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One point functions of fermionic operators in the Super Sine Gordon model

C. Babenko^a, F. Smirnov^{a,1}

^a*Sorbonne Universite, UPMC Univ Paris 06
CNRS, UMR 7589, LPTHE
F-75005, Paris, France*

Abstract

We describe the integrable structure of the space of local operators for the supersymmetric sine-Gordon model. Namely, we conjecture that this space is created by acting on the primary fields by fermions and a Kac-Moody current. We proceed with the computation of the one-point functions. In the UV limit they are shown to agree with the alternative results obtained by solving the reflection relations.

Keywords: SUSY CFT, super sine-Gordon model, one-point functions, reflection relations

1. Introduction

The importance of the one-point functions for the computation of correlation functions in the framework of the Perturbed Conformal Field Theory (PCFT) is well-known [1]. For the sine-Gordon model at finite temperature the one-point functions were computed in [2] using the fermionic basis of the space of local operators. This basis was found first on the lattice for the (inhomogeneous) six-vertex model [3]. Since the expectation values in the fermionic basis are rather simple the scaling limit is not very difficult to consider. One of the main achievements is the exact relation between the local operators in the fermionic basis and their counterparts in the UV Conformal Field Theory (CFT). We would like also to cite the recent paper [4] where more general matrix elements are considered by similar methods.

An alternative approach to the one-point functions uses the reflection relations [5, 6, 7] which are based on two reflections (Heisenberg and Virasoro). We do not go into the details which, in addition to the original papers mentioned above, are discussed in [8]. This way of doing includes certain subtleties with the analytical continuation with respect to the coupling constant. However, if the final goal is restricted to finding a basis in the CFT, invariant under the two reflections, one should not worry because the problem can be considered as a purely algebraic one. The reflection relations are equivalent to a certain Riemann-Hilbert problem, and for a long time it was unclear how to solve it. The synthesis of the two methods, the fermionic basis on the one hand and the reflection relations on the other, was made in [8]. In this paper it was shown that the known examples of the fermionic basis (up to level 8) solve the reflection relations. Moreover, making a qualitative

¹Membre du CNRS

assumption of existence of the fermionic basis one can use the reflection relations in order to compute the fermionic basis quantitatively.

It is interesting to apply a similar procedure to other integrable models. For the models related to higher ranks the problem does not look very realistic for the moment. However, the $\widehat{\mathfrak{sl}}_2$ (or rather $U_q(\widehat{\mathfrak{sl}}_2)$) symmetric case allows a highly non trivial extension to the Fateev model, symmetric under the exceptional algebra $U_q(\widehat{D}(2|1; \alpha))$ [9]. This model deserves the most profound study. It allows numerous particular cases and restrictions. The simplest of them is the sine-Gordon model (sG) and the next in complexity is the supersymmetric sine-Gordon model (ssG). The latter is the subject of the present paper.

Similarly to the sG case we begin the study of the ssG model by considering its lattice regularization which is the inhomogeneous 19-vertex model introduced by Fateev and Zamolodchikov [10] in other words the model based on the spin-1 evaluation representations of $U_q(\widehat{\mathfrak{sl}}_2)$. By the method close to that of the fermionic basis this model was considered in [11] (this paper relies on the previous research [12]). Namely, it was shown that the space of (quasi)-local operators allows a basis created by fermions and a Kac-Moody (KM) current on level one. It is easy to guess that for the integrable lattice models related to higher representations of spin s of $U_q(\widehat{\mathfrak{sl}}_2)$ the space of (quasi)-local operators is generated by currents with all half-integer spins up to s . In the scaling limit these models produce the parafermionic sine-Gordon models. If we learn how to treat them in their totality it will bring us very close to the general case of the Fateev model.

As has been said, in the present paper we consider the ssG model starting with the 19 vertex lattice model. We explain the basis of (quasi)-local operators created by fermions and a KM current at the lattice level. Then we proceed to the scaling limit. This provides the basis of local operators for the ssG model created by fermions and a KM current. Our consideration relies heavily on the numerical study of scaling equations for the function $\Omega(\theta, \theta')$. The equation for this function is not rigorously derived, so, it is considered as a conjecture and should be checked against alternative data. Using the function $\Omega(\theta, \theta')$ it is straightforward to compute the one-point functions on the cylinder of radius R (at finite temperature) for the purely fermionic part of the basis. We restrict our attention to these operators leaving the KM contributions for future study. We consider the UV limit $R \rightarrow 0$ in order to find agreement with the corresponding CFT.

The UV limit is studied using the numerical data and interpolating with respect to the coupling constant, the quasi-momentum and the parameter of the primary field. There is a difference with the sG case for which this kind of data allowed to obtain exact relation to the Virasoro descendants up to the level 6. Then, an important check of the entire procedure consisted in verifying that the results satisfy the reflection relations. In the ssG case only level 2 is available by these means. This case agrees with the reflection relations but we would like to proceed at least a little further. We reverse the procedure following [8], namely, assuming that there are local operators created by fermions which transform simply under the reflection, and compute the elements of the fermionic basis up to the level 6. Needless to say that the reflection relations are considered as a conjecture which is hard to justify rigorously.

Finally, it is possible to compare with the results obtained by the interpolation of numerical data finding a perfect agreement. This is the main result of the present paper: two kind of formulae whose derivations are based on completely different conjectures agree.

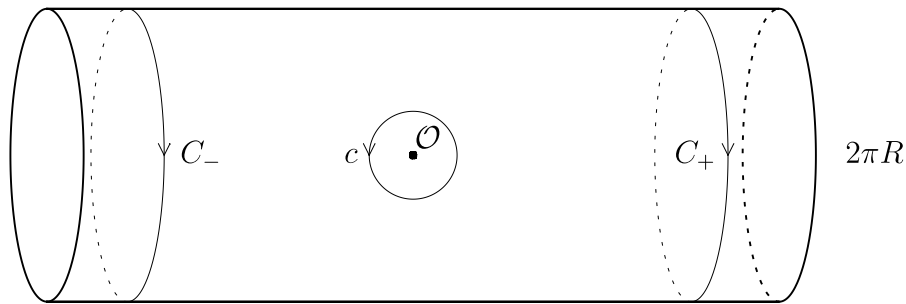


Figure 1: Cylinder with insertion.

The paper is organized as follows. In Section 2 we introduce the ssG model and recall some previous results, in Section 3 we describe the 19-vertex model, and introduce the spin 1 fermions and the KM currents. The function Ω is defined on the lattice and its normalisation is checked. In Section 4 the scaling equation for Ω is given, and the first one point functions of fermionic operators obtained by numerical interpolation are presented. Finally, in Section 5 we describe the alternative approach to the one point functions that uses the reflection relations, and verify the results obtained in Section 4.

2. Supersymmetric sine-Gordon model

The $N = 1$ supersymmetric sine-Gordon model (ssG) is described by the (Euclidean) action

$$\mathcal{A} = \int \left[\left(\frac{1}{4\pi} \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{1}{2\pi} (\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}) - 2\mu \bar{\psi} \psi \cos\left(\frac{\beta}{\sqrt{2}} \varphi\right) \right) \right] d^2 z. \quad (2.1)$$

Here we take the CFT which include massless free boson and Majorana fermion and perturb it by the operator of dimension $\Delta = \frac{1}{2}(1 + \beta^2)$. We consider the domain $0 < \beta < 1$ where the perturbation is relevant. An additional term needed for the supersymmetry in the classical case is omitted for known reasons [13, 14].

The subject of the present paper are the one-point functions, this corresponds to the geometry of the cylinder (that we take of radius R) with a local insertion. Correspondingly we consider three types of contours: the contour c encircling the local insertion and two contours C_{\mp} which go around the cylinder to the right and to the left of the insertion. We shall use the notation C talking about any of C_{\mp} .

The cylinder is infinite, its generatrix is called Space direction, its directrix is called Matsubara direction. In the present context by the Matsubara transfer-matrix we understand an operator acting from the Matsubara Hilbert space to itself which is graphically represented as a slice of our cylinder of small Space length ϵ . Since the cylinder is infinite, both transfer-matrices to the left and to the right of the insertion are replaced by the one-dimensional projectors on the same eigenvector with maximal eigenvalue. Since the potential is invariant under $\varphi \rightarrow \varphi + 2\sqrt{2}/\beta$ we can introduce additional parameter P which is the Floquet index of the Matsubara wave-function. The one-point function (partition function with insertion) is denoted by

$$\langle \mathcal{O}(0) \rangle_{P,R}. \quad (2.2)$$

The ssG model can be formally considered as the perturbation of the conformal complex super-Liouville model

$$\mathcal{A}_L = \int \left[\left(\frac{1}{4\pi} \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{1}{2\pi} (\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}) - \mu \bar{\psi} \psi e^{-i \frac{\beta}{\sqrt{2}} \varphi} \right) \right] d^2 z, \quad (2.3)$$

by the relevant operator

$$W = \mu \bar{\psi} \psi e^{i \frac{\beta}{\sqrt{2}} \varphi},$$

whose scaling dimension is $\Delta = \beta^2$.

Let us concentrate on the conformal model. The central charge of the complex super-Liouville model is

$$c = \frac{3}{2} \hat{c}, \quad \hat{c} = 1 - 2(\beta^{-1} - \beta)^2.$$

In this paper we consider only the NS sector. In the conformal case we can easily change the scale to have $R = 1$. According to the usual argument the operator \mathcal{O} with scaling dimensions $(\Delta_{\mathcal{O}}, \bar{\Delta}_{\mathcal{O}})$ in generic position has an uniquely defined counterpart in perturbed theory. We do not distinguish the two notationally, the UV limit is

$$\lim_{r \rightarrow 0} r^{\Delta_{\mathcal{O}} + \bar{\Delta}_{\mathcal{O}}} \langle \mathcal{O}(0) \rangle_{P,R} = \langle \mathcal{O}(0) \rangle_P, \quad (2.4)$$

where $r \sim R$ is a dimensionless quantity proportional to R , see (2.15) for details. By a change of variables from the cylinder to the sphere the CFT one-point function $\langle \mathcal{O}(0) \rangle_P$ is mapped to the three point function for the image of the operator \mathcal{O} (for descendants this image can have rather complicated expression in terms of \mathcal{O}) and two primary fields with dimensions

$$\delta_P = P^2 + \frac{\hat{c} - 1}{16}.$$

The superconformal algebra is generated by the operators $T(z)$, $S(z)$ with the OPE's

$$T(z)T(w) = -\frac{c}{12} \chi'''(z-w) - 2\chi'(z-w)T(w) + \chi(z-w)T'(w) + O(1), \quad (2.5)$$

$$T(z)S(w) = -\frac{3}{2} \chi'(z-w)S(w) + \chi(z-w)S'(w) + O(1),$$

$$S(z)S(w) = -2\xi(z-w)T(w) - \frac{c}{3} \xi''(z-w) + O(1),$$

where the functions $\chi(z)$, $\xi(z)$ are chosen to be compatible with our geometry and with the NS (anti)-periodicity conditions:

$$\chi(z) = \frac{1}{2} \coth\left(\frac{z}{2}\right), \quad \xi(z) = \frac{1}{2 \sinh\left(\frac{z}{2}\right)}.$$

There are two kinds of the primary fields parametrised by $\alpha \in \mathbb{C}$

$$V_{\alpha} = e^{i\alpha(\beta^{-1}-\beta)\frac{1}{2\sqrt{2}}\varphi}, \quad W_{\alpha} = \bar{\psi}\psi e^{i\alpha(\beta^{-1}-\beta)\frac{1}{2\sqrt{2}}\varphi}. \quad (2.6)$$

The scaling dimension of V_{α} is

$$\Delta_{\alpha} = \frac{1}{8}(\beta^{-1} - \beta)^2 \alpha(\alpha - 2).$$

The scaling dimension of W_α equals $\Delta_\alpha + 1/2$. Later we shall use the OPE's:

$$\begin{aligned} T(z)V_\alpha(w) &= -\Delta_\alpha\chi'(z-w)V_\alpha(w) + \chi(z-w)V'_\alpha(w) + O(1), \\ T(z)W_\alpha(w) &= -(\Delta_\alpha + \frac{1}{2})\chi'(z-w)V_\alpha(w) + \chi(z-w)V'_\alpha(w) + O(1), \\ S(z)V_\alpha(w) &= -\xi(z-w)W_\alpha(w) + O(1), \\ S(z)W_\alpha(w) &= 2\Delta_\alpha\xi'(z-w)V_\alpha(w) - \xi(z-w)V'_\alpha(w) + O(1), \end{aligned} \quad (2.7)$$

Consider the chiral component of the energy momentum tensor $T(z)$. The geometry of the problem makes it natural to consider two superconformal algebras which we call local and global

$$\mathbf{l}_n\mathcal{O}(y) = \frac{1}{2\pi i} \oint_c (z-y)^{n+1} T(z)\mathcal{O}(y) \frac{dz}{2\pi i}, \quad \mathbf{s}_k\mathcal{O}(y) = \frac{1}{2\pi i} \oint_c (w-y)^{k+\frac{1}{2}} S(w)\mathcal{O}(y) \frac{dw}{2\pi i}, \quad (2.8)$$

$$L_n = \frac{1}{2\pi} \oint_C e^{-nz} \left(T(z) + \frac{c}{24} \right) \frac{dz}{2\pi i}, \quad S_k = \frac{1}{2\pi} \oint_C e^{-(k+\frac{1}{2})z} S(z) \frac{dz}{2\pi i},$$

where $n \in \mathbb{Z}$, $k \in \mathbb{Z}/2$. The operators \mathbf{l}_n , \mathbf{s}_k act on the local operators inserted at $z = 0$, the operators L_n , S_k act on the Matsubara Hilbert space.

The convenient way of finding the CFT one-point functions consists in using the OPE and the asymptotical conditions

$$\lim_{\text{Re}(z) \rightarrow \pm\infty} T(z) = \delta_P - \frac{\hat{c}}{16}, \quad \lim_{\text{Re}(z) \rightarrow \pm\infty} S(z) = 0. \quad (2.9)$$

In Section 5.2 we shall give some examples of computations with these formulae.

The super-Liouville model, in addition to the super-conformal symmetry, possesses the structure of an integrable model, namely, it allows an infinite number of local integrals of motion with chiral local densities $T_{2j}(z), \bar{T}_{2j}(\bar{z})$. In our geometry there are two facets of the local integrals of motion: they act either on the Matsubara Hilbert space or on the local operators inserted at the point $z = 0$ being respectively

$$I_{2j-1} = \oint_C T_{2j}(z) \frac{dz}{2\pi i}, \quad (\mathbf{i}_{2j-1}\mathcal{O})(0) = \oint_c T_{2j}(z)\mathcal{O}(0) \frac{dz}{2\pi i}, \quad (2.10)$$

and similarly for the other chirality.

Let us write explicitly the first two densities:

$$T_2(z) = T(z), \quad T_4(z) = T(z)^2 - \frac{1}{4}S(z)S'(z). \quad (2.11)$$

The formula for $T_2(z)$ means simply that the light cone component of the energy-momentum tensor is the first integral, the formula for $T_4(z)$ is the most important: it is well-known that higher local integrals of motion are completely defined by requirement of commutativity with the density $T_4(z)$.

Let us return to the perturbed model. It has been said that at least for irrational α the local operator V_α and its super-Virasoro descendants (W_α in particular) have uniquely defined counterparts in the perturbed theory, which we do not distinguish notationally. The local integrals of motion survive the perturbation, and get rise to an infinite series of

pairs of operators $(T_{2j}(z, \bar{z}), \Theta_{2j-1}(z, \bar{z}))$, $(\bar{T}_{2j}(z, \bar{z}), \bar{\Theta}_{2j-1}(z, \bar{z}))$ satisfying the continuity equations

$$\partial_{\bar{z}} T_{2j}(z, \bar{z}) = \partial_z \Theta_{2j-1}(z, \bar{z}), \quad \partial_z \bar{T}_{2j}(z, \bar{z}) = \partial_{\bar{z}} \bar{\Theta}_{2j-1}(z, \bar{z}).$$

Consider $(T_{2j}(z, \bar{z}), \Theta_{2j-1}(z, \bar{z}))$, the other pair being treated quite similarly. The action on the local operators is

$$I_{2j-1} = \frac{1}{2\pi i} \oint_C T_{2j}(z) dz + \Theta_{2j-1}(z) d\bar{z},$$

$$(\mathbf{i}_{2j-1} \mathcal{O})(0) = \frac{1}{2\pi i} \oint_c T_{2j}(z) \mathcal{O}(0) dz + \Theta_{2j-1}(z) \mathcal{O}(0) d\bar{z},$$

The operators I_1, \bar{I}_1 are the light-cone components of the energy-momentum tensor.

The equivalence $c = C_- - C_+$ implies that the one-point function of a local operators obtained by the action of $\mathbf{i}_{2j-1}, \bar{\mathbf{i}}_{2j-1}$ vanishes. So, like in [15] we work with the quotient space $\mathcal{V}_\alpha^{\text{quo}} \otimes \bar{\mathcal{V}}_\alpha^{\text{quo}}$ obtained from the tensor product of two super conformal Verma modules by factoring out the descendants of the integrals of motion. The quotient space $\mathcal{V}_\alpha^{\text{quo}}$ will be realised as the one obtained by the action on V_α of all \mathbf{s}_{-k} and \mathbf{l}_{-m} with even m only.

The particle content of the ssG model consists of solitons and, for $\beta^2 < 1/2$, their bound states. There is an exact formula relating the mass of soliton M to the dimensional coupling constant μ :

$$M = \frac{4(1 - \beta^2)}{\pi\beta^2} \left(\frac{\pi}{2} \mu \gamma \left(\frac{1 - \beta^2}{2} \right) \right)^{\frac{1}{1 - \beta^2}}, \quad (2.12)$$

here and later we use the traditional notation

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1 - x)}.$$

The free energy of the model is defined by the maximal eigenvalue of the Matsubara transfer-matrix. This eigenvalue is found via the Suzuki equations [16, 17, 18]

$$\log y(\theta) = \int_{-\infty}^{\infty} 2\text{Re} \left[L(\theta - \theta' + \pi i \gamma) \log B(\theta' - \pi i \gamma) \right] d\theta', \quad (2.13)$$

$$\log b(\theta - \pi i \gamma) = -2\pi M R \cosh(\theta - \pi i \gamma) - \frac{\pi i \sqrt{2}}{\beta} P + \frac{1}{2} \log 2 \quad (2.14)$$

$$+ \int_{-\infty}^{\infty} L(\theta - \theta' + \pi i \gamma) \log \left(\frac{1}{2} Y(\theta') \right) d\theta'$$

$$+ \int_{-\infty}^{\infty} \left[G(\theta - \theta') \log B(\theta' - \pi i \gamma) - G(\theta - \theta' + \pi i(1 - 2\gamma)) \log \overline{B(\theta' - \pi i \gamma)} \right] d\theta',$$

where $B(\theta) = 1 + b(\theta)$, $Y(\theta) = 1 + y(\theta)$. The kernels are

$$L(\theta) = \frac{1}{2\pi \cosh \theta},$$

$$G(\theta) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sinh \left(\frac{3\beta^2 - 1}{2(1 - \beta^2)} \pi k \right)}{\sinh \left(\frac{\beta^2}{1 - \beta^2} \pi k \right) \cosh \left(\frac{1}{2} \pi k \right)} e^{ik\theta} dk.$$

It is convenient to parametrise R by the dimensionless θ_0 :

$$R = \frac{\beta}{\sqrt{2}} \left(\frac{\pi}{2} \mu \gamma \left(\frac{1 - \beta^2}{2} \right) \right)^{-\frac{1}{1-\beta^2}} e^{-\theta_0}, \quad (2.15)$$

The eigenvalues of the local integrals of motion I_{2k-1} are denoted by i_{2k-1} , they are found from the asymptotics at $\theta \rightarrow \infty$

$$\log y(\theta) \simeq \sum_{k=1}^{\infty} C_{2k-1} i_{2k-1}(\theta_0) e^{-(2k-1)(\theta-\theta_0)}, \quad (2.16)$$

and similarly for the asymptotics $\theta \rightarrow -\infty$ related to another set of integrals \bar{I}_{2k-1} . In the limit $\theta_0 \rightarrow \infty$ the eigenvalue $i_{2k-1}(\theta_0)$ goes to its CFT value i_{2k-1} , the constants are chosen as

$$C_m = -\sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{1}{1-\beta^2}m\right)}{m! \left(\frac{m+1}{2}\right)! \beta \Gamma\left(\frac{\beta^2}{1-\beta^2}m\right)}. \quad (2.17)$$

in order to allow the conventional normalisation of the integrals of motion

$$i_{2k-1} = P^{2k} + \dots$$

Exact formulae for i_1, i_3, i_5 can be found in [18]. Notice that contrary to the sG case all the Γ -functions collapse in the formula (2.17) and many similar formulae which we shall have later.

3. Expectation values in lattice model

3.1. General structure

In the lattice case we historically use for the coupling constant

$$\nu = \frac{1 - \beta^2}{2}.$$

The paper [11] considers an (inhomogeneous) 19 vertex Fateev-Zamolodchikov model on a cylinder or equivalently arbitrary generalised Gibbs ensemble for the (inhomogeneous) spin-1 integrable spin chain. In what follows we closely follow the notations of [19, 11] with one exception: we switch from the multiplicative spectral parameter to the additive one. In this respect we are close to the notations of the paper [20] which was at the origin of presenting the correlation functions with temperature and magnetic field in factorised form. Let us present some basic formulae. As usual we combine the 19 vertices of the model into the L -operator

$$\mathcal{L}(\theta) = \begin{pmatrix} a(\theta) & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & 0 \\ 0 & b(\theta) & 0 & | & c(\theta) & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & f(\theta) & | & 0 & d(\theta) & 0 & | & h(\theta) & 0 & 0 \\ \hline 0 & c(\theta) & 0 & | & b(\theta) & 0 & 0 & | & 0 & 0 & 0 \\ 0 & 0 & d(\theta) & | & 0 & e(\theta) & 0 & | & d(\theta) & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & b(\theta) & | & 0 & c(\theta) & 0 \\ \hline 0 & 0 & h(\theta) & | & 0 & d(\theta) & 0 & | & f(\theta) & 0 & 0 \\ 0 & 0 & 0 & | & 0 & 0 & c(\theta) & | & 0 & b(\theta) & 0 \\ 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & a(\theta) \end{pmatrix},$$

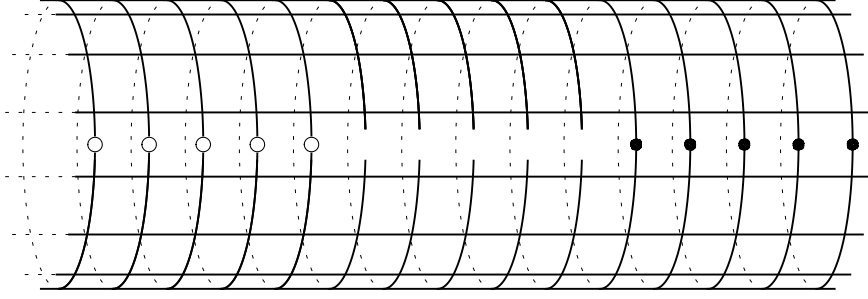


Figure 2: 19 vertex model on a cylinder with quasi-local insertion.

where

$$\begin{aligned}
 a(\theta) &= \sinh \nu(\theta + \frac{\pi i}{2}) \sinh \nu\theta, & b(\theta) &= \sinh \nu\pi(\theta - \frac{\pi i}{2}) \sinh \nu\theta, & c(\theta) &= \sinh \nu\pi i \sinh \nu\theta, \\
 d(\theta) &= \sinh \nu(\theta - \frac{\pi i}{2}) \sinh \nu\pi i, & f(\theta) &= \sinh \nu(\theta - \frac{\pi i}{2}) \sinh \nu(\theta - \pi i), \\
 e(\theta) &= \cosh \nu(\theta + \frac{\pi i}{2}) \cosh \nu(\theta - \pi i) - \cosh \frac{\nu\pi i}{2}, & h(\theta) &= \sinh \frac{\nu\pi i}{2} \sinh \nu\pi i.
 \end{aligned}$$

We introduce an infinite Space chain of length N and the Matsubara chain of length L . Introduce the rectangular monodromy matrix

$$T_{\mathbf{S},\mathbf{M}} = \prod_{j=-N/2+1}^{\widehat{N/2}} T_{j,\mathbf{M}}, \quad T_{j,\mathbf{M}} = \prod_{m=1}^{\widehat{L}} \mathcal{L}_{j,m},$$

where both Space and Matsubara chains can be inhomogeneous,

$$\mathcal{L}_{j,m} = \mathcal{L}_{j,m}(\xi_j - \tau_m), \tag{3.1}$$

$\xi_j, (\tau_m)$ are Space (Matsubara) inhomogenities. The indices j, m in the right hand side have double meaning: they count inhomogenities and the copies in the tensor product. These notations are standard. Eventually we take the limit $N \rightarrow \infty$, the Space inhomogenities are suppose to follow some regular pattern in the limit.

Introduce the operators

$$H(j) = \sum_{k=-N/2+1}^j H_k, \quad H = H(N/2),$$

with H_j being the Cartan generator acting on j -th Space site. Consider the ‘‘primary field’’ $q^{\alpha H(0)}$, and an operator \mathcal{O} acting nontrivially on a finite number of Space sites. The operators $q^{\alpha H(0)}\mathcal{O}$ are called quasi-local. The main object of our study is

$$z_L^\kappa \{q^{\alpha H(0)}\mathcal{O}\} = \lim_{N \rightarrow \infty} \frac{\text{Tr}_{\mathbf{S}} \text{Tr}_{\mathbf{M}} (T_{\mathbf{S},\mathbf{M}} q^{\kappa H + \alpha H(0)} \mathcal{O})}{\text{Tr}_{\mathbf{S}} \text{Tr}_{\mathbf{M}} (T_{\mathbf{S},\mathbf{M}} q^{\kappa H + \alpha H(0)})}, \tag{3.2}$$

with κ being a parameter. Graphically this is represented on *Fig. 2*.

The main result of [11] is that an effective way of computation goes through the introduction of eight families of creation operators acting on the space of quasi-local operators.

These families are fermions $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$, $\tilde{\mathbf{b}}^*(\theta)$, $\tilde{\mathbf{c}}^*(\theta)^2$, level 1 Kac-Moody currents $\mathbf{j}^+(\theta)$, $\mathbf{j}^-(\theta)$, $\mathbf{j}^0(\theta)$, and an operator lying in the centre of the entire algebra $\mathbf{t}^*(\theta)$. To be more precise the generating functions of the quasi-local operators are produced by normally ordered products of fermions and Kac-Moody currents (the central operator $\mathbf{t}^*(\theta)$ does not need normal ordering). This is explained in [11, 21]. Since the most significant results of the present paper concern the quasi-local operators created by fermions only, in which case the normal ordering is not needed, we shall not go into details of the normal ordering which is standard anyway.

In the case of homogeneous Space ($\xi_j = 0, \forall j$) the creation operators are understood as power series in θ . We shall be interested in the case when the Space inhomogenieties are staggering: ξ at even sites and $-\xi$ at odd one. In that case every of above operators give rise to two "chiral" families defined as power series in $\theta - \xi$, $\theta + \xi$. All that is absolutely parallel to [2] so we do not go into much details.

The main advantage of our creation operators is that on the descendants which they create acting on the "primary field", the functional \mathcal{Z}_L^κ takes simple form. We shall describe a formal prescription for the computation, detailed explanations being given in [11]. Introduce the creation operators $b^*(\theta)$, $c^*(\theta)$, $t^*(\theta)$, $n(\theta)$ (the first two are fermions, the last two are bosons) which (anti)-commute among themselves. Prescribe the following values of the functional \mathcal{Z}_L^κ :

$$\begin{aligned} & \mathcal{Z}_L^\kappa \{ b^*(\theta_1^+) \cdots b^*(\theta_k^+) c^*(\theta_k^-) \cdots b^*(\theta_1^-) t^*(\theta_1^0) \cdots t^*(\theta_m^0) n(\sigma_1) \cdots n(\sigma_n) q^{\alpha H(0)} \} \\ &= \prod_{j=1}^n \frac{1}{\mathcal{N}(\sigma_j)} \prod_{j=1}^m \rho(\theta_j^0) \det (\omega(\theta_i^+, \theta_j^-))_{i,j=1,\dots,k} , \end{aligned}$$

where the functions $\mathcal{N}(\theta)$, $\rho(\theta)$, $\omega(\theta, \theta')$ depending on the Matsubara data will be defined soon. The expectation values of the operators created by \mathbf{j}^+ , \mathbf{j}^0 , \mathbf{j}^- , \mathbf{b}^* , \mathbf{c}^* , $\tilde{\mathbf{b}}^*$, $\tilde{\mathbf{c}}^*$ are computed using the identification

$$\begin{aligned} \mathbf{j}^+(\theta) &= n(\theta) b^*(\theta + \frac{\pi i}{2}) b^*(\theta - \frac{\pi i}{2}) , \\ \mathbf{j}^-(\theta) &= n(\theta) c^*(\theta - \frac{\pi i}{2}) c^*(\theta + \frac{\pi i}{2}) , \\ \mathbf{j}^0(\theta) &= n(\theta) (b^*(\theta + \frac{\pi i}{2}) c^*(\theta - \frac{\pi i}{2}) + c^*(\theta + \frac{\pi i}{2}) b^*(\theta - \frac{\pi i}{2})) , \\ \mathbf{b}^*(\theta) &= n(\theta) (b^*(\theta + \frac{\pi i}{2}) t^*(\theta - \frac{\pi i}{2}) + b^*(\theta - \frac{\pi i}{2})) , \\ \mathbf{c}^*(\theta) &= n(\theta) (c^*(\theta + \frac{\pi i}{2}) t^*(\theta - \frac{\pi i}{2}) + c^*(\theta - \frac{\pi i}{2})) , \\ \tilde{\mathbf{b}}^*(\theta) &= n(\theta) (b^*(\theta + \frac{\pi i}{2}) + t^*(\theta + \frac{\pi i}{2}) b^*(\theta - \frac{\pi i}{2})) , \\ \tilde{\mathbf{c}}^*(\theta) &= n(\theta) (c^*(\theta + \frac{\pi i}{2}) + t^*(\theta + \frac{\pi i}{2}) c^*(\theta - \frac{\pi i}{2})) . \end{aligned} \tag{3.3}$$

We had one more operator: $\mathbf{t}^*(\theta)$, it is similar to $t^*(\theta)$ with $\rho(\theta)$ being replaced by $P(\theta)$, this function will be given soon. The operator $\mathbf{t}^*(\theta)$ is in the center, so, we manipulate it as a \mathbb{C} -number.

3.2. Basic functions

The functions $\omega(\theta, \theta')$, $\rho(\theta)$, $P(\theta)$ are defined by the Matsubara data. The latter consists of the length L chain, with inhomogenieties τ_j , right and left twists κ , $\kappa + \alpha$, and

²These operators were denoted by $\bar{\mathbf{b}}^*(\theta)$, $\bar{\mathbf{c}}^*(\theta)$ in [11], but we prefer to keep the "bars" for other, more important, use.

the eigenvectors with maximal eigenvalues of the right and left transfer-matrices:

$$T_{\mathbf{M}}(\theta|\kappa) = \text{Tr}_j (T_{j,\mathbf{M}}(\theta)q^{\kappa H_j}) , \quad T_{\mathbf{M}}(\theta|\kappa + \alpha) = \text{Tr}_j (T_{j,\mathbf{M}}(\theta)q^{(\kappa+\alpha)H_j}) .$$

Denote the maximal eigenvalues by $T(\theta|\kappa)$, $T(\theta|\kappa + \alpha)$. Then we are ready to define the first of our functions:

$$P(\theta) = \frac{T(\theta|\kappa + \alpha)}{T(\theta|\kappa)} . \quad (3.4)$$

We shall need the eigenvalues of the two Baxter Q -operators [22]

$$Q^\pm(\theta, \kappa) = e^{\pm\nu\kappa\theta} \prod_{j=1}^m \sinh \nu(\theta - \sigma_j(\kappa)) , \quad (3.5)$$

and similarly for $\kappa + \alpha$. The Bethe roots are denoted by $\sigma_j(\kappa)$. If κ is not too large the maximal eigenvalue corresponds to $m = L/2$. We have the relation between T and Q^\pm :

$$\begin{aligned} T(\theta, \kappa) &= a(\theta + \pi i/2)a(\theta - \pi i/2) \frac{Q^\pm(\theta + 3\pi i/2, \kappa)}{Q^\pm(\theta - \pi i/2, \kappa)} \\ &+ a(\theta + \pi i/2)d(\theta - \pi i/2) \frac{Q^\pm(\theta - 3\pi i/2, \kappa)Q^\pm(\theta + 3\pi i/2, \kappa)}{Q^\pm(\theta - \pi i/2, \kappa)Q^\pm(\theta + \pi i/2, \kappa)} \\ &+ d(\theta + \pi i/2)d(\theta - \pi i/2) \frac{Q^\pm(\theta - 3\pi i/2, \kappa)}{Q^\pm(\theta + \pi i/2, \kappa)} , \end{aligned}$$

where

$$a(\theta) = s(\theta - \pi i), \quad d(\theta) = s(\theta + \pi i), \quad s(\theta) = \prod_{j=1}^L \sinh \nu(\theta - \tau_j) .$$

We shall need the eigenvalues of the transfer-matrix with the two-dimensional auxiliary space $t(\theta|\kappa)$, for which

$$\begin{aligned} t(\theta, \kappa)Q^\pm(\theta, \kappa) &= a(\theta)Q^\pm(\theta + \pi i, \kappa) + d(\theta)Q^\pm(\theta - \pi i, \kappa) , \\ T(\theta, \kappa) &= t(\theta - \pi i/2, \kappa)t(\theta + \pi i/2, \kappa) - f(\theta) , \quad f(\theta) = s(\theta - 3\pi i/2)s(\theta + 3\pi i/2) . \end{aligned}$$

Denote

$$y(\theta) = \frac{T(\theta|\kappa)}{f(\theta)} . \quad (3.6)$$

We do not explicitly indicate the dependence of $y(\theta)$ on κ because it will be never used for another value of twist.

Now we are ready to define two more functions

$$\rho(\theta) = \frac{t(\theta|\kappa + \alpha)}{t(\theta|\kappa)} , \quad \mathcal{N}(\theta) = \frac{y(\theta)}{1 + y(\theta)} . \quad (3.7)$$

Recall the definition of the function ω from [19]. This function splits in two parts:

$$\omega(\theta, \theta') = \omega_{\text{hol}}(\theta, \theta') + \omega_{\text{sing}}(\theta, \theta') , \quad (3.8)$$

where $\omega_{\text{hol}}(\theta, \theta')$ as a function of θ has no other singularities but simple poles at the zeros of $T(\theta, \kappa)$, and ω_{sing} is its singular part given by :

$$\begin{aligned} \omega_{\text{sing}}(\theta, \theta') &= \frac{1}{t(\theta|\kappa)t(\theta'|\kappa)} \left(a(\theta)d(\theta')\psi(\theta - \theta' + \pi i, \alpha) - d(\theta)a(\theta')\psi(\theta - \theta' - \pi i, \alpha) \right) \\ &+ (1 + \rho(\theta)\rho(\theta'))\phi(\theta - \theta', \alpha) - \rho(\theta)\phi(\theta - \theta' + \pi i, \alpha) - \rho(\theta')\phi(\theta - \theta' - \pi i, \alpha), \end{aligned} \quad (3.9)$$

where

$$\psi(\theta, \alpha) = 2\nu \frac{e^{\alpha\nu\theta}}{e^{2\nu\theta} - 1}, \quad (3.10)$$

and ϕ is defined as a solution of the difference equation:

$$\Delta_\theta \phi(\theta, \alpha) = \phi(\theta + i\pi, \alpha) - \phi(\theta - i\pi, \alpha) = \psi(\theta, \alpha). \quad (3.11)$$

We shall remind the normalisation conditions for the function ω . Start by defining the function φ :

$$\varphi(\theta) = \prod_{j=1}^L \frac{1}{\sinh \nu(\theta - \tau_j - \pi i) \sinh \nu(\theta - \tau_j) \sinh \nu(\theta - \tau_j + \pi i)},$$

satisfying

$$d(\theta + \pi i)\varphi(\theta + \pi i) = a(\theta)\varphi(\theta),$$

and the measure

$$d\mu^\pm(\theta) = Q^\mp(\theta, \kappa + \alpha)Q^\pm(\theta, \kappa)\varphi(\theta)d\theta. \quad (3.12)$$

The poles of φ come in triplets reflecting the fact that the Matsubara chain consists of spin-1 representations. Let the contour Γ_j go around the three points $\tau_j, \tau_j \pm \pi i$. The normalisation conditions on the function $\omega(\theta, \eta)$ from [19] are given by :

$$\int_{\Gamma_j} t(\theta, \kappa)\omega(\theta, \eta)d\mu^+(\theta) = 0, \quad (3.13)$$

The equations (3.9) , (3.13) define $\omega(\theta, \theta')$ completely. Due to the deformed Riemann bilinear identity the following relation is automatic:

$$\int_{\Gamma_j} t(\theta, \kappa)\omega(\eta, \theta)d\mu^-(\theta) = 0.$$

In order to make the further formulae more readable we shall denote by τ without index any of inhomogenieties τ_j .

For future use we rewrite the normalisation condition as

$$\omega(\tau + \pi i, \eta) + Y(\tau)\omega(\tau, \eta) + X(\tau)\omega(\tau - \pi i, \eta) = 0, \quad (3.14)$$

with

$$X(\theta) = \frac{T(\theta + \pi i/2, \kappa + \alpha)}{a(\theta)d(\theta + \pi i)}, \quad Y(\theta) = \frac{1}{\rho(\theta)} (1 + X(\theta)).$$

Similarly,

$$\omega(\eta, \tau + \pi i) + Y(\tau)\omega(\eta, \tau) + X(\tau)\omega(\eta, \tau - \pi i) = 0, \quad (3.15)$$

3.3. Rewriting normalisation conditions

Introduce

$$\begin{aligned}
F^+(\theta, \eta) &= \langle \mathbf{b}^*(\theta) c^*(\eta) \rangle = \frac{1}{\mathcal{N}(\theta)} \left(\omega(\theta + \pi i/2, \eta) \rho(\theta - \pi i/2) + \omega(\theta - \pi i/2, \eta) \right), \\
\tilde{F}^+(\theta, \eta) &= \langle \tilde{\mathbf{b}}^*(\theta) c^*(\eta) \rangle = \frac{1}{\mathcal{N}(\theta)} \left(\omega(\theta + \pi i/2, \eta) + \rho(\theta + \pi i/2) \omega(\theta - \pi i/2, \eta) \right), \\
F^-(\eta, \theta) &= \langle b^*(\eta) \mathbf{c}^*(\theta) \rangle = \frac{1}{\mathcal{N}(\theta)} \left(\omega(\eta, \theta + \pi i/2) \rho(\theta - \pi i/2) + \omega(\eta, \theta - \pi i/2) \right), \\
\tilde{F}^-(\eta, \theta) &= \langle b^*(\eta) \tilde{\mathbf{c}}^*(\theta) \rangle = \frac{1}{\mathcal{N}(\theta)} \left(\omega(\eta, \theta + \pi i/2) + \rho(\theta + \pi i/2) \omega(\eta, \theta - \pi i/2) \right).
\end{aligned}$$

The functions describe pairings between the fused operators $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$ with unfused ones c^* , b^* . Clearly knowledge of these pairings is sufficient to compute any expectation value containing $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$. So, the analytical properties of $F^+(\theta, \eta)$ etc characterise in the weak sense the analytical properties of $\mathbf{b}^*(\theta)$, $\mathbf{c}^*(\theta)$.

Similarly, in order to understand the analytical properties of $\mathbf{j}^+(\theta)$, $\mathbf{j}^0(\theta)$, $\mathbf{j}^-(\theta)$ we introduce

$$\begin{aligned}
G^+(\theta, \eta_1, \eta_2) &= \langle \mathbf{j}^+(\theta) c^*(\eta_2) c^*(\eta_1) \rangle = \frac{1}{\mathcal{N}(\theta)} \begin{vmatrix} \omega(\theta + \pi i/2, \eta_1) & \omega(\theta + \pi i/2, \eta_2) \\ \omega(\theta - \pi i/2, \eta_1) & \omega(\theta - \pi i/2, \eta_2) \end{vmatrix}, \\
G^-(\eta_1, \eta_2, \theta) &= \langle b^*(\eta_1) b^*(\eta_2) \mathbf{j}^-(\theta) \rangle = \frac{1}{\mathcal{N}(\theta)} \begin{vmatrix} \omega(\eta_1, \theta + \pi i/2) & \omega(\eta_2, \theta + \pi i/2) \\ \omega(\eta_1, \theta - \pi i/2) & \omega(\eta_2, \theta - \pi i/2) \end{vmatrix}, \\
G^0(\theta, \eta_1, \eta_2) &= \langle \mathbf{j}^0(\theta) b^*(\eta_1) c^*(\eta_2) \rangle \\
&= \frac{1}{\mathcal{N}(\theta)} \left((\omega(\theta + \pi i/2, \theta - \pi i/2) - \omega(\theta - \pi i/2, \theta + \pi i/2)) \omega(\eta_1, \eta_2) \right. \\
&\quad \left. + \omega(\theta - \pi i/2, \eta_2) \omega(\eta_1, \theta + \pi i/2) - \omega(\theta + \pi i/2, \eta_2) \omega(\eta_1, \theta - \pi i/2) \right),
\end{aligned}$$

where in the last line we imply

$$\omega(\theta + \pi i/2, \theta - \pi i/2) - \omega(\theta - \pi i/2, \theta + \pi i/2) = \lim_{\theta' \rightarrow \theta} \left(\omega(\theta + \pi i/2, \theta' - \pi i/2) - \omega(\theta - \pi i/2, \theta' + \pi i/2) \right).$$

We want to rewrite the normalisation conditions in terms of these functions and $P(\theta)$ only. As before let τ be any inhomogeneity. Then we claim that

$$\begin{aligned}
F^+(\tau + \pi i/2, \eta) + P(\tau + \pi i/2) \tilde{F}^+(\tau - \pi i/2, \eta) &= 0, \\
\tilde{F}^+(\tau + \pi i/2, \eta) + P(\tau + \pi i/2) F^+(\tau - \pi i/2, \eta) &= 0, \\
F^-(\eta, \tau + \pi i/2) + P(\tau + \pi i/2) \tilde{F}^-(\eta, \tau - \pi i/2) &= 0, \\
\tilde{F}^-(\eta, \tau + \pi i/2) + P(\tau + \pi i/2) F^-(\eta, \tau - \pi i/2) &= 0, \\
G^+(\tau + \pi i/2, \eta_1, \eta_2) - P(\tau + \pi i/2) G^+(\tau - \pi i/2, \eta_1, \eta_2) &= 0, \\
G^-(\tau + \pi i/2, \eta_1, \eta_2) - P(\tau + \pi i/2) G^-(\tau - \pi i/2, \eta_1, \eta_2) &= 0, \\
G^0(\tau + \pi i/2, \eta_1, \eta_2) - P(\tau + \pi i/2) G^0(\tau - \pi i/2, \eta_1, \eta_2) &= 0.
\end{aligned} \tag{3.16}$$

Let us prove the first of these identities, others are checked similarly.

We begin with some useful identities. Using

$$a(\tau + \pi i) = 0, \quad d(\tau - \pi i) = 0,$$

we find

$$\begin{aligned}
t(\tau + \pi i, \kappa) &= d(\tau + \pi i) \frac{Q^\pm(\tau, \kappa)}{Q^\pm(\tau + \pi i, \kappa)}, \\
t(\tau - \pi i, \kappa) &= a(\tau - \pi i) \frac{Q^\pm(\tau, \kappa)}{Q^\pm(\tau - \pi i, \kappa)}, \\
T(\tau + \pi i/2, \kappa) &= d(\tau + \pi i) d(\tau) \frac{Q^\pm(\tau - \pi i, \kappa)}{Q^\pm(\tau + \pi i, \kappa)}, \\
T(\tau - \pi i/2, \kappa) &= a(\tau - \pi i) a(\tau) \frac{Q^\pm(\tau + \pi i, \kappa)}{Q^\pm(\tau - \pi i, \kappa)}.
\end{aligned} \tag{3.17}$$

For (3.16) we have

$$\begin{aligned}
&F^+(\tau + \pi i/2, \eta) + P(\tau + \pi i/2) \tilde{F}^+(\tau - \pi i/2, \eta) \\
&= \frac{1}{\mathcal{N}(\tau + \pi i/2)} \left(\omega(\tau + \pi i, \eta) \rho(\tau) + \omega(\tau, \eta) \left(1 + \frac{\mathcal{N}(\tau + \pi i/2)}{\mathcal{N}(\tau - \pi i/2)} P(\tau + \pi i/2) \right) \right. \\
&\quad \left. + \omega(\tau - \pi i, \eta) \rho(\tau) \frac{\mathcal{N}(\tau + \pi i/2)}{\mathcal{N}(\tau - \pi i/2)} P(\tau + \pi i/2) \right).
\end{aligned}$$

Using (3.17) we compute

$$\frac{\mathcal{N}(\tau + \pi i/2)}{\mathcal{N}(\tau - \pi i/2)} P(\tau + \pi i/2) = \frac{d(\tau) Q^-(\tau - \pi i, \kappa + \alpha)}{a(\tau) Q^-(\tau + \pi i, \kappa + \alpha)} = X(\tau).$$

Using the latter identity we evaluate

$$\begin{aligned}
&F^+(\tau + \pi i/2, \eta) + P(\tau + \pi i/2) \tilde{F}^+(\tau - \pi i/2, \eta) \\
&= \frac{\rho(\tau)}{\mathcal{N}(\tau + \pi i/2)} \left(\omega(\tau + \pi i, \eta) + \omega(\tau, \eta) \frac{Q^-(\tau, \kappa + \alpha) T_1(\tau, \kappa + \alpha)}{a(\tau) \rho(\tau) Q^-(\tau + \pi i, \kappa + \alpha)} \right. \\
&\quad \left. + X(\tau) \omega(\tau - \pi i, \eta) \right) \\
&= \frac{\rho(\tau)}{\mathcal{N}(\tau + \pi i/2)} \left(\omega(\tau + \pi i, \eta) + Y(\tau) \omega(\tau, \eta) + X(\tau) \omega(\tau - \pi i, \eta) \right) = 0,
\end{aligned}$$

due to (3.13).

3.4. The case $\alpha = 0$

In the case $\alpha = 0$ the left and right eigenstates coincide, hence $\rho(\theta) = 1$ and in the weak there is no difference between $\mathbf{b}^*, \mathbf{c}^*$ at one hand and $\tilde{\mathbf{b}}^*, \tilde{\mathbf{c}}^*$ on the other. So, all the expectation values containing only fermions are expressed via one function

$$\begin{aligned}
\Omega(\theta, \theta') &= \frac{1}{\mathcal{N}(\theta) \mathcal{N}(\theta')} \\
&\times \left(\omega\left(\theta + \frac{\pi i}{2}, \theta' + \frac{\pi i}{2}\right) + \omega\left(\theta + \frac{\pi i}{2}, \theta' - \frac{\pi i}{2}\right) + \omega\left(\theta - \frac{\pi i}{2}, \theta' + \frac{\pi i}{2}\right) + \omega\left(\theta - \frac{\pi i}{2}, \theta' - \frac{\pi i}{2}\right) \right),
\end{aligned} \tag{3.18}$$

We want to find an independent way of defining this function. As explained in [19] for $\alpha = 0$ there is an important analogy between the function $\omega(\theta, \theta')$ and the normalised

second kind differential on a hyperelliptic Riemann surface. The normalisation condition (3.13) is the analogue of the requirement of vanishing of the a -periods.

We set

$$\tau = \tau_j.$$

Consider the function

$$\tilde{\omega}(\theta) = \frac{\delta}{\delta\tau} \left\{ \log t(\theta) - \log \left(\frac{s(\theta - \pi i)s(\theta + \pi i)}{s(\theta)} \right) \right\}.$$

Notice that

$$\frac{\delta}{\delta\tau} \log \left(\frac{s(\theta - \pi i)s(\theta + \pi i)}{s(\theta)} \right) = (\delta_\theta^+)^{-1} \frac{\delta}{\delta\tau} \log (s(\theta - 2\pi i)s(\theta + \pi i)),$$

where $\delta_\theta^+ f(\theta) = f(\theta) + f(\theta - \pi i)$.

We want to show that $\tilde{\omega}(\theta)$ is a normalised differential. First we prove that

$$\int_{\Gamma_k} t(\theta) \tilde{\omega}(\theta) d\mu^\pm(\theta) = 0, \quad k \neq j.$$

The case $k = j$ is special, instead of direct computation for this case we consider $\Gamma_{\pm\infty} = [\pm\Lambda, \pm\Lambda + \pi i/\nu]$ for $|\lambda| > \max(|\tau_k|)$. For $\Gamma_{\pm\infty}$ the computation is exactly the same as for Γ_k , $k \neq j$.

Recall that (in the case $\alpha = 0$ we have $d\mu^+ = d\mu^- = d\mu$) :

$$d\mu(\theta) = Q^+(\theta)Q^-(\theta)\varphi(\theta)d\theta.$$

We have two identities [19]:

$$\begin{aligned} \int_{\Gamma_k} t(\theta)(\delta_\theta^+)^{-1} f(\theta) d\mu(\theta) &= \int_{\Gamma_k} d(\theta) f(\theta) Q^+(\theta - \pi i) Q^-(\theta) \varphi(\theta) d\theta, \\ &= \int_{\Gamma_k} a(\theta) f(\theta + \pi i) Q^+(\theta + \pi i) Q^-(\theta) \varphi(\theta) d\theta. \end{aligned}$$

Using these identities we derive

$$\begin{aligned} &\int_{\Gamma_k} t(\theta) \tilde{\omega}(\theta) d\mu(\theta) \\ &= \int_{\Gamma_k} \left\{ Q^+(\theta) \frac{\delta}{\delta\tau} t(\theta) - Q^+(\theta + \pi i) \frac{\delta}{\delta\tau} a(\theta) - Q^+(\theta - \pi i) \frac{\delta}{\delta\tau} d(\theta) \right\} Q^-(\theta) \varphi(\theta) d\theta \\ &= \int_{\Gamma_k} \left\{ a(\theta) \frac{\delta}{\delta\tau} Q^+(\theta + \pi i) + d(\theta) \frac{\delta}{\delta\tau} Q^+(\theta - \pi i) - t(\theta) \frac{\delta}{\delta\tau} Q^+(\theta) \right\} Q^-(\theta) \varphi(\theta) d\theta \\ &= \int_{\Gamma_k} \left\{ a(\theta) Q^-(\theta) \frac{\delta}{\delta\tau} Q^+(\theta + \pi i) - d(\theta) Q^-(\theta - \pi i) \frac{\delta}{\delta\tau} Q^+(\theta) \right\} \varphi(\theta) d\theta \\ &+ \int_{\Gamma_k} \left\{ d(\theta) Q^-(\theta) \frac{\delta}{\delta\tau} Q^+(\theta - \pi i) - a(\theta) Q^-(\theta + \pi i) \frac{\delta}{\delta\tau} Q^+(\theta) \right\} \varphi(\theta) d\theta = 0. \end{aligned}$$

As a normalised differential $\tilde{\omega}(\theta)$ must be expressible as a linear combination of $\omega(\theta, \eta_j)$ for some set $\{\eta_j\}$. The structure of singularities of $\tilde{\omega}(\theta)$ suggests that this set is just $\tau, \tau + \pi i$. To be precise we claim that

$$\tilde{\omega}(\zeta) = \frac{1}{\mathcal{N}(\tau + \frac{\pi i}{2})} (\omega(\zeta, \tau) + \omega(\zeta, \tau + \pi i)). \quad (3.19)$$

Let us prove this. We have

$$\begin{aligned} \omega(\theta, \tau) + \omega(\theta, \tau + \pi i) &= \omega_{\text{hol}}(\theta, \tau) + \omega_{\text{hol}}(\theta, \tau + \pi i) + \omega_{\text{sing}}(\theta, \tau) + \omega_{\text{sing}}(\theta, \tau + \pi i), \\ \text{where } \omega_{\text{hol}}(\theta, \eta) &\text{ as function of } \theta \text{ has no other singularities but simple poles at zeros of } t(\theta), \\ \omega_{\text{sing}}(\theta, \eta) &= \delta_\theta^- \delta_\eta^- \Delta_\theta^{-1} (\nu \coth \nu(\theta - \eta)) \\ &+ \frac{1}{t(\theta)t(\eta)} (a(\theta)d(\eta)\nu \coth \nu(\theta - \eta + \pi i) - d(\theta)a(\eta)\nu \coth \nu(\theta - \eta - \pi i)), \end{aligned} \quad (3.20)$$

which implies

$$\begin{aligned} \omega_{\text{sing}}(\theta, \tau) + \omega_{\text{sing}}(\theta, \tau + \pi i) &= \nu \coth \nu(\theta - \tau - \pi i) - \nu \coth \nu(\theta - \tau) \\ &+ \frac{a(\theta)d(\tau)}{t(\theta)t(\tau)} \nu \coth \nu(\theta - \tau + \pi i) - \frac{d(\theta)a(\tau)}{t(\theta)t(\tau)} \nu \coth \nu(\theta - \tau - \pi i) \\ &+ \frac{a(\theta)d(\tau + \pi i)}{t(\theta)t(\tau + \pi i)} \nu \coth \nu(\theta - \tau). \\ &= \frac{a(\theta)d(\tau)}{t(\theta)t(\tau)} \nu \coth \nu(\theta - \tau + \pi i) + \frac{t(\theta)t(\tau) - d(\theta)a(\tau)}{t(\theta)t(\tau)} \nu \coth \nu(\theta - \tau - \pi i) \\ &- \frac{t(\theta)t(\tau + \pi i) - a(\theta)d(\tau + \pi i)}{t(\theta)t(\tau + \pi i)} \nu \coth \nu(\theta - \tau). \end{aligned}$$

Using this identity one finds

$$\begin{aligned} \text{res}_{\theta=\tau-\pi i} (\omega_{\text{sing}}(\theta, \tau) + \omega_{\text{sing}}(\theta, \tau + \pi i)) &= \frac{a(\tau - \pi i)d(\tau)}{t(\tau - \pi i)t(\tau)} = \mathcal{N}(\tau + \frac{\pi i}{2}), \\ \text{res}_{\theta=\tau+\pi i} (\omega_{\text{sing}}(\theta, \tau) + \omega_{\text{sing}}(\theta, \tau + \pi i)) &= \frac{T_2(\tau + \frac{\pi i}{2})}{t(\tau + \pi i)t(\tau)} = \mathcal{N}(\tau + \frac{\pi i}{2}), \\ \text{res}_{\theta=\tau} (\omega_{\text{sing}}(\theta, \tau) + \omega_{\text{sing}}(\theta, \tau + \pi i)) &= -\frac{T_2(\tau + \frac{\pi i}{2})}{t(\tau + \pi i)t(\tau)} = -\mathcal{N}(\tau + \frac{\pi i}{2}), \end{aligned}$$

This finishes the proof.

Now we obtain the most important relation of this section :

$$\begin{aligned} &\frac{\delta}{\delta\tau} \log \left(\frac{T_2(\theta)}{f(\theta)} \right) \quad (3.21) \\ &= \frac{1}{\mathcal{N}(\theta)} \left(\frac{\delta}{\delta\tau} \log T_1(\theta + \pi i/2) + \frac{\delta}{\delta\tau} \log T_1(\theta - \pi i/2) \right) - \left(\frac{f(\theta)}{T_2(\theta)} + 1 \right) \frac{\delta}{\delta\tau} \log f(\theta) \\ &= \frac{1}{\mathcal{N}(\theta)} \left(\frac{\delta}{\delta\tau} \log T_1(\theta + \pi i/2) + \frac{\delta}{\delta\tau} \log T_1(\theta - \pi i/2) - \frac{\delta}{\delta\tau} \log f(\theta) \right) \\ &= \frac{1}{\mathcal{N}(\theta)\mathcal{N}(\tau + \frac{\pi i}{2})} \left(\omega(\theta + \frac{\pi i}{2}, \tau) + \omega(\theta + \frac{\pi i}{2}, \tau + \pi i) + \omega(\theta - \frac{\pi i}{2}, \tau) + \omega(\theta - \frac{\pi i}{2}, \tau + \pi i) \right). \\ &= \Omega(\theta, \tau + \frac{\pi i}{2}). \end{aligned}$$

4. Scaling limit

In considering the scaling limit, we want, similarly to [15, 2], to combine two seemingly inconsistent requirements: $\alpha \neq 0$ and $\rho(\theta) = P(\theta) = 1$. In fact this can be achieved for a discrete set of α 's introducing the fermionic screening operators [15], and then invoking the analytical continuation. As will be clear later our definition is consistent rather with the understanding on the model in terms of the action (2.3).

The scaling limit consists in taking in both Space and Matsubara directions staggering inhomogeneities $\tau_j = (-1)^j \tau$, and considering the limit

$$\tau \rightarrow \infty, \quad L \rightarrow \infty, \quad 2Le^{-\tau} \rightarrow 2\pi MR \text{ finite},$$

where R is the radius of the cylinder, M is the mass of the soliton (2.12).

For $\rho(\theta) = P(\theta) = 1$ in the weak sense the operators $\tilde{\mathbf{b}}^*(\zeta)$, $\tilde{\mathbf{c}}^*(\zeta)$ coincide with the operators $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$. Similarly to [15, 2] the relations (3.16) hint that the asymptotics for $\theta \rightarrow \pm\infty$ of the fermions (KM currents) are anti-periodic (periodic) in θ . Explicitly we assume

$$\begin{aligned} \mathbf{b}^*(\theta) &\underset{\theta \rightarrow \pm\infty}{\simeq} \sum_{j=1}^{\infty} e^{\mp(2j-1)\theta} \mathbf{b}_{2j-1}^*, & \mathbf{c}^*(\theta) &\underset{\theta \rightarrow \pm\infty}{\simeq} \sum_{j=1}^{\infty} e^{\mp(2j-1)\theta} \mathbf{c}_{2j-1}^*, \\ \mathbf{j}^\sigma(\theta) &\underset{\theta \rightarrow \pm\infty}{\simeq} \sum_{j=1}^{\infty} e^{\mp 2j\theta} \mathbf{j}_{2j}^\sigma, & \sigma &= 0, \pm. \end{aligned}$$

The Suzuki equations (2.14) are obtained by this procedure from the corresponding lattice equations, which have the same structure as (2.14), but differ only by the driving term. In the case of the lattice it is given by :

$$D(\theta) = 2 \sum_j \log \left(\tanh \frac{1}{2}(\theta - \tau_j - i0) \right) - \frac{\pi i \nu \kappa}{1 - 2\nu}, \quad (4.1)$$

for which we have in the scaling limit

$$D(\theta) \rightarrow -2\pi MR \cosh(\theta - \pi i \gamma) - \frac{\pi i \sqrt{2}}{\beta} P.$$

The identification between κ and P is

$$\sqrt{2}\beta P = \nu \kappa. \quad (4.2)$$

4.1. Equations for Ω

Now we shall present a conjecture for the scaling limit of $\Omega(\theta, \theta')$ in the case $\alpha \neq 0$ and $\rho(\theta) = P(\theta) = 1$ and provide some justifications for it :

$$\begin{aligned} \Omega(\theta, \theta') &= \int_{-\infty}^{\infty} L(\theta - \eta + \pi i \gamma) \mathfrak{G}(\eta - \pi i \gamma, \theta') dm_b(\eta - \pi i \gamma) \\ &+ \int_{-\infty}^{\infty} L(\theta - \eta - \pi i \gamma) \bar{\mathfrak{G}}(\eta + \pi i \gamma, \theta') d\bar{m}_b(\eta + \pi i \gamma), \end{aligned} \quad (4.3)$$

where for the auxiliary functions we have the linear equations

$$\begin{aligned} \mathfrak{G}(\theta - \pi i \gamma, \theta') &= L(\theta - \theta' - \pi i \gamma) + \int_{-\infty}^{\infty} L(\theta - \eta - \pi i \gamma) \Omega(\eta, \theta') dm_y(\eta) \\ &+ \int_{-\infty}^{\infty} G_\alpha(\theta - \eta) \mathfrak{G}(\eta - \pi i \gamma, \theta') dm_b(\eta - \pi i \gamma) \\ &- \int_{-\infty}^{\infty} G_\alpha(\theta - \eta + \pi i(1 - 2\gamma)) \bar{\mathfrak{G}}(\eta + \pi i \gamma, \theta') d\bar{m}_b(\eta + \pi i \gamma), \end{aligned} \quad (4.4)$$

$$\begin{aligned} \bar{\mathfrak{G}}(\theta + \pi i \gamma, \theta') &= L(\theta - \theta' + \pi i \gamma) + \int_{-\infty}^{\infty} L(\theta - \eta + \pi i \gamma) \Omega(\eta, \theta') dm_y(\eta) \\ &+ \int_{-\infty}^{\infty} G_\alpha(\theta - \eta - \pi i(1 - 2\gamma)) \mathfrak{G}(\eta - \pi i \gamma, \theta') dm_b(\eta - \pi i \gamma) \\ &- \int_{-\infty}^{\infty} G_\alpha(\theta - \eta) \bar{\mathfrak{G}}(\eta + \pi i \gamma, \theta') d\bar{m}_b(\eta + \pi i \gamma), \end{aligned} \quad (4.5)$$

and we defined

$$\begin{aligned} dm_y(\theta) &= \frac{y(\theta)}{1 + y(\theta)}, \quad dm_b(\theta) = \frac{b(\theta)}{1 + b(\theta)}, \quad d\bar{m}_b(\theta) = \frac{\bar{b}(\theta)}{1 + \bar{b}(\theta)}, \\ L(\theta) &= \frac{1}{2\pi \cosh \theta}, \quad G_\alpha(\theta) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\sinh\left(\frac{3\beta^2 - 1}{2(1 - \beta^2)} \pi k + \frac{\pi i \alpha}{2}\right)}{\sinh\left(\frac{\beta^2}{1 - \beta^2} \pi k + \frac{\pi i \alpha}{2}\right) \cosh\left(\frac{1}{2} \pi k\right)} e^{ik\theta} dk. \end{aligned}$$

The shift γ is an arbitrary real number from the interval $(0, \pi/2)$.

The most important support for this definition is provided by the case $\alpha = 0$ for which the requirements $\rho(\theta) = P(\theta) = 1$ are automatic and do not require additional work even on the lattice. In that case we have (3.21)

$$\frac{\delta}{\delta \tau} \log y(\theta) = \Omega(\theta, \tau + \frac{\pi i}{2}). \quad (4.6)$$

Using the Suzuki equations (2.14) with the driving term replaced by $D(\theta)$ (4.1) one readily compute the variation with respect to any τ_j finding agreement with (4.3). Strictly speaking even for $\alpha = 0$ to combine the equations (4.6) for all τ_j , we do not have enough conditions to assert (4.3) for all θ' , but this is a very natural conjecture to make.

The next question is how did we incorporate α into the equations (4.3), (4.4), (4.5). This was done due to the experience with equations of this kind [15, 2] and some meditation. Our choice is supported by computation of the residue at $\theta = \theta + \pi i$. After some rather tedious computation we obtain the following result

$$\text{res}_{\theta = \theta' + \pi i} \Omega(\theta, \theta') = \frac{1}{2\pi i} \frac{y(\theta')y(\theta' + \pi i) - 1}{y(\theta')y(\theta' + \pi i)},$$

which coincides with the expected result from the definition (3.18) and known singularities of $\omega(\theta, \theta')$ [19].

4.2. Numerical results by interpolation

Our method of numerical investigation of the equations (2.14) was explained in [18]. With these results at hand the numerical solution to the linear equations (4.3), (4.4), (4.5) is rather straightforward. The most interesting thing to study is the limit $\theta_0 \rightarrow \infty$ where we make contact with the UV CFT. We begin with the case $\theta \rightarrow \infty, \theta' \rightarrow \infty$ for which we assume

$$\Omega(\theta, \theta') \simeq \sum_{i,j=1}^{\infty} e^{-(2i-1)\theta} e^{-(2j-1)\theta'} D_{2i-1}(\alpha) D_{2j-1}(2-\alpha) \Omega_{2i-1,2j-1}(\theta_0). \quad (4.7)$$

The coefficients $D_{2i-1}(\alpha)$ are not hard to guess from (2.17) and by analogy with [15]:

$$D_m(\alpha) = i^m \sqrt{\frac{\pi}{2}} \frac{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{1}{1-\beta^2}m + \frac{\alpha}{2}\right)}{(m-1)! \left(\frac{m-1}{2}\right)! \Gamma\left(\frac{\beta^2}{(1-\beta^2)}m + \frac{\alpha}{2}\right)}. \quad (4.8)$$

Additional support for this formula will be given below by considering the reflection relations. We have further

$$\lim_{\theta_0 \rightarrow \infty} e^{-2(i+j-1)\theta_0} \Omega_{2i-1,2j-1}(\theta_0) \rightarrow \Omega_{2i-1,2j-1},$$

$\Omega_{2i-1,2j-1}$ is a polynomial of P of degree $2i + 2j - 2$ with the leading coefficient equal to $1/(i + j - 1)$.

We proceed with numerical checks of these assumptions. For $\theta_0 = 15$ we obtain already perfect agreement with the scaling behaviour. The values of P should not be too large, we take $P \leq 0.2$. Considering an important amount of numerical data with different P, α, ν we come with the following conjectures for the exact forms of the first several $\Omega_{m,n}$:

$$\Omega_{1,1} = P^2 - \frac{1}{16} - \frac{1}{8} \Delta_\alpha. \quad (4.9)$$

$$2 \cdot \Omega_{1,3} = P^4 - P^2 \frac{5}{48} (2\Delta_\alpha + 3) + \frac{\hat{c} + 8}{1536} (4\Delta_\alpha + 3) + \frac{1}{128} \Delta_\alpha^2 \mp \frac{1}{96} d_\alpha \Delta_\alpha. \quad (4.10)$$

$$3 \cdot \Omega_{3,3} = P^6 - P^4 \frac{1}{64} (18\Delta_\alpha + 47) + P^2 \left(\frac{27}{1280} \Delta_\alpha^2 + \frac{23\hat{c} + 378}{3840} \Delta_\alpha + \frac{46\hat{c} + 881}{5120} \right) \quad (4.11)$$

$$- \frac{1}{2048} \Delta_\alpha^3 - \frac{40\hat{c} + 21}{61440} \Delta_\alpha^2 - \frac{5\hat{c}^2 + 52\hat{c} + 222}{81920} (2\Delta_\alpha + 3).$$

$$3 \cdot \Omega_{1,5} = P^6 - P^4 \left(\frac{35}{48} + \frac{7}{24} \Delta_\alpha \right) + P^2 \left(\frac{89}{3840} \Delta_\alpha^2 + \frac{23\hat{c} + 514}{15360} (4\Delta_\alpha + 5) \right) \quad (4.12)$$

$$- \frac{1}{2048} \Delta_\alpha^3 - \frac{10\hat{c} + 479}{61440} \Delta_\alpha^2 - \frac{6\hat{c}^2 + 83\hat{c} + 386}{245760} (6\Delta_\alpha + 5)$$

$$\mp d_\alpha \Delta_\alpha \left(P^2 \frac{23}{960} - \frac{1}{512} \Delta_\alpha - \frac{83 + 12\hat{c}}{15360} \right).$$

where

$$d_\alpha = \frac{1}{4} (\beta^{-2} - \beta^2) (\alpha - 1). \quad (4.13)$$

In Table 1,2,3 we give some examples of comparison between numerical results and the analytical conjectures above.

P	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$	
	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.	$\Omega_{1,1}$ comp.	$\Omega_{1,1}$ analyt.
0.02	-0.059287494	-0.0592875	-0.057099995	-0.0571	-0.055537495	-0.0555375
0.04	-0.058087495	-0.0580875	-0.055899995	-0.0559	-0.054337495	-0.0543375
0.06	-0.056087495	-0.0560875	-0.053899995	-0.0539	-0.052337495	-0.0523375
0.08	-0.053287495	-0.0532875	-0.051099995	-0.0511	-0.049537496	-0.0495375
0.1	-0.049687495	-0.0496875	-0.047499996	-0.0475	-0.045937496	-0.0459375
0.12	-0.045287496	-0.0452875	-0.043099996	-0.0431	-0.041537497	-0.0415375
0.14	-0.04008745	-0.0400875	-0.037899997	-0.0379	-0.03633745	-0.0363375
0.16	-0.034087497	-0.0340875	-0.031899998	-0.0319	-0.030337498	-0.0303375
0.18	-0.027287498	-0.0272875	-0.025099998	-0.0251	-0.023537499	-0.0235375
0.2	-0.019687499	-0.0196875	-0.017499999	-0.0175	-0.015937400	-0.0159375

Table 1: Coefficient $\Omega_{1,1}$ and $\beta^2 = \frac{1}{2}$

P	$\alpha = 0.2$		$\alpha = 0.4$		$\alpha = 0.6$	
	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.	$\Omega_{1,3}$ comp.	$\Omega_{1,3}$ analyt.
0.02	0.01612247	0.01612249	0.01591101	0.01591102	0.01577159	0.01577160
0.04	0.01575287	0.01575289	0.01554374	0.01554376	0.01540599	0.01540600
0.06	0.01514328	0.01514329	0.01493803	0.01493804	0.01480306	0.01480307
0.08	0.01430328	0.01430329	0.0141035	0.01410349	0.01397239	0.01397240
0.1	0.01324632	0.01324633	0.01305352	0.01305353	0.01292743	0.01292744
0.12	0.01198969	0.01198969	0.01180544	0.01180545	0.01168546	0.01168547
0.14	0.01055449	0.01055449	0.01038035	0.01038036	0.01026760	0.01026760
0.16	0.008965691	0.008965693	0.008803224	0.008803226	0.008698802	0.008698804
0.18	0.007252093	0.007252093	0.007102848	0.007102848	0.007007872	0.007007871
0.2	0.005446336	0.005446333	0.005311868	0.005311867	0.005227447	0.005227444

Table 2: Coefficient $\Omega_{1,3}$ and $\beta^2 = \frac{3}{5}$

The scaling limit of (3.2) is supposed to give the ratio

$$\frac{\langle \mathcal{O}_\alpha(0) \rangle_{P,R}}{\langle V_\alpha(0) \rangle_{P,R}},$$

for some operator \mathcal{O}_α . In the case under consideration this operator is supposed to be a chiral descendant of V_α (recall that we do not distinguish between the CFT operators and their perturbed counterparts). To be more precise $\Omega_{2i-1,2j-1}(\theta_0)$ should be related to a descendant on the level $2i+2j-2$. The determinants made of $\Omega_{2i-1,2j-1}(\theta_0)$ correspond to other descendants but we shall not discuss them here restricting ourselves to the simplest case.

All together we must have

$$\lim_{\theta_0 \rightarrow \infty} e^{-2(i+i-1)\theta_0} \Omega_{2i-1,2j-1}(\theta_0) = \frac{\langle \mathcal{P}_{2i-1,2j-1}(\{\mathbf{s}_k, \mathbf{l}_{2m}\}) V_\alpha \rangle_P}{\langle V_\alpha(0) \rangle_P}, \quad (4.14)$$

where $\mathcal{P}_{2i-1,2j-1}(\{\mathbf{s}_k, \mathbf{l}_{2m}\}) V_\alpha$ is an element of the Verma module generated by V_α quotiented by the action of local integrals of motion, this will be discussed in Section 5.

P	$\alpha = 0.4$		$\alpha = 0.6$	
	$\Omega_{3,3}$ comp.	$\Omega_{3,3}$ analyt.	$\Omega_{3,3}$ comp.	$\Omega_{3,3}$ analyt.
0.02	-0.0078464501	-0.0078464506	-0.0077795388	-0.0077795392
0.04	-0.0076463818	-0.0076463822	-0.0075809093	-0.0075809097
0.06	-0.0073175266	-0.0073175270	-0.0072544297	-0.0072544302
0.08	-0.0068666923	-0.0068666926	-0.0068068739	-0.0068068743
0.1	-0.0063032481	-0.0063032484	-0.006247564	-0.0062475644
0.12	-0.0056389640	-0.0056389643	-0.0055882093	-0.0055882096
0.14	-0.0048878029	-0.0048878031	-0.0048426982	-0.0048426985
0.16	-0.0040656674	-0.0040656675	-0.0040268458	-0.0040268459
0.18	-0.0031901003	-0.0031901004	-0.0031580934	-0.0031580936
0.2	-0.0022799390	-0.0022799391	-0.0022551639	-0.002255164

Table 3: Coefficient $\Omega_{3,3}$ and $\beta^2 = \frac{1}{2}$

The expressions like the one in the right hand side of (4.14) can be computed for any $\mathcal{P}_{2i-1,2j-1}$, we shall give some examples in the next section. However, trying to find $\mathcal{P}_{2i-1,2j-1}$ from (4.14) we encounter more problems than in the usual Virasoro case [15]. The point is that the universal enveloping algebra of the super conformal algebra contains much more elements than that of the Virasoro algebra. The coefficients of the polynomials $\mathcal{P}_{2i-1,2j-1}$ do not depend on P , and actually the appearance of different degrees of P is the source (the only one) of different equations. When the level grows the number of coefficients of $\mathcal{P}_{2i-1,2j-1}$ grows much faster than the degree of the left hand side in P . For the Virasoro case we still could define the coefficients up to the level 6, and for levels 2 and 4 the systems of equations were even overdetermined, the fact that they allowed solutions was considered as an important check of our procedure. In the super conformal case the only possibility to find the coefficients occurs on the level 2: we have two descendants created by \mathbf{l}_{-2} and $\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}$ and two coefficients of the polynomial in P in the left hand side. Starting from the level 4 we do not have enough equations.

One way out of this difficulty would be to allow descendants in the asymptotic states like it was done in [23] for the level 8 in the Virasoro case. This would be too hard, and not necessary: we have another, similar to that of [8], way of fixing the polynomials $\mathcal{P}_{2i-1,2j-1}$ based on the reflection relations [5, 6, 7]. We shall explain this in the next section. When the polynomials $\mathcal{P}_{2i-1,2j-1}$ are defined from the reflection relations, the formulae (4.14), (4.9), (4.10), (4.12), (4.11) can be used for checks. Since both our equation for $\Omega(\theta, \theta')$ and the reflection relations have the status of conjectures the fact that the results of their application are in agreement provides a very solid support for both.

4.3. Primary fields

Let us now consider the asymptotics $\theta \rightarrow -\infty$, $\theta' \rightarrow \infty$. We have

$$\Omega(\theta, \theta') \simeq \sum_{i,j=1}^{\infty} e^{(2i-1)\theta} e^{-(2j-1)\theta'} \Omega_{-(2i-1),2j-1}(\theta_0).$$

We suspect that similarly to [2] the $\Omega_{-1,1}(\theta_0)$ is related to the ratio of the expectation values of two shifted primary fields. The question is: which primary fields exactly? Now

we have two of them: V_α, W_α . Solving numerically our equations we find that for fixed β, α, P

$$\log \Omega_{-1,1}(\theta_0) \simeq 2\theta_0 \left(\Delta_{\alpha + \frac{2\beta^2}{1-\beta^2}} + 1/2 - \Delta_\alpha \right).$$

Let us give an example. Consider the normalised expression:

$$R(\theta_0) = \exp \left\{ -2\theta_0 \left(\Delta_{\alpha + \frac{2\beta^2}{1-\beta^2}} + 1/2 - \Delta_\alpha \right) \right\} \Omega_{-1,1}(\theta_0).$$

For $\alpha = 1/2, \beta^2 = 1/2, P = 0.1$ we have

θ_0	12	13	14	15	16
$R(\theta_0)$	0.16825979	0.16825580	0.16825433	0.16825379	0.16825359

So, we see that the scaling is achieved with great precision.

This suggests that $\Omega_{-1,1}(\theta_0)$ is proportional to the ratio of the expectation values of $W_{\alpha + \frac{2\beta^2}{1-\beta^2}}$ and V_α . Let us check the limiting value against the CFT. First, we have to normalise the primary fields

$$\hat{V}_\alpha = \frac{1}{F(\alpha)} V_\alpha, \quad \hat{W}_\alpha = \frac{1}{F(\alpha)} W_\alpha,$$

where $F(\alpha)$ is the one-point function of the operator V_α on the plane (for $R = \infty$) [24]. For the operator W_α the one-point function on the plane vanishes since this operator is a super Poincare descendant of V_α and the vacuum is super Poincare invariant. Nevertheless we normalise W_α by the same function $F(\alpha)$. The reason for that is in the reflection relations as explained in the next section. Denote by $c(\alpha, P)$ ($\tilde{c}(\alpha, P)$) the CFT one-point functions of the normalised operator \hat{V}_α (\hat{W}_α) on the cylinder with our usual asymptotic conditions. In the next section we compute

$$\begin{aligned} \frac{\tilde{c}(\alpha + \frac{2\beta^2}{1-\beta^2}, P)}{c(\alpha, P)} &= \frac{\pi^2}{1-\beta^2} \beta^{\frac{1}{2}(\alpha\beta^2 - 2\beta^2 - \alpha)} \frac{\gamma(\frac{1}{2}(1-\beta^2)(2-\alpha))}{\gamma(\frac{1}{4}(1-\beta^2)(2-\alpha))^2} \\ &\times \gamma(\frac{1}{2}(1+\beta^2) + (1-\beta^2)\alpha - \beta P) \gamma(\frac{1}{2}(1+\beta^2) + (1-\beta^2)\alpha + \beta P). \end{aligned} \quad (4.15)$$

Consider the ratio

$$R_1(\theta_0) = R(\theta_0) \frac{c(\alpha, P)}{\tilde{c}(\alpha + \frac{2\beta^2}{1-\beta^2}, P)}. \quad (4.16)$$

For $\theta_0 = 15$ and a random choice of ν, α, P we have

data	$\beta^2 = \frac{1}{2}, \alpha = \frac{2}{5}, P = 0.2$	$\beta^2 = \frac{3}{5}, \alpha = \frac{2}{3}, P = 0.1$	$\beta^2 = \frac{1}{3}, \alpha = \frac{1}{2}, P = 0.15$
$R_1(15)$	1.00000211	1.00009870	0.99999998

The agreement is very good.

5. Reflection relations and three-point functions in super CFT

Long ago Al. Zamolodchikov did a remarkable observation that the one-point functions for sine-Gordon and sinh-Gordon model are related by analytical continuation. This is

very different from other properties of these models, for example the particle content is quite different. Nevertheless the Al. Zamolodchikov's observation proved to be correct in many cases. Here we shall apply it to the ssG model relating it to the super sinh-Gordon with the action

$$\mathcal{A}'_L = \int \left[\left(\frac{1}{4\pi} \partial_z \varphi \partial_{\bar{z}} \varphi + \frac{1}{2\pi} (\psi \partial_{\bar{z}} \psi + \bar{\psi} \partial_z \bar{\psi}) - \mu \bar{\psi} \psi e^{\frac{b}{\sqrt{2}} \varphi} \right) - \mu \bar{\psi} \psi e^{-\frac{b}{\sqrt{2}} \varphi} \right] d^2 z. \quad (5.1)$$

We shall use the habitual notation

$$Q = b + b^{-1}.$$

The analytical continuation to the ssG case corresponds to

$$\beta = ib, \quad \alpha = \frac{2a}{Q}. \quad (5.2)$$

Slightly abusing the notation we will write the primary fields (2.6) as V_a and W_a . The idea behind the reflection relations is that the physical quantities must be invariant under the two reflections:

$$\sigma_1 : a \rightarrow -a, \quad \sigma_2 : a \rightarrow Q - a, \quad (5.3)$$

The first of them reflects simply the C -reflection of the action (5.1) while the second one is inherited from the symmetry of the super Liouville model. The reflection relations can be applied to the calculation of one point functions. For the primary fields it is rather direct, since their one point functions are invariant under σ_1 and their transformation rule under σ_2 is inherited from a remarkable property of the (super) Liouville three point function. This will be explained in more details in Section 5.1. The situation is more complicated for descendants fields : a Virasoro descendants has a manifest σ_2 symmetry, but its behaviour for σ_1 is unclear. This explains the necessity to construct a passage matrix $U(a)$ that relates the Virasoro and Heisenberg descendants in order to use the action of the two reflections simultaneously. Recall that $\mathcal{V}_a^{\text{quo}}$ is the quotient of the Verma module by the action of the local integrals of motion. Consider $V(a) \in (\mathcal{V}_a^{\text{quo}})^*$. The reflection relations [5] can be presented as the following Riemann-Hilbert problem (see [8] for more details):

$$V(a + Q) = S(a)V(a), \quad S(a) = U(-a)U(a)^{-1}. \quad (5.4)$$

Let us apply this idea.

5.1. Primary fields

We begin with the primary fields for which the three-point function on a sphere on the one hand and the one-point function on the cylinder (with our usual asymptotical conditions) on the other hand coincide. The main reference for this subsection is [25]. Consider the three-point function of the fields V_{a_j} , $j = 1, 2, 3$:

$$C(a_1, a_2, a_3) = \left(\frac{1}{2} \pi \mu \gamma \left(\frac{1}{2} b Q \right) b^{-1-b^2} \right)^{\frac{Q-a}{b}} \Upsilon_{\text{NS}}(2a_1) \quad (5.5)$$

$$\times \frac{\Upsilon'_{\text{NS}}(0) \Upsilon_{\text{NS}}(2a_2) \Upsilon_{\text{NS}}(2a_3)}{\Upsilon_{\text{NS}}(a-Q) \Upsilon_{\text{NS}}(a_1+2-3) \Upsilon_{\text{NS}}(a_2+3-1) \Upsilon_{\text{NS}}(a_3+1-2)},$$

where $a = a_1 + a_2 + a_3$, $a_{1+2-3} = a_1 + a_2 - a_3$ etc,

$$\Upsilon_{\text{NS}}(x) = \Upsilon\left(\frac{x}{2}\right) \Upsilon\left(\frac{x+Q}{2}\right), \quad \Upsilon_{\text{R}}(x) = \Upsilon\left(\frac{x+b}{2}\right) \Upsilon\left(\frac{x+b^{-1}}{2}\right),$$

and the well-known function $\Upsilon(x)$ satisfies the identities

$$\frac{\Upsilon(x-b)}{\Upsilon(x)} = b^{2bx-2b^2-1} \gamma(1+b^2-bx), \quad \frac{\Upsilon(x-b^{-1})}{\Upsilon(x)} = \gamma(1+b^{-2}-xb^{-1}) b^{1-2x/b+2b^{-2}}.$$

$$\Upsilon(Q-x) = \Upsilon(x).$$

We recall that $\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$. One can use the following integral representation for $\log \Upsilon$ in the range $0 < \text{Re}(x) < Q$:

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left\{ \left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\text{sh}^2\left(\left(\frac{Q}{2} - x\right)\frac{t}{2}\right)}{\text{sh}\left(\frac{tb}{2}\right)\text{sh}\left(\frac{t}{2b}\right)} \right\}. \quad (5.6)$$

The function $\Upsilon_{\text{R}}(x)$ was introduced in order to be able to write down the three-point function for the fields $W_{a_1}, V_{a_2}, V_{a_3}$:

$$\begin{aligned} \tilde{C}(a_1, a_2, a_3) &= \left(\frac{1}{2}\pi\mu\gamma\left(\frac{1}{2}bQ\right)b^{-1-b^2}\right)^{\frac{Q-a}{b}} \Upsilon_{\text{NS}}(2a_1) \\ &\times \frac{2i\Upsilon'_{\text{NS}}(0)\Upsilon_{\text{NS}}(2a_2)\Upsilon_{\text{NS}}(2a_3)}{\Upsilon_{\text{R}}(a-Q)\Upsilon_{\text{R}}(a_{1+2-3})\Upsilon_{\text{R}}(a_{2+3-1})\Upsilon_{\text{R}}(a_{3+1-2})}. \end{aligned} \quad (5.7)$$

In the formulae (5.5), (5.7) we separated the multiplier in the first line from the rest because this is the only one which is not invariant under $a_1 \rightarrow Q - a_1$. This gives the possibility to compute the reflection coefficient relating

$$\begin{aligned} V_a &= R(a)V_{Q-a}, \quad W_a = R(a)W_{Q-a}, \\ R(a) &= \left(\pi\mu\gamma(b^2)\right)^{\frac{Q-2a}{b}} b^{-2} \gamma(2ab-b^2)\gamma(2ab^{-1}-b^{-2}-1). \end{aligned}$$

The one-point function of V_a in infinite volume for super sinh-Gordon is invariant under both reflections (5.3), hence it satisfies

$$F(a) = F(a-Q)R(a).$$

The operators

$$\hat{V}_a = \frac{1}{F(a)}V_a, \quad \hat{W}_a = \frac{1}{F(a)}W_a,$$

are invariant under both reflections. For our goals we do not need $F(a)$ but rather the ratio $f(a) = \frac{F(a-b)}{F(a)}$ for which

$$f(a-Q) = f(a)\frac{R(a)}{R(a-b)}.$$

We compute and rewrite the result in a useful for us way

$$\frac{R(a)}{R(a-b)} = \left(\frac{1}{2}\pi\mu\gamma\left(\frac{1}{2}bQ\right)\right)^{-2} \frac{\gamma\left(\frac{1}{2} + \frac{1}{2}b(2a-b)\right)}{\gamma\left(\frac{1}{2} + \frac{1}{2}b(2(a-b)-b)\right)}.$$

This equality implies

$$f(a) = C(b) \left(\frac{1}{2} \pi \mu \gamma \left(\frac{1}{2} b Q \right) \right)^{\frac{2}{bQ} (\Delta_a - b + \frac{1}{2} - \Delta_a)} \gamma \left(\frac{1}{2} b (Q - 2a) \right), \quad (5.8)$$

where $C(b)$ is a constant depending on b only. To finish the consideration of primary fields let us we give the expression for the ratio

$$\frac{\tilde{C}(a-b, Q/2+k, Q/2-k)}{C(a, Q/2+k, Q/2-k)} = \left(\frac{1}{2} \pi \mu \gamma \left(\frac{1}{2} b Q \right) b^{-1} \right) \frac{\gamma^2 \left(\frac{1}{2} (1+ab-b^2) \right) \gamma \left(\frac{1}{2} b (Q-2a) \right)}{\gamma(ab-b^2)} \quad (5.9)$$

$$\times \gamma \left(\frac{1}{2} (1-b^2+ab) + bk \right) \gamma \left(\frac{1}{2} (1-b^2+ab) - bk \right).$$

Divide (5.9) by $f(a)$ (5.8) (there is an important cancellation) and change the variables by (5.2) and

$$bk = \beta P,$$

after some simplification this gives (4.15).

5.2. Descendants

Here we find it more convenient to begin with the one-point functions for CFT on the cylinder. This is not absolutely trivial for the descendants. In the case of Super CFT, one should consider both descendants created by the stress energy tensor T and by the super current S . Denote

$$\langle \mathbf{l}_{-n_1} \dots \mathbf{l}_{-n_p} \mathbf{s}_{-r_1} \dots \mathbf{s}_{-r_q} V_a(y) \rangle = \frac{\langle \Delta_- | \mathbf{l}_{-n_1} \dots \mathbf{l}_{-n_p} \mathbf{s}_{-r_1} \dots \mathbf{s}_{-r_q} V_a(y) | \Delta_+ \rangle}{\langle \Delta_- | V_a(y) | \Delta_+ \rangle}, \quad (5.10)$$

where Δ_{\pm} are primary states, located at the extremities $\pm\infty$ of the cylinder, and $\mathbf{l}_m, \mathbf{s}_r$ are defined in (2.8). The main idea is to follow the route of [15], where Ward-Takahashi identities have been used to obtain the values of the same kind of correlation functions but containing purely Virasoro generators. Using Ward-Takahashi identities to express the correlation functions $\langle T(z_1) \dots T(z_p) S(w_1) \dots S(w_q) V_a(y) \rangle$ we can then obtain (5.10) by a successive application of (2.8) :

$$\langle \mathbf{l}_{-n_1} \dots \mathbf{l}_{-n_p} \mathbf{s}_{-r_1} \dots \mathbf{s}_{-r_q} V_a \rangle = \oint_{c_{z_1}} dz_1 \dots \oint_{c_{z_p}} dz_p \oint_{c_{w_1}} dw_1 \dots \oint_{c_{w_q}} dw_q \quad (5.11)$$

$$\times \langle T(z_1) \dots T(z_p) S(w_1) \dots S(w_q) V_a(y) \rangle,$$

with the notation:

$$\oint_{c_{z_k}} dz_k = \oint_{c_{z_k}} \frac{dz_k}{2\pi i (z_k - y)^{n_k - 1}}, \quad \oint_{c_{w_j}} dw_j = \oint_{c_{w_j}} \frac{dw_j}{2\pi i (w_j - y)^{r_j - \frac{1}{2}}}, \quad (5.12)$$

and the contours being small concentric circles around the point y : $c_{z_1} \subset \dots \subset c_{w_q}$. Using the OPEs (2.5), (2.7), we can deduce the following simple correlation functions between the fields (we present here only the specific identities that we shall need later) :

$$\langle S(x) V_a(y) \rangle = 0,$$

$$\langle S(x) W_a(y) \rangle = (2\Delta_a \xi'(x-y) - \xi(x-y)(\Delta_+ - \Delta_-)) \langle V_a(y) \rangle,$$

$$\langle S(x_2) S(x_1) V_a(y) \rangle = -\xi(x_1 - y) \langle S(x_2) W_a(y) \rangle - 2\xi(x_1 - x_2) \langle T(x_2) V_a(y) \rangle +$$

$$- \frac{c}{3} \xi''(x_1 - x_2) \langle V_a(y) \rangle.$$

And the more complicated :

$$\begin{aligned}
& \langle T(x_3)S(x_2)S(x_1)V_a(y) \rangle = \\
& \left(-\frac{3}{2}\chi'(x_3 - x_2) + (\chi(x_3 - x_2) - \chi(x_3 - y))\frac{\partial}{\partial x_2} \right) \langle S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-\frac{3}{2}\chi'(x_3 - x_1) + (\chi(x_3 - x_1) - \chi(x_3 - y))\frac{\partial}{\partial x_1} \right) \langle S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-\Delta_a\chi'(x_3 - y) + \chi(x_3 - y)(\Delta_+ - \Delta_-) + \left(\frac{\Delta_+ + \Delta_-}{2} - \frac{c}{24} \right) \right) \langle S(x_2)S(x_1)V_a(y) \rangle , \\
& \langle T(x_4)T(x_3)S(x_2)S(x_1)V_a(y) \rangle = \\
& - \frac{c}{12}\chi'''(x_4 - x_3) \langle S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-2\chi'(x_4 - x_3) + (\chi(x_4 - x_3) - \chi(x_4 - y))\frac{\partial}{\partial x_3} \right) \langle T(x_3)S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-\frac{3}{2}\chi'(x_4 - x_2) + (\chi(x_4 - x_2) - \chi(x_4 - y))\frac{\partial}{\partial x_2} \right) \langle T(x_3)S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-\frac{3}{2}\chi'(x_4 - x_1) + (\chi(x_4 - x_1) - \chi(x_4 - y))\frac{\partial}{\partial x_1} \right) \langle T(x_3)S(x_2)S(x_1)V_a(y) \rangle \\
& + \left(-\Delta_a\chi'(x_4 - y) + \chi(x_4 - y)(\Delta_+ - \Delta_-) + \left(\frac{\Delta_+ + \Delta_-}{2} - \frac{c}{24} \right) \right) \langle T(x_3)S(x_2)S(x_1)V_a(y) \rangle
\end{aligned}$$

As has been explained we apply this formulae in the case of equal boundary conditions $\Delta_+ = \Delta_- = \delta_P$. The calculation of one point functions of descendants on the cylinder are then given by the application of (5.11). As examples we present the results at level 2

$$\langle \mathbf{1}_{-2}V_a \rangle = \delta_P - \frac{c}{24} - \frac{\Delta_a}{12}, \quad \langle \mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}V_a \rangle = \frac{\Delta_a}{12}, \quad (5.13)$$

and at level 4 :

$$\begin{aligned}
\langle \mathbf{1}_{-2}^2V_a \rangle &= \left(\delta_P - \frac{c}{24} \right)^2 - \frac{1}{6} \left(\delta_P - \frac{c}{24} \right) - \frac{\Delta_a}{6} \left(\delta_P - \frac{c}{24} \right) + \frac{\Delta_a^2}{144} + \frac{7}{360}\Delta_a, \quad (5.14) \\
\langle \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}V_a \rangle &= -\Delta_a \frac{7}{960}, \quad \langle \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}V_a \rangle = -\frac{1}{12} \left(\delta_P - \frac{c}{24} \right) + \Delta_a \frac{17}{960} + \frac{7c}{2880}, \\
\langle \mathbf{1}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}} \rangle &= \frac{\Delta_a}{12} \left(\delta_P - \frac{c}{24} - \frac{\delta_a}{12} \right) - \frac{1}{144}\Delta_a, \quad \langle \mathbf{1}_{-4}V_a \rangle = \frac{\Delta_a}{240}.
\end{aligned}$$

We also will need the one point functions at level 6. Since the results are quite long, we prefer to display them in due time.

5.3. Super Virasoro and super Heisenberg algebras

We would like to have an independent check of the results (4.9) - (4.12). In order to do so, we should interpret the expressions obtained for $\beta_{2m-1}^*\gamma_{2m-1}^*V_a$ as decompositions of the fermionic operators on the Super Virasoro basis, and check that this decomposition is compatible with the reflection relations. As has been explained above and is clear from the interpretation of the reflections, it is first important to make the connection between

the Super-Virasoro and the Super-Heisenberg algebras, that is to construct the passage matrix $U(a)$. This is our goal in this subsection.

The expression of the stress energy tensor and the super current in terms of the fields in the action (5.1) are given by :

$$\begin{aligned} T(z) &= -\frac{1}{4}(\partial_z \varphi)^2 + \frac{Q}{2\sqrt{2}}\partial^2 \varphi - \frac{1}{2}\psi\partial\psi, \\ S(z) &= i \left(\frac{1}{\sqrt{2}}\psi\partial\varphi - Q\partial\psi \right). \end{aligned}$$

In order to exhibit the Heisenberg basis, we split the field $\varphi(z, \bar{z}) = \phi(z) + \phi(\bar{z})$ in chiral parts and expand in modes :

$$\phi(z) = \phi_0 - 2i\pi_0 + i \sum_{k \in \mathbb{Z}^*} \frac{a_k}{k} z^{-k},$$

where the Heisenberg algebra is :

$$[a_k, a_l] = 2k\delta_{k,-l}, \quad [\phi_0, \pi_0] = i. \quad (5.15)$$

The same analysis holds for the fermionic field

$$\psi(z) = \sum_{r \in \mathbb{Z}} b_{r+\frac{1}{2}} z^{-r-1},$$

with the fermionic algebra defined by :

$$\{b_r, b_s\} = \delta_{r,-s}. \quad (5.16)$$

We will call the combination of (5.15) and (5.16) the super Heisenberg algebra (together with the commutation relation $[a_k, b_r] = 0$). The primary field $e^{\frac{a}{\sqrt{2}}\phi(0)}$ is identified with the highest weight vector of the super Heisenberg algebra :

$$e^{\frac{a}{\sqrt{2}}\phi(0)} \iff e^{\frac{a}{\sqrt{2}}\phi_0} |0\rangle, \quad a_k |0\rangle = b_r |0\rangle = 0, \quad k, r > 0$$

In the general case we should then take :

$$V_a = e^{\frac{a}{\sqrt{2}}(\phi_0 + \bar{\phi}_0)} |0\rangle \otimes \overline{|0\rangle}. \quad (5.17)$$

The calculation for the two chiralities being independent, we will work only with the holomorphic one. We can now introduce the generators of the super Virasoro algebra :

$$\begin{aligned} \mathbf{l}_m &= \frac{1}{4} \sum_{k \neq 0, m} : a_k a_{m-k} : + (\pi_0^2 + i\pi_0 \frac{Q}{\sqrt{2}}) \delta_{m,0} + \\ & (\pi_0 + i \frac{Q}{2\sqrt{2}}(m+1)) a_m (1 - \delta_{m,0}) + \frac{1}{2} \sum_{k \in \tilde{\mathbb{Z}}} : b_{m-k} b_k : (k + \frac{1}{2}), \end{aligned}$$

and the modes of the super current :

$$\mathbf{s}_r = \frac{1}{\sqrt{2}} \sum_{k \in \tilde{\mathbb{Z}'}} b_k a_{r-k} + \left(\sqrt{2}\pi_0 + iQ(r + \frac{1}{2}) \right) b_r.$$

Here the symbol $: \dots :$ means normal order. These generators satisfy the super Virasoro algebra

$$\begin{aligned} [\mathbf{l}_m, \mathbf{l}_n] &= (m - n)\mathbf{l}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n}, \\ \{\mathbf{s}_r, \mathbf{s}_s\} &= 2\mathbf{l}_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}, \end{aligned}$$

with $c = \frac{3}{2}(1 + 2Q^2)$, and since S is a primary field of conformal dimension $\Delta = \frac{3}{2}$ we also have the relation :

$$[\mathbf{l}_m, \mathbf{s}_r] = \left(\frac{m}{2} - r\right)\mathbf{s}_{m+r}.$$

Finally, the natural identity holds :

$$\mathbf{l}_0 V_a = \Delta_a V_a, \quad \Delta_a = \frac{1}{2}a(Q - a).$$

We are now ready to compute the passage matrix between the Super-Virasoro and the super Heisenberg basis. Recall that we work modulo the action of local integrals of motion. For our calculations (up to level 6), the integrals of motion that will be involved are just the first two given by the densities (2.11). Explicitly :

$$\mathbf{i}_1 = \mathbf{l}_{-1}, \tag{5.18}$$

$$\mathbf{i}_3 = 2 \sum_{k=-1}^{\infty} \mathbf{l}_{-3-k} \mathbf{l}_k + \frac{1}{2} \sum_{k=-\frac{1}{2}}^{\infty} \mathbf{s}_{-3-k} \mathbf{s}_k \left(k + \frac{3}{2}\right). \tag{5.19}$$

5.3.1. Level 2.

At level 2 there is only one integral of motion to take into account :

$$\mathbf{i}_1^2 V_a = \mathbf{l}_{-1}^2 V_a = 0.$$

We define $U^{(2)}$ to be the passage matrix between the base $\{\mathbf{l}_{-2}, \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}}\}$ and $\{a_{-1}^2, b_{-\frac{3}{2}} b_{-\frac{1}{2}}\}$ which is found to be :

$$U^{(2)} = \begin{pmatrix} \frac{1}{4}(2a^2 + Qa + 1) & \frac{1}{2} \\ \frac{a^2}{2} & -a(a + Q) \end{pmatrix}.$$

Its determinant factorises and gives as expected the null-vector conditions :

$$\det(U^{(2)}) = -\frac{1}{4}a(2a + b + b^{-1})(a + b)(a + b^{-1}).$$

5.3.2. Level 4.

At this level there are 10 operators in total, but working modulo integrals of motion (in this case also only \mathbf{i}_1) we need to keep only 5 of them, that we choose to be

$$\mathbf{l}_{-2}^2, \quad \mathbf{l}_{-4}, \quad \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{l}_{-2} \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}}.$$

On the other hand, we select the following operators to describe the states at level 4 from the super Heisenberg algebra point of view :

$$a_{-2}^2, \quad a_{-3} a_{-1}, \quad b_{-\frac{7}{2}} b_{-\frac{1}{2}}, \quad b_{-\frac{5}{2}} b_{-\frac{3}{2}}, \quad a_{-1}^2 b_{-\frac{3}{2}} b_{-\frac{1}{2}}.$$

We find for the matrix $U^{(4)}$:

$$U^{(4)} = \begin{pmatrix} U_{11}^{(4)} & U_{12}^{(4)} & \frac{29a+12Q}{4a} & \frac{5a+4Q}{4a} & \frac{1}{4}(2a^2+Qa+1) \\ \frac{1}{4} & \frac{1}{12}(2a^2+3Qa+6) & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{a^2}{6} & -a^2-3Qa+3 & 1 & \frac{a^2}{2} \\ 0 & \frac{1}{6}(a^2+Qa+3) & -\frac{4(2a+3Q)}{a} & U_{44}^{(4)} & \frac{1}{2}(-a^2-Qa+1) \\ \frac{1}{4}(-2a^2-Qa-1) & \frac{1}{6}(2a^2-3) & U_{53}^{(4)} & U_{54}^{(4)} & U_{55}^{(4)} \end{pmatrix}, \quad (5.20)$$

where the lengthiest coefficients are :

$$\begin{aligned} U_{11}^{(4)} &= -\frac{4a^4 + 4Qa^3 + Q^2a^2 + 4a^2 + 2Qa + 3}{8a^2}, & U_{12}^{(4)} &= \frac{4a^4 + 2Qa^3 - 6a^2 - 6Qa - 9}{12a^2}, \\ U_{44}^{(4)} &= -\frac{a^3 + 3Qa^2 + 2Q^2a + 3a + 4Q}{a}, & U_{53}^{(4)} &= \frac{1}{2}(-17a^2 - 23Qa - 6Q^2 + 6), \\ U_{54}^{(4)} &= \frac{1}{2}(-a^2 - 3Qa - 2Q^2 + 2), & U_{55}^{(4)} &= -\frac{1}{4}a(2a^3 + 3Qa^2 + Q^2a + Q). \end{aligned}$$

Its determinant can be factorised :

$$\det(U^{(4)}) = \frac{1}{384} \frac{D_V^{(4)}(\Delta, c)}{D_H^{(4)}(a^2, Q^2)} N^{(4)}(a, b).$$

The contribution from the null vectors is :

$$\begin{aligned} N^{(4)}(a, b) &= a^4(a+b)^2(a+b^{-1})^2(a+2b)(a+2b^{-1})(a+3b)(a+3b^{-1}) \\ &\times (2a+b+b^{-1})(2a+b+3b^{-1})(2a+3b+b^{-1}), \end{aligned} \quad (5.21)$$

and we have :

$$D_V^{(4)}(\Delta, c) = 1, \quad D_H^{(4)}(a^2, Q^2) = a^2. \quad (5.22)$$

5.3.3. Level 6.

We proceed through the same analysis. At level 6 we will need to factor out the action of both \mathbf{i}_1 and \mathbf{i}_3 . There are 28 Virasoro operators at level 6, but the factorisation of the action of the integrals of motion leaves only 10, that we choose to be :

$$\begin{aligned} \mathbf{l}_{-2}^3, \quad \mathbf{l}_{-6}, \quad \mathbf{l}_{-3}^2, \quad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}}, \quad \mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}}, \\ \mathbf{l}_{-2}^2\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}. \end{aligned}$$

These are expressed on the super Heisenberg basis :

$$\begin{aligned} a_{-1}^6, \quad a_{-1}^4 a_{-2}, \quad a_{-3}^2, \quad b_{-\frac{7}{2}} b_{-\frac{5}{2}}, \quad b_{-\frac{9}{2}} b_{-\frac{3}{2}}, \quad b_{-\frac{11}{2}} b_{-\frac{1}{2}}, \\ a_{-1} a_{-2} b_{-\frac{5}{2}} b_{-\frac{1}{2}}, \quad a_{-1}^2 b_{-\frac{7}{2}} b_{-\frac{1}{2}}, \quad a_{-1} a_{-3} b_{-\frac{3}{2}} b_{-\frac{1}{2}}, \quad a_{-1}^2 b_{-\frac{5}{2}} b_{-\frac{1}{2}}. \end{aligned}$$

The passage matrix $U^{(6)}$ is too big to be presented here, but we can give its determinant :

$$\det(U^{(6)}) = -\frac{1}{212336640} N^{(6)}(a, b) \frac{D_V^{(6)}(\Delta, c)}{D_H^{(6)}(a^2, Q^2)}. \quad (5.23)$$

with :

$$\begin{aligned}
N^{(6)}(a, b) &= a^2(a+b)^5(a+b^{-1})^5(a+2b)^2(a+3b)^2(a+2b^{-1})^2(a+3b^{-1})^2 \\
&\times (a+4b)(a+5b)(a+4b^{-1})(a+5b^{-1}) \\
&\times (a+b+b^{-1})(2a+b+b^{-1})^5(2a+b+3b^{-1})^2(2a+3b+b^{-1})^2 \\
&\times (2a+5b+b^{-1})(2a+b+5b^{-1}),
\end{aligned} \tag{5.24}$$

the null vector contribution, and

$$D_H^{(6)}(a^2, Q^2) = a^2(-15 + 3a^2 - 10Q^2), \quad D_V^{(6)}(\Delta, c) = 1. \tag{5.25}$$

5.4. Reflections relations

We claim that similarly to [8], the action of both reflections σ_1 and σ_2 implies that the fermions transform as :

$$\beta_{2j-1}^* \rightarrow \gamma_{2j-1}^*, \quad \gamma_{2j-1}^* \rightarrow \beta_{2j-1}^*. \tag{5.26}$$

This means that we can use the coefficients (4.8) to redefine the elements of the fermionic basis and obtain purely CFT objects :

$$\beta_{2m-1}^* = D_{2m-1}(a)\beta_{2m-1}^{\text{CFT}*}, \quad \gamma_{2m-1}^* = D_{2m-1}(Q-a)\gamma_{2m-1}^{\text{CFT}*}. \tag{5.27}$$

For $\beta_{2m-1}^{\text{CFT}*}$ and $\gamma_{2m-1}^{\text{CFT}*}$ we have clear transformation rules under $\sigma_{1,2}$. As in the non-super symmetric case for σ_2

$$\beta_{2m-1}^{\text{CFT}*} \rightarrow \gamma_{2m-1}^{\text{CFT}*}, \quad \gamma_{2m-1}^{\text{CFT}*} \rightarrow \beta_{2m-1}^{\text{CFT}*}. \tag{5.28}$$

For σ_1 we must consider an additional term coming from the change in the passage from $D_{2m-1}(a)$ to $D_{2m-1}(Q-a)$:

$$D_{2m-1}(Q-a) = D_{2m-1}(-a) \left(\frac{a - (2m-1)b^{-1}}{a + (2m-1)b} \right), \tag{5.29}$$

which implies

$$\begin{aligned}
\beta_{2m-1}^{\text{CFT}*} &\rightarrow \left(\frac{a - (2m-1)b}{a + (2m-1)b^{-1}} \right) \gamma_{2m-1}^{\text{CFT}*}, \\
\gamma_{2m-1}^{\text{CFT}*} &\rightarrow \left(\frac{a - (2m-1)b^{-1}}{a + (2m-1)b} \right) \beta_{2m-1}^{\text{CFT}*}.
\end{aligned} \tag{5.30}$$

The main conclusion drawn from Section 4.2, is that the fermionic basis should be decomposable on the super Virasoro basis in the following way :

$$\beta_{I^+}^{\text{CFT}*} \gamma_{I^-}^{\text{CFT}*} V_a = C_{I^+, I^-} (P_{I^+, I^-}^E(\{\mathbf{1}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) + d_a P_{I^+, I^-}^O(\{\mathbf{1}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c)) V_a, \tag{5.31}$$

where C_{I^+, I^-} is the Cauchy determinant and d_a is the function (4.13) rewritten in the variables a, b :

$$d_a = \frac{1}{8} \sqrt{(9 - \widehat{c})(16\Delta_\alpha + 1 - \widehat{c})} = \frac{1}{4} (b - b^{-1})(Q - 2a). \tag{5.32}$$

The functions P_{I^+,I^-}^E and P_{I^+,I^-}^O (E, O subscripts stand respectively for even and odd) are polynomials in the modes of the super Virasoro algebra, depending rationally on the parameters Δ, c . They are defined modulo the local integrals \mathbf{i}_{2k-1} and satisfy the symmetry relations :

$$P_{I^+,I^-}^E = P_{I^-,I^+}^E, \quad P_{I^+,I^-}^O = -P_{I^-,I^+}^O. \quad (5.33)$$

The decomposition (5.31), as well as the transformation rules (5.28) and (5.30), imply a relation of the type

$$\begin{aligned} \beta_{I^+}^{\text{CFT}^*} \gamma_{I^-}^{\text{CFT}^*} V_a &= C_{I^+,I^-} \prod_{2j-1 \in I^+} (a + (2j-1)b^{-1}) \prod_{2j-1 \in I^-} (a + (2j-1)b) \\ &\times (Q_{I^+,I^-}^E(\{a_{-2k}, b_{-r}\}, b_r, a^2, Q^2) + g_a Q_{I^+,I^-}^O(\{a_{-2k}, b_{-r}\}, a^2, Q^2)) V_a, \end{aligned} \quad (5.34)$$

with

$$g_a = a(b - b^{-1}),$$

and $Q_{I^+,I^-}^E, Q_{I^+,I^-}^O$ polynomials in the super Heisenberg algebra, depending rationally on a^2 and Q^2 . In the following we are going to verify this conjecture level by level.

Level 2

Let us start with the simplest case of level 2 :

$$\beta_1^{\text{CFT}^*} \gamma_1^{\text{CFT}^*} V_a = \Omega_{1,1} = P^2 - \frac{1}{16} - \frac{\Delta_a}{8}. \quad (5.35)$$

On this level only two operators \mathbf{l}_{-2} and $\mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}}$ are present. The calculation of one point functions on the cylinder was explained in Subsection 5.2 and gave in this case (5.13):

$$\langle \mathbf{l}_{-2} V_a \rangle = \delta_P - \frac{c}{24} - \frac{\Delta_a}{12}, \quad \langle \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}} \rangle = \frac{\Delta_a}{12}, \quad (5.36)$$

Hence it is not difficult to compare with (5.35) to obtain :

$$\beta_1^{\text{CFT}^*} \gamma_1^{\text{CFT}^*} V_a = \left(\mathbf{l}_{-2} - \frac{1}{2} \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}} \right) V_a. \quad (5.37)$$

Using (5.20), one can rewrite the combination (5.37) as :

$$\beta_1^{\text{CFT}^*} \gamma_1^{\text{CFT}^*} V_a = \frac{1}{4} (a+b)(a+b^{-1}) \left(a_{-1}^2 + 2b_{-\frac{3}{2}} b_{-\frac{1}{2}} \right) V_a. \quad (5.38)$$

This neat factorisation of the term $(a+b)(a+b^{-1})$ is a check of our conjecture, and the above shows that : $Q_{\{1,1\}}^E = \frac{1}{4} \left((a_{-1})^2 + 2b_{-\frac{3}{2}} b_{-\frac{1}{2}} \right)$.

The main difference with the usual Liouville case, is that at higher levels, we do not know a priori the decompositions of the type (5.31) (recall the discussion at the end of the Section 4.2). To overcome this difficulty, we shall proceed as in [8] and obtain the decomposition by solving the reflection constraints implied by (5.34). Let us briefly recall the main steps.

Consider that at an (even) level $k = |I^+| + |I^-|$ we have a basis of super Virasoro generators $\{\mathbf{v}_1^{(k)}, \dots, \mathbf{v}_d^{(k)}\}$ (by convention we consider that $\mathbf{v}_1^{(k)} = \mathbf{I}_{-2}^{\frac{k}{2}}$) that are related to the super Heisenberg basis $\{\mathbf{h}_1^{(k)}, \dots, \mathbf{h}_d^{(k)}\}$ modulo the action of integrals of motions by :

$$\mathbf{v}_i^{(k)} = \sum_{k=1}^d U_{i,j}^{(k)}(a) \mathbf{h}_j^{(k)},$$

with $U^{(k)}(a)$ the passage matrix, whose determinant is factorisable :

$$\det(U^{(k)}(a)) = C^{(k)} N^{(k)}(a, b) \frac{D_V^{(k)}(\Delta, c)}{D_H^{(k)}(a^2, Q^2)}, \quad (5.39)$$

where $N^{(k)}(a, b)$ is the null vector contribution. We look for $P_{I^+, I^-}^E, P_{I^+, I^-}^O$ in the form :

$$P_{I^+, I^-}^E = \mathbf{v}_1 + \frac{1}{D_V^{(k)}(\Delta, c)} \sum_{i=2}^d X_{I^+, I^-, i}(\Delta, c) \mathbf{v}_i,$$

$$P_{I^+, I^-}^O = \frac{1}{D_V^{(k)}(\Delta, c)} \sum_{i=2}^d Y_{I^+, I^-, i}(\Delta, c) \mathbf{v}_i,$$

where $X_{I^+, I^-, i}(\Delta, c), Y_{I^+, I^-, i}(\Delta, c)$ are polynomials of some degree D to be determined. Also introduce the polynomials :

$$T_{I^+ I^-}^+(a) = \frac{1}{2} \left(\prod_{j \in I^+} (a + jb^{-1}) \prod_{j \in I^-} (a + jb) + \prod_{j \in I^+} (a + jb) \prod_{j \in I^-} (a + jb^{-1}) \right),$$

$$T_{I^+ I^-}^-(a) = \frac{1}{2(b - b^{-1})} \left(\prod_{j \in I^+} (a + jb^{-1}) \prod_{j \in I^-} (a + jb) - \prod_{j \in I^+} (a + jb) \prod_{j \in I^-} (a + jb^{-1}) \right).$$

Then (5.34) gives strong conditions on the structure of $X_{I^+, I^-, i}(\Delta, c), Y_{I^+, I^-, i}(\Delta, c)$ (see [8] for details). For any $1 \leq j \leq d$ we must have

$$D_V^{(k)}(\Delta(-a), c) D_H^{(k)}(a^2, Q^2) \quad (5.40)$$

$$\times \{T_{I^+ I^-}^+(-a) \left(D_V^{(k)}(\Delta, c) U_{1,j}^{(k)}(a) + \sum_{i=2}^d X_{I^+, I^-, i} U_{i,j}^{(k)}(a) \right)$$

$$- (Q^2 - 4)(Q - 2a) T_{I^+ I^-}^-(-a) \sum_{i=2}^d Y_{I^+, I^-, i} U_{i,j}^{(k)}(a) \} \quad \text{is even in } a,$$

and

$$D_V^{(k)}(\Delta(-a), c) D_H^{(k)}(a^2, Q^2) \quad (5.41)$$

$$\times \{-T_{I^+ I^-}^-(-a) \left(D_V^{(k)}(\Delta, c) U_{1,j}^{(k)}(a) + \sum_{i=2}^d X_{I^+, I^-, i} U_{i,j}^{(k)}(a) \right)$$

$$+ (Q^2 - 4)(Q - 2a) T_{I^+ I^-}^+(-a) \sum_{i=2}^d Y_{I^+, I^-, i} U_{i,j}^{(k)}(a) \} \quad \text{is odd in } a,$$

Taking the degree D appropriately large, we obtain enough linear equations on the coefficients of $X_{I^+, I^-, i}(\Delta, c)$, $Y_{I^+, I^-, i}(\Delta, c)$. Now we demonstrate how this procedure works at higher levels.

Level 4

Consider the set up described in 5.3.2. Recall that at this level there are 5 operators in total (modulo the action of \mathbf{i}_1), that are :

$$\mathbf{l}_{-2}^2, \quad \mathbf{l}_{-4}, \quad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}.$$

We solve the constraints (5.40) and (5.41) with the use of (5.20) and (5.22), and obtain the following expressions :

$$\begin{aligned} P_{\{1,3\}}^E &= \mathbf{l}_{-2}^2 + \left(\frac{-45 + 4c}{18} - \frac{\Delta_a}{3} \right) \mathbf{l}_{-4} + \left(\frac{45 - 4c}{36} + \frac{\Delta_a}{6} \right) \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}} + \\ &\quad \frac{1}{4}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}} - \frac{1}{2}\mathbf{l}_{-2}\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}, \\ P_{\{1,3\}}^O &= \frac{1}{3}\mathbf{l}_{-4} - \frac{1}{6}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}, \end{aligned}$$

as well as the mirror polynomials $P_{\{3,1\}}^E, P_{\{3,1\}}^O$. One can now compute the one point function of $\left(P_{\{1,3\}}^E \mp d_a P_{\{1,3\}}^O \right) V_a$ (all the individual contributions of descendants at level 4 are given in (5.14)) recovering exactly the value of $\Omega_{1,3}^{\substack{3,1}}$ obtained in (4.10) by interpolation. Summarising :

$$\beta_{\substack{1 \\ 3}}^{\text{CFT}^*} \gamma_{\substack{3 \\ 1}}^{\text{CFT}^*} V_a = \frac{1}{2} \left(P_{\{1,3\}}^E(\{\mathbf{l}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) \mp d_a P_{\{1,3\}}^O(\{\mathbf{l}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) \right) V_a. \quad (5.42)$$

This is an independent argument in favor of (4.10).

Level 6

We proceed through the same analysis. Recall that we have (modulo the action of \mathbf{i}_1 and \mathbf{i}_3) 10 Virasoro operators, that we took to be

$$\begin{aligned} \mathbf{l}_{-2}^3, \quad \mathbf{l}_{-6}, \quad \mathbf{l}_{-3}^2, \quad \mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{5}{2}}, \quad \mathbf{s}_{-\frac{9}{2}}\mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{s}_{-\frac{11}{2}}\mathbf{s}_{-\frac{1}{2}}, \\ \mathbf{l}_{-2}^2\mathbf{s}_{-\frac{3}{2}}\mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{7}{2}}\mathbf{s}_{-\frac{1}{2}}, \quad \mathbf{l}_{-2}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{3}{2}}, \quad \mathbf{l}_{-3}\mathbf{s}_{-\frac{5}{2}}\mathbf{s}_{-\frac{1}{2}}. \end{aligned}$$

Using the explicit value of $U^{(6)}$ and the factors (5.25), the reflection constraints bring the following results :

$$\begin{aligned}
P_{\{3,3\}}^E &= \mathbf{l}_{-2}^3 + \frac{1}{480} (572\Delta_a^2 + 1976\Delta_a - 80c^2 - 96c\Delta_a + 2076c - 18381) \mathbf{l}_{-6} + \\
&\frac{1}{96} (12\Delta_a^2 + 228\Delta_a - 16c\Delta_a - 12c - 27) \mathbf{l}_{-3}^2 + \\
&\frac{1}{192} (-28\Delta_a^2 + 192\Delta_a - 16c\Delta_a - 20c + 117) \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{5}{2}} + \\
&\frac{1}{64} (-4\Delta_a^2 + 92\Delta_a - 8c\Delta_a - 8c + 105) \mathbf{s}_{-\frac{9}{2}} \mathbf{s}_{-\frac{3}{2}} + \\
&\frac{1}{960} (28\Delta_a^2 + 404\Delta_a + 56c\Delta_a - 136c + 6021) \mathbf{s}_{-\frac{11}{2}} \mathbf{s}_{-\frac{1}{2}} + \\
&-\frac{1}{2} \mathbf{l}_{-2}^2 \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}} + \frac{1}{12} (4\Delta_a - 2c + 27) \mathbf{l}_{-2} \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{1}{2}} + \\
&\frac{1}{16} (9 - 2\Delta_a) \mathbf{l}_{-2} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{3}{2}} + \frac{1}{192} (4\Delta_a^2 - 68\Delta_a + 8c\Delta_a - 8c + 291) \mathbf{l}_{-3} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{1}{2}},
\end{aligned}$$

as well as

$$\begin{aligned}
P_{\{1,5\}}^E &= \mathbf{l}_{-2}^3 + \frac{1}{90} (79\Delta_a^2 + 1004\Delta_a - 12c^2 - 98c\Delta_a + 322c - 2855) \mathbf{l}_{-6} + \\
&\frac{1}{12} (2\Delta_a^2 + 12\Delta_a - 2c\Delta_a - c - 2) \mathbf{l}_{-3}^2 + \frac{1}{36} (\Delta_a^2 + 7\Delta_a - 2c\Delta_a - 3c + 18) \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{5}{2}} + \\
&\frac{1}{8} (2\Delta_a - c + 11) \mathbf{s}_{-\frac{9}{2}} \mathbf{s}_{-\frac{3}{2}} + \frac{1}{180} (-19\Delta_a^2 - 804\Delta_a + 2c^2 + 88c\Delta_a - 72c + 1315) \mathbf{s}_{-\frac{11}{2}} \mathbf{s}_{-\frac{1}{2}} + \\
&-\frac{1}{2} \mathbf{l}_{-2}^2 \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}} + \frac{1}{9} (2\Delta_a - c + 14) \mathbf{l}_{-2} \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{1}{2}} + \\
&\frac{1}{2} \mathbf{l}_{-2} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{3}{2}} + \frac{1}{36} (-\Delta_a^2 + \Delta_a + 2c\Delta_a - c + 38) \mathbf{l}_{-3} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{1}{2}}.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
P_{\{1,5\}}^O &= \frac{1}{30} (-136\Delta_a - 12c + 335) \mathbf{l}_{-6} + -\Delta_a \mathbf{l}_{-3}^2 + \frac{1}{12} (-4\Delta_a - 3) \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{5}{2}} + \\
&-\frac{3}{4} \mathbf{s}_{-\frac{9}{2}} \mathbf{s}_{-\frac{3}{2}} + \frac{1}{60} (76\Delta_a - 8c - 115) \mathbf{s}_{-\frac{11}{2}} \mathbf{s}_{-\frac{1}{2}} + -\frac{2}{3} \mathbf{l}_{-2} \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{1}{2}} + \frac{1}{12} (4\Delta_a - 5) \mathbf{l}_{-3} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{1}{2}}.
\end{aligned}$$

We also find the same expressions of the polynomials $P_{\{5,1\}}^E$ and $P_{\{5,1\}}^O$ (up to a relevant minus sign for $P_{\{5,1\}}^O$). Then one can proceed and calculate the relevant one point functions of descendants on the cylinder (see Subsection 5.2). We summarize here the results :

$$\begin{aligned}
\langle \mathbf{l}_{-6} V_a \rangle &= -\frac{\Delta_a}{6048}, & \langle \mathbf{l}_{-3}^2 V_a \rangle &= \frac{72\Delta_a + 31c - 504\delta_P}{30240}, \\
\langle \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{5}{2}} V_a \rangle &= \frac{604\Delta_a + 457c - 3528\delta_P}{483840}, & \langle \mathbf{s}_{-\frac{9}{2}} \mathbf{s}_{-\frac{3}{2}} V_a \rangle &= \frac{-1371\Delta_a - 457c + 3528\delta_P}{1451520}, \\
\langle \mathbf{s}_{-\frac{11}{2}} \mathbf{s}_{-\frac{1}{2}} V_a \rangle &= \frac{31\Delta_a}{96768}, & \langle \mathbf{l}_{-2} \mathbf{s}_{-\frac{7}{2}} \mathbf{s}_{-\frac{1}{2}} V_a \rangle &= \frac{294\Delta_a^2 + 1252\Delta_a + 147c\Delta_a - 3528\Delta_a\delta_P}{483840}, \\
\langle \mathbf{l}_{-3} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{1}{2}} V_a \rangle &= \frac{17\Delta_a}{60480},
\end{aligned}$$

as well as the most complex results :

$$\begin{aligned}
\langle \mathbf{1}_{-2}^3 V_a \rangle &= \frac{1}{483840} \left(-280\Delta_a^3 - 2352\Delta_a^2 - 3968\Delta_a - 35c^3 - 210c^2\Delta_a + 2520c^2\right. \\
&\delta_P - 462c^2 - 420c\Delta_a^2 - 2100c\Delta_a + 10080c\Delta_a\delta_P - 60480c\delta_P^2 + 21168c\delta_P - 1504c + \\
&10080\Delta_a^2\delta_P - 120960\Delta_a\delta_P^2 + 48384\Delta_a\delta_P + 483840\delta_P^3 - 241920\delta_P^2 + 32256\delta_P \left. \right) \\
\langle \mathbf{1}_{-2}^2 \mathbf{s}_{-\frac{3}{2}} \mathbf{s}_{-\frac{1}{2}} V_a \rangle &= \frac{\Delta_a}{241920} \left(140\Delta_a^2 + 672\Delta_a + 35c^2 + 140c\Delta_a - 1680c\delta_P + \right. \\
&294c - 3360\Delta_a\delta_P + 20160\delta_P^2 - 6720\delta_P + 544 \left. \right), \\
\langle \mathbf{1}_{-2} \mathbf{s}_{-\frac{5}{2}} \mathbf{s}_{-\frac{3}{2}} V_a \rangle &= \frac{1}{483840} \left(-714\Delta_a^2 - 3588\Delta_a - 119c^2 - \right. \\
&595c\Delta_a + 4536c\delta_P - 1196c + 11928\Delta_a\delta_P - 40320\delta_P^2 + 18144\delta_P \left. \right).
\end{aligned}$$

Using these values for the one point functions, we recover exactly the expressions (4.11) and (4.12). That is we check that :

$$\begin{aligned}
\beta_5^{\text{CFT}*} \gamma_5^{\text{CFT}*} V_a &= \frac{1}{3} \left(P_{\substack{\{1,5\} \\ \{5,1\}}}^E (\{\mathbf{1}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) \mp d_a P_{\substack{\{1,5\} \\ \{5,1\}}}^O (\{\mathbf{1}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) \right) V_a, \\
\beta_3^{\text{CFT}*} \gamma_3^{\text{CFT}*} V_a &= \frac{1}{3} P_{\{3,3\}}^E (\{\mathbf{1}_{-2k}, \mathbf{s}_{-r}\}, \Delta, c) V_a.
\end{aligned}$$

This strongly confirms the results obtained by interpolation.

6. Conclusion

The achievement of this work is the computation of the one point functions of fermionic operators in the ssG model. They are constructed out of a single function Ω , defined by a set of scaling equations, and which origin is traced to the computation of vacuum expectation values of lattice operators on the underlying 19-vertex model. On one hand, the analysis of the scaling equations in the conformal regime allowed to compute the one point functions of specific fermionic operators in the UV limit, and to establish the correspondence between the usual Virasoro description of CFT and the fermionic part of the fermion-current description. On the other hand, these results have been checked by an alternative method that relies on the reflection symmetry of the ssG model. We emphasize again that both techniques completely differ in their nature and are both based on conjectures. The matching of the results from both sides is a very strong assertion for both of them.

Concerning the primary fields notice that we have obtained the most important for applications quantity. Indeed, we argued that the simplest non-chiral fermionic descendant provides the ratio of one point functions of the operators $W_{\alpha + \frac{2\beta^2}{1-\beta^2}}$ and V_α . The former operator is exactly the most relevant contribution occurring in the OPE of the latter one with the perturbing operator $W_{\frac{2\beta^2}{1-\beta^2}}$. In other words the ratio of one-point functions in question provides the most important contribution to the conformal perturbation theory.

We need to consider the entire space of local operators adding those created by the KM currents. The one-point functions of the latter include the function $\omega(\theta, \theta')$. Recall the equation (3.18). Using this equation and known $\Omega(\theta, \theta')$ one can, in principle, reconstruct

$\omega(\theta, \theta')$. The result is not unique, one has to find a way of fixing the quasi-constants (anti-periodic with period πi functions of θ, θ'). Following this numerically it is hard to achieve a good precision which makes it difficult to put forward a conjecture based on the interpolation. This is a technical difficulty which we hope to overcome in future.

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