$ and the two equalities holds when this constraint on μ is fulfilled. The analytic uniqueness theorem shows that the left-hand-side and the right-hand-side are actually equal for any $\mu < 0$. Since N_k is negligible, we keep the equality of the L^2-norms. Therefore, for $-\Lambda < \mu < 0$ we get

$$\begin{aligned}&\left\|\partial_\omega^\beta \mathscr{F}(\phi_{k,\mu}\gamma_\delta)\right\|_{L^2(\,\mathrm{d}\omega)} \\ &\leq \frac{2}{\kappa}\left\|\partial_\omega^\beta \mathscr{F}(a_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)} \|\mathscr{F}(\gamma_\delta)\|_{L^1(\,\mathrm{d}\omega)} \\ &\qquad + \frac{2}{\kappa}\left(\sum_{j=0}^{\beta-1}\left\|\partial_\omega^{\beta-j}\mathscr{F}(K^0_{k,\mu})\right\|_{L^\infty(\,\mathrm{d}\omega)} \left\|\partial_\omega^j \mathscr{F}(\phi_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)}\right)\|\mathscr{F}(\gamma_\delta)\|_{L^1(\,\mathrm{d}\omega)} \\ &= \frac{2}{\kappa}\left\|\partial_\omega^\beta \mathscr{F}(a_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)} + \frac{2}{\kappa}\sum_{j=0}^{\beta-1}\left\|\partial_\omega^{\beta-j}\mathscr{F}(K^0_{k,\mu})\right\|_{L^\infty(\,\mathrm{d}\omega)}\left\|\partial_\omega^j \mathscr{F}(\phi_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)}.\end{aligned}$$

Multiplying by $|k|^\beta$, we apply Lemma A.15-(iv) and we obtain

$$\begin{aligned}\left\| |k|^\beta \partial_\omega^\beta \mathscr{F}(\phi_{k,\mu}\gamma_\delta)\right\|_{L^2(\,\mathrm{d}\omega)} &\leq \frac{2}{\kappa}\left\| |k|^\beta \partial_\omega^\beta \mathscr{F}(a_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)} \\ &\qquad + \frac{2}{\kappa}\sum_{j=0}^{\beta-1}\left\| |k|^{\beta-j}\partial_\omega^{\beta-j}\mathscr{F}(K^0_{k,\mu})\right\|_{L^\infty(\,\mathrm{d}\omega)}\left\| |k|^j \partial_\omega^j \mathscr{F}(\phi_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)} \\ &\leq \frac{2}{\kappa}\left\| |k|^\beta \partial_\omega^\beta \mathscr{F}(a_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)} + \frac{2C^{ste}}{\kappa}\sum_{j=0}^{\beta-1}\left\| |k|^j \partial_\omega^j \mathscr{F}(\phi_{k,\mu})\right\|_{L^2(\,\mathrm{d}\omega)},\end{aligned}$$

which can be cast as

$$\left\| t\mapsto |tk|^\beta \phi_{k,\mu}(t)\gamma_\delta(t)\right\|_{L^2(\,\mathrm{d}t)} \leq \frac{2}{\kappa}\left\| t\mapsto |tk|^\beta a_{k,\mu}(t)\right\|_{L^2(\,\mathrm{d}t)} + \frac{2C^{ste}}{\kappa}\sum_{j=0}^{\beta-1}\left\| t\mapsto |tk|^j \phi_{k,\mu}(t)\right\|_{L^2(\,\mathrm{d}\omega)}.$$

We now let μ, and next δ, both tend to 0 and we arrive at

$$\begin{aligned}\left\| t\mapsto |tk|^\beta \phi_k(t)\right\|_{L^2(\,\mathrm{d}t)} &\leq \frac{2}{\kappa}\left\| t\mapsto |tk|^\beta a_k(t)\right\|_{L^2(\,\mathrm{d}t)} + \frac{2C^{ste}}{\kappa}\sum_{j=0}^{\beta-1}\left\| t\mapsto |tk|^j \phi_k(j)\right\|_{L^2(\,\mathrm{d}t)} \\ &\leq C(\beta,\kappa)\left\| t\mapsto \langle tk\rangle^\beta a_k(t)\right\|_{L^2(\,\mathrm{d}t)}\end{aligned}$$

Step 2. We go back to the equation satisfied by ϕ_k; we use the previous step to deduce the L^∞ estimate on $(\phi_k)_k$. To this end, observe that

$$\begin{aligned}(t|k|)^\beta \phi_k(t) &= (t|k|)^\beta a_k(t) + \int_0^{+\infty}[(t-\tau)|k| + \tau|k|]^\beta K^0_k(t-\tau)\phi_k(\tau)\,\mathrm{d}\tau \\ &= (t|k|)^\beta a_k(t) + \sum_{j=0}^{\beta}\binom{\beta}{j}\int_0^{+\infty}|k|^{\beta-j}(t-\tau)^{\beta-j}K^0_k(t-\tau)|k|^j\tau^j\phi_k(\tau)\,\mathrm{d}\tau.\end{aligned}$$

It yields

$$\begin{aligned}&\left\| t\mapsto |tk|^\beta \phi_k(t)\right\|_{L^\infty(\,\mathrm{d}t)} \\ \leq\ & \left\| t\mapsto |tk|^\beta a_k(t)\right\|_{L^\infty(\,\mathrm{d}t)} + \sum_{j=0}^{\beta}\binom{\beta}{j}\left\| t\mapsto |tk|^{\beta-j}K^0_k(t)\right\|_{L^2(\,\mathrm{d}t)}\left\| t\mapsto |tk|^j\phi_k(t)\right\|_{L^2(\,\mathrm{d}t)} \\ \leq\ & \left\| t\mapsto |tk|^\beta a_k(t)\right\|_{L^\infty(\,\mathrm{d}t)} + \sum_{j=0}^{\beta}\left\| t\mapsto |tk|^{\beta-j}K^0_k(t)\right\|_{L^2(\,\mathrm{d}t)}\left\| t\mapsto \langle tk\rangle^j a_k(t)\right\|_{L^2(\,\mathrm{d}t)}.\end{aligned}$$

Like in the analytic framework, we check that this estimate does not depend on k. We combine the case $\beta = 0$ and $\beta = \min(m-1, m_0-3)$ and we conclude that

$$|\varphi_k(t)| \leq C\langle tk\rangle^{\min(m-1,m_0-3)}.$$

■

Remark A.16 *As mentioned in the analytic case, the L^∞ estimate of φ_k is the main argument for proving the* linearized *Landau damping, but it crucially relies on the preliminary L^2 estimate. The latter can be rewritten (with $m_\star = \min(m-1, m_0-3)$)*

$$\int_0^{+\infty} |tk|^{2m_\star} |\varphi_k(t)|^2 \,\mathrm{d}t \le C(m_\star, \kappa) \int_0^{+\infty} \langle tk\rangle^{2m_\star} |a_k(t)|^2 \,\mathrm{d}t.$$

This estimate becomes the main ingredient for studying the non linear *problem. Modifying $a_k(t)$ into $a_k(t)\mathbf{1}_{0\le t\le T}$, we can equally obtain*

$$\int_0^{T} |tk|^{2m_\star} |\varphi_k(t)|^2 \,\mathrm{d}t \le C(m_\star, \kappa) \int_0^{T} \langle tk\rangle^{2m_\star} |a_k(t)|^2 \,\mathrm{d}t.$$

Similarly, replacing $\phi_k(t)$ by $|k|^{1/2}\langle k\rangle\phi_k(t)$ and $a_k(t)$ by $|k|^{1/2}\langle k\rangle a_k(t)$ leads to

$$\int_0^{T} |k|(\langle k\rangle + |tk|)^{2m_\star} |\varphi_k(t)|^2 \,\mathrm{d}t \le C \int_0^{T} |k|(\langle k\rangle + \langle tk\rangle)^{2m_\star} |a_k(t)|^2 \,\mathrm{d}t.$$

Since $\langle k, tk\rangle \lesssim \langle k\rangle + |k|t \lesssim \langle k, tk\rangle$, it yields

$$\int_0^{T} |k|\langle k, tk\rangle^{2m_\star} |\varphi_k(t)|^2 \,\mathrm{d}t \le C \int_0^{T} |k|\langle k, tk\rangle^{2m_\star} |a_k(t)|^2 \,\mathrm{d}t.$$

This estimate is at the heart of the analysis of the non linear damping.

A.2.2 The Vlasov-Wave case

Now, we assume

(A:H3b) $|a_k(t)| \le \alpha\langle tk\rangle^{-m}$,

(A:H4b) $K_k(t) = |\widehat{\sigma}_1(k)|^2|k|^2 \int_0^t p_c(t-\tau)\tau\widehat{\mathscr{M}}(\tau k)\,\mathrm{d}\tau\mathbf{1}_{t\ge 0}$ and $|\widehat{\mathscr{M}}(\eta)| \le C_0\langle\eta\rangle^{-m_0}$,

(A:Lvw) There exists $\kappa > 0$ such that for any $k \in \mathbb{X}^{\star d}\setminus\{0\}$ and any $\tilde{\omega} \in \mathbb{R}$

$$|\mathscr{F}K_k(|k|\tilde{\omega}) - 1| \ge \kappa,$$

(A:c) For any $\alpha \in [0, m_0]$, we have $\int_0^{+\infty} \langle t\rangle^\alpha |p_c(t)|\,\mathrm{d}t < +\infty$ and $\|p_c\|_{L^\infty} < +\infty$,

(A:d) $|\widehat{\sigma}_1(k)| \lesssim \langle k\rangle^{-2-m_0}$.

Remark A.17 *Assumption **(A:d)** is the analog in finite regularity of **(A:b)**. We point out however that in this framework p_c is not necessarily supposed to be compactly supported: a slow decay, related to the regularity of the equilibrium state $\mathscr{M}$, is enough.*

Remark A.18 *Like for the analytic case, the assumptions* ***(A:H4b)***, ***(A:c)*** *and* ***(A:d)*** *ensures that the behavior of the kernel* K_k *remains close to the pure Vlasov case; namely, we have*

$$\begin{aligned}
|K_k(t)| &\leq C_0|\widehat{\sigma}_1(k)|^2|k|^2\int_0^t |p_c(\tau)||t-\tau|\langle(t-\tau)k\rangle^{-m_0}\,\mathrm{d}\tau \\
&\leq C_0|\widehat{\sigma}_1(k)|^2|k|^2 t\left(\int_0^{t/2}|p_c(\tau)|\langle(t-\tau)k\rangle^{-m_0}\,\mathrm{d}\tau+\int_{t/2}^t|p_c(\tau)|\,\mathrm{d}\tau\right) \\
&\leq C_0|\widehat{\sigma}_1(k)|^2|k|^2 t \\
&\qquad\times\left(\langle tk/2\rangle^{-m_0}\int_0^{t/2}|p_c(\tau)|\,\mathrm{d}\tau+\langle tk/2\rangle^{-m_0}\int_{t/2}^t|p_c(\tau)|\langle tk/2\rangle^{m_0}\,\mathrm{d}\tau\right) \\
&\lesssim C_0|\widehat{\sigma}_1(k)|^2|k|^2 t\langle tk\rangle^{-m_0}\left(\int_0^{t/2}|p_c(\tau)|\,\mathrm{d}\tau+\int_{t/2}^t|p_c(\tau)|\langle\tau k\rangle^{m_0}\,\mathrm{d}\tau\right).
\end{aligned}$$

Since $\langle\tau k\rangle^{m_0}\lesssim 1+(\tau|k|)^{m_0}\lesssim\langle\tau k\rangle^{m_0}$, *we are led to*

$$\int_0^{t/2}|p_c(\tau)|\,\mathrm{d}\tau+\int_{t/2}^t|p_c(\tau)|\langle\tau k\rangle^{m_0}\,\mathrm{d}\tau\leq\left(\int_0^{+\infty}\mathrm{d}\tau+\int_0^{+\infty}|p_c(\tau)|\langle\tau\rangle^{m_0}\right)\langle k\rangle^{m_0}.$$

We conclude with ***(A:c)–(A:d)***. *As a matter of fact, the (rescaled) Laplace transform of* K_k *is well defined, for any* $\omega\in\mathbb{C}$ *tel que* $\mathrm{Re}(\omega)\leq 0$ *:*

$$\mathscr{L}K(\omega,k)=\int_0^{+\infty}e^{-|k|\omega t}K_k(t)\,\mathrm{d}t.$$

We remind the reader that

$$\mathscr{L}K(\omega,k)=\mathscr{L}p_c(\omega,k)\mathscr{L}\tilde{K}(\omega,k),$$

with $\tilde{K}$ *similar to the pure Vlasov case.*

Remark A.19 *For the Vlasov-Wave problem, the data* a_k *is the sum of two contributions*

$$a_k(t)=f_0(t,tk)-|k|^2\int_0^t\widehat{\phi}_I(\tau,k)(t-\tau)\widehat{\mathscr{M}}((t-\tau)k)\,\mathrm{d}\tau$$

with ϕ_I *defined from the solution of the free wave equation*

$$\widehat{\phi}_I(t,k)=\widehat{\sigma}_1(k)\int_{\mathbb{R}^n}\widehat{\sigma}_2(\zeta)\widehat{\psi}_I(t,k,\zeta)\,\mathrm{d}\zeta.$$

With ***(H1)***, ***(H2)*** *and* ***(H3)***, $t\mapsto\widehat{\phi}_I(t,k)$ *is compactly supported and uniformly dominated with respect to* t *and* k. *In particular, we get*

$$|a_k(t)|\leq\alpha\langle tk\rangle^{-m},$$

see Lemma 3.2 and its proof.

Theorem A.20 *Assume **(A:H3b)**–**(A:d)**. Let $m_\star = \min(m-1, m_0-3)$. Then, there exists $C > 0$ such that for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and any $t \geq 0$, we have*

$$|\varphi_k(t)| \leq C\langle |k|t\rangle^{-m_\star}.$$

Let us collect the necessary preliminary statements about the locations of the zeroes of $\mathscr{L}K(\omega, k) - 1$.

Lemma A.21 *Assume **(A:H3b)**–**(A:d)**. There exists $\Lambda > 0$ such that, for any $k \in \mathbb{X}^{\star d} \setminus \{0\}$ and $\omega \in \mathbb{C}$,*

$$\textit{if } 0 \leq \mathrm{Re}(\omega) \leq \Lambda, \textit{ then } |\mathscr{L}K(\omega, k) - 1| \geq \frac{\kappa}{2}.$$

Proof. The proof is an adaptation of the analytic case, where, again, we need to pay attention that the obtained constant Λ does not depend on k. ■

Lemma A.22 *For any $k \in \mathbb{X}^{\star d} \setminus \{0\}$, the open set*

$$\Omega = \{\omega \in \mathbb{C} \textit{ such that } \Lambda < \mathrm{Re}(\omega)\}$$

contains at most a countable set of zeroes of $\omega \mapsto \mathscr{L}K(\omega, k) - 1$.

Lemma A.23 *The following assertions hold.*

(i) If $\beta \leq m$, then the function $t \mapsto (t|k|)^\beta a_k(t)$ is bounded uniformly with respect to k and t;

(ii) If $\beta \leq m-1$, then the function $t \mapsto (t|k|)^\beta a_k(t)$ is square integrable and

$$\Big\| t \mapsto (t|k|)^\beta a_k(t) \Big\|_{L^2} \lesssim \frac{1}{\sqrt{|k|}};$$

(iii) If $\beta \leq m_0 - 2$, then the function $t \mapsto (t|k|)^\beta K_k(t)$ is square integrable and

$$\Big\| t \mapsto (t|k|)^\beta K_k(t) \Big\|_{L^2} \lesssim \sqrt{|k|};$$

(iv) For $\mu \leq 0$, let $K^0_{k,\mu}(t) = K_k(t) e^{\mu|k|t} \mathbf{1}_{t\geq 0}$. If $\beta \leq m_0 - 3$, then the function $\omega \in \mathbb{R} \mapsto |k|^\beta \partial^\beta_{\tilde\omega} \mathscr{F}(K^0_{k,\mu})(\tilde\omega)$ is bounded uniformly with respect to k and μ; namely

$$\Big| |k|^\beta \partial^\beta_{\tilde\omega} \mathscr{F}(K^0_{k,\mu})(\tilde\omega) \Big| \leq C(\sigma_1, p_c) \int_0^{+\infty} u^{\beta+1} \langle u \rangle^{-m_0} \, \mathrm{d}u < +\infty.$$

Proof. The statement follows by direct evaluation. As a matter of fact, properties (iii) and (iv) on the kernel K_k are consequences of the obervations in Remark A.18. ■

Having these statements at hand we can repeat the arguments of the Vlasov case.

B Penrose criterion

For the usual Vlasov equation, a "practical " condition on the equilibrium $\mathscr{M}$, the Penrose criterion, see [28, Condition (c) in Proposition 2.1], can be exhibited to ensure the linearized stability. Hence, by following a similar approach, we expect to find a criterion with the same flavor on the equilibrium $\mathscr{M}$ and the coefficients of the problem, for the Vlasov-Wave system.

B.1 Towards a Landau-Penrose criterion

The stability criterion (**L**) is absolutely crucial for justifying the Landau damping; however, it is not easy to check it in practice. We already know that a large wave speed guarantees the damping, see Proposition 3.4. For the Vlasov equation, a practical criterion, referred to as the *Penrose criterion* can be devised: the real and imaginary parts of $\mathscr{L}K$ decouple which leads to a simple way of checking that $\mathscr{L}K$ remains far from 1. We will discuss a similar criterion for the Vlasov-Wave problem; however the real/imaginary splitting is not that simple, due to the role of the convolution with respect to time with p_c. As a preliminary, we detail why it suffices to check that $\mathscr{L}(\omega, k)$ does not reach 1 on the imaginary axis.

B.1.1 The Vlasov case

Throughout this Section, we assume that **(A:H4)** since it covers more general cases than **(A:H2)**.

Proposition B.1 (Periodic framework) *Let $\mathbb{X}^{\star d} \setminus \{0\} = \mathbb{Z}^d \setminus \{0\}$. Suppose that $\mathscr{L}K(i\tilde{\omega}, k) \neq 1$ for any $k \in \mathbb{Z}^d \setminus \{0\}$ and $\tilde{\omega} \in \mathbb{R}$. Then, there exists $\kappa > 0$ such that*

$$|\mathscr{L}K(i\tilde{\omega}, k) - 1| \geq \kappa$$

holds for any $k \in \mathbb{Z}^d \setminus \{0\}$ and $\tilde{\omega} \in \mathbb{R}$

Proof. It suffices to consider a finite set of $k \in \mathbb{Z}^d \setminus \{0\}$ since

$$|\mathscr{L}K(i\tilde{\omega}, k)| \leq |\widehat{W}(k)| C_0 \int_0^{+\infty} \langle u \rangle^{-m_0} u \, \mathrm{d}u \underset{|k| \to +\infty}{\longrightarrow} 0.$$

This asymptotic behavior follows from the Riemann-Lebesgue Lemma; it is uniform with respect to $\tilde{\omega}$. Since $u \mapsto \widehat{\mathscr{M}}(ku/|k|)u$ lies in L^1, the Riemann-Lebesgue Lemma also implies

$$|\mathscr{L}K(i\tilde{\omega}, k)| \leq \|W\|_{L^1} \left| \int_0^{+\infty} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right) u \, \mathrm{d}u \right| \underset{|\tilde{\omega}| \to +\infty}{\longrightarrow} 0$$

uniformly with respect to k since $k/|k|$ takes a finite number of values when k spans $\mathbb{Z}^d \setminus \{0\}$. This observation equally permits us to restrict to a compact set for $\tilde{\omega} \in \mathbb{R}$. The Lebesgue Theorem shows that $\tilde{\omega} \in \mathbb{R} \mapsto \mathscr{L}K(i\tilde{\omega}, k)$ is continuous, which allows to conclude. ∎

Proposition B.2 (Free space problem) *Let $\mathbb{X}^{\star d} \setminus \{0\} = \mathbb{R}^d \setminus \{0\}$. Suppose that $\mathscr{L}K(i\tilde{\omega}, k) \neq 1$ for any $k \in \mathbb{R}^d \setminus \{0\}$ and $\tilde{\omega} \in \mathbb{R}$. Moreover, suppose that*

$$-\widehat{W}(0) \int_0^{+\infty} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}(\zeta u) u \,\mathrm{d}u \neq 1,$$

for any $\zeta \in \mathbb{S}^{d-1}$ and any $\tilde{\omega} \in \mathbb{R}$. Then, there exists $\kappa > 0$ such that

$$|\mathscr{L}K(i\omega, k) - 1| \geq \kappa,$$

holds for any $k \in \mathbb{Z}^d \setminus \{0\}$ and $\tilde{\omega} \in \mathbb{R}$

Remark B.3 *With $\mathbb{X}^{\star d} \setminus \{0\} = \mathbb{R}^d \setminus \{0\}$, k can be arbitrarily close to 0, which motivates the additional condition. It would be tempting to write $\mathscr{L}K(i\tilde{\omega}, 0) \neq 1$, but this quantity is not well defined. Thus, we obtain the condition by letting k go to 0, for fixed $\tilde{\omega}$: with $(k_n)_{n\in\mathbb{N}}$ converging to 0 and $(k_n/|k_n|)_n$ converging to a certain $\zeta \in \mathbb{S}^{d-1}$, we have*

$$\lim_{n\to\infty} \mathscr{L}K(i\tilde{\omega}, k_n) = -\widehat{W}(0) \int_0^{+\infty} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}(\zeta u) u \,\mathrm{d}u.$$

Proof. The Riemann-Lebesgue Lemma yields

$$|\mathscr{L}K(i\tilde{\omega}, k)| \leq |\widehat{W}(k)| C_0 \int_0^{+\infty} \langle u \rangle^{-m_0} u \,\mathrm{d}u \underset{|k|\to+\infty}{\longrightarrow} 0,$$

uniformly with respect to $\tilde{\omega}$. Hence we can restrict to a bounded subset of $k \in \mathbb{R}^d \setminus \{0\}$. Like in the previous proof, we obtain

$$|\mathscr{L}K(i\tilde{\omega}, k)| \leq \|W\|_{L^1} \left| \int_0^{+\infty} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}\left(\frac{k}{|k|} u\right) u \,\mathrm{d}u \right| \underset{|\tilde{\omega}|\to+\infty}{\longrightarrow} 0.$$

However, now, we cannot conclude directly that this convergence holds uniformly with respect to k. In order to handle this difficulty we introduce the function

$$g : (\tilde{\omega}, \zeta) \in \mathbb{R} \times \mathbb{S}^{d-1} \longmapsto \int_0^{+\infty} e^{-i\tilde{\omega}u} \widehat{\mathscr{M}}(\zeta u) u \,\mathrm{d}u.$$

We already know that, for any $\zeta \in \mathbb{S}^{d-1}$, we can find $R_\zeta > 0$ such that, for any $\tilde{\omega} \in \mathbb{R}$, we have

$$\text{if } |\tilde{\omega}| \geq R_\zeta \text{ then } \|W\|_{L^1} |g(\tilde{\omega}, \zeta)| \leq \frac{1}{4}.$$

By using the Lebesgue Theorem, we check that g is continuous. Actually, the continuity with respect to ζ is uniform with respect to $\tilde{\omega}$. Indeed, this follows from the inequality

$$|g(\tilde{\omega}, \zeta_1) - g(\tilde{\omega}, \zeta_2)| \leq \int_0^{+\infty} \left| \widehat{\mathscr{M}}(\zeta_1 u) - \widehat{\mathscr{M}}(\zeta_2 u) \right| u \,\mathrm{d}u,$$

which holds for any $\tilde{\omega} \in \mathbb{R}$ and $\zeta_1, \zeta_2 \in \mathbb{S}^{d-1}$. The integrand is a continuous function of the variable ζ, that can be dominated independently of ζ by an integrable function. Since $\mathbb{S}^{d-1}$ is compact, by virtue of Heine's theorem, we conclude that, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $\zeta_1, \zeta_2 \in \mathbb{S}^d$ satisfying $|\zeta_1 - \zeta_2| < \delta_\epsilon$ we have

$$\int_0^{+\infty} \left| \widehat{\mathscr{M}}(\zeta_1 u) - \widehat{\mathscr{M}}(\zeta_2 u) \right| u \,\mathrm{d}u \leq \varepsilon.$$

From the covering $\mathbb{S}^{d-1} \subset \bigcup_{\zeta\in\mathbb{S}^{d-1}} B(\zeta,\delta_\varepsilon)$, we can extract a finite covering

$$\mathbb{S}^{d-1} \subset \bigcup_{i=1}^{J_\epsilon} B(\zeta_i,\delta_\varepsilon).$$

Then let $R_\epsilon = \max_{i=1,\dots,J_\epsilon} R_{\zeta_i}$. For any $\tilde{\omega}\in\mathbb{R}$, $|\tilde{\omega}|\geq R_\epsilon$ and any $\zeta\in\mathbb{S}^{d-1}$, we get

$$\|W\|_{L^1}|g(\tilde{\omega},\zeta)| \leq \|W\|_{L^1}\left(|g(\tilde{\omega},\zeta)-g(\tilde{\omega},\zeta_{i_0})|+|g(\tilde{\omega},\zeta_{i_0})|\right) \leq \varepsilon\|W\|_{L^1}+\frac{1}{4}\leq\frac{1}{2}.$$

We have shown that $\mathscr{L}K(i\tilde{\omega},k)$ remains far from 1 for large enough $\tilde{\omega}\in\mathbb{R}$, uniformly with respect to k; to be specific, we have just found $R>0$ such that, for any $|\tilde{\omega}|\geq R$ and any $k\in\mathbb{X}^{d\star}$, we have $|\mathscr{L}K(i\tilde{\omega},k)|\leq\|W\|_{L^1}|g(\tilde{\omega},k/|k|)|\leq 1/2$.

Therefore we can restrict to a compact set of $\tilde{\omega}\in\mathbb{R}$. Since the function $(\tilde{\omega},k)\mapsto\widehat{W}(k)g(\tilde{\omega},k/|k|)$ is continuous $\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$, for any compact set $K\subset\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$, we have

$$\inf_{(\tilde{\omega},k)\in K}|\mathscr{L}K(i\tilde{\omega},k)-1|>0.$$

We end the proof by arguing by contradiction. Suppose that

$$\inf_{(\tilde{\omega},k)\in\mathbb{R}\times\mathbb{R}^d\setminus\{0\}}|\mathscr{L}K(\tilde{\omega},k)-1|=0.$$

Then, we can find a sequence $\left(\tilde{\omega}_n,k_n\right)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$ such that $\mathscr{L}K(i\tilde{\omega}_n,k_n)$ tends to 1 as $n\to+\infty$. The previous step tells us that the sequence $\left(\tilde{\omega}_n\right)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ while $\left(k_n\right)_{n\in\mathbb{N}}$ tends to 0. Possibly at the price of extracting a subsequence, we can suppose that

$$\tilde{\omega}_n \underset{n\to+\infty}{\longrightarrow} \tilde{\omega}_\infty \qquad k_n \underset{n\to+\infty}{\longrightarrow} 0, \qquad \frac{k_n}{|k_n|} \underset{n\to+\infty}{\longrightarrow} \zeta_\infty.$$

By continuity, it yields

$$1=\lim_{n\to+\infty}\mathscr{L}K(i\tilde{\omega}_n,k_n)=-\widehat{W}(0)g(\tilde{\omega}_\infty,\zeta_\infty),$$

which is known to differ form 1, a contradiction. ■

B.1.2 The Vlasov-Wave case

We consider the set of assumptions **(A:H4b)**, **(A:c)**, **(A:d)** which are more general than **(A:H3b)**, **(A:a)**, **(A:b)**.

Remark B.4 *In fact* ***(A:c)*** *and* ***(A:d)*** *can be slightly relaxed, for instance dealing with $p_c\in L^1(\mathrm{d}t)$ with $\sigma_1,\nabla\sigma_1\in L^1(\mathrm{d}x)$, which are enough to ensure that K_k is integrable for $k\in\mathbb{X}^{\star d}\setminus\{0\}$) and*

$$\mathscr{L}K(i\tilde{\omega},k)=|\widehat{\sigma}_1(k)|^2\left(\int_0^{+\infty}e^{-i\tilde{\omega}|k|t}p_c(t)\,\mathrm{d}t\right)\left(\int_0^{+\infty}e^{-i\omega u}u\widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right)\,\mathrm{d}u\right).$$

Proposition B.5 (Periodic framework) *Let $\mathbb{X}^{\star d}\setminus\{0\}=\mathbb{Z}^d\setminus\{0\}$. If $\mathscr{L}K(i\omega,k)\neq 1$, for any $k\in\mathbb{Z}^d\setminus\{0\}$ and any $\tilde\omega\in\mathbb{R}$, then there exists $\kappa>0$ such that*

$$|\mathscr{L}K(i\tilde\omega,k)-1|\geq\kappa,$$

holds for any $k\in\mathbb{Z}^d\setminus\{0\}$ and $\tilde\omega\in\mathbb{R}$.

Proof. The proof involves a few modifications compared to the Vlasov case. We can restrict to a finite set of $k\in\mathbb{Z}^d\setminus\{0\}$ since

$$|\mathscr{L}K(i\tilde\omega,k)|\leq|\widehat{\sigma}_1(k)|^2\|p_c\|_{L^1}C_0\int_0^{+\infty}\langle u\rangle^{-m_0}u\,\mathrm{d}u\underset{|k|\to+\infty}{\longrightarrow}0,$$

as a consequence of the Riemann-Lebesgue Lemma. The convergence holds uniformly with respect to $\tilde\omega$. Since $u\mapsto\widehat{\mathscr{M}}(ku/|k|)u$ is integrable, the Riemann-Lebesgue Lemma also leads to

$$|\mathscr{L}K(i\tilde\omega,k)|\leq\|\sigma_1\|^2_{L^1}\|p_c\|_{L^1}\left|\int_0^{+\infty}e^{-i\tilde\omega u}\widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right)u\,\mathrm{d}u\right|\underset{|\tilde\omega|\to+\infty}{\longrightarrow}0.$$

The convergence holds uniformly with respect to k since $k/|k|$ takes only a finite number of values when k spans $\mathbb{Z}^d\setminus\{0\}$. We can equally restrict to a compact set for $\tilde\omega\in\mathbb{R}$. With the Lebesgue Theorem, we conclude that $\tilde\omega\in\mathbb{R}\mapsto\mathscr{L}K(i\tilde\omega,k)$ is continuous. ■

Proposition B.6 (Free space problem) *Let $\mathbb{X}^{\star d}\setminus\{0\}=\mathbb{R}^d\setminus\{0\}$. Suppose that $\mathscr{L}K(i\tilde\omega,k)\neq 1$ for any $k\in\mathbb{R}^d\setminus\{0\}$ and $\tilde\omega\in\mathbb{R}$. Moreover suppose that*

$$|\widehat{\sigma}_1(0)|^2\left(\int_0^{+\infty}p_c(t)\,\mathrm{d}t\right)\left(\int_0^{+\infty}e^{-i\tilde\omega u}\widehat{\mathscr{M}}(\zeta u)u\,\mathrm{d}u\right)\neq 1,$$

holds for any $\zeta\in\mathbb{S}^d$ and any $\tilde\omega\in\mathbb{R}$. Then, there exists $\kappa>0$ such that

$$|\mathscr{L}K(i\omega,k)-1|\geq\kappa$$

holds for any $k\in\mathbb{R}^d\setminus\{0\}$ and any $\tilde\omega\in\mathbb{R}$.

Proof. The half convolution with p_c requires some adaptations from the Vlasov case. The Riemann-Lebesgue Lemma yields

$$|\mathscr{L}K(i\tilde\omega,k)|\leq|\widehat{\sigma}_1(k)|^2\|p_c\|_{L^1}C_0\int_0^{+\infty}\langle u\rangle^{-m_0}u\,\mathrm{d}u\underset{|k|\to+\infty}{\longrightarrow}0,$$

where the convergence holds uniformly with respect to $\tilde\omega$. Thus we can restrict to a bounded set of $k\in\mathbb{R}^d\setminus\{0\}$. Next, we obtain

$$|\mathscr{L}K(i\tilde\omega,k)|\leq\|\sigma_1\|^2_{L^1}\|p_c\|_{L^1}\left|\int_0^{+\infty}e^{-i\tilde\omega u}\widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right)u\,\mathrm{d}u\right|\underset{|\tilde\omega|\to+\infty}{\longrightarrow}0.$$

That the convergence holds uniformly with respect to k is not direct, but we can reproduce the arguments of the Vlasov case to justify this property. It allows us to restrict to a compact set of $\tilde\omega\in\mathbb{R}$. The application

$$(\tilde\omega,k)\mapsto|\widehat{\sigma}_1(k)|^2\left(\int_0^{+\infty}e^{-i\tilde\omega|k|t}p_c(t)\,\mathrm{d}t\right)\left(\int_0^{+\infty}e^{-i\tilde\omega u}u\widehat{\mathscr{M}}\left(\frac{k}{|k|}u\right)\,\mathrm{d}u\right)$$

is continuous over $\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$. Therefore, for any compact set $K\subset\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$, we get

$$\inf_{(\tilde\omega,k)\in K}|\mathscr{L}K(i\tilde\omega,k)-1|>0.$$

Suppose that

$$\inf_{(\tilde\omega,k)\in\mathbb{R}\times\mathbb{R}^d\setminus\{0\}}|\mathscr{L}K(\tilde\omega,k)-1|=0.$$

Then, we can find a sequence $\big(\tilde\omega_n,k_n\big)_{n\in\mathbb{N}}$ in $\mathbb{R}\times\mathbb{R}^d\setminus\{0\}$ such that $\mathscr{L}K(i\tilde\omega_n,k_n)\to 1$ as $n\to+\infty$. We infer that $\big(\tilde\omega_n\big)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ while $\big(k_n\big)_{n\in\mathbb{N}}$ converges to 0. Extracting a subsequence if necessary, we can suppose that

$$\tilde\omega_n\underset{n\to+\infty}{\longrightarrow}\tilde\omega_\infty\qquad k_n\underset{n\to+\infty}{\longrightarrow}0\qquad \frac{k_n}{|k_n|}\underset{n\to+\infty}{\longrightarrow}\zeta_\infty.$$

By continuity, we are led to

$$1=\lim_{n\to+\infty}\mathscr{L}K(i\tilde\omega_n,k_n)=|\widehat{\sigma}_1(0)|^2\left(\int_0^{+\infty}p_c(t)\,\mathrm{d}t\right)\left(\int_0^{+\infty}e^{-i\tilde\omega_\infty u}u\widehat{\mathscr{M}}(\zeta_\infty u)\,\mathrm{d}u\right),$$

which is known to be different from 1, a contradiction. ■

B.2 Computations of Laplace transform for the Penrose criterion

In order to find an expression for the stability criterion, we compute $\mathscr{L}\mathscr{K}(\omega|k|,k)$ on the imaginary axis: namely, with $\beta\in\mathbb{R}$, we consider

$$\begin{aligned}\mathscr{L}\mathscr{K}(i\beta|k|,k)&=\lim_{\substack{\alpha\to0\\\alpha>0}}\mathscr{L}\mathscr{K}((\alpha+i\beta)|k|,k)\\&=\rho_0|\widehat{\sigma_1}(k)|^2\lim_{\substack{\alpha\to0\\\alpha>0}}\mathscr{L}p_c((\alpha+i\beta)|k|)\times\lim_{\substack{\alpha\to0\\\alpha>0}}\mathscr{L}(|k|^2t\widehat{M}(kt))((\alpha+i\beta)|k|).\end{aligned}$$

We write, for $\mathrm{Re}(\omega)>0$,

$$\mathscr{L}\mathscr{K}(\omega|k|,k)=\rho_0|\widehat{\sigma_1}(k)|^2\mathscr{L}p_c(\omega|k|)\times\mathscr{L}(|k|^2t\widehat{M}(kt))(\omega|k|).$$

There are several useful expressions for this quantity

$$\begin{aligned}\mathscr{L}(|k|^2t\widehat{M}(kt))(\omega)&=\int_0^\infty|k|t\widehat{M}(kt)e^{-\omega t}|k|\,\mathrm{d}t\\&=\int_0^\infty s\widehat{M}\Big(\frac{k}{|k|}s\Big)e^{-\omega s/|k|}\,\mathrm{d}s\\&=\int_0^\infty\int_{\mathbb{R}^d}M(v)se^{-i\frac{k}{|k|}s\cdot v}e^{-\omega s/|k|}\,\mathrm{d}v\,\mathrm{d}s.\end{aligned}$$

We use the change of variable $v=r\frac{k}{|k|}+v_\perp$, with $v_\perp\cdot k=0$, so that

$$\mathscr{L}(|k|^2t\widetilde{M}(kt))(\omega)=\int_0^\infty\int_{\mathbb{R}}\left(\int_{v_\perp\cdot k=0}M\Big(r\frac{k}{|k|}+v_\perp\Big)\,\mathrm{d}v_\perp\right)i\frac{\mathrm{d}}{\mathrm{d}r}e^{-irs}\;e^{-\omega s/|k|}\,\mathrm{d}r\,\mathrm{d}s.$$

Let us set

$$r \longmapsto \mu_{k/|k|}(r) = \int_{v_\perp\cdot k=0} M\Big(r\frac{k}{|k|} + v_\perp\Big)\,\mathrm{d}v_\perp.$$

We arrive at

$$\mathscr{L}(|k|^2 t\widetilde{M}(kt))(\omega) = -i\int_0^\infty \int_{\mathbb{R}} e^{-irs}\ e^{-\omega s/|k|}\mu'_{k/|k|}(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{62}$$

It yields

$$\begin{aligned}\mathscr{L}(|k|^2 t\widehat{M}(kt))((\alpha+i\beta)|k|,k) &= -i\int_{\mathbb{R}}\mu'_{k/|k|}(r)\left(\int_0^\infty e^{-i(r+\beta)s-\alpha s}\,\mathrm{d}s\right)\mathrm{d}r\\ &= -i\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{\alpha+i(r+\beta)}\,\mathrm{d}r = -\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{(r+\beta)-i\alpha}\,\mathrm{d}r.\end{aligned}$$

Next, the Laplace transform of p_c can be determined by using the classical result [31, Formula (VI,2;13)]

$$\mathscr{L}(\mathbf{1}_{t\geq 0}\sin(\theta t))(\omega) = \frac{\theta}{\omega^2+\theta^2}, \qquad \text{for } \mathrm{Re}(\omega) > 0.$$

For $\alpha > 0$, $\beta\in\mathbb{R}$, we thus get

$$\mathscr{L}p_c((\alpha+i\beta)|k|) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{|\widehat{\sigma}_2(\zeta)|^2\,\mathrm{d}\zeta}{(\alpha+i\beta)^2|k|^2+c^2|\zeta|^2}.$$

Since σ_2 is radially symmetric, its Fourier transform is radially symmetric too and we can write

$$\mathscr{L}p_c((\alpha+i\beta)|k|) = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty\frac{(r')^{n-1}|\widehat{\sigma}_2(r')|^2\,\mathrm{d}r'}{(\alpha^2-\beta^2)|k|^2+c^2|r'|^2+2i\alpha\beta|k|^2}.$$

In order to find the expression of the Laplace transform on the imaginary axis, we shall need the following claims.

Lemma B.7 (Plemelj formula) *Let $f:\mathbb{R}\to\mathbb{R}$ be in $L^1\cap W^{1,\infty}(\mathbb{R})$. Then, we have*

$$\lim_{\lambda\to 0}\int_{\mathbb{R}}\frac{f(x)}{x+\kappa-i\lambda}\,\mathrm{d}x = \mathrm{P.V.}\int_{\mathbb{R}}\frac{f(x)}{x+\kappa}\,\mathrm{d}x + i\pi f(-\kappa).$$

We refer the reader for instance to [14, Example 5.2]. An adaptation of the proof leads to the following useful statement.

Lemma B.8 *Let $n\geq 3$. Let $f:\mathbb{R}\to\mathbb{R}$ be Schwartz class. We have*

$$\lim_{\lambda\to 0}\int_0^\infty\frac{r^{n-1}f(r)}{r^2-\kappa^2+i\lambda+\lambda^2}\,\mathrm{d}r = \mathrm{P.V.}\int_0^\infty\frac{r^{n-1}f(r)}{r^2-\kappa^2}\,\mathrm{d}r - i\frac{\pi}{2}\kappa^{n-2}f(\kappa).$$

Proof. Let us denote by $I(\lambda)$ the quantity under consideration and $f(r) = g(r^2)$; with the change of variable $u = r^2$ we get

$$I(\lambda) = \frac{1}{2}\int_0^{+\infty} \frac{u^{n/2-1}g(u)}{u-\kappa^2+i\lambda+\lambda^2}\,\mathrm{d}u.$$

We adapt the computations that lead to Plemelj's formula. Let $\gamma(u) = u^{n/2-1}g(u)$. It is crucial to remark that

$$\gamma(0) = 0, \qquad \gamma' \in L^p((0,\infty)) \text{ for some } p < 2. \tag{63}$$

(At worst, $\gamma'(u)$ has the same singularity as $1/\sqrt{u}$ as $u \to 0$.) We start with

$$I(\lambda) = \frac{1}{2}\int_0^{+\infty} \frac{\gamma(u)}{(u-\kappa^2+\lambda^2)^2+\lambda^2}(u-\kappa^2+\lambda^2)\,\mathrm{d}u - \frac{i\lambda}{2}\int_0^{+\infty} \frac{\gamma(u)}{(u-\kappa^2+\lambda^2)^2+\lambda^2}\,\mathrm{d}u.$$

Setting $v = u-\kappa^2+\lambda^2$, and $v/\lambda = w$, the second term recasts as

$$-\frac{i}{2}\int_{-\kappa^2+\lambda^2}^{+\infty} \frac{\gamma(v+\kappa^2-\lambda^2)}{(v/\lambda)^2+1}\frac{\mathrm{d}v}{\lambda} = -\frac{i}{2}\int_{-\kappa^2/\lambda+\lambda}^{+\infty} \frac{\gamma(\lambda w+\kappa^2-\lambda^2)}{w^2+1}\,\mathrm{d}w$$

which tends to

$$-\frac{i}{2}\gamma(\kappa^2) \times \begin{cases} \pi & \text{if } \kappa \neq 0, \\ \pi/2 & \text{if } \kappa = 0, \end{cases}$$

as $\lambda \to 0$. Since $\gamma(0) = 0$ we can actually use a single formula. Similarly, we consider

$$J(\lambda) = \int_{-\kappa^2+\lambda^2}^{+\infty} \frac{\gamma(v+\kappa^2-\lambda^2)}{v^2+\lambda^2}v\,\mathrm{d}v.$$

We start with the case $\kappa = 0$. Owing to (63), we have $0 \leq \gamma(u) = \int_0^u \gamma'(y)\,\mathrm{d}y \leq \|\gamma'\|_{L^p}|u|^{1-1/p}$ so that the function $v \mapsto \frac{v\gamma(v)}{v^2}$ lies in $L^1((0,\infty))$. It allows us to apply the Lebesgue theorem and to conclude that

$$\lim_{\lambda\to 0}\int_{\lambda^2}^{+\infty} \frac{\gamma(v-\lambda^2)}{v^2+\lambda^2}v\,\mathrm{d}v = \int_0^{\infty} \frac{\gamma(v)}{v}\,\mathrm{d}v.$$

Next, let $\kappa \neq 0$. Since λ is intended to tend to 0, we can consider $\kappa^2 \gg \lambda^2 > 0$ Given $0 < \delta < \kappa^2 - \lambda^2$, we split into 2 parts

$$J(\lambda) = \int_{|v|>\delta} \ldots\,\mathrm{d}v + \int_{-\delta}^{+\delta} \ldots\,\mathrm{d}v = J^\delta(\lambda) + J_\delta(\lambda).$$

First, we show that $J_\delta(\lambda)$ tends to 0 as $\delta \to 0$, uniformly with respect to λ. Indeed, since $v \mapsto \frac{v}{v^2+\lambda^2}$ is odd, we have

$$\begin{aligned} |J_\delta(\lambda)| &= \left|\int_{-\delta}^{+\delta} \frac{\gamma(v+\kappa^2-\lambda^2)-\gamma(\kappa^2-\lambda^2)}{v^2+\lambda^2}v\,\mathrm{d}v\right| \\ &\leq \|\gamma'\|_{L^p}\int_{-\delta}^{+\delta} \frac{\mathrm{d}v}{|v|^{1/p}} \xrightarrow[\delta\to 0]{} 0. \end{aligned}$$

By dominated convergence, we get

$$\begin{aligned}
\lim_{\lambda\to 0} J^\delta(\lambda) &= \int_{|v|>\delta} \mathbf{1}_{v\geq -\kappa^2} \frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v \\
&= \int_{-\kappa^2}^{-\delta} \frac{\gamma(v+\kappa^2)-\gamma(\kappa^2)}{v}\,\mathrm{d}v + \int_{\delta}^{\kappa^2} \frac{\gamma(v+\kappa^2)-\gamma(\kappa^2)}{v}\,\mathrm{d}v \\
&\quad + \int_{\kappa^2}^{+\infty} \frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v
\end{aligned}$$

The same reasoning shows that this quantity admits a limit as δ goes 0, that we write with the shorthand notation

$$\lim_{\delta\to 0}\lim_{\lambda\to 0} J^\delta(\lambda) = \mathrm{P.V.} \int_{-\kappa^2}^{\infty} \frac{\gamma(v+\kappa^2)}{v}\,\mathrm{d}v.$$

■

We now come back to the explicit computation of $\mathscr{L}\mathscr{K}(\omega|k|,k)$ on the imaginary axis. On the one hand, we get

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}(|k|^2 t\widehat{M}(kt))((\alpha+i\beta)|k|,k) = -\mathrm{P.V.}\int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r - i\pi\mu'_{k/|k|}(-\beta).$$

For the latter, we use Lemma B.7, which eventually leads to

$$\begin{aligned}
&\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}p_c((\alpha+i\beta)|k|) \\
&\quad = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\left(\mathrm{P.V.}\int_0^\infty (r')^{n-1}\frac{|\widehat{\sigma}_2(r')|^2}{c^2|r'|^2-\beta^2|k|^2}\,\mathrm{d}r' - \frac{i\pi}{2c^2}\Big(\frac{|\beta k|}{c}\Big)^{n-2}\ \Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2\right).
\end{aligned}$$

Remark B.9 *In the case $\beta = 0$ a direct application of the dominated convergence theorem allows us to obtain*

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} \mathscr{L}p_c(\alpha|k|) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\widehat{\sigma}_2(\xi)|^2 \frac{\mathrm{d}\xi}{c^2|\xi|^2} = \frac{\kappa}{c^2}.$$

Therefore, we obtain the following expression for $\mathscr{L}\mathscr{K}(i\beta|k|,k)$ which identifies the real and imaginary parts

$$\begin{aligned}
\mathscr{L}\mathscr{K}(i\beta|k|,k) &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\big(\mathscr{R}(\beta|k|,k) + i\mathscr{I}(\beta|k|,k)\big), \\
\mathscr{R}(\beta|k|,k) &= -\rho_0|\widehat{\sigma_1}(k)|^2\Bigg\{\mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r \times \mathrm{P.V.}\int_0^\infty (r')^{n-1}\frac{|\widehat{\sigma}_2(r')|^2}{c^2|r'|^2-\beta^2|k|^2}\,\mathrm{d}r' \\
&\qquad + \frac{\pi^2}{2c^2}\mu'_{k/|k|}(-\beta)\times\Big(\frac{|\beta k|}{c}\Big)^{n-2}\ \Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2\Bigg\} \\
\mathscr{I}(\beta|k|,k) &= \pi\rho_0|\widehat{\sigma_1}(k)|^2\Bigg\{\frac{1}{2c^2}\Big(\frac{|\beta k|}{c}\Big)^{n-2}\ \Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2 \times \mathrm{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r+\beta}\,\mathrm{d}r \\
&\qquad - \mu'_{k/|k|}(-\beta)\times\mathrm{P.V.}\int_0^\infty (r')^{n-1}\frac{|\widehat{\sigma}_2(r')|^2}{c^2|r'|^2-\beta^2|k|^2}\,\mathrm{d}r'\Bigg\}.
\end{aligned}$$

(Note that the formula applies for $\beta = 0$ as well.) It leads to the *Penrose stability criterion*, hereafter denoted **(P)**:

If

$$\frac{1}{2c^2}\Big(\frac{|\beta k|}{c}\Big)^{n-2}\ \Big|\widehat{\sigma}_2\Big(\frac{|\beta k|}{c}\Big)\Big|^2 \times \text{P.V.}\int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r-\beta}\,\mathrm{d}r$$
$$= \mu'_{k/|k|}(\beta) \times \text{P.V.}\int_0^\infty (r')^{n-1}\frac{|\widehat{\sigma}_2(r')|^2}{c^2|r'|^2-\beta^2|k|^2}\,\mathrm{d}r',$$

then

$$-\rho_0|\widehat{\sigma_1}(k)|^2\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\Bigg\{\text{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r-\beta}\,\mathrm{d}r \times \text{P.V.}\int_0^\infty (r')^{n-1}\frac{|\widehat{\sigma}_2(r')|^2}{c^2|r'|^2-\beta^2|k|^2}\,\mathrm{d}r'$$
$$-\frac{\pi^2}{2c^2}\mu'_{k/|k|}(\beta)\times\Big(\frac{|k\beta|}{c}\Big)^{n-2}\ \Big|\widehat{\sigma}_2\Big(\frac{|k\beta|}{c}\Big)\Big|^2\Bigg\} \neq 1.$$

When $\mathbb{X}^d = \mathbb{R}^d$, the Penrose criterion **(P)** has to be completed with the following criterion (hereafter denoted **(P')**):

$$\text{if } \mu'_\omega(\beta) = 0 \text{ then } -\rho_0\,|\widehat{\sigma}_1(0)|^2\,\text{P.V.}\int_{\mathbb{R}}\frac{\mu'_\omega(r)}{r-\beta}\,\mathrm{d}r \neq 1,$$

(for all $\omega \in \mathbb{S}^d$).

We conclude that, when **(P)** (resp. **(P)** and **(P')**) is satisfied, then (**L**) holds, which, in turn, implies that the decay properties stated for the linearized problem in Section 3 hold. This criterion is much more involved than the Penrose criterion for the Vlasov equation, because the memory term p_c completely changes the evaluation of the symbol $\mathscr{L}\mathscr{K}$ and does not keep a simple separation between the real and imaginary parts.

Remark B.10 *Let us rescale the problem as in [9]: roughly speaking, it amounts to replace the wave equation by*

$$\partial^2_{tt}\psi - c^2\Delta_z\psi = -c^2\sigma_2\ \sigma_1\star\rho.$$

Letting c run to ∞, the problem looks like the Vlasov equation where the self-consistent potential is defined by the convolution $-\kappa\sigma_1\star\sigma_1\star\rho$. According to [28], the stability criterion for this limiting problem reads

$$\textit{if } \mu'_{k/|k|}(\beta) = 0,\ \textit{then } -\rho_0\kappa|\widehat{\sigma_1}(k)|^2\text{P.V.}\int_{\mathbb{R}}\frac{\mu'_{k/|k|}(r)}{r-\beta}\,\mathrm{d}r \neq 1,$$

which corresponds to the limit $c\to\infty$ in the rescaled version of **(P)**. *In particular, mind the minus sign in front of the coefficient $\rho_0|\widehat{\sigma_1}(k)|^2$: it makes the situation very similar to those of the* attractive *Vlasov-system.*

B.3 Stable and unstable states

The criterion **(P)** is a bit ugly and not that practical. Nevertheless, some relevant information can be extracted from the formula, showing again the similarity with the attractive Vlasov-Poisson equation.

Proposition B.11 *Let $\mathbb{X}^d = \mathbb{R}^d$ with $d \geq 3$. Let $\mathscr{M}$ be a spatially homogeneous and radially symmetric equilibrium. Then, there exists a threshold for the wave speed $c_0(\mathscr{M}, \sigma_1, \sigma_2) > 0$ such that for any $0 < c < c_0(\mathscr{M}, \sigma_1, \sigma_2)$, $\mathscr{M}$ in an unstable equilibrium state.*

Proof. We find k and β such that $\mathscr{L}\mathscr{K}(i\beta|k|, k) = 1$. To this end, we use the fact that $\mathscr{L}p_c(i\beta|k|)$ belongs to $\mathbb{R}$ for $\beta = 0$ and the radial symmetry of $\mathscr{M}$ which implies that $\mathscr{L}(|k|^2 t\widehat{M}(tk))(i\beta|k|, k)$ is real too when $\beta = 0$:

$$\mathscr{L}\mathscr{K}(0, k) = -\rho_0 \, |\widehat{\sigma}_1(k)|^2 \left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r \right) \frac{\kappa}{c^2}. \tag{64}$$

Moreover, the symmetry of $\mathscr{M}$ (and the condition on the dimension d, see Remark B.12 below) also ensures (except for $\mathscr{M} = 0$, but 0 is obviously a stable state)

$$-\left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r \right) > 0.$$

Now let us pick a vector k_0 such that $\widehat{\sigma}_1(k_0) \neq 0$. As far as c is small enough, we have $\mathscr{L}\mathscr{K}(0, k_0) > 1$. Next,

$$\mathscr{L}\mathscr{K}(0, \lambda k_0) \underset{\lambda \to +\infty}{\longrightarrow} 0$$

and the continuity of $\lambda \in \mathbb{R} \mapsto \widehat{\sigma}_1(\lambda k_0)$ (observe that $\lambda k_0/|\lambda k_0|$ does not depend on λ and thus only $\widehat{\sigma}_1$ depends on λ in the expression of $\mathscr{L}\mathscr{K}(0, \lambda k_0)$), allow us to exhibit a $\lambda_0 \in \mathbb{R}$ such that $\mathscr{L}\mathscr{K}(0, \lambda_0 k_0) = 1$. ■

Remark B.12 *The condition $d \geq 3$ ensures that all marginals of a non negative radially symmetric function $\mathscr{M}$ are non increasing function of $|v|$, see [28, Remark 2.2], which yields*

$$-\left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r \right) \geq 0. \tag{65}$$

When $d = 1$ or $d = 2$ this does not hold in full generality. Nevertheless, Proposition B.11 still holds provided (65) *is fulfiilled.*

Remark B.13 *When $\mathbb{X}^d = \mathbb{T}^d$, the same proof shows that, for any spatially homogeneous and radially symmetric equilibrium, we can find some wave speed c such that $\mathscr{M}$ is unstable. However, since $k \in \mathbb{Z}^d$, it is not clear that we can exhibit a non trivial* interval $[0, c_0(\mathscr{M})]$ *such that instability occurs. To identify a threshold on c determining whether or not the stability criterion holds can be interpreted by means of Jeans' criterion, a standard criterion for the Vlasov-Poisson system, see [28, Proposition 2.1 & Remark 2.2]). To be more specific, let us consider a form function σ_1 defined on $\mathbb{R}^d$, the Fourier transform of which has a singularity at $\xi = 0$: typically $\widehat{\sigma}_1(k) = |k|^{-\alpha}$ for some $\alpha > 1$. Of course, such singular potential are beyond the analysis detailed in this paper; we only use this assumption to establish a parallel with the usual Jeans' criterion. Let $\sigma_1^{(L)}$ be the periodic potential defined on $\mathbb{T}^d_L = (\mathbb{R}/(2\pi L\,\mathbb{Z}))^d$ by*

$$\sigma_1^{(L)}(x) = \sum_{k \in \mathbb{Z}^d} \sigma_1(x + 2\pi L\, k).$$

Observing that $\widehat{\sigma_1^{(L)}}(k) = \widehat{\sigma}_1(k/L)$, (64) *becomes*

$$\mathscr{L}\mathscr{K}(0,k) = -\rho_0 \frac{L^{2\alpha}}{|k|^{2\alpha}} \left(\text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r \right) \frac{\kappa}{c^2},$$

where L has a role similar to $1/c$. In particular, for any spatially homogeneous equilibrium $\mathscr{M}$, there exists a critical length L_J beyond which the equilibrium can be unstable, this defines Jeans' length.

Remark B.14 *Denoting $\mathscr{M} = \rho_0 M$, with M being normalized, we can equally say (with the same arguments) that, for any fixed wave speed c we can find a mass threshold $m_0(M, c, \sigma_1, \sigma_2) > 0$ such that for any $\rho_0 > m_0(M, c, \sigma_1, \sigma_2)$, $\mathscr{M}$ is unstable. Nevertheless we point out that, for c fixed, the mass ϱ_0 of the profile $\mathscr{M}$ is not the unique quantity that governs the stability of $\mathscr{M}$, as indicated by the following claim*

Proposition B.15 *Let $\mathscr{M}$ be a spatially homogeneous equilibrium. We can find two positive constants $C_1 = C_1(c, \sigma_1, \sigma_2)$ and $C_2 = C_2(c, \sigma_1, \sigma_2)$ such that*

$$\text{if, for any } \omega \in \mathbb{S}^d, \text{ we have } \int_0^{+\infty} u \left| \widehat{\mathscr{M}}(u\omega) \right| \mathrm{d}u \le C_1(c, \sigma_1, \sigma_2), \text{ then } \mathscr{M} \text{ is stable,}$$

$$\text{if there exists } \omega \in \mathbb{S}^d \text{ such that } \int_0^{+\infty} u \widehat{\mathscr{M}}(u\omega) \, \mathrm{d}u \ge C_2(c, \sigma_1, \sigma_2), \text{ then } \mathscr{M} \text{ is unstable.}$$

This statement can be interpreted as follows. For fixed c, σ_1 and σ_2 there always exist stable spatially homogeneous equilibria with an arbitrarily large mass (resp. kinetic energy), and there always exist unstable spatially homogeneous equilibria with an arbitrarily small mass (resp. kinetic energy). This comes from the fact that the constant C_1 and C_2 in Proposition B.15 are left invariant by the rescaling $M \to M_\lambda(v) = \lambda^{d-2}\mathscr{M}(\lambda v)$, while the associated mass (resp. kinetic energy) is invariant for the scaling $M \to \lambda^d \mathscr{M}(\lambda v)$ (resp. $M \to \lambda^{d+2}\mathscr{M}(\lambda v)$). These findings will be investigated on numerical grounds in [18].

Proof. The first part of the statement is a direct consequence of Proposition 3.4, which tells us that a given profile $\mathscr{M}$ is stable provided c is large enough. The second part of the statement is a direct consequence of Proposition B.11 and it comes from the formula

$$\mathscr{L}(|k|^2 t \widehat{M}(tk))(0,k) = \text{P.V.} \int_{\mathbb{R}} \frac{\mu'_{k/|k|}(r)}{r} \, \mathrm{d}r = \int_0^{+\infty} u \widehat{\mathscr{M}}(u\omega) \, \mathrm{d}u.$$

■

C Analytic Cauchy theory for the Vlasov-Wave system

In this Section, we go back to the Cauchy problem, addressed in the functional framework of Section 5. We are going to justify Theorem 5.6. The discussion is based on general arguments presented in [24, 29, 30]. Throghout this section we suppose **(H1)**–**(H3)** and **(R2)**.

C.1 Local analysis

We write the problem in the form

$$\begin{cases} \partial_t g(t,x,v) = \mathscr{N}(g)(t,x,v) \\ g(0,x,v) = f_0(x,v) \end{cases} \tag{66}$$

where

$$\mathscr{N}(g)(t,x,v) = \nabla\sigma_1 \star (\mathscr{F}_I + \sigma_1 \star \mathscr{G}_\varrho)(t, x + t\,v) \cdot (\nabla_v - t\,\nabla_x)(\mathscr{M} + g)(t,x,v),$$

$$\varrho(t,x) = \int_{\mathbb{R}^d} g(t, x - tv, v)\,\mathrm{d}v.$$

We start with an abstract statement about the local existence of analytic solutions for (66).

Theorem C.1 *Let $P > d/2$ be an integer and let $\sigma > d/2$. For any $\mathscr{M}, f_0 \in \mathcal{G}_P^{\lambda_0,\sigma;1}$ with $\lambda_0 < \min(\lambda_1/\langle 2R_2/c\rangle, 2\lambda_1/\langle S_0\rangle)$, there exists $\varepsilon > 0$ such that, for any $0 < T < \varepsilon$ the mapping*

$$\Phi : g \longmapsto \left(t \mapsto f_0 + \int_0^t \mathscr{N}(g)(\tau)\mathrm{d}\tau\right)$$

admits a fixed point in the set $B_T^{\lambda_0}$, made of functions $(t,x,v) \mapsto g(t,x,v)$ such that

$$\|g\|_{B_T^{\lambda_0}} := \sup_{0<\lambda<\lambda_0} \left(\sup_{t\in[0,T(\lambda_0-\lambda))} \left[1 - \frac{t}{T(\lambda_0-\lambda)}\right] \|g(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \right)$$

is finite.

Remark C.2 *The constraint on λ_0 comes from the fact that the proof uses Proposition 5.2. When $\lambda_0 \geq \min(\lambda_1/\langle 2R_2/c\rangle, 2\lambda_1/\langle S_0\rangle)$, we can still conclude that Φ admits a fixed point, which now lies in $B_T^{\min(\lambda_1/\langle 2R_2/c\rangle, 2\lambda_1/\langle S_0\rangle)}$. Up to choosing a smaller λ_0 (or, equivalently, working with larger wave speeds c), we can still suppose that $\lambda_0 < \min(\lambda_1/\langle 2R_2/c\rangle, 2\lambda_1/\langle S_0\rangle)$ is satisfied. In what follows, we will always assume implicitly this condition.*

Remark C.3 *The proof of this statement provides further information: there exists $R > 0$ such that for any $0 < \lambda < \lambda_0$and $t \in [0, T(\lambda_0 - \lambda))$, we have*

$$\|g(t) - f_0\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \leq R.$$

Before starting the proof, let us explain why it is somehow natural to deal with the spaces $B_T^{\lambda_0}$. First of all, remark that the operator $\mathscr{N}$ involves first order derivatives with respect to space and velocity, and thus the mapping Φ does not map $\mathcal{G}_P^{\lambda_0,\sigma;1}$ into itself, but has its range in $\mathcal{G}_{\mathcal{P}}^{\lambda,\sigma;\infty}$ with $0 < \lambda < \lambda_0$, possibly arbitrarily close to λ_0. For this reason, we work instead with a space that involves all the norms $\mathcal{G}_P^{\lambda,\sigma;1}$ for $\lambda \in (0, \lambda_0)$. However, Lemma C.5 suggests that $\|\mathcal{N}(g)(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}$ blows up as $\lambda \nearrow \lambda_0$, and this viewpoint is not sufficient. We should also take advantage of the time integration in order to control this blow up. This leads to incorporate a suitable weight with respect to time

$$w(t) = 1 - \frac{t}{T(\lambda_0 - \lambda)}$$

and then to consider the supremum over $t \in [0, T(\lambda_0 - \lambda))$. These norms are a bit unusual, nevertheless the following claim shows that most of the analysis can be performed in more natural functional spaces.

Corollary C.4 *Let $P > d/2$ be an integer and let $\sigma > d/2$. For any $\mathscr{M}, f_0 \in \mathcal{G}_P^{\widetilde{\lambda_0},0;1}$, there exists $T^\star > 0$ and a function $0 < \widetilde{\lambda}(t) < \widetilde{\lambda_0}$, continuous and decreasing, such that (66) has a unique solution g in $C^0([0, T^\star); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;s})$. Moreover, if for some $0 < T \le T^\star$, we have*

$$\begin{cases} \limsup\limits_{t \nearrow T} \|g(t)\|_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} < +\infty \\ \lim\limits_{t \nearrow T} \widetilde{\lambda}(t) > 0, \end{cases}$$

then $T < T^\star$.

The proof of Theorem C.1 uses the estimates (49), (50) and (53a) (see Section 5) together with the following claim.

Lemma C.5 *Let $g = g(t, x, v) \in \mathcal{G}_P^{\lambda,\sigma;s}$. The, for any $0 \le \lambda' < \lambda$, the function $(\nabla_v - t\,\nabla_x)g(t)$ defines an element of $\mathcal{G}_P^{\lambda',\sigma;s}$; we have*

$$\|(\nabla_v - t\,\nabla_x)g(t)\|_{\mathcal{G}_P^{\lambda',\sigma;s}} \lesssim \frac{\langle t\rangle}{(\lambda - \lambda')^{1/s}} \|g(t)\|_{\mathcal{G}_P^{\lambda,\sigma;s}}. \tag{67}$$

Proof. Since

$$\|\,(\nabla_v - t\,\nabla_x)g(t)\|^2_{\mathcal{G}_P^{\lambda',\sigma;s}} = \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda'\langle k,\xi\rangle^s} \left|\mathrm{D}_\xi^\alpha(\xi \mapsto (\xi - tk)\widehat{g}(t,k,\xi))\right|^2 \mathrm{d}\xi$$

$$\lesssim \langle t\rangle^2 \sum_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|\le P}} \sum_{\substack{j\in\mathbb{N}^d\\ j\le\alpha\,;\,|j|\le 1}} \sum_{k\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\lambda\langle k,\xi\rangle^s} \left|\mathrm{D}_\xi^{\alpha-j}\widehat{g}(t,k,\xi)\right|^2 \langle k,\xi\rangle^2 e^{-2(\lambda-\lambda')\langle k,\xi\rangle^s}\,\mathrm{d}\xi,$$

we are led to identify the supremum over $[0,\infty)$ of the function $x \mapsto x^2\exp(-2(\lambda - \lambda')x^s)$. It is reached at $1/(s[\lambda-\lambda'])^{1/s}$ and its value is $\exp(-2/s)/(s[\lambda-\lambda'])^{2/S}$. This ends the proof. ∎

Proof of Theorem C.1. We split the proof into three steps.

- *Step 1.* Fix $R > 0$. We introduce the subset $E^{\lambda_0}_{T,R}$ of $B^{\lambda_0}_T$ defined by

  $$E^{\lambda_0}_{T,R} := \left\{ g \in B^{\lambda_0}_T \text{ s.t. } \forall \lambda \in (0, \lambda_0), \forall t \in [0, T(\lambda_0 - \lambda)),\ \|g(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \leq R \right\}.$$

  If g lies in $E^{\lambda_0}_{T,R}$, then $\Phi(g)$ belongs to $B^{\lambda_0}_T$. To be more specific, we have

  $$\|\Phi(g)\|_{B^{\lambda_0}_T} \leq \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} + C_1\, T \langle T\lambda_0 \rangle \left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} \right) \left(\|\mathscr{M}\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} + \|g\|_{B^{\lambda_0}_T} \right).$$

- *Step 2.* If g and h belong to $E^{\lambda_0}_{T,R}$, then, we have

  $$\begin{aligned} \|\Phi(g) - \Phi(h)\|_{B^{\lambda_0}_T} \quad &\leq \quad C_2\, T \langle T\lambda_0 \rangle \sqrt{T\lambda_0} \left(\|\mathscr{M}\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} + R + \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} \right) \|g - h\|_{B^{\lambda_0}_T} \\ &\quad + C_3\, T \langle T\lambda_0 \rangle \left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} \right) \|g - h\|_{B^{\lambda_0}_T}. \end{aligned}$$

  With these estimates, we cannot apply directly the standard Banach-Picard fixed point theorem since the range of $E^{\lambda_0}_{T,R}$ by Φ is not necessarily included in $E^{\lambda_0}_{T,R}$. However, for any $0 < T' < T$, we have $\Phi(E^{\lambda_0}_{T,R}) \subset E^{\lambda_0}_{T',R}$. We are going to exploit this observation to construct a fixed point.

- *Step 3.* We introduce the following sequence of times

  $$T_k = \delta \prod_{j=0}^{k} \left(1 - \frac{1}{(j+2)^2} \right)$$

  (where $\delta > 0$ can be chosen arbitrarily small), and we define a sequence of functions by the recursion formula

  $$\begin{cases} g_0 = f_0 \\ g_{k+1} = f_0 + \displaystyle\int_0^t \mathscr{N}(g_k)(\tau) \mathrm{d}\tau = \Phi(g_k). \end{cases}$$

  Provided δ is small enough, we can show that, for any $k \in \mathbb{N}$, we have

  a) $g_k \in E^{\lambda_0}_{T_k,R}$.

  b) $\mu_k := \|g_{k+1} - g_k\|_{B^{\lambda_0}_{T_k}} \leq C\delta \frac{1}{(k+3)^4}$ where $C > 0$ is a certain constant that will be made precise later on.

  Consequently, $(g_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $B^{\lambda_0}_{\delta T^\infty}$ (with $T^\infty = \prod_{k=0}^{+\infty}(1 - (k+2)^{-2}) > 0$) and it converges to g in $B^{\lambda_0}_{\delta T^\infty}$, which is a fixed point of Φ.

Let us now detail the justification of each of these steps.
Step 1. Remark that

$$\|\Phi(g)(t)\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \leq \|f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} + \int_0^t \|\mathscr{N}(g)(\tau)\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \,\mathrm{d}t.$$

Then, we are going to estimate $\|\mathscr{N}(g)(\tau)\|_{\mathcal{G}^{\lambda,\sigma;1}_P}$. We use the σ-ring property (49), the estimate (53b) and the embedding (50) for the left hand side, and Lemma C.5 for the right hand side. We obtain, for any $0<\lambda<\lambda'<\lambda_0$ and $0\le\tau\le t<T(\lambda_0-\lambda)$:

$$\begin{aligned}
\|\mathcal{N}(g)(\tau)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} &\lesssim \|\nabla\sigma_1\star(\mathscr{F}_I(\tau)-\sigma_1\star\mathscr{G}_\varrho(\tau))\|_{\mathcal{F}^{\lambda,\sigma;1}}\|(\nabla_v-\tau\nabla_x)(\mathscr{M}+g(\tau))\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\\
&\lesssim \left(\mathscr{E}_I+\sup_{\tau\in[0,T(\lambda_0-\lambda))}\|g(\tau)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\right)\frac{\langle\tau\rangle}{\lambda'-\lambda}\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda',\sigma;1}}.
\end{aligned}$$

Moreover, since g lies in $E_{T,R}^{\lambda_0}$ and possibly by adapting the choice of λ' as a function of τ, we get

$$\|\mathcal{N}(g)(\tau)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\lesssim\left(\mathscr{E}_I+R+\|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\frac{\langle T\lambda_0\rangle}{\lambda'(\tau)-\lambda}\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}}.$$

Consequently, for any $0<\lambda<\lambda_0$ and $t\in[0,T(\lambda_0-\lambda))$, we are led to

$$\begin{aligned}
&\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\|\Phi(g)(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\\
&\qquad\lesssim\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\|f_0\|_{\mathcal{G}_P^{\lambda,\sigma;1}}+\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\int_0^t\|\mathcal{N}(u)(\tau)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\,\mathrm{d}\tau\\
&\qquad\lesssim\|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}+\langle T\lambda_0\rangle\left(\mathscr{E}_I+R+\|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\\
&\qquad\qquad\times\int_0^t\frac{\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}}}{\lambda'(\tau)-\lambda}\,\mathrm{d}\tau.
\end{aligned}$$

Let $\lambda'(\tau)=(\lambda_0-\tau/T+\lambda)/2$ so that both conditions $\lambda<\lambda'(\tau)<\lambda_0$ and $\tau\le T(\lambda_0-\lambda'(\tau))$ sare satisfied for $0\le\tau\le t<T(\lambda_0-\lambda)$, we can make use of the assumption $g\in B_T^{\lambda_0}$ and we obtain

$$\begin{aligned}
&\int_0^t\frac{\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}}}{\lambda'(\tau)-\lambda}\,\mathrm{d}\tau\\
&\qquad\le\int_0^t\frac{\left[1-\frac{\tau}{T(\lambda_0-\lambda'(\tau))}\right]\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}}}{(\lambda'(\tau)-\lambda)\left[1-\frac{\tau}{T(\lambda_0-\lambda'(\tau))}\right]}\,\mathrm{d}\tau\le\int_0^t\frac{\|\mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}+\|g\|_{B_T^{\lambda_0}}}{(\lambda'(\tau)-\lambda)\left[1-\frac{\tau}{T(\lambda_0-\lambda'(\tau))}\right]}\,\mathrm{d}\tau.
\end{aligned}$$

Finally, since

$$\lambda'(\tau)-\lambda=\frac{1}{2T}[T(\lambda_0-\lambda)-\tau]$$

and

$$T(\lambda_0-\lambda'(\tau))=\frac{1}{2}[T(\lambda_0-\lambda)+\tau]\le\frac{1}{2}[T(\lambda_0-\lambda)+t]\le T(\lambda_0-\lambda),$$

we arrive at

$$\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\int_0^t \frac{1}{(\lambda'(\tau)-\lambda)\left[1-\frac{\tau}{T(\lambda_0-\lambda'(\tau))}\right]}\,\mathrm{d}\tau$$

$$= \left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\int_0^t \frac{T(\lambda_0-\lambda'(\tau))}{(\lambda'(\tau)-\lambda)\left[T(\lambda_0-\lambda'(\tau))-\tau\right]}\,\mathrm{d}\tau$$

$$\le \left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\int_0^t \frac{T(\lambda_0-\lambda)}{\frac{1}{4T}[T(\lambda_0-\lambda)-\tau]^2}\,\mathrm{d}\tau = 4T[T(\lambda_0-\lambda)]\int_0^t \frac{1}{[T(\lambda_0-\lambda)-\tau]^2}\,\mathrm{d}\tau$$

$$= 4T\frac{t}{T(\lambda_0-\lambda)} \le 4T.$$

It allows us to conclude that

$$\|\Phi(g)\|_{B_T^{\lambda_0}} \lesssim \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + 4T\langle T\lambda_0\rangle \left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\left(\|\mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + \|g\|_{B_T^{\lambda_0}}\right).$$

Step 2. Like in Step 1, we introduce two real numbers $0<\lambda<\lambda'<\lambda_0$, two times $0\le\tau\le t<T(\lambda_0-\lambda)$ and we estimate

$$\|\mathcal{N}(g)(t)-\mathcal{N}(h)(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}$$

$$\le \|(x,v)\mapsto \nabla\Sigma\star\mathscr{G}_{\varrho_g-\varrho_h}(\tau,x+\tau v)\cdot(\nabla_v-\tau\nabla_x)\left(\mathscr{M}(v)+g(\tau,x,v)\right)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}$$
$$+\|(x,v)\mapsto \nabla\sigma_1 * (\mathscr{F}_I-\sigma_1\star\mathscr{G}_{\varrho_h})(\tau,x+\tau v)\cdot(\nabla_v-\tau\nabla_x)(g(\tau,x,v)-h(\tau,x,v))\|_{\mathcal{G}_P^{\lambda,\sigma;1}}.$$

The second term can be treated as in Step 1. For the first term, we apply (49) again, with (53b) and (50) combined to Lemma C.5, and we obtain

$$\|(x,v)\mapsto \nabla\Sigma\star\mathscr{G}_{\varrho_g-\varrho_h}(\tau,x+\tau v)\cdot(\nabla_v-\tau\nabla_x)\left(\mathscr{M}(v)+g(\tau,x,v)\right)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}$$

$$\lesssim \left(\int_0^\tau \|g(s)-h(s)\|^2_{\mathcal{G}_P^{\lambda,\sigma;1}}\,\mathrm{d}s\right)^{1/2}\frac{\langle T\lambda_0\rangle}{\lambda'(\tau)-\lambda}\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}}.$$

Since $0\le s<T(\lambda_0-\lambda)$, we can appeal to the assumption $g,h\in B_T^{\lambda_0}$, so that

$$\int_0^\tau \|g(s)-h(s)\|^2_{\mathcal{G}_P^{\lambda,\sigma;1}}\,\mathrm{d}s$$

$$= \int_0^\tau \frac{\left[1-\frac{s}{T(\lambda_0-\lambda)}\right]^2\|g(s)-h(s)\|^2_{\mathcal{G}_P^{\lambda,\sigma;1}}}{\left[1-\frac{s}{T(\lambda_0-\lambda)}\right]^2}\,\mathrm{d}s \le \|g-h\|^2_{B_T^{\lambda_0}}\int_0^\tau \frac{1}{\left[1-\frac{s}{T(\lambda_0-\lambda)}\right]^2}\,\mathrm{d}s$$

$$= \|g-h\|^2_{B_T^{\lambda_0}}\left[\frac{T^2(\lambda_0-\lambda)^2}{T(\lambda_0-\lambda)-\tau}-\frac{T^2(\lambda_0-\lambda)^2}{T(\lambda_0-\lambda)}\right] \le \|g-h\|^2_{B_T^{\lambda_0}}\frac{T^2(\lambda_0-\lambda)^2}{T(\lambda_0-\lambda)-\tau}.$$

Moreover, still with $\lambda'(\tau)=(\lambda_0-\tau/T+\lambda)/2$ (the conditions $\lambda<\lambda'(\tau)<\lambda_0$ and $\tau\le T(\lambda_0-\lambda'(\tau))$ are thus fulfilled for $0\le\tau\le t<T(\lambda_0-\lambda)$), we make use of the

assumption $g \in E^{\lambda_0}_{T,R}$which yields

$$\| \mathscr{M} + g(\tau)\|_{\mathcal{G}_P^{\lambda'(\tau),\sigma;1}} \leq \| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}.$$

Therefore, this discussion leads to

$$\begin{aligned}
&\|(x,v) \mapsto \nabla\Sigma \star \mathscr{G}_{\varrho_g-\varrho_h}(\tau, x+\tau\, v)\cdot(\nabla_v - \tau\nabla_x)\left(\mathscr{M}(v)+g(\tau,x,v)\right)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \\
&\lesssim \|g-h\|_{B_T^{\lambda_0}} \frac{T(\lambda_0-\lambda)}{\sqrt{T(\lambda_0-\lambda)-\tau}} \frac{\langle T\lambda_0\rangle}{\lambda'(\tau)-\lambda}\left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)
\end{aligned}$$

Integrating over $[0,t]$ and multiplying by $(1-t/[T(\lambda_0-\lambda)])$, we get

$$\begin{aligned}
&\left[1-\frac{t}{T(\lambda_0-\lambda)}\right]\int_0^t \|(x,v) \mapsto \nabla\Sigma \star \mathscr{G}_{\varrho_g-\varrho_h}(\tau, x+\tau\, v)\cdot(\nabla_v - \tau\nabla_x)(\mathscr{M}(v)+g(\tau,x,v)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\,\mathrm{d}\tau \\
&\lesssim \langle T\lambda_0\rangle \left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)[T(\lambda_0-\lambda)-t] \\
&\qquad\qquad \left(\int_0^t \frac{2T}{[T(\lambda_0-\lambda)-\tau]^{3/2}}\,\mathrm{d}\tau\right)\|g-h\|_{B_T^{\lambda_0}} \\
&\lesssim \langle T\lambda_0\rangle \left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)[T(\lambda_0-\lambda)-t] \\
&\qquad\qquad \left(\frac{2T}{\sqrt{T(\lambda_0-\lambda)-t}} - \frac{2T}{\sqrt{T(\lambda_0-\lambda)}}\right)\|g-h\|_{B_T^{\lambda_0}} \\
&\lesssim \langle T\lambda_0\rangle \left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)2T\sqrt{T(\lambda_0-\lambda)-t}\|g-h\|_{B_T^{\lambda_0}}.
\end{aligned}$$

We conclude with

$$\begin{aligned}
\|\Phi(g)-\Phi(h)\|_{B_T^{\lambda_0}} \quad &\lesssim \quad 2T\langle T\lambda_0\rangle\sqrt{T\lambda_0}\left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\|g-h\|_{B_T^{\lambda_0}} \\
&\qquad + 4T\langle T\lambda_0\rangle\left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\|g-h\|_{B_T^{\lambda_0}}.
\end{aligned}$$

Step 3. Let $R>0$, $\delta_0>0$ and introduce $C = C(R,\delta_0,\mathscr{E}_I,\mathscr{M},f_0)>0$ such that

$$\begin{cases}
C_1\langle\delta_0\lambda_0\rangle\left(\mathscr{E}_I + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\left(\| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right) \leq C\dfrac{1}{3^4}, \\
\langle\delta_0\lambda_0\rangle\left(C_2\sqrt{\delta_0\lambda_0}+C_3\right)\left(\mathscr{E}_I + \| \mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right) \leq C.
\end{cases}$$

(The C_j's are the constants that appear in the estimates established in the first two steps.) We introduce the sequences defined by

$$T_k = \delta\prod_{j=0}^k\left(1-\frac{1}{(j+2)^2}\right) \quad ; \quad \begin{cases} g_0 = f_0 \\ g_{k+1} = \Phi(g_k)\end{cases} \quad ; \quad \mu_k = \|g_{k+1}-g_k\|_{B_{T_k}}$$

where $\delta > 0$ is such that

$$\begin{cases} \delta \leq \delta_0, \\ C\delta \displaystyle\sum_{k=0}^{+\infty} \frac{1}{(k+3)^2} \leq R, \\ C\delta \displaystyle\sup_{x\geq 0} \left(\frac{x+4}{x+3}\right)^4 \leq 1. \end{cases}$$

We are going to show that, with this definition of δ, we have, for any $k \in \mathbb{N}$,

$$g_k \in E^{\lambda_0}_{T_k,R} \ \text{ et } \ \mu_k \leq C\delta \frac{1}{(k+3)^4} \tag{68}$$

We start by establishing that the sequence $(T_k)_{k\in\mathbb{N}}$ is decreasing and that

$$\delta T^\infty \leq T_k \leq T_0 < \delta_0.$$

Initialisation. Since $g_0 = f_0 \in \mathcal{G}^{\lambda_0,\sigma;1}_P$ does not depend on time, we obviously have $g_0 \in E^{\lambda_0}_{T_0,R}$. Step 1 tells us that $g_1 = \Phi(g_0) \in B^{\lambda_0}_{T_0}$. More precisely, we have

$$\|g_1 - g_0\|_{B^{\lambda_0}_{T_0}} = \|\Phi(g_0) - f_0\|_{B^{\lambda_0}_{T_0}} \leq C_1\, \delta\langle\delta_0\lambda_0\rangle \left(\mathscr{E}_I + \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P}\right)\left(\|\mathscr{M}\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P} + \|f_0\|_{\mathcal{G}^{\lambda_0,\sigma;1}_P}\right).$$

The definition of C ensures that

$$\mu_0 \leq C\delta \frac{1}{(0+3)^4}.$$

Recursion. Suppose that (68) holds up to a certain step N. Then, for any $0 < \lambda < \lambda_0$ and $t \in [0, T_{N+1}(\lambda_0 - \lambda)]$, we get

$$\|g_{N+1}(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \leq \frac{1 - \frac{t}{T_N(\lambda_0-\lambda)}}{1 - \frac{t}{T_N(\lambda_0-\lambda)}} \|g_{N+1}(t) - g_N(t)\|_{\mathcal{G}^{\lambda,\sigma;1}_P} + \|g_N(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P}$$

$$\leq \frac{1}{1 - \frac{t}{T_N(\lambda_0-\lambda)}}\mu_N + \|g_N(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \leq \frac{1}{1 - \frac{T_{N+1}}{T_N}}\mu_N + \|g_N(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P}$$

$$\leq \sum_{k=0}^{N} \frac{\mu_k}{1 - \frac{T_{k+1}}{T_k}} + \|g_0 - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} = \sum_{k=0}^{N} (k+3)^2 \mu_k$$

$$\leq \sum_{k=0}^{N} (k+3)^2 C\delta \frac{1}{(k+3)^4} \leq C\delta \sum_{k=0}^{+\infty} \frac{1}{(k+3)^2}.$$

The definition of δ implies

$$\|g_{N+1}(t) - f_0\|_{\mathcal{G}^{\lambda,\sigma;1}_P} \leq R.$$

Since $g_{N+1} = \Phi(g_N)$ and $g_N \in E^{\lambda_0}_{T_N,R}$, Step 1 and the previous computation show that $g_{N+1} \in E^{\lambda_0}_{T_{N+1},R}$. Applying Step 2, we obtain (owing to the definition adopted for C)

$$\begin{aligned} \mu_{N+1} &= \|\Phi(g_{N+1}) - \Phi(g_N)\|_{B^{\lambda_0}_{T_{N+1}}} \leq C\delta \|g_{N+1} - g_N\|_{B^{\lambda_0}_{T_{N+1}}} \\ &\leq \mu_N \leq C\delta \left[C\delta \left(\tfrac{N+4}{N+3}\right)^4\right] \tfrac{1}{(N+4)^4}. \end{aligned}$$

Finally, the constraints imposed on δ are such that

$$\mu_{N+1} \leq C\delta \frac{1}{(N+4)^4},$$

which ends the proof.

Step 4: Conclusion. Let g denote the limit of the sequence $(g_k)_{k\in\mathbb{N}}$ in $B^{\lambda_0}_{\delta T^\infty}$. Let us show that $g \in E^{\lambda_0}_{\delta T^\infty, R}$. Let $0<\lambda<\lambda_0$ and $t \in [0, \delta T^\infty(\lambda_0-\lambda))$. Of course, we have, for any $N \in \mathbb{N}$,

$$\|g(t)-f_0\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \leq \frac{1}{1-\frac{t}{\delta T^\infty(\lambda_0-\lambda)}} \|g-g_N\|_{B^{\lambda_0}_{\delta T^\infty}} + \|g_N(t)-f_0\|_{\mathcal{G}_P^{\lambda,\sigma;1}}.$$

Let $\varepsilon>0$. There exists $N\in\mathbb{N}$ (that depends on t, λ and ε) such that

$$\|g-g_N\|_{B^{\lambda_0}_{\delta T^\infty}} \leq \left[1-\frac{t}{\delta T^\infty(\lambda_0-\lambda)}\right]\varepsilon.$$

Using this in the previous estimate yields

$$\|g(t)-f_0\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \leq \varepsilon + R,$$

which thus holds for any $\varepsilon>0$. We conclude that $g \in E^{\lambda_0}_{\delta T^\infty,R}$, by letting ε go to 0. Next, we can apply Step 2 and we conclude that g is a fixed point of Φ:

$$\begin{aligned}\|g-\Phi(g)\|_{B_T} &\leq \|g-g_k\|_{B_T} + \|g_k-\Phi(g_k)\|_{B_T} + \|\Phi(g_k)-\Phi(g)\|_{B_T} \\ &\lesssim \|g-g_k\|_{B_T} + \|g_k-g_{k+1}\|_{B_T} + \|g_k-g\|_{B_T} \underset{k\to+\infty}{\longrightarrow} 0.\end{aligned}$$

■

Proof of Corollary C.4. Since $f_0, \mathscr{M} \in \mathcal{G}_P^{\widetilde{\lambda_0},0;1}$, for $0<\lambda_0<\widetilde{\lambda_0}$ arbitrarily close to $\widetilde{\lambda_0}$, we have $f_0,\mathscr{M} \in \mathcal{G}_P^{\lambda_0,\sigma;1}$, too. Therefore, we can appeal to Theorem C.1: there exist $T>0$ and $g \in B_T^{\lambda_0}$ solution of (66). We also know that there exists $R>0$ such that $g\in E^{\lambda_0}_{T,R}$.

We are going to show that $g \in C^0([0,T(\lambda_0-\lambda));\mathcal{G}_P^{\lambda,\sigma;1})$ for any $0<\lambda<\lambda_0$. By using an argument of composition of continuous functions, it follows that we can work with $\lambda=\lambda(t)$ such that $0\leq t<T(\lambda_0-\lambda(t))$ on a time interval $[0,T_f]$, and we have $g\in C^0([0,T_f];\mathcal{G}_P^{\lambda(t),\sigma;1})$.

Let us pick $0<\lambda<\lambda_0$ and a time $t\in[0,T(\lambda_0-\lambda))$. Remark that, for any $h>0$ with $t+h<T(\lambda_0-\lambda)$,we can find $\lambda<\lambda'<\lambda_0$ verifying $t+h<T(\lambda_0-\lambda')$ and we can choose λ' (depending on $h:\lambda'=\lambda'_h$) so that λ'_h does not converge to λ as h tends to 0. Going back to the beginning of the proof of Theorem C.1, we get

$$\|g(t+h)-g(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} = \|\Phi(g)(t+h)-\Phi(g)(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \leq \int_t^{t+h} \|\mathcal{N}(g)(\tau)\|_{\mathcal{G}_P^{\lambda,\sigma;1}}\,\mathrm{d}\tau$$

$$\lesssim \int_t^{t+h} \left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right)\frac{\langle\tau\rangle}{\lambda'-\lambda}\|\mathscr{M}+g(\tau)\|_{\mathcal{G}_P^{\lambda',\sigma;1}}\,\mathrm{d}\tau.$$

Since $\tau \leq t+h < T(\lambda_0 - \lambda')$ and $g \in E^{\lambda_0}_{T,R}$, we are led to

$$\|g(t+h)-g(t)\|_{\mathcal{G}_P^{\lambda,\sigma;1}} \lesssim \left(\mathscr{E}_I + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right) \frac{\langle T\lambda_0\rangle}{\lambda' - \lambda} \left(\|\mathscr{M}\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}} + R + \|f_0\|_{\mathcal{G}_P^{\lambda_0,\sigma;1}}\right) h \xrightarrow[h\to 0]{} 0.$$

Let us end the discussion with a few hints on the extension criterion. We are going to show that, if $g \in C^0([0,T); \mathcal{G}_P^{\lambda(t),\sigma;1})$ (with $0 < \lambda(t) \leq \lambda_0$ continuous and decreasing) is a solution of (66) such that

$$\begin{cases} \limsup\limits_{t\nearrow T} \|g(t)\|_{\mathcal{G}_P^{\lambda(t),\sigma;1}} < +\infty \\ \lim\limits_{t\nearrow T} \lambda(t) > 0 \end{cases}$$

then, possibly at the price of replacing $\lambda(t)$ by another function $\widetilde{\lambda}(t)$ such that $0 < \widetilde{\lambda}(t) \leq \lambda(t)$ on $[0,T)$, we can extend g into a solution of (66) on $[0,T')$, with $g \in C^0([0,T'); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1})$.

To this end, we apply Theorem C.1 with $g(t)$ as initial data for any $t \in [0,T)$. For each of these data, there exists T'_t and a solution of (66) in $B^{\lambda(t)}_{T'_t}$. But the proof of Theorem C.1 shows that T'_t depends (among other things) on the norm $\mathcal{G}_P^{\lambda,\sigma;1}$ of the initial data and on the coefficient λ (see the role of the constants C and δ). Here, we know that there exists $A > 0$ such that, for any $t \in [0,T)$,

$$\|g(t)\|_{\mathcal{G}_P^{\lambda(t),\sigma;1}} \leq A$$

holds, and $\lambda(t) \leq \lambda_0$. Hence, the times T'_t can be chosen independently of the data $g(t)$: $T'_t = T'$. Furthermore, we also know that there exists a constant $a > 0$ such that, for any $t \in [0,T)$, $\lambda(t) \geq a$. Thus, there also exists $t^\star > 0$ such that $t + T'\lambda(t) > T$ holds for any $t \in [t^\star, T)$. This allows us to extend the solution; we refer the reader to Fig. 2 and Fig. 3 for guiding the intuition. ■

C.2 Extension of the strong analycity property

We wish to prove Proposition 5.6. To this end, we are going to combine Corollary C.4 to the following statement.

Proposition C.6 *Let $P > d/2$ be an integer and let $\sigma > d/2$ be a real number. If $g \in C^0([0,T); \mathcal{G}_P^{\lambda(t),\sigma;1})$ is a solution of* (66) *on $[0,T)$ that satisfies*

$$\limsup_{t\nearrow T} \|g(t)\|_{H^\sigma_P} < +\infty,$$

then, there exists a function $\widetilde{\lambda}(t) > 0$ continuous and decreasing such that $g \in C^0\left([0,T); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}\right)$, and, for any $t \in [0,T)$, w ehave

$$\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} \leq \|g(0)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(0),\sigma;1}} + 1 + 2\int_0^t 1 + \theta(\tau)\,\mathrm{d}\tau$$

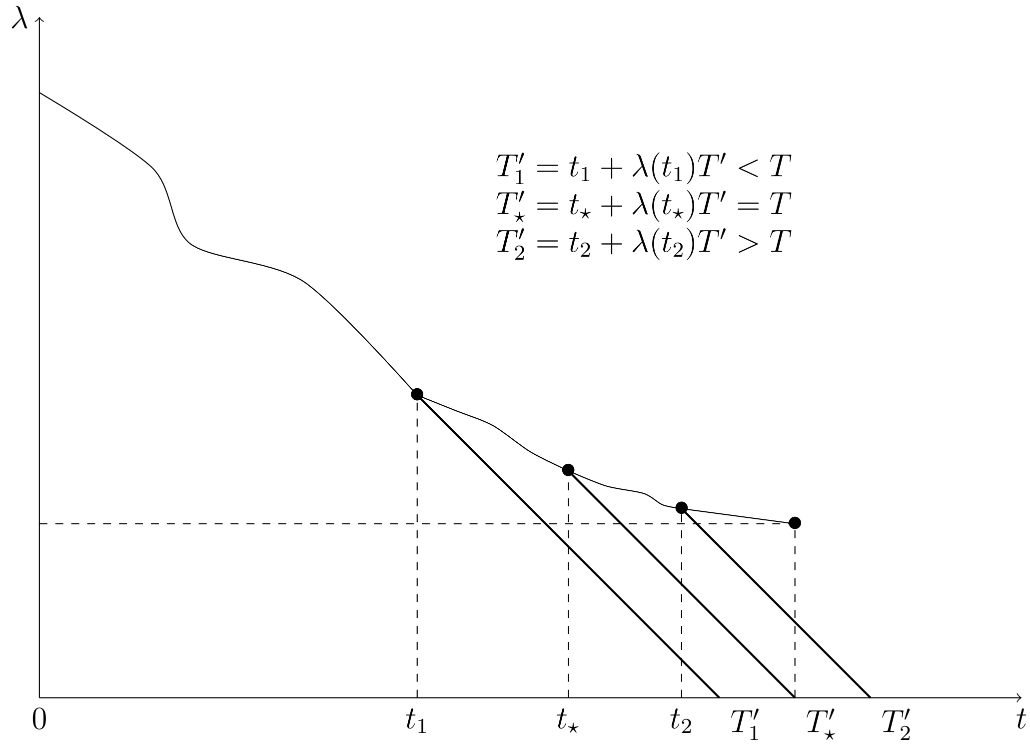

Figure 2: Analycity radius, as a function of the time variable

where $\theta(t)$ depends on g only through the following Sobolev norms

$$\theta(t) = \left(C_2 \langle T \rangle \|g(t)\|_{H_P^\sigma} + C_3(T, \widetilde{\lambda}(0)) \|\nabla_v \mathscr{M}\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}} \right) \left(\mathscr{E}_I + \int_0^t \|g(\tau)\|_{H_P^\sigma}^2 \, \mathrm{d}\tau \right)^{1/2} \|g(t)\|_{H_P^\sigma}. \tag{69}$$

and the constant C_i do not depend on g.

Remark C.7 *The proof provides an explicit formula for $\widetilde{\lambda}$. In particular, it justifies that $\widetilde{\lambda}(T) > 0$. The proof of Theorem 5.6 then follows readlily form Corollary C.4 and Proposition C.6.*

Let us start by establishing the following *a priori* estimate.

Lemma C.8 *Let $P > d/2$ be an integer and let $\sigma > d/2$ be a real number. If $g \in C^0([0,T); \mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1})$ is a solution of(66) on $[0,T)$ such that*

$$\limsup_{t \nearrow T} \|g(t)\|_{H_P^\sigma} < +\infty,$$

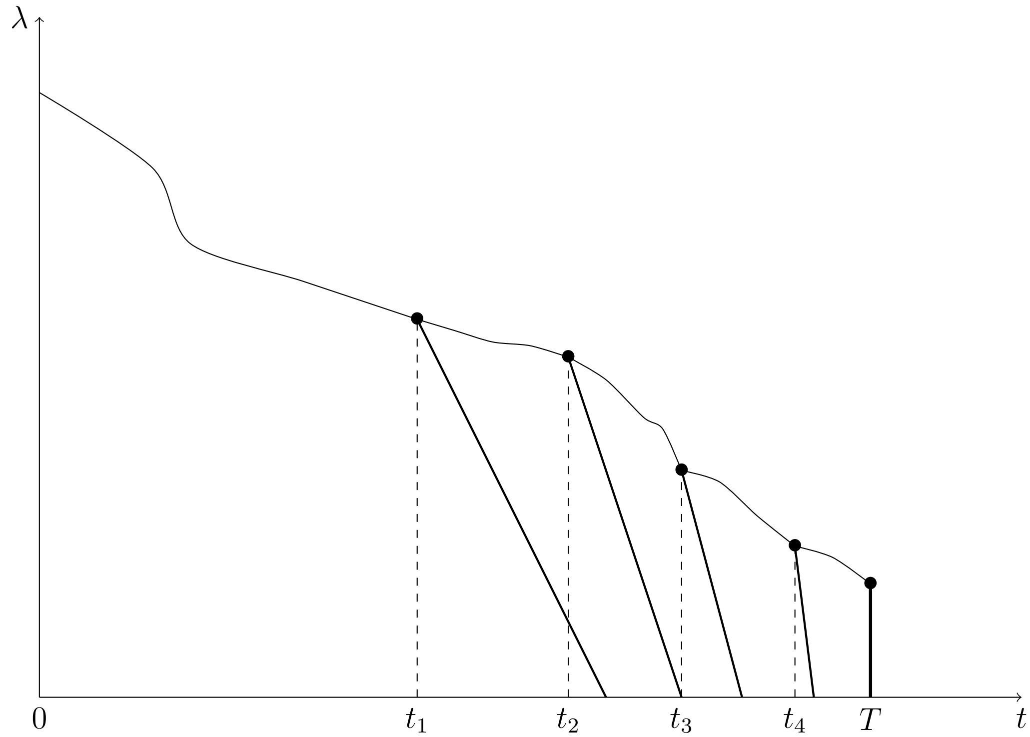

Figure 3: Analycity radius, as a function of the time variable: critical case with T' depending on t

where $0 < \widetilde{\lambda}(t)$ *is derivable and decreasing function, then, for any* $t \in [0, T)$*, we have*

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} \quad &\leq \quad \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda}(t)\right)\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}} \\ &+\widetilde{\lambda}(t)C_1\langle T\rangle^2\left(\mathscr{E}_I + \int_0^t \|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(\tau),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}} + \theta(t) \end{aligned}$$

where $\theta(t)$ *is defined by* (69)*.*

The proof uses in several places the following claim. (We do not detail its proof, which reduces to repeated applications of the Cauchy-Schwarz inequality.)

Lemma C.9 *For any* $\widetilde{\sigma} > d/2$*, we have*

$$\left|\sum_{k,n\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\widehat{f}(k,\xi)\widehat{g}(n)\widehat{h}(k-n,\xi-tn)\,\mathrm{d}\xi\right| \lesssim \|f\|_{L^2_{x,v}}\left(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2\widetilde{\sigma}}|\widehat{g}(n)|^2\right)^{1/2}\|h\|_{L^2_{x,v}}.$$

Proof of Lemma C.8. Since

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda}(t)\right)\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+\frac{1}{2};1}} + \Re\left\langle g(t),\partial_t g(t)\right\rangle_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}}$$

with

$$\Re \langle g(t), \partial_t g(t)\rangle_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} = \sum_{\substack{\alpha\in\mathbb{N}^d\\ |\alpha|\le P}} \underbrace{\Re \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\widetilde{\lambda}(t)\langle k,\xi\rangle} \mathrm{D}_\xi^\alpha \overline{\widehat{g}(t,k,\xi)} \mathrm{D}_\xi^\alpha \partial_t \widehat{g}(t,k,\xi)\,\mathrm{d}\xi}_{:=I(\alpha)},$$

we fix $\alpha\in\mathbb{N}^d$, $|\alpha|\le P$ and estimate $I(\alpha)$. Let us write

$$\begin{aligned}
\mathrm{D}_\xi^\alpha \partial_t \widehat{g}(t,k,\xi) &= \mathrm{D}_\xi^\alpha \widehat{\mathcal{N}}(g)(t,k,\xi)\\
&= -k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma_1}(k)\widehat{\mathscr{G}_\varrho}(t,k)) \cdot \mathrm{D}_\xi^\alpha \left(\xi \mapsto (\xi - tk)\widehat{\mathscr{M}}(\xi-tk)\right)\\
&\quad - \sum_{n\in\mathbb{Z}^d} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma_1}(n)\widehat{\mathscr{G}_\varrho}(t,n)) \cdot \mathrm{D}_\xi^\alpha \left(\xi \mapsto (\xi-tk)\widehat{g}(t,k-n,\xi-tk)\right).
\end{aligned}$$

Next we split $I(\alpha)$ as follows

$$\begin{aligned}
I(\alpha) &= -\Re\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\widetilde{\lambda}(t)\langle k,\xi\rangle} \mathrm{D}_\xi^\alpha \overline{\widehat{g}(t,k,\xi)} k\widehat{\sigma_1}(k)(\widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma_1}(k)\widehat{\mathscr{G}_\varrho}(t,k))\\
&\qquad\qquad \cdot \mathrm{D}_\xi^\alpha \left(\xi\mapsto (\xi-tk)\widehat{\mathscr{M}}(\xi-tk)\right)\,\mathrm{d}\xi\\
&\quad -\Re\sum_{k,n\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{2\sigma} e^{2\widetilde{\lambda}(t)\langle k,\xi\rangle} \mathrm{D}_\xi^\alpha \overline{\widehat{g}(t,k,\xi)} n\widehat{\sigma_1}(n)(\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma_1}(n)\widehat{\mathscr{G}_\varrho}(t,n))\\
&\qquad\qquad \cdot \mathrm{D}_\xi^\alpha \left(\xi\mapsto (\xi-tk)\widehat{g}(t,k-n,\xi-tk)\right)\,\mathrm{d}\xi\\
&= I_1(\alpha) + I_2(\alpha).
\end{aligned}$$

Estimate of $I_1(\alpha)$. With the Cauchy-Schwarz inequality we obtain the rough inequality

$$\begin{aligned}
I_1(\alpha) &\lesssim \sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{\sigma} \left|\mathrm{D}_\xi^\alpha \widehat{g}(t,k,\xi)\right| \langle k,tk\rangle^\sigma e^{2\widetilde{\lambda}(t)\langle k,tk\rangle} |k||\widehat{\sigma_1}(k)| \left|\widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma_1}(k)\widehat{\mathscr{G}_\varrho}(t,k)\right|\\
&\qquad \times \langle \xi - tk\rangle^\sigma e^{2\widetilde{\lambda}(t)\langle\xi-tk\rangle} \left|\mathrm{D}_\xi^\alpha \left(\xi\mapsto(\xi-tk)\widehat{\mathscr{M}}(\xi-tk)\right)\right|\,\mathrm{d}\xi\\
&\lesssim \left(\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\langle k,\xi\rangle^{2\sigma}\left|\mathrm{D}_\xi^\alpha\widehat{g}(t,k,\xi)\right|^2\mathrm{d}\xi\right)^{1/2}\\
&\qquad \times\left(\sum_{k\in\mathbb{Z}^d}\int_{\mathbb{R}^d_\xi}\langle k,tk\rangle^{2\sigma}e^{4\widetilde{\lambda}(t)\langle k,tk\rangle}|k|^2|\widehat{\sigma_1}(k)|^2\left|\widehat{\mathscr{F}_I}(t,k)-\widehat{\sigma_1}(k)\widehat{\mathscr{G}_\varrho}(t,k)\right|^2\right.\\
&\qquad\qquad \left.\langle\xi-tk\rangle^{2\sigma}e^{4\widetilde{\lambda}(t)\langle\xi-tk\rangle}\left|\mathrm{D}_\xi^\alpha\left(\xi\mapsto(\xi-tk)\widehat{\mathscr{M}}(\xi-tk)\right)\right|^2\mathrm{d}\xi\right)^{1/2}\\
&\lesssim \|(x,v)\mapsto v^\alpha g(t,x,v)\|_{H^\sigma}\|\nabla\sigma_1\star(\mathscr{F}_I(t)-\sigma_1\star\mathscr{G}_\varrho(t))\|_{\mathcal{F}^{2\widetilde{\lambda}(t),\sigma;1}}\|v\mapsto v^\alpha\nabla_v\mathscr{M}\|_{\mathcal{G}^{2\widetilde{\lambda}(t),\sigma;1}}\\
&\lesssim \|g(t)\|_{H^\sigma_P}\|\nabla\sigma_1\star(\mathscr{F}_I(t)-\sigma_1\star\mathscr{G}_\varrho(t))\|_{\mathcal{F}^{2\widetilde{\lambda}(t),\sigma;1}}\|\nabla_v\mathscr{M}\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}}.
\end{aligned}$$

By using (54), we get

$$
\begin{aligned}
&\left\| \nabla \sigma_1 \star (\mathscr{F}_I(t) - \sigma_1 \star \mathscr{G}_\varrho(t)) \right\|^2_{\mathcal{F}^{2\widetilde{\lambda}(t),\sigma;1}} \\
&= \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{4\widetilde{\lambda}(t)\langle k,tk\rangle} |k|^2 |\widehat{\sigma}_1(k)|^2 \left| \widehat{\mathscr{F}_I}(t,k) - \widehat{\sigma}_1(k)\widehat{\mathscr{G}_\varrho}(t,k)\right|^2 \\
&\lesssim \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma} \langle t\rangle^{2\sigma} e^{4\widetilde{\lambda}(0)\langle k\rangle\langle t\rangle} |k|^2 |\widehat{\sigma}_1(k)|^2 \mathscr{E}_I \mathbf{1}_{0\le t\le S_0} \\
&\qquad\qquad + \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} e^{4\widetilde{\lambda}(0)\langle k\rangle\langle T\rangle} |k|^2 |\widehat{\sigma}_1(k)|^4 |\widehat{\mathscr{G}_\varrho}(t,k)|^2 \\
&\lesssim \left(\langle S_0\rangle^{2\sigma} \sum_{k\in\mathbb{Z}^d} \langle k\rangle^{2\sigma} e^{4\widetilde{\lambda}(0)\langle S_0\rangle\langle k\rangle} |k|^2 e^{-2\lambda_1|k|} \right) \mathscr{E}_I \\
&\qquad\qquad + \sup_{n\in\mathbb{Z}^d} \left(e^{4\widetilde{\lambda}(0)\langle T\rangle\langle n\rangle} |n|^2 e^{-4\lambda_1|n|} \right) \sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} |\widehat{\mathscr{G}_\varrho}(t,k)|^2
\end{aligned}
$$

and

$$
\sum_{k\in\mathbb{Z}^d} \langle k, tk\rangle^{2\sigma} |\widehat{\mathscr{G}_\varrho}(t,k)|^2 \lesssim \int_0^t |p_c(t-\tau)| \langle t-\tau\rangle^{2\sigma} \left(\sum_{k\in\mathbb{Z}^d} \langle k, \tau k\rangle^{2\sigma} |\widehat{\varrho}(\tau,k)|^2 \right) \mathrm{d}\tau \lesssim \int_0^t \|g(\tau)\|^2_{H^\sigma_P} \, \mathrm{d}\tau.
$$

We deduce that

$$
I_1(\alpha) \le K_1(T, \widetilde{\lambda}(0)) \|g(t)\|_{H^\sigma_P} \left(\mathscr{E}_I + \int_0^t \|g(\tau)\|^2_{H^\sigma_P} \, \mathrm{d}\tau \right)^{1/2} \|\nabla_v \mathscr{M}\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}}.
$$

Remark C.10 *That K_1 remains finite make some constraints on $\widetilde{\lambda}(0)$ appear:*

$$
2\widetilde{\lambda}(0) < \widetilde{\lambda}_0 \;\; ; \;\; 4\widetilde{\lambda}(0)\langle S_0 < 2\lambda_1 \quad et \quad 4\widetilde{\lambda}(0)\langle T\rangle < 4\lambda_1.
$$

Therefore we should pay attention to the following facts

- $\widetilde{\lambda}(0)$ *depends on T. In particular, as T tends to $+\infty$, $\widetilde{\lambda}(0)$ should not converge to* 0 *(since we need $\widetilde{\lambda}(0) > 0$);*
- *the constant K_1 depends on $\widetilde{\lambda}(0)$. In what follows, we should check that $\widetilde{\lambda}(0)$ can be chosen independently of the value of $K_1(T, \widetilde{\lambda}(0))$.*

Estimate of $I_2(\alpha)$. Applying (48) leads to

$$
\begin{aligned}
&I_2(\alpha) = \\
&-\Re \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \mathrm{D}^\alpha_\xi \overline{\widehat{g}(t,k,\xi)} \left[\langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} - \langle k-n,\xi-tn\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle}\right] \\
&\qquad \times n\widehat{\sigma}_1(n)\left(\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right)\cdot \mathrm{D}^\alpha_\xi\left(\xi\mapsto(\xi-tk)\widehat{g}(t,k-n,\xi-tn)\right)\mathrm{d}\xi \\
&\le \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \left|\langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} - \langle k-n,\xi-tn\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle}\right| \\
&\qquad \times |n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|\left|\mathrm{D}^\alpha_\xi\left(\xi\mapsto(\xi-tk)\widehat{g}(t,k-n,\xi-tn)\right)\right|\mathrm{d}\xi.
\end{aligned}
$$

Next, we apply the following statement (for further details, we refer the reader to [24, Lemma 9]).

Lemma C.11 *For any* $r,\tau,x,y\ge 0$*, we have*

$$
|x^r e^{\tau x} - y^r e^\tau y| \le c(r)|x-y|\left(|x-y|^{r-1} + |y|^{r-1} + \tau\left[|x-y|^r + |y|^r\right] e^{\tau|x-y|}e^{\tau|y|}\right).
$$

Set $r=\sigma$, $\tau=\widetilde{\lambda}(t)$, $x=\langle k,\xi\rangle$ and $y=\langle k-n,\xi-tn\rangle$. We obtain (remark that $|x-y|\le\langle n,tn\rangle$)

$$
\begin{aligned}
&I_2(\alpha) \lesssim \\
&\sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle\left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right) \\
&\qquad \times |n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|\left|\mathrm{D}^\alpha_\xi\left(\xi\mapsto(\xi-tk)\widehat{g}(t,k-n,\xi-tn)\right)\right|\mathrm{d}\xi \\
&+\widetilde{\lambda}(t)\sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \\
&\qquad \times \langle n,tn\rangle\left(\langle n,tn\rangle^\sigma + \langle k-n,\xi-tn\rangle^\sigma\right) e^{\widetilde{\lambda}(t)\langle n,tn\rangle} e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle} \\
&\qquad \times |n||\widehat{\sigma}_1(n)|\left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|\left|\mathrm{D}^\alpha_\xi\left(\xi\mapsto(\xi-tk)\widehat{g}(t,k-n,\xi-tn)\right)\right|\mathrm{d}\xi \\
&= I_{21}(\alpha) + \widetilde{\lambda}(t) I_{22}(\alpha).
\end{aligned}
$$

Next, we use

$$
\langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \le \langle k,\xi\rangle^\sigma\left(1+\widetilde{\lambda}(t)\langle k,\xi\rangle e^{\widetilde{\lambda}(t)\langle k,\xi\rangle}\right)
$$

for dealing with $I_{21}(\alpha)$; we get

$$
\begin{aligned}
&I_{21}(\alpha) \leq \\
&\sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle \left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right) \\
&\qquad \times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \left|\mathrm{D}^\alpha_\xi \left(\xi \mapsto (\xi - tk)\widehat{g}(t,k-n,\xi-tn)\right)\right| \mathrm{d}\xi \\
&+\widetilde{\lambda}(t) \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^{\sigma+1} e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle \left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right) \\
&\qquad \times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \left|\mathrm{D}^\alpha_\xi \left(\xi \mapsto (\xi - tk)\widehat{g}(t,k-n,\xi-tn)\right)\right| \mathrm{d}\xi \\
&= I_{211}(\alpha) + \widetilde{\lambda}(t) I_{212}(\alpha).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\langle k,\xi\rangle &\left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right) \\
&\lesssim \left(\langle n,tn\rangle + \langle k-n,\xi-tn\rangle\right) \left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right) \\
&\lesssim \langle n,tn\rangle^\sigma + \langle k-n,\xi-tn\rangle^\sigma.
\end{aligned}
$$

Hence $I_{212}(\alpha) \lesssim I_{22}(\alpha)$ and thus $I_2(\alpha) \lesssim I_{211}(\alpha) + \widetilde{\lambda}(t) I_{22}(\alpha)$.

Estimate of $I_{211}(\alpha)$. We remind the reader that

$$
\begin{aligned}
&\mathrm{D}^\alpha_\xi \left(\xi \mapsto (\xi - tk)\widehat{g}(t,k-n,\xi-tn)\right) \\
&\qquad = (\xi - tk)\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn) + \sum_{\substack{j\in\mathbb{N}^d \\ j\leq\alpha\, |j|=1}} \binom{\alpha}{j} j \mathrm{D}^{\alpha-j}_\xi \widehat{g}(t,k-n,\xi-tn).
\end{aligned}
$$

Then, we have

$$
\left|\mathrm{D}^\alpha_\xi \left(\xi \mapsto (\xi - tk)\widehat{g}(t,k-n,\xi-tn)\right)\right| \lesssim \langle t\rangle \langle k-n,\xi-tn\rangle \sum_{\substack{\beta\in\mathbb{N}^d \\ |\beta|\leq P}} \left|\mathrm{D}^\beta_\xi \widehat{g}(t,k-n,\xi-tn)\right|. \tag{70}
$$

It allows us to obtain

$$I_{211}(\alpha) \lesssim$$

$$\sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le P}} \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle \left(\langle n,tn\rangle^{\sigma-1} + \langle k-n,\xi-tn\rangle^{\sigma-1}\right)$$

$$\times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \langle t\rangle\langle k-n,\xi-tn\rangle \left|\mathrm{D}^\beta_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}\xi$$

$$\lesssim \langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le P}} \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle^\sigma \langle k-n,\xi-tn\rangle^\sigma$$

$$\times |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right| \left|\mathrm{D}^\beta_\xi \widehat{g}(t,k-n,\xi-tn)\right| \,\mathrm{d}\xi$$

$$= \langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\le P}} J_1(\alpha,\beta).$$

For all $J_1(\alpha,\beta)$ we apply Lemma C.9 and we arrive at

$$J_1(\alpha,\beta) \lesssim \|(x,v)\mapsto v^\alpha g(t,x,v)\|_{H^\sigma} \|(x,v)\mapsto v^\beta g(t,x,v)\|_{H^\sigma}$$

$$\times \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{\widetilde{2\sigma}} |n|^2 |\widehat{\sigma}_1(n)|^2 \langle n,tn\rangle^{2\sigma} \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right|^2\right)^{1/2}.$$

Since

$$\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{\widetilde{2\sigma}} |n|^2 |\widehat{\sigma}_1(n)|^2 \langle n,tn\rangle^{2\sigma} \left|\widehat{\mathscr{F}_I}(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}_\varrho}(t,n)\right|^2$$

$$\lesssim \langle S_0\rangle^{2\sigma} \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{\widetilde{2\sigma}} |n|^2 |\widehat{\sigma}_1(n)|^2 \langle n\rangle^{2\sigma}\right) \mathscr{E}_I + \left(\sup_{k\in\mathbb{Z}^d} \langle k\rangle^{\widetilde{2\sigma}} |k|^2 |\widehat{\sigma}_1(k)|^2\right) \left(\sum_{n\in\mathbb{Z}^d} \langle n,tn\rangle^{2\sigma} |\widehat{\mathscr{G}_\varrho}(t,n)|^2\right)$$

$$\lesssim \mathscr{E}_I + \int_0^t \|g(\tau)\|^2_{H^\sigma_P} \,\mathrm{d}\tau,$$

we are led to

$$I_{211}(\alpha) \lesssim \langle T\rangle \left(\mathscr{E}_I + \int_0^t \|g(\tau)\|^2_{H^\sigma_P} \,\mathrm{d}\tau\right)^{1/2} \|g(t)\|^2_{H^\sigma_P}.$$

Estimate of $I_{22}(\alpha)$. Again, we apply (70). We obtain

$$I_{22}(\alpha) \lesssim$$

$$\langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\leq P}} \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle \left(\langle n,tn\rangle^\sigma + \langle k-n,\xi-tn\rangle^\sigma\right) e^{\widetilde{\lambda}(t)\langle n,tn\rangle}$$

$$\times e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle} |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \langle k-n,\xi-tn\rangle \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \mathrm{d}\xi$$

$$\lesssim \langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\leq P}} \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle^{\sigma+1} \langle k-n,\xi-tn\rangle e^{\widetilde{\lambda}(t)\langle n,tn\rangle}$$

$$\times e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle} |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \mathrm{d}\xi$$

$$+\langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\leq P}} \sum_{k,n\in\mathbb{Z}^d} \int_{\mathbb{R}^d_\xi} \langle k,\xi\rangle^\sigma e^{\widetilde{\lambda}(t)\langle k,\xi\rangle} \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k,\xi)\right| \langle n,tn\rangle \langle k-n,\xi-tn\rangle^{\sigma+1} e^{\widetilde{\lambda}(t)\langle n,tn\rangle}$$

$$\times e^{\widetilde{\lambda}(t)\langle k-n,\xi-tn\rangle} |n||\widehat{\sigma}_1(n)| \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right| \left|\mathrm{D}^\alpha_\xi \widehat{g}(t,k-n,\xi-tn)\right| \mathrm{d}\xi$$

$$= \langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\leq P}} J_2(\alpha,\beta) + \langle T\rangle \sum_{\substack{\beta\in\mathbb{N}^d\\|\beta|\leq P}} J_3(\alpha,\beta).$$

We estimate $J_2(\alpha,\beta)$ by using Lemma C.9, again. We get

$$J_2(\alpha,\beta) \lesssim \|(x,v)\mapsto v^\alpha g(t,x,v)\|_{\mathcal{G}^{\widetilde{\lambda}(t),\sigma;1}} \|(x,v)\mapsto v^\beta g(t,x,v)\|_{\mathcal{G}^{\widetilde{\lambda}(t),1;1}}$$

$$\times \left(\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2\widetilde{\sigma}} |n|^2 |\widehat{\sigma}_1(n)|^2 \langle n,tn\rangle^{2\sigma+2} e^{2\widetilde{\lambda}(t)\langle n,tn\rangle} \left|\widehat{\mathscr{F}}_I(t,n) - \widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2\right)^{1/2}.$$

Since

$$\sum_{n\in\mathbb{Z}^d} \langle n\rangle^{2\widetilde{\sigma}}|n|^2|\widehat{\sigma}_1(n)|^2\langle n,tn\rangle^{2\sigma+2}e^{2\widetilde{\lambda}(t)\langle n,tn\rangle}\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2$$

$$\lesssim \langle S_0\rangle^{2\sigma+2}\left(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2\widetilde{\sigma}}|n|^2|e^{-2\lambda_1|n|}\langle n\rangle^{2\sigma+2}e^{2\widetilde{\lambda}(0)\langle n\rangle\langle S_0\rangle}\right)\mathscr{E}_I$$

$$+\langle T\rangle^2\left(\sup_{k\in\mathbb{Z}^d}\langle k\rangle^{2\widetilde{\sigma}}|k|^2|\widehat{\sigma}_1(k)|^2\langle k\rangle^2\right)\sum_{n\in\mathbb{Z}^d}\langle n,tn\rangle^{2\sigma}e^{2\widetilde{\lambda}(t)\langle n,tn\rangle}\left|\widehat{\mathscr{G}}_\varrho(t,n)\right|^2$$

$$\lesssim \langle T\rangle^2\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(\tau),\sigma;1}}\,\mathrm{d}\tau\right),$$

we are led to

$$J_2(\alpha,\beta)\lesssim\langle T\rangle\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}.$$

Eventually, we turn to $J_3(\alpha,\beta)$. We write

$$\langle k-n,\xi-tn\rangle^{\sigma+1}\leq\langle k,\xi\rangle^{1/2}\langle n,tn\rangle^{1/2}\langle k-n,\xi-tn\rangle^{\sigma+1/2}$$

and we apply Lemma C.9

$$J_3(\alpha,\beta)\lesssim\|(x,v)\mapsto v^\alpha g(t,x,v)\|_{\mathcal{G}^{\widetilde{\lambda}(t),\sigma+1/2;1}}\|(x,v)\mapsto v^\beta g(t,x,v)\|_{\mathcal{G}^{\widetilde{\lambda}(t),\sigma+1/2;1}}$$

$$\times\left(\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2\widetilde{\sigma}}|n|^2|\widehat{\sigma}_1(n)|^2\langle n,tn\rangle^3e^{2\widetilde{\lambda}(t)\langle n,tn\rangle}\left|\widehat{\mathscr{F}}_I(t,n)-\widehat{\sigma}_1(n)\widehat{\mathscr{G}}_\varrho(t,n)\right|^2\right)^{1/2}.$$

We obtain

$$J_3(\alpha,\beta)\lesssim\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}.$$

It follows that

$$I_{22}(\alpha)\lesssim\langle T\rangle^2\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}.$$

Recap. We have found constants C_1, C_2 et $C_3(T,\widetilde{\lambda}(0))$ such that

$$\Re\langle g(t),\partial_t g(t)\rangle\widetilde{\lambda}(t)\leq C_1\langle T\rangle^2\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}+\theta(t)$$

with

$$\theta(t)=\left(C_2\langle T\rangle\|g(t)\|_{H^\sigma_P}+C_3(T,\widetilde{\lambda}(0))\|\nabla_v\mathscr{M}\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}}\right)\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{H^\sigma_P}\,\mathrm{d}\tau\right)^{1/2}\|g(t)\|_{H^\sigma_P}.$$

■

Proof of Proposition C.6. We wish to apply Lemma C.8. However, the function g does not satisfy the required assumptions; we thus need to introduce a regularization

$$g_\varepsilon(t) = \chi_\varepsilon \star g(t) \;\text{ avec }\; \widehat{\chi}_\varepsilon(k,\xi) = e^{-\varepsilon|k,\xi|^2},$$

so that, for any $\lambda > 0$, $g_\varepsilon(t) \in \mathcal{G}_P^{\lambda,\sigma+1/2;1}$. We still cannot apply Lemma C.8 to g_ε since g_ε is not a solution of (66). Nevertheless, we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda}(t)\right)\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}} + \Re\langle g_\varepsilon(t), \partial_t g_\varepsilon\rangle.$$

Next, $\partial_t g_\varepsilon$ can be cast as

$$\begin{aligned}
&\partial_t \widehat{g}_\varepsilon(t,k,\xi) = \widehat{\chi}(k,\xi)\partial_t\widehat{g}(t,k,\xi)\\
&= -\frac{\widehat{\chi}(k,\xi)}{\widehat{\chi}(k,tk)\widehat{\chi}(\xi-tk)} k\widehat{\sigma}_1(k)\left(\widehat{\chi}(k,tk)\widehat{F}_I(t,k) - \widehat{\chi}(k,tk)\widehat{\sigma}_1(k)\widehat{G}_\varrho(t,k)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\xi - tk)\widehat{\chi}(\xi-tk)\widehat{\mathscr{M}}(\xi-tk)\\
&- \sum_{n\in\mathbb{Z}^d}\frac{\widehat{\chi}(k,\xi)}{\widehat{\chi}(n,tn)\widehat{\chi}(k-n,\xi-tn)} n\widehat{\sigma}_1(n)\left(\widehat{\chi}(n,tn)\widehat{F}_I(t,n) - \widehat{\chi}(n,tn)\widehat{\sigma}_1(n)\widehat{G}_\varrho(t,n)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\xi-tk)\widehat{\chi}(k-n,\xi-tn)\widehat{g}(t,k-n,\xi-tn)\\
&= -\frac{\widehat{\chi}(k,\xi)}{\widehat{\chi}(k,tk)\widehat{\chi}(\xi-tk)} k\widehat{\sigma}_1(k)\left(\widehat{\chi}(k,tk)\widehat{F}_I(t,k) - \widehat{\sigma}_1(k)\int_0^t p_c(t-\tau)\frac{\widehat{\chi}(k,tk)}{\widehat{\chi}(k,\tau k)}\widehat{\varrho}_\varepsilon(\tau,k)\,\mathrm{d}\tau\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\xi-tk)\widehat{\mathscr{M}_\varepsilon}(\xi-tk)\\
&- \sum_{n\in\mathbb{Z}^d}\frac{\widehat{\chi}(k,\xi)}{\widehat{\chi}(n,tn)\widehat{\chi}(k-n,\xi-tn)} n\widehat{\sigma}_1(n)\left(\widehat{\chi}(n,tn)\widehat{F}_I(t,n) - \widehat{\sigma}_1(n)\int_0^t \frac{\widehat{\chi}(n,tn)}{\widehat{\chi}(n,\tau n)}\widehat{\varrho}_\varepsilon(\tau,n)\,\mathrm{d}\tau\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot(\xi-tk)\widehat{g}_\varepsilon(t,k-n,\xi-tn).
\end{aligned}$$

Remarking that

$$\widehat{\chi}(k,\xi) \le 1, \qquad \frac{\widehat{\chi}(k,tk)}{\widehat{\chi}(k,\tau k)} \le 1 \quad\text{and}\quad \frac{\widehat{\chi}(k+n,\xi+\zeta)}{\widehat{\chi}(k,\xi)\widehat{\chi}(n,\zeta)} \le 1,$$

holds, we go back to the proof of Lemma C.8 and we conclude that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} &\leq \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda}(t)\right)\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}} \\
&\quad +\widetilde{\lambda}(t)C_1\langle T\rangle^2\left(\mathscr{E}_I+\int_0^t\|g_\varepsilon(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(\tau),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}+\theta_\varepsilon(t) \\
&< \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\lambda}(t)\right)\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}} \\
&\quad +\widetilde{\lambda}(t)C_1\langle T\rangle^2\left(\mathscr{E}_I+\int_0^t\|g_\varepsilon(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(\tau),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma+1/2;1}}+1+\theta_\varepsilon(t)
\end{aligned}$$

holds with

$$\theta_\varepsilon(t)=\left(C_2\langle T\rangle\|g_\varepsilon(t)\|_{H_P^\sigma}+C_3(T,\widetilde{\lambda}(0))\|\nabla_v\mathscr{M}_\varepsilon\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}}\right)\left(\mathscr{E}_I+\int_0^t\|g_\varepsilon(\tau)\|^2_{H_P^\sigma}\,\mathrm{d}\tau\right)^{1/2}\|g_\varepsilon(t)\|_{H_P^\sigma}.$$

where the constants C_i do not depend on ε. Let us introduce the function

$$Y_\varepsilon(t)=\|g_\varepsilon(0)\|^2_{\mathcal{G}_P^{\mu_0,\sigma;1}}+1+2\int_0^t 1+\theta_\varepsilon(\tau)\,\mathrm{d}\tau,$$

where $\mu_0>0$ will be precised later on. We apply Lemma C.8 to g_ε with $\widetilde{\lambda}(t)=\widetilde{\lambda}_\varepsilon(t)$ defined by

$$\widetilde{\lambda}_\varepsilon(t)=\widetilde{\lambda}_\varepsilon(0)\exp\left\{-\int_0^t\left[C_1\langle T\rangle^2\left(\mathscr{E}_I+\int_0^t Y_\varepsilon(\tau)\,\mathrm{d}\tau\right)^{1/2}\right]\right\}. \tag{71}$$

We are led to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma;1}}-Y_\varepsilon(t)\right) \\
&< C_1\langle T\rangle^2\widetilde{\lambda}_\varepsilon(t)\left[\left(\mathscr{E}_I+\int_0^t\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(\tau),\sigma;1}}\,\mathrm{d}\tau\right)^{1/2}-\left(\mathscr{E}_I+\int_0^t Y_\varepsilon(\tau)\,\mathrm{d}\tau\right)^{1/2}\right]\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma+1/2;1}} \\
&< \frac{C_1\langle T\rangle^2}{2\sqrt{E_I}}\widetilde{\lambda}_\varepsilon(t)\left[\int_0^t\left(\|g(\tau)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(\tau),\sigma;1}}-Y_\varepsilon(\tau)\right)\mathrm{d}\tau\right]\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma+1/2;1}}.
\end{aligned}$$

Since

$$\|g_\varepsilon(0)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(0),\sigma;1}}-Y_\varepsilon(0)=-1<0\ \text{ et }\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma;1}}-Y_\varepsilon(t)\right)|_{t=0}<0,$$

it is now possible to check that

$$\|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma;1}}<Y_\varepsilon(t).$$

It is worth commenting the value of $\widetilde{\lambda}_\varepsilon(0)$. We remind the reader that we wish to take the limit $\varepsilon \to 0$; hence, we should choose $\widetilde{\lambda}_\varepsilon(0)$ so that $\lim_{\varepsilon\to 0} \widetilde{\lambda}_\varepsilon(0) > 0$ (beware that C_3 depends on $\widetilde{\lambda}_\varepsilon(0)$, and thus C_3 can now we considered as a function of ε; we should check that C_3 remains bounded as $\varepsilon \to 0$). The constraints on $\widetilde{\lambda}_\varepsilon(0)$ issued from the proof of Lemma C.8 do not depend on ε, it is therefore possible to choose $\widetilde{\lambda}_\varepsilon(0)$ independently of ε. Observe that the definition (71) involves C_3 (and thus $\widetilde{\lambda}_\varepsilon(0)$) in the argument of the exponential, but this does not impose further constraint on $\widetilde{\lambda}_\varepsilon(0)$. In what follows, we thus fix this quantity as to be independent on ε; from now on it is denoted μ_0 (and the issue of the dependence of C_3 with respect to ε is equally answered).

We conclude by observing that

$$\begin{aligned}
\theta_\varepsilon(t) \underset{\varepsilon\to 0^+}{\longrightarrow} \theta(t) &= \left(C_2\langle T\rangle \|g(t)\|_{H^\sigma_P} + C_3(T,\widetilde{\lambda}(0)) \|\nabla_v \mathscr{M}\|_{\mathcal{G}_P^{2\widetilde{\lambda}(0),\sigma;1}} \right) \\
&\qquad \times \left(\mathscr{E}_I + \int_0^t \|g(\tau)\|^2_{H^\sigma_P} \,\mathrm{d}\tau \right)^{1/2} \|g(t)\|_{H^\sigma_P}, \\
Y_\varepsilon(t) \underset{\varepsilon\to 0^+}{\longrightarrow} Y(t) &= \|g(0)\|^2_{\mathcal{G}_P^{\mu_0,\sigma;1}} + 2\int_0^t \theta(\tau)\,\mathrm{d}\tau, \\
\widetilde{\lambda}_\varepsilon(t) \underset{\varepsilon\to 0^+}{\longrightarrow} \widetilde{\lambda}(t) &= \mu_0 \exp\left\{ -\int_0^t \left[C_1\langle T\rangle^2 \left(\mathscr{E}_I + \int_0^t Y(\tau)\,\mathrm{d}\tau \right)^{1/2} \right] \right\}.
\end{aligned}$$

By applying Fatou's lemma we finally obtain

$$\|g(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}(t),\sigma;1}} \le \liminf_{\varepsilon\to 0^+} \|g_\varepsilon(t)\|^2_{\mathcal{G}_P^{\widetilde{\lambda}_\varepsilon(t),\sigma;1}} \le \liminf_{\varepsilon\to 0^+} Y_\varepsilon(t) = Y(t).$$

■

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