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EXPLICIT DEGREE BOUNDS FOR RIGHT FACTORS OF LINEAR DIFFERENTIAL OPERATORS

A. BOSTAN, T. RIVOAL, AND B. SALVY

Abstract. If a linear differential operator with rational function coefficients is reducible, its factors may have coefficients with numerators and denominators of very high degree. When the base field is $\mathbb{C}$, we give a completely explicit bound for the degrees of the monic right factors in terms of the degree and the order of the original operator, as well as the largest modulus of the local exponents at all its singularities. As a consequence, if a differential operator L has rational function coefficients over a number field, we get degree bounds for its monic right factors in terms of the degree, the order and the height of L, and of the degree of the number field.

1. Introduction

Context. We are interested in factorizations of linear differential operators in $\mathbb{K}(z)[\frac{d}{dz}]$, where $\mathbb{K}$ is either $\mathbb{C}$ or $\overline{\mathbb{Q}}$ (embedded into $\mathbb{C}$). In the latter case, there is no loss of generality in assuming that $\mathbb{K}$ is a number field (because the coefficients all live in such a number field) and in this case we denote its degree by $\kappa := [\mathbb{K} : \mathbb{Q}]$.

Without loss of generality, we assume that $L \in \mathbb{K}[z][\frac{d}{dz}]$, i.e., it has the form

$$L = \sum_{j=0}^{m} p_j(z)\Big(\frac{d}{dz}\Big)^j$$

for some polynomials $p_j(z) \in \mathbb{K}[z]$, which are assumed to be coprime, with $p_m \neq 0$. We call $m \geq 1$ the order of L, and $q := \deg_z(L)$ the degree of L, where the degree of an operator such as L is understood as $\max_j(\delta(p_j)) \geq 0$, with $\delta(r)$ defined as $\max(\deg(u), \deg(v))$ for a rational function $r = u/v \in \mathbb{K}(z)$ with coprime $u, v \in \mathbb{K}[z]$.

Assume given some factorization $L = NM$ with $M, N \in \mathbb{K}(z)[\frac{d}{dz}]$, where

$$M = \sum_{j=0}^{r} a_j(z)\Big(\frac{d}{dz}\Big)^j,$$

for $a_j(z) \in \mathbb{K}(z)$ and $a_r \neq 0$. For our purposes, there will be no loss of generality in further assuming that M is monic, i.e. $a_r(z) = 1$. Let $n := \deg_z(M) = \max_j(\delta(a_j))$ be its degree.

Obviously $r \leq m$, but it is well known that n can be much larger than q, and it is in fact notoriously difficult to control n in terms of L. To the best of our knowledge, the first

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(and so far the only) written bound for n has been given by Grigoriev [15, Theorem 1.2]. On the one hand, Grigoriev's bound holds for any factor, not only for right factors. But on the other hand, it is only an asymptotic bound; for instance, with respect to the input degree m, it writes $\exp\left(2^{m\cdot o(2^m)}\right)$. The asymptotic nature of this bound is unsatisfactory for the applications we have in mind.

Main result. Here, we seek entirely explicit bounds holding for all m and for *any* operator $L \in \overline{\mathbb{Q}}[z][\frac{d}{dz}]$. As we will see, such a bound is a consequence of the following result.

Theorem 1. *For any $L \in \mathbb{C}[z][\frac{d}{dz}]$ of order m and degree q, if M is a monic right factor of L of order r, then its degree n satisfies*

$$n \leq r^2(q+1)\mathcal{E}(L) + r(q+1)\mathcal{N}(L) + rq + \frac{1}{2}r^2(r-1)(3q+1), \tag{1}$$

where $\mathcal{E}(L) \geq 0$ is the largest modulus of the local generalized exponents at the singularities of L, and $\mathcal{N}(L) \geq 0$ is the largest of all the slopes of L at its singularities and at ∞.

Moreover, if M is known to be Fuchsian, then

$$n \leq r^2(q+1)\mathcal{E}(L) + rq + \frac{1}{2}r^2(r-1)(q-1). \tag{2}$$

The notions of slopes and generalized exponents of a differential operator are recalled in §2.2. When L is Fuchsian, then M is necessarily Fuchsian and $\mathcal{N}(L) = 0$, thus the bound (1) is very close to (2) in that case, and both even coincide when $r = 1$. If $\mathcal{N}(L) \geq 1$, then on the right-hand side of (1), rq can be replaced by $r(q-1)$; see the discussion following inequalities (19) in §2.2. We refer the reader to the comments made after the proof of Theorem 1 concerning the choice of $\mathbb{C}$ (which is important for applications) as the base field instead of an arbitrary algebraically closed field of characteristic 0.

Bounding the degree of M in terms of the degree, the order and the height of L. Note that $\mathcal{N}(L) \leq m+q$, so that Theorem 1 reduces the problem of bounding n to the determination of an explicit upper bound for $\mathcal{E}(L)$. Bounds are known in the case where $L \in \mathbb{K}(z)[\frac{d}{dz}]$, where $\mathbb{K}$ is a number field of degree κ, embedded into $\mathbb{C}$. Grigoriev exhibited such a bound in that case, but again his result [15, Corollary, p. 21] is only asymptotic in the order m of L, see below. The first entirely explicit bound for $\mathcal{E}(L)$ was obtained in 2004 by Bertrand, Chirskii and Yebbou [4]. Their approach was based in part on Malgrange's truncation method, which was eventually published in [13, pp. 97–107]. In terms of a quantity called the height $H(L)$ of the operator L [4, p. 246 and p. 252], their bound reads

$$\mathcal{E}(L) \leq 2^{(36(q+1)m\kappa)^{9(q+1)^2 m^{3m}}} H(L)^{(5\kappa(q+1)m)^{9(q+1)^2 m^{3m}}}. \tag{3}$$

The inequalities (1) and (3), together with the bounds $r \leq m$ and $\mathcal{N}(L) \leq m+q$, completely solve the problem of finding an explicit upper bound for the degree of any monic right factor M of L, when $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$. It seems to be the first of this type in the literature. We have chosen to formulate Theorem 1 in terms of $\mathcal{E}(L)$ as a parameter because the upper bound in (3) seems pessimistic and any improvement of it would implicitly

improve Theorem 1. On the other hand, the other terms on the right-hand side of (1) are already polynomial in the parameters and are thus probably only marginally improvable.

Asymptotic comparisons. Below, we let $\mathcal{P}(X)$ denote different polynomials in $\mathbb{Z}[X]$, with degree and coefficients independent of κ, m and q. With our notations, Grigoriev obtains the asymptotic estimate $\mathcal{E}(L) \leq H(L)^{\mathcal{P}(\kappa q m)^m}$ as $m \to +\infty$, which is much better than (3), which reads $\mathcal{E}(L) \leq H(L)^{\mathcal{P}(\kappa q m)^{q^2 m^{3m}}}$ as $m \to +\infty$. When L is Fuchsian, Grigoriev's method as well as [4, p. 254] provide better bounds, which turn out to be both of the form $\mathcal{E}(L) \leq H(L)^{\mathcal{P}(\kappa q m)}$; one may wonder if this is asymptotically optimal as $m \to +\infty$. In the general case, it would obviously be interesting to close the gap between the uniform bound (3) and Grigoriev's asymptotic bound for $\mathcal{E}(L)$. It would also be interesting to do so in intermediate cases where some properties of L are known in advance. For instance, for applications related to E-functions (see [1]), L may have only two singularities: $z = 0$ which is regular, and $z = \infty$ which is irregular with slopes in $\{0, 1\}$.

Quasi-optimality of the bound in Theorem 1. For any integer $k \geq 1$, the second-order operator $L = z\left(\frac{d}{dz}\right)^2 + (2-z)\frac{d}{dz} + k$ admits the right factor $M = \frac{d}{dz} - \frac{H'(z)}{H(z)}$, where $H(z)$ is the confluent hypergeometric Kummer polynomial $H(z) = {}_1F_1(-k;\, 2; z) = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{(-z)^\ell}{(\ell+1)!}$. Thus, $m = 2$, $q = 1$, $r = 1$ and $n = k$, and it is easy to check that $\mathcal{E}(L) = k$ and $\mathcal{N}(L) = 1$. Therefore the bound of Theorem 1 writes $n \leq 2k + 2$ (using the slight improvement in the case $\mathcal{N}(L) \geq 1$). The bound is thus optimal up to the multiplicative constant 2.

Degrees of left factors. Taking formal adjoints exchanges left and right factors: if $L = NM$, then $L^\star = M^\star N^\star$, see e.g. [19, p. 39–40]. Therefore, one can effectively bound the degrees of the left factor N as well, by applying Theorem 1 to $N^\star$ and using the fact that all the quantities (order, degree, largest slope, maximal exponent modulus), involved in the inequality (1) for $L^\star$ and $N^\star$ are easily related to the corresponding quantities for L and N.

Minimal differential equations. Besides its own interest, one of our motivations to study this factorization problem comes from combinatorics [8] and number theory [1, 14], where certain D-finite power series in $\overline{\mathbb{Q}}[[z]]$, called E- and G-functions, are under study. In both cases, it is useful to be able to perform the following task efficiently: given $f(z) \in \overline{\mathbb{Q}}[[z]]$ and $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ such that $Lf(z) = 0$, determine $M \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ such that $Mf(z) = 0$ and M is of minimal order with this property. Obviously, M is then a right factor of L and Theorem 1 applies to it. Assume $L \in \mathbb{K}(z)[\frac{d}{dz}]$ with the same data as above and $\mathbb{K}$ a number field, and let $f(z) \in \mathbb{K}[[z]]$ be a solution of the differential equation $Ly(z) = 0$. The power series f needs not be convergent. For any integers r, n such that $1 \leq r \leq m$ and $n \geq 0$, let us define

$$R(z) := \sum_{j=0}^{r} P_j(z) f^{(j)}(z),$$

where $P_j(z) \in \mathbb{K}[z]$ are all of degree at most n. Then $R(z) = \sum_{k=0}^{\infty} r_k z^k$ is a formal power series in $\mathbb{K}[[z]]$, and we denote by N its valuation (or order) at $z = 0$, i.e. N is the smallest

integer $k \geq 0$ such that $r_k \neq 0$. A key inequality is the following upper bound on N [4]: either $R(z)$ is identically zero or

$$N \leq r(n+1) + 2(q+1)^2 m^3 + 2(q+1)m^2(\mathcal{E}(L)+1). \tag{4}$$

This is proved by putting together results by Shidlovskii [20, Lemma 8, p. 83 and Eq. (83) p. 99] and Bertrand, Chirskii, Yebbou [4, Thm. 1.2 p. 245]. With our notations, this yields $N \leq r(n+1) + n_0(L)$ where $n_0(L)$ is a quantity bounded above by $2(q+1)m^2(\mathcal{R}(L)+1)$, with $\mathcal{R}(L) \leq \mathcal{E}(L) + (q+1)m$, see [4, p. 252].

Now, given n and $r+1$ polynomials P_j, not all zero, letting N denote the upper bound in Eq. (4), if the first $N+1$ Taylor coefficients of $R(z)$ are all 0, then $R(z)$ is proven identically zero, which means that $f(z)$ is a solution of

$$M := \sum_{j=0}^{r} P_j(z)\Big(\frac{d}{dz}\Big)^j \in \mathbb{K}(z)\Big[\frac{d}{dz}\Big] \setminus \{0\},$$

and thus M is a right factor of L.

This remark was used by Adamczewski and the second author [1] to give an algorithm that computes a non-zero operator M such that $Mf(z) = 0$ and M is of minimal order with this property. The input is $L \in \mathbb{K}(z)[\frac{d}{dz}]$ and sufficiently many initial Taylor coefficients of f, so that the following ones can be computed using L. Let $\widehat{n}$ be the quantity on the right-hand side of the inequality (1). The algorithm first sets $r = 1$ and looks for R with order r and degree $\lfloor \widehat{n} \rfloor$ by requiring that its first $N+1$ Taylor coefficients all be 0 (this amounts to solving a homogeneous linear system with algebraic coefficients given by the Taylor coefficients of f). If no non-zero solution is found, r is increased and the same procedure is repeated, and so on up to $r = m$ if necessary. In the end, $M \neq 0$ minimal for f will be found.

This algorithm is not very efficient in practice. Moreover, the inequalities (3) and (4), as well as Grigoriev's Theorem 1.2 are all used to ensure the termination of the algorithm. It is important however to use Theorem 1 instead of Grigoriev's, as it holds for arbitrary differential operators L and $M \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ and not only asymptotically. A much more efficient minimization algorithm is under development [9].

Related works. The proof of Theorem 1 does not use Grigoriev's method [15], which relies on a subtle analysis of Beke's classical factorization algorithm [19, p. 118, §4.2.1], see also [21]. Instead, our method is inspired by van Hoeij's factorization algorithm [22, 23]. This algorithm internally computes, on any input operator L, upper bounds for the number of apparent singularities and the degree for right factors of L using the generalized Fuchs relation between local exponents. He did not give any explicit *a priori* degree bound, valid for any operator L. Our main contribution here is an explicit *a priori* degree bound when the base field is a number field. It is difficult to trace back exactly when in the 80's the (generalized) Fuchs relation was found to be relevant in this type of problem. In the Fuchsian case, it was used by Chudnovsky [11] to bound the number of apparent singularities in order to obtain an effective multiplicity estimate. See also [12, p. 364, Example 2.7] for a similar use of Fuchs' relation. Chudnovsky's result was adapted by

Bertrand and Beukers to the general case with the help of the generalized Fuchs relation [3]. They obtained a multiplicity estimate in which the effectivity of one specific constant was not completely clear. This effectivity issue was eventually solved by Bertrand, Chirskii and Yebbou [4].

2. Proof of Theorem 1

From this point on, we write ∂_z for $\frac{d}{dz}$. Consider a monic operator

$$R = \sum_{j=0}^{\mu} a_j(z)\partial_z^j \in \mathbb{C}(z)[\partial_z]. \tag{5}$$

We write $a_j = A_j/B$, with $A_j, B \in \mathbb{C}[z]$, the A_j's are coprime and B is the monic common denominator of lowest degree of the a_j's; we have $A_\mu = B$ and $\deg(a_j) = \deg(A_j) - \deg(B)$ and $\deg_z(R) := \max_j(\delta(a_j)) \leq \max(\deg(A_0), \ldots, \deg(A_{\mu-1}), \deg(B))$. By definition, the set $\mathrm{Sing}(R)$ of finite singularities of R is the set of roots of B. (Equivalently, for an operator with relatively prime polynomial coefficients such as L, this set is the set of roots of the leading coefficient.) The point ∞ may or may not be a singularity of R. Amongst the finite singularities of R, we denote by $\alpha(R)$ the set of the *apparent* ones, i.e. those at which R admits a local basis of power series solutions. Note that an apparent singularity ρ is necessarily a regular one. We denote by $\sigma(R)$ the set of finite singularities of R which are not in $\alpha(R)$, so that $\sigma(R)$ and $\alpha(R)$ form a partition of $\mathrm{Sing}(R)$. In a factorization $L = NM$, we have $\sigma(M) \subset \sigma(L) \subset \mathrm{Sing}(L)$ but $\alpha(M)$ may have no common element with $\mathrm{Sing}(L)$. Because of this, the main difficulty in the method presented below is to bound the number of apparent singularities of a right factor of L.

We split the proof of the theorem into two parts. We start with the Fuchsian case because it is simpler but at the same time it contains essentially all the ideas needed to prove the general case.

2.1. Fuchsian case.

Let us assume that we have a factorization $L = NM$ with operators N, M in $\mathbb{C}(z)[\partial_z]$ for which the operator M is Fuchsian and monic. Note that L need not necessarily be Fuchsian itself. We compute an explicit upper bound on $n := \deg_z(M)$ in terms of $\mathcal{E}(L)$. Our strategy is inspired by van Hoeij's approach [23], itself based on ideas by Chudnovsky [11] and Bertrand-Beukers [3], see also [17].

The Fuchsianity of M implies that it can be written

$$M = \partial_z^r + \frac{A_1(z)}{A(z)}\partial_z^{r-1} + \cdots + \frac{A_r(z)}{A(z)^r}, \qquad r \leq m := \mathrm{ord}(L),$$

where $A(z)$ is squarefree and $\deg\big(A_i(z)\big) \leq \deg\big(A(z)^i\big) - i$; see [18, Chap. V, §20, p. 77]. All we now have to do is to bound from above the degree of A. This is done in two steps. The polynomial $A(z)$ can be factored in $\mathbb{C}[z]$ as $A(z) = A_{\mathsf{sing}}(z)A_{\mathsf{app}}(z)$, where the roots of A_{sing} are the elements of $\sigma(M)$, while those of A_{app} are the elements of $\alpha(M)$.

Since A_{sing} is squarefree, its degree is equal to $\#\sigma(M) \leq \#\mathrm{Sing}(L)$. Thus $\deg(A_{\mathsf{sing}}) \leq \deg_z(L) = q$.

The degree of $A_{\mathsf{app}}(z)$ is equal to $\#\alpha(M)$ and it can be bounded above using the Fuchs relation [19, p. 138], which we now recall. We set

$$S_\rho(M) = \sum_{j=1}^{r} e_j(\rho) - \frac{r(r-1)}{2} \tag{6}$$

where the $e_j(\rho)$'s are the local exponents of M at the point ρ, so that clearly $S_\rho(M) = 0$ when $\rho \in \mathbb{C} \cup \{\infty\}$ is an ordinary point of M. The Fuchs relation shows that these local quantities obey a global relation:

$$\sum_{\rho \in \mathbb{C} \cup \{\infty\}} S_\rho(M) = \sum_{\rho \in \mathrm{Sing}(M) \cup \{\infty\}} S_\rho(M) = -r(r-1). \tag{7}$$

Now, the main observation is that if $\rho \in \alpha(M)$, then $S_p(M) \in \mathbb{N} \setminus \{0\}$ by [18, Chap. V, §18, p. 69] [1], so that

$$\#\alpha(M) \leq \sum_{\rho \in \alpha(M)} S_\rho(M)$$

and by (7) this implies that

$$\#\alpha(M) \leq -r(r-1) - \sum_{\rho \in \sigma(M) \cup \{\infty\}} S_p(M).$$

Since M is a right divisor of L, we have $\sigma(M) \subset \sigma(L) \subset \mathrm{Sing}(L)$ and for any such singularity $\rho \in \sigma(M)$, the exponents of M at ρ are also exponents of L at ρ, so that $|S_\rho(M)| \leq r\mathcal{E}(L) + r(r-1)/2$ by (6). Therefore,

$$\#\alpha(M) + r(r-1) \leq \sum_{\rho \in \sigma(L) \cup \{\infty\}} |S_\rho(M)| \leq (q+1)\left(r\mathcal{E}(L) + \frac{r(r-1)}{2}\right)$$

and

$$\#\alpha(M) \leq (q+1)r\mathcal{E}(L) + \frac{1}{2}r(r-1)(q-1).$$

Hence,

$$\deg(A) \leq (q+1)r\mathcal{E}(L) + \frac{1}{2}r(r-1)(q-1) + q$$

and finally

$$n = \deg_z(M) \leq (q+1)r^2\mathcal{E}(L) + qr + \frac{1}{2}r^2(r-1)(q-1).$$

This concludes the proof of Inequality (2) in Theorem 1.

2.2. **General case.** Again, we follow a strategy similar to that of van Hoeij [23], replacing the Fuchs identity by a generalization due to Bertrand and Beukers [3].

[1] *Stricto sensu*, [18] proves this under an *a priori* stronger definition of an apparent singularity p, which requires the *holomorphy* of the basis of solutions at p. Note, however, that the proof is algebraic and does not use this assumption, see also [19, p. 187–188].

Newton polygons. Part of the information on the degrees of factors come from patching up local information at each singularity that can be read off the Newton polygons of the operators. We first recall their main definitions and properties (see [19, p. 90, §3.3]).

Let $R = \sum_{j=0}^{\mu} a_j(z)\partial_z^j \in \mathbb{C}(z)[\partial_z]$ denote a monic differential operator in $\mathbb{C}(z)[\partial_z]$, ie. with $a_\mu \equiv 1$. As in (5), we write $a_j = A_j/B$, with $A_j, B \in \mathbb{C}[z]$ where B is the (normalized) common denominator of lowest degree of the a_j. The *Newton polygon of R at 0* is obtained by first rewriting $R = \sum_{j=0}^{\mu} a_j(z) z^{-j} P_j(\theta_z)$, where $\theta_z := z\partial_z$ and the $P_j(z) \in \mathbb{C}[z]$, $P_\mu \equiv 1$, and then taking the lower-left boundary of the convex hull of the points $(i,j) \in \mathbb{R}^2$ such that the coefficient of $z^i\theta^j$ is nonzero. The Newton polygon at another finite point $\rho \in \mathbb{C}$ is obtained similarly with $\theta_{\rho,z} = (z-\rho)\partial_z$ and coefficients in $\mathbb{C}((z-\rho))$, while the Newton polygon at infinity is the Newton polygon at 0 of the operator $\widetilde{R}$ obtained from R by changing z into $1/z$. By definition, the slopes of the Newton polygon at ρ are all ≥ 0 and they are 0 if and only if R is regular singular at ρ.

In this work, we only use the largest slope of R at $\rho \in \mathbb{C} \cup \{\infty\}$, that we denote by $\mathcal{N}_\rho(R)$; this is also known as the *Katz rank* of R at ρ, see [2, pp. 229–231] and [6]. The following two properties will be important in what follows.

(**1**) For any $R \in \mathbb{C}(z)[\partial_z]$, we have

$$\sum_{\rho \in \mathrm{Sing}(R) \cup \{\infty\}} \big(\mathcal{N}_\rho(R) + 1\big) \leq 2\deg_z(R) + 2, \tag{8}$$

by the arguments used in the proof of [3, p. 185, Lemme 2bis].

(**2**) Let $L, M, N, \in \mathbb{C}(z)[\partial_z]$ be such that $L = NM$. Then, for any $\rho \in \mathbb{C} \cup \{\infty\}$, we have

$$\mathcal{N}_\rho(M) \leq \mathcal{N}_\rho(L). \tag{9}$$

Indeed, a fundamental property is that the Newton polygon of a product of operators is the Minkowski sum of their Newton polygons [19, p. 92, Lemma 3.45]. Hence, the slopes of M at any point ρ form a subset of those of L at ρ.

We now assume $R \in \mathbb{C}(z)[\partial_z]$ to be monic and of the form (5). Let $v_j := \mathrm{val}_{z=0}(a_j(z))$ for $j \leq \mu$. Note that $v_\mu = 0$. Then for any $j \in \{0, \ldots, \mu-1\}$, we have

$$\mathcal{N}_0(R) \geq \frac{(v_\mu - \mu) - (v_j - j)}{\mu - j} = -1 - \frac{v_j}{\mu - j}. \tag{10}$$

It follows that for any $j \in \{0, \ldots, \mu-1\}$

$$\mathrm{val}_{z=0}\big(a_j(z)\big) \geq -\mu(\mathcal{N}_0(R) + 1).$$

By a similar reasoning, for any finite $\rho \in \mathbb{C}$ and any $j \in \{0, \ldots, \mu-1\}$,

$$\mathrm{val}_{z=\rho}\big(a_j(z)\big) \geq -\mu(\mathcal{N}_\rho(R) + 1). \tag{11}$$

If $\rho = \infty$, since $\theta_{1/z} = -\theta_z$, we have

$$\widetilde{R} = \sum_{j=0}^{\mu} \big(z^j a_j(1/z)\big) Q_j(\theta_z)$$

where $Q_j(X) := P_j(-X)$. In view of $\mathrm{val}_{z=0}(z^j a_j(1/z)) = j - \deg(a_j(z))$, the analogue of the inequality (10) is then

$$\mathcal{N}_\infty(R) \geq \frac{\mu - (j - \deg(a_j))}{\mu - j} = 1 + \frac{\deg(a_j)}{\mu - j}, \qquad j = 0, \dots, \mu - 1,$$

leading to the bound

$$\deg(a_j) \leq \mu(\mathcal{N}_\infty(R) - 1), \qquad j = 0, \dots, \mu - 1. \tag{12}$$

Any finite singularity ρ of R is a root of B and there exists $j_\rho \in \{0, \dots, \mu - 1\}$ such that ρ is not a root of A_{j_ρ}, so that $\mathrm{val}_{z=\rho}\big(a_{j_\rho}(z)\big) = -\mathrm{val}_{z=\rho}\big(B(z)\big)$. Using (11) with $j = j_\rho$, we thus deduce that

$$\mathrm{val}_{z=\rho}\big(B(z)\big) \leq \mu(\mathcal{N}_\rho(R) + 1). \tag{13}$$

A similar reasoning at infinity gives

$$\deg(A_j) \leq \deg(B) + \mu(\mathcal{N}_\infty(R) - 1), \qquad j = 0, \dots, \mu - 1. \tag{14}$$

Let now $L = NM$ be a factorization of L with $M \in \mathbb{C}(z)[\frac{d}{dz}]$ monic. We apply the above bounds to $R := M$ and $\mu := r$. We set

$$\mathcal{N}(L) = \max_{\rho \in \mathrm{Sing}(L) \cup \{\infty\}} \mathcal{N}_\rho(L) \qquad \text{and} \qquad M = \sum_{j=0}^{r} \frac{A_j(z)}{B(z)} \partial_z^j$$

where the A_j's and B are as in (5). In particular, by (9), for any $j = 0, \dots, r - 1$,

$$\begin{aligned} \deg(A_j) &\leq \deg(B) + r\mathcal{N}_\infty(M) - r \\ &\leq \deg(B) + r\mathcal{N}(L) - r. \end{aligned} \tag{15}$$

If $\rho \in \mathrm{Sing}(M)$, then Eq. (13) gives

$$\mathrm{val}_{z=\rho}\big(B(z)\big) \leq r(\mathcal{N}_\rho(M) + 1). \tag{16}$$

If furthermore $\rho \in \alpha(M)$, then in particular it is a regular singularity, so that $\mathcal{N}_\rho(M) = 0$ and this bound reduces to

$$\mathrm{val}_{z=\rho}\big(B(z)\big) \leq r. \tag{17}$$

It follows from (16) and (17) that

$$\begin{aligned} \deg(B) &\leq r(\mathcal{N}(L) + 1) \cdot \#\mathrm{Sing}(L) + r \cdot \#\alpha(M) \\ &\leq r(\mathcal{N}(L) + 1)q + r \cdot \#\alpha(M) \end{aligned} \tag{18}$$

where $q := \deg_z(L)$. From (15) and (18), we see that an explicit upper bound for

$$\begin{aligned} \deg_z(M) &:= \max{}_j(\delta(a_j)) \\ &\leq \max(\deg(A_0), \dots, \deg(A_{r-1}), \deg(B)) \\ &\leq \deg(B) + r\mathcal{N}(L) \\ &\leq rq + r(q+1)\mathcal{N}(L) + r \cdot \#\alpha(M) \end{aligned} \tag{19}$$

will again be obtained from an explicit upper bound for $\#\alpha(M)$. (If $\mathcal{N}(L) \geq 1$, then the right-hand side of (19) can be improved to $\deg(B) + r\mathcal{N}(L) - r$, with corresponding improvements in subsequent equations.)

Generalized Fuchs' relation. For any $R \in \mathbb{C}(z)[\partial_z]$ of order r, we consider the D-module $\widehat{R} := \mathbb{C}(z)[\partial_z]/(\mathbb{C}(z)[\partial_z]R)$. The generalization of (7) given in [6, Appendice, p. 84] and [7, p. 53, Theorem 2] is

$$\sum_{\rho \in \mathrm{Sing}(R) \cup \{\infty\}} \Big(S_\rho(R) - \frac{1}{2} \mathrm{irr}_\rho\big(\mathrm{End}(\widehat{R})\big) \Big) = -r(r-1), \tag{20}$$

where as before

$$S_\rho(R) = \sum_{j=1}^{r} e_j(\rho) - \frac{r(r-1)}{2} \tag{21}$$

but now the $e_j(\rho)$'s are the *generalized local exponents* of R at the point $\rho \in \mathbb{C} \cup \{\infty\}$ (see [6, Appendice, pp. 82-83] for their definition).

Given a differential operator R in $\mathbb{C}(z)[\partial_z]$, its *Malgrange's irregularity* [16], denoted $\mathrm{irr}_\rho(\widehat{R})$, is a non-negative integer which measures the defect of Fuchsianity of R at ρ. Precisely, $\mathrm{irr}_\rho(\widehat{R}) = 0$ if and only if R is singular regular at ρ. Then $\mathrm{End}(\widehat{R})$ in (20) is isomorphic to the D-module $\widehat{R} \otimes \widehat{R^*}$, where R^* is the adjoint of R. By [6, Appendice, p. 84], the integer $\mathrm{irr}_\rho\big(\mathrm{End}(\widehat{R})\big)$ can be bounded in terms of $\mathcal{N}_\rho(R)$ as follows: for any $\rho \in \mathbb{C} \cup \{\infty\}$,

$$\mathrm{irr}_\rho\big(\mathrm{End}(\widehat{R})\big) \leq r(r-1)\mathcal{N}_\rho(R). \tag{22}$$

If $\rho \in \mathbb{C} \cup \{\infty\}$ is an ordinary point or a regular singularity of R, we have $\mathcal{N}_\rho(R) = 0$ and *a fortiori* $\mathrm{irr}_\rho\big(\mathrm{End}(\widehat{R})\big) = 0$ as well; we thus recover the usual Fuchs relation (7) when R is Fuchsian.

We are now ready to bound $\#\alpha(M)$ in any factorization $L = NM$. We recall that $\alpha(M)$ and $\sigma(M)$ form a partition of $\mathrm{Sing}(M)$, and that $\sigma(M) \subset \sigma(L)$, so that $\sigma(M) \subset \mathrm{Sing}(L)$. Therefore,

$$r(r-1) + \sum_{\rho \in \alpha(M)} S_\rho(M) = -\sum_{\rho \in \sigma(M) \cup \{\infty\}} S_\rho(M) + \frac{1}{2} \sum_{\rho \in \sigma(M) \cup \{\infty\}} \mathrm{irr}_\rho\big(\mathrm{End}(\widehat{M})\big).$$

Now, $S_\rho(M) \in \mathbb{N} \setminus \{0\}$ for any $\rho \in \alpha(M)$ (again by [18, Chap. V, §18, p. 69]) and $|S_\rho(M)| \leq r\mathcal{E}(L) + \frac{r(r-1)}{2}$ for any $\rho \in \sigma(M) \cup \{\infty\}$ by (21).

It follows that

$$\begin{aligned}
&\#\alpha(M) + r(r-1) \qquad (23)\\
&\quad \leq \#\sigma(M)\cdot\Big(r\mathcal{E}(L) + \frac{r(r-1)}{2}\Big) + \frac{r(r-1)}{2}\sum_{\rho\in\sigma(M)\cup\{\infty\}}\big(\mathcal{N}_\rho(M)+1\big)\\
&\quad \leq (q+1)\Big(r\mathcal{E}(L) + \frac{r(r-1)}{2}\Big) + \frac{r(r-1)}{2}\sum_{\rho\in\mathrm{Sing}(L)\cup\{\infty\}}\big(\mathcal{N}_\rho(L)+1\big)\\
&\quad \leq (q+1)\Big(r\mathcal{E}(L) + \frac{r(r-1)}{2}\Big) + r(r-1)(q+1). \qquad (24)
\end{aligned}$$

In this sequence of inequalities, the first one follows from (22), the second one uses (9), and the last one relies on (8). Hence,

$$\#\alpha(M) \leq (q+1)r\mathcal{E}(L) + \frac{1}{2}r(r-1)(3q+1).$$

It follows that

$$n := \deg_z(M) \leq r^2(q+1)\mathcal{E}(L) + r(q+1)\mathcal{N}(L) + \frac{1}{2}r^2(r-1)(3q+1) + rq. \qquad (25)$$

This completes the proof of Theorem 1.

The proof of (8) given in [3] is analytic but it is possible that it could be replaced by an algebraic one (with holomorphic series replaced by formal series). Alternatively, we can use a direct estimate to bound the left-hand side of (8), so that the last term $r(r-1)(q+1)$ in (24) has to be replaced by the quantity $r(r-1)(q+1)(\mathcal{N}(L)+1)/2$. Therefore, it seems at first sight that it should be possible to adapt the proof of Theorem 1 to any algebraically closed field $\mathbb{K}$ of characteristic 0, instead of $\mathbb{C}$. This is the case as long as *equalities* are used. However, the deductions made from the generalized Fuchs relation (which holds for such a field $\mathbb{K}$) are based on various *inequalities* and the argument might in principle be adapted when $\mathbb{K}$ is also endowed with an archimedean absolute value. But by Ostrowski's Theorem [10, p. 33, Theorem 1.1], such a field can be embedded into a sub-field of $\mathbb{C}$ endowed with an absolute value given by a positive power of the modulus, and we would in fact gain nothing. Finally, it is not clear to us how a bound similar to (25) could be obtained with this method for a field $\mathbb{K}$ not endowed with an archimedean absolute value.

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