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Abstract

The present paper develops a simple theoretical setup to examine the role of the tax-spending mix of fiscal adjustments on aggregate (in)stability in indebted economies. To this end, we build an AK endogenous growth model with public debt dynamics. If the adjustment of the government's budget constraint is based on a single instrument (taxes or public spending), the economy converges towards a high-growth path. With mixed adjustment, however, another equilibrium appears (the no-growth path) that can be locally over-determine (unstable) or under-determined (stable). A hopf bifurcation can occurs at the border between the last two cases, which leads to cyclical dynamics. We also show that global indeterminacy is likely to emerge if fiscal adjustment is mainly based on public spending. A calibration of the model shows that area of indeterminacy covers reasonable values for parameters.

Keywords: Endogenous growth; indeterminacy; balanced-budget rules; hopf bifurcation

1. Introduction

In response to the Great Recession and to the long-lasting increase of public debt since four decades, many OECD countries implemented fiscal consolidation programmes. However, empirical researches on the macroeconomic effects of these programmes remain unsettled. Some authors find expansionary austerity episodes (Giavazzi and Pagano, 1990; Briotti, 2005), while others join traditional textbook Keynesian models highlighting the adverse effects of fiscal austerity on economic growth (Perotti, 2011).

Beyond these conflicting views, the current consensus emerging from recent empirical research is that the composition of fiscal consolidations (tax increases vs spending cuts) matters. Typically, robust evidences suggest that consolidations based on tax-increases generate larger fluctuations and output losses compared to consolidations relying on reductions in government spending, including both public investment and government consumption or transfers.[2] From a theoretical perspective, such findings can be related to

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[2] This result is established using VAR (Perotti and Alesina, 1995; Alesina and Ardagna, 2010), or with the narrative IMF data of Pescatori et al. (2011), as Alesina and Ardagna (2013); Alesina et al. (2017, 2018). This result contrast with standard Keynesian works predicting that spending cuts are always recessionary and that multiplier for spending are higher than for taxes (Galí et al., 2007; DeLong and Summers, 2012).

the aggregate fluctuation in the form of belief-driven fluctuations in neoclassical growth models. Indeed, in these models, a tax-based (TB) fiscal adjustment can produce aggregate instability (Schmitt-Grohé and Uribe, 1997), while there is a stable staddle-path to the steady state under a expenditure-based (EB) adjustment (provided that public spending is useless, Guo and Harrison, 2004). However, this theoretical literature has two shortcomings. First, it rests on the assumption of a balanced-budget rule (BBR), without accounting for public debt. Yet, public deficits and debts characterize most developed countries since the mid-1970, and the implemented consolidation plans mostly aim at reaching a sustainable public debt path.[3] Second, these models only consider a single instrument-based adjustment (taxes *or* spending), contrasting with effective fiscal adjustment plans, which are often complex policy packages that closely associate the tax and spending sides.[4]

The goal of this paper is to provide a simple theoretical setup to examine the role of the tax-spending mix of fiscal adjustments on aggregate (in)stability in indebted economies. To this end, we build a continuous-time one-sector endogenous growth model with two innovative features that reflects stylized facts of fiscal adjustments.

On the one hand, we consider a generalized BBR, which allows taking account of public debt. Indeed, the economies that adopt a BBR generally have a positive debt at the time they implement the rule (and, as Lledó et al., 2017, shows, the implementation of the BBR mostly results from the presence of a high public debt).[5] This generalized BBR can generate a complex dynamics of the debt-to-output *ratio*, even if the public debt *level* is constant over time.

On the other hand, we specify a general adjustment scheme, such that the debt-burden is covered both by cuts in public spending and rises in taxes, and we carefully examine the effect of the sharing of fiscal adjustment between the two instruments. Indeed, historical evidences show that both expenditure and revenue items contribute to fiscal adjustment. For example, Alesina and Ardagna find contributions around 35% for EB and 65% for TB adjustment in OECD contractionary episodes (1970-2007). Based

[3]The deficit-to-GDP ratio was around 2.5% on average in OECD countries in the period 1970-2005, and this ratio increased since the Great Recession (according to the 2017 IMFs World Economic Outlook, average general government gross debt in ratio of GDP in developed countries rose from around 72% in 2007 to roughly 105% in 2007).

[4]For example, Alesina et al. (2015) identify TB (resp. EB) fiscal adjustments, as episodes such that (announced or unexpected) changes in taxes (resp. expenditures) are larger than changes in expenditures (resp. taxes). They conclude that fiscal adjustments mostly mix changes in taxes and expenditures: in the 60 plans documented, around 40% consist in years of TB and 60% in years of EB adjustments.

[5]A number of recent works have shown that endogenous growth setups are a useful framework for analyzing the effects of a continuous grow of public debt in the long run; see, e.g., Minea and Villieu (2012); Nishimura et al. (2015a); Boucekkine et al. (2015); Nishimura et al. (2015b); Menuet et al. (2018a). Albeit we focus here on BBR regimes, our model can be extended to deficit rules without qualitative changes (see, in particular, Minea and Villieu, 2012; Menuet et al., 2018a).

on a simple small-scale (two dimensional) dynamic system, we notably show that small changes in the tax-spending mix generate radical shifts in the dynamic properties of the economy, i.e., bifurcations.

Our results are as follows.

(i) If the fiscal adjustment is based on a single instrument, there is a unique well-determined positive balanced-growth path (BGP) in the long run. With EB adjustment, the steady state is unique, while under TB adjustment, the interaction between the government's budget constraint and households optimal saving behavior gives birth to a pair of BGPs: a high-growth path with zero debt and a no-growth trap with high debt. However,the latter is unstable and can be removed thanks to local dynamic analysis.

(ii) With mixed adjustement of taxes and public spending, multiplicity cannot be excluded. Indeed, in this case, while the high equilibrium is always saddle-path stable, the topological behavior of the no-growth trap is more complex. Depending on the relative weigh of TB adjustment (that we use as a bifurcation parameter), the no-growth trap can be locally over-determined (unstable), or under-determined (stable). Effectively a subcritical Hopf bifurcation can occur, leading to cyclical dynamics.

(iii) The simplicity of our framework allows fully characterizing the global dynamics of the economy. We notably show that global indeterminacy is likely to emerge if fiscal adjustment is mainly based on public spending. A calibration of the model show that area of indeterminacy covers reasonable values for parameters, since the share of TB adjustement that gives rise to the Hopf bifurcation is around 40%, close to Alesina et al. and Dvies empirical findings.

Although stylized, our model addresses major long-lasting topics in macroeconomics.

First, our paper complements the fast-growing literature on indeterminacy in endogenous growth models.[6] Starting from the seminal paper of Matsuyama (1991), local and global indeterminacy come from public capital externality (productive or welfare-enhancing public spending), increasing returns, or interactions in two-sector frameworks[7]. In contrast, in our model, global indeterminacy is established in a simple one-sector model with wasteful public spending, and does not fundamentally rest on increasing return in production. Indeed, under a BBR, the non-trivial dynamics of the debt-to-capital ratio give rise to complex interactions between the government's budget constraint and the households' saving behavior.[8]

[6]See the surveys of Benhabib and Farmer (1999), chap. 6, or Mino et al. (2008) regarding the local indeterminacy.

[7]See, e.g., Benhabib et al. (1994); Benhabib and Nishimura (1998); Matsuyama (1999); Benhabib et al. (2000); Brito and Venditti (2010); Mattana et al. (2012); Nishimura et al. (2013), among others.

[8]Some papers have shown that endogenous growth models with public debt generate indeterminacy (Minea and Villieu, 2012; Nishimura et al., 2015a; Menuet et al., 2018a).

Second, in our model, two instruments (taxes and pubic spending) can adjust jointly the government's budget constraint. This is an important feature because, to the best of our knowlged, this is the first paper that provides a theoretical basis to the large empirical literature emphasizing that fiscal adjustments closely associate both tax and spending sides.[9]

The remainder of the paper is organized as follows. Section 2 presents the model, section 3 studies local dynamics, section 4 provides a numerical example, section 5 explores the global dynamics, section 6 discusses findings in term of economic policy and concludes.

2. The model

We consider a simple continuous-time endogenous-growth model with a representative individual, who consists of a household and a competitive firm, and a government. All agents are infinitely-lived and have perfect foresight.

2.1. Households

The representative household starts at the initial period with a positive stock of capital (K_0) and a given dotation of time that is inelastically devoted to labor (thus, labor supply L is exogenous). He chooses the path of consumption $\{C_t\}_{t\geq 0}$, and capital $\{K_t\}_{t>0}$, so as to maximize the present discount value of its lifetime utility.

$$U = \int_0^\infty e^{-\rho t} u(C_t) dt, \tag{1}$$

where $\rho \in (0, 1)$ is the subjective discount rate

As usual, we define a constant-elasticity of substitution (CES) utility function

$$u(C_t) = \begin{cases} \frac{S}{S-1}\left\{(C_t)^{\frac{S-1}{S}} - 1\right\}, & \text{if } S \neq 1, \\ \log(C_t), & \text{if } S = 1, \end{cases} \tag{2}$$

with $S := -u''(C_t)C_t/u'(C_t) > 0$ the elasticity of intertemporal substitution in consumption.

Households use their income ($Y_t = r_t K_t + w_t L_t$) to consume (C_t), invest ($\dot{K}_t$), buy government bonds (B_t), with a real expected return $\tilde{R}_t$, and pay taxes ($\tau_t Y_t$, where τ_t is a proportional income tax rate); hence the following budget constraint

$$\dot{K}_t + \dot{B}_t = \tilde{R}_t B_t + (1 - \tau_t)(r_t K_t + w_t L) - C_t + X_t. \tag{3}$$

[9]In previous theoretical literature, the fiscal adjustment is based on a single instrument: the distortionary taxation with a fixed exogenous spending (Schmitt-Grohé and Uribe, 1997; Giannitsarou, 2007), or the public spending with a fixed tax rate (Guo and Harrison, 2004, 2008).

X_t is a transfer from the government, to be defined below.

When studying the dynamics of public debt, it is important to distinguish between the return of capital r_t and the the return of public debt, say R_t. Effectively, history has shown that substantial risk premia on public debt can appear at high public-debt ratios. To introduce this element in our setup without complexify the model with an explicit treatment of financial imperfections, we imagine the following story. We suppose that, at the instant they make portfolio choices (i.e., at the beginning of the period), households expect that a fraction $\chi_t \in (0, 1)$ of public debt may not be repaid. Thus, in the budget constraint (6), the return of public debt must be weighed by the probability of a future "haircut" $(1 - \chi_t)$. If the real return of public debt is R_t, the expected return for households is only $\tilde{R}_t := R_t(1 - \chi_t)$. However, in equilibrium, the government will always honor his commitments, so that the totality of public debt will be repaid. To describes this fact, we consider that households receive, at equilibrium (i.e., at the end of the period), a lump-sum transfer X_t that corresponds to the remaining part of interest payments.[10] Therefore, even if the government does not default in equilibrium, households do not exclude the possibility of default at the time they make their portfolio choice.

Such a framework generates a risk premium on public debt, without considering explicit microfoundations of risk. Effectively, the trade-off between public bonds and private capital provides the following condition: $(1 - \chi_t)R_t = (1 - \tau_t)r_t$, with $1/(1 - \chi_t) \geq 1$ being the risk premium. In order to endogenize this premium, we consider that χ_t is an increasing function of the ratio of aggregate public debt to GDP, namely: $\chi_t = \chi(\bar{B}_t/\bar{Y}_t)$, with $\chi'(\cdot) \geq 0$, where $\bar{B}_t$ and $\bar{Y}_t$ represent global equilibrium variables that the household takes as given in its program (aggregate externality). Then, by defining $\theta(\bar{B}_t/\bar{Y}_t) := 1/(1 - \chi(\bar{B}_t/\bar{Y}_t))$, we have

$$R_t = \theta(\bar{B}_t/\bar{Y}_t)(1 - \tau_t)r_t.$$

The term $\theta(\bar{B}_t/\bar{Y}_t) \geq 1$ represents the risk premium on public debt, which positively depends on the public debt ratio.

The first order condition for the maximization of the household's programme gives rise to the Keynes-Ramsey relationship

$$\frac{\dot{C}_t}{C_t} = S\left((1 - \tau_t)r_t - \rho\right). \tag{4}$$

[10]Hence, at equilibrium, each household receives a lump-sum transfer $X_t = \chi_t R_t \bar{B}_t/N$, where $\bar{B}_t$ represents total public debt issued by the government and N the number of households in the economy (that we have normalized to $N = 1$).

In addition, the optimal path has to verify the set of transversality conditions

$$\lim_{t\to+\infty}\{\exp(-\rho t)\ u'(C_t)\,K_t\}=0 \text{ and } \lim_{t\to+\infty}\{\exp(-\rho t)\ u'(C_t)\,B_t\}=0,$$

ensuring that lifetime utility U is bounded.[11]

2.2. Firms

Output (Y_t) is produced using a constant returns-to-scale technology with a capital externality, namely

$$Y_t = AK_t^{\alpha}(L_t\bar{K}_t)^{1-\alpha},$$

where $A>\rho/\alpha$ is a scale parameter (that ensures positive growth solutions) and $\alpha\in(0,1)$ is the elasticity of output to private capital. K_t stands for private capital and $\bar{K}_t$ is the economy-wide level of capital that generates positive technological spillovers onto firm's productivity (Romer, 1986).

The first order conditions for profit maximization (relative to private factors) are

$$r_t = \alpha\frac{Y_t}{K_t}, \tag{5}$$

$$w_t = (1-\alpha)\frac{Y_t}{L_t}. \tag{6}$$

with, at equilibrium, $L_t = L$. We henceforth normalize $L = 1$.

2.3. The government

The government provides public expenditures G_t, levies taxes T_t, and borrows from households. Fiscal deficit is financed by issuing debt ($\dot{B}_t$); hence, the following budget constraint

$$\dot{B}_t = \tilde{R}_tB_t + G_t - T_t - X_t = R_tB_t + G_t - T_t, \tag{7}$$

Without loss of generality, we define tax and public spending ratios as $\tau_t = T_t/Y_t$, and $g_t = G_t/Y_t$, respectively. At this stage, the government has three instruments: the tax rate (τ_t), the public spending ratio (g_t), and the public debt path ($\dot{B}_t$).

The paper aims to study the implications of the BBR. Without public debt, such a rule corresponds to $G_t = T_t$. However, the economies that adopt a BBR do not necessarily have zero debt at the time of adoption. On the contrary, almost all countries having adopted a BBR were (sometimes highly) indebted countries. For an economy starting with an initial public debt B_0, the BBR (i.e. $\dot{B}_t = 0 \Leftrightarrow B_t = B_0, \forall t$) thus corresponds to

$$R_tB_0 + G_t = T_t \Rightarrow R_tB_0 + g_tY_t = \tau_tY_t. \tag{8}$$

[11] On the BGP associated to constant growth and interest rates (γ^* and r^*, respectively), the transversality conditions correspond to the no-Ponzi game constraint $\gamma^* < r^*$. Such condition ensures that public debt will be repaid in the long run, and does not precludes the possibility that $\gamma > r$ in the short run.

The presence of a positive public debt level (B_0) has crucial implications, since our model can exhibit complex dynamics of the public debt *ratio* (B_0/Y_t), even in the presence of the BBR.

2.4. Equilibrium

At equilibrium, we have $K_t = \bar{K}_t$, which in turn leads to the simple social technology

$$Y_t = AK_t. \tag{9}$$

Thanks to constant-returns at the social level, endogenous growth can emerge, despite decreasing returns of private capital from the individual firm's perspective. Therefore, using (5), the real interest rate becomes, at equilibrium

$$r_t = \alpha A. \tag{10}$$

To obtain long-run stationary ratios, we deflate consumption and public debt by output and we use minuscule letters to depict ratios, namely: $c_t := C_t/Y_t$ and $b_t = B_0/Y_t$. Thus, the return of government bonds: $R_t = \theta(b_t)(1-\tau_t)r_t$.

The path of the capital stock is given by the goods market equilibrium

$$\frac{\dot{K}_t}{K_t} = A(1 - g_t - c_t). \tag{11}$$

From (7), we obtain

$$\frac{\dot{b}_t}{b_t} = \frac{\dot{B}_t}{B_t} - \frac{\dot{K}_t}{K_t} = R_t b_t + g_t y_t - \tau_t y_t - \frac{\dot{K}_t}{K_t},$$

hence, under the BBR (8),

$$\dot{b}_t = -b_t \frac{\dot{K}}{K}. \tag{12}$$

From (4), (10), (11), and (12), the reduced-form of the model is

$$\begin{cases} \dfrac{\dot{c}_t}{c_t} = S[\alpha(1-\tau_t)A - \rho] - A(1 - g_t - c_t) & \text{(a)}, \\ \\ \dot{b}_t = -Ab_t(1 - g_t - c_t) & \text{(b)}. \end{cases} \tag{13}$$

Considering the BBR, the government must select the set of policy instrument $\{\tau_t, g_t\}_{t\geq 0}$ to balance its budget each period. Thus, we must specify an adjustment scheme for government's finance. Deflating (8) by Y_t and using (10), we have

$$(1-\tau_t)x(b_t) + g_t = \tau_t, \tag{14}$$

where $x(b_t) := \alpha A\theta(b_t)b_t$ is the (gross) debt burden, with $x'(b_t) \geq 0$.

Therefore, any increase of the public debt ratio (b_t) requires a lower public spending ratio (g_t) and/or a higher the tax rate (τ_t). Let us introduce a general adjustment scheme, such that the debt burden is shared between the two instruments, namely

$$\tau_t = \tau(b_t), \text{ and } g_t = g(b_t),$$

where $g, \tau : \mathbb{R}^+ \mapsto [0, 1]$ are $\mathcal{C}^2$-functions, with $g(0) = \tau(0) =: \tau_0 \in (0, \bar{\tau})$, where $\bar{\tau} := 1 - \rho/\alpha A \in (0, 1)$, and $-R'(b_t)b_t - R(b_t) \leq g'(b_t) \leq 0$. The latter assumption means that debt-burden-increases are partially covered by cuts in public spending.[12] This ensures that $\tau'(b_t) \geq 0$, namely that the residual part of the debt burden is covered by tax-increases.[13]

2.5. Steady states

We define a BGP as a path on which consumption, capital, and output grow at the same (endogenous) rate, namely (we henceforth omit time indexes)

$$\gamma^* := \dot{C}/C = \dot{K}/K = \dot{Y}/Y.$$

Proposition 1. *There is a non-empty set of parameters, such that*

i. *There are two candidate long-run solutions: a high steady state (H), characterized by positive growth ($\gamma^H > 0$) and zero debt ($b^H = 0$), and a low steady state (L), characterized by zero growth ($\gamma^L = 0$) and positive debt ($b^L > 0$).*
ii. *If $\tau' = 0$, only the high steady-state solution emerges,*
iii. *If $\tau' > 0$, the two solutions are feasible (multiplicity).*

Proof.

(i) By setting $\dot{b} = 0$ in (13b), we have either $b > 0 \Rightarrow \gamma = 0$ – this defines the low BGP (L) –, or $\gamma > 0 \Rightarrow b = 0$ – this defines the high BGP (H).

(ii) If $\tau' = 0$, i.e. $\tau(b_t) = \tau_0$, for any $b_t \geq 0$, the rate of economic growth is: $\gamma^H = S[\alpha A(1 - \tau_0) - \rho]$. As $\tau_0 < \bar{\tau}$, we have $\gamma^H > 0$; hence $b^H = 0$. Therefore, $c^H = 1 - g(0) - \gamma^H/A$. This solution is well defined if $c^H > 0$. As $g(0) = \tau(0) = \tau_0$, this requires that $S < \bar{S}$, with $\bar{S} := (1 - \tau_0)/[(1 - \tau_0) - \rho/A] > 1$; which is true for usual values of the elasticity of intertemporal substitution in consumption ($S \leq 1$).

(iii) If $\tau' > 0$, there are two kinds of solutions. First, according to point (i) we find the same solution as in point (ii), namely: $\gamma^H = S[\alpha A(1-\tau(0))-\rho] = S[\alpha A(1-\tau_0)-\rho]$. Second, we have a zero-growth solution at $\gamma^L = 0 \Leftrightarrow c^L = 1-g(b^L)$, where, by introducing in (13a): $b^L = \tau^{-1}(1 - \rho/\alpha A)$. This solution is well defined if $\rho < \alpha A$, and $g(b^L) < 1$.

[12]Indeed: $\frac{d}{db_t}\{R_t b_t + g_t\} \geq 0 \Leftrightarrow -R'(b_t)b_t - R(b_t) \leq g'(b_t)$.

[13]Effectively, under the BBR, $\tau_t = g_t + R(b_t)b_t$; hence $\tau'(b_t) = R'(b_t)b_t + R(b_t) + g'(b_t) \geq 0$. Noteworthy, since $\theta(0) < +\infty$, the government's budget constraint imposes that $\tau(0) = g(0)$ when $b_t = 0$.

$\square$

The multiplicity comes from the interaction between the government's budget constraint and household's saving behaviour. Especially, if public spending is the only adjustment variable in government's budget constraint ($\tau' = 0$), there is one unique steady-state solution (the high BGP).[14] Indeed, in this case, the long-run rate of economic growth (as defined in the Keynes-Ramsey relationship) is positive and independent of public debt, such that the no-growth solution cannot happen.

With a time-varying tax-rate ($\tau' > 0$), however, the net return of capital depends on public debt in the Keynes-Ramsey rule. In steady state, the BBR is then consistent with two situations. In the first case, public debt is zero in the long-run, which implies a zero debt burden. As a result, the tax-rate is low, thus leading to the high BGP. In the second case, in contrast, expected long-run public debt is high and generates a high risk premium that forces the government to set a high tax-rate. Then, in steady state, the long-run public debt ratio is such that its burden completely stifles economic growth, given the tax-rate that must be imposed. In this case, the economy is trapped into a no-growth BGP.

3. Local Dynamics

By linearization, in the neighborhood of steady-state i, $i \in \mathscr{S} = \{L, H\}$, the system (13) behaves according to $(\dot{c}_t, \dot{b}_t) = \mathbf{J}^i(c_t - c^i, b_t - b^i)$, where $\mathbf{J}^i$ is the Jacobian matrix. The reduced-form includes one jump variable (the consumption ratio c_0) and one predetermined variable (the public-debt ratio b_0, since initial stocks of public debt B_0 and private capital K_0 are predetermined). Thus, for BGP i to be well determined, $\mathbf{J}^i$ must contain two opposite-sign eigenvalues. Using (13), we compute

$$\mathbf{J}^i = \begin{pmatrix} CC^i & CB^i \\ BC^i & BB^i \end{pmatrix},$$

where

$$CC^i = Ac^i, \tag{15}$$

$$CB^i = Ac^i[g'(b^i) - \alpha S\tau'(b^i)], \tag{16}$$

$$BB^i = -\gamma^i + Ab^i g'(b^i), \tag{17}$$

$$BC^i = Ab^i. \tag{18}$$

[14] However, the multiplicity would appear again with productive public spending Menuet et al. (2018a), or with endogenous labor supply (Menuet et al., 2018b).

Hence, the trace and the determinant of the Jacobian matrix are, respectively

$$\mathrm{Tr}(\mathbf{J}^i) = Ac^i - \gamma^i + Ab^i g'(b^i), \tag{19}$$
$$\det(\mathbf{J}^i) = -A\gamma^i c^i + \alpha A^2 S c^i b^i \tau'(b^i). \tag{20}$$

The following theorem establishes the topological behaviour of each steady-state.

Theorem 1. *(Local Stability)*

- *The high BGP is locally determined (saddle-point stable).*
- *The topological behaviour of the low BGP is the following.*
    - *If $c^L > -b^L g'(b^L)$, L is locally over-determined (unstable),*
    - *If $c^L = -b^L g'(b^L)$, a Hopf bifurcation occurs,*
    - *If $c^L < -b^L g'(b^L)$, L is locally under-determined (stable).*

Proof.

(i) $\det(\mathbf{J}^H) = -A\gamma^H c^H < 0$, namely there are two opposite-sign eigenvalues. Consequently, H is saddle-point stable.

(ii) $\det(\mathbf{J}^L) = \alpha A^2 S c^L b^L \tau'(b^L) > 0$, and $\mathrm{Tr}(\mathbf{J}^L) = A(c^L + b^L g'(b^L))$. Therefore, if $c^L < (>) - b^L g'(b^L)$, there are two eigenvalues with negative (positive) real part, so that L is stable (instable). At $c^L = -b^L g'(b^L)$, the Hopf bifurcation arises, and a periodic orbit through a local change in the stability properties of L appears. As $c^L = 1 - g(b^L)$, the Hopf bifurcation occurs at a point b_h^L such that $1 - g(b_h^L) = -b_h^L g'(b_h^L)$. By defining the elasticity $e(b) := -bg'(b)/g(b)$, the Hopf bifurcation arises at $g(b_h^L) = 1/(1 + e(b_h^L))$. This, in turn, defines a critical value for some parameter included in b_h^L (given existence and uniqueness restrictions, see our numerical results in section 4).

□

The different stability properties of the two equilibria comes from the dynamics of the public debt. In the neighborhood of the high steady state, economic growth is high enough to overcome the unstable dynamic of the public debt-to-output ratio. Hence, the topological behaviour of the high BGP does not depend on the adjustment scheme of public finance. Effectively, this steady state is saddle-path stable, independently on the specification of functions $g(\cdot)$ and $\tau(\cdot)$.

In contrast, the local determinacy of the low BGP crucially depends on the form of the adjustment scheme of public finance, through functions $g(\cdot)$ and $\tau(\cdot)$. Especially, a necessary condition for the Hopf bifurcation to occur is that public expenditures (taxes) negatively (positively) react to public debt, as establishes the following lemma.

Lemma 1. *If (i) $\tau' = 0$ or (ii) $g' = 0$, the model is well determined.*

Proof. (i) If $\tau' = 0$ (i.e. $\tau_t =: \tau_0, \forall t$), there is a unique positive BGP, namely $\gamma^H = S[\alpha(1-\tau_0)A - \rho]$. This BGP is locally well determined, because, from (20), $\det(\mathbf{J}) = -A\gamma^H c^H < 0$.

(ii) If $g' = 0$ (and $\tau' > 0$), then $c^L > -b^L g'(b^L) = 0$, the low BGP is unstable and indeterminacy is removed. □

Contrary to the high BGP, around the no-growth solution (if this solution exists, i.e. $\tau' > 0$), the snowball effect of the debt burden cannot be avoided, giving rise to the emergence of a cyclical dynamics. If public spending does not react to the debt burden or lowly reacts, this dynamics is explosive, while in the opposite case, the no-growth solution becomes locally stable and can be reached, but at the price of (possibly large) oscillations during the transition path. This finding stresses the importance of having both an adjustment of public spending and resources in the dynamics of the model.

4. A numerical exploration

To assess the dynamics of the low BGP, it is necessary to characterize the nature of the Hopf bifurcation. Effectively, depending on the value of the first Lyapunov coefficient, the bifurcation can be subcritical or supercritical. To compute this coefficient, we must characterize explicitly the adjustment scheme of government's finance, i.e. define explicit functions $\tau(\cdot)$ and $g(\cdot)$. From Eq.(14), public spending and taxes must adjust to changes in the public debt burden, as we have seen. We consider here that a share $\eta \in (0,1)$ of debt-burden increases are covered by tax-increases, and a share $1-\eta$ by a cuts in public spending, namely

$$\tau_t = \eta R_t b_t + \tau_0, \text{ and } g_t = \tau_0 - (1-\eta) R_t b_t,$$

where $\tau_0 \geq 0$ is a constant (that corresponds to the long-run tax rate in the high BGP) that ensures $g_t \geq 0$. Obviously, if $\eta = 0$ (resp. $\eta = 1$), taxes (resp. public spending) is the only adjustment variable to the debt burden. Then, the equilibrium can be fully characterized by considering a specific function for the (gross) debt burden. In the following, we consider a iso-elastic function: $x(b_t) := \alpha A b_t^{\varepsilon}$, with $\varepsilon > 1$.

In this case, the value of η that gives rise to the Hopf bifurcation is (see Appendix B)

$$\eta_h = \frac{(\alpha_0 - 1)(\varepsilon - \alpha_0)}{\alpha_0 + (\alpha_0 - 1)(\varepsilon - \alpha_0)}, \tag{21}$$

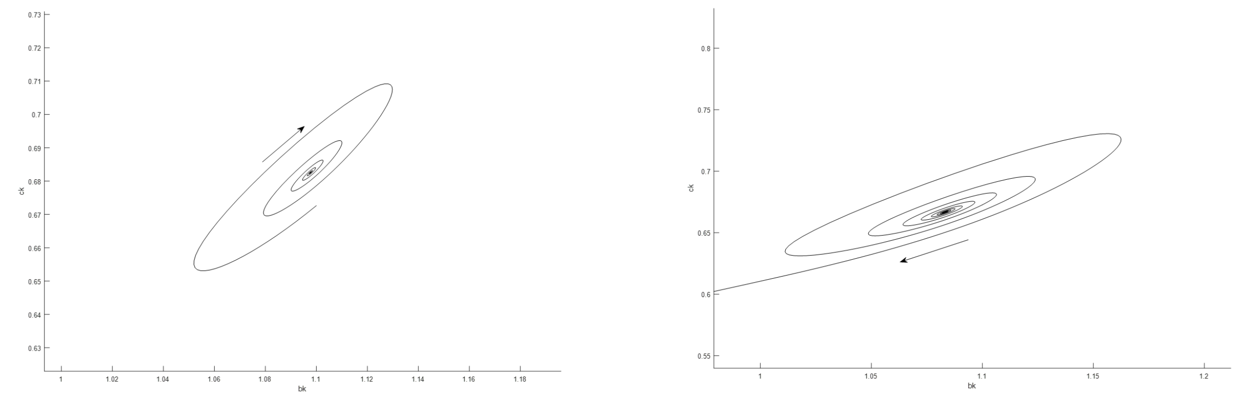
where $\alpha_0 := \alpha A(1-\tau_0)/\rho > 1$.[15] We can remark that $\eta_h \in (0,1)$ under the sufficient condition that $\varepsilon > \alpha_0$.[16]

[15] $\alpha_0 > 1$ is a necessary condition for the public debt to be positive.

[16] η_h is positively related to the elasticity of the debt burden function, with $\eta_h = 0$ if $\varepsilon = \alpha_0$ and $\lim_{\varepsilon \to \infty}(\eta_h) = 1$.

Our numerical results are based on reasonable values of parameters. Regarding household's preferences, we choose $\rho = 0.05$,[17] and a logarithmic utility function ($S = 1$). Regarding the technology, we set $A = 0.3$ to obtain realistic rates of economic growth and real interest rate, and the capital share in the production function is $\alpha = 0.3$. Regarding the government's behavior, the long-run value of the tax-rate is fixed at $\tau_0 = 0.4$ in the high BGP, corresponding to long-run average values in the US or OECD from 1950 to 2015, and the elasticity of the debt burden is chosen to be $\varepsilon = 10$. The benchmark value of η is 0.5, but this parameter will be scanned over a large range of values to verify the presence (or not) of a Hopf bifurcation. For these parameters' values, the corresponding growth rate is $\gamma^H \simeq 0.4\%$ in the high BGP (with $\gamma^L = 0$ in the low BGP), and the associated public debt ratios are $b^H = 0$ and $b^L \simeq 1.06$.

The Hopf bifurcation occurs at $\eta_h \simeq 0.38$ corresponding to a public debt ratio of $b^L \simeq 1.09$. For values of the elasticity less than η_h, the low BGP is stable, as in Figure 1a, while it is unstable for values above η_h (Figure 1b). With $\eta_h = 0.38$, the corresponding value for the risk premium is around 5%, which seems reasonable for high indebted countries.



1a – $\eta = 0.35 < \eta_h$

1b – $\eta = 0.4 > \eta_h$

Figure 1 – The subcritical Hopf bifurcation

Simulations using © matcont show that the first Lyapunov coefficient is positive (approximatively 1.00), thus defining a subcritical Hopf bifurcation. At $\eta = \eta_h$ the low BGP is neither stable or unstable, but for slightly lower values of η, closed orbits arise that enclose the low BGP. These orbits becomes larger the lower the value of η (see Figure 2). However, these orbits are instable and do not define limit-cycles. Hence, inside the closed orbit, all paths converge towards the low BGP. Consequently, the area of stability

[17]Since we ignore depreciation, this term can reflect the sum of the risk free (real) interest rate plus depreciation.

of the low BGP becomes larger as η decreases.[18]

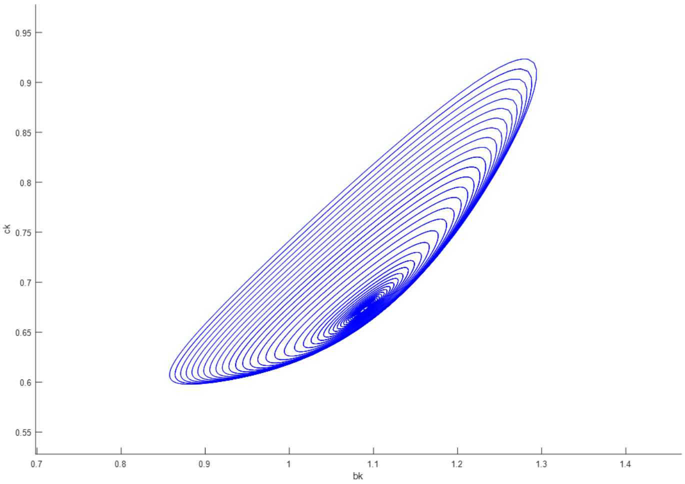


Figure 2: The family of closed orbits as η declines

Thanks to local analysis, we can now turn to global dynamics.

5. Global Dynamics

The simplicity of our two-dimensional model allows fully characterizing the global dynamics of the system. We can distinguish two cases, depending on the topological behaviour of the low BGP.

In the first configuration, that arises if $\eta > \eta_h$, the low BGP is unstable and only the high BGP can be reached. Figure 4a depicts the phase portrait in this case. There is a heteroclinic connexion between the low and the high BGPs, and the system is both locally and globally well-determined (local and global determinacy).

[18]The expansion of the closed orbits is limited by the non-negativity condition on public spending. For some value $\eta = \bar{\eta}$, we have g=0, hence defining the maximum feasible η and the maximal amplitude in the family of closed orbits.

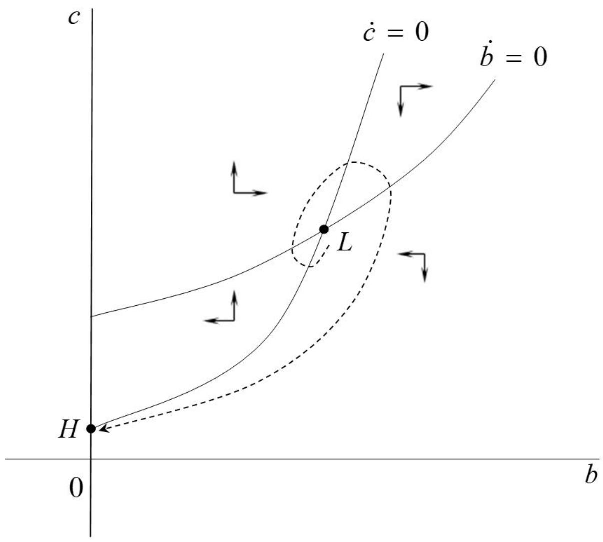


Figure 4a: Global Dynamics ($\eta > \eta_h$)

At $\eta = \eta_h$, the Hopf bifurcation arises, and the low BGP is neither unstable or stable. Beyond the Hopf bifurcation, the low BGP becomes stable and there is a closed orbit that encloses it. This gives rise to the second configuration, as depicted in Figure 4b. In this case, $\eta < \eta_h$, and the high BGP is still staddle-path stable and can therefore be reached by a unique well-determined manifold, but the low BGP now is stable, and thus characterized by local indeterminacy. Effectively, inside the closed orbit, all paths converge towards the low BGP, so that the initial consumption ratio and transitory dynamics are undetermined. In addition, if the initial public debt ratio $b_0 \in (\underline{b}_0, \bar{b}_0)$, where $\underline{b}_0$ and $\bar{b}_0$ are, respectively, the leftmost and the rightmost points of the closed orbit, there is global indeterminacy, because either the low or the high BGPs can be reached in the long-run, following an adequate initial jump of the consumption ratio. In this configuration (local and global indeterminacy), both the transition path and the long run equilibrium are subject to "animal spirits".

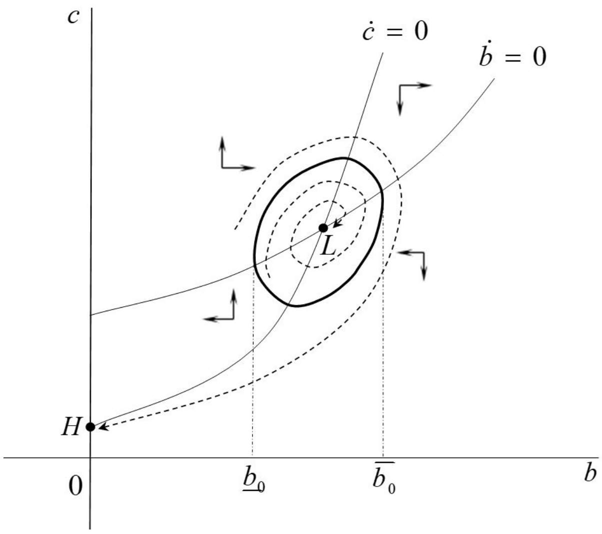


Figure 4b: Global Dynamics ($\eta < \eta_h$)

Indeed, the coexistence of multiple feasible equilibrium paths illustrates the possibility of self-fulfilling prophecies: if households think that the economy will end up on the high BGP, then it will; whereas if the low BGP is expected, then this equilibrium will be attained. In such a case, the transition path and the long-run solution of the model are subject to optimistic or pessimistic views on the future. Do the agents expect strong economic growth, the economy will reach the high BGP; do they anticipate a low-growth trap, the economy will be condemned to the low BGP.

Interestingly, as η decreases, the area of indetermination widens, since the values $\underline{b}_0$ and $\bar{b}_0$ deviate from each other. Indeterminacy thus can arise for realistic values of the public debt ratio, since, as we have shown in the example of Figure 2, for reasonable parameter values, the largest closed orbit (consistent with a positive public spending ratio) in our benchmark calibration is obtained for $\underline{b}_0 \simeq 85\%$ and $\bar{b}_0 \simeq 135\%$.

The intuition of such an indeterminacy is the following. For a given initial public debt ratio (b_0), if zero public debt is expected in the future, the after-tax return of private investment is expected to be high, and, at the initial time, households increase their savings, such that the initial consumption ratio c_0 is low. This means that, in equilibrium, initial private investment and economic growth will be high, and that the debt ratio will effectively decline in the future, generating a (self-fulfilling) high growth path in the steady-state. On the contrary, if households expect a high debt ratio in the future, the risk premium on public debt will be high, so as the tax rate to finance the debt burden, and the after-tax return of capital will be low. Thus, households initially choose a high initial consumption ratio c_0 because the (perfectly expected) return on their savings is expected to be low. In equilibrium, such a consumption ratio crowds out private investment and the initial economic growth is low, which does not allows to reduce the public debt ratio in the long-run, thus confirming household's expectations. Therefore, the economy goes towards the no-growth trap.

As we have shown in Lemma 1, indeterminacy crucially depends on the fact that the tax-rate adjusts to the debt burden. This generates the multiplicity of equilibrium paths, because the long-run achievable rate of economic growth depends on the after-tax return of capital.[19] But the elasticity of public spending to the debt burden is also a crucial feature, because it allows stabilizing the low BGP. Effectively, if public spending was independent of the public debt ratio, the low BGP would be unstable, regardless the value of other parameters, and indeterminacy could be removed.

6. Discussion and concluding remarks

This paper shows that local and global indeterminacy can appear when wasteful pubic spending and taxes adjust jointly the government's budget constraint. Fundamentally, such findings come from the nature of the BBR, which is consistent with non trivial dynamics of the debt ratio when considering endogenous growth. In term of economic policy, our results are mixed.

On the one hand, from the point of view of the high BGP, there is no difference between the two adjustment schemes, because public debt is zero. Furthermore, this BGP is locally well determined, irrespective to the composition of fiscal adjustment. However (partial of full) TB adjustment generates multiplicity of BGPs, with the emergence of a no-growth trap. Such multiplicity can be avoided if public spending fully responds to the debt burden ($\eta = 0$). In this case, the tax rate is constant at τ_0 and the after-tax expected return of capital is constant, thus removing sunspot equilibria: the no-growth solution vanishes, and the unique long-run BGP is such that public debt is zero, as described in Lemma 1.

This situation pleads for the implementation of full EB adjustments, without any move of the tax rate. Nevertheless, fiscal adjustments exclusively based on expenditure can imply very large cuts in the initial public spending ratio ($g_0 = \tau_0 - (1-\tau_0)\alpha A b_0^{\varepsilon}$), especially if the initial public debt is high. Such cuts in public spending could be very costly for households (if, e.g. public expenditures exert a positive externality on households' welfare), or could simply not be feasible, because public spending cannot take negative values. For countries that are initially highly indebted (in our model, if the initial public debt ratio is larger than $\bar{b}_0 := (\alpha A(1-\tau_0)/\tau_0)^{-1/\varepsilon}$), a complete fiscal adjustment with spending-cuts would not be implementable, i.e. an adjustment of the tax ratio to the public debt burden is needed.

On the other hand, if the tax rate partially adjusts to the debt burden ($\eta > 0$), results change dramatically. As we have seen, TB adjustments give birth to a no-growth

[19] In the context of exogenous growth, Schmitt-Grohé and Uribe (1997) and Guo and Harrison (2004) find similarly that the adjustment of the tax-rate is a condition for aggregate instability to emerge.

solution and to multiplicity. From the determinacy perspective, a large adjustement of taxes is then required ($\eta > \eta_h$). Effectively, the lower the share of TB adjustment (η), the larger the area of local stability of the no-growth trap and the more probable the emergence of global indeterminacy. The latter can be removed only if the composition of the adjustment sufficiently relies on taxes (i.e. $\eta > \eta_h$). But strong response of taxes to the debt burden is likely to affect the net return of investment and lower economic growth during the transition path.[20] Aggregate instability can therefore be viewed as the price to be paid for obtaining higher transitional growth.

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[20] Effectively, the rate of consumption growth is, during the transition, $\dot{C}_t/C_t = S[\alpha A(1-\tau_t) - \rho] = S[\alpha A(1-\tau_0)/(1+\eta x(b_t)) - \rho]$. Hence, for any predetermined public-debt ratio: $d(\dot{C}_t/C_t)/d\eta < 0$.

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Appendix A. Construction of the phase portrait

To build the phase portrait, we consider that a share $\eta \in (0,1)$ of debt burden increases are covered by tax-increases, and a share $1-\eta$ by a cuts in public spending, namely

$$\tau = \eta R b + \tau_0, \text{ and } g = \tau_0 - (1-\eta) R b,$$

where $\tau_0 \geq 0$ is a constant that ensures $g \geq 0$. Defining $\alpha A b \theta(b) =: x(b)$, and using (5), this leads to the following functional specifications (that will be considered in the numerical section)

$$\tau(b) := \frac{\eta x(b) + \tau_0}{1 + \eta x(b)}, \tag{A.1}$$

and

$$g(b) := \frac{\tau_0(1 + x(b)) - (1 - \eta)x(b)}{1 + \eta x(b)}. \tag{A.2}$$

From (13.a), we have

$$\dot{c} = 0 \Leftrightarrow c = 1 - g - S[\alpha(1-\tau) - \rho/A], \text{ namely } c = \frac{(1 - \tau_0)[(1 + x(b)) - \alpha S]}{1 + \eta x(b)} + S\rho/A =: \Phi_1(b),$$

and, from (13.b),

$$\dot{b} = 0 \Leftrightarrow b = 0 \text{ or } c = 1 - g, \text{ namely } c = \frac{(1 - \tau_0)(1 + x(b))}{1 + \eta x(b)} =: \Phi_2(b).$$

Clearly, Φ_1 and Φ_2 are monotonic increasing continuous function for $b \geq 0$, with $\Phi_1(0) = 1 - \tau_0 - S[\alpha(1 - \tau_0) - \rho/A]$, and $\Phi_2(0) = 1 - \tau_0$. Besides, $\Phi_1(b) = \Phi_2(b) \Leftrightarrow b = \hat{b} := x^{-1}(\frac{1}{\eta}[\frac{\alpha A(1-\tau_0)}{\rho} - 1])$. Therefore, if $\hat{b} > 0 \Leftrightarrow \Phi_1(0) < \Phi_2(0)$, $b^L = \hat{b}$ is the unique positive steady-state.

Figures ?? are built by the way of system (13).

Computation of the Hopf-bifurcation coefficient

From Theorem 1, the Hopf bifurcation is obtained for

$$1 - g(b^L) = -b^L g'(b^L).$$

With $g(b)$ defined in Eq(.2), and using a iso-elastic debt burden function $x(\cdot)$, this condition amounts to

$$(1 + x(b^L))(1 + \eta x(b^L)) = \varepsilon(1 - \eta)x(b^L), \tag{A.3}$$

where b^L is such that $\gamma^L = S[\alpha A(1 - \tau(b^L) - \rho] = 0$; hence $1 - \tau(b^L) = \rho/\alpha A$.

Thus, we can compute, from (.1), $1 - \tau(b^L) := (1 - \tau_0)/(1 + \eta x(b^L)) = \rho/\alpha A$; hence: $1 + \eta x(b^L) = 1/\alpha_0$, and $(1 + x(b^L))/x(b^L) = (\eta + \alpha_0 - 1)/(\alpha_0 - 1)$, where $\alpha_0 := \alpha A(1 - \tau_0)/\rho > 1$. By reintroducing these value in (.3) we obtain the value η_h that gives rise the the Hopf bifurcation, namely

$$\eta_h = \frac{(\alpha_0 - 1)(\varepsilon - \alpha_0)}{\alpha_0 + (\alpha_0 - 1)(\varepsilon - \alpha_0)}. \tag{A.4}$$