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Asymptotic representations of augmented q -Onsager algebra and boundary K -operators related to Baxter Q-operators

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Abstract

We consider intertwining relations of the augmented q -Onsager algebra introduced by Ito and Terwilliger, and obtain generic (diagonal) boundary K -operators in terms of the Cartan element of $U_q(sl_2)$. These K -operators solve reflection equations. Taking appropriate limits of these K -operators in Verma modules, we derive K -operators for Baxter Q-operators and corresponding reflection equations.

Keywords: augmented q -Onsager algebra, Baxter Q-operator, K-operator, reflection equation, asymptotic representation, L-operator, universal R-matrix

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1 Introduction

In the context of quantum integrable systems with periodic boundary conditions, the Baxter Q-operator [1] is an important object. It contains the information about the eigenfunctions and Bethe roots of the Hamiltonian and transfer matrix that are studied within a Bethe ansatz approach. Importantly, a key ingredient in the construction of Baxter Q-operators are L-operators. In the context of the representation theory, the L-operators that are suitable for Baxter Q-operators are obtained as certain homomorphic

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images of the universal R-matrix for a given quantum affine algebra. The auxiliary spaces of these L-operators are q-oscillator representations of one of the Borel subalgebras of the quantum affine algebra. This ‘q-oscillator construction’ of the Q-operators was proposed by Bazhanov, Lukyanov and Zamolodchikov [2], and developed by a number of authors (for example, [3, 4, 6, 8, 7, 9] and references therein ¹). It is known ² (cf. [2, 3]) that these q-oscillator representations of one of the Borel subalgebras are given as limits of representations of them (they are sometimes called ‘asymptotic representations’). A systematic study of this from the point of view of the representation theory was done in [13]. Thus, taking limits of representations of one of the Borel algebras is basically enough to derive the L-operators for Q-operators associated with integrable systems with periodic boundary condition. However, it is important to stress that these representations cannot be straightforwardly ³ extended to those of the whole quantum affine algebra. Instead, the extended representations could be interpreted [8] as representations of contracted algebras of the original quantum affine algebra.

By analogy, for models with non-periodic integrable boundary conditions [15, 16] the explicit construction of Baxter Q-operators is an interesting problem that deserves to be further studied [17, 18, 19] ⁴. From the algebraic point of view, for these models the relevant algebras are the reflection equation algebras and related coideal subalgebras of quantum affine algebras [20]. Given a certain representation, integrable boundary conditions are classified according to solutions - the so-called K-matrices - to the reflection and dual reflection equations [15, 16]. In order to construct Q-operators for those integrable models with boundaries, a crucial step is the construction of K-operators associated with q-oscillator representations.

In the present paper, we focus on a class of K-operators associated with integrable models with diagonal boundary conditions and $U_q(\widehat{sl_2})$ R-matrix. In this case, the coideal subalgebras of $U_q(\widehat{sl_2})$ that are relevant in the analysis are related with the so-called augmented q-Onsager algebra [21] (see also [22]), and its contracted versions introduced in this paper. For certain homomorphic images of two different coideal subalgebras of $U_q(\widehat{sl_2})$, we determine the K-operators. Certain limits are then considered, from which K-operators for Q-operators of models with diagonal boundary conditions can be derived. In contrast to the periodic boundary conditions case, we have to deal with limits of representations of the whole $U_q(\widehat{sl_2})$ in the spirit of [8] since the augmented q-Onsager algebra is realized by the generators of the whole $U_q(\widehat{sl_2})$ rather than one of the Borel subalgebras. The intertwining relations of our K-operators for Q-operators are no longer the ones for the augmented q-Onsager algebra but the ones for a contracted version of it.

The paper is organized as follows. In Section 2, we recall the definitions of the quan-

¹See also [10, 11] for the rational ($q = 1$) case, and [12, 17] for a different approach.

²This might be a sort of common knowledge or folklore among a part of the experts. Thus, there were cases where detailed explanations on limits were omitted and only the final expressions of q-oscillator representations were explicitly written in papers. Set aside Q-operators, a relation between a q-oscillator algebra and a limit of a highest weight representation of $U_q(sl_2)$ was discussed in [31].

³As for elliptic quantum groups, where the notion of the Borel subalgebras is obscure, an extension to the whole algebra might be necessary (cf. [14]).

⁴Among these, [19] is the only paper relevant to us in the sense that it deals with an ‘oscillator construction’ ($q = 1$ case) of Q-operators.

tum algebra $U_q(\widehat{sl_2})$, $U_q(sl_2)$ and two q-oscillator algebras that will be needed for our purpose. L-operators and their limits are recalled in Section 3. In Section 4, we introduce the augmented q-Onsager algebra through generators and relations. Two different types of realizations of the augmented q-Onsager algebra are considered, namely as a right or left coideal subalgebra $U_q(\widehat{sl_2})$. The two corresponding intertwining relations are considered and solved, giving an explicit expression for K-operators. The reflection and dual reflection equations they satisfy are displayed. Certain limits of those K-operators are considered, that are required for the construction of Q-operators. The rational limit ($q \rightarrow 1$) of these K-operators for Q-operators correspond to the K-operators found in [19]. In Appendices, we give some material that is needed for the main discussion. In Appendix A, to make the text self-contained we give a brief review on the universal R-matrix. In Appendix B, the contracted algebras associated with $U_q(\widehat{sl_2})$ and corresponding L-operators are reviewed. A universal form of the intertwining relations among L-operators for Q-operators is presented. In Appendix C, contractions of the augmented q-Onsager algebra are introduced and corresponding K-operators are described. In Appendices D,E,F, miscellaneous results are collected. In Appendix G, definitions of universal T- and Q-operators in terms K-operators text are explained. Throughout this paper, we will work on the general gradation of $U_q(\widehat{sl_2})$. This does not produce particularly new results since the L-operators in the general gradation can easily be obtained from the ones in a particular gradation by similarity transformation and rescaling of the spectral parameter. However, we expect that this may clarify some relations rather ambiguously treated in literatures.

Notation 1. *In the text, $q \in \mathbb{C}$ is assumed not to be a root of unity. We introduce the q -commutator $[X, Y]_q = XY - qYX$. In particular, we denote $[X, Y] = [X, Y]_1$. Also, we use the notation:*

$$[n]_q = (q^n - q^{-n})/(q - q^{-1}), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

2 Quantum algebras

In this section, basic definitions that will be used in the next sections are introduced. Successively, we recall the definitions of the quantum affine algebra $U_q(\widehat{sl_2})$, the quantum algebra $U_q(sl_2)$ and two q-oscillator algebras through generators and relations. Coproduct, automorphisms and certain finite dimensional representations are also displayed. We will follow the style of presentation in [9].

2.1 The quantum affine algebra $U_q(\widehat{sl_2})$

The quantum affine algebra $U_q(\widehat{sl_2})$ is a Hopf algebra generated by the generators e_i, f_i, h_i, d , where $i \in \{0, 1\}$. For $i, j \in \{0, 1\}$, the defining relations of the algebra $U_q(\widehat{sl_2})$

are given by

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad (2.1)$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad (2.2)$$

$$[e_i, [e_i, [e_i, e_j]_{q^2}]]_{q^{-2}} = [f_i, [f_i, [f_i, f_j]_{q^{-2}}]]_{q^2} = 0 \quad i \neq j, \quad (2.3)$$

where $(a_{ij})_{0 \leq i, j \leq 1}$ is the Cartan matrix

$$(a_{ij})_{0 \leq i, j \leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The algebra has automorphisms σ and τ defined by

$$\begin{aligned} \sigma(e_0) &= e_1, & \sigma(f_0) &= f_1, & \sigma(h_0) &= h_1, \\ \sigma(e_1) &= e_0, & \sigma(f_1) &= f_0, & \sigma(h_1) &= h_0. \end{aligned} \quad (2.4)$$

and

$$\tau(e_i) = f_i, \quad \tau(f_i) = e_i, \quad \tau(h_i) = -h_i, \quad i = 0, 1. \quad (2.5)$$

We use the following co-product $\Delta : U_q(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2}) \otimes U_q(\widehat{sl_2})$:

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + q^{-h_i} \otimes e_i, \\ \Delta(f_i) &= f_i \otimes q^{h_i} + 1 \otimes f_i, \\ \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i. \end{aligned} \quad (2.6)$$

We will also utilize an opposite co-product defined by

$$\Delta' = \mathbf{p} \circ \Delta, \quad \mathbf{p} \circ (X \otimes Y) = Y \otimes X, \quad X, Y \in U_q(\widehat{sl_2}). \quad (2.7)$$

The automorphisms (2.4) and (2.5) are related to the co-product as

$$(\sigma \otimes \sigma) \circ \Delta = \Delta \circ \sigma, \quad (\tau \otimes \tau) \circ \Delta = \Delta' \circ \tau. \quad (2.8)$$

We always assume that the central element $h_0 + h_1$ is zero. Anti-pode, co-unit and grading element d are not explicitly used in this paper.

The Borel subalgebra \mathcal{B}_+ (resp. \mathcal{B}_-) is generated by e_i, h_i (resp. f_i, h_i), where $i \in \{0, 1\}$. For complex numbers $c_i \in \mathbb{C}$ which obey the relation $\sum_{i=0}^1 c_i = 0$, the transformation

$$\tau_{c_1}(h_i) = h_i + c_i, \quad i = 0, 1, \quad (2.9)$$

gives the shift automorphism of \mathcal{B}_+ (or of \mathcal{B}_-). Here we omit the unit element multiplied by the above complex numbers.

There exists a unique element [23, 24] \mathcal{R} in a completion of $\mathcal{B}_+ \otimes \mathcal{B}_-$ called the universal R-matrix which satisfies the following relations

$$\begin{aligned}\Delta'(a) \mathcal{R} &= \mathcal{R} \Delta(a) \quad \text{for } \forall a \in U_q(\widehat{sl_2}), \\ (\Delta \otimes 1) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{23}, \\ (1 \otimes \Delta) \mathcal{R} &= \mathcal{R}_{13} \mathcal{R}_{12}\end{aligned}\tag{2.10}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, $\mathcal{R}_{13} = (\mathfrak{p} \otimes 1) \mathcal{R}_{23}$. We will use the relation

$$(\xi^{h_1} \otimes \xi^{h_1}) \mathcal{R} = \mathcal{R}(\xi^{h_1} \otimes \xi^{h_1}) \quad \text{for } \xi \in \mathbb{C} \setminus \{0\},\tag{2.11}$$

which follows from the first relation in (2.10). The Yang-Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12},\tag{2.12}$$

is a corollary of these relations (2.10). For $\overline{\mathcal{R}} = \mathcal{R}_{21} = (\mathfrak{p} \otimes 1) \mathcal{R}_{12}$, the relations (2.10) become

$$\begin{aligned}\Delta(a) \overline{\mathcal{R}} &= \overline{\mathcal{R}} \Delta'(a) \quad \text{for } \forall a \in U_q(\widehat{sl_2}), \\ (\Delta \otimes 1) \overline{\mathcal{R}} &= \overline{\mathcal{R}}_{23} \overline{\mathcal{R}}_{13}, \\ (1 \otimes \Delta) \overline{\mathcal{R}} &= \overline{\mathcal{R}}_{12} \overline{\mathcal{R}}_{13}\end{aligned}\tag{2.13}$$

One can also check that \mathcal{R}_{21}^{-1} (resp. \mathcal{R}^{-1}) satisfies (2.10) (resp. (2.13)). The universal R-matrix can be written in the form

$$\mathcal{R} = \tilde{\mathcal{R}} q^{\mathcal{K}}, \quad \mathcal{K} = \frac{1}{2} h_1 \otimes h_1.\tag{2.14}$$

Here $\tilde{\mathcal{R}}$ is the reduced universal R-matrix, which is a series in $e_j \otimes 1$ and $1 \otimes f_j$ and does not contain Cartan elements. Thus the reduced universal R-matrix is unchanged under the shift automorphism τ_{c_1} of \mathcal{B}_+ , see (2.9), while the prefactor \mathcal{K} is shifted as

$$\mathcal{K} \mapsto \mathcal{K} + \frac{c_1}{2} (1 \otimes h_1).\tag{2.15}$$

The universal R-matrix is invariant under $\sigma \otimes \sigma$:

$$(\sigma \otimes \sigma) \mathcal{R} = \mathcal{R}\tag{2.16}$$

Then we will use the following relation, which follows from this:

$$(\sigma \otimes 1) \mathcal{R} = (1 \otimes \sigma^{-1}) \mathcal{R}.\tag{2.17}$$

2.2 The quantum algebra $U_q(sl_2)$

The algebra $U_q(sl_2)$ is generated by the elements E, F, H . The defining relations are

$$\begin{aligned}[H, E] &= 2E, & [H, F] &= -2F, \\ [E, F] &= \frac{q^H - q^{-H}}{q - q^{-1}}.\end{aligned}\tag{2.18}$$

The following elements are central in $U_q(sl_2)$:

$$C = FE + \frac{q^{H+1} + q^{-H-1}}{(q - q^{-1})^2} = EF + \frac{q^{H-1} + q^{-H+1}}{(q - q^{-1})^2}. \quad (2.19)$$

Note that the following map gives an automorphism of the algebra.

$$\nu : \quad E \mapsto F, \quad F \mapsto E, \quad H \mapsto -H. \quad (2.20)$$

We will also use an anti-automorphism defined by

$$^t : \quad E \mapsto q^{-H-1}F, \quad F \mapsto Eq^{H+1}, \quad H \mapsto H, \quad (2.21)$$

where $(ab)^t = b^t a^t$ holds for any $a, b \in U_q(sl_2)$. There is an evaluation map $\mathbf{ev}_x : U_q(\widehat{sl_2}) \mapsto U_q(sl_2)$:

$$\begin{aligned} e_0 &\mapsto x^{s_0}F, & f_0 &\mapsto x^{-s_0}E, & h_0 &\mapsto -H, \\ e_1 &\mapsto x^{s_1}E, & f_1 &\mapsto x^{-s_1}F, & h_1 &\mapsto H, \end{aligned} \quad (2.22)$$

where $x \in \mathbb{C}$ is a spectral parameter and $s_0, s_1 \in \mathbb{C}$. We set $s = s_0 + s_1$. If we apply the similarity transformation $\mathbf{ev}_x(a) \mapsto x^{\frac{s_0-s_1}{4}H} \mathbf{ev}_x(a) x^{-\frac{s_0-s_1}{4}H}$ (resp. $\mathbf{ev}_x(a) \mapsto x^{-\frac{s_1}{2}H} \mathbf{ev}_x(a) x^{\frac{s_1}{2}H}$) for $a \in U_q(\widehat{sl_2})$ and the rescaling of the spectral parameter $x \mapsto x^{\frac{2}{s}}$ (resp. $x \mapsto x^{\frac{1}{s}}$), we will obtain the principal gradation $s_0 = s_1 = 1$ (resp. the homogeneous gradation $s_0 = 1, s_1 = 0$). Let π_μ^+ be the Verma module over $U_q(sl_2)$ with the highest weight μ . In a basis $\{v_n | n \in \mathbb{Z}_{\geq 0}\}$, we have

$$Hv_n = (\mu - 2n)v_n, \quad Ev_n = [n]_q[\mu - n + 1]_q v_{n-1}, \quad Fv_n = v_{n+1}. \quad (2.23)$$

For $\mu \in \mathbb{Z}_{\geq 0}$, the finite dimensional irreducible module π_μ with the highest weight μ is given as quotient of Verma modules:

$$\pi_\mu^+ / \pi_{-\mu-2}^+ \simeq \pi_\mu. \quad (2.24)$$

In particular, $\pi_1(E) = E_{12}$, $\pi_1(F) = E_{21}$ and $\pi_1(H) = E_{11} - E_{22}$ gives the fundamental representation of $U_q(sl_2)$, where E_{ij} is a 2×2 matrix unit whose (k, l) -element is $\delta_{i,k} \delta_{j,l}$. In this case, (2.21) coincides with transposition of matrices.

Then the compositions $\pi_\mu^+(x) = \pi_\mu^+ \circ \mathbf{ev}_x$ and $\pi_\mu(x) = \pi_\mu \circ \mathbf{ev}_x$ give evaluation representations of $U_q(\widehat{sl_2})$.

2.3 The q-oscillator algebras

We introduce two kinds of oscillator algebras Osc_i ($i = 1, 2$). They are generated by the elements $\mathbf{h}_i, \mathbf{e}_i, \mathbf{f}_i$ which obey the following relations:

$$\begin{aligned} [\mathbf{h}_1, \mathbf{h}_1] &= 0, & [\mathbf{h}_1, \mathbf{e}_1] &= 2\mathbf{e}_1, & [\mathbf{h}_1, \mathbf{f}_1] &= -2\mathbf{f}_1, \\ \mathbf{f}_1 \mathbf{e}_1 &= q \frac{1 - q^{\mathbf{h}_1}}{(q - q^{-1})^2}, & \mathbf{e}_1 \mathbf{f}_1 &= q \frac{1 - q^{\mathbf{h}_1-2}}{(q - q^{-1})^2}, \end{aligned} \quad (2.25)$$

$$\begin{aligned}
[\mathbf{h}_2, \mathbf{h}_2] &= 0, & [\mathbf{h}_2, \mathbf{e}_2] &= 2\mathbf{e}_2, & [\mathbf{h}_2, \mathbf{f}_2] &= -2\mathbf{f}_2, \\
\mathbf{f}_2\mathbf{e}_2 &= q^{-1} \frac{1 - q^{-\mathbf{h}_2}}{(q - q^{-1})^2}, & \mathbf{e}_2\mathbf{f}_2 &= q^{-1} \frac{1 - q^{-\mathbf{h}_2+2}}{(q - q^{-1})^2},
\end{aligned} \tag{2.26}$$

Note that Osc_1 and Osc_2 can be swapped by the transformation $q \mapsto q^{-1}$. One can prove the following corollaries of (2.25) and (2.26):

$$[\mathbf{e}_1, \mathbf{f}_1] = \frac{q^{\mathbf{h}_1}}{q - q^{-1}}, \quad [\mathbf{e}_2, \mathbf{f}_2] = -\frac{q^{-\mathbf{h}_2}}{q - q^{-1}}, \tag{2.27}$$

$$[\mathbf{e}_1, \mathbf{f}_1]_{q^{-2}} = \frac{1}{q - q^{-1}}, \quad [\mathbf{e}_2, \mathbf{f}_2]_{q^2} = -\frac{1}{q - q^{-1}}. \tag{2.28}$$

We will use anti-automorphisms of Osc_i , which are analogues of (2.21), defined by

$$^t : \quad \mathbf{e}_i \mapsto q^{-\mathbf{h}_i-1} \mathbf{f}_i, \quad \mathbf{f}_i \mapsto \mathbf{e}_i q^{\mathbf{h}_i+1}, \quad \mathbf{h}_i \mapsto \mathbf{h}_i, \tag{2.29}$$

where $(ab)^t = b^t a^t$ holds for any $a, b \in \text{Osc}_i$, $i = 1, 2$. Relations (2.27) are nothing but contractions⁵ of the commutation relation (2.18). On the other hand, the relations (2.28) are conditions that central elements take constant values. The following limits of the generators of $U_q(\mathfrak{sl}_2)$ for the Verma module⁶ π_μ^+

$$\begin{aligned}
\lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(H - \mu) &= \mathbf{h}_1, & \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(q^{-H-\mu}) &= 0, \\
\lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(Fq^{-\frac{s_0\mu}{s}}) &= \mathbf{f}_1, & \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(Eq^{-\frac{s_1\mu}{s}}) &= \mathbf{e}_1,
\end{aligned} \tag{2.30}$$

$$\begin{aligned}
\lim_{q^\mu \rightarrow 0} \pi_\mu^+(H - \mu) &= \mathbf{h}_2, & \lim_{q^\mu \rightarrow 0} \pi_\mu^+(q^{H+\mu}) &= 0, \\
\lim_{q^\mu \rightarrow 0} \pi_\mu^+(Fq^{\frac{s_0\mu}{s}}) &= \mathbf{f}_2, & \lim_{q^\mu \rightarrow 0} \pi_\mu^+(Eq^{\frac{s_1\mu}{s}}) &= \mathbf{e}_2,
\end{aligned} \tag{2.31}$$

realize the q -oscillator algebras Osc_1 and Osc_2 , respectively. Once these limits are taken in formulas, one may forget about the representations, and consider only algebraic relations defined in (2.25) or (2.26).

3 L-operators and limits

In this section, we first recall the definition of the Lax operators which follows from the universal R -matrix. In the context of quantum integrable systems, it is known that certain limits (cf. [2, 3]) of L -operators provide the basic ingredient for the construction of Baxter Q -operators associated with the Yang-Baxter algebra (see [5, 6, 7, 8] for examples of L -operators for Q -operators, and [10] for examples of the rational case). By analogy, here we consider different limits of L -operators that will be useful in the construction of

⁵Contractions of a quantum algebra and its relation to a q -oscillator algebra was discussed in [31].

⁶for (2.30): on the renormalized basis $v'_n = q^{-\frac{\mu s_0 n}{s}} v_n$; for (2.31): on the renormalized basis $v'_n = q^{\frac{\mu s_0 n}{s}} v_n$

Q -operators associated with the reflection equation algebra. Technically, we follow the presentation of [9]. From now on we denote $\lambda = q - q^{-1}$. We set

$$\mathbf{L}(x) = \begin{pmatrix} q^{\frac{H}{2}} - q^{-1}x^s q^{-\frac{H}{2}} & \lambda x^{s_0} F q^{-\frac{H}{2}} \\ \lambda x^{s_1} E q^{\frac{H}{2}} & q^{-\frac{H}{2}} - q^{-1}x^s q^{\frac{H}{2}} \end{pmatrix}, \quad (3.1)$$

$$\bar{\mathbf{L}}(x) = \begin{pmatrix} q^{\frac{H}{2}} - q^{-1}x^{-s} q^{-\frac{H}{2}} & \lambda x^{-s_1} F q^{-\frac{H}{2}} \\ \lambda x^{-s_0} E q^{\frac{H}{2}} & q^{-\frac{H}{2}} - q^{-1}x^{-s} q^{\frac{H}{2}} \end{pmatrix}. \quad (3.2)$$

These are images of the universal R-matrix (see Appendix A)

$$\mathbf{L}(x/y) = \mathbf{L}(x, y) = \phi(x/y)(\mathbf{ev}_x \otimes \pi_1(y))\mathcal{R}, \quad (3.3)$$

$$\bar{\mathbf{L}}(x/y) = \bar{\mathbf{L}}(x, y) = \phi(y/x)(\mathbf{ev}_x \otimes \pi_1(y))\mathcal{R}_{21}, \quad x, y \in \mathbb{C}, \quad (3.4)$$

where the overall factor is defined by $\phi(x) = e^{-\Lambda(x^s q^{-1})}$, $\Lambda(x) = \sum_{k=1}^{\infty} \frac{C_k}{k(q^k + q^{-k})} x^k$ and (A.21). One can check

$$\mathbf{L}(x)\bar{\mathbf{L}}(x) = \bar{\mathbf{L}}(x)\mathbf{L}(x) = q^{-1}(\lambda^2 C - x^s - x^{-s}), \quad (3.5)$$

$$\begin{aligned} g_2 \mathbf{L}(x q^{\frac{4}{s}})^{t_2} g_2^{-1} \bar{\mathbf{L}}(x)^{t_2} &= \mathbf{g}_1^{-1} \mathbf{L}(x q^{\frac{4}{s}})^{t_2} \mathbf{g}_1 \bar{\mathbf{L}}(x)^{t_2} = \\ &= \bar{\mathbf{L}}(x)^{t_2} g_2 \mathbf{L}(x q^{\frac{4}{s}})^{t_2} g_2^{-1} = \bar{\mathbf{L}}(x)^{t_2} \mathbf{g}_1^{-1} \mathbf{L}(x q^{\frac{4}{s}})^{t_2} \mathbf{g}_1 = q(\lambda^2 C - q^2 x^s - q^{-2} x^{-s}), \end{aligned} \quad (3.6)$$

where $\mathbf{g}_1 = \mathbf{g} \otimes 1$, $\mathbf{g} = q^{\frac{(s_0 - s_1)H}{s}}$, $g_2 = 1 \otimes g$, $g = \pi_1(\mathbf{g}) = \text{diag}(q^{\frac{s_0 - s_1}{s}}, q^{-\frac{s_0 - s_1}{s}})$, and t_2 is the transposition in the second component of the tensor product. Evaluating the first space of these L-operators for the fundamental representation, we obtain R-matrices of the 6-vertex model.

$$R(x) = q^{\frac{1}{2}}(\pi_1 \otimes 1)\mathbf{L}(x) = \begin{pmatrix} q - q^{-1}x^s & 0 & 0 & 0 \\ 0 & 1 - x^s & \lambda x^{s_1} & 0 \\ 0 & \lambda x^{s_0} & 1 - x^s & 0 \\ 0 & 0 & 0 & q - q^{-1}x^s \end{pmatrix}, \quad (3.7)$$

$$\bar{R}(x) = q^{\frac{1}{2}}(\pi_1 \otimes 1)\bar{\mathbf{L}}(x) = \begin{pmatrix} q - q^{-1}x^{-s} & 0 & 0 & 0 \\ 0 & 1 - x^{-s} & \lambda x^{-s_0} & 0 \\ 0 & \lambda x^{-s_1} & 1 - x^{-s} & 0 \\ 0 & 0 & 0 & q - q^{-1}x^{-s} \end{pmatrix}, \quad (3.8)$$

3.1 L-operators for Q-operators

Applying the limits (2.30) and (2.31) to (3.1) and (3.2), we define four type of L-operators as ^{7 8}

$$\mathbf{L}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \mathbf{L}(xq^{-\frac{\mu}{s}}) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} = \begin{pmatrix} q^{\frac{h_1}{2}} & \lambda x^{s_0} \mathbf{f}_1 q^{-\frac{h_1}{2}} \\ \lambda x^{s_1} \mathbf{e}_1 q^{\frac{h_1}{2}} & q^{-\frac{h_1}{2}} - q^{-1} x^s q^{\frac{h_1}{2}} \end{pmatrix}, \quad (3.10)$$

$$\mathbf{L}^{(2)}(x) = \lim_{q^{\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \mathbf{L}(xq^{\frac{\mu}{s}}) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} = \begin{pmatrix} q^{\frac{h_2}{2}} - q^{-1} x^s q^{-\frac{h_2}{2}} & \lambda x^{s_0} \mathbf{f}_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_1} \mathbf{e}_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} \end{pmatrix}, \quad (3.11)$$

$$\begin{aligned} \bar{\mathbf{L}}^{(1)}(x) &= \lim_{q^{-\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \left(\left(q^{\frac{(s_0-s_1)\mu H}{2s}} \otimes 1 \right) \bar{\mathbf{L}}(xq^{\frac{\mu}{s}}) \left(q^{-\frac{(s_0-s_1)\mu H}{2s}} \otimes 1 \right) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \right) \\ &= \begin{pmatrix} q^{\frac{h_1}{2}} & \lambda x^{-s_1} \mathbf{f}_1 q^{-\frac{h_1}{2}} \\ \lambda x^{-s_0} \mathbf{e}_1 q^{\frac{h_1}{2}} & q^{-\frac{h_1}{2}} - q^{-1} x^{-s} q^{\frac{h_1}{2}} \end{pmatrix}, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \bar{\mathbf{L}}^{(2)}(x) &= \lim_{q^{\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \left(\left(q^{\frac{(s_1-s_0)\mu H}{2s}} \otimes 1 \right) \bar{\mathbf{L}}(xq^{-\frac{\mu}{s}}) \left(q^{-\frac{(s_1-s_0)\mu H}{2s}} \otimes 1 \right) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \right) \\ &= \begin{pmatrix} q^{\frac{h_2}{2}} - q^{-1} x^{-s} q^{-\frac{h_2}{2}} & \lambda x^{-s_1} \mathbf{f}_2 q^{-\frac{h_2}{2}} \\ \lambda x^{-s_0} \mathbf{e}_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} \end{pmatrix}. \end{aligned} \quad (3.13)$$

As mentioned above, the L-operators (3.10) and (3.11) are essential ingredients in the construction of Baxter Q-operators associated with the Yang-Baxter algebra. For instance, for a spin chain with periodic boundary conditions, the corresponding Q-operators are defined as trace over product of these L-operators:

$$\mathbb{Q}^{(a)}(x) = (\mathbb{Z}^{(a)})^{-1} (\text{tr}_{W_a} \otimes 1^{\otimes L}) \left(q^{\alpha \mathbf{h}_a} \mathbf{L}_{0L}^{(a)}(x\xi_L^{-1}) \cdots \mathbf{L}_{02}^{(a)}(x\xi_1^{-1}) \mathbf{L}_{01}^{(a)}(x\xi_1^{-1}) \right), \quad a = 1, 2, \quad (3.14)$$

where $\xi_1, \dots, \xi_L \in \mathbb{C} \setminus \{0\}$ are inhomogeneities on the spectral parameter in the quantum space; $\alpha \in \mathbb{C}$; the trace is taken over the auxiliary space (a Fock space W_a for Osc_a denoted as 0). Here the normalization operator $\mathbb{Z}^{(a)}$ is defined by

$$\mathbb{Z}^{(a)} = (\pi_1(\xi_1) \otimes \cdots \otimes \pi_1(\xi_L)) \Delta^{\otimes(L-1)} (\text{tr}_{W_a} \otimes 1) (1 \otimes z)^{\mathbf{h}_a \otimes 1}, \quad z = q^{\alpha + \frac{1}{2} h_1}. \quad (3.15)$$

⁷The factor $q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}}$ came from (2.15) for $c_1 = -\mu$.

⁸ We could also use automorphisms of $U_q(\widehat{sl_2})$ or $U_q(sl_2)$ to derive various L-operators: for example,

$$\mathbf{L}^{(2)'}(x) = \lim_{q^{-\mu} \rightarrow 0} (\pi_{\mu}^{+} \circ \sigma \otimes 1) \mathbf{L}(xq^{-\frac{\mu}{s}}) q^{-\frac{\mu \otimes \pi_1(1)(h_0)}{2}} = \begin{pmatrix} q^{-\frac{h_1}{2}} - q^{-1} x^s q^{\frac{h_1}{2}} & \lambda x^{s_0} \mathbf{e}_1 q^{\frac{h_1}{2}} \\ \lambda \mathbf{f}_1 x^{s_1} q^{-\frac{h_1}{2}} & q^{\frac{h_1}{2}} \end{pmatrix}. \quad (3.9)$$

This could be a substitute of (3.11) (cf. [7]). Instead of using automorphisms, we will use Chevalley like generators of the q-oscillator algebras and take various different limits of the L-operators (as already demonstrated in [9]). A merit for this is that the resultant L-operators become just reductions of the original L-operators (for example, compare (3.1) with (3.10), (3.16) and (3.18)). This is also the case with the intertwining relations and the K-operators.

In Appendix G, we will propose Q-operators associated with different types of reflection equation algebra (cf. eqs. (4.53), (4.72)). Such operators are useful in the analysis of spin chains with open diagonal boundary conditions. For this purpose, we need additional L-operators (3.12) and (3.13) that are introduced as follows. Observe that the pair of L-operators (3.10) and (3.12) (or (3.11) and (3.13)) no longer satisfies relations corresponding to (3.5) and (3.6). For this reason, consider the following L-operators:

$$\begin{aligned}\check{\mathbf{L}}^{(1)}(x) &= \lim_{q^{-\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) q^{\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \left(q^{\frac{(s_0-s_1)\mu H}{2s}} \otimes 1 \right) \mathbf{L}(xq^{\frac{\mu}{s}}) \left(q^{-\frac{(s_0-s_1)\mu H}{2s}} \otimes 1 \right) q^{-\mu} \\ &= \begin{pmatrix} q^{\frac{h_1}{2}} - q^{-1}x^s q^{-\frac{h_1}{2}} & \lambda x^{s_0} \mathbf{f}_1 q^{-\frac{h_1}{2}} \\ \lambda x^{s_1} \mathbf{e}_1 q^{\frac{h_1}{2}} & -q^{-1}x^s q^{\frac{h_1}{2}} \end{pmatrix},\end{aligned}\quad (3.16)$$

$$\begin{aligned}\check{\bar{\mathbf{L}}}^{(1)}(x) &= \lim_{q^{-\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \left(q^{\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \bar{\mathbf{L}}(xq^{-\frac{\mu}{s}}) q^{-\mu} \right) \\ &= \begin{pmatrix} q^{\frac{h_1}{2}} - q^{-1}x^{-s} q^{-\frac{h_1}{2}} & \lambda x^{-s_1} \mathbf{f}_1 q^{-\frac{h_1}{2}} \\ \lambda x^{-s_0} \mathbf{e}_1 q^{\frac{h_1}{2}} & -q^{-1}x^{-s} q^{\frac{h_1}{2}} \end{pmatrix},\end{aligned}\quad (3.17)$$

$$\begin{aligned}\check{\mathbf{L}}^{(2)}(x) &= \lim_{q^{\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) q^{\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \left(q^{\frac{(s_1-s_0)\mu H}{2s}} \otimes 1 \right) \mathbf{L}(xq^{-\frac{\mu}{s}}) \left(q^{-\frac{(s_1-s_0)\mu H}{2s}} \otimes 1 \right) q^{\mu} \\ &= \begin{pmatrix} -q^{-1}x^s q^{-\frac{h_2}{2}} & \lambda x^{s_0} \mathbf{f}_2 q^{-\frac{h_2}{2}} \\ \lambda x^{s_1} \mathbf{e}_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} - q^{-1}x^s q^{\frac{h_2}{2}} \end{pmatrix},\end{aligned}\quad (3.18)$$

$$\begin{aligned}\check{\bar{\mathbf{L}}}^{(2)}(x) &= \lim_{q^{\mu} \rightarrow 0} (\pi_{\mu}^{+} \otimes 1) \left(q^{\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \bar{\mathbf{L}}(xq^{\frac{\mu}{s}}) q^{\mu} \right) \\ &= \begin{pmatrix} -q^{-1}x^{-s} q^{-\frac{h_2}{2}} & \lambda x^{-s_1} \mathbf{f}_2 q^{-\frac{h_2}{2}} \\ \lambda x^{-s_0} \mathbf{e}_2 q^{\frac{h_2}{2}} & q^{-\frac{h_2}{2}} - q^{-1}x^{-s} q^{\frac{h_2}{2}} \end{pmatrix}.\end{aligned}\quad (3.19)$$

Then the limits of (3.5)-(3.6) are given by

$$\mathbf{L}^{(1)}(x) \check{\bar{\mathbf{L}}}^{(1)}(x) = \check{\bar{\mathbf{L}}}^{(1)}(x) \mathbf{L}^{(1)}(x) = q^{-1}(q - x^{-s}), \quad (3.20)$$

$$\check{\mathbf{L}}^{(1)}(x) \bar{\mathbf{L}}^{(1)}(x) = \bar{\mathbf{L}}^{(1)}(x) \check{\mathbf{L}}^{(1)}(x) = q^{-1}(q - x^s), \quad (3.21)$$

$$\mathbf{L}^{(2)}(x) \check{\bar{\mathbf{L}}}^{(2)}(x) = \check{\bar{\mathbf{L}}}^{(2)}(x) \mathbf{L}^{(2)}(x) = q^{-1}(q^{-1} - x^{-s}), \quad (3.22)$$

$$\check{\mathbf{L}}^{(2)}(x) \bar{\mathbf{L}}^{(2)}(x) = \bar{\mathbf{L}}^{(2)}(x) \check{\mathbf{L}}^{(2)}(x) = q^{-1}(q^{-1} - x^s), \quad (3.23)$$

$$g_2 \mathbf{L}^{(1)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} \check{\bar{\mathbf{L}}}^{(1)}(x)^{t_2} = \check{\bar{\mathbf{L}}}^{(1)}(x)^{t_2} g_2 \mathbf{L}^{(1)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} = q(q - q^{-2}x^{-s}), \quad (3.24)$$

$$g_2 \check{\mathbf{L}}^{(1)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} \bar{\mathbf{L}}^{(1)}(x)^{t_2} = \bar{\mathbf{L}}^{(1)}(x)^{t_2} g_2 \check{\mathbf{L}}^{(1)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} = q(q - q^2x^s), \quad (3.25)$$

$$g_2 \mathbf{L}^{(2)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} \check{\bar{\mathbf{L}}}^{(2)}(x)^{t_2} = \check{\bar{\mathbf{L}}}^{(2)}(x)^{t_2} g_2 \mathbf{L}^{(2)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} = q(q^{-1} - q^{-2}x^{-s}), \quad (3.26)$$

$$g_2 \check{\mathbf{L}}^{(2)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} \bar{\mathbf{L}}^{(2)}(x)^{t_2} = \bar{\mathbf{L}}^{(2)}(x)^{t_2} g_2 \check{\mathbf{L}}^{(2)}(xq^{\frac{4}{s}})^{t_2} g_2^{-1} = q(q^{-1} - q^2x^s), \quad (3.27)$$

where the following relations are used:

$$\lim_{q^{-\mu} \rightarrow 0} \pi_{\mu}^{+}(Cq^{-\mu}) = \mathbf{e}_1 \mathbf{f}_1 + \frac{q^{\mathbf{h}_1 - 1}}{\lambda^2} = \frac{q}{\lambda^2}, \quad (3.28)$$

$$\lim_{q^{\mu} \rightarrow 0} \pi_{\mu}^{+}(Cq^{\mu}) = \mathbf{e}_2 \mathbf{f}_2 + \frac{q^{-\mathbf{h}_2 + 1}}{\lambda^2} = \frac{q^{-1}}{\lambda^2}. \quad (3.29)$$

These relations are among the conditions that are necessary to establish the commutativity of T- and Q-operators.

The intertwining relations for these L-operators have unusual form (cf. [8]). For example, (3.10) satisfies

$$\begin{aligned} (\mathbf{h}_1 \otimes 1 + 1 \otimes (E_{11} - E_{22})) \mathbf{L}^{(1)}(x/y) &= \mathbf{L}^{(1)}(x/y) (\mathbf{h}_1 \otimes 1 + 1 \otimes (E_{11} - E_{22})), \\ (1 \otimes y^{s_0} E_{21} + x^{s_0} \mathbf{f}_1 \otimes q^{E_{11} - E_{22}}) \mathbf{L}^{(1)}(x/y) &= \mathbf{L}^{(1)}(x/y) (x^{s_0} \mathbf{f}_1 \otimes 1 + q^{\mathbf{h}_1} \otimes y^{s_0} E_{21}), \\ (1 \otimes y^{s_1} E_{12} + x^{s_1} \mathbf{e}_1 \otimes q^{-E_{11} + E_{22}}) \mathbf{L}^{(1)}(x/y) &= \mathbf{L}^{(1)}(x/y) (x^{s_1} \mathbf{e}_1 \otimes 1 + q^{-\mathbf{h}_1} \otimes y^{s_1} E_{12}), \\ (x^{-s_0} \mathbf{e}_1 \otimes 1) \mathbf{L}^{(1)}(x/y) &= \mathbf{L}^{(1)}(x/y) (x^{-s_0} \mathbf{e}_1 \otimes q^{-E_{11} + E_{22}} + 1 \otimes y^{-s_0} E_{12}), \\ (q^{\mathbf{h}_1} \otimes y^{-s_1} E_{21} + x^{-s_1} \mathbf{f}_1 \otimes 1) \mathbf{L}^{(1)}(x/y) &= \mathbf{L}^{(1)}(x/y) (x^{-s_1} \mathbf{f}_1 \otimes q^{E_{11} - E_{22}}). \end{aligned} \quad (3.30)$$

They are derived from the first relation in (2.10) or (2.13) by taking the limits involved in the definitions (3.10)-(3.19) and (3.16)-(3.13). For more details, see Appendix B.

4 The augmented q -Onsager algebra

In this section, we first recall the definition of the augmented q -Onsager algebra [21, 22] through generators and relations. Realizations of the augmented q -Onsager algebra as either right or left coideal subalgebras of $U_q(\widehat{sl_2})$ are then introduced, and co-actions maps are given. Correspondingly, two different intertwiners of the augmented q -Onsager algebra are constructed explicitly. They solve a reflection equation and dual reflection in $U_q(sl_2) \otimes U_q(sl_2)$. Under the specialization π_1 , known results are recovered.

The augmented q -Onsager algebra - denoted below \mathcal{O}_q^{aug} - is generated by four generators $K_0, K_1, Z_1, \tilde{Z}_1$ subject to the defining relations [22]:

$$\begin{aligned} [K_0, K_1] &= 0, \\ K_0 Z_1 &= q^{-2} Z_1 K_0, \quad K_0 \tilde{Z}_1 = q^2 \tilde{Z}_1 K_0, \\ K_1 Z_1 &= q^2 Z_1 K_1, \quad K_1 \tilde{Z}_1 = q^{-2} \tilde{Z}_1 K_1, \\ [Z_1, [Z_1, [Z_1, \tilde{Z}_1]_{q^2}]_{q^{-2}}] &= \rho_{\text{diag}} Z_1 (K_1 K_1 - K_0 K_0) Z_1, \\ [\tilde{Z}_1, [\tilde{Z}_1, [\tilde{Z}_1, Z_1]_{q^2}]_{q^{-2}}] &= \rho_{\text{diag}} \tilde{Z}_1 (K_0 K_0 - K_1 K_1) \tilde{Z}_1 \end{aligned} \quad (4.1)$$

with

$$\rho_{\text{diag}} = \frac{(q^3 - q^{-3})(q^2 - q^{-2})^3}{q - q^{-1}}. \quad (4.2)$$

This algebra can be embedded into $U_q(\widehat{sl_2})$. Below, we will introduce two different realizations of the algebra \mathcal{O}_q^{aug} . They are related each other via the automorphism (2.5) of $U_q(\widehat{sl_2})$.

4.1 The first realization

In this subsection, the augmented q -Onsager algebra is realized as a right coideal subalgebra of $U_q(\widehat{sl_2})$. According to the coaction map, an intertwiner $\mathbf{K}(x)$ is explicitly constructed.

4.1.1 Right coideal subalgebra of $U_q(\widehat{sl_2})$ and the intertwiner $\mathbf{K}(x)$

A realization of the augmented q -Onsager algebra \mathcal{O}_q^{aug} , as a right coideal subalgebra of $U_q(\widehat{sl_2})$ is known [22]. Let ϵ_{\pm} be non-zero scalars. It is given by ⁹:

$$\begin{aligned} \mathbf{K}_0 &= \epsilon_+ q^{-h_0}, & \mathbf{K}_1 &= \epsilon_- q^{-h_1}, \\ \mathbf{Z}_1 &= (q^2 - q^{-2})(\epsilon_- q f_1 q^{-h_1} + \epsilon_+ e_0), \\ \tilde{\mathbf{Z}}_1 &= (q^2 - q^{-2})(\epsilon_- e_1 + \epsilon_+ q f_0 q^{-h_0}). \end{aligned} \quad (4.3)$$

Note that the automorphism (2.4) of $U_q(\widehat{sl_2})$ also gives the automorphism of \mathcal{O}_q^{aug} ,

$$\sigma : \quad \mathbf{K}_0 \mapsto \mathbf{K}_1, \quad \mathbf{K}_1 \mapsto \mathbf{K}_0, \quad \mathbf{Z}_1 \mapsto \tilde{\mathbf{Z}}_1, \quad \tilde{\mathbf{Z}}_1 \mapsto \mathbf{Z}_1, \quad (4.4)$$

under the condition $\sigma(\epsilon_{\pm}) = \epsilon_{\mp}$. The co-action map $\Delta : \mathcal{O}_q^{aug} \mapsto \mathcal{O}_q^{aug} \otimes U_q(\widehat{sl_2})$ that is compatible with the relations (4.1) corresponds to the restriction of the co-product (2.6) of $U_q(\widehat{sl_2})$ to \mathcal{O}_q^{aug} under the realization (4.3). It is such that:

$$\begin{aligned} \Delta(\mathbf{K}_0) &= \mathbf{K}_0 \otimes q^{-h_0}, & \Delta(\mathbf{K}_1) &= \mathbf{K}_1 \otimes q^{-h_1}, \\ \Delta(\mathbf{Z}_1) &= \mathbf{Z}_1 \otimes 1 + (q^2 - q^{-2})(\mathbf{K}_1 \otimes q f_1 q^{-h_1} + \mathbf{K}_0 \otimes e_0), \\ \Delta(\tilde{\mathbf{Z}}_1) &= \tilde{\mathbf{Z}}_1 \otimes 1 + (q^2 - q^{-2})(\mathbf{K}_1 \otimes e_1 + \mathbf{K}_0 \otimes q f_0 q^{-h_0}). \end{aligned} \quad (4.5)$$

On the other hand, the restriction of the opposite co-product Δ' of $U_q(\widehat{sl_2})$ to \mathcal{O}_q^{aug} gives the co-action map $\Delta' : \mathcal{O}_q^{aug} \mapsto U_q(\widehat{sl_2}) \otimes \mathcal{O}_q^{aug}$.

The restriction of the evaluation map (2.22) to \mathcal{O}_q^{aug} under (4.3) produces the evaluation map $\mathcal{O}_q^{aug} \mapsto U_q(sl_2)$:

$$\begin{aligned} \text{ev}_x(\mathbf{K}_0) &= \epsilon_+ q^H, & \text{ev}_x(\mathbf{K}_1) &= \epsilon_- q^{-H}, \\ \text{ev}_x(\mathbf{Z}_1) &= (q^2 - q^{-2})F(\epsilon_- x^{-s_1} q^{1-H} + \epsilon_+ x^{s_0}), \\ \text{ev}_x(\tilde{\mathbf{Z}}_1) &= (q^2 - q^{-2})E(\epsilon_- x^{s_1} + \epsilon_+ x^{-s_0} q^{H+1}). \end{aligned}$$

⁹ $(\mathbf{K}_0, \mathbf{K}_1, \mathbf{Z}_1, \tilde{\mathbf{Z}}_1)$ in (4.3) corresponds to $(\bar{\mathbf{K}}_0, \bar{\mathbf{K}}_1, -\bar{\tilde{\mathbf{Z}}}_1, -\bar{\mathbf{Z}}_1)$ in eq. (3.24) in [22] under the transformations $q \mapsto q^{-1}$ and $\epsilon_{\pm} \mapsto \bar{\epsilon}_{\mp}$.

Let us now consider the following intertwining relations associated with the first realization of the augmented q -Onsager algebra. They read:

$$\text{ev}_{x^{-1}}(a)\mathbf{K}(x) = \mathbf{K}(x)\text{ev}_x(a) \quad \text{for any } a \in \{\mathbf{K}_0, \mathbf{K}_1, \mathbf{Z}_1, \tilde{\mathbf{Z}}_1\}. \quad (4.6)$$

The equations for $a \in \{\mathbf{K}_0, \mathbf{K}_1\}$ imply that $[\mathbf{K}(x), q^H] = 0$. The equations for $a \in \{\mathbf{Z}_1, \tilde{\mathbf{Z}}_1\}$ give:

$$F(\epsilon_+ x^{-s_0} + \epsilon_- x^{s_1} q^{-H+1})\mathbf{K}(x) = \mathbf{K}(x)F(\epsilon_+ x^{s_0} + \epsilon_- x^{-s_1} q^{-H+1}), \quad (4.7)$$

$$E(\epsilon_- x^{-s_1} + \epsilon_+ x^{s_0} q^{H+1})\mathbf{K}(x) = \mathbf{K}(x)E(\epsilon_- x^{s_1} + \epsilon_+ x^{-s_0} q^{H+1}). \quad (4.8)$$

4.1.2 Solutions of the intertwining relations

According to the intertwining relations (4.7) and (4.8), solutions are defined up to a function $f(H)$ of the Cartan element ($f(x)$ is a function of $x \in \mathbb{C}$ with $f(x+2) = f(x)$). We find various different solutions with different non-trivial prefactors. Here we present two typical examples of them: ¹⁰

$$\mathbf{K}(x) = x^{s_0 H} \frac{\left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-H-1}; q^{-2}\right)_\infty}{\left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-H-1}; q^{-2}\right)_\infty} \quad \text{for } |q| > 1, \quad (4.9)$$

$$= x^{s_0 H} \frac{\left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-H+1}; q^2\right)_\infty}{\left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-H+1}; q^2\right)_\infty} \quad \text{for } |q| < 1, \quad (4.10)$$

and

$$\mathbf{K}(x) = x^{-s_1 H} \frac{\left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{H-1}; q^{-2}\right)_\infty}{\left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{H-1}; q^{-2}\right)_\infty} \quad \text{for } |q| > 1, \quad (4.11)$$

$$= x^{-s_1 H} \frac{\left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{H+1}; q^2\right)_\infty}{\left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{H+1}; q^2\right)_\infty} \quad \text{for } |q| < 1. \quad (4.12)$$

These solutions (4.9)-(4.12) satisfy

$$\mathbf{K}(x)\mathbf{K}(x^{-1}) = \mathbf{K}(x^{-1})\mathbf{K}(x) = 1, \quad \mathbf{K}(1) = 1. \quad (4.13)$$

Note that other expressions are given in Appendix E.

¹⁰After we obtained these solutions, we were informed by S. Belliard that he obtained a solution for the rational case $Y(sl_2)$.

4.1.3 Reflection equations

Let us define the R-operators in $U_q(sl_2) \otimes U_q(sl_2)$ by $\mathbf{R}_{12}(x, y) = (\mathbf{ev}_x \otimes \mathbf{ev}_y)\mathcal{R}$ and $\mathbf{R}_{21}(x, y) = (\mathbf{ev}_x \otimes \mathbf{ev}_y)\mathcal{R}_{21}$. Then the first relations in (2.10) and (2.13) produce the following intertwining relations

$$\begin{aligned} ((\mathbf{ev}_x \otimes \mathbf{ev}_y)\Delta'(a))\mathbf{R}_{12}(x, y) &= \mathbf{R}_{12}(x, y)((\mathbf{ev}_x \otimes \mathbf{ev}_y)\Delta(a)), \\ ((\mathbf{ev}_x \otimes \mathbf{ev}_y)\Delta(a))\mathbf{R}_{21}(x, y) &= \mathbf{R}_{21}(x, y)((\mathbf{ev}_x \otimes \mathbf{ev}_y)\Delta'(a)) \quad \text{for any } a \in U_q(\widehat{sl_2}). \end{aligned} \quad (4.14)$$

The intertwining relations (4.6) and (4.14) imply the following reflection equation in $U_q(sl_2) \otimes U_q(sl_2)$:

$$\mathbf{R}_{12}(x^{-1}, y^{-1})\mathbf{K}_1(x)\mathbf{R}_{21}(x, y^{-1})\mathbf{K}_2(y) = \mathbf{K}_2(y)\mathbf{R}_{12}(x^{-1}, y)\mathbf{K}_1(x)\mathbf{R}_{21}(x, y), \quad (4.15)$$

where we set $\mathbf{K}_1(x) = \mathbf{K}(x) \otimes 1$, $\mathbf{K}_2(y) = 1 \otimes \mathbf{K}(y)$. In fact, the intertwining relations $\mathbf{r}_i((\mathbf{ev}_x \otimes \mathbf{ev}_y)\Delta'(a)) = ((\mathbf{ev}_{x^{-1}} \otimes \mathbf{ev}_{y^{-1}})\Delta'(a))\mathbf{r}_i$ for any $a \in \mathcal{O}_q^{aug}$ follow¹¹ from (4.6) and (4.14), where the right hand side and the left hand side of (4.15) are denoted as \mathbf{r}_1 and \mathbf{r}_2 , respectively. Evaluating (4.15) for $1 \otimes \pi_1$, we obtain the following reflection equation for the L-operators

$$\mathbf{L}\left(\frac{y}{x}\right)\mathbf{K}_1(x)\overline{\mathbf{L}}(xy)K_2(y) = K_2(y)\mathbf{L}\left(\frac{1}{xy}\right)\mathbf{K}_1(x)\overline{\mathbf{L}}\left(\frac{x}{y}\right), \quad (4.16)$$

where we set $K(x) = \pi_1(\mathbf{K}(x))$ and used the difference property with respect to the spectral parameters. Expanding (4.16) with respect to the spectral parameter y , one recognizes the intertwining relations (4.6).

Specialization to π_1 : Evaluating (4.16) further for $\pi_1 \otimes 1$, we obtain the following reflection equation for the R-matrices.

$$R\left(\frac{y}{x}\right)K_1(x)\overline{R}(xy)K_2(y) = K_2(y)R\left(\frac{1}{xy}\right)K_1(x)\overline{R}\left(\frac{x}{y}\right). \quad (4.17)$$

The solution of (4.17) is given by

$$K(x) = \pi_1(\mathbf{K}(x)) = \kappa(x) \begin{pmatrix} x^{s_0}\epsilon_+ + x^{-s_1}\epsilon_- & 0 \\ 0 & x^{-s_0}\epsilon_+ + x^{s_1}\epsilon_- \end{pmatrix}. \quad (4.18)$$

Here $\kappa(x)$ is an overall factor. In case one uses (4.9) for $|q| > 1$, it reads

$$\kappa(x) = \frac{\left(-\frac{\epsilon_-}{\epsilon_+}x^s q^{-2}; q^{-2}\right)_\infty}{\epsilon_+ \left(-\frac{\epsilon_-}{\epsilon_+}x^{-s}; q^{-2}\right)_\infty}. \quad (4.19)$$

¹¹This is not a substitute of a proof of (4.15). One will be able to prove this on the level of irreducible representations of \mathcal{O}_q^{aug} by using the Schur's lemma (which fixes $\mathbf{r}_1 = \text{scalar} \times \mathbf{r}_2$) and an assumption on the behavior of \mathbf{r}_i with respect to the spectral parameters x, y (which determines $\text{scalar} = 1$). We do not have a universal K-matrix relevant to our discussion. Thus we do not have a proof of (4.15) on the level of the algebra. On the other hand, we have a proof of the reflection equation for the L-operators (4.16), which follows from this generic reflection equation (4.15).

Note that the solution (4.18) is a special case of the most general scalar solution¹² (F.1) of the reflection equation (4.17) [25, 26]. In the context of quantum integrable systems, it characterizes systems with arbitrary diagonal boundary conditions.

4.2 The second realization

Next, the augmented q -Onsager algebra is realized as a left coideal subalgebra of $U_q(\widehat{sl_2})$. An intertwiner $\overline{\mathbf{K}}(x)$ is explicitly constructed.

4.2.1 Left coideal subalgebra of $U_q(\widehat{sl_2})$ and the intertwiner $\overline{\mathbf{K}}(x)$

Using the automorphism (2.5) of $U_q(\widehat{sl_2})$, a second realization of the augmented q -Onsager algebra \mathcal{O}_q^{aug} , now as a left-coideal subalgebra of $U_q(\widehat{sl_2})$. Let $\bar{\epsilon}_{\pm}$ be non-zero scalars. It is given by¹³:

$$\begin{aligned}\overline{\mathbf{K}}_0 &= \tau(\mathbf{K}_0) = \bar{\epsilon}_- q^{h_0}, & \overline{\mathbf{K}}_1 &= \tau(\mathbf{K}_1) = \bar{\epsilon}_+ q^{h_1}, \\ \overline{\mathbf{Z}}_1 &= \tau(\mathbf{Z}_1) = (q^2 - q^{-2})(\bar{\epsilon}_+ q e_1 q^{h_1} + \bar{\epsilon}_- f_0), \\ \tilde{\overline{\mathbf{Z}}}_1 &= \tau(\tilde{\mathbf{Z}}_1) = (q^2 - q^{-2})(\bar{\epsilon}_+ f_1 + \bar{\epsilon}_- q e_0 q^{h_0}).\end{aligned}\tag{4.21}$$

where $\tau(\epsilon_{\pm}) = \bar{\epsilon}_{\mp}$ is assumed. Note that the automorphism (2.4) of $U_q(\widehat{sl_2})$ also gives the automorphism of \mathcal{O}_q^{aug} ,

$$\sigma : \quad \overline{\mathbf{K}}_0 \mapsto \overline{\mathbf{K}}_1, \quad \overline{\mathbf{K}}_1 \mapsto \overline{\mathbf{K}}_0, \quad \overline{\mathbf{Z}}_1 \mapsto \tilde{\overline{\mathbf{Z}}}_1, \quad \tilde{\overline{\mathbf{Z}}}_1 \mapsto \overline{\mathbf{Z}}_1,\tag{4.22}$$

under the condition $\sigma(\bar{\epsilon}_{\pm}) = \bar{\epsilon}_{\mp}$.

The co-action map $\Delta : \mathcal{O}_q^{aug} \mapsto U_q(\widehat{sl_2}) \otimes \mathcal{O}_q^{aug}$, that is compatible with the relations (4.1) corresponds to the restriction of the co-product (2.6) of $U_q(\widehat{sl_2})$ to \mathcal{O}_q^{aug} under the realization (4.21). It is such that:

$$\begin{aligned}\Delta(\overline{\mathbf{K}}_0) &= q^{h_0} \otimes \overline{\mathbf{K}}_0, & \Delta(\overline{\mathbf{K}}_1) &= q^{h_1} \otimes \overline{\mathbf{K}}_1, \\ \Delta(\overline{\mathbf{Z}}_1) &= 1 \otimes \overline{\mathbf{Z}}_1 + (q^2 - q^{-2})(q e_1 q^{h_1} \otimes \overline{\mathbf{K}}_1 + f_0 \otimes \overline{\mathbf{K}}_0), \\ \Delta(\tilde{\overline{\mathbf{Z}}}_1) &= 1 \otimes \tilde{\overline{\mathbf{Z}}}_1 + (q^2 - q^{-2})(f_1 \otimes \overline{\mathbf{K}}_1 + q e_0 q^{h_0} \otimes \overline{\mathbf{K}}_0).\end{aligned}\tag{4.23}$$

¹²Here the word ‘scalar’ means the matrix elements of the solution are not operators but scalar quantities.

¹³ $(\overline{\mathbf{K}}_0, \overline{\mathbf{K}}_1, \overline{\mathbf{Z}}_1, \tilde{\overline{\mathbf{Z}}}_1)$ in (4.21) corresponds to $(\mathbf{K}_0, \mathbf{K}_1, -\tilde{\mathbf{Z}}_1, -\mathbf{Z}_1)$ in eq. (3.21) in [22] under the transformations $q \mapsto q^{-1}$ and $\bar{\epsilon}_{\pm} \mapsto \epsilon_{\mp}$. In addition, instead of using the automorphism (2.5), one can also use an anti-automorphism $\bar{\tau}$ defined by

$$\bar{\tau}(e_i) = -f_i, \quad \bar{\tau}(f_i) = -e_i, \quad \bar{\tau}(h_i) = h_i, \quad \bar{\tau}(q) = q^{-1}, \quad i = 0, 1.\tag{4.20}$$

Recall the evaluation map (2.22). Define ¹⁴ $\overline{\mathbf{ev}}_x = \mathbf{ev}_x|_{(s_0, s_1) \rightarrow (-s_1, -s_0)}$. It follows:

$$\begin{aligned}\overline{\mathbf{ev}}_x(\overline{\mathbf{K}}_0) &= \bar{\epsilon}_- q^{-H}, \quad \overline{\mathbf{ev}}_x(\overline{\mathbf{K}}_1) = \bar{\epsilon}_+ q^H, \\ \overline{\mathbf{ev}}_x(\overline{\mathbf{Z}}_1) &= (q^2 - q^{-2})E(\bar{\epsilon}_+ x^{-s_0} q^{H+1} + \bar{\epsilon}_- x^{s_1}), \\ \overline{\mathbf{ev}}_x(\tilde{\overline{\mathbf{Z}}}_1) &= (q^2 - q^{-2})F(\bar{\epsilon}_+ x^{s_0} + \bar{\epsilon}_- x^{-s_1} q^{-H+1}).\end{aligned}$$

We now consider the following intertwining relations associated with the second realization of the augmented q -Onsager algebra:

$$\overline{\mathbf{ev}}_{x^{-1}q^{-\frac{2}{s}}}(a) \mathbf{g}\overline{\mathbf{K}}(x)^t = \mathbf{g}\overline{\mathbf{K}}(x)^t \overline{\mathbf{ev}}_{xq^{\frac{2}{s}}}(a) \quad \text{for any } a \in \{\overline{\mathbf{K}}_0, \overline{\mathbf{K}}_1, \overline{\mathbf{Z}}_1, \tilde{\overline{\mathbf{Z}}}_1\}, \quad (4.24)$$

where $\mathbf{g} = q^{H(s_0 - s_1)/s}$. Here t is the transposition. One may drop it from (4.24) since the \mathbf{K} -operator here is a diagonal operator. The equations for $a \in \{\overline{\mathbf{K}}_0, \overline{\mathbf{K}}_1\}$ imply that $[\overline{\mathbf{K}}(x)^t, q^H] = 0$. The equations for $a \in \{\overline{\mathbf{Z}}_1, \tilde{\overline{\mathbf{Z}}}_1\}$ give:

$$E(\bar{\epsilon}_+ q x^{s_0} q^{H+1} + \bar{\epsilon}_- q^{-1} x^{-s_1}) \overline{\mathbf{K}}(x)^t = \overline{\mathbf{K}}(x)^t E(\bar{\epsilon}_+ q^{-1} x^{-s_0} q^{H+1} + \bar{\epsilon}_- q x^{s_1}), \quad (4.25)$$

$$F(\bar{\epsilon}_+ q^{-1} x^{-s_0} + \bar{\epsilon}_- q x^{s_1} q^{-H+1}) \overline{\mathbf{K}}(x)^t = \overline{\mathbf{K}}(x)^t F(\bar{\epsilon}_+ q x^{s_0} + \bar{\epsilon}_- q^{-1} x^{-s_1} q^{-H+1}). \quad (4.26)$$

4.2.2 Solutions of the intertwining relations

The solutions of the intertwining relations (4.24) follow from the ones for the first realization (4.6) under the identification

$$\mathbf{K}(x) = \mathbf{g}\overline{\mathbf{K}}(xq^{-\frac{2}{s}})^t|_{\bar{\epsilon}_{\pm} = \epsilon_{\pm}}. \quad (4.27)$$

4.2.3 Reflection equations

The intertwining relations (4.24) and (4.14) imply the following dual reflection equation in $U_q(sl_2) \otimes U_q(sl_2)$:

$$\begin{aligned}\mathbf{R}_{12}(x^{-1}q^{-\frac{2}{s}}, y^{-1}q^{-\frac{2}{s}}) \mathbf{g}_1 \overline{\mathbf{K}}_1(x)^{t_1} \mathbf{R}_{21}(xq^{\frac{2}{s}}, y^{-1}q^{-\frac{2}{s}}) \mathbf{g}_2 \overline{\mathbf{K}}_2(y)^{t_2} = \\ = \mathbf{g}_2 \overline{\mathbf{K}}_2(y)^{t_2} \mathbf{R}_{12}(x^{-1}q^{-\frac{2}{s}}, yq^{\frac{2}{s}}) \mathbf{g}_1 \overline{\mathbf{K}}_1(x)^{t_1} \mathbf{R}_{21}(xq^{\frac{2}{s}}, yq^{\frac{2}{s}}).\end{aligned} \quad (4.28)$$

This also follows from (4.15) under the identification (4.27). Evaluating (4.28) for $1 \otimes \pi_1$, we obtain the following dual reflection equation for the L-operators [16]:

$$\mathbf{L}\left(\frac{y}{x}\right) \overline{\mathbf{K}}_1(x)^{t_1} g_2^{-1} \overline{\mathbf{L}}\left(xyq^{\frac{4}{s}}\right) g_2 \overline{\mathbf{K}}_2(y)^{t_2} = \overline{\mathbf{K}}_2(y)^{t_2} g_2 \mathbf{L}\left(\frac{q^{-\frac{4}{s}}}{xy}\right) g_2^{-1} \overline{\mathbf{K}}_1(x)^{t_1} \overline{\mathbf{L}}\left(\frac{x}{y}\right), \quad (4.29)$$

where $g = \pi_1(\mathbf{g}^{-1})$ and $\overline{\mathbf{K}}(x) = \pi_1(\overline{\mathbf{K}}(x))$. Taking appropriate limits in the variable y of the reflection equation (4.16), one recovers the intertwining relations (4.25)-(4.26).

¹⁴One may also interpret this as $\overline{\mathbf{ev}}_x = \mathbf{ev}_x \circ \sigma \circ \tau|_{\epsilon_{\pm} = \bar{\epsilon}_{\pm}}$, $\tau(\bar{\epsilon}_{\pm}) = \epsilon_{\mp}$.

Specialization to π_1 : Specializing to the two-dimensional representation of $U_q(sl_2)$, the solution of the intertwining relations is unique (up to an overall factor). It reads:

$$\overline{K}(x) = \pi_1(\overline{\mathbf{K}}(x)) = \overline{\kappa}(x) \begin{pmatrix} qx^{s_0}\overline{\epsilon}_+ + q^{-1}x^{-s_1}\overline{\epsilon}_- & 0 \\ 0 & q^{-1}x^{-s_0}\overline{\epsilon}_+ + qx^{s_1}\overline{\epsilon}_- \end{pmatrix}. \quad (4.30)$$

Here $\overline{\kappa}(x)$ is an overall factor. In case one uses (4.9) for $|q| > 1$ and (4.27), it reads

$$\overline{\kappa}(x) = \kappa(xq^{\frac{2}{s}})|_{\epsilon_{\pm}=\overline{\epsilon}_{\pm}} = \frac{\left(-\frac{\overline{\epsilon}_-}{\overline{\epsilon}_+}x^s; q^{-2}\right)_{\infty}}{\overline{\epsilon}_+ \left(-\frac{\overline{\epsilon}_-}{\overline{\epsilon}_+}x^{-s}q^{-2}; q^{-2}\right)_{\infty}}. \quad (4.31)$$

By construction, it solves the specialization of the dual reflection equation¹⁵ (4.29):

$$R\left(\frac{y}{x}\right) \overline{K}_1(x)^{t_1} g_1 \overline{R}\left(xyq^{\frac{4}{s}}\right) g_1^{-1} \overline{K}_2(y)^{t_2} = \overline{K}_2(y)^{t_2} g_1^{-1} R\left(\frac{q^{-\frac{4}{s}}}{xy}\right) g_1 \overline{K}_1(x)^{t_1} \overline{R}\left(\frac{x}{y}\right). \quad (4.32)$$

The solution (4.30) is a special case of the general scalar solution (F.2) of the dual reflection equation. Note that the solutions of the reflection and dual reflection equations are related by the following transformation (see [16]):

$$\overline{K}(x) = K^t(xq^{2/s}) g^{-1}|_{\overline{\epsilon}_{\pm}=\epsilon_{\pm}}.$$

4.3 Limit of intertwining relations and their solutions: K-operators for Q-operators

In this subsection, we consider the limit $q^{\mp\mu} \rightarrow 0$ of the intertwining relations and their solutions in the Verma module π_{μ}^{+} . In order to avoid divergences, we have to renormalize the generators of the augmented q-Onsager algebra and the K-operators. The resulting K-operators serve as building blocks of Q-operators in that they solve reflection equations for the L-operators for Q-operators. Similar K-operators for the rational case can be found in [19].

4.3.1 The first realization

The limit $q^{-\mu} \rightarrow 0$ under the shift $x \rightarrow xq^{\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.7)-(4.8) and multiply the factors $x^{-s_0\mu}q^{-\frac{\mu s_0 H}{s}-\frac{2s_0\mu}{s}}$ (for (4.7)) and $x^{-s_0\mu}q^{-\frac{\mu s_0 H}{s}-\frac{2s_1\mu}{s}}$ (for (4.8)) from the left. We find that the limit $q^{-\mu} \rightarrow 0$ for this in π_{μ}^{+} produces

$$\mathbf{f}_1(\epsilon_+x^{-s_0} + \epsilon_-x^{s_1}q^{-\mathbf{h}_1+1})\mathbf{K}^{(1)}(x) = \mathbf{K}^{(1)}(x)\mathbf{f}_1(\epsilon_+x^{s_0}), \quad (4.33)$$

$$\mathbf{e}_1(\epsilon_+x^{s_0}q^{\mathbf{h}_1+1})\mathbf{K}^{(1)}(x) = \mathbf{K}^{(1)}(x)\mathbf{e}_1(\epsilon_+x^{-s_0}q^{\mathbf{h}_1+1} + \epsilon_-x^{s_1}), \quad (4.34)$$

¹⁵One can modify this by the relations $g_1\overline{R}\left(xyq^{-\frac{4}{s}}\right)g_1^{-1} = g_2^{-1}\overline{R}\left(xyq^{-\frac{4}{s}}\right)g_2$, $g_1^{-1}R\left(\frac{q^{\frac{4}{s}}}{xy}\right)g_1 = g_2R\left(\frac{q^{\frac{4}{s}}}{xy}\right)g_2^{-1}$.

where we set

$$\mathbf{K}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} x^{-s_0\mu} \pi_\mu^+ (q^{-\frac{s_0\mu H}{s}} \mathbf{K}(q^{\frac{\mu}{s}} x)). \quad (4.35)$$

Similarly, we derive the limit of the equations (4.6) for the renormalized generators $\mathbf{K}_0 q^{-\mu}$ and $\mathbf{K}_1 q^\mu$:

$$q^{\pm \mathbf{h}_1} \mathbf{K}^{(1)}(x) = \mathbf{K}^{(1)}(x) q^{\pm \mathbf{h}_1}. \quad (4.36)$$

Then a solution of (4.33), (4.34) and (4.36) is given by the limit of (4.9) or (4.10):

$$\begin{aligned} \mathbf{K}^{(1)}(x) &= x^{s_0 \mathbf{h}_1} \left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-\mathbf{h}_1-1}; q^{-2} \right)_\infty \quad \text{for } |q| > 1, \\ &= x^{s_0 \mathbf{h}_1} \left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-\mathbf{h}_1+1}; q^2 \right)_\infty^{-1} \quad \text{for } |q| < 1. \end{aligned} \quad (4.37)$$

The limit $q^{-\mu} \rightarrow 0$ under the shift $x \rightarrow xq^{-\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.7)-(4.8) and multiply the factor $x^{-s_0\mu} q^{\frac{\mu s_0 H}{s}}$ (for (4.7)) and $x^{-s_0\mu} q^{\frac{\mu s_0 H}{s} - 2\mu}$ (for (4.8)) from the left. We find that the limit $q^{-\mu} \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{f}_1(\epsilon_+ x^{-s_0}) \check{\mathbf{K}}^{(1)}(x) = \check{\mathbf{K}}^{(1)}(x) \mathbf{f}_1(\epsilon_+ x^{s_0} + \epsilon_- x^{-s_1} q^{-\mathbf{h}_1+1}), \quad (4.38)$$

$$\mathbf{e}_1(\epsilon_- x^{-s_1} + \epsilon_+ x^{s_0} q^{\mathbf{h}_1+1}) \check{\mathbf{K}}^{(1)}(x) = \check{\mathbf{K}}^{(1)}(x) \mathbf{e}_1(\epsilon_+ x^{-s_0} q^{\mathbf{h}_1+1}), \quad (4.39)$$

where we set

$$\check{\mathbf{K}}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} x^{-s_0\mu} \pi_\mu^+ (q^{\frac{s_0\mu H}{s}} \mathbf{K}(q^{-\frac{\mu}{s}} x)). \quad (4.40)$$

Similarly, we derive the limit of the equations (4.6) for the renormalized generators $\mathbf{K}_0 q^{-\mu}$ and $\mathbf{K}_1 q^\mu$:

$$q^{\pm \mathbf{h}_1} \check{\mathbf{K}}^{(1)}(x) = \check{\mathbf{K}}^{(1)}(x) q^{\pm \mathbf{h}_1}. \quad (4.41)$$

Then a solution of (4.38), (4.39) and (4.41) is given by the limit of (4.9) or (4.10):

$$\begin{aligned} \check{\mathbf{K}}^{(1)}(x) &= x^{s_0 \mathbf{h}_1} \left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-\mathbf{h}_1-1}; q^{-2} \right)_\infty^{-1} \quad \text{for } |q| > 1, \\ &= x^{s_0 \mathbf{h}_1} \left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-\mathbf{h}_1+1}; q^2 \right)_\infty \quad \text{for } |q| < 1. \end{aligned} \quad (4.42)$$

The limit $q^\mu \rightarrow 0$ under the shift $x \rightarrow xq^{-\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.7)-(4.8) and multiply the factors $x^{s_1\mu} q^{-\frac{\mu s_1 H}{s} + \frac{2s_0\mu}{s}}$ (for (4.7))

and $x^{s_1\mu}q^{-\frac{\mu s_1 H}{s} + \frac{2s_1\mu}{s}}$ (for (4.8)) from the left. We find that the limit $q^\mu \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{f}_2(\epsilon_- x^{s_1} q^{-\mathbf{h}_2+1}) \mathbf{K}^{(2)}(x) = \mathbf{K}^{(2)}(x) \mathbf{f}_2(\epsilon_+ x^{s_0} + \epsilon_- x^{-s_1} q^{-\mathbf{h}_2+1}), \quad (4.43)$$

$$\mathbf{e}_2(\epsilon_+ x^{s_0} q^{\mathbf{h}_2+1} + \epsilon_- x^{-s_1}) \mathbf{K}^{(2)}(x) = \mathbf{K}^{(2)}(x) \mathbf{e}_2(\epsilon_- x^{s_1}), \quad (4.44)$$

where we define

$$\mathbf{K}^{(2)}(x) = \lim_{q^\mu \rightarrow 0} x^{s_1\mu} \pi_\mu^+(q^{-\frac{s_1\mu H}{s}} \mathbf{K}(q^{-\frac{\mu}{s}} x)). \quad (4.45)$$

Similarly, we derive the limit of the equations (4.6) for the renormalized generators $\mathbf{K}_0 q^{-\mu}$ and $\mathbf{K}_1 q^\mu$:

$$q^{\pm \mathbf{h}_2} \mathbf{K}^{(2)}(x) = \mathbf{K}^{(2)}(x) q^{\pm \mathbf{h}_2}. \quad (4.46)$$

Then a solution of (4.43), (4.44) and (4.46) is given by the limit of (4.11) or (4.12) in π_μ^+ :

$$\begin{aligned} \mathbf{K}^{(2)}(x) &= x^{-s_1 \mathbf{h}_2} \left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{\mathbf{h}_2-1}; q^{-2} \right)_\infty \quad \text{for } |q| > 1, \\ &= x^{-s_1 \mathbf{h}_2} \left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{\mathbf{h}_2+1}; q^2 \right)_\infty^{-1} \quad \text{for } |q| < 1. \end{aligned} \quad (4.47)$$

The limit $q^\mu \rightarrow 0$ under the shift $x \rightarrow xq^{\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.7)-(4.8) and multiply the factor $x^{s_1\mu} q^{\frac{\mu s_1 H}{s} + 2\mu}$ (for (4.7)) and $x^{s_1\mu} q^{\frac{\mu s_1 H}{s}}$ (for (4.8)) from the left. We find that the limit $q^\mu \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{f}_2(\epsilon_+ x^{-s_0} + \epsilon_- x^{s_1} q^{-\mathbf{h}_2+1}) \check{\mathbf{K}}^{(2)}(x) = \check{\mathbf{K}}^{(2)}(x) \mathbf{f}_2(\epsilon_- x^{-s_1} q^{-\mathbf{h}_2+1}), \quad (4.48)$$

$$\mathbf{e}_2(\epsilon_- x^{-s_1}) \check{\mathbf{K}}^{(2)}(x) = \check{\mathbf{K}}^{(2)}(x) \mathbf{e}_2(\epsilon_- x^{s_1} + \epsilon_+ x^{-s_0} q^{\mathbf{h}_2+1}), \quad (4.49)$$

where we define

$$\check{\mathbf{K}}^{(2)}(x) = \lim_{q^\mu \rightarrow 0} x^{s_1\mu} \pi_\mu^+(q^{\frac{s_1\mu H}{s}} \mathbf{K}(q^{\frac{\mu}{s}} x)). \quad (4.50)$$

Similarly, we derive the limit of the equations (4.6) for the renormalized generators $\mathbf{K}_0 q^{-\mu}$ and $\mathbf{K}_1 q^\mu$:

$$q^{\pm \mathbf{h}_2} \check{\mathbf{K}}^{(2)}(x) = \check{\mathbf{K}}^{(2)}(x) q^{\pm \mathbf{h}_2}. \quad (4.51)$$

Then a solution of (4.48), (4.49) and (4.51) is give by the limit of (4.11) or (4.12) in π_μ^+ :

$$\begin{aligned} \check{\mathbf{K}}^{(2)}(x) &= x^{-s_1 \mathbf{h}_2} \left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{\mathbf{h}_2-1}; q^{-2} \right)_\infty^{-1} \quad \text{for } |q| > 1, \\ &= x^{-s_1 \mathbf{h}_2} \left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{\mathbf{h}_2+1}; q^2 \right)_\infty \quad \text{for } |q| < 1. \end{aligned} \quad (4.52)$$

Renormalizing (4.16) and taking the limits $q^{\mp\mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain the reflection equations¹⁶ for L-operators for Q-operators:

$$\mathbf{L}^{(a)}\left(\frac{y}{x}\right) \mathbf{K}_1^{(a)}(x) \bar{\mathbf{L}}^{(a)}(xy) K_2(y) = K_2(y) \mathbf{L}^{(a)}\left(\frac{1}{xy}\right) \mathbf{K}_1^{(a)}(x) \bar{\mathbf{L}}^{(a)}\left(\frac{x}{y}\right), \quad (4.53)$$

$$\check{\mathbf{L}}^{(a)}\left(\frac{y}{x}\right) \check{\mathbf{K}}_1^{(a)}(x) \check{\bar{\mathbf{L}}}^{(a)}(xy) K_2(y) = K_2(y) \check{\mathbf{L}}^{(a)}\left(\frac{1}{xy}\right) \check{\mathbf{K}}_1^{(a)}(x) \check{\bar{\mathbf{L}}}^{(a)}\left(\frac{x}{y}\right), \quad a = 1, 2. \quad (4.54)$$

4.3.2 The second realization

The limit $q^{-\mu} \rightarrow 0$ under the shift $x \rightarrow xq^{\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.25)-(4.26) and multiply the factors $x^{-s_0\mu} q^{-\frac{\mu s_0 H}{s} - \mu - \frac{2s_1\mu}{s}}$ (for (4.25)) and $x^{-s_0\mu} q^{-\frac{\mu s_0 H}{s} - \mu - \frac{2s_0\mu}{s}}$ (for (4.26)) from the left. We find that the limit $q^{-\mu} \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{e}_1(\bar{\epsilon}_+ q x^{s_0} q^{\mathbf{h}_1+1}) \bar{\mathbf{K}}^{(1)}(x)^t = \bar{\mathbf{K}}^{(1)}(x)^t \mathbf{e}_1(\bar{\epsilon}_+ q^{-1} x^{-s_0} q^{\mathbf{h}_1+1} + \bar{\epsilon}_- q x^{s_1}), \quad (4.55)$$

$$\mathbf{f}_1(\bar{\epsilon}_+ q^{-1} x^{-s_0} + \bar{\epsilon}_- q x^{s_1} q^{-\mathbf{h}_1+1}) \bar{\mathbf{K}}^{(1)}(x)^t = \bar{\mathbf{K}}^{(1)}(x)^t \mathbf{f}_1(\bar{\epsilon}_+ q x^{s_0}), \quad (4.56)$$

where we set

$$\bar{\mathbf{K}}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} x^{-s_0\mu} \pi_\mu^+(\bar{\mathbf{K}}(q^{\frac{\mu}{s}} x) q^{-\frac{s_0\mu H}{s} - \mu}). \quad (4.57)$$

¹⁶ The limit does not change the form of the reflection equation for the L-operators (4.16). Making the shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.16), and multiplying $(\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))(q^{-\mu h_1} \otimes q^{-\mu h_1}) = q^{-\mu H} \otimes q^{-\mu(E_{11} - E_{22})}$ from the left and $(q^{\frac{s_1\mu H}{s}} \otimes 1)$ from the right, we obtain

$$\begin{aligned} & \mathbf{L}\left(\frac{yq^{-\frac{\mu}{s}}}{x}\right) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \left(q^{-\frac{s_0\mu H}{s}} \mathbf{K}(xq^{\frac{\mu}{s}}) \otimes 1\right) \left(q^{\frac{(s_0-s_1)\mu H}{2s}} \otimes 1\right) \bar{\mathbf{L}}\left(q^{\frac{\mu}{s}} xy\right) \left(q^{-\frac{(s_0-s_1)\mu H}{2s}} \otimes 1\right) \\ & \quad \times q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} K_2(y) = \\ & = K_2(y) \mathbf{L}\left(\frac{q^{-\frac{\mu}{s}}}{xy}\right) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}} \left(q^{-\frac{s_0\mu H}{s}} \mathbf{K}(xq^{\frac{\mu}{s}}) \otimes 1\right) \left(q^{\frac{(s_0-s_1)\mu H}{2s}} \otimes 1\right) \bar{\mathbf{L}}\left(\frac{xq^{\frac{\mu}{s}}}{y}\right) \\ & \quad \times \left(q^{-\frac{(s_0-s_1)\mu H}{2s}} \otimes 1\right) q^{-\frac{\mu \otimes \pi_1(1)(h_1)}{2}}. \end{aligned}$$

Here we used the fact that $\pi_1(y^{-1})(h_1)$ is independent of y , and the commutativity (2.11). Multiplying $x^{-s_0\mu}$ and taking the limit $q^{-\mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we arrive at (4.53) for $a = 1$. Expanding (4.53) with respect to the spectral parameter y , one can reproduce (4.33), (4.34) and (4.36). Similarly, making the shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.16), multiplying $x^{-s_0\mu} q^{-2\mu} (\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))(q^{-\frac{s_1\mu h_1}{s}} \otimes 1)(q^{\mu h_1} \otimes q^{\mu h_1})$ from the left, and taking the limit $q^{-\mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.54) for $a = 1$; making the shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.16), multiplying $x^{s_1\mu} (\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))(q^{-\mu h_1} \otimes q^{-\mu h_1})$ from the left and $(q^{\frac{s_0\mu H}{s}} \otimes 1)$ from the right, and taking the limit $q^\mu \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.53) for $a = 2$; making the shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.16), multiplying $x^{s_1\mu} q^{2\mu} (\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))(q^{-\frac{s_0\mu h_1}{s}} \otimes 1)(q^{\mu h_1} \otimes q^{\mu h_1})$ from the left, and taking the limit $q^\mu \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.54) for $a = 2$.

Similarly, we derive the limit of the equations (4.24) for the renormalized generators $\bar{K}_0 q^\mu$ and $\bar{K}_1 q^{-\mu}$:

$$q^{\mp h_1} \bar{K}^{(1)}(x) = \bar{K}^{(1)}(x) q^{\mp h_1}. \quad (4.58)$$

The limit $q^{-\mu} \rightarrow 0$ under the shift $x \rightarrow xq^{-\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.25)-(4.26) and multiply the factors $x^{-s_0\mu} q^{\frac{\mu s_0 H}{s} - 3\mu}$ (for (4.25)) and $x^{-s_0\mu} q^{\frac{\mu s_0 H}{s} - \mu}$ (for (4.26)) from the left. We find that the limit $q^{-\mu} \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{e}_1(\bar{\epsilon}_+ q x^{s_0} q^{h_1+1} + \bar{\epsilon}_- q^{-1} x^{-s_1}) \check{\bar{K}}^{(1)}(x)^t = \check{\bar{K}}^{(1)}(x)^t \mathbf{e}_1(\bar{\epsilon}_+ q^{-1} x^{-s_0} q^{h_1+1}), \quad (4.59)$$

$$\mathbf{f}_1(\bar{\epsilon}_+ q^{-1} x^{-s_0}) \check{\bar{K}}^{(1)}(x)^t = \check{\bar{K}}^{(1)}(x)^t \mathbf{f}_1(\bar{\epsilon}_+ q x^{s_0} + \bar{\epsilon}_- q^{-1} x^{-s_1} q^{-h_1+1}), \quad (4.60)$$

where we set

$$\check{\bar{K}}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} x^{-s_0\mu} \pi_\mu^+(\bar{K}(q^{-\frac{\mu}{s}} x) q^{\frac{s_0\mu H}{s} - \mu}). \quad (4.61)$$

Similarly, we derive the limit of the equations (4.24) for the renormalized generators $\bar{K}_0 q^\mu$ and $\bar{K}_1 q^{-\mu}$:

$$q^{\mp h_1} \check{\bar{K}}^{(1)}(x) = \check{\bar{K}}^{(1)}(x) q^{\mp h_1}. \quad (4.62)$$

The limit $q^\mu \rightarrow 0$ under the shift $x \rightarrow xq^{-\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.25)-(4.26) and multiply the factors $x^{s_1\mu} q^{-\frac{\mu s_1 H}{s} + \frac{2s_1\mu}{s} + \mu}$ (for (4.25)) and $x^{s_1\mu} q^{-\frac{\mu s_1 H}{s} + \frac{2s_0\mu}{s} + \mu}$ (for (4.26)) from the left. We find that the limit $q^\mu \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{e}_2(\bar{\epsilon}_+ q x^{s_0} q^{h_2+1} + \bar{\epsilon}_- q^{-1} x^{-s_1}) \bar{K}^{(2)}(x)^t = \bar{K}^{(2)}(x)^t \mathbf{e}_2(\bar{\epsilon}_- q x^{s_1}), \quad (4.63)$$

$$\mathbf{f}_2(\bar{\epsilon}_- q x^{s_1} q^{-h_2+1}) \bar{K}^{(2)}(x)^t = \bar{K}^{(2)}(x)^t \mathbf{f}_2(\bar{\epsilon}_+ q x^{s_0} + \bar{\epsilon}_- q^{-1} x^{-s_1} q^{-h_2+1}), \quad (4.64)$$

where we set

$$\bar{K}^{(2)}(x) = \lim_{q^\mu \rightarrow 0} x^{s_1\mu} \pi_\mu^+(\bar{K}(q^{-\frac{\mu}{s}} x) q^{-\frac{s_1\mu H}{s} + \mu}). \quad (4.65)$$

Similarly, we derive the limit of the equations (4.24) for the renormalized generators $\bar{K}_0 q^\mu$ and $\bar{K}_1 q^{-\mu}$:

$$q^{\mp h_2} \bar{K}^{(2)}(x) = \bar{K}^{(2)}(x) q^{\mp h_2}. \quad (4.66)$$

The limit $q^\mu \rightarrow 0$ under the shift $x \rightarrow xq^{\frac{\mu}{s}}$: Let us make a shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.25)-(4.26) and multiply the factors $x^{s_1\mu}q^{\frac{\mu s_1 H}{s}+\mu}$ (for (4.25)) and $x^{s_1\mu}q^{\frac{\mu s_1 H}{s}+3\mu}$ (for (4.26)) from the left. We find that the limit $q^\mu \rightarrow 0$ for this in π_μ^+ produces

$$\mathbf{e}_2(\bar{\epsilon}_-q^{-1}x^{-s_1})\check{\mathbf{K}}^{(2)}(x)^t = \check{\mathbf{K}}^{(2)}(x)^t \mathbf{e}_2(\bar{\epsilon}_+q^{-1}x^{-s_0}q^{\mathbf{h}_2+1} + \bar{\epsilon}_-qx^{s_1}), \quad (4.67)$$

$$\mathbf{f}_2(\bar{\epsilon}_+q^{-1}x^{-s_0} + \bar{\epsilon}_-qx^{s_1}q^{-\mathbf{h}_2+1})\check{\mathbf{K}}^{(2)}(x)^t = \check{\mathbf{K}}^{(2)}(x)^t \mathbf{f}_2(\bar{\epsilon}_-q^{-1}x^{-s_1}q^{-\mathbf{h}_2+1}), \quad (4.68)$$

where we set

$$\check{\mathbf{K}}^{(2)}(x) = \lim_{q^\mu \rightarrow 0} x^{s_1\mu} \pi_\mu^+(\bar{\mathbf{K}}(q^{\frac{\mu}{s}}x)q^{\frac{s_1\mu H}{s}+\mu}). \quad (4.69)$$

Similarly, we derive the limit of the equations (4.24) for the renormalized generators $\bar{\mathbf{K}}_0q^\mu$ and $\bar{\mathbf{K}}_1q^{-\mu}$:

$$q^{\mp \mathbf{h}_2} \check{\mathbf{K}}^{(2)}(x) = \check{\mathbf{K}}^{(2)}(x) q^{\mp \mathbf{h}_2}. \quad (4.70)$$

Explicit expressions of the second intertwiner $\bar{\mathbf{K}}(x)$ are given by (4.27) with (4.9)-(4.12). Thus explicit expressions of (4.57), (4.61), (4.65) and (4.69) are obtained via

$$\bar{\mathbf{K}}^{(a)}(x) = \mathbf{K}^{(a)}(xq^{\frac{2}{s}})^t q^{-\frac{(s_0-s_1)\mathbf{h}_a}{s}}|_{\epsilon_\pm=\bar{\epsilon}_\pm}, \quad \check{\mathbf{K}}^{(a)}(x) = \check{\mathbf{K}}^{(a)}(xq^{\frac{2}{s}})^t q^{-\frac{(s_0-s_1)\mathbf{h}_a}{s}}|_{\epsilon_\pm=\bar{\epsilon}_\pm}, \quad a = 1, 2, \quad (4.71)$$

and (4.37), (4.42), (4.47) and (4.52).

Renormalizing ¹⁷ (4.29) and taking the limits $q^{\mp \mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain the dual reflection equations for L-operators for Q-operators:

$$\begin{aligned} \mathbf{L}^{(a)}\left(\frac{y}{x}\right) \bar{\mathbf{K}}_1^{(a)}(x)^{t_1} g_2^{-1} \bar{\mathbf{L}}^{(a)}\left(xyq^{\frac{4}{s}}\right) g_2 \bar{K}_2(y)^{t_2} = \\ = \bar{K}_2(y)^{t_2} g_2 \mathbf{L}^{(a)}\left(\frac{q^{-\frac{4}{s}}}{xy}\right) g_2^{-1} \bar{\mathbf{K}}_1^{(a)}(x)^{t_1} \bar{\mathbf{L}}^{(a)}\left(\frac{x}{y}\right), \end{aligned} \quad (4.72)$$

$$\begin{aligned} \check{\mathbf{L}}^{(a)}\left(\frac{y}{x}\right) \check{\mathbf{K}}_1^{(a)}(x)^{t_1} g_2^{-1} \check{\mathbf{L}}^{(a)}\left(xyq^{\frac{4}{s}}\right) g_2 \bar{K}_2(y)^{t_2} = \\ = \bar{K}_2(y)^{t_2} g_2 \check{\mathbf{L}}^{(a)}\left(\frac{q^{-\frac{4}{s}}}{xy}\right) g_2^{-1} \check{\mathbf{K}}_1^{(a)}(x)^{t_1} \check{\mathbf{L}}^{(a)}\left(\frac{x}{y}\right), \quad a = 1, 2. \end{aligned} \quad (4.73)$$

¹⁷ Making the shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.29), multiplying $x^{-s_0\mu}q^{-\mu}(\mathbf{ev}_x \otimes \pi_1(y))((q^{\frac{s_1\mu h_1}{s}} \otimes 1)(q^{-\mu h_1} \otimes q^{-\mu h_1}))$ from the right, and taking the limit $q^{-\mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.72) for $a = 1$; making the shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.29), multiplying $x^{-s_0\mu}q^{-2\mu}(\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))((q^{\mu h_1} \otimes q^{\mu h_1})(q^{-\frac{s_1\mu h_1}{s}} \otimes 1))$ from the left, and taking the limit $q^{-\mu} \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.73) for $a = 1$; making the shift $x \rightarrow xq^{-\frac{\mu}{s}}$ on the spectral parameter in (4.29), multiplying $x^{s_1\mu}q^\mu(\mathbf{ev}_x \otimes \pi_1(y))((q^{\frac{s_0\mu h_1}{s}} \otimes 1)(q^{-\mu h_1} \otimes q^{-\mu h_1}))$ from the right, and taking the limit $q^\mu \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.72) for $a = 2$; making the shift $x \rightarrow xq^{\frac{\mu}{s}}$ on the spectral parameter in (4.29), multiplying $x^{s_1\mu}q^{2\mu}(\mathbf{ev}_{x^{-1}} \otimes \pi_1(y^{-1}))((q^{\mu h_1} \otimes q^{\mu h_1})(q^{-\frac{s_0\mu h_1}{s}} \otimes 1))$ from the left, and taking the limit $q^\mu \rightarrow 0$ in $\pi_\mu^+ \otimes 1$, we obtain (4.73) for $a = 2$.

4.3.3 Rational limit $q \rightarrow 1$

One can take the rational limit $q \rightarrow 1$ of the formulas in this paper easily. The q -gamma function (see for example, [27]) is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x} \quad \text{for} \quad |q| < 1. \quad (4.74)$$

This reduces to the normal gamma function in the rational limit.

$$\lim_{q \rightarrow 1} \Gamma_q(x) = \Gamma(x). \quad (4.75)$$

Let us define the rational limit of the generators of the q -oscillator algebra Osc_1 by

$$\mathbf{a} = \lim_{q \rightarrow 1} \lambda \mathbf{e}_1, \quad \mathbf{a}^\dagger = \lim_{q \rightarrow 1} \mathbf{f}_1, \quad \mathbf{n} = \lim_{q \rightarrow 1} \frac{1 - \mathbf{h}_1}{2}. \quad (4.76)$$

Then these generators satisfy $[\mathbf{a}, \mathbf{a}^\dagger] = 1$, $\mathbf{a}\mathbf{a}^\dagger = \mathbf{n} + \frac{1}{2}$, $\mathbf{a}^\dagger\mathbf{a} = \mathbf{n} - \frac{1}{2}$. Let $q^{2p} = -\epsilon_-/\epsilon_+$. Then one can take the rational limit of renormalized versions of (4.9)-(4.12), (3.10), (3.12), (4.18) and (4.37) as

$$\lim_{|q| \rightarrow 1+0} \mathbf{K}(q^{2u})(1 - q^{-2})^{-2su} = \frac{\Gamma(-p + su + \frac{H+1}{2})}{\Gamma(-p - su + \frac{H+1}{2})} \quad \text{for} \quad (4.9), \quad (4.77)$$

$$\lim_{|q| \rightarrow 1-0} \mathbf{K}(q^{2u})(1 - q^2)^{-2su} = \frac{\Gamma(p + su - \frac{H-1}{2})}{\Gamma(p - su - \frac{H-1}{2})} \quad \text{for} \quad (4.10), \quad (4.78)$$

$$\lim_{|q| \rightarrow 1+0} \mathbf{K}(q^{2u})(1 - q^{-2})^{-2su} = \frac{\Gamma(p + su - \frac{H-1}{2})}{\Gamma(p - su - \frac{H-1}{2})} \quad \text{for} \quad (4.11), \quad (4.79)$$

$$\lim_{|q| \rightarrow 1-0} \mathbf{K}(q^{2u})(1 - q^2)^{-2su} = \frac{\Gamma(-p + su + \frac{H+1}{2})}{\Gamma(-p - su + \frac{H+1}{2})} \quad \text{for} \quad (4.12), \quad (4.80)$$

$$\lim_{q \rightarrow 1} \mathbf{L}^{(1)}(q^{2u})(1 \otimes (E_{11} - q^{-1}\lambda^{-1}E_{22})) = \begin{pmatrix} 1 & -\mathbf{a}^\dagger \\ \mathbf{a} & su - \mathbf{n} \end{pmatrix}, \quad (4.81)$$

$$\lim_{q \rightarrow 1} \bar{\mathbf{L}}^{(1)}(q^{2u})(1 \otimes (E_{11} - q^{-1}\lambda^{-1}E_{22})) = \begin{pmatrix} 1 & -\mathbf{a}^\dagger \\ \mathbf{a} & -su - \mathbf{n} \end{pmatrix}, \quad (4.82)$$

$$\lim_{q \rightarrow 1} K(q^{2u})(1 - q^{-2})^{-2su} = -\frac{\Gamma(su - p)}{\Gamma(1 - su - p)} \begin{pmatrix} p - su & 0 \\ 0 & p + su \end{pmatrix}, \quad (4.83)$$

$$\lim_{q \rightarrow 1} (q^2; q^2)_\infty (1 - q^2)^{-p-su+\frac{h_1+1}{2}} \mathbf{K}^{(1)}(q^{2u}) = \Gamma(p + su + \mathbf{n}), \quad u \in \mathbb{C}. \quad (4.84)$$

It is important to note that the above limits keep the reflection equation (4.53) unchanged. This is because of the relation (2.11) for $\xi = (1 - q^2)^{\frac{h_1}{2}}$ and

$$\rho_x^{(1)}((1 - q^2)^{\frac{h_1}{2}}) = (1 - q^2)^{\frac{h_1}{2}}, \quad \pi_1(y)((1 - q^2)^{\frac{h_1}{2}}) = (1 - q^2)^{\frac{1}{2}}(E_{11} - q^{-1}\lambda^{-1}E_{22}), \quad (4.85)$$

where $\rho_x^{(1)}$ is defined in (B.12). The rational limit of all the other K - and L -operators can be taken in the same way. In this way, we have recovered K -operators for Q -operators for rational (XXX-) models which are similar to the ones in [19].

5 Concluding remarks

In the context of quantum groups and related coideal subalgebras, finding a universal product formula for the K-matrix by analogy with the known product formula (A.1) for the universal R-matrix proposed in [29] is an interesting problem. In this direction, a universal formulation of the reflection equation algebra and related intertwining relations associated with a given coideal subalgebra are highly desirable (see recent progress in [28]). In the present paper, we have focused on homomorphic images of two different coideal subalgebras of $U_q(\widehat{sl_2})$ (onto $U_q(sl_2)$), that are related with the augmented q-Onsager algebra first introduced by Ito and Terwilliger in [21] (see also [22]). Based on the intertwining relations, product formulae for the K-matrix solutions in terms of the generators of $U_q(sl_2)$ are derived. They solve certain reflection and dual reflection equations associated with L-operators. In the second part of the paper, certain limits of these K-operators are studied. For these limits, contracted versions of the augmented q-Onsager algebra are considered and q-oscillator representations are constructed. Importantly, these K-operators are the basic ingredient for the construction of Q-operators that are relevant in the analysis of quantum integrable models with non-periodic diagonal boundary conditions. An interesting problem would be to extend the analysis presented here to the case of the q-Onsager algebra [35, 36], which is isomorphic to the fixed point subalgebra of $U_q(\widehat{sl_2})$ under the action of the Chevalley involution [37]. A product formula in this case is an open problem, that should find applications to the analysis of integrable models with non-diagonal boundary conditions.

It is known that L-operators for Verma modules of the quantum affine algebra (or the Yangian) factorize with respect to L-operators for the Q-operators. Examples for such factorization formulas appeared in a number of papers (see for example, [12, 10] and references therein). In [9], such factorization formulas were reconsidered in relation to properties of the universal R-matrix, and a universal factorization formula, which is independent of the quantum space, was proposed. One of our motivations was to generalize the universal factorization formula [9] to the case of open boundary conditions in the light of the augmented q-Onsager algebra [22]. The main obstacle for this is that we do not have the defining relations of the universal K-matrix corresponding to the relations (2.10) for the universal R-matrix of $U_q(\widehat{sl_2})$. This is a reason why we focused our discussions only on one of the most essential objects, the K-matrices, without application to Q-operators and their properties. Still, in appendix G we suggest universal T- and Q- operators. For a class of representations, the commutativity is proven. It is desirable to reconsider the problem after formulation of the universal K-matrix for the universal R-matrix of $U_q(\widehat{sl_2})$ in the future.

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A The universal R-matrix

In this section, we briefly review the product expression of the universal R-matrix given by Khoroshkin and Tolstoy in [29]. Their universal R-matrix was already reviewed by several authors [30, 7]. Here we basically follow these.

Let $\{\alpha + k\delta\}_{k=0}^{\infty} \cup \{k\delta\}_{k=1}^{\infty} \cup \{\delta - \alpha + k\delta\}_{k=0}^{\infty}$ be a positive root system of \widehat{sl}_2 in the notation of [29]. We choose the root ordering as $\alpha + (k-1)\delta \prec \alpha + k\delta \prec l\delta \prec (l+1)\delta \prec \delta - \alpha + m\delta \prec \delta - \alpha + (m-1)\delta$ for any $k, l, m \in \mathbb{Z}_{\geq 1}$. In this case, the universal R-matrix has the following expression:

$$\mathcal{R} = \overline{\mathcal{R}}^+ \overline{\mathcal{R}}^0 \overline{\mathcal{R}}^- q^K, \quad (\text{A.1})$$

$$\overline{\mathcal{R}}^+ = \prod_{k=0}^{\overrightarrow{\infty}} \exp_{q^{-2}} (\lambda e_{\alpha+k\delta} \otimes f_{\alpha+k\delta}), \quad (\text{A.2})$$

$$\overline{\mathcal{R}}^0 = \exp \left(\lambda \sum_{k=1}^{\infty} \frac{k}{[2k]_q} e_{k\delta} \otimes f_{k\delta} \right), \quad (\text{A.3})$$

$$\overline{\mathcal{R}}^- = \prod_{k=0}^{\overleftarrow{\infty}} \exp_{q^{-2}} (\lambda e_{\delta-\alpha+k\delta} \otimes f_{\delta-\alpha+k\delta}), \quad (\text{A.4})$$

where we use notations

$$\exp_q(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{(k)_q!}, \quad (k)_q! = (1)_q(2)_q \cdots (k)_q, \quad (k)_q = \frac{1-q^k}{1-q}.$$

Let $e_{\alpha} = e_1$, $e_{\delta-\alpha} = e_0$, $f_{\alpha} = f_1$, $f_{\delta-\alpha} = f_0$. Then the other root vectors are defined by the following recursion relations:

$$e_{\alpha+k\delta} = [2]_q^{-1} [e_{\alpha+(k-1)\delta}, e'_{\delta}], \quad (\text{A.5})$$

$$e'_{k\delta} = [e_{\alpha+(k-1)\delta}, e_{\delta-\alpha}]_{q^{-2}}, \quad (\text{A.6})$$

$$e_{\delta-\alpha+k\delta} = [2]_q^{-1} [e'_{\delta}, e_{\delta-\alpha+(k-1)\delta}], \quad (\text{A.7})$$

$$f_{\alpha+k\delta} = [2]_q^{-1} [f'_{\delta}, f_{\alpha+(k-1)\delta}], \quad (\text{A.8})$$

$$f'_{k\delta} = [f_{\delta-\alpha}, f_{\alpha+(k-1)\delta}]_{q^2}, \quad (\text{A.9})$$

$$f_{\delta-\alpha+k\delta} = [2]_q^{-1} [f_{\delta-\alpha+(k-1)\delta}, f'_{\delta}], \quad k \in \mathbb{Z}_{\geq 1}. \quad (\text{A.10})$$

where the root vectors with prime are given by the following generating functions.

$$\lambda \sum_{k=1}^{\infty} e_{k\delta} z^{-k} = \log \left(1 + \lambda \sum_{k=1}^{\infty} e'_{k\delta} z^{-k} \right), \quad (\text{A.11})$$

$$-\lambda \sum_{k=1}^{\infty} f_{k\delta} z^{-k} = \log \left(1 - \lambda \sum_{k=1}^{\infty} f'_{k\delta} z^{-k} \right), \quad z \in \mathbb{C}. \quad (\text{A.12})$$

In general, root vectors contain many commutators. However, simplification occurs under the evaluation map.

$$\text{ev}_x(e_{\alpha+k\delta}) = (-1)^k x^{ks+s_1} q^{-kH} E, \quad (\text{A.13})$$

$$\text{ev}_x(e_{\delta-\alpha+k\delta}) = (-1)^k x^{ks+s_0} F q^{-kH}, \quad (\text{A.14})$$

$$\text{ev}_x(f_{\alpha+k\delta}) = (-1)^k x^{-ks-s_1} F q^{kH}, \quad (\text{A.15})$$

$$\text{ev}_x(f_{\delta-\alpha+k\delta}) = (-1)^k x^{-ks-s_0} q^{kH} E \quad \text{for } k \in \mathbb{Z}_{\geq 0}, \quad \text{and} \quad (\text{A.16})$$

$$\text{ev}_x(e'_{k\delta}) = (-1)^{k-1} x^{ks} q^{-(k-1)H} [E, F]_{q^{-2k}}, \quad (\text{A.17})$$

$$\text{ev}_x(e_{k\delta}) = \frac{(-1)^{k-1} q^{-k} x^{ks}}{(q - q^{-1})k} (C_k - (q^k + q^{-k}) q^{-kH}), \quad (\text{A.18})$$

$$\text{ev}_x(f'_{k\delta}) = (-1)^{k-1} x^{-ks} [E, F]_{q^{2k}} q^{(k-1)H}, \quad (\text{A.19})$$

$$\text{ev}_x(f_{k\delta}) = -\frac{(-1)^{k-1} q^k x^{-ks}}{(q - q^{-1})k} (C_k - (q^k + q^{-k}) q^{kH}) \quad \text{for } k \in \mathbb{Z}_{\geq 1}, \quad (\text{A.20})$$

where the central elements C_k are defined by

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} C_k}{k} z^{-k} = \log(1 + \lambda^2 C z^{-1} + z^{-2}), \quad z \in \mathbb{C}. \quad (\text{A.21})$$

Inserting these¹⁸ into (A.1), we obtain $\mathbf{R}(x, y)$. In order to obtain $\mathbf{R}_{21}(x, y)$, one has to swap the first and the second components of the tensor product in (A.1) beforehand. Based on these product formulas, one can check

$$(\nu \otimes \nu) \mathbf{R}(x, y) = \mathbf{R}_{21}(x^{-1}, y^{-1}). \quad (\text{A.22})$$

B Contraction of the quantum affine algebra

A systematic study of the asymptotic representation theory of the Borel subalgebras of quantum affine algebras was given in [13]. How to evaluate the universal R-matrix for the purpose of Q-operators was explained in detail in [7]. We nevertheless review the subject in the spirit of [8], which is inspired by earlier discussions [2, 3, 32, 33, 6]. We are interested in considering limits of representations of the whole quantum affine algebra rather than those of its Borel subalgebras. In particular, we will present the universal form of the intertwining relations for the L-operators for Q-operators (B.23)-(B.26).

¹⁸For the second component of the tensor product, one has to replace x with y beforehand.

B.1 The contracted algebra $U_q(\widehat{sl}(2; I))$

Let I be a subset¹⁹ of the set $\{0, 1\}$, and define $\theta(\text{True}) = 1, \theta(\text{False}) = 0$. The contracted algebra $\widetilde{U}_q(\widehat{sl}(2; I))$ is an algebra generated by the generators²⁰ e_i, f_i, h_i , where $i \in \{0, 1\}$. For $i, j \in \{0, 1\}$, the defining relations of the algebra $\widetilde{U}_q(\widehat{sl}(2; I))$ are given by

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad (\text{B.1})$$

$$[e_i, f_j] = \delta_{ij} \frac{\theta(i+1 \in I)q^{h_i} - \theta(i \in I)q^{-h_i}}{q - q^{-1}}, \quad (\text{B.2})$$

$$[e_i, [e_i, [e_i, e_j]_{q^2}]]_{q^{-2}} = [f_i, [f_i, [f_i, f_j]_{q^{-2}}]]_{q^2} = 0, \quad i \neq j, \quad (\text{B.3})$$

where $(a_{ij})_{0 \leq i, j \leq 1}$ is the Cartan matrix of \widehat{sl}_2 , and $2 \equiv 0$ in I . Note that $\widetilde{U}_q(\widehat{sl}(2; \{0, 1\}))$ coincides with $U_q(\widehat{sl}_2)$. We use the following co-products $\Delta, \Delta', \overline{\Delta}, \overline{\Delta}' : \widetilde{U}_q(\widehat{sl}(2; I)) \rightarrow \widetilde{U}_q(\widehat{sl}(2; I)) \otimes U_q(\widehat{sl}_2)$:

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + q^{-h_i} \otimes e_i, & \Delta'(e_i) &= 1 \otimes e_i + e_i \otimes q^{-h_i}, \\ \Delta(f_i) &= f_i \otimes q^{h_i} + \theta(i \in I)(1 \otimes f_i), & \Delta'(f_i) &= \theta(i+1 \in I)(q^{h_i} \otimes f_i) + f_i \otimes 1, \\ \Delta(h_i) &= \Delta'(h_i) = h_i \otimes 1 + 1 \otimes h_i. \end{aligned} \quad (\text{B.4})$$

and

$$\begin{aligned} \overline{\Delta}(e_i) &= e_i \otimes 1 + \theta(i \in I)(q^{-h_i} \otimes e_i), & \overline{\Delta}'(e_i) &= \theta(i+1 \in I)(1 \otimes e_i) + e_i \otimes q^{-h_i}, \\ \overline{\Delta}(f_i) &= f_i \otimes q^{h_i} + 1 \otimes f_i, & \overline{\Delta}'(f_i) &= q^{h_i} \otimes f_i + f_i \otimes 1, \\ \overline{\Delta}(h_i) &= \overline{\Delta}'(h_i) = h_i \otimes 1 + 1 \otimes h_i. \end{aligned} \quad (\text{B.5})$$

We define a smaller contracted algebra²¹ $U_q(\widehat{sl}(2; I))$ by imposing

$$[e_0, [e_0, e_1]_{q^2}] = [e_1, [e_1, e_0]_{q^{-2}}] = 0 \quad \text{for } I = \{0\}, \quad (\text{B.6})$$

$$[f_0, [f_0, f_1]_{q^{-2}}] = [f_1, [f_1, f_0]_{q^2}] = 0 \quad \text{for } I = \{0\}, \quad (\text{B.7})$$

$$[e_0, [e_0, e_1]_{q^{-2}}] = [e_1, [e_1, e_0]_{q^2}] = 0 \quad \text{for } I = \{1\}, \quad (\text{B.8})$$

$$[f_0, [f_0, f_1]_{q^2}] = [f_1, [f_1, f_0]_{q^{-2}}] = 0 \quad \text{for } I = \{1\}, \quad (\text{B.9})$$

on $U_q(\widehat{sl}(2; I))$. Note that (B.3) follows from (B.6)-(B.9). $U_q(\widehat{sl}(2; \{0\}))$ and $U_q(\widehat{sl}(2; \{1\}))$ are sort of coupled q-oscillator algebras. The map (2.5), which preserves the defining

¹⁹This came from a notation in [34], where 2^{M+N} Q-functions for $U_q(\widehat{gl}(M|N))$ are classified in terms of all the subsets I of the set $\{1, 2, \dots, M+N\}$. The number '0' in I corresponds to '2' (for $(M, N) = (2, 0)$ case) in the [34]. In our present paper, there are $2^2 = 4$ Q-operators. Two of them, which correspond to $I = \{0, 1\}, \emptyset$, are identity operators in the normalization of the universal R-matrix.

²⁰In this paper, we do not use the derivation d .

²¹The coproduct (B.4) (resp. (B.5)) does not keep (B.6) and (B.8) (resp. (B.7) and (B.9)). Then there is an option to consider algebras bigger than $U_q(\widehat{sl}(2; I))$, where (B.7) and (B.9) with (B.4), or (B.6) and (B.8) with (B.5) are imposed on $\widetilde{U}_q(\widehat{sl}(2; I))$. However, we focus on $U_q(\widehat{sl}(2; I))$ since (B.6)-(B.9) always hold true for the q-oscillator representations in this paper.

relations of $U_q(\widehat{sl}(2; \{0, 1\}))$, swaps the defining relations $U_q(\widehat{sl}(2; \{0\}))$ and $U_q(\widehat{sl}(2; \{1\}))$ one another. The fact that Serre-type relations for q-oscillator representations for the Q-operators can be simpler than the original ones was pointed out first by Bazhanov et al. in [3] for \mathcal{B}_+ of $U_q(\widehat{sl}(3))$. In [8], this phenomenon was observed for the contracted algebras $U_q(\widehat{gl}(M|N; I))$ of $U_q(\widehat{gl}(M|N))$. It is desirable to study this systematically for the Drinfeld's second realization of the quantum affine (super)algebras. For the case of the Yangian $Y(sl(2))$, a degenerated algebra \mathcal{A} for Q-operators was studied in terms of the Drinfeld's second realization [11]. Their co-products correspond to the case $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{A}$. On the other hand, our co-product (B.4), which was used for the intertwining relations for L-operators for Q-operators [8], might be related to the case $\Delta : \mathcal{A} \mapsto \mathcal{A} \otimes Y(sl(2))$ if an appropriate rational limit was taken. Thus, it will be interesting to consider the case $\Delta : U_q(\widehat{sl}(2; I)) \mapsto U_q(\widehat{sl}(2; I)) \otimes U_q(\widehat{sl}(2; I))$.

The Borel subalgebras of $\widetilde{U}_q(\widehat{sl}(2; I))$ and $U_q(\widehat{sl}_2)$ share the same defining relations. $U_q(\widehat{sl}(2; I))$ is a subalgebra of $\widetilde{U}_q(\widehat{sl}(2; I))$. Taking note of this fact, we purposely use the same symbols for the generators of these algebras.

The contracted algebra $U_q(\widehat{sl}(2; I))$ has subalgebras $U_q(sl(2; I))$ generated by the generators E, F, H obeying the relations,

$$[H, E] = 2E, \quad [H, F] = -2F, \quad (\text{B.10})$$

$$[E, F] = \frac{\theta(0 \in I)q^H - \theta(1 \in I)q^{-H}}{q - q^{-1}}. \quad (\text{B.11})$$

This reduces to $U_q(sl(2))$ for $I = \{0, 1\}$ and to q-oscillator algebras for $I = \{0\}$ or $I = \{1\}$. The q-oscillator algebra Osc_1 (resp. Osc_2) corresponds to $U_q(sl(2; \{0\}))$ (resp. $U_q(sl(2; \{1\}))$) with a fixed value of the central element $[E, F]_{q^{-2}} = 1/(q - q^{-1})$ (resp. $[E, F]_{q^2} = -1/(q - q^{-1})$). Contractions of a quantum algebra was previously discussed in [31]. Contracted quantum algebras in relation to L-operators for Q-operators were previously discussed in [32] and developed in [33, 8].

B.2 Universal L -operators for Q-operators and their intertwining relations

The L -operators (3.10)-(3.13) can also be presented as homomorphic images of the universal R -matrix under the homomorphism $\rho_x^{(i)} : \mathcal{B}_+ \rightarrow \text{Osc}_i, i = 1, 2$ defined by the relations

$$\rho_x^{(i)}(e_0) = x^{s_0} \mathbf{f}_i, \quad \rho_x^{(i)}(e_1) = x^{s_1} \mathbf{e}_i, \quad \rho_x^{(i)}(h_0) = -\mathbf{h}_i, \quad \rho_x^{(i)}(h_1) = \mathbf{h}_i, \quad (\text{B.12})$$

or the homomorphism $\rho_x^{(i)} : \mathcal{B}_- \rightarrow \text{Osc}_i, i = 1, 2$ defined by the relations

$$\rho_x^{(i)}(f_0) = x^{-s_0} \mathbf{e}_i, \quad \rho_x^{(i)}(f_1) = x^{-s_1} \mathbf{f}_i, \quad \rho_x^{(i)}(h_0) = -\mathbf{h}_i, \quad \rho_x^{(i)}(h_1) = \mathbf{h}_i. \quad (\text{B.13})$$

We remark that the maps $\rho_x^{(i)}$ cannot be straightforwardly extended to the whole algebra $U_q(\widehat{sl}_2)$. $\rho_x^{(1)}$ (resp. $\rho_x^{(2)}$) should be regarded as a map from the contracted algebra $U_q(\widehat{sl}(2; \{0\}))$ or $\widetilde{U}_q(\widehat{sl}(2; \{0\}))$ (resp. $U_q(\widehat{sl}(2; \{1\}))$ or $\widetilde{U}_q(\widehat{sl}(2; \{1\}))$) to Osc_1 (resp. Osc_2). Namely, they preserve the relations (B.1)-(B.3), (B.6)-(B.9), (2.1) and (2.3), but do not

keep the relation (2.2) unchanged. Let \mathcal{N}_+ (resp. \mathcal{N}_-) be the nilpotent subalgebra of $U_q(\widehat{sl_2})$ generated by e_i (resp. f_i), $i = 0, 1$. The maps $\rho_x^{(1)}$ and $\rho_x^{(2)}$ can be realized as limits of shifted representations of \mathcal{B}_\pm and representations ²² of \mathcal{N}_\mp . We are interested in the following realizations ²³:

$$\rho_x^{(1)}(a) = \begin{cases} \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(xq^{-\frac{\mu}{s}})(\tau_{-\mu}(a)) & \text{for } a \in \mathcal{B}_+ \\ \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(xq^{-\frac{\mu}{s}})(q^{-\deg(a)\mu}a) & \text{for } a \in \mathcal{N}_-, \end{cases} \quad (\text{B.14})$$

$$\rho_x^{(2)}(a) = \begin{cases} \lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}})(\tau_{-\mu}(a)) & \text{for } a \in \mathcal{B}_+ \\ \lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}})(q^{\deg(a)\mu}a) & \text{for } a \in \mathcal{N}_-, \end{cases} \quad (\text{B.15})$$

or ²⁴

$$\rho_x^{(1)}(a) = \begin{cases} \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}}) \left(q^{-\deg(a)\mu + \frac{(s_0-s_1)\mu h_1}{2s}} a q^{-\frac{(s_0-s_1)\mu h_1}{2s}} \right) & \text{for } a \in \mathcal{N}_+ \\ \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}}) \left(q^{\frac{(s_0-s_1)\mu h_1}{2s}} \tau_{-\mu}(a) q^{-\frac{(s_0-s_1)\mu h_1}{2s}} \right) & \text{for } a \in \mathcal{B}_-, \end{cases} \quad (\text{B.16})$$

$$\rho_x^{(2)}(a) = \begin{cases} \lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{-\frac{\mu}{s}}) \left(q^{\deg(a)\mu + \frac{(s_1-s_0)\mu h_1}{2s}} a q^{-\frac{(s_1-s_0)\mu h_1}{2s}} \right) & \text{for } a \in \mathcal{N}_+ \\ \lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{-\frac{\mu}{s}}) \left(q^{\frac{(s_1-s_0)\mu h_1}{2s}} \tau_{-\mu}(a) q^{-\frac{(s_1-s_0)\mu h_1}{2s}} \right) & \text{for } a \in \mathcal{B}_-, \end{cases} \quad (\text{B.17})$$

where \deg is a linear operator which evaluates the degree of the monomials of the generators (for example, $\deg(f_i) = 1$, $\deg(e_i e_j e_k) = 3$).

We define ²⁵ universal L-operators as homomorphic image of the universal R-matrix:

$$\mathcal{L}^{(i)}(x) = (\rho_x^{(i)} \otimes 1)\mathcal{R}, \quad \overline{\mathcal{L}}^{(i)}(x) = (\rho_x^{(i)} \otimes 1)\overline{\mathcal{R}}, \quad (\text{B.18})$$

$$\check{\mathcal{L}}^{(i)}(x) = (\rho_x^{(i)} \otimes 1)\overline{\mathcal{R}}^{-1}, \quad \check{\overline{\mathcal{L}}}^{(i)}(x) = (\rho_x^{(i)} \otimes 1)\mathcal{R}^{-1}. \quad (\text{B.19})$$

For example, one can calculate ²⁶

$$\mathcal{L}^{(1)}(x) = \exp_{q^{-2}}(\lambda x^{s_1} \mathbf{e}_1 \otimes f_\alpha) \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{sk}}{[2k]_q} \otimes f_{k\delta} \right) \exp_{q^{-2}}(\lambda x^{s_0} \mathbf{f}_1 \otimes f_{\delta-\alpha}) q^{\frac{1}{2} \mathbf{h}_1 \otimes h_1}, \quad (\text{B.20})$$

²²Note that \mathcal{N}_\mp is invariant under $\tau_{-\mu}$.

²³To be precise, the letter a denotes an element of the quantum affine algebra on the right hand side, while that of a contracted algebra on the left hand side.

²⁴Similarity transformations by the Cartan element are used to renormalize the generators: $\xi e_0 \xi^{-1} = q^{-\frac{(s_0-s_1)\mu}{s}} e_0$, $\xi e_1 \xi^{-1} = q^{\frac{(s_0-s_1)\mu}{s}} e_1$, $\xi f_0 \xi^{-1} = q^{\frac{(s_0-s_1)\mu}{s}} f_0$, $\xi f_1 \xi^{-1} = q^{-\frac{(s_0-s_1)\mu}{s}} f_1$, where $\xi = q^{\frac{(s_0-s_1)\mu h_1}{2s}}$.

²⁵One may also define (B.19) as $\check{\mathcal{L}}^{(i)}(x) = (\rho_x^{(i)} \circ S \otimes 1)\overline{\mathcal{R}}$, $\check{\overline{\mathcal{L}}}^{(i)}(x) = (\rho_x^{(i)} \circ S \otimes 1)\mathcal{R}$, where S is the anti-pode satisfying $(S \otimes 1)\mathcal{R} = \mathcal{R}^{-1} = (1 \otimes S^{-1})\mathcal{R}$.

²⁶One can directly plug (B.12) into (A.1), or apply (B.14) to (A.1) by way of (A.13), (A.14), (A.18), (A.21) with the help of (2.15), (2.30) and (3.28). As remarked in [9], $\mathcal{L}^{(1)}(x)$ contains only two q-exponentials, while $\mathcal{L}^{(2)}(x)$ contains infinitely many. One could use the universal version of (3.9), $\mathcal{L}^{(2)'}(x) = (\rho_x^{(1)} \circ \sigma \otimes 1)\mathcal{R} = (1 \otimes \sigma^{-1})\mathcal{L}^{(1)}(x)$ instead of (B.21) to avoid the infinite product.

$$\begin{aligned}
\mathcal{L}^{(2)}(x) &= \prod_{k=0}^{\overrightarrow{\infty}} \exp_{q^{-2}} \left((-1)^k \lambda x^{ks+s_1} q^{-k\mathbf{h}_2} \mathbf{e}_2 \otimes f_{\alpha+k\delta} \right) \\
&\quad \times \exp \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^{-k} x^{ks}}{[2k]_q} (q^{-k} - (q^k + q^{-k}) q^{-k\mathbf{h}_2}) \otimes f_{k\delta} \right) \\
&\quad \times \prod_{k=0}^{\overleftarrow{\infty}} \exp_{q^{-2}} \left((-1)^k \lambda x^{ks+s_0} \mathbf{f}_2 q^{-k\mathbf{h}_2} \otimes f_{\delta-\alpha+k\delta} \right), \tag{B.21}
\end{aligned}$$

where the relation

$$\lim_{q^{\mp} \rightarrow 0} \pi_{\mu}^{+}(C_k q^{\mp k\mu}) = \left(\lambda^2 \lim_{q^{\mp} \rightarrow 0} \pi_{\mu}^{+}(C q^{\mp \mu}) \right)^k = q^{\pm k} \quad \text{for } k \in \mathbb{Z}_{\geq 1}, \tag{B.22}$$

which follows from (A.21), (3.28) and (3.29), is used. Then the L-operators (3.10)-(3.13) and (3.16)-(3.19) are given by $\mathbf{L}^{(i)}(x) = \phi^{(i)}(x)(1 \otimes \pi_1(1))\mathcal{L}^{(i)}(x)$, $\overline{\mathbf{L}}^{(i)}(x) = \phi^{(i)}(x^{-1})(1 \otimes \pi_1(1))\overline{\mathcal{L}}^{(i)}(x)$, $\check{\mathbf{L}}^{(i)}(x) = \check{\phi}^{(i)}(x^{-1})(1 \otimes \pi_1(1))\check{\mathcal{L}}^{(i)}(x)$, $\check{\overline{\mathbf{L}}}^{(i)}(x) = \check{\phi}^{(i)}(x)(1 \otimes \pi_1(1))\check{\overline{\mathcal{L}}}^{(i)}(x)$, where ²⁷ $\phi^{(i)}(x) = e^{-\Phi(x^s q^{-2\delta_{i,2}})}$, $\check{\phi}^{(i)}(x) = (-x^{-s} q^{-1})e^{-\Phi(x^s q^{2\delta_{i,1}})}$, $\Phi(x) = \sum_{k=1}^{\infty} \frac{1}{k(q^k + q^{-k})} x^k$.

The intertwining relations for these operators are given ²⁸ by

$$((\rho_x^{(i)} \otimes 1)\Delta'(a))\mathcal{L}^{(i)}(x) = \mathcal{L}^{(i)}(x)((\rho_x^{(i)} \otimes 1)\Delta(a)), \tag{B.23}$$

$$((\rho_x^{(i)} \otimes 1)\overline{\Delta}(a))\overline{\mathcal{L}}^{(i)}(x) = \overline{\mathcal{L}}^{(i)}(x)((\rho_x^{(i)} \otimes 1)\overline{\Delta}'(a)), \tag{B.24}$$

$$((\rho_x^{(i)} \otimes 1)\overline{\Delta}'(a))\check{\mathcal{L}}^{(i)}(x) = \check{\mathcal{L}}^{(i)}(x)((\rho_x^{(i)} \otimes 1)\overline{\Delta}(a)), \tag{B.25}$$

$$((\rho_x^{(i)} \otimes 1)\Delta(a))\check{\overline{\mathcal{L}}}^{(i)}(x) = \check{\overline{\mathcal{L}}}^{(i)}(x)((\rho_x^{(i)} \otimes 1)\Delta'(a)), \tag{B.26}$$

$$\text{for } a \in \widetilde{U}_q(\widehat{sl}(2; I)), \quad I = \{i-1\}, \quad i \in \{1, 2\}.$$

The relation (B.23) follows from (2.10), (B.14) and (B.15); (B.24) follows from (2.13), (B.16) and (B.17); (B.25) follows from (2.13), (B.16) and (B.17); (B.26) follows from (2.10), (B.14) and (B.15). For example, let us multiply $q^{-\mu - \frac{1}{2}\mu \otimes h_1}$ from the right of the first equation in (2.10) for f_1 :

$$(q^{h_1 - \mu} \otimes f_1 + f_1 q^{-\mu} \otimes 1) \mathcal{R} q^{-\frac{1}{2}\mu \otimes h_1} = \mathcal{R} q^{-\frac{1}{2}\mu \otimes h_1} (f_1 q^{-\mu} \otimes q^{h_1} + q^{-2\mu} \otimes f_1). \tag{B.27}$$

Evaluating this for $\pi_{\mu}^{+}(xq^{-\frac{\mu}{s}}) \otimes 1$, we obtain

$$\begin{aligned}
&\left(\pi_{\mu}^{+}(q^{H-\mu}) \otimes f_1 + x^{-s_1} \pi_{\mu}^{+}(F q^{-\frac{s_0\mu}{s}}) \otimes 1 \right) \left((\pi_{\mu}^{+}(xq^{-\frac{\mu}{s}})\tau_{-\mu} \otimes 1) \mathcal{R} \right) \\
&= \left((\pi_{\mu}^{+}(xq^{-\frac{\mu}{s}})\tau_{-\mu} \otimes 1) \mathcal{R} \right) \left(x^{-s_1} \pi_{\mu}^{+}(F q^{-\frac{s_0\mu}{s}}) \otimes q^{h_1} + q^{-2\mu} \otimes f_1 \right), \tag{B.28}
\end{aligned}$$

²⁷There is a useful identity $\phi^{(i)}(x)\check{\phi}^{(i)}(x) = q^{-1}(q^{1-2\delta_{i,2}} - x^{-s})$, $i = 1, 2$.

²⁸We may interpret the first space of these as a composition of natural evaluation maps $\widetilde{U}_q(\widehat{sl}(2; I)) \mapsto U_q(\widehat{sl}(2; I)) \mapsto \text{Osc}_i$.

where (2.15) for $c_1 = -\mu$ is used. Then the limit $q^{-\mu} \rightarrow 0$ produces

$$(q^{h_1} \otimes f_1 + x^{-s_1} \mathbf{f}_1 \otimes 1) ((\rho_x^{(1)} \otimes 1) \mathcal{R}) = ((\rho_x^{(1)} \otimes 1) \mathcal{R}) (x^{-s_1} \mathbf{f}_1 \otimes q^{h_1}), \quad (\text{B.29})$$

where (B.14) for \mathcal{B}_+ is applied to the first component of the tensor product in $\mathcal{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$, and (2.30) is used. Taking note on (B.4) and (B.13), we arrive at (B.23) for $i = 1$, $a = f_1 \in \tilde{U}_q(\widehat{sl}(2; \{0\}))$. The other relations in (B.23)-(B.26) can be derived in a similar manner. Evaluating these relations (B.23)-(B.26) for $1 \otimes \pi_1(1)$, one can derive the intertwining relations for the L-operators (3.10)-(3.13) and (3.16)-(3.19).

C Contraction of the augmented q -Onsager algebra

Let I be a subset of the set $\{0, 1\}$, and define $\theta(\text{True}) = 1, \theta(\text{False}) = 0$. The contracted augmented q -Onsager algebra - denoted below $\tilde{\mathcal{O}}(I)_q^{aug}$ - is generated by four generators $K_0, K_1, Z_1, \tilde{Z}_1$ subject to the defining relations:

$$\begin{aligned} [K_0, K_1] &= 0, \\ K_0 Z_1 &= q^{-2} Z_1 K_0, & K_0 \tilde{Z}_1 &= q^2 \tilde{Z}_1 K_0, \\ K_1 Z_1 &= q^2 Z_1 K_1, & K_1 \tilde{Z}_1 &= q^{-2} \tilde{Z}_1 K_1, \\ [Z_1, [Z_1, [Z_1, \tilde{Z}_1]_{q^2}]_{q^{-2}}] &= \rho_{\text{diag}} Z_1 (\theta(1 \in I) K_1 K_1 - \theta(0 \in I) K_0 K_0) Z_1, \\ [\tilde{Z}_1, [\tilde{Z}_1, [\tilde{Z}_1, Z_1]_{q^2}]_{q^{-2}}] &= \rho_{\text{diag}} \tilde{Z}_1 (\theta(0 \in I) K_0 K_0 - \theta(1 \in I) K_1 K_1) \tilde{Z}_1 \end{aligned} \quad (\text{C.1})$$

with

$$\rho_{\text{diag}} = \frac{(q^3 - q^{-3})(q^2 - q^{-2})^3}{q - q^{-1}}. \quad (\text{C.2})$$

Note that $\tilde{\mathcal{O}}(I)_q^{aug}$ coincides with \mathcal{O}_q^{aug} for $I = \{0, 1\}$. This algebra can be embedded into coideal subalgebras of $U_q(\widehat{sl}_2)$. We introduce the smaller contracted augmented q -Onsager algebra $\mathcal{O}(I)_q^{aug}$ for $I = \{0\}, \{1\}$ by imposing the following additional relations on the the algebras $\tilde{\mathcal{O}}(I)_q^{aug}$:

$$\begin{aligned} [Z_1, [Z_1, \tilde{Z}_1]_{q^2}] &= \bar{\rho}_{\text{diag}} q^2 K_0 K_0 Z_1, & [\tilde{Z}_1, [\tilde{Z}_1, Z_1]_{q^{-2}}] &= \bar{\rho}_{\text{diag}} \tilde{Z}_1 K_0 K_0 & \text{for } I = \{0\}, \\ [Z_1, [Z_1, \tilde{Z}_1]_{q^{-2}}] &= \bar{\rho}_{\text{diag}} Z_1 K_1 K_1, & [\tilde{Z}_1, [\tilde{Z}_1, Z_1]_{q^2}] &= \bar{\rho}_{\text{diag}} q^2 K_1 K_1 \tilde{Z}_1 & \text{for } I = \{1\}, \end{aligned} \quad (\text{C.3})$$

with

$$\bar{\rho}_{\text{diag}} = -\frac{q(q^2 - q^{-2})^3}{q - q^{-1}}. \quad (\text{C.4})$$

Note that the last two relations in (C.1) automatically hold²⁹ true under (C.3). This algebra can be embedded into $U_q(\widehat{sl}(2; I))$. Below, we will introduce four different realizations of the algebra $\mathcal{O}(I)_q^{aug}$.

²⁹One has to take care the relations of the form: $[A, [A, [A, B]_{q^2}]_{q^{-2}}] = [A, [A, [A, B]_{q^2}]]_{q^{-2}} = A^3 B - (q^2 + 1 + q^{-2}) A^2 B A + (q^2 + 1 + q^{-2}) A B A^2 - B A^3$, which are symmetric with respect to $q \leftrightarrow q^{-1}$.

C.1 The first realization

In this subsection, the contracted augmented q -Onsager algebra is embedded into $U_q(\widehat{sl}(2; I))$. We consider two types of realization of the contracted augmented q -Onsager algebra $\mathcal{O}_q^{aug}(I)$, as a subalgebra of $U_q(\widehat{sl}(2; I))$. The realizations of $\mathcal{O}_q^{aug}(\{0\})$ in terms of $U_q(\widehat{sl}(2; \{0\}))$ are given by

$$\begin{aligned} K_0^{(1,-)} &= \epsilon_+ q^{-h_0}, & K_1^{(1,-)} &= \epsilon_- q^{-h_1}, \\ Z_1^{(1,-)} &= (q^2 - q^{-2})(\epsilon_+ e_0), \\ \tilde{Z}_1^{(1,-)} &= (q^2 - q^{-2})(\epsilon_- e_1 + \epsilon_+ q f_0 q^{-h_0}), \end{aligned} \quad (C.5)$$

$$\begin{aligned} K_0^{(1,+)} &= \epsilon_+ q^{-h_0}, & K_1^{(1,+)} &= \epsilon_- q^{-h_1}, \\ Z_1^{(1,+)} &= (q^2 - q^{-2})(\epsilon_- q f_1 q^{-h_1} + \epsilon_+ e_0), \\ \tilde{Z}_1^{(1,+)} &= (q^2 - q^{-2})(\epsilon_+ q f_0 q^{-h_0}), \end{aligned} \quad (C.6)$$

and the realizations of $\mathcal{O}_q^{aug}(\{1\})$ in terms of $U_q(\widehat{sl}(2; \{1\}))$ are given by

$$\begin{aligned} K_0^{(2,-)} &= \epsilon_+ q^{-h_0}, & K_1^{(2,-)} &= \epsilon_- q^{-h_1}, \\ Z_1^{(2,-)} &= (q^2 - q^{-2})(\epsilon_- q f_1 q^{-h_1} + \epsilon_+ e_0), \\ \tilde{Z}_1^{(2,-)} &= (q^2 - q^{-2})(\epsilon_- e_1), \end{aligned} \quad (C.7)$$

$$\begin{aligned} K_0^{(2,+)} &= \epsilon_+ q^{-h_0}, & K_1^{(2,+)} &= \epsilon_- q^{-h_1}, \\ Z_1^{(2,+)} &= (q^2 - q^{-2})(\epsilon_- q f_1 q^{-h_1}), \\ \tilde{Z}_1^{(2,+)} &= (q^2 - q^{-2})(\epsilon_- e_1 + \epsilon_+ q f_0 q^{-h_0}). \end{aligned} \quad (C.8)$$

Here we attach symbols $(1, +)$, $(1, -)$, $(2, +)$, $(2, -)$ on the generators to distinguish different realizations of the algebras. We remark that $\tilde{\mathcal{O}}_q^{aug}(I)$ is realized by (C.5)-(C.8) even if $\{e_0, e_1, f_0, f_1, h_0, h_1\}$ are generators of $U_q(\widehat{sl}(2))$, while $\mathcal{O}_q^{aug}(I)$ is realized only by the generators of $U_q(\widehat{sl}(2; I))$.

Limits of the renormalized generators of the augmented q -Onsager algebra in Verma modules are related to the images of the contracted augmented q -Onsager algebra under the maps (B.12)-(B.13) as follows:

$$\lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(x q^{\frac{\mu}{s}})(Z_1 q^{-\frac{2s_0\mu}{s}}) = \rho_x^{(1)}(Z_1^{(1,-)}), \quad \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(x q^{\frac{\mu}{s}})(\tilde{Z}_1 q^{-\frac{2s_1\mu}{s}}) = \rho_x^{(1)}(\tilde{Z}_1^{(1,-)}), \quad (C.9)$$

$$\lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(x q^{-\frac{\mu}{s}})(Z_1) = \rho_x^{(1)}(Z_1^{(1,+)}), \quad \lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(x q^{-\frac{\mu}{s}})(\tilde{Z}_1 q^{-2\mu}) = \rho_x^{(1)}(\tilde{Z}_1^{(1,+)}), \quad (C.10)$$

$$\lim_{q^{-\mu} \rightarrow 0} \pi_\mu^+(x q^{\pm \frac{\mu}{s}})(\tau_{-\mu}(K_a)) = \rho_x^{(1)}(K_a^{(1,\mp)}), \quad a = 0, 1, \quad (C.11)$$

$$\lim_{q^{\mu} \rightarrow 0} \pi_\mu^+(x q^{-\frac{\mu}{s}})(Z_1 q^{\frac{2s_0\mu}{s}}) = \rho_x^{(2)}(Z_1^{(2,-)}), \quad \lim_{q^{\mu} \rightarrow 0} \pi_\mu^+(x q^{-\frac{\mu}{s}})(\tilde{Z}_1 q^{\frac{2s_1\mu}{s}}) = \rho_x^{(2)}(\tilde{Z}_1^{(2,-)}), \quad (C.12)$$

$$\lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}})(Z_1 q^{2\mu}) = \rho_x^{(2)}(Z_1^{(2,+)}), \quad \lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{\frac{\mu}{s}})(\tilde{Z}_1) = \rho_x^{(2)}(\tilde{Z}_1^{(2,+)}), \quad (\text{C.13})$$

$$\lim_{q^\mu \rightarrow 0} \pi_\mu^+(xq^{\pm \frac{\mu}{s}})(\tau_{-\mu}(\mathbf{K}_a)) = \rho_x^{(2)}(\mathbf{K}_a^{(2,\pm)}), \quad a = 0, 1. \quad (\text{C.14})$$

Based on these relations and the commutation relations (4.1), one can show³⁰ that the contracted commutation relations (C.1) hold under the maps (B.12)-(B.13). The limit of the intertwining relations associated with the first realization of the augmented q -Onsager algebra given in the main text can now be compactly summarized, in terms of the contracted augmented q -Onsager algebra, as

$$\rho_{x^{-1}}^{(i)}(a^{(i,+)})\mathbf{K}^{(i)}(x) = \mathbf{K}^{(i)}(x)\rho_x^{(i)}(a^{(i,-)}), \quad \rho_{x^{-1}}^{(i)}(a^{(i,-)})\check{\mathbf{K}}^{(i)}(x) = \check{\mathbf{K}}^{(i)}(x)\rho_x^{(i)}(a^{(i,+)})$$

for any $a \in \{\mathbf{K}_0, \mathbf{K}_1, Z_1, \tilde{Z}_1\}$. (C.15)

C.2 The second realization

In this subsection, the contracted augmented q -Onsager algebra is also embedded into $U_q(\widehat{sl}(2; I))$. We consider two types of realization of the contracted augmented q -Onsager algebra $\mathcal{O}_q^{aug}(\bar{I})$ with $\bar{I} = \{0, 1\} \setminus I$, as subalgebra of $U_q(\widehat{sl}(2; I))$. The realizations of $\mathcal{O}_q^{aug}(\{1\})$ in terms of $U_q(\widehat{sl}(2; \{0\}))$ are given by

$$\begin{aligned} \bar{\mathbf{K}}_0^{(1,-)} &= \tau(\mathbf{K}_0^{(2,-)}) = \bar{\epsilon}_- q^{h_0}, & \bar{\mathbf{K}}_1^{(1,-)} &= \tau(\mathbf{K}_1^{(2,-)}) = \bar{\epsilon}_+ q^{h_1}, \\ \bar{Z}_1^{(1,-)} &= \tau(Z_1^{(2,-)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ q e_1 q^{h_1} + \bar{\epsilon}_- f_0), \\ \tilde{Z}_1^{(1,-)} &= \tau(\tilde{Z}_1^{(2,-)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ f_1), \end{aligned} \quad (\text{C.16})$$

$$\begin{aligned} \bar{\mathbf{K}}_0^{(1,+)} &= \tau(\mathbf{K}_0^{(2,+)}) = \bar{\epsilon}_- q^{h_0}, & \bar{\mathbf{K}}_1^{(1,+)} &= \tau(\mathbf{K}_1^{(2,+)}) = \bar{\epsilon}_+ q^{h_1}, \\ \bar{Z}_1^{(1,+)} &= \tau(Z_1^{(2,+)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ q e_1 q^{h_1}), \\ \tilde{Z}_1^{(1,+)} &= \tau(\tilde{Z}_1^{(2,+)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ f_1 + \bar{\epsilon}_- q e_0 q^{h_0}), \end{aligned} \quad (\text{C.17})$$

where $\tau(\epsilon_\pm) = \bar{\epsilon}_\mp$ is assumed. The realizations of $\mathcal{O}_q^{aug}(\{0\})$ in terms of $U_q(\widehat{sl}(2; \{1\}))$ are given by

$$\begin{aligned} \bar{\mathbf{K}}_0^{(2,-)} &= \tau(\mathbf{K}_0^{(1,-)}) = \bar{\epsilon}_- q^{h_0}, & \bar{\mathbf{K}}_1^{(2,-)} &= \tau(\mathbf{K}_1^{(1,-)}) = \bar{\epsilon}_+ q^{h_1}, \\ \bar{Z}_1^{(2,-)} &= \tau(Z_1^{(1,-)}) = (q^2 - q^{-2})(\bar{\epsilon}_- f_0), \\ \tilde{Z}_1^{(2,-)} &= \tau(\tilde{Z}_1^{(1,-)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ f_1 + \bar{\epsilon}_- q e_0 q^{h_0}), \end{aligned} \quad (\text{C.18})$$

³⁰For example, multiplying the equation in the 4-th line in (4.1) by $q^{-2\mu - \frac{4s_0\mu}{s}}$, one obtains $[Z_1 q^{-\frac{2s_0\mu}{s}}, [Z_1 q^{-\frac{2s_0\mu}{s}}, [Z_1 q^{-\frac{2s_0\mu}{s}}, \tilde{Z}_1 q^{-\frac{2s_1\mu}{s}}]_{q^2}]_{q^{-2}}] = \rho_{\text{diag}} Z_1 q^{-\frac{2s_0\mu}{s}} ((\mathbf{K}_1 q^\mu)^2 q^{-4\mu} - (\mathbf{K}_0 q^{-\mu})^2) Z_1 q^{-\frac{2s_0\mu}{s}}$. Then take the limit $q^{-\mu} \rightarrow 0$ in the representation $\pi_\mu^+(xq^{\frac{\mu}{s}})$. Thanks to (C.9) and (C.11), one finds that the equation in the 4-th line in (C.1) for $I = \{0\}$ is satisfied under the map $\rho_x^{(1)}$. The other relations in (C.1) can be checked in the same way.

$$\begin{aligned}
\bar{K}_0^{(2,+)} &= \tau(K_0^{(1,+)}) = \bar{\epsilon}_- q^{h_0}, & \bar{K}_1^{(2,+)} &= \tau(K_1^{(1,+)}) = \bar{\epsilon}_+ q^{h_1}, \\
\bar{Z}_1^{(2,+)} &= \tau(Z_1^{(1,+)}) = (q^2 - q^{-2})(\bar{\epsilon}_+ q e_1 q^{h_1} + \bar{\epsilon}_- f_0), \\
\tilde{\bar{Z}}_1^{(2,+)} &= \tau(\tilde{Z}_1^{(1,+)}) = (q^2 - q^{-2})(\bar{\epsilon}_- q e_0 q^{h_0}).
\end{aligned} \tag{C.19}$$

We remark that $\tilde{\mathcal{O}}_q^{aug}(\bar{I})$ is realized by (C.16)-(C.19) even if $\{e_0, e_1, f_0, f_1, h_0, h_1\}$ are generators of $U_q(\widehat{sl}(2))$, while $\mathcal{O}_q^{aug}(\bar{I})$ is realized only by the generators of $U_q(\widehat{sl}(2; I))$.

Limits of the renormalized generators of the augmented q -Onsager algebra in the Verma modules are related to the images of the contracted augmented q -Onsager algebra under the maps (B.12)-(B.13) as follows:

$$\lim_{q^{-\mu} \rightarrow 0} \bar{\pi}_\mu^+(x q^{\frac{\mu}{s}})(\bar{Z}_1 q^{-\frac{2s_1\mu}{s}}) = \bar{\rho}_x^{(1)}(\bar{Z}_1^{(1,-)}), \quad \lim_{q^{-\mu} \rightarrow 0} \bar{\pi}_\mu^+(x q^{\frac{\mu}{s}})(\tilde{\bar{Z}}_1 q^{-\frac{2s_0\mu}{s}}) = \bar{\rho}_x^{(1)}(\tilde{\bar{Z}}_1^{(1,-)}), \tag{C.20}$$

$$\lim_{q^{-\mu} \rightarrow 0} \bar{\pi}_\mu^+(x q^{-\frac{\mu}{s}})(\bar{Z}_1 q^{-2\mu}) = \bar{\rho}_x^{(1)}(\bar{Z}_1^{(1,+)}), \quad \lim_{q^{-\mu} \rightarrow 0} \bar{\pi}_\mu^+(x q^{-\frac{\mu}{s}})(\tilde{\bar{Z}}_1) = \bar{\rho}_x^{(1)}(\tilde{\bar{Z}}_1^{(1,+)}), \tag{C.21}$$

$$\lim_{q^{-\mu} \rightarrow 0} \bar{\pi}_\mu^+(x q^{\pm \frac{\mu}{s}})(\tau_{-\mu}(\bar{K}_a)) = \bar{\rho}_x^{(1)}(\bar{K}_a^{(1,\mp)}), \quad a = 0, 1, \tag{C.22}$$

$$\lim_{q^\mu \rightarrow 0} \bar{\pi}_\mu^+(x q^{-\frac{\mu}{s}})(\bar{Z}_1 q^{\frac{(s_1-s_0)\mu}{s}}) = \bar{\rho}_x^{(2)}(\bar{Z}_1^{(2,-)}), \quad \lim_{q^\mu \rightarrow 0} \bar{\pi}_\mu^+(x q^{-\frac{\mu}{s}})(\tilde{\bar{Z}}_1 q^{\frac{(s_0-s_1)\mu}{s}}) = \bar{\rho}_x^{(2)}(\tilde{\bar{Z}}_1^{(2,-)}), \tag{C.23}$$

$$\lim_{q^\mu \rightarrow 0} \bar{\pi}_\mu^+(x q^{\frac{\mu}{s}})(\bar{Z}_1 q^{-\mu}) = \bar{\rho}_x^{(2)}(\bar{Z}_1^{(2,+)}), \quad \lim_{q^\mu \rightarrow 0} \bar{\pi}_\mu^+(x q^{\frac{\mu}{s}})(\tilde{\bar{Z}}_1 q^\mu) = \bar{\rho}_x^{(2)}(\tilde{\bar{Z}}_1^{(2,+)}), \tag{C.24}$$

$$\lim_{q^\mu \rightarrow 0} \bar{\pi}_\mu^+(x q^{\pm \frac{\mu}{s}})(\tau_{-\mu}(\bar{K}_a)) = \bar{\rho}_x^{(2)}(\bar{K}_a^{(2,\pm)}), \quad a = 0, 1, \tag{C.25}$$

where $\bar{\pi}_\mu^+(x) = \pi_\mu^+(x)|_{(s_0, s_1) \mapsto (-s_1, -s_0)}$ and $\bar{\rho}_x^{(i)} = \rho_x^{(i)}|_{(s_0, s_1) \mapsto (-s_1, -s_0)}$.

The limit of the intertwining relations associated with the first realization of the augmented q -Onsager algebra in the main text can now be compactly summarized, in terms of the contracted augmented q -Onsager algebra, as

$$\begin{aligned}
\bar{\rho}_{x^{-1}q^{-\frac{2}{s}}}^{(i)}(a^{(i,+)}) \mathbf{g}^{(i)} \bar{\mathbf{K}}^{(i)}(x)^t &= \mathbf{g}^{(i)} \bar{\mathbf{K}}^{(i)}(x)^t \bar{\rho}_{xq^{\frac{2}{s}}}^{(i)}(a^{(i,-)}), \\
\bar{\rho}_{x^{-1}q^{-\frac{2}{s}}}^{(i)}(a^{(i,-)}) \mathbf{g}^{(i)} \tilde{\bar{\mathbf{K}}}^{(i)}(x)^t &= \mathbf{g}^{(i)} \tilde{\bar{\mathbf{K}}}^{(i)}(x)^t \bar{\rho}_{xq^{\frac{2}{s}}}^{(i)}(a^{(i,+)}) \\
&\text{for any } a \in \{\bar{K}_0, \bar{K}_1, \bar{Z}_1, \tilde{\bar{Z}}_1\}, \tag{C.26}
\end{aligned}$$

where $\mathbf{g}^{(i)} = q^{\mathbf{h}_i(s_0-s_1)/s} \in \text{Osc}_i$.

D Inversion relations

$$\begin{aligned}
g_1^{-1} R(x q^{\frac{4}{s}})^{t_1} g_1 (R(x)^{-1})^{t_1} &= (R(x)^{-1})^{t_1} g_1^{-1} R(x q^{\frac{4}{s}})^{t_1} g_1 = \\
&= g_2 R(x q^{\frac{4}{s}})^{t_2} g_2^{-1} (R(x)^{-1})^{t_2} = (R(x)^{-1})^{t_2} g_2 R(x q^{\frac{4}{s}})^{t_2} g_2^{-1} = \frac{(x^s - 1)(q^4 x^s - 1)}{(x^s - q^2)(x^s - q^{-2})}, \\
g_1 &= \begin{pmatrix} q^{\frac{s_0 - s_1}{s}} & 0 \\ 0 & q^{-\frac{s_0 - s_1}{s}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} q^{\frac{s_0 - s_1}{s}} & 0 \\ 0 & q^{-\frac{s_0 - s_1}{s}} \end{pmatrix}. \quad (\text{D.1})
\end{aligned}$$

E Various expressions of the solutions

Up to an overall factor ³¹, a formal solution of (4.7) and (4.8) is given by

$$\mathbf{K}(x) = \prod_{j=0}^{\infty} \left(\epsilon_- x^{-s_1} q^{-\frac{H-1}{2}+j} + \epsilon_+ x^{s_0} q^{\frac{H-1}{2}-j} \right) \left(\epsilon_- x^{s_1} q^{-\frac{H+1}{2}-j} + \epsilon_+ x^{-s_0} q^{\frac{H+1}{2}+j} \right). \quad (\text{E.1})$$

In order to make this converge, we have to rewrite this in various different form with different prefactors. In addition to the ones in the main text, we find the following expressions

$$\mathbf{K}(x) = \left(\frac{\epsilon_-}{\epsilon_+} x^{s_0 - s_1} q^{-\frac{H}{2}} \right)^{\frac{H}{2}} \left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{H-1}; q^{-2} \right)_{\infty} \left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-H-1}; q^{-2} \right)_{\infty} \quad \text{for } |q| > 1, \quad (\text{E.2})$$

$$= \left(\frac{\epsilon_-}{\epsilon_+} x^{s_0 - s_1} q^{-\frac{H}{2}} \right)^{\frac{H}{2}} \left(-\frac{\epsilon_+}{\epsilon_-} x^s q^{H+1}; q^2 \right)_{\infty}^{-1} \left(-\frac{\epsilon_-}{\epsilon_+} x^s q^{-H+1}; q^2 \right)_{\infty}^{-1} \quad \text{for } |q| < 1, \quad (\text{E.3})$$

and

$$\mathbf{K}(x) = \left(\frac{\epsilon_+}{\epsilon_-} x^{s_0 - s_1} q^{\frac{H}{2}} \right)^{\frac{H}{2}} \left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-H-1}; q^{-2} \right)_{\infty}^{-1} \left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{H-1}; q^{-2} \right)_{\infty}^{-1} \quad \text{for } |q| > 1, \quad (\text{E.4})$$

$$= \left(\frac{\epsilon_+}{\epsilon_-} x^{s_0 - s_1} q^{\frac{H}{2}} \right)^{\frac{H}{2}} \left(-\frac{\epsilon_-}{\epsilon_+} x^{-s} q^{-H+1}; q^2 \right)_{\infty} \left(-\frac{\epsilon_+}{\epsilon_-} x^{-s} q^{H+1}; q^2 \right)_{\infty} \quad \text{for } |q| < 1. \quad (\text{E.5})$$

F General scalar K-matrices

Let $k_{\pm}, \bar{k}_{\pm}, \epsilon_{\pm}, \bar{\epsilon}_{\pm}$ be scalars. The most general solutions of the reflection equation (4.17) and the dual one (4.32) are given, respectively, by (see [25, 26, 16])

$$K(x) = \begin{pmatrix} x^{s_0} \epsilon_+ + x^{-s_1} \epsilon_- & \frac{k_+(x^s - x^{-s})}{q - q^{-1}} \\ \frac{k_-(x^s - x^{-s})}{q - q^{-1}} & x^{-s_0} \epsilon_+ + x^{s_1} \epsilon_- \end{pmatrix}, \quad (\text{F.1})$$

³¹The overall factor has the form $f(H)$, where $f(x)$ is a function of $x \in \mathbb{C}$ with $f(x+2) = f(x)$.

$$\overline{K}(x) = \begin{pmatrix} qx^{s_0}\overline{\epsilon}_+ + q^{-1}x^{-s_1}\overline{\epsilon}_- & \frac{\overline{k}_+(q^2x^s - q^{-2}x^{-s})}{q - q^{-1}} \\ \frac{\overline{k}_-(q^2x^s - q^{-2}x^{-s})}{q - q^{-1}} & q^{-1}x^{-s_0}\overline{\epsilon}_+ + qx^{s_1}\overline{\epsilon}_- \end{pmatrix}. \quad (\text{F.2})$$

Note that (F.2) is related to (F.1) via

$$\overline{K}(x) = K^t(xq^{2/s})g^{-1}|_{\epsilon_{\pm}=\overline{\epsilon}_{\pm}, k_{\pm}q^{\mp\frac{s_0-s_1}{s}}=\overline{k}_{\mp}}.$$

Moreover, (F.1) and (F.2) reduce to (4.18) and (4.30) at $k_{\pm} = \overline{k}_{\pm} = 0$, respectively.

G Universal T- and Q-operators

In this appendix, we propose several version of operators in terms of L- and K-operators in the main text, which are candidate of universal T- and Q-operators for integrable systems with open boundary conditions associated with $U_q(\widehat{sl}_2)$, and mention merit and demerit of them. We only sketch our idea on the definition of them and do not discuss convergence of the trace, explicit rational limit, functional relations among T- and Q-operators, Bethe equations, etc, which we prefer to consider in a separate publication (if there is an opportunity).

We define universal L-operators as

$$\mathcal{L}(x) = (\text{ev}_x \otimes 1)\mathcal{R}, \quad \overline{\mathcal{L}}(x) = (\text{ev}_x \otimes 1)\overline{\mathcal{R}}, \quad x \in \mathbb{C}, \quad (\text{G.1})$$

and define universal dressed K-operators as ³²

$$\mathcal{K}(x) = \mathcal{L}(x^{-1})(\mathbf{K}(x) \otimes 1)\mathcal{L}^{-1}(x). \quad (\text{G.2})$$

$$\tilde{\mathcal{K}}(x) = \mathcal{L}(x^{-1})(\mathbf{K}(x) \otimes 1)\overline{\mathcal{L}}(x) \quad (\text{G.3})$$

Note that (G.2) is an element of $U_q(sl_2) \otimes \mathcal{B}_-$ and (G.3) is an element of $U_q(sl_2) \otimes U_q(\widehat{sl}_2)$. One can show ³³ that (G.2) satisfies the following dressed reflection equation

$$\mathbf{R}_{12}(x^{-1}, y^{-1})\mathcal{K}_{13}(x)\mathbf{R}_{21}(x, y^{-1})\mathcal{K}_{23}(y) = \mathcal{K}_{23}(y)\mathbf{R}_{12}(x^{-1}, y)\mathcal{K}_{13}(x)\mathbf{R}_{21}(x, y), \quad (\text{G.4})$$

under (4.15). In contrast, we have no proof (or disproof) that (G.3) satisfies

$$\mathbf{R}_{12}(x^{-1}, y^{-1})\tilde{\mathcal{K}}_{13}(x)\mathbf{R}_{21}(x, y^{-1})\tilde{\mathcal{K}}_{23}(y) = \tilde{\mathcal{K}}_{23}(y)\mathbf{R}_{12}(x^{-1}, y)\tilde{\mathcal{K}}_{13}(x)\mathbf{R}_{21}(x, y), \quad (\text{G.5})$$

even if we assume (4.15) since we do not have an analogue of (3.5) for the universal L-operators. Evaluating ³⁴ (G.4) and (G.5) for $1 \otimes \pi_1 \otimes 1$, we obtain the following dressed

³²As remarked in subsection 2.1, $\overline{\mathcal{R}}^{-1} = \mathcal{R}_{21}^{-1}$ also satisfies the relation (2.10). Thus one may also define a universal dressed K-operator as $\mathcal{K}(x) = \overline{\mathcal{L}}^{-1}(x^{-1})(\mathbf{K}(x) \otimes 1)\overline{\mathcal{L}}(x)$, where $\overline{\mathcal{L}}(x) = (\text{ev}_x \otimes 1)\overline{\mathcal{R}}$. In this case, $\mathcal{K}(x)$ is an element of $U_q(sl_2) \otimes \mathcal{B}_+$.

³³One has to use $\mathbf{R}_{12}(x, y)\mathcal{L}_{13}(x)\mathcal{L}_{23}(y) = \mathcal{L}_{23}(y)\mathcal{L}_{13}(x)\mathbf{R}_{12}(x, y)$ and $\mathbf{R}_{21}(x, y)\mathcal{L}_{23}(y)\mathcal{L}_{13}(x) = \mathcal{L}_{13}(x)\mathcal{L}_{23}(y)\mathbf{R}_{21}(x, y)$, which follow from (2.12), and (4.15).

³⁴Here we abuse notation and use the same expression for both the image of $\mathcal{K}_{23}(y)$ for $1 \otimes \pi_1 \otimes 1$ and the original one.

reflection equations for the L-operators

$$\mathbf{L}_{12}(x^{-1}y) \mathcal{K}_{13}(x) \bar{\mathbf{L}}_{12}(xy) \mathcal{K}_{23}(y) = \mathcal{K}_{23}(y) \mathbf{L}_{12}(x^{-1}y^{-1}) \mathcal{K}_{13}(x) \bar{\mathbf{L}}_{12}(xy^{-1}). \quad (\text{G.6})$$

$$\mathbf{L}_{12}(x^{-1}y) \tilde{\mathcal{K}}_{13}(x) \bar{\mathbf{L}}_{12}(xy) \tilde{\mathcal{K}}_{23}(y) = \tilde{\mathcal{K}}_{23}(y) \mathbf{L}_{12}(x^{-1}y^{-1}) \tilde{\mathcal{K}}_{13}(x) \bar{\mathbf{L}}_{12}(xy^{-1}). \quad (\text{G.7})$$

One can also prove (G.6) independent of (G.4) since (G.6) is a dressed version of (4.16). As for (G.7), we can only prove the image of it for (tensor product of) the fundamental representation in the 3rd space based on the relation (3.5). We define universal T-operators as

$$\mathcal{T}_\pi(x) = (\text{tr}_\pi \otimes 1) \left((\bar{\mathbf{K}}(q^{-\frac{4}{s}}x^{-1}) \mathbf{g}^2 \otimes 1) \mathcal{K}(x) \right), \quad (\text{G.8})$$

$$\tilde{\mathcal{T}}_\pi(x) = (\text{tr}_\pi \otimes 1) \left((\bar{\mathbf{K}}(q^{-\frac{4}{s}}x^{-1}) \mathbf{g}^2 \otimes 1) \tilde{\mathcal{K}}(x) \right), \quad (\text{G.9})$$

where π is any representation of $U_q(sl_2)$ for which the trace converges. Thanks to the relation (2.24), the T-operators for a finite dimensional representation are expressed in terms of those for Verma modules:

$$\mathcal{T}_{\pi_\mu}(x) = \mathcal{T}_{\pi_\mu^+}(x) - \mathcal{T}_{\pi_{-\mu-2}^+}(x) \quad \text{for } \mu \in \mathbb{Z}_{\geq 0}, \quad (\text{G.10})$$

$$\tilde{\mathcal{T}}_{\pi_\mu}(x) = \tilde{\mathcal{T}}_{\pi_\mu^+}(x) - \tilde{\mathcal{T}}_{\pi_{-\mu-2}^+}(x) \quad \text{for } \mu \in \mathbb{Z}_{\geq 0}. \quad (\text{G.11})$$

In a similar way as for T-operators, we define universal L-operators for Q-operators as

$$\mathcal{L}^{(a)}(x) = (\rho_x^{(a)} \otimes 1) \mathcal{R}, \quad \bar{\mathcal{L}}^{(a)}(x) = (\rho_x^{(a)} \otimes 1) \bar{\mathcal{R}}, \quad x \in \mathbb{C}, \quad a = 1, 2, \quad (\text{G.12})$$

and universal dressed K-operators as

$$\mathcal{K}^{(a)}(x) = \mathcal{L}^{(a)}(x^{-1}) (\mathbf{K}^{(a)}(x) \otimes 1) \mathcal{L}^{(a)}(x)^{-1}. \quad (\text{G.13})$$

$$\tilde{\mathcal{K}}^{(a)}(x) = \mathcal{L}^{(a)}(x^{-1}) (\mathbf{K}^{(a)}(x) \otimes 1) \bar{\mathcal{L}}^{(a)}(x), \quad a = 1, 2. \quad (\text{G.14})$$

One can prove that (G.13) satisfy dressed reflection equations

$$\mathbf{L}_{12}^{(a)}(x^{-1}y) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy) \mathcal{K}_{23}(y) = \mathcal{K}_{23}(y) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1}) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy^{-1}), \quad a = 1, 2. \quad (\text{G.15})$$

In contrast, we have no proof (or disproof) that (G.14) satisfy

$$\mathbf{L}_{12}^{(a)}(x^{-1}y) \tilde{\mathcal{K}}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy) \tilde{\mathcal{K}}_{23}(y) = \tilde{\mathcal{K}}_{23}(y) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1}) \tilde{\mathcal{K}}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy^{-1}), \quad a = 1, 2. \quad (\text{G.16})$$

As (G.16) are limit of (G.7), we can prove (G.16) only for (tensor product of) the fundamental representation in the 3rd space at the moment. We also define universal Q-

operators³⁵ as

$$\mathcal{Q}^{(a)}(x) = (\text{tr}_{W_a} \otimes 1) \left((\check{\mathbf{K}}^{(a)}(q^{-\frac{4}{s}}x^{-1})(\mathbf{g}^{(a)})^2 \otimes 1) \mathcal{K}^{(a)}(x) \right), \quad (\text{G.17})$$

$$\tilde{\mathcal{Q}}^{(a)}(x) = (\text{tr}_{W_a} \otimes 1) \left((\check{\mathbf{K}}^{(a)}(q^{-\frac{4}{s}}x^{-1})(\mathbf{g}^{(a)})^2 \otimes 1) \tilde{\mathcal{K}}^{(a)}(x) \right) \quad \text{for } a = 1, 2, \quad (\text{G.18})$$

where $\mathbf{g}^{(a)} = q^{\frac{(s_0-s_1)\mathbf{h}_a}{s}}$ and W_a are Fock spaces generated by Osc_a . Note that (G.17) are elements of \mathcal{B}_- and (G.18) are elements of $U_q(\widehat{sl}_2)$. We remark that (G.18) are limit of the universal T-operator (G.9):

$$\tilde{\mathcal{Q}}^{(1)}(x) = \lim_{q^{-\mu} \rightarrow 0} \tilde{\mathcal{T}}_{\pi_\mu}^+(q^{\frac{\mu}{s}}x) q^{\mu-\mu h_1}, \quad (\text{G.19})$$

$$\tilde{\mathcal{Q}}^{(2)}(x) = \lim_{q^\mu \rightarrow 0} \tilde{\mathcal{T}}_{\pi_\mu}^+(q^{-\frac{\mu}{s}}x) q^{-\mu-\mu h_1}, \quad (\text{G.20})$$

where (B.14)-(B.17) are used. In contrast, we can not interpret (G.17) as a straightforward limit of (G.8). Evaluating these for various representations of \mathcal{B}_- or $U_q(\widehat{sl}_2)$, we obtain a wide class of T- and Q-operators. For example, T- and Q-operators acting on $(\mathbb{C}^2)^{\otimes L}$ are given by

$$\begin{aligned} \mathbf{T}_{\pi_\mu}(x) &= (\pi_1(\xi_1) \otimes \cdots \otimes \pi_1(\xi_L)) \Delta^{\otimes(L-1)} \mathcal{T}_{\pi_\mu}(x), \\ &= \Psi(x, \{\xi_i\}) (\text{tr}_{\pi_\mu} \otimes 1^{\otimes L}) \left(\left(\overline{\mathbf{K}} \left(q^{-\frac{4}{s}}x^{-1} \right) \mathbf{g}^2 \otimes 1^{\otimes L} \right) \mathbf{L}_{0L}(x^{-1}\xi_L^{-1}) \cdots \mathbf{L}_{01}(x^{-1}\xi_1^{-1}) \right. \\ &\quad \left. \times (\mathbf{K}(x) \otimes 1^{\otimes L}) \overline{\mathbf{L}}_{01}(x\xi_1^{-1}) \cdots \overline{\mathbf{L}}_{0L}(x\xi_L^{-1}) \right), \quad (\text{G.21}) \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{T}}_{\pi_\mu}(x) &= (\pi_1(\xi_1) \otimes \cdots \otimes \pi_1(\xi_L)) \Delta^{\otimes(L-1)} \tilde{\mathcal{T}}_{\pi_\mu}(x), \\ &= \tilde{\Psi}(x, \{\xi_i\}) (\text{tr}_{\pi_\mu} \otimes 1^{\otimes L}) \left(\left(\overline{\mathbf{K}} \left(q^{-\frac{4}{s}}x^{-1} \right) \mathbf{g}^2 \otimes 1^{\otimes L} \right) \mathbf{L}_{0L}(x^{-1}\xi_L^{-1}) \cdots \mathbf{L}_{01}(x^{-1}\xi_1^{-1}) \right. \\ &\quad \left. \times (\mathbf{K}(x) \otimes 1^{\otimes L}) \overline{\mathbf{L}}_{01}(x\xi_1^{-1}) \cdots \overline{\mathbf{L}}_{0L}(x\xi_L^{-1}) \right), \quad (\text{G.22}) \end{aligned}$$

$$\begin{aligned} \mathbf{Q}^{(a)}(x) &= (\pi_1(\xi_1) \otimes \cdots \otimes \pi_1(\xi_L)) \Delta^{\otimes(L-1)} \mathcal{Q}^{(a)}(x), \\ &= \Psi^{(a)}(x, \{\xi_i\}) (\text{tr}_{W_a} \otimes 1^{\otimes L}) \left(\left(\check{\mathbf{K}}^{(a)} \left(q^{-\frac{4}{s}}x^{-1} \right) (\mathbf{g}^{(a)})^2 \otimes 1^{\otimes L} \right) \mathbf{L}_{0L}^{(a)}(x^{-1}\xi_L^{-1}) \cdots \mathbf{L}_{01}^{(a)}(x^{-1}\xi_1^{-1}) \right. \\ &\quad \left. \times (\mathbf{K}^{(a)}(x) \otimes 1^{\otimes L}) \check{\mathbf{L}}_{01}^{(a)}(x\xi_1^{-1}) \cdots \check{\mathbf{L}}_{0L}^{(a)}(x\xi_L^{-1}) \right), \quad a = 1, 2, \quad (\text{G.23}) \end{aligned}$$

³⁵Another possible definition will be $\check{\mathcal{Q}}^{(a)}(x) = (\text{tr}_{W_a} \otimes 1) \left((\check{\mathbf{K}}^{(a)}(q^{-\frac{4}{s}}x^{-1})(\mathbf{g}^{(a)})^2 \otimes 1) \check{\mathcal{K}}^{(a)}(x) \right)$, $\check{\mathcal{K}}^{(a)}(x) = \check{\mathcal{L}}^{(a)}(x^{-1})(\check{\mathbf{K}}^{(a)}(x) \otimes 1) \check{\mathcal{L}}^{(a)}(x)^{-1}$, $a = 1, 2$.

$$\begin{aligned}
\tilde{\mathbf{Q}}^{(a)}(x) &= (\pi_1(\xi_1) \otimes \cdots \otimes \pi_1(\xi_L)) \Delta^{\otimes(L-1)} \tilde{\mathbf{Q}}^{(a)}(x), \\
&= \tilde{\Psi}^{(a)}(x, \{\xi_i\}) (\text{tr}_{W_a} \otimes 1^{\otimes L}) \left(\left(\tilde{\mathbf{K}}^{(a)} \left(q^{-\frac{4}{s}} x^{-1} \right) (\mathbf{g}^{(a)})^2 \otimes 1^{\otimes L} \right) \mathbf{L}_{0L}^{(a)}(x^{-1} \xi_L^{-1}) \cdots \mathbf{L}_{01}^{(a)}(x^{-1} \xi_1^{-1}) \right. \\
&\quad \left. \times (\mathbf{K}^{(a)}(x) \otimes 1^{\otimes L}) \bar{\mathbf{L}}_{01}^{(a)}(x \xi_1^{-1}) \cdots \bar{\mathbf{L}}_{0L}^{(a)}(x \xi_L^{-1}) \right), \quad a = 1, 2, \quad (\text{G.24})
\end{aligned}$$

where $\xi_1, \dots, \xi_L \in \mathbb{C} \setminus \{0\}$ are inhomogeneities on the spectral parameter in the quantum space; the trace is taken over the auxiliary space (denoted as 0); the relations (3.5), (3.20) and (3.22) are applied. The overall factors³⁶ are given by

$$\Psi(x, \{\xi_i\}) = \prod_{k=1}^L \pi_\mu \left(\frac{\phi(x \xi_k^{-1})}{\varphi(x \xi_k^{-1}) \phi(x^{-1} \xi_k^{-1})} \right), \quad (\text{G.25})$$

$$\tilde{\Psi}(x, \{\xi_i\}) = \prod_{k=1}^L \pi_\mu \left(\frac{1}{\phi(x^{-1} \xi_k^{-1}) \phi(x^{-1} \xi_k)} \right), \quad \varphi(x) := q^{-1}(\lambda^2 C - x^s - x^{-s}), \quad (\text{G.26})$$

$$\Psi^{(a)}(x, \{\xi_i\}) = \prod_{k=1}^L \left(\frac{\phi^{(a)}(x \xi_k^{-1})}{\varphi^{(a)}(x \xi_k^{-1}) \phi^{(a)}(x^{-1} \xi_k^{-1})} \right), \quad \varphi^{(a)}(x) := q^{-1}(q^{3-2a} - x^{-s}), \quad (\text{G.27})$$

$$\tilde{\Psi}^{(a)}(x, \{\xi_i\}) = \prod_{k=1}^L \left(\frac{1}{\phi^{(a)}(x^{-1} \xi_k^{-1}) \phi^{(a)}(x^{-1} \xi_k)} \right), \quad a = 1, 2. \quad (\text{G.28})$$

The T-operators (G.21) and (G.22) are essentially the same object if the representation for the auxiliary space is irreducible, while the corresponding Q-operators (G.23) and (G.24) are substantially different each other. We expect that (G.23) or (G.24) give Q-operators for the XXZ-model. In fact our Q-operators (G.24) reduce to Q-operators for the XXX-model similar to the ones in [19] in the rational limit $q \rightarrow 1$. In contrast, we can not take the rational limit of (G.23) straightforwardly³⁷. In this sense, (G.24) might be more promising than (G.23). However, (G.23) still deserve further study as they have good properties on commutativity of operators.

We expect that these operators constitute mutually commuting family of operators. In fact, we have proven commutativity of the universal T-operators (G.8) for the fundamental representation in the auxiliary space and the universal Q-operators (G.17):

$$\mathcal{T}_{\pi_1}(x) \mathcal{T}_{\pi_1}(y) = \mathcal{T}_{\pi_1}(y) \mathcal{T}_{\pi_1}(x), \quad \mathcal{Q}^{(a)}(x) \mathcal{T}_{\pi_1}(y) = \mathcal{T}_{\pi_1}(y) \mathcal{Q}^{(a)}(x), \quad a = 1, 2, \quad x, y \in \mathbb{C}. \quad (\text{G.29})$$

At the moment, we do not have a proof of commutativity among T-operators for the generic representations in the auxiliary space and commutativity among Q-operators. As for (G.9) and (G.18), we have proof³⁸ of commutativity for only a particular representa-

³⁶These factors cannot be taken outside of the trace if the representations are not irreducible.

³⁷One may have to generalize (G.23) to interpolate (G.23) and a rational analogue of it (to use renormalized operators, which appear for example in the left hand side of (4.84)). This remains to be clarified.

³⁸The second relation also follows from limit of $\tilde{\mathbf{T}}_{\pi_\mu^+}(x) \tilde{\mathbf{T}}_{\pi_1}(y) = \tilde{\mathbf{T}}_{\pi_1}(y) \tilde{\mathbf{T}}_{\pi_\mu^+}(x)$.

tion in the quantum space.

$$\tilde{\mathbf{T}}_{\pi_1}(x)\tilde{\mathbf{T}}_{\pi_1}(y) = \tilde{\mathbf{T}}_{\pi_1}(y)\tilde{\mathbf{T}}_{\pi_1}(x), \quad \tilde{\mathbf{Q}}^{(a)}(x)\tilde{\mathbf{T}}_{\pi_1}(y) = \tilde{\mathbf{T}}_{\pi_1}(y)\tilde{\mathbf{Q}}^{(a)}(x), \quad a = 1, 2, \quad x, y \in \mathbb{C}. \quad (\text{G.30})$$

A proof of the second relation in (G.29) is given as follows³⁹. First, we rewrite the reflection equation (4.73) in terms of $\overline{\mathbf{K}}'(x) = \overline{\mathbf{K}}(q^{-\frac{4}{s}}x)g^2$ and $\check{\overline{\mathbf{K}}}^{(a)'}(x) = \check{\overline{\mathbf{K}}}^{(a)}(q^{-\frac{4}{s}}x)(\mathbf{g}^{(a)})^2$ as

$$\begin{aligned} \check{\mathbf{L}}^{(a)}\left(\frac{y}{x}\right)\check{\overline{\mathbf{K}}}_1^{(a)'}(x)^{t_1}g_2\check{\mathbf{L}}^{(a)}\left(xyq^{-\frac{4}{s}}\right)g_2^{-1}\overline{\mathbf{K}}_2'(y)^{t_2} = \\ = \overline{\mathbf{K}}_2'(y)^{t_2}g_2^{-1}\check{\mathbf{L}}^{(a)}\left(\frac{q^{\frac{4}{s}}}{xy}\right)g_2\check{\overline{\mathbf{K}}}_1^{(a)'}(x)^{t_1}\check{\mathbf{L}}^{(a)}\left(\frac{x}{y}\right), \quad a = 1, 2. \end{aligned} \quad (\text{G.31})$$

Any object we consider under the trace is a linear combination of elements of the form $\mathcal{J}_{m,n} = E^m F^n q^{\xi H} \in U_q(sl_2)$ (or $\mathcal{J}_{m,n} = \mathbf{e}_a^m \mathbf{f}_a^n q^{\xi \mathbf{h}_a} \in \text{Osc}_a$, $a = 1, 2$) for $m, n \in \mathbb{Z}_{\geq 0}$, $\xi \in \mathbb{C}$. In particular, only the terms for $m = n$ ($\mathcal{J} := \mathcal{J}_{m,m}$) contribute to the trace. Then $\text{Tr} \mathcal{J}^t = \text{Tr} \mathcal{J}$ holds if the trace converges and the cyclicity of the trace holds. In the following, we assume⁴⁰ this. One can check the following relations for the L-operators⁴¹:

$$\begin{aligned} \mathbf{L}(x)^{t_1 t_2} &= \overline{\mathbf{L}}(x^{-1}), & \overline{\mathbf{L}}(x)^{t_1 t_2} &= \mathbf{L}(x^{-1}), \\ \mathbf{L}^{(a)}(x)^{t_1 t_2} &= \overline{\mathbf{L}}^{(a)}(x^{-1}), & \overline{\mathbf{L}}^{(a)}(x)^{t_1 t_2} &= \mathbf{L}^{(a)}(x^{-1}), \\ \check{\mathbf{L}}^{(a)}(x)^{t_1 t_2} &= \check{\overline{\mathbf{L}}}^{(a)}(x^{-1}), & \check{\overline{\mathbf{L}}}^{(a)}(x)^{t_1 t_2} &= \check{\mathbf{L}}^{(a)}(x^{-1}), \quad a = 1, 2. \end{aligned} \quad (\text{G.32})$$

Let $\varphi^{(a)}(x) = q^{-1}(q^{3-2a} - x^{-s})$, $\tilde{\varphi}^{(a)}(x) = q(q^{3-2a} - q^{-2}x^{-s})$ for $a = 1, 2$. Then we use a refinement of the Sklyanin's method [15]:

$$\begin{aligned} \mathcal{Q}^{(a)}(x)\mathcal{T}_{\pi_1}(y) &= \text{tr}_1 \left(\check{\overline{\mathbf{K}}}_1^{(a)}(q^{-\frac{4}{s}}x^{-1})(\mathbf{g}_1^{(a)})^2 \mathcal{K}_{13}^{(a)}(x) \right) \text{tr}_2 \left(\overline{\mathbf{K}}_2(q^{-\frac{4}{s}}y^{-1})\mathbf{g}_2^2 \mathcal{K}_{23}(y) \right) \\ &= \text{tr}_1 \left(\check{\overline{\mathbf{K}}}_1^{(a)'}(x^{-1})\mathcal{K}_{13}^{(a)}(x) \right) \text{tr}_2 \left(\overline{\mathbf{K}}_2'(y^{-1})\mathcal{K}_{23}(y) \right) \\ &= \text{tr}_1 \left(\check{\overline{\mathbf{K}}}_1^{(a)'}(x^{-1})^{t_1} \mathcal{K}_{13}^{(a)}(x)^{t_1} \right) \text{tr}_2 \left(\overline{\mathbf{K}}_2'(y^{-1})\mathcal{K}_{23}(y) \right) \\ &= \text{tr}_{12} \left(\check{\overline{\mathbf{K}}}_1^{(a)'}(x^{-1})^{t_1} \overline{\mathbf{K}}_2'(y^{-1})\mathcal{K}_{13}^{(a)}(x)^{t_1} \mathcal{K}_{23}(y) \right) \\ &= \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \text{tr}_{12} \left(\check{\overline{\mathbf{K}}}_1^{(a)'}(x^{-1})^{t_1} g_2 \check{\mathbf{L}}_{12}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right) g_2^{-1} \overline{\mathbf{K}}_2'(y^{-1})^{t_2} \right)^{t_2} \end{aligned}$$

³⁹The first relation in (G.29) can be proven similarly (easier than the second relation).

⁴⁰This is not a trivial issue in particular for infinite dimensional representations.

⁴¹The anti-automorphism t defined by (2.21) or (2.29) might not be enough for more general R-operators. A relation corresponding to (G.32) (similar to (A.22)) may not hold true in general situation. However, it is enough for the L-operators discussed here.

$$\begin{aligned}
& \times \left(\mathcal{K}_{13}^{(a)}(x) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1})^{t_1 t_2} \mathcal{K}_{23}(y) \right)^{t_1} \quad [\text{by (3.24), (3.26)}] \\
& = \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \text{tr}_{12} \left(\check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} g_2 \check{\mathbf{L}}_{12}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right) g_2^{-1} \check{\mathbf{K}}_2'(y^{-1})^{t_2} \right)^{t_1 t_2} \\
& \quad \times \left(\mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy) \mathcal{K}_{23}(y) \right) \quad [\text{by (G.32)}] \\
& = \varphi^{(a)}(x^{-1}y)^{-1} \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \\
& \quad \times \text{tr}_{12} \left(\check{\mathbf{L}}_{12}^{(a)}(x^{-1}y)^{t_1 t_2} \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} g_2 \check{\mathbf{L}}_{12}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right) g_2^{-1} \check{\mathbf{K}}_2'(y^{-1})^{t_2} \right)^{t_1 t_2} \\
& \quad \times \left(\mathbf{L}_{12}^{(a)}(x^{-1}y) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy) \mathcal{K}_{23}(y) \right) \quad [\text{by (3.20), (3.22)}] \\
& = \varphi^{(a)}(x^{-1}y)^{-1} \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \\
& \quad \times \text{tr}_{12} \left(\check{\mathbf{L}}_{12}^{(a)}(xy^{-1}) \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} g_2 \check{\mathbf{L}}_{12}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right) g_2^{-1} \check{\mathbf{K}}_2'(y^{-1})^{t_2} \right)^{t_1 t_2} \\
& \quad \times \left(\mathbf{L}_{12}^{(a)}(x^{-1}y) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy) \mathcal{K}_{23}(y) \right) \quad [\text{by (G.32)}] \\
& = \varphi^{(a)}(x^{-1}y)^{-1} \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \\
& \quad \times \text{tr}_{12} \left(\check{\mathbf{K}}_2'(y^{-1})^{t_2} g_2^{-1} \check{\mathbf{L}}_{12}^{(a)} \left(xyq^{\frac{4}{s}} \right) g_2 \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} \check{\mathbf{L}}_{12}^{(a)}(x^{-1}y) \right)^{t_1 t_2} \\
& \quad \times \left(\mathcal{K}_{23}(y) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1}) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy^{-1}) \right) \quad [\text{by (G.15), (G.31)}] \\
& = \varphi^{(a)}(x^{-1}y)^{-1} \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \\
& \quad \times \text{tr}_{12} \check{\mathbf{L}}_{12}^{(a)}(xy^{-1}) \left(\check{\mathbf{K}}_2'(y^{-1})^{t_2} g_2^{-1} \check{\mathbf{L}}_{12}^{(a)} \left(xyq^{\frac{4}{s}} \right) g_2 \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} \right)^{t_1 t_2} \\
& \quad \times \left(\mathcal{K}_{23}(y) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1}) \mathcal{K}_{13}^{(a)}(x) \bar{\mathbf{L}}_{12}^{(a)}(xy^{-1}) \right) \quad [\text{by (G.32)}] \\
& = \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \text{tr}_{12} \left(\check{\mathbf{K}}_2'(y^{-1})^{t_2} g_2^{-1} \check{\mathbf{L}}_{12}^{(a)} \left(xyq^{\frac{4}{s}} \right) g_2 \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} \right)^{t_2} \\
& \quad \times \left(\mathcal{K}_{23}(y) \mathbf{L}_{12}^{(a)}(x^{-1}y^{-1}) \mathcal{K}_{13}^{(a)}(x) \right)^{t_1} \quad [\text{by (3.21), (3.23)}] \\
& = \tilde{\varphi}^{(a)} \left(x^{-1}y^{-1}q^{-\frac{4}{s}} \right)^{-1} \text{tr}_{12} \left(g_2 \check{\mathbf{L}}_{12}^{(a)} \left(xyq^{\frac{4}{s}} \right)^{t_2} g_2^{-1} \check{\mathbf{K}}_2'(y^{-1}) \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} \right. \\
& \quad \left. \times \mathcal{K}_{23}(y) \mathcal{K}_{13}^{(a)}(x)^{t_1} \bar{\mathbf{L}}_{12}^{(a)}(xy)^{t_2} \right) \quad [\text{by (G.32)}] \\
& = \text{tr}_{12} \left(\check{\mathbf{K}}_2'(y^{-1}) \check{\mathbf{K}}_1^{(a)'}(x^{-1})^{t_1} \mathcal{K}_{23}(y) \mathcal{K}_{13}^{(a)}(x)^{t_1} \right) \quad [\text{by (3.25), (3.27)}]
\end{aligned}$$

$$\begin{aligned}
&= \text{tr}_2 \left(\overline{\mathbf{K}}'_2(y^{-1}) \mathcal{K}_{23}(y) \right) \text{tr}_1 \left(\check{\overline{\mathbf{K}}}_1^{(a)'}(x^{-1})^{t_1} \mathcal{K}_{13}^{(a)}(x)^{t_1} \right) \\
&= \mathcal{T}_{\pi_1}(y) \mathcal{Q}^{(a)}(x).
\end{aligned} \tag{G.33}$$

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