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# Limit theorem for perturbed random walks

Hoang-Long Ngo\*    Marc Peigné †

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**Abstract** We consider random walks perturbed at zero which behave like (possibly different) random walks with i.i.d. increments on each half lines and restarts at 0 whenever they cross that point. We show that the perturbed random walk, after being rescaled in a proper way, converges to a skew Brownian motion whose parameter is defined by renewal functions of the simple random walks and the transition probabilities from 0.

**Keywords** Invariance principle, Reflected Brownian motion, Renewal function, Skew Brownian motion

**Mathematics Subject Classification (2010)** 60F17 60M50

## 1 Introduction

Let  $(S(n))_{n \geq 0}$  be a random walks on  $\mathbb{Z}$  starting from 0 whose increments  $\xi_k, k \geq 1$ , are i.i.d with finite second moment. A continuous time process  $(S(t))_{t \geq 0}$  can be constructed from the sequence  $(S(n))_{n \geq 0}$  by using the linear interpolation between the values at integer points. According to the well-known Donsker's theorem, the sequence of stochastic processes  $(S_n(t))_{n \geq 1}$ , defined by

$$S_n(t) := \frac{1}{\sqrt{n}} S(nt), \quad n \geq 0,$$

weakly converges in the space continuous functions on  $[0, 1]$  to the Wiener process, as  $n \rightarrow +\infty$ .

In this paper, we consider the more general cases of spatially inhomogeneous random walks  $(X(n))_{n \geq 0}$  on  $\mathbb{Z}$  which model some discrete time diffusion in a one dimensional space with two different media  $\mathbb{Z}^-$  and  $\mathbb{Z}^+$  and a barrier  $\{0\}$ . Let us state the main result of the present article.

**Theorem 1.** *Assume that*

1.  $(\xi_n)_{n \geq 1}$  and  $(\xi'_n)_{n \geq 1}$  are sequences of  $\mathbb{Z}$ -valued, centered and i.i.d. random variables with finite second moments and respective variances  $\sigma^2$  and  $\sigma'^2$ ;

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2. the supports of the distribution of the  $\xi_n$  and  $\xi'_n$  are not included in the coset of a proper subgroup of  $\mathbb{Z}$  (aperiodicity condition);
3.  $(\eta_n)_{n \geq 1}$  is a sequence of  $\mathbb{Z}$ -valued and i.i.d. random variables such that  $\mathbb{E}[|\eta_n|] < +\infty$ , and  $\mathbb{P}[\eta_n = 0] < 1$ ;
4. the sequences  $(\xi_n)_{n \geq 1}$ ,  $(\xi'_n)_{n \geq 1}$  and  $(\eta_n)_{n \geq 1}$  are independent.

Let  $(X(n))_{n \geq 0}$  be the  $\mathbb{Z}$ -valued process defined by:  $X(0) = 0$  and, for  $n \geq 1$ ,

$$X(n) = \begin{cases} X(n-1) + \xi_n & \text{if } X(n-1) > 0 \text{ and } X(n-1) + \xi_n > 0, \\ 0 & \text{if } X(n-1) > 0 \text{ and } X(n-1) + \xi_n \leq 0, \\ \eta_n & \text{if } X(n-1) = 0, \\ X(n-1) + \xi'_n & \text{if } X(n-1) < 0 \text{ and } X(n-1) + \xi'_n < 0, \\ 0 & \text{if } X(n-1) < 0 \text{ and } X(n-1) + \xi'_n \geq 0. \end{cases} \quad (1)$$

Let  $(X(t))_{t \geq 0}$  be the continuous time process constructed from the sequence  $(X(n))_{n \geq 0}$  by linear interpolation between the values at integer points, and, for any  $n \geq 1, t \geq 0$ , set

$$X_n(t) := \begin{cases} \frac{1}{\sigma\sqrt{n}}X(nt) & \text{when } X(nt) \geq 0, \\ \frac{1}{\sigma'\sqrt{n}}X(nt) & \text{when } X(nt) \leq 0. \end{cases}$$

Then, as  $n \rightarrow +\infty$ , the sequence of stochastic processes  $(X_n(t))_{n \geq 1}$  weakly converges in the space of continuous functions on  $[0, +\infty)$  to the skew Brownian motion  $(B^\alpha(t))_{t \geq 0}$  on  $\mathbb{R}$  with parameter  $\alpha$  depending on the distribution of the  $\xi_n, \xi'_n$  and  $\eta_n$  as follows:

$$\alpha = \frac{c_1 \mathbb{E}[h(\eta_1) \mathbf{1}_{\{\eta_1 > 0\}}]}{c_1 \mathbb{E}[h(\eta_1) \mathbf{1}_{\{\eta_1 > 0\}}] + c'_1 \mathbb{E}[h'(-\eta_1) \mathbf{1}_{\{\eta_1 < 0\}}]}, \quad (2)$$

where

1.  $h$  is the “descending renewal function”<sup>1</sup> of the random walk  $(S(n))_{n \geq 0}$  with increments  $\xi_k$ ;
2.  $c_1 = \frac{1}{\sqrt{\pi}} \exp \left( \sum_{k=1}^{+\infty} \frac{1}{k} \left( \mathbb{P}[S(k) \geq 0] - \frac{1}{2} \right) \right)$ ;
3.  $h'$  is the “ascending renewal function” of the random walk  $(S'(n))_{n \geq 0}$  with increments  $\xi'_k$ ;
4.  $c'_1 = \frac{1}{\sqrt{\pi}} \exp \left( \sum_{k=1}^{+\infty} \frac{1}{k} \left( \mathbb{P}[S'(k) \leq 0] - \frac{1}{2} \right) \right)$ .

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<sup>1</sup>We refer to sections 2.2 and 4 for the definition of the “renewal function” of an oscillating random walk.

The Markov chain  $(X(n))_{n \geq 0}$  has been the object of several studies in various contexts.

It first appeared in [10] where the skew Brownian motion (SBM) on  $\mathbb{R}$  is obtained as the weak limit of a normalized simple random walk on  $\mathbb{Z}$  which has special behavior only at the origin; in this seminal work, the authors consider the case when the  $\xi_n$  and  $\xi'_n$  are Bernoulli symmetric random variables and the variables  $\eta_k$  are also  $\{-1, 1\}$  valued, but with respective probabilities  $1 - \alpha$  and  $\alpha$ . The key step of the proof in [10] relies on the reflection principle (but the convergence of the finite dimensional distributions is not done in details), which is valid only for Bernoulli symmetric random walks. Let us cite also the long and rich article [15] by A. Lejay on the construction of the skew Brownian motion, where a slightly different argument is also presented: the approximation of the SBM( $\alpha$ ) by Bernoulli symmetric random walks is obtained starting from a trajectory of the SBM( $\alpha$ ) and constructing recursively a suitable sequence of stopping times. Both proofs only work for Bernoulli symmetric random walks. Note also that Harrison and Shepp mentioned without a proof in [10] that such a result holds for arbitrary integrable random variables  $\eta_n$ ; Theorem 1 covers (and extends) this case.

More recently, in [18], the SBM appears as the weak limit of a random walk in  $\mathbb{Z}$  whose transition probabilities coincide with those of a symmetric random walk with unit steps throughout except for a fixed neighborhood  $\{-m, \dots, m\}$  of zero, called a “membrane”. Assuming that, from the membrane, the chain jumps to an arbitrary point of the set  $\{-m - 1, \dots, m + 1\}$ , the authors describes the different possible weak limit of the suitably scale process according to the fact that the sites  $-m - 1$  and  $m + 1$  are essential or not. All possible limits for the corresponding random walks are described. In [12], it is proved that this convergence holds in fact for random walks such that the absolute value  $|\xi_k|$  of the steps outside the membrane are bounded by  $2m + 1$ ; this last condition ensures that the random walk cannot jump over the membrane without passing through it. A. Iksanov and A. Pilipenko offer in [12] a new approach which is based on the martingale characterization of the SBM. In the present paper, the membrane is reduced to  $\{0\}$  and the assumption of “bounded jumps out of the membrane” is replaced by the absorption condition at 0; nevertheless, Theorem 1 may be extended to a finite membrane, with similar conditions as in [18].

Let us mention that the expression of the parameter  $\alpha$  is different of the one proposed in [18]; it takes into account in an explicit form the inhomogeneity of the random walk on  $\mathbb{Z}$ . In the case when the two random walks are Bernoulli symmetric<sup>2</sup>, it holds  $h(-x) = h'(x) = x$  for any  $x \geq 0$  and  $c_1 = c'_1$ ; thus,  $\alpha = \frac{\mathbb{E}[(\eta_1)^+]}{\mathbb{E}[|\eta_1|]}$  as announced in [10].

When  $\mathcal{L}(\xi_1) = \mathcal{L}(-\xi'_1)$  and  $\eta_1$  is symmetric (ie  $\mathcal{L}(\eta_1) = \mathcal{L}(-\eta_1)$ ), the process  $(|X(n)|)_{n \geq 0}$  above coincides with the process  $(Y(n))_{n \geq 0}$  defined by  $Y(0) = 0$  and, for  $n \geq 1$ ,

$$Y(n) = \begin{cases} Y(n-1) + \xi_n & \text{if } Y(n-1) > 0 \text{ and } Y(n-1) + \xi_n > 0, \\ 0 & \text{if } Y(n-1) > 0 \text{ and } Y(n-1) + \xi_n \leq 0, \\ \gamma_n & \text{if } Y(n-1) = 0 \end{cases} \quad (3)$$

---

<sup>2</sup>The Bernoulli symmetric random walk is not aperiodic and thus does not fall exactly within the scope of Theorem 1; nevertheless, statements of Lemmas 3 and 4 still hold in this case and the rest of our proof works. The general non aperiodic case remains an open question; in particular, as far as we know, Lemmas 3 and 4 are still unknown in the non aperiodic case, except for particular distributions.

where  $(\gamma_k)_{k \geq 1}$  is a sequence of i.i.d.  $\mathbb{N}$ -valued random variables, independent on the sequence  $(\xi_n)_{n \geq 1}$ . This process  $(Y(n))_{n \geq 0}$  is a variation of the so-called *Lindley process*  $(L(n))_{n \geq 0}$ , which appears in queuing theory and corresponds to the case when the random variables  $\xi_n$  and  $\gamma_n$  are equal. Recall that the Lindley process is a fundamental example of processes governed by iterated functions systems ([7], [17]); indeed, setting  $f_a(x) := \max(x + a, 0)$  for any  $a, x \in \mathbb{R}$ , the quantity  $L(n)$  equals (recall  $\xi_n = \gamma_n$  for the Lindely process)

$$L(n) = f_{\xi_n} \circ \cdots \circ f_{\xi_2} \circ f_{\xi_1}(0).$$

The composition of the maps  $f_a$  is not commutative, this introduces several difficulties. Namely, between two consecutive visits of 0, the Lindley process behaves like a classical random walk  $(S(n))_{n \geq 1}$ ; thus, its study relies on fluctuations of random walks on  $\mathbb{Z}$ . In the case of  $(Y(n))_{n \geq 0}$ , the fact that after each visit of 0 the increment is governed by another distribution, the one of the  $\gamma_k$ , introduces some kind of inhomogeneity we have to control.

**Corollary 2.** *Assume that*

1.  $(\xi_n)_{n \geq 1}$  is a sequence of  $\mathbb{Z}$ -valued i.i.d. random variables, and  $\mathbb{E}[\xi_n] = 0$  and  $\mathbb{E}[\xi_n^2] = \sigma^2 < \infty$ ;
2. the support of the distribution of  $\xi_n$  is not included in the coset of a proper subgroup of  $\mathbb{Z}$ ;
3.  $(\gamma_n)_{n \geq 1}$  is a sequence of  $\mathbb{N}$ -valued and i.i.d. random variables such that  $\mathbb{E}[\gamma_n] < +\infty$ , and  $\mathbb{P}[\gamma_n = 0] < 1$ ;
4. the sequences  $(\xi_n)_{n \geq 1}$  and  $(\gamma_n)_{n \geq 1}$  are independent.

Let  $(Y(t))_{t \geq 0}$  be the continuous time process constructed from the sequence  $(Y(n))_{n \geq 0}$  by linear interpolation between the values at integer points. Then, as  $n \rightarrow +\infty$ , the sequence of stochastic processes  $(Y_n(t))_{n \geq 1}$ , defined by

$$Y_n(t) := \frac{1}{\sigma\sqrt{n}}Y(nt), \quad n \geq 1, 0 \leq t \leq 1,$$

weakly converges in the space of continuous functions on  $[0, 1]$  to the absolute value  $(|B(t)|)_{t \geq 0}$  of the Brownian motion on  $\mathbb{R}$ .

This process has been already considered in the literature. Let us cite for instance [19] where the jumps  $\xi_k$  are bounded from below by  $-1$  but the  $\gamma_k$  are  $\mathbb{N}$ -valued with distribution in the domain of attraction of some stable distribution (possible without first moment).

We follow the classical strategy to prove invariance principles [4]. Firstly, we show that the finite dimensional distribution of the processes  $(X_n)_{n \geq 0}$  and  $(Y_n)_{n \geq 0}$  do converge to the suitable limit, then the tightness of these sequences of processes. This ‘‘pedestrian’’ approach is of interest as soon as we have a precise control of both the fluctuations of the random walks on each half line  $\mathbb{Z}^-$  and  $\mathbb{Z}^+$  and the steps starting from 0; in particular, it is quite flexible to study processes whose trajectories cannot be decomposed exactly into independent pieces, as for instance the reflected random walk on  $\mathbb{Z}^+$  with elastic reflections at 0.

We feel it is easier to prove Corollary 2, in which case there is only the random walks  $(S(n))_{n \geq 0}$  to consider; this reduces considerably the notations. Thus, the paper is organized as follows: in Section 2 we recall some classical results on fluctuations of random walks and their consequences, Section 3 is devoted to the proof of Corollary 2 and in Section 4 we explain how to adapt this argument to prove the main theorem.

## 2 Notations and auxiliary estimates

### 2.1 Notations

Firstly, let us fix notations. We denote  $S = (S(n))_{n \geq 0}$  the classical random walks with steps  $\xi_k$  defined by  $S(0) = 0$  and  $S(n) = \xi_1 + \dots + \xi_n$  for any  $n \geq 1$  and by  $\mathcal{F}_n$  the  $\sigma$ -field generated by the random variables  $\xi_1, \gamma_1, \dots, \xi_n, \gamma_n$  for  $n \geq 1$ ; by convention  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ .

Since the random variables  $\xi_k$  are centered, the process  $(Y(n))_{n \geq 0}$  visits 0 infinitely often; thus, we introduce the following sequences  $(\tau_l^Y)_{l \geq 0}$  of stopping times with respect to the filtration  $(\mathcal{F}_n)_{n \geq 0}$  defined by  $\tau_0^Y = 0$  and, for any  $l \geq 0$ ,

$$\tau_{l+1}^Y = \inf\{n > \tau_l^Y \mid Y(n) = 0\}.$$

In order to establish an invariance principle for the process  $(Y(n))_{n \geq 0}$ , we control the excursions of this process between two visits of 0; after each visit, the first transition jump has distribution  $\mathcal{L}(\gamma_1)$ , it is thus natural to introduce the following random variables  $U_{k,n}$ ,  $0 \leq k \leq n$ , defined by

$$U_{k,n} := \gamma_{k+1} + \xi_{k+2} + \dots + \xi_n.$$

Furthermore, the excursions of  $(Y(n))_{n \geq 0}$  between two visits of 0 coincide with some parts of the trajectory of  $(S(n))_{n \geq 0}$ , thus their study is related to the fluctuations of this random walk. Hence, we need also to introduce the sequence of strictly descending ladder epochs  $(\ell_l)_{l \geq 0}$  of the random walk  $S = (S(n))_{n \geq 0}$  defined inductively by  $\ell_0 = 0$  and, for any  $l \geq 1$ ,

$$\ell_{l+1} := \min\{n > \ell_l \mid S(n) < S(\ell_l)\}.$$

The random variables  $\ell_1, \ell_2 - \ell_1, \ell_3 - \ell_2, \dots$  are  $\mathbb{P}$ -a.s. finite and i.i.d.. The same property holds for the random variables  $S(\ell_1), S(\ell_2) - S(\ell_1), S(\ell_3) - S(\ell_2), \dots$ ; in other words, the processes  $(\ell_l)_{l \geq 0}$  and  $(S(\ell_l))_{l \geq 0}$  are random walks on  $\mathbb{N}$  and  $\mathbb{Z}$  with respective distribution  $\mathcal{L}(\ell_1)$  and  $\mathcal{L}(S(\ell_1))$ .

We denote by  $p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$  the Gaussian density. The transition density of a skew Brownian motion with parameter  $\alpha \in [0, 1]$  is given by

$$\begin{aligned} p_t^\alpha(x, y) &= \mathbf{1}_{\{x=0\}} [2\alpha p_t(y) \mathbf{1}_{\{y>0\}} + 2(1-\alpha) p_t(y) \mathbf{1}_{\{y<0\}}] \\ &+ \mathbf{1}_{\{x>0\}} [(p_t(y-x) + (2\alpha-1)p_t(x+y)) \mathbf{1}_{\{y>0\}} + 2(1-\alpha)p_t(y-x) \mathbf{1}_{\{y<0\}}] \\ &+ \mathbf{1}_{\{x<0\}} [(p_t(y-x) + (1-2\alpha)p_t(y+x)) \mathbf{1}_{\{y<0\}} + 2\alpha p_t(y-x) \mathbf{1}_{\{y>0\}}], \end{aligned} \quad (4)$$

(see [20], page 87).

At last, in order to simplify the text, we use the following notations. Let  $u = (u_n)_{n \geq 0}$  and  $v = (v_n)_{n \geq 0}$  be two sequences of positive reals; we write

- $u \stackrel{c}{\preceq} v$  (or simply  $u \preceq v$ ) when  $u_n \leq cv_n$  for some constant  $c > 0$  and  $n$  large enough;
- $u_n \sim v_n$  if  $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$ .
- $u_n \approx v_n$  if  $\lim_{n \rightarrow \infty} (u_n - v_n) = 0$ .

## 2.2 On the fluctuation of random walks

Let  $h$  be the Green function of the random walk  $(S(\ell_l))_{l \geq 0}$ , called sometimes the “descending renewal function” of  $S$ , defined by

$$h(x) = \begin{cases} 1 + \sum_{l=1}^{+\infty} \mathbb{P}[S(\ell_l) \geq -x] & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

The function  $h$  is harmonic for the random walk  $(S(n))_{n \geq 0}$  killed when it reaches the negative half line  $(-\infty; 0]$ ; namely, for any  $x \geq 0$ ,

$$\mathbb{E}[h(x + \xi_1); x + \xi_1 > 0] = h(x).$$

This holds for any oscillating random walk, possible without finite second moment.

Moreover, the function  $h$  is increasing,  $h(0) = 1$  and  $h(x) = O(x)$  as  $x \rightarrow \infty$  (see [2], p. 648).

We have also to take into account that the random walk  $S$  may start from another point than the origin; hence, for any  $x \geq 0$ , we set  $\tau^S(x) := \inf\{n \geq 1 : x + S(n) \leq 0\}$ ; it holds

$$[\tau^S(x) > n] = [L_n > -x]$$

where  $L_n = \min(S(1), \dots, S(n))$ .

The following result is a consequence of Theorem A in [14] and Theorem II.6 in [16]. Recall that  $c_1$  was defined in Theorem 1.

**Lemma 3.** *For any  $x \geq 0$ , it holds that*

$$\mathbb{P}[\tau^S(x) > n] \sim c_1 \frac{h(x)}{\sqrt{n}} \quad \text{as } n \rightarrow +\infty. \quad (6)$$

Moreover, there exists a constant  $C_1 > 0$  such that for any  $x \geq 0$  and  $n \geq 1$ ,

$$\mathbb{P}[\tau^S(x) > n] \leq C_1 \frac{h(x)}{\sqrt{n}}. \quad (7)$$

Let us emphasize that, despite appearances, inequality (7) is more difficult to obtain than equivalence (6). Let denote by  $\tilde{h}$  the ascending renewal function of  $S$ , that is also the descending renewal function of the random walk  $\tilde{S} = -S$  defined as in (5). The following statement holds.

**Lemma 4.** *There exist constants  $c_2, C_2 > 0$  such that, for any  $x, y \geq 0$ ,*

$$\mathbb{P}[\tau^S(x) > n, x + S(n) = y] \sim c_2 \frac{h(x)\tilde{h}(y)}{n^{3/2}} \quad \text{as } n \rightarrow +\infty, \quad (8)$$

and, for any  $n \geq 1$ ,

$$\mathbb{P}[\tau^S(x) > n, x + S(n) = y] \leq C_2 \frac{h(x)\tilde{h}(y)}{n^{3/2}}. \quad (9)$$

The constants  $c_1$  and  $c_2$  are closely connected, see (10) below.

*Proof.* The first assertion follows from Proposition 11 in [9] (see also Theorem II.7 in [16]). Inequality (9) may be deduced from Lemma 3 following the strategy of Proposition 2.3 in [1]. Let us detail this argument.

For any  $n \geq 1$  and  $1 \leq k \leq n$ , set  $\tilde{S}_k = -(\xi_n + \dots + \xi_{n-k+1})$  and  $\tilde{L}_k = \min(\tilde{S}_1, \dots, \tilde{S}_k)$ . We set  $A_{n,x} = [L_{[n/3]} > -x]$ ,  $\tilde{A}_{n,y} = [\tilde{L}_{[n/3]} > -y]$  and  $\bar{A}_{n,x,y} = [x + S(n) = y]$ . It holds

$$\begin{aligned} [\tau^S(x) > n, x + S(n) = y] &= [S(1) > -x, \dots, S(n) > -x, x + S(n) = y] \\ &\subset A_{n,x} \cap \tilde{A}_{n,y} \cap \bar{A}_{n,x,y}. \end{aligned}$$

On the one hand, the events  $A_{n,x}$  and  $\tilde{A}_{n,y}$  are independent and measurable with respect to the  $\sigma$ -field  $\mathcal{A}_n$  generated by  $\xi_1, \dots, \xi_{[n/3]}$  and  $\xi_{[2n/3]+1}, \dots, \xi_n$ . On the other hand, the random variable  $T_n = S(n) + S([n/3]) - S([2n/3])$  is  $\mathcal{A}_n$ -measurable and the event  $\bar{A}_{n,x,y}$  may be rewritten as

$$\bar{A}_{n,x,y} = [S([2n/3]) - S([n/3]) = -x + y - T_n].$$

By the classical local limit theorem in  $\mathbb{R}$ , there exists a constant  $C > 0$  such that

$$\mathbb{P}[\bar{A}_{n,x,y} \mid \mathcal{A}_n] \leq \sup_{z \in \mathbb{R}} \mathbb{P}[S([2n/3]) - S([n/3]) = z] \leq \frac{C}{\sqrt{n}}.$$

Moreover, by Lemma 3,

$$\mathbb{P}[A_{n,x}] = \mathbb{P}[\tau^S(x) > [n/3]] \leq 2C_1 \frac{h(x)}{\sqrt{n}},$$

and

$$\mathbb{P}[\tilde{A}_{n,y}] = \mathbb{P}[\tau^{\tilde{S}}(y) > [2n/3] - [n/3]] \leq 2C_1 \frac{\tilde{h}(y)}{\sqrt{n}}.$$

Inequality (9) follows with  $C_2 = 4CC_1^2$ . □

Lemma 4 is the key point to control the behavior as  $n \rightarrow +\infty$  of the probability of the events  $[\tau^S(x) = n], x \geq 0$ . The following result is more precise than Lemma 3.

**Lemma 5.** *For any  $x \geq 0$ , as  $n \rightarrow +\infty$ ,*

$$\mathbb{P}[\tau^S(x) = n] \sim \frac{c_1}{2} h(x) \frac{1}{n^{3/2}} \quad \text{as } n \rightarrow +\infty,$$

and there exists a constant  $C_3 > 0$  such that, for any  $x \geq 0$  and  $n \geq 1$ ,

$$\mathbb{P}[\tau^S(x) = n] \leq C_3 \frac{h(x)}{n^{3/2}}.$$



*Proof.* We write

$$\begin{aligned}\mathbb{P}[\tau^S(x) = n] &= \sum_{y \geq 1} \mathbb{P}[\tilde{\tau}^S(x) > n - 1, x + S(n - 1) = y, y + \xi_n \leq 0] \\ &= \sum_{y \geq 1} \mathbb{P}[\tau^S(x) > n - 1, x + S(n - 1) = y] \mathbb{P}[y + \xi_1 \leq 0].\end{aligned}$$

From Lemma 4 and the fact that  $\tilde{h}(y) = O(y)$  as  $y \rightarrow +\infty$ , for any  $y \geq 1$ ,

$$\begin{aligned}n^{3/2} \mathbb{P}[\tau^S(x) > n - 1, x + S(n - 1) = y] \mathbb{P}[\xi_1 \leq -y] \\ \leq h(x) \tilde{h}(y) \mathbb{P}[y + \xi_1 \leq 0] \\ \leq h(x) (1 + y) \mathbb{P}[y + \xi_1 \leq 0],\end{aligned}$$

with  $\sum_{y \geq 1} (1 + y) \mathbb{P}[y + \xi_1 \leq 0] < +\infty$  since  $\mathbb{E}(\xi_1^2) < +\infty$ .

Thus, the second assertion of the Lemma holds; with (9) and using the dominated convergence theorem, it yields

$$\begin{aligned}\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}[\tau^S(x) = n] \\ = \lim_{n \rightarrow +\infty} n^{3/2} \sum_{y \geq 1} \mathbb{P}[\tau^S(x) > n - 1, S(n - 1) + x = y] \mathbb{P}[y + \xi_1 \leq 0] \\ = c_2 h(x) \sum_{y \geq 1} \tilde{h}(y) \mathbb{P}[y + \xi_1 \leq 0].\end{aligned}$$

Comparing with Lemma 3, the constants  $c_1$  and  $c_2$  satisfy the equality

$$c_1 = 2c_2 \sum_{y \geq 1} \tilde{h}(y) \mathbb{P}[y + \xi_1 \leq 0] \quad (10)$$

and the first assertion of the Lemma follows. □

### 2.3 On the fluctuation of the perturbed random walk $Y$

Firstly, notice that the sequence  $(\tau_l^Y)_{l \geq 0}$  is a strictly increasing random walk on  $\mathbb{N}$ , with i.i.d. increments distributed as  $\tau_1^Y$ . Thus, its potential  $\sum_{l=1}^{+\infty} \mathbb{P}[\tau_l^Y = n]$  is finite for any  $n \geq 0$ . In this subsection, we prove that the distributions of  $\tau^S(x)$  and  $\tau_1^Y$  have the same behavior at infinity and we describe the behavior as  $n \rightarrow +\infty$  of the potential of the random walk  $(\tau_l^Y)_{l \geq 0}$ .

**Lemma 6.** *It holds, as  $n \rightarrow +\infty$ ,*

$$\mathbb{P}[\tau_1^Y > n] \sim c_1 \frac{\mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}]}{\sqrt{n}}.$$

*More precisely,*

$$\mathbb{P}[\tau_1^Y = n] \sim \frac{c_1}{2} \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}] \frac{1}{n^{3/2}}.$$

*Proof.* Firstly

$$\begin{aligned}
\mathbb{P}[\tau_1^Y > n] &= \mathbb{P}[\gamma_1 > 0, \gamma_1 + \xi_2 > 0, \dots, \gamma_1 + \xi_2 + \dots + \xi_n > 0] \\
&= \sum_{x>0} \mathbb{P}[x + \xi_2 > 0, \dots, x + \xi_2 + \dots + \xi_n > 0] \mathbb{P}[\gamma_1 = x] \\
&= \sum_{x>0} \mathbb{P}[x + S(1) > 0, \dots, x + S(n-1) > 0] \mathbb{P}[\gamma_1 = x] \\
&= \sum_{x>0} \mathbb{P}[\tau^S(x) > n-1] \mathbb{P}[\gamma_1 = x].
\end{aligned}$$

Recall that  $h(x) = O(x)$  as  $x \rightarrow +\infty$  and  $\mathbb{E}[\gamma_1] < +\infty$ ; then, by Lemma 3, it holds

$$\sum_{x>0} \sup_{n \geq 1} \sqrt{n} \mathbb{P}[\tau^S(x) > n-1] \mathbb{P}[\gamma_1 = x] < 2C_1 \mathbb{E}[h(\gamma_1)] < +\infty,$$

and Lebesgue's dominated convergence theorem yields

$$\lim_{n \rightarrow +\infty} \sqrt{n} \mathbb{P}[\tau_1^Y > n] = c_1 \sum_{x>0} h(x) \mathbb{P}[\gamma_1 = x] = c_1 \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}].$$

To prove the second assertion, and for the convenience, we overestimate  $\mathbb{P}[\tau_1^Y = n+2]$ . It holds, for any  $n \geq 0$ ,

$$\begin{aligned}
\mathbb{P}[\tau_1^Y = n+2] &= \mathbb{P}[\gamma_1 > 0, U_{0,2} > 0, \dots, U_{0,n+1} > 0, U_{0,n+2} \leq 0] \\
&= \sum_{x>0} \mathbb{P}[\gamma_1 = x] \mathbb{P}[x + S(1) > 0, \dots, x + S(n) > 0, x + S(n+1) \leq 0] \\
&= \sum_{x>0} \mathbb{P}[\gamma_1 = x] \sum_{y \leq -1} \mathbb{P}[\xi_{n+1} = y] \mathbb{P}[x + S(1) > 0, \dots, x + S(n) > 0, x + S(n) \leq |y|] \\
&= \sum_{x>0} \mathbb{P}[\gamma_1 = x] \sum_{y \leq -1} \mathbb{P}[\xi_1 = y] \sum_{z=1}^{|y|} \mathbb{P}[\tau^S(x) > n, x + S(n) = z].
\end{aligned}$$

Lemma 4 and the fact that  $\tilde{h}$  is increasing, imply, for any  $y \leq -1$ ,

$$\sum_{z=1}^{|y|} \mathbb{P}[\tau^S(x) > n, x + S(n) = z] \leq C_2 \frac{h(x) |y| \tilde{h}(|y|)}{n^{3/2}},$$

with  $\mathbb{E}[h(\gamma_1)] < +\infty$  and  $\mathbb{E}[|\xi_1| \tilde{h}(|\xi_1|)] < +\infty$ .

Eventually, Lemma 4 and the dominated convergence theorem yield

$$\begin{aligned}
\lim_{n \rightarrow +\infty} n^{3/2} \mathbb{P}[\tau_1^Y = n+2] &= \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}] \sum_{y \leq -1} \left( \sum_{z=1}^{|y|} \tilde{h}(z) \right) \mathbb{P}[\xi_1 = y] \\
&= \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}] \sum_{y \geq 1} \tilde{h}(y) \mathbb{P}[y + \xi_1 \leq 0].
\end{aligned}$$

□

As a direct consequence, we get the following estimation of the Green function of the random walk  $(\tau_l)_{l \geq 0}$ .

**Lemma 7.** *As  $n \rightarrow +\infty$ ,*

$$\Sigma_n := \sum_{l=0}^{+\infty} \mathbb{P}[\tau_l^Y = n] \sim \frac{1}{c_1 \pi \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\gamma_1 > 0\}}]} \frac{1}{\sqrt{n}}.$$

*Proof.* We apply Theorem B in [8]<sup>3</sup> and have to check that

$$\sup_{n \geq 0} \frac{n \mathbb{P}[\tau_1^Y = n]}{\mathbb{P}[\tau_1^Y > n]} < +\infty.$$

This follows from Lemma 6. □

## 2.4 Conditional limit theorem

We recall the following result, which is a consequence of Lemma 2.3 in [2]. The symbol “ $\Rightarrow$ ” means “weak convergence”.

**Lemma 8.** *Let  $(S_t)_{t \geq 0}$  be the continuous time process constructed from the sequence  $(S(n))_{n \geq 0}$  by using the linear interpolation between the values at integer points. Then*

$$\mathcal{L}\left(\left(\frac{S(\lfloor nt \rfloor)}{\sigma \sqrt{n}}\right)_{0 \leq t \leq 1} \mid \min\{S(1), \dots, S(n)\} \geq -x\right) \Rightarrow \mathcal{L}(L^+) \quad \text{as } n \rightarrow +\infty,$$

where  $L^+$  is the Brownian meander.

As a consequence, we propose the following statement.

**Lemma 9.** *Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded, Lipschitz continuous function. Then, for any  $t \in [0, 1]$  and  $x \geq 0$ , it holds*

$$\lim_{n \rightarrow +\infty} \mathbb{E}\left[\varphi\left(\frac{x + S(n - \lfloor nt \rfloor)}{\sigma \sqrt{n}}\right) \mid \tau^S(x) > n - \lfloor nt \rfloor\right] = \int_0^{+\infty} \varphi(z \sqrt{1-t}) z e^{-z^2/2} dz.$$

*Proof.* We first note that

$$\begin{aligned} \frac{1}{n} \mathbb{E}[S(n) \mid \tau^S(x) > n] &= \frac{\mathbb{E}[S(n), \tau^S(x) > n]}{n \mathbb{P}[\tau^S(x) > n]} \\ &\leq \frac{\sqrt{\mathbb{E}[S(n)^2] \mathbb{P}[\tau^S(x) > n]}}{n \mathbb{P}[\tau^S(x) > n]} = \frac{\sigma}{\sqrt{n \mathbb{P}[\tau^S(x) > n]}}. \end{aligned}$$

Applying Lemma 3, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \mathbb{E}[S(n) \mid \tau^S(x) > n] = 0. \tag{11}$$

---

<sup>3</sup>The limit presented in Theorem B in [8] is not correct; in fact, Theorem B is a corollary of Theorem 3 where, after a tedious calculus, the constant  $\pi$  does appear. See also [3], page 2.

Since  $\varphi$  is Lipschitz continuous, we may write, denoting  $[\varphi]$  the Lipschitz coefficient of  $\varphi$ ,

$$\begin{aligned}
& \left| \mathbb{E} \left[ \varphi \left( \frac{x + S(n - [nt])}{\sigma \sqrt{n}} \right) \mid \tau^S(x) > n - [nt] \right] \right. \\
& \quad \left. - \mathbb{E} \left[ \varphi \left( \frac{S(n - [nt])}{\sigma \sqrt{n - [nt]}} \sqrt{1 - t} \right) \mid \tau^S(x) > n - [nt] \right] \right| \\
& \leq [\varphi] \mathbb{E} \left[ \frac{x}{\sigma \sqrt{n}} + \frac{S(n - [nt])}{\sigma \sqrt{n - [nt]}} \left( \sqrt{1 - \frac{[nt]}{n}} - \sqrt{1 - t} \right) \mid \tau^S(x) > n - [nt] \right] \\
& \leq [\varphi] \frac{(x + 1)}{\sigma \sqrt{n}} \left( 1 + \mathbb{E} \left[ \frac{S(n - [nt])}{\sqrt{n - [nt]}} \mid \tau^S(x) > n - [nt] \right] \right) \rightarrow 0 \quad \text{as } n \rightarrow +\infty,
\end{aligned}$$

which derives from (11). Applying Lemma 8, we get the desired result.  $\square$

## 2.5 An useful identity

We achieve this section with a classical result which we use in the sequel.

**Lemma 10.** (Page 17 in [13]) *For any  $\alpha, \beta$  positive, it holds*

$$\int_0^{+\infty} \frac{1}{\sqrt{t}} \exp \left( -\alpha t - \frac{\beta}{t} \right) dt = \sqrt{\frac{\pi}{\alpha}} e^{-2\sqrt{\alpha\beta}}.$$

## 3 Proof of Corollary 2

The proof uses the classical argument for weak convergence in the space  $C[0, 1]$  of continuous functions on  $[0, 1]$  (see [4], chapter 2). Firstly, we prove that the finite dimensional distributions of the process  $(Y_n(t); 0 \leq t \leq 1)$  converge weakly to those of the absolute value  $(|W(t)|)_{t \geq 0}$  of the Brownian motion on  $\mathbb{R}$ , then the tightness of the distributions of this sequence of processes.

Throughout this paper,  $\varphi, \varphi_1$  and  $\varphi_2$  are bounded and Lipschitz continuous functions on  $\mathbb{R}$ .

### 3.1 One-dimensional distribution

In this section, we prove the following lemma concerning the weak convergence of the one-dimensional distributions of the sequence  $(Y_n(t))_{n \geq 1}$ .

**Lemma 11.** *For any  $t \in [0, 1]$ , it holds*

$$\lim_{n \rightarrow +\infty} \mathbb{E} [\varphi(Y_n(t))] = \int_0^{+\infty} \varphi(u) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du = \mathbb{E}[\varphi(|B_t|)],$$

where  $B$  is a standard Brownian motion.

*Proof.* We fix  $t \in (0, 1)$  and firstly decompose the expectation  $\mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right]$  as follows:

$$\begin{aligned}
& \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] \\
& \approx \sum_{k=0}^{[nt]-1} \sum_{l=0}^{+\infty} \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right); \tau_l^Y = k, Y(k+1) > 0, \dots, Y([nt]) > 0 \right] \\
& = \sum_{k=0}^{[nt]-1} \sum_{l=0}^{+\infty} \mathbb{E} \left[ \varphi \left( \frac{\gamma_{k+1} + \xi_{k+2} + \dots + \xi_{[nt]}}{\sigma\sqrt{n}} \right); \tau_l^Y = k, \gamma_{k+1} > 0, \right. \\
& \qquad \qquad \qquad \left. \gamma_{k+1} + \xi_{k+2} > 0, \dots, \gamma_{k+1} + \xi_{k+2} + \dots + \xi_{[nt]} > 0 \right].
\end{aligned}$$

Since the random variables  $\tau_l^Y$  are stopping times, the event  $[\tau_l^Y = k]$  is independent of the variables  $\gamma_{k+1}, \xi_{k+2}, \dots, \xi_{[nt]}$ ; furthermore  $\mathcal{L}(\gamma_{k+1}, \xi_{k+2}, \dots, \xi_{[nt]}) = \mathcal{L}(\gamma_1, \xi_1, \dots, \xi_{[nt]-k-1})$ . Hence, recalling that  $\Sigma_k = \sum_{l=0}^{+\infty} \mathbb{P}[\tau_l^Y = k]$ ,

$$\begin{aligned}
& \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] \\
& \approx \sum_{k=0}^{[nt]-1} \Sigma_k \mathbb{E} \left[ \varphi \left( \frac{\gamma_1 + S([nt] - k - 1)}{\sigma\sqrt{n}} \right); \gamma_1 > 0, \gamma_1 + S(1) > 0, \dots, \right. \\
& \qquad \qquad \qquad \left. \gamma_1 + S([nt] - k - 1) > 0 \right] \\
& = \sum_{k=0}^{[nt]-1} \Sigma_k \mathbb{E} \left[ \mathbb{E} \left( \varphi \left( \frac{x + S([nt] - k - 1)}{\sigma\sqrt{n}} \right); \tau^S(x) > [nt] - k - 1 \right) \Big|_{\gamma_1=x>0} \right] \\
& = \sum_{k=0}^{[nt]-1} \Sigma_k \mathbb{E} \left[ \mathbb{E} \left( \varphi \left( \frac{x + S([nt] - k - 1)}{\sigma\sqrt{n}} \right) \Big| \tau^S(x) > [nt] - k - 1 \right) \right. \\
& \qquad \qquad \qquad \left. \times \mathbb{P}[\tau^S(x) > [nt] - k - 1] \Big|_{\gamma_1=x>0} \right].
\end{aligned}$$

From now on, we use the notation  $\mathbb{E}[\phi(x)|_{\gamma_1=x>0}] := \mathbb{E}[\phi(\gamma_1)\mathbf{1}_{\{\gamma_1>0\}}]$  for any bounded function  $\phi : \mathbb{Z} \rightarrow \mathbb{R}$ . By Lemmas 3 and 7, each term in this sum is  $\mathcal{O}(\frac{1}{\sqrt{n}})$  and thus converges to 0 as  $n \rightarrow +\infty$ . Thus, it is sufficient to control the sum above from  $k = 2$  to  $k = [nt] - 4$ , writing it as the integral of the function  $f_n : (0, t) \rightarrow \mathbb{R}$  defined by: for each  $k = 2, \dots, [nt] - 4$  and

any  $s \in [\frac{k}{n}, \frac{k+1}{n})$ ,

$$\begin{aligned} f_n(s) &= n\Sigma_k \mathbb{E} \left[ \mathbb{E} \left( \varphi \left( \frac{x + S([nt] - k - 1)}{\sigma\sqrt{n}} \right) \middle| \tau^S(x) > [nt] - k - 1 \right) \right. \\ &\quad \left. \times \mathbb{P}[\tau^S(x) > [nt] - k - 1] \middle|_{\gamma_1=x>0} \right] \\ &= n\Sigma_k \mathbb{E} \left[ \mathbb{E} \left( \varphi \left( \frac{x + S([nt] - [ns] - 1)}{\sigma\sqrt{n}} \right) \middle| \tau^S(x) > [nt] - [ns] - 1 \right) \right. \\ &\quad \left. \times \mathbb{P}[\tau^S(x) > [nt] - [ns] - 1] \middle|_{\gamma_1=x>0} \right], \end{aligned}$$

and  $f_n(s) = 0$  on  $[0, \frac{2}{n})$  and  $[\frac{[nt]-1}{n}, t)$ . Hence,

$$\mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] = \int_0^t f_n(s) ds + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).$$

Applying Lemmas 3, 7 and 9, we have: for any  $s \in (0, t)$ ,

$$\lim_{n \rightarrow +\infty} f_n(s) = \frac{1}{\pi} \frac{1}{\sqrt{s(t-s)}} \int_0^{+\infty} \varphi(z\sqrt{t-s}) z e^{-z^2/2} dz.$$

On the other hand, for  $2 \leq [ns] \leq [nt] - 4$ ,

$$\begin{aligned} |f_n(s)| &\leq C_1 \|\varphi\|_\infty \mathbb{E}[h(\gamma_1)] \frac{n}{\sqrt{[ns]([nt] - [ns] - 1)}} \sqrt{[ns] \Sigma_{[ns]}} \\ &\preceq \frac{n}{\sqrt{[ns]([nt] - [ns] - 1)}} \\ &\leq \frac{2}{\sqrt{s(t-s)}} \end{aligned}$$

(it is in the last inequality that we use the fact that  $2 \leq [ns] \leq [nt] - 4$ ). Therefore, from Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] &= \lim_{n \rightarrow +\infty} \int_0^t f_n(s) ds \\ &= \frac{1}{\pi} \int_0^t \frac{1}{\sqrt{s(t-s)}} \left( \int_0^{+\infty} \varphi(z\sqrt{t-s}) z e^{-z^2/2} dz \right) ds \end{aligned}$$

Denote  $s = ta$ , we have

$$\begin{aligned} &\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] \\ &= \frac{1}{\pi} \int_0^1 \frac{da}{\sqrt{a(1-a)}} \int_0^{+\infty} \varphi(z\sqrt{t(1-a)}) z e^{-z^2/2} dz \\ &= \frac{1}{\pi} \int_0^1 \frac{da}{\sqrt{a(1-a)}} \int_0^{+\infty} \varphi(x) \frac{x}{t(1-a)} \exp \left( -\frac{x^2}{2t(1-a)} \right) dx \\ &= \frac{1}{\pi} \int_0^{+\infty} \varphi(x) \frac{x}{t} dx \int_0^1 \frac{da}{\sqrt{a(1-a)}^3} \exp \left( -\frac{x^2}{2t(1-a)} \right) da, \end{aligned}$$

where we change the variable  $x = z\sqrt{t(1-a)}$ . Using Lemma 10, we get

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] = \int_0^{+\infty} \varphi(x) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du.$$

We achieve the proof of Lemma 11 by noting that since  $\varphi$  is Lipschitz continuous, it holds that

$$\begin{aligned} \left| \mathbb{E} \left[ \varphi \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] - \mathbb{E} [\varphi(Y_n(t))] \right| &\leq [\varphi] \mathbb{E} \left[ \left| \frac{Y([nt])}{\sigma\sqrt{n}} - Y_n(t) \right| \right] \\ &\leq \frac{1}{\sigma\sqrt{n}} [\varphi] \mathbb{E} [|\xi_{[nt]+1}| + |\gamma_{[nt]+1}|] \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned} \quad (12)$$

□

### 3.2 Two-dimensional distribution

The convergence of the finite-dimensional distributions of  $(Y_n(t))_{n \geq 1}$  is more delicate. We detail the argument for two-dimensional ones, the general case may be treated in a similar way.

Let us fix  $0 < s < t, n \geq 1$  and denote

$$\kappa = \kappa(n, s) = \min\{k > [ns] : Y(k)=0\}.$$

We write

$$\mathbb{E} \left[ \varphi_1 \left( \frac{Y([ns])}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \right] = A_1(n) + A_2(n),$$

where

$$\begin{aligned} A_1(n) &= \sum_{k=[ns]+1}^{[nt]} \mathbb{E} \left[ \varphi_1 \left( \frac{Y([ns])}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \mathbf{1}_{\{\kappa=k\}} \right], \\ A_2(n) &= \mathbb{E} \left[ \varphi_1 \left( \frac{Y([ns])}{\sigma\sqrt{n}} \right) \varphi_2 \left( \frac{Y([nt])}{\sigma\sqrt{n}} \right) \mathbf{1}_{\{\kappa > [nt]\}} \right]. \end{aligned}$$

The term  $A_1(n)$  deals with the trajectories of  $Y_k, 0 \leq k \leq n$ , which visit 0 between  $[ns] + 1$  and  $[nt]$  while  $A_2(n)$  concerns the others trajectories.

### 3.2.1 Estimate of $A_1(n)$

As in the previous section, we decompose  $A_1(n)$  as

$$\begin{aligned}
A_1(n) &= \sum_{k_2=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \varphi_1 \left( \frac{Y(\lfloor ns \rfloor)}{\sigma\sqrt{n}} \right) \mathbf{1}_{\{\kappa=k_2\}} \right] \mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - k_2)}{\sigma\sqrt{n}} \right) \right] \\
&\approx \sum_{l=0}^{+\infty} \sum_{k_1=0}^{\lfloor ns \rfloor-1} \sum_{k_2=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - k_2)}{\sigma\sqrt{n}} \right) \right] \\
&\quad \times \mathbb{E} \left[ \varphi_1 \left( \frac{Y(\lfloor ns \rfloor)}{\sigma\sqrt{n}} \right); \tau_l^Y = k_1, \kappa = k_2; \gamma_{k_1+1} > 0, U_{k_1, k_1+2} > 0, \dots, U_{k_1, \lfloor ns \rfloor} > 0 \right] \\
&= \sum_{l=0}^{+\infty} \sum_{k_1=0}^{\lfloor ns \rfloor-1} \sum_{k_2=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbb{P}[\tau_l^Y = k_1] \mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - k_2)}{\sigma\sqrt{n}} \right) \right] \\
&\quad \times \mathbb{E} \left[ \varphi_1 \left( \frac{Y(\lfloor ns \rfloor - k_1)}{\sigma\sqrt{n}} \right); \gamma_1 > 0, U_{0,2} > 0, \dots, U_{0, k_2 - k_1 - 1} > 0, U_{0, k_2 - k_1} \leq 0 \right] \\
&= \sum_{k_1=0}^{\lfloor ns \rfloor-1} \sum_{k_2=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - k_2)}{\sigma\sqrt{n}} \right) \right] \\
&\quad \times \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma\sqrt{n}} \right); \tau^S(x) = k_2 - k_1 - 1 \right] \Big|_{\gamma_1=x>0} \right].
\end{aligned}$$

For any  $2 \leq k_1 < \lfloor ns \rfloor - 6$  and  $\lfloor ns \rfloor \leq k_2 \leq \lfloor nt \rfloor$  and any  $s_1 \in [\frac{k_1}{n}, \frac{k_1+1}{n})$  and  $s_2 \in [\frac{k_2}{n}, \frac{k_2+1}{n})$ , we write

$$\begin{aligned}
f_n(s_1, s_2) &= n^2 \sum_{k_1} \mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - \lfloor ns_2 \rfloor)}{\sigma\sqrt{n}} \right) \right] \\
&\quad \times \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - \lfloor ns_1 \rfloor - 1)}{\sigma\sqrt{n}} \right); \tau^S(x) = \lfloor ns_2 \rfloor - \lfloor ns_1 \rfloor - 1 \right] \Big|_{\gamma_1=x>0} \right],
\end{aligned}$$

and  $f_n(s_1, s_2) = 0$  for the others values of  $k_1$ , such that  $0 \leq k_1 \leq \lfloor ns \rfloor$ . Hence,

$$A_1(n) = \int_0^s ds_1 \int_s^t ds_2 f_n(s_1, s_2) + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right).$$

Let us study the pointwise limit of the sequence  $(f_n)_{n \geq 1}$  as  $n \rightarrow +\infty$ . This is here that we apply the fact that  $\gamma$  and  $\xi$  take integer values, in order to apply Caravenna-Chaumont's result [6]. We may also consider the case when  $\gamma$  and  $\xi$  have absolutely continuous distributions as well; in both cases, it is possible to fix the arrival point of the random walk  $S$  at time  $\lfloor ns_1 \rfloor - \lfloor ns_2 \rfloor - 2$  and to consider its rescaled limit as  $n \rightarrow +\infty$ . We write  $k_1 = \lfloor ns_1 \rfloor$  and



$k_2 = \lfloor ns_2 \rfloor$ . For any  $x > 0$ , it holds that

$$\begin{aligned}
& \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right); \tau^S(x) = k_2 - k_1 - 1 \right] \\
&= \sum_{y=1}^{+\infty} \mathbb{P}[y + \xi_{k_2 - k_1 - 1} \leq 0] \\
&\times \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right); \tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y \right] \\
&= \sum_{y=1}^{+\infty} \mathbb{P}[\tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y] \times \mathbb{P}[y + \xi_1 \leq 0] \\
&\times \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right) \mid \tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y \right] \\
&= \sum_{y=1}^{+\infty} \mathbb{P}[\tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y] \times \mathbb{P}[y + \xi_1 \leq 0] \\
&\times \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right) \mid \tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y \right].
\end{aligned}$$

By Corollary 2.5 in [6], the random walk bridge conditioned to stay positive, starting at  $x$  and ending at  $y$ , under linear interpolation and diffusive rescaling, converges in distribution on  $C([0, 1], \mathbb{R})$  toward the normalized Brownian excursion; in other words, for each  $x, y \geq 1$ ,

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi_1 \left( \frac{x + S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right) \mid \tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y \right] \\
&= \int_0^{+\infty} 2\varphi_1(u\sqrt{s_2 - s_1}) \exp \left( -\frac{u^2}{2 \frac{s-s_1}{s_2-s_1} \frac{s_2-s}{s_2-s_1}} \right) \frac{u^2}{\sqrt{2\pi \frac{(s-s_1)^3 (s_2-s)^3}{(s_2-s_1)^3}}} du \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} \varphi_1(v) \exp \left( -\frac{v^2}{2 \frac{(s-s_1)(s_2-s)}{s_2-s_1}} \right) \frac{v^2}{\sqrt{\frac{(s-s_1)^3 (s_2-s)^3}{(s_2-s_1)^3}}} dv.
\end{aligned}$$

By Lemma 11,  $\mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - \lfloor ns_2 \rfloor)}{\sigma \sqrt{n}} \right) \right]$  converges to  $\int_0^{+\infty} \varphi_2(u) \frac{2e^{-u^2/2(t-s_2)}}{\sqrt{2\pi(t-s_2)}} du$ .

To apply the dominated convergence theorem, notice that the quantities  $\mathbb{E} \left[ \varphi_1 \left( \frac{x+S(\lfloor ns \rfloor - k_1 - 1)}{\sigma \sqrt{n}} \right) \mid \tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y \right]$  and  $\mathbb{E} \left[ \varphi_2 \left( \frac{Y(\lfloor nt \rfloor - k_2)}{\sigma \sqrt{n}} \right) \right]$  are bounded by  $|\varphi_1|_\infty$  and  $|\varphi_2|_\infty$  respectively. Furthermore, on the one hand, by Lemmas 5 it holds, for any  $x, y > 0$ , that

$$\begin{aligned}
& n^{3/2} \mathbb{P}[\tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y] \\
&= n^{3/2} \mathbb{P}[\tau^S(x) > k_2 - k_1 - 2; x + S(k_2 - k_1 - 2) = y] \\
&\leq \frac{C_2 h(x) \tilde{h}(y)}{(s_2 - s_1)^{3/2}},
\end{aligned}$$

and

$$\lim_{n \rightarrow \infty} n^{3/2} \mathbb{P}[\tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y] = \frac{c_2 h(x) \tilde{h}(y)}{(s_2 - s_1)^{3/2}}.$$

Moreover,

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{3/2} \sum_{y=1}^{\infty} \mathbb{P}[y + \xi_1 \leq 0] \mathbb{P}[\tau^S(x) > k_2 - k_1 - 3, x + S(k_2 - k_1 - 2) = y] \\ &= \lim_{n \rightarrow \infty} n^{3/2} \mathbb{P}[x + S(1) > 0, \dots, x + S(k_2 - k_1 - 2) > 0, x + S(k_2 - k_1 - 1) \leq 0] \\ &= \lim_{n \rightarrow \infty} n^{3/2} \mathbb{P}[\tau^S(x) = [ns_2] - [ns_1]] = \frac{c_1}{2(s_2 - s_1)^{3/2}} h(x). \end{aligned}$$

Eventually, since  $\sum_{y \geq 1} \tilde{h}(y) \mathbb{P}(y + \xi_1 \leq 0) < +\infty$ , the Lebesgue dominated convergence theorem and identity (10) imply that, for any  $0 < s_1 < s_2 < t$ , the sequence  $(f_n)_{n \geq 1}$  converges to  $f$  given by

$$\begin{aligned} f(s_1, s_2) &= \frac{1}{\pi^2 \sqrt{s_1}} \int_0^{+\infty} \varphi_1(v) \exp\left(-\frac{v^2}{2 \frac{(s_2-s)(s-s_1)}{s_2-s_1}}\right) \frac{v^2}{\sqrt{(s-s_1)^3 (s_2-s)^3}} dv \\ &\quad \times \int_0^{+\infty} \varphi_2(u) \frac{e^{-u^2/2(t-s_2)}}{\sqrt{t-s_2}} du. \end{aligned}$$

By the same argument, the function  $f_n$  may be dominated as follows: for  $2 \leq k_1 < [ns] - 6$ ,  $[ns] \leq k_2 \leq [nt]$  and  $s_1 \in [\frac{k_1}{n}, \frac{k_1+1}{n})$  and  $s_2 \in [\frac{k_2}{n}, \frac{k_2+1}{n})$

$$\begin{aligned} |f_n(s_1, s_2)| &\leq \frac{n^2}{\sqrt{[ns_1]}([ns_2] - [ns_1] - 2)^{3/2}} \\ &\leq \frac{1}{\sqrt{s_1}(s_2 - s_1)^{3/2}}, \end{aligned}$$

with  $\int_0^s ds_1 \int_s^t ds_2 \frac{1}{\sqrt{s_1}(s_2 - s_1)^{3/2}} < +\infty$ . Hence

$$\begin{aligned} \lim_{n \rightarrow +\infty} A_1(n) &= \int_0^s ds_1 \int_s^t ds_2 f(s_1, s_2) \\ &= \frac{1}{\pi^2} \int_0^s \frac{ds_1}{\sqrt{s_1}} \int_s^t ds_2 \int_0^{+\infty} \varphi_1(v) \exp\left(-\frac{v^2}{2 \frac{(s_2-s)(s-s_1)}{s_2-s_1}}\right) \frac{v^2}{\sqrt{(s-s_1)^3 (s_2-s)^3}} \\ &\quad \times \int_0^{+\infty} \varphi_2(u) \frac{e^{-u^2/2(t-s_2)}}{\sqrt{t-s_2}} dudv. \end{aligned}$$

Set  $a = s_1/s$  and  $b = (s_2 - s)/(t - s)$ ; we obtain  $\lim_{n \rightarrow +\infty} A_1(n) = A_1$  with

$$\begin{aligned} A_1 &= \frac{1}{\pi^2} \frac{1}{s(t-s)} \int_0^{+\infty} du \int_0^{+\infty} dv \int_0^1 \frac{da}{\sqrt{a(1-a)^3}} \int_0^1 \frac{db}{\sqrt{b^3(1-b)}} \varphi_1(v) \varphi_2(u) \\ &\quad \times \exp\left(-\frac{u^2}{2(t-s)(1-b)}\right) v^2 \exp\left(-\frac{v^2}{2(t-s)b}\right) \exp\left(-\frac{v^2}{2s(1-a)}\right). \end{aligned}$$

Note that (with the change of variable  $x = \frac{a}{1-a}$ )

$$\int_0^1 \frac{1}{\sqrt{a(1-a)^3}} \frac{v}{\sqrt{s}} \exp\left(-\frac{v^2}{2s(1-a)}\right) da = \sqrt{2\pi} e^{-v^2/2s}.$$

Now we compute

$$I = \int_0^1 \frac{v}{\sqrt{b^3(1-b)}} \exp\left(-\frac{u^2}{2(t-s)(1-b)}\right) \exp\left(-\frac{v^2}{2(t-s)b}\right) db.$$

Denote  $z = \frac{b}{1-b}$ , we get

$$I = \exp\left(-\frac{u^2 + v^2}{2(t-s)}\right) \int_0^{+\infty} \frac{v}{z^{3/2}} \exp\left(-\frac{u^2 z}{2(t-s)} - \frac{v^2}{2(t-s)z}\right) dz.$$

Change  $z$  by  $1/z$ , we get

$$I = \exp\left(-\frac{u^2 + v^2}{2(t-s)}\right) \int_0^{+\infty} \frac{v}{z^{1/2}} \exp\left(-\frac{u^2}{2(t-s)z} - \frac{v^2 z}{2(t-s)}\right) dz.$$

By using Lemma 10, we get

$$\begin{aligned} I &= \exp\left(-\frac{u^2 + v^2}{2(t-s)}\right) \sqrt{2(t-s)\pi} \exp\left(-\frac{uv}{t-s}\right) \\ &= \sqrt{2(t-s)\pi} \exp\left(-\frac{(u+v)^2}{2(t-s)}\right). \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} A_1(n) = \frac{2}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(v) \varphi_2(u) e^{-v^2/2s} e^{-\frac{(u+v)^2}{2(t-s)}} dudv. \quad (13)$$

### 3.2.2 Estimate of $A_2(n)$

We decompose  $A_2(n)$  as

$$\begin{aligned} & \sum_{k \leq [ns]} \sum_{l \geq 0} \mathbb{E} \left[ \varphi_1 \left( \frac{Y([ns])}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{Y([nt])}{\sigma \sqrt{n}} \right); \tau_l = k, \gamma_{k+1} > 0, U_{k,k+2} > 0, \dots, U_{k,[nt]} > 0 \right] \\ &= \sum_{k \leq [ns]} \sum_{l \geq 0} \mathbb{E} \left[ \varphi_1 \left( \frac{U_{k,[ns]}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{U_{k,[nt]}}{\sigma \sqrt{n}} \right); \tau_l = k, \gamma_{k+1} > 0, U_{k,k+2} > 0, \dots, U_{k,[nt]} > 0 \right] \\ &= \sum_{k \leq [ns]} \sum_{l \geq 0} \mathbb{E} \left[ \varphi_1 \left( \frac{U_{0,[ns]-k}}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{U_{0,[nt]-k}}{\sigma \sqrt{n}} \right); \tau_1 > [nt] - k \right] \mathbb{P}[\tau_l = k] \\ &\approx \sum_{k < [ns]} \Sigma_k \times \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( \frac{x + S([ns] - k - 1)}{\sigma \sqrt{n}} \right) \right. \right. \\ &\quad \left. \left. \times \varphi_2 \left( \frac{x + S([nt] - k - 1)}{\sigma \sqrt{n}} \right); \tau^S(x) > [nt] - k - 1 \right] \Big|_{\gamma_1 = x > 0} \right]. \end{aligned}$$

For  $u \in (0, s]$ , we denote

$$g_n(u) = n\Sigma_{[nu]}\mathbb{E}\left[\mathbb{E}\left[\varphi_1\left(\frac{x + S([ns] - [nu] - 1)}{\sigma\sqrt{n}}\right)\right.\right. \\ \left.\left.\times \varphi_2\left(\frac{x + S([nt] - [nu] - 1)}{\sigma\sqrt{n}}\right); \tau^S(x) > [nt] - [nu] - 1\right]\right]_{\gamma_1=x>0}.$$

By Lemmas 3 and 7, it is clear that  $0 \leq g_n(u) \leq \frac{1}{\sqrt{u(t-u)}}$ .

Now, let us compute the pointwise limit on  $(0, s]$  of the sequence  $(g_n)_{n \geq 1}$ . We write  $g_n(u)$  as

$$g_n(u) = n\Sigma_{[nu]}\mathbb{E}\left[\mathbb{E}\left[\varphi_1\left(\frac{x + S([ns] - [nu] - 1)}{\sigma\sqrt{n}}\right)\right.\right. \\ \left.\left.\times \varphi_2\left(\frac{x + S([nt] - [nu] - 1)}{\sigma\sqrt{n}}\right); \tau^S(x) > [nt] - [nu] - 1\right]\right]_{\gamma_1=x>0} \\ = n\Sigma_{[nu]}\mathbb{E}\left[\mathbb{E}\left[\varphi_1\left(\frac{x + S([ns] - [nu] - 1)}{\sigma\sqrt{n}}\right)\right] \varphi_2\left(\frac{x + S([nt] - [nu] - 1)}{\sigma\sqrt{n}}\right) \mid \right. \\ \left. \tau^S(x) > [nt] - [nu] - 1\right]_{\gamma_1=x>0} \mathbb{P}[\tau^S(x) > [nt] - [nu] - 1]_{\gamma_1=x>0}.$$

Since  $\varphi_1, \varphi_2$  are bounded and continuous on  $\mathbb{R}$ , it follows from Theorem 3.2 in [5] and The-

orems 2.23 and 3.4 in [11] that<sup>4</sup>

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi_1 \left( \frac{x + S([ns] - [nu] - 1)}{\sigma \sqrt{n}} \right) \right. \\
& \quad \times \left. \varphi_2 \left( \frac{x + S([nt] - [nu] - 1)}{\sigma \sqrt{n}} \right) \middle| \tau^S(x) > [nt] - [nu] - 1 \right] \\
&= \lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi_1 \left( \frac{x + S([ns] - [nu] - 1)}{\sigma \sqrt{[nt] - [nu] - 1}} \frac{\sqrt{[nt] - [nu] - 1}}{\sqrt{n}} \right) \right. \\
& \quad \times \left. \varphi_2 \left( \frac{x + S([nt] - [nu] - 1)}{\sigma \sqrt{[nt] - [nu] - 1}} \frac{\sqrt{[nt] - [nu] - 1}}{\sqrt{n}} \right) \middle| \tau^S(x) > [nt] - [nu] - 1 \right] \\
&= \int_0^{+\infty} \int_0^{+\infty} \varphi_1(y\sqrt{t-u}) \varphi_2(z\sqrt{t-u}) \left( \frac{t-u}{s-u} \right)^{3/2} y e^{-\frac{t-u}{2(s-u)}y^2} \\
& \quad \times \frac{e^{-\frac{1}{2}\frac{t-u}{t-s}(z-y)^2} - e^{-\frac{1}{2}\frac{t-u}{t-s}(z+y)^2}}{\sqrt{2\pi} \left( 1 - \frac{s-u}{t-u} \right)} dy dz \\
&= \frac{1}{\sqrt{2\pi(t-s)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(y') \varphi_2(z') \frac{\sqrt{t-u}}{(s-u)^{3/2}} y' e^{-\frac{y'^2}{2(s-u)}} \\
& \quad \times \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right) dy' dz'.
\end{aligned}$$

This fact together with Lemma 3, Lemma 7 and the Lebesgue dominated convergence theorem implies that the sequence  $(g_n)_{n \geq 1}$  pointwise converges to  $g$  with

$$\begin{aligned}
g(u) &= \frac{1}{\pi^{3/2} \sqrt{2(t-s)}} \frac{1}{\sqrt{u(s-u)^3}} \\
& \quad \times \int_0^{+\infty} \int_0^{+\infty} \varphi_1(y') \varphi_2(z') y' e^{-\frac{y'^2}{2(s-u)}} \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right) dy' dz'.
\end{aligned}$$

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<sup>4</sup>In [11] the author needed the third order moment of the increment is finite; in fact, it only requires finite second moment [5].

Using again Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} A_2(n) \\
&= \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k \leq [ns]} g_n(k/n) = \int_0^s g(u) du \\
&= \frac{1}{\pi^{3/2} \sqrt{2(t-s)}} \int_0^s du \int_0^{+\infty} dy' \int_0^{+\infty} dz' \\
&\quad \times \varphi_1(y') \varphi_2(z') \frac{e^{-\frac{y'^2}{2(s-u)}}}{\sqrt{u(s-u)^3}} \frac{y'}{\sqrt{2\pi(t-s)}} \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right) \\
&= \frac{1}{\pi^{3/2} s \sqrt{2(t-s)}} \int_0^{+\infty} dy' \int_0^{+\infty} dz' \varphi_1(y') \varphi_2(z') \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right) \\
&\quad \times \left( \int_0^1 \frac{y'}{\sqrt{v(1-v)^3}} e^{-\frac{y'^2}{2s(1-v)}} dv \right) \\
&= \frac{1}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} dy' \int_0^{+\infty} dz' \varphi_1(y') \varphi_2(z') e^{-y'^2/2s} \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right). \tag{14}
\end{aligned}$$

Therefore, it follows from (13) and (14) that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \varphi_1 \left( \frac{Y([ns])}{\sigma \sqrt{n}} \right) \varphi_2 \left( \frac{Y([nt])}{\sigma \sqrt{n}} \right) \right] = \mathbb{E}[\varphi_1(|B_s|) \varphi_2(|B_t|)].$$

Using a similar estimate as the one in (12), we get

$$\lim_{n \rightarrow \infty} \mathbb{E} [\varphi_1(Y_n(s)) \varphi_2(Y_n(t))] = \mathbb{E}[\varphi_1(|B_s|) \varphi_2(|B_t|)],$$

which concludes the convergence of  $(Y_n)$  in two-dimensional marginal distribution to a reflected Brownian motion.

### 3.3 Tightness

Now we verify the tightness of the sequence of processes  $(Y_n(t))$ . Using Theorem 7.3 in [4], it is sufficient to show that these two conditions hold

- (i) For each positive  $\gamma$ , there exist an  $a$  and an  $n_0$  such that

$$\mathbb{P} [|Y_n(0)| \geq a] \leq \gamma, \quad n \geq n_0.$$

- (ii) For each positive  $\epsilon$  and  $\gamma$ , there exist a  $\delta \in (0, 1)$ , and an  $n_0$  such that

$$\mathbb{P} [w_{Y_n}(\delta) \geq \epsilon] \leq \gamma, \quad n \geq n_0$$

where  $w_{Y_n}(\delta) = \sup\{|Y_n(t) - Y_n(s)| : t, s \in [0, 1], |t - s| \leq \delta\}$  is the modulus of continuity of  $Y_n$ .

The first condition is trivial since  $Y_n(0) = Y(0) = 0$ . For the second condition, we have

$$w_{Y_n}(\delta) \leq \frac{3}{\sigma\sqrt{n}} \sup_{1 \leq i < j \leq n; |i-j| \leq n\delta} |S(i) - S(j)| + \frac{1}{\sigma\sqrt{n}} \sup_{1 \leq l \leq n} \gamma_l \mathbf{1}_{\{\tau_l^Y \leq n\}}.$$

Denote  $N_n = \sum_{l=1}^{\infty} \mathbf{1}_{\{\tau_l^Y \leq n\}}$  the number of times that  $Y(k)$ ,  $1 \leq k \leq n$ , visits zeros. We have

$$\begin{aligned} & \mathbb{P} \left[ \frac{1}{\sqrt{n}} \sup_{1 \leq l \leq n} \gamma_l \mathbf{1}_{\{\tau_l^Y \leq n\}} \geq \frac{\epsilon}{2} \right] \\ &= \mathbb{P} \left[ \sup_{1 \leq l \leq N_n} \gamma_l \geq \frac{\sqrt{n}\epsilon}{2} \right] \\ &= \mathbb{P} \left[ \sup_{1 \leq l \leq N_n} \gamma_l \geq \frac{\sqrt{n}\epsilon}{2}; N_n \leq a\sqrt{n} \right] + \mathbb{P} \left[ \sup_{1 \leq l \leq N_n} \gamma_l \geq \frac{\sqrt{n}\epsilon}{2}; N_n > a\sqrt{n} \right] \\ &\leq \mathbb{P} \left[ \sup_{1 \leq l \leq a\sqrt{n}} \gamma_l \geq \frac{\sqrt{n}\epsilon}{2} \right] + \mathbb{P} [N_n > a\sqrt{n}], \end{aligned}$$

for any positive constant  $a$ . Using the simple estimate that  $(1-x)^\alpha \geq 1-\alpha x$  for any  $\alpha \geq 1$  and  $x \in (0, 1)$ , we get

$$\begin{aligned} \mathbb{P} \left[ \sup_{1 \leq l \leq a\sqrt{n}} \gamma_l \geq \frac{\sqrt{n}\epsilon}{2} \right] &= 1 - \left( 1 - \mathbb{P}[\gamma_1 \geq \frac{\sqrt{n}\epsilon}{2}] \right)^{[a\sqrt{n}]} \\ &\leq a\sqrt{n} \mathbb{P}[\gamma_1 \geq \frac{\sqrt{n}\epsilon}{2}] \\ &\leq \frac{2a}{\epsilon} \mathbb{E}[\gamma_1 \mathbf{1}_{\{\gamma_1 \geq \frac{\sqrt{n}\epsilon}{2}\}}]. \end{aligned}$$

The last term tends to zeros as  $n \rightarrow \infty$ , since  $\gamma_1$  is integrable.

It follows from Lemma 7 that there exists a positive constant  $C_4$  such that for all  $n > 0$ , it holds that

$$\sum_{l=0}^{\infty} \mathbb{P}[\tau_l^Y = n] \leq \frac{C_4}{\sqrt{n}}.$$

Then we get

$$\mathbb{E}[N_n] = \mathbb{E} \left[ \sum_{l=0}^{\infty} \sum_{m=1}^n \mathbf{1}_{\{\tau_l^Y = m\}} \right] = \sum_{m=1}^n \sum_{l=0}^{\infty} \mathbb{P}[\tau_l^Y = m] \leq \sum_{m=1}^n \frac{C_4}{\sqrt{m}} \leq 3C_4\sqrt{n}.$$

This fact together with Markov's inequality implies that

$$\mathbb{P}[N_n > a\sqrt{n}] \leq \frac{\mathbb{E}[N_n]}{a\sqrt{n}} \leq \frac{3C_4}{a}.$$

Therefore,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left[ \frac{1}{\sqrt{n}} \sup_{1 \leq l \leq n} \gamma_l \mathbf{1}_{\{\tau_l^Y \leq n\}} \geq \frac{\epsilon}{2} \right] \leq \limsup_{n \rightarrow \infty} \frac{3C_4}{a},$$

for any  $a > 0$ . It implies that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[ \frac{1}{\sqrt{n}} \sup_{1 \leq l \leq n} \gamma_l \mathbf{1}_{\{\tau_l^Y \leq n\}} \geq \frac{\epsilon}{2} \right] = 0. \quad (15)$$

Moreover, it follows from the argument in Chapter 7 of [4] that

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P} \left[ \frac{1}{\sigma \sqrt{n}} \sup_{1 \leq i < j \leq n: |i-j| \leq n\delta} |S(i) - S(j)| \geq \frac{\epsilon}{6} \right] = 0.$$

This fact together with (15) implies that condition (ii) holds. Therefore the sequence of processes  $(Y_n(\cdot))$  is tight. This concludes the proof of Corollary 2.

## 4 Proof of Theorem 1

The proof of Theorem 1 follows the same strategy as the one of Corollary 2.

Let us fix briefly the notations.

- $\mathcal{F}_n$  is the  $\sigma$ -field generated by the random variables  $\xi_1, \xi'_1, \eta_1, \dots, \xi_n, \xi'_n, \eta_n$  for  $n \geq 1$  (as before,  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  by convention);
- $S' = (S'(n))_{n \geq 0}$  denotes the random walks with steps  $\xi'_k$ ;
- $(\ell'_l)_{l \geq 0}$  is the sequence of ascending ladder epochs of the random walk  $S'$  defined inductively by  $\ell'_0 = 0$  and, for any  $l \geq 1$ ,

$$\ell'_{l+1} = \min\{n > \ell'_l \mid S'(n) > S'(\ell'_l)\};$$

- $h'$  denotes the ‘‘ascending renewal function’’ of  $S'$  defined by

$$h'(x) = 1 + \sum_{l=1}^{+\infty} \mathbb{P}[S'(\ell'_l) \leq x] \quad \text{if } x \geq 0 \quad \text{and} \quad h'(x) = 0 \quad \text{otherwise.}$$

The function  $h'$  is increasing and  $h'(x) = O(x)$  as  $x \rightarrow \infty$ .

- $(\tau_l^X)_{l \geq 0}$  is the sequence of stopping times with respect to the filtration  $(\mathcal{F}_n)_{n \geq 0}$  defined by

$$\tau_0^X = 0 \quad \text{and} \quad \tau_{l+1}^X = \inf\{n > \tau_l^X : X(n) = 0\} \quad \text{for any } l \geq 0;$$

- for  $n > k \geq 1$ ,

$$\begin{cases} U_{k,n} &= \eta_{k+1} + \xi_{k+2} + \dots + \xi_n \text{ when } \eta_{k+1} > 0; \\ U'_{k,n} &= \eta_{k+1} + \xi'_{k+2} + \dots + \xi'_n \text{ when } \eta_{k+1} < 0. \end{cases}$$

### 4.1 On the fluctuation of the perturbed random walk $X$

**Lemma 12.** *It holds, as  $n \rightarrow +\infty$ ,*

$$\mathbb{P}[\tau_1^X > n] \sim \frac{c_1 \mathbb{E}[h(\gamma_1) \mathbf{1}_{\{\eta_1 > 0\}}] + c'_1 \mathbb{E}[h'(-\eta_1) \mathbf{1}_{\{\eta_1 < 0\}}]}{\sqrt{n}}, \quad (16)$$



and

$$\mathbb{P}[\tau_1^X = n] \sim \frac{c_1 \mathbb{E}[h(\eta_1) \mathbf{1}_{\{\eta_1 > 0\}}] + c'_1 \mathbb{E}[h'(-\eta_1) \mathbf{1}_{\{\eta_1 < 0\}}]}{2n^{3/2}}. \quad (17)$$

Moreover,

$$\Sigma_n(= \Sigma^X) := \sum_{l=0}^{+\infty} \mathbb{P}[\tau_l^X = n] \sim \frac{1}{c_1 \mathbb{E}[h(\eta_1) \mathbf{1}_{\{\eta_1 > 0\}}] + c'_1 \mathbb{E}[h'(-\eta_1) \mathbf{1}_{\{\eta_1 < 0\}}]} \frac{1}{\pi \sqrt{n}}. \quad (18)$$

*Proof.* We write

$$\begin{aligned} \mathbb{P}[\tau_1^X > n] &= \mathbb{P}[\eta_1 > 0, \eta_1 + \xi_2 > 0, \dots, \eta_1 + \xi_2 + \dots + \xi_n > 0] \\ &\quad + \mathbb{P}[\eta_1 < 0, \eta_1 + \xi'_2 < 0, \dots, \eta_1 + \xi'_2 + \dots + \xi'_n < 0] \\ &= \sum_{x>0} \left( \mathbb{P}[\tau^S(x) > n-1] \mathbb{P}[\eta_1 = x] + \mathbb{P}[\tau^{S''}(x) > n-1] \mathbb{P}[\eta_1 = -x] \right). \end{aligned}$$

Using a similar argument as the one in Lemma 6, we get (16). The relation (17) can be obtained by first writing

$$\begin{aligned} \mathbb{P}[\tau_1^X = n+2] &= \sum_{x>0} \mathbb{P}[\eta_1 = x] \sum_{y \leq -1} \mathbb{P}[\xi_1 = y] \sum_{z=1}^{|y|} \mathbb{P}[\tau^S(x) > n, x + S(n) = z] \\ &\quad + \sum_{x>0} \mathbb{P}[\eta_1 = -x] \sum_{y \leq -1} \mathbb{P}[-\xi'_1 = y] \sum_{z=1}^{|y|} \mathbb{P}[\tau^{S''}(x) > n, x + S''_n = z], \end{aligned}$$

and then repeating the argument in the proof of Lemma 6. Finally, using again Theorem B in [8], we get (18).  $\square$

In the following we show that the one and two marginal distributions of  $Y_n$  converge to the ones of a skew Brownian motion with parameter  $\alpha$  defined in (2). The convergence of higher marginal distributions can be proved by induction with similar arguments. Moreover, we suppose that  $\sigma = \sigma' = 1$  in order to ease our notations. The proof for general  $\sigma$  and  $\sigma'$  follows the same line.

## 4.2 One-dimensional distribution

Note that

$$\alpha = \lim_{n \rightarrow \infty} \frac{\mathbb{P}[\tau_1^X > n, \eta_1 > 0]}{\mathbb{P}[\tau_1^X > n]}.$$

For a function  $\varphi$ , we denote  $\hat{\varphi}(u) = \alpha \varphi(u) + (1 - \alpha) \varphi(-u)$ .

**Lemma 13.** *For any  $t \in [0, 1]$ , it holds*

$$\lim_{n \rightarrow +\infty} \mathbb{E}[\varphi(X_n(t))] = \int_0^{+\infty} \hat{\varphi}(u) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du = \int_{-\infty}^{+\infty} \varphi(u) p_t^\alpha(0, u) du.$$

*Proof.* We fix  $x \geq 0$  and  $t \in (0, 1)$ . Follows the argument in the proof of Lemma 11, we write  $\mathbb{E} \left[ \varphi \left( \frac{X([nt])}{\sqrt{n}} \right) \right] = A_{0+}(n) + A_{0-}(n)$ , where

$$A_{0+}(n) = \sum_{k=0}^{[nt]} \sum_{l=0}^{+\infty} \mathbb{E} \left[ \varphi \left( \frac{X([nt])}{\sqrt{n}} \right); \tau_l^X = k, X(k+1) > 0, \dots, X([nt]) > 0 \right],$$

$$A_{0-}(n) = \sum_{k=0}^{[nt]} \sum_{l=0}^{+\infty} \mathbb{E} \left[ \varphi \left( \frac{X([nt])}{\sqrt{n}} \right); \tau_l^X = k, X(k+1) < 0, \dots, X([nt]) < 0 \right].$$

We have

$$A_{0-}(n) \approx \sum_{k=0}^{[nt]-1} \Sigma_k \mathbb{E} \left[ \mathbb{E} \left( \varphi \left( -\frac{x + S''_{[nt]-k-1}}{\sqrt{n}} \right) \middle| \tau^{S''}(x) > [nt] - k - 1 \right) \right. \\ \left. \times \mathbb{P}[\tau^{S''}(x) > [nt] - k - 1] \middle|_{\eta_1 = -x < 0} \right].$$

Using Lemmas 3, 9 and 12 and the dominated convergence theorem as in the proof of Lemma 11, we get

$$\lim_{n \rightarrow +\infty} A_{0-}(n) = (1 - \alpha) \int_0^t \frac{1}{\pi \sqrt{s(t-s)}} \left( \int_0^{+\infty} \varphi(-z\sqrt{t-s}) z e^{-z^2/2} dz \right) ds \\ = \alpha \int_0^{+\infty} \varphi(-u) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du.$$

A similar computation yields

$$\lim_{n \rightarrow +\infty} A_{0+}(n) = \alpha \int_0^{+\infty} \varphi(u) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du.$$

Therefore,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[ \varphi \left( \frac{X([nt])}{\sqrt{n}} \right) \right] = \int_0^{+\infty} \hat{\varphi}(u) \frac{2e^{-u^2/2t}}{\sqrt{2\pi t}} du.$$

Using (4) and an argument similar to (12), we conclude Lemma 13.  $\square$

### 4.3 Finite dimensional distribution

The convergence of finite-dimensional distributions of  $(X_n(t))_{n \geq 1}$  is more delicate. We detail the argument for two-dimensional ones.

Let us fix  $0 < s < t, n \geq 1$  and denote

$$\kappa = \kappa(n, s) = \min\{k > [ns] : X(k) = 0\}.$$

We write

$$\mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right) \right] = A_1(n) + A_2(n),$$

where

$$A_1(n) = \sum_{k=[ns]+1}^{[nt]} \mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right) \mathbf{1}_{\{\kappa=k\}} \right],$$

$$A_2(n) = \mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right) \mathbf{1}_{\{\kappa>[nt]\}} \right].$$

### 4.3.1 Estimate of $A_1(n)$

As in the previous section, based on the sign of  $X([ns])$ ,  $A_1(n)$  can be decomposed as  $A_{1+}(n) + A_{1-}(n)$  where

$$A_{1+}(n) = \sum_{k_1=0}^{[ns]} \Sigma_{k_1} \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E} \left[ \varphi_2 \left( \frac{X([nt] - k_2)}{\sqrt{n}} \right) \right]$$

$$\times \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( \frac{x + S([ns] - k_1 - 1)}{\sqrt{n}} \right); \tau^S(x) = k_2 - k_1 - 1 \right] \Big|_{\eta_1=x>0} \right],$$

$$A_{1-}(n) = \sum_{k_1=0}^{[ns]} \Sigma_{k_1} \sum_{k_2=[ns]+1}^{[nt]} \mathbb{E} \left[ \varphi_2 \left( \frac{X([nt] - k_2)}{\sqrt{n}} \right) \right]$$

$$\times \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( -\frac{-x + S''_{[ns]-k_1-1}}{\sqrt{n}} \right); \tau^{S''}(-x) = k_2 - k_1 - 1 \right] \Big|_{\eta_1=x<0} \right].$$

Using Lemmas 12, 13 and repeating the argument in Section 3.2.1 we get

$$\lim_{n \rightarrow \infty} A_{1+}(n) = \frac{2\alpha}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(v) \hat{\varphi}_2(u) e^{-v^2/2s} e^{-\frac{(u+v)^2}{2(t-s)}} dudv,$$

$$\lim_{n \rightarrow \infty} A_{1-}(n) = \frac{2(1-\alpha)}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} \int_0^{+\infty} \varphi_1(-v) \hat{\varphi}_2(u) e^{-v^2/2s} e^{-\frac{(u+v)^2}{2(t-s)}} dudv.$$

Therefore,

$$\lim_{n \rightarrow \infty} A_1(n) = 4 \int_0^{+\infty} \int_0^{+\infty} \hat{\varphi}_1(v) \hat{\varphi}_2(u) p_s(v) p_{t-s}(u+v) dudv. \quad (19)$$

### 4.3.2 Estimate of $A_2(n)$

We decompose  $A_2(n) = A_{2+}(n) + A_{2-}(n)$ , where

$$A_{2+}(n) = \sum_{k \leq [ns]} \sum_{l \geq 0} \mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right); \right. \\ \left. \tau_l^X = k, \eta_{k+1} > 0, U_{k,k+2} > 0, \dots, U_{k,[nt]} > 0 \right],$$

$$A_{2-}(n) = \sum_{k \leq [ns]} \sum_{l \geq 0} \mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right); \right. \\ \left. \tau_l^X = k, \eta_{k+1} < 0, U'_{k,k+2} < 0, \dots, U'_{k,[nt]} < 0 \right].$$

We have

$$A_{2-}(n) = \mathbb{E} \left[ \varphi_1 \left( \frac{U'_{k,[ns]}}{\sqrt{n}} \right) \varphi_2 \left( \frac{U'_{k,[nt]}}{\sqrt{n}} \right); \tau_l^X = k, U'_{k,k+2} < 0, \dots, U'_{k,[nt]} < 0 \right] \\ = \sum_{k \leq [ns]} \Sigma_k \mathbb{E} \left[ \mathbb{E} \left[ \varphi_1 \left( -\frac{-x + S''_{[ns]-k-1}}{\sqrt{n}} \right) \varphi_2 \left( -\frac{-x + S''_{[nt]-k-1}}{\sqrt{n}} \right); \right. \right. \\ \left. \left. \tau^{S''}(x) > [nt] - k - 1 \right] \Bigg|_{\eta_1 = x < 0} \right].$$

By adapting the argument in Section 3.2.2 we get

$$\lim_{n \rightarrow +\infty} A_{2-}(n) = \frac{1 - \alpha}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} dy' \int_0^{+\infty} dz' \varphi_1(-y') \varphi_2(-z') \\ \times e^{-y'^2/2s} \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right).$$

Similar argument yields,

$$\lim_{n \rightarrow +\infty} A_{2+}(n) = \frac{\alpha}{\pi \sqrt{s(t-s)}} \int_0^{+\infty} dy' \int_0^{+\infty} dz' \varphi_1(y') \varphi_2(z') \\ \times e^{-y'^2/2s} \left( e^{-\frac{(z'-y')^2}{2(t-s)}} - e^{-\frac{(z'+y')^2}{2(t-s)}} \right).$$

Therefore,

$$\lim_{n \rightarrow +\infty} A_2(n) = 2 \int_0^{+\infty} \int_0^{+\infty} (\alpha \varphi_1(y') \varphi_2(z') + (1 - \alpha) \varphi_1(-y') \varphi_2(-z')) p_s(y') \\ \times (p_{t-s}(y' - z') - p_{t-s}(y' + z')) dy' dz'. \quad (20)$$

Finally, it follows from (19), (20) and (4) that for all  $0 \leq s < t \leq 1$ , it holds that

$$\lim_{n \rightarrow \infty} \mathbb{E} [\varphi_1(X_n(s)) \varphi_2(X_n(t))] = \lim_{n \rightarrow \infty} \mathbb{E} \left[ \varphi_1 \left( \frac{X([ns])}{\sqrt{n}} \right) \varphi_2 \left( \frac{X([nt])}{\sqrt{n}} \right) \right] \\ = \mathbb{E} [\varphi_1(B_s^\alpha) \varphi_2(B_t^\alpha)],$$

where  $B^\alpha$  is a skew Brownian motion with parameter  $\alpha$ .

## 4.4 Tightness

Note that the modulus of continuity  $w_{X_n}$  of  $X_n$  satisfies

$$w_{X_n}(\delta) \leq \frac{3}{\sqrt{n}} \sup_{1 \leq i < j \leq n; |i-j| \leq n\delta} |S(i) - S(j)| + \frac{3}{\sqrt{n}} \sup_{1 \leq i < j \leq n; |i-j| \leq n\delta} |S'(i) - S'(j)| \\ + \frac{2}{\sqrt{n}} \sup_{1 \leq l \leq n} |\eta_l| \mathbf{1}_{\{\tau_l^X \leq n\}}.$$

Therefore, the tightness of  $X$  can be proved in a similar way as the one of  $Y$ . We achieve the proof of Theorem 1 using Theorem 7.1 in [4].

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## References

- [1] Afanasyev, V. I., Böinghoff, C., Kersting, G., Vatutin, V. A: Limit theorems for weakly subcritical branching processes in random environment, *Journal of Theoretical Probability*, Vol. 25 , No 3, 703–732 (2012).
- [2] Afanasyev, V. I., Geiger, J., Kersting, G., Vatutin, V. A.: Criticality for branching processes in random environment, *The Annals of probability*, Vol. 33, No. 2, 645–673 (2005).
- [3] Alexander, K. S., Berger, Q.: Local limit theorems and renewal theory with no moments, *Electronic Journal of Probability*, Vol. 21, No. 66, 1–18 (2016).
- [4] Billingsley, P.: *Convergence of Probability Measures*, Willey, New York. (1968)
- [5] Bolthausen, E.: On a Functional Central Limit Theorem for Random Walks Conditioned to Stay Positive, *Annals of Probability*, Vol. 4, No. 3, 480–485 (1976).
- [6] Caravenna, F., Chaumont, L.: An invariance principle for random walk bridges conditioned to stay positive, *Electron. J. Probab.*, Vol.18, No. 60, 1-32 (2013).
- [7] Diaconis P, Freedman D.: Iterated Random Functions , *SIAM review*, Vol. 41, No. 1, 45–76 (1999).
- [8] Doney, R. A.: One-sided local large deviation and renewal theorems in the case of infinite mean, *Probab. Theory and Related Fields*, Vol. 107, No. 4, 451–465 (1997).

- [9] Doney, R. A.: Local behaviour of first passage probabilities. *Probability Theory and Related Fields*, 152(3-4), 559–588 (2012).
- [10] Harrison, J. M. , L. A. Shepp: On Skew Brownian Motion, *The Annals of Probability*, Vol. 9, No. 2, 608–619 (1981).
- [11] Iglehart, D. L.: Functional central limit theorems for random walks conditioned to stay positive, *The Annals of Probability*, Vol. 2, No. 4, 608–619 (1974).
- [12] Iksanov, A., Pilipenko A.: A functional limit theorem for locally perturbed random walks, *Probability and Mathematical Statistics*, Vol. 36, Fasc. 2, 353–368 (2016).
- [13] Itô, K., McKean, H.P. Jr.: *Diffusion processes and their sample paths*, Springer Science Business Media (2012).
- [14] Kozlov, M.V.: On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment, *Theory Probab. Appl.*, Vol. 21, No. 4, 791–804 (1976).
- [15] Lejay, A.: On the constructions of the skew Brownian motion, *Probab. Surveys* Vol. 3, 413–466 (2006).
- [16] Le Page, E., Peigné, M.: A local limit theorem on the semi-direct product of  $\mathbb{R}^{*+}$  and  $\mathbb{R}^d$ , *Annales de l’I.H.P. Probab. et Stat.* Vol. 33, No. 2, 223–252 (1997).
- [17] Peigné, M., Woess. W.: Stochastic dynamical systems with weak contractivity I. Strong and local contractivity, *Colloquium Mathematicum*, Vol. 125, 1–54 (2011).
- [18] Pilipenko, A. Yu., Pryhod’ko, Y. E.: Limit behavior of the symmetric random walks with a membrane, *Theor. Probability and Math. Statist.* Vol. 85, 93–105 (2012).
- [19] Pilipenko, A. Yu., Pryhod’ko, Y. E.: Limit behavior of a simple random walks with non integrable jumps from a barrier, *Theory of Stochastic Processes* Vol. 19 (35), No. 1, 52–61 (2014).
- [20] Revuz, D., Yor, M.: *Continuous Martingales and Brownian Motion*, Third Edition. Springer (2005).