



HAL
open science

Stability of the Laughlin phase against long-range interactions

Alessandro Olgiati, Nicolas Rougerie

► **To cite this version:**

Alessandro Olgiati, Nicolas Rougerie. Stability of the Laughlin phase against long-range interactions. 2019. hal-02153307v1

HAL Id: hal-02153307

<https://hal.science/hal-02153307v1>

Preprint submitted on 12 Jun 2019 (v1), last revised 2 Mar 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STABILITY OF THE LAUGHLIN PHASE AGAINST LONG-RANGE INTERACTIONS

ALESSANDRO OLGIATI AND NICOLAS ROUGERIE

ABSTRACT. A natural, “perturbative”, problem in the modelization of the fractional quantum Hall effect is to minimize a classical energy functional within a variational set based on Laughlin’s wave-function. We prove that, for small enough pair interactions, and asymptotically for large particle numbers, a minimizer can always be looked for in the particular form of uncorrelated quasi-holes superimposed to Laughlin’s wave-function.

CONTENTS

1. Introduction	1
2. Setting and theorems	4
2.1. Main results	4
2.2. Remarks	7
2.3. Sketch of proof	8
3. Solid-phase minimizers of the flocking problem	10
3.1. Existence of minimizers	10
3.2. Solid phase	12
3.3. Slightly perturbed problems	14
4. Energy lower bound	18
5. Energy upper bound	23
6. Convergence of densities	28
6.1. The Hewitt-Savage and Diaconis-Freedman Theorems	28
6.2. Energy estimates on the support of the limit measure	30
6.3. Incompressibility estimate for the limit measure and proof of Corollary 2.4 ..	32
References	35

1. INTRODUCTION

The fractional quantum Hall effect (FQHE) occurs when a gas of interacting electrons is confined to two space dimensions and subjected to a strong external magnetic field, perpendicular to the confinement plane [14, 11, 12, 37, 17]. Under these conditions, the Hall resistance exhibits a quantized behavior by forming a series of plateaus in correspondence

to particular rational values of the filling factor¹

$$\nu = \frac{hc}{e_0} \frac{\rho}{B}.$$

While the integer quantum Hall effect (IQHE, integer values of the filling factor) can, in first approximation, be understood in terms of non-interacting electrons, the origin of the FQHE (special fractional values of the filling factor) lies in inter-particle interactions. It was first recognized by Laughlin [15, 16] that the origin of the FQHE is a very remarkable strongly correlated phase that the gas exhibits in its ground state, for certain special values of ν . It was then proposed, and later confirmed experimentally [36, 6, 26] that within this phase the gas presents quasiparticle excitations with fractional charge. They are also believed to have fractional statistics [1, 25].

At the basis of our theoretical understanding of the FQHE is Laughlin's wave-function which, for a system with filling factor $\nu = 1/\ell$ with $\ell \in \mathbb{N}$, and subjected to an external magnetic field B , reads

$$\Psi_{\text{Lau}}(z_1, \dots, z_N) = c_{\text{Lau}} \prod_{i < j} (z_i - z_j)^\ell e^{-B \sum_{i=1}^N |z_i|^2/4}. \quad (1.1)$$

The variables z_1, \dots, z_N are complex coordinates of N particles in \mathbb{R}^2 , which is here identified with the complex plane, and c_{Lau} is a L^2 -normalization factor. Ψ_{Lau} was proposed as a variational trial wave function for the ground state of the many-body Hamiltonian²

$$H_N^{\text{QM}} = \sum_{i=1}^N \left[\left(-i\nabla_{x_j} - \frac{B}{2} x_j^\perp \right)^2 + V(x_j) \right] + \lambda \sum_{i < j} W(x_i - x_j) \quad (1.2)$$

acting on the Hilbert space $L^2(\mathbb{R}^{2N})$. Our assumptions on the external (describing trapping and/or the effect of impurities in the sample) potential $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ and on the interaction $W : \mathbb{R}^2 \rightarrow \mathbb{R}$ will be specified later, together with the role of the coupling constant λ .

The argument leading to (1.1) is based on the following two requirements:

- For a very strong external magnetic field B , the leading scale is the magnetic kinetic energy. Hence, it is energetically convenient for each particle to occupy the lowest Landau level (LLL), i.e. the ground eigenspace of the magnetic Laplacian. This allows to restrict the action of H_N^{QM} from $L^2(\mathbb{R}^{2N})$ to $\otimes^N \mathfrak{H}$ where

$$\mathfrak{H} = \{ \psi(z) = f(z) e^{-B|z|^2/4} \in L^2(\mathbb{R}^2) \mid f \text{ analytic} \}. \quad (1.3)$$

- Once the kinetic energy has been fixed by the reduction to the LLL, one assumes that the next leading scale is given by the interaction W . If it is repulsive and strong enough at short range, then it forces the wave function to vanish whenever two particles meet. In order for this prescription to be compatible with (1.3) the behavior for $z_i \sim z_j$ must be $\sim (z_i - z_j)^\ell$ for some³ integer ℓ .

¹ ρ is the electron density, B the magnetic field, the rest universal constants.

²Spins are assumed to all be polarized by the magnetic field.

³Notice that the parity of the integer ℓ allows to differentiate the statistics of particles, namely, a Laughlin wave function with even ℓ describes bosons, while odd values of ℓ correspond to fermions.

We remark that these prescriptions to derive (1.1) actually yield the larger space

$$\mathcal{L}_{\ell,B}^N = \left\{ \Psi_F = F(z_1, \dots, z_N) \Psi_{\text{Lau}}(z_1, \dots, z_N) \in L^2(\mathbb{R}^{2N}) \mid F \text{ analytic and symmetric} \right\}. \quad (1.4)$$

Indeed, any $\Psi_F \in \mathcal{L}_{\ell,B}^N$ belongs to $\bigotimes^N \mathfrak{H}$ and vanishes at least as $(z_i - z_j)^\ell$ when $z_i \sim z_j$. We shall refer to F as a correlation factor.

The energy of a generic Ψ_F is (we subtract the magnetic kinetic energy, which is just a constant within the LLL)

$$\mathcal{E}_{N,\lambda}[\Psi_F] = \left\langle \Psi_F \left| \sum_{j=1}^N V(x_j) + \lambda \sum_{i<j} W(x_i - x_j) \right| \Psi_F \right\rangle \quad (1.5)$$

and we define ($B > 0$ and ℓ a positive integer will be fixed throughout the paper)

$$E(N, \lambda) = \inf \left\{ \mathcal{E}_{N,\lambda}[\Psi_F] \mid \Psi_F \in \mathcal{L}_{\ell,B}^N, \int_{\mathbb{R}^{2N}} |\Psi_F|^2 = 1 \right\}. \quad (1.6)$$

The motivation for the above variational problem is as follows: assuming that the two previous requirements are indeed imposed by the energy scales of the problem, what kind of ground state does the system actually choose, within the huge space (1.4)? This has a flavor of degenerate perturbation theory, where one looks for the lowest eigenstate of a perturbed Hamiltonian, within the ground eigenspace of the unperturbed part.

In Laughlin's theory, his wave-function (1.1) plays the role of a new "vacuum". Crucial to the arguments is the nature of the excitations of this vacuum, and their quasi-particle nature. It is argued that such quasi-particles excitations can be described by wave-functions of the form

$$\Psi_f(z_1, \dots, z_N) = \prod_{j=1}^N f(z_j) \Psi_{\text{Lau}}(z_1, \dots, z_N) \quad (1.7)$$

where f is a polynomial in a single variable. The function Ψ_f clearly vanishes in correspondence of the zeroes of the polynomial f . This is interpreted as having added a quasi-hole [15, 16] on top of the Laughlin wave function, the latter being interpreted as the vacuum state for quasi-holes. These quasi-holes carry a fractional charge corresponding to $1/\ell$ of the electron's charge, and they are expected to behave as anyons [1, 25] with statistics parameter $-1/\ell$. The aim of this work is to show that the energy (1.6) can be reached (asymptotically, for large particle numbers) by restricting to the above sub-class, consistently with the physical picture of quasi-holes generation.

We define the minimal energy within this class as

$$e(N, \lambda) = \inf \left\{ \mathcal{E}_{N,\lambda}[\Psi_f] \mid \Psi_f \text{ of the form (1.7)}, \int_{\mathbb{R}^{2N}} |\Psi_f|^2 = 1 \right\}. \quad (1.8)$$

Notice that, clearly,

$$E(N, \lambda) \leq e(N, \lambda), \quad (1.9)$$

since $e(N, \lambda)$ is the minimal energy in a smaller domain than the one corresponding to $E(N, \lambda)$. We aim at proving that actually

$$\boxed{E(N, \lambda) \simeq e(N, \lambda) \quad \text{as } N \rightarrow \infty \text{ with } \lambda \text{ fixed.}} \quad (1.10)$$

This means that it is never favorable to add correlations through F in order to minimize the external potential energy and the interaction energy. Furthermore, the lowest energy is attainable with uncorrelated quasi-holes exciting the Laughlin wave function.

In previous papers [22, 23, 35], the case $\lambda = 0$ was considered, i.e. it was assumed that restricting the variational set to (1.4) rendered the interaction energy negligible compared to the potential energy. There (1.10) was proved for external potentials varying on the characteristic scale of the Laughlin function (a technical assumption that we shall also make, see below for further discussion).

In the present contribution we consider the effect of W , and prove (1.10) for suitably scaled potentials V and W and a sufficiently small coupling constant λ . This means that the Laughlin phase (perturbation of the Laughlin function by uncorrelated quasi-holes, as in (1.7)) is stable against both external potentials and the long-range part of the interaction (the short-range part is supposed to have been taken into account by restricting to (1.4)).

Acknowledgments: We thank Jakob Yngvason for useful discussions. Funding from the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No 758620) is gratefully acknowledged.

2. SETTING AND THEOREMS

2.1. Main results. We now present the precise setting in which we prove (1.10) and discuss further this main result and its corollaries.

For any $\Psi_F \in \mathcal{L}_{\ell, B}^N$ let us introduce the associated reduced densities

$$\rho_F^{(k)}(z_1, \dots, z_k) = \binom{N}{k} \int_{\mathbb{R}^{2(N-k)}} |\Psi_F(z_1, \dots, z_k, z_{k+1}, \dots, z_N)|^2 dz_{k+1} \dots dz_N \quad (2.1)$$

for $k = 1, \dots, N$. Notice that we are choosing the convention

$$\int_{\mathbb{R}^{2k}} \rho_F^{(k)} = \binom{N}{k}.$$

It was proven in [23] (see also [33, 34] for earlier partial results) that the Laughlin fluid satisfies an incompressibility estimate. More precisely, for any $\Psi_F \in \mathcal{L}_{\ell, B}^N$, the associated one-particle density satisfies

$$\rho_F^{(1)}(z) \lesssim \frac{B}{2\pi\ell} \quad (2.2)$$

in a suitable sense of local averages. This is a rather nontrivial rigidity property: no matter how complicated the correlation factor F might be, it cannot locally compress the liquid's density beyond a universal threshold. The proof of (2.2) relies on the *plasma analogy*, a useful mapping of the problem known since Laughlin's work [15, 16]: the many-body density of Ψ_F can be mapped to a classical Hamiltonian for N particles in 2D. Within this framework, a generic F , due to its analyticity, plays the role of the potential generated by a *positive* charge distribution (see [23, Section 2.3]). Since the N classical particles are also (by convention) positively charged in the plasma analogy, one can expect that the overall effect of F is to *decrease* the density $\rho_F^{(1)}$ with respect to the case $F = 1$. The analogon of (2.2) for this latter case had already been argued [15], and then rigorously derived [31, 32].

We remark that the Laughlin phase naturally occupies a length scale of order $N^{1/2}$, as can easily be seen by combining $\int \rho_F^{(1)} = N$ with (2.2). We then define new densities with rescaled lengths according to

$$\mu_F^{(k)}(x_1, \dots, x_k) = \frac{N^k}{\binom{N}{k}} \rho_F^{(k)}(\sqrt{N}x_1, \dots, \sqrt{N}x_k). \quad (2.3)$$

Note that we set

$$\int_{\mathbb{R}^{2k}} \mu_F^{(k)} = 1.$$

We also introduce rescaled potentials v and w through

$$v(x) = V(\sqrt{N}x) \quad (2.4)$$

and

$$\frac{w(x)}{N} = W(\sqrt{N}x). \quad (2.5)$$

From now on we will only work in the scaled variables $x = z/\sqrt{N}$. The notation we introduced allows to bring (1.5) to the form

$$\mathcal{E}_{N,\lambda}[\Psi_F] = N \left[\int_{\mathbb{R}^2} v(x) \mu_F^{(1)}(x) dx + \frac{\lambda}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu_F^{(2)}(x,y) dx dy \right]. \quad (2.6)$$

The definition of w in (2.5) effectively puts us in the mean-field regime, by making the two contributions to (2.6) formally of the same order. Recall that the reduced many-body ground state energy is

$$E(N, \lambda) = \inf \left\{ \mathcal{E}_{N,\lambda}[\Psi_F] \mid \Psi_F \in \mathcal{L}_{\ell,B}^N, \int_{\mathbb{R}^{2N}} |\Psi_F|^2 = 1 \right\}$$

and the energy within states with uncorrelated quasi-holes is

$$e(N, \lambda) = \inf \left\{ \mathcal{E}_{N,\lambda}[\Psi_f] \mid \Psi_f \text{ of the form (1.7)}, \int_{\mathbb{R}^{2N}} |\Psi_f|^2 = 1 \right\}.$$

As customary in the study of asymptotic properties of many-body systems, a crucial role is played by the associated effective mean-field model. For any probability measure μ on \mathbb{R}^2 , define the functional

$$\mathcal{E}^{\text{MF}}[\mu] = \int_{\mathbb{R}^2} v(x) \mu(x) dx + \frac{\lambda}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu(x) \mu(y) dx dy. \quad (2.7)$$

This is the mean-field version of (2.6) (having ignored the N multiplicative factor), in that the two-body density has been replaced by the uncorrelated product of one-body densities.

We will show that the minimal energy attained by \mathcal{E}^{MF} is related to both $E(N, \lambda)$ and $e(N, \lambda)$. However, the mere minimization of \mathcal{E}^{MF} under the sole mass constraint $\int \mu = 1$ would be insensitive of the fact that variational states for $E(N, \lambda)$ and $e(N, \lambda)$ satisfy the incompressibility bound (2.2). For this reason, we define the ‘flocking’ energy as

$$E^{\text{flo}} = \inf \left\{ \mathcal{E}^{\text{MF}}[\mu] \mid 0 \leq \mu \leq \frac{B}{2\pi\ell}, \int_{\mathbb{R}^2} \mu = 1 \right\}. \quad (2.8)$$

The upper constraint is precisely the one in (2.2). We use the word ‘flocking’, since (2.8) is the minimal energy for a functional where confining and repulsive terms compete, and

there is an overall constraint on the maximal density. This mechanism resembles the one occurring in models for bird flocks.

Minimization problems similar to (2.8) have been studied in [5] and [10], although within a slightly different setting, namely when $v = |\cdot|^p * \mu$ for some $p > 0$, hence with a dependence on μ . Depending on the parameters of the problem, the behavior of a minimizer μ_m can exhibit three phases:

- ‘Liquid phase’: $|\{\mu_m = B/2\pi\ell\}| = 0$,
- ‘Intermediate phase’: $|\{0 < \mu_m < B/2\pi\ell\}| > 0$ and $|\{\mu_m = B/2\pi\ell\}| > 0$,
- ‘Solid phase’: $|\{0 < \mu_m < B/2\pi\ell\}| = 0$.

It was proven in [5] and [10] that, for $v = |\cdot|^p * \mu$ and for suitable values of p , the value of the mass constraint $\int \mu = M$ discriminates between the phases: for small enough M any minimizer is in the liquid phase, while for large enough M any minimizer is in the solid phase. In our setting, the parameter which governs the behavior of minimizers is the coupling constant. The flocking energy (2.8) with $\lambda = 0$ coincides with the classical *bathtub* energy [21, Theorem 1.14], namely the minimal possible energy in an external potential achievable by densities of fixed mass with upper and lower constraints. It is well-known that for this problem the minimizers are always in the solid phase, and saturate the upper constraint almost everywhere on their support.

We now present our assumptions on the potentials v and w .

Asumption 2.1 (The external potential).

There exists a fixed function $v \in C^2(\mathbb{R}^2, \mathbb{R}^+)$ such that

$$V(z) = v\left(\frac{z}{\sqrt{N}}\right).$$

We assume that v has a finite number of critical points, all of which are non-degenerate. We also assume that v has polynomial growth, i.e.,

$$|x|^s \leq v(x) \leq |x|^t \quad \text{for } |x| \text{ large enough} \quad (2.9)$$

for some $t \geq s \geq 1$.

Asumption 2.2 (The interaction potential).

There exists a fixed function $w \in W^{2,\infty}(\mathbb{R}^2, \mathbb{R})$ (two bounded derivatives) such that

$$W(z) = \frac{1}{N} w\left(\frac{z}{\sqrt{N}}\right).$$

Our main result is the following:

Theorem 2.3 (Energy of the Laughlin phase).

For V and W satisfying the assumptions presented above there exists $\lambda_0 > 0$ such that, for $|\lambda| \leq \lambda_0$,

$$\lim_{N \rightarrow \infty} \frac{E(N, \lambda)}{e(N, \lambda)} = 1. \quad (2.10)$$

From the proof we can deduce properties of (quasi-)minimizers:

Corollary 2.4 (Convergence of densities).

Let F be a (sequence of) correlation factors such that the associated $\Psi_F \in \mathcal{L}_{\ell, B}^N$ satisfy

$$\mathcal{E}_N[\Psi_F] = E(N, \lambda) + o(N) \quad (2.11)$$

as $N \rightarrow \infty$. Assume $|\lambda| \leq \lambda_0$ as in Theorem 2.3. Then there exists a probability measure P supported on the set of minimizers of the flocking problem (2.8) such that, for every $k \in \mathbb{N}$,

$$\mu_F^{(k)} \rightharpoonup \int_{\{\mu \mid \mathcal{E}^{\text{MF}}[\mu] = E^{\text{flo}}\}} \mu^{\otimes k} dP(\mu)$$

weakly as probability measures as $N \rightarrow \infty$.

2.2. Remarks. As explained before, the scaling we chose in (2.5) is motivated by the fact that it renders potential and interaction energies of the same order of magnitude, so that Theorem 2.3 is not a perturbative statement. On the other hand, the regularity assumptions on the data of the problem are mostly of a technical nature. In this section we further discuss these aspects.

Scaling of the problem. Although our Assumptions 2.1-2.2 do not cover it (see below), it is instructive to think of the original, unscaled potentials as

$$W(x) = \frac{1}{|x|}, \quad V(x) = -\rho_{\text{ext}} \star |\cdot|^{-1}$$

i.e. the interactions are electrostatic⁴ and the external potential is generated by a fixed density of charge $-\rho_{\text{ext}}$. Since the system lives on a thermodynamic length scale $\propto \sqrt{N}$ it is natural to scale ρ_{ext} as

$$\rho_{\text{ext}} = \mu_{\text{ext}}(N^{-1/2}x).$$

A scaling of length units, using (2.3), then gives, for these choices,

$$\mathcal{E}_{N,\lambda}[\Psi_F] = N^{3/2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x-y|} \left[-\mu_{\text{ext}}(x) \mu_F^{(1)}(y) + \frac{\lambda}{2} \mu_F^{(2)}(x,y) \right] dx dy. \quad (2.12)$$

The scaling (2.6) used for our main theorem mimics the above, where the two terms are also of the same order.

Link with filling factor. In (2.12), increasing λ is equivalent to decreasing μ_{ext} . Neutrality (or almost-neutrality) considerations for the full system (ρ^{ext} minus the electrons' charge) make it natural to expect (or demand) that the latter be in relation with the electron density, so that increasing λ (decreasing μ_{ext}) is related to decreasing the filling factor. From this point of view it is important that the system stays of the form ‘‘Laughlin plus quasi-holes’’ for $|\lambda| \leq \lambda_0$. This is one of the ingredients of Laughlin’s explanation of the quantization of the Hall conductivity for filling factors close to $1/n$ (and not just equal to $1/n$). This is indeed what we prove, in a simplified model at least. It is also natural to expect the restriction to small (but independent of N) values of λ to be necessary: for larger values/lower filling factors, the system should form another FQH ground state (e.g. a Laughlin state with higher exponent), corresponding to another plateau of the Hall conductivity.

Singular interactions. The main restrictive assumption we make is the smoothness of w in Assumption 2.2. This is certainly not satisfied in the Coulomb case we just described. There are a few reasons why our main results are nevertheless relevant for 2D electron gases:

⁴Recall that we consider 3D electrons confined to 2D, so the Coulomb potential should be the 3D one.

- The finite (although small) thickness of the gas in the direction perpendicular to the plane yields a smoothing of the Coulomb interaction's singularity at short distances, see e.g. the discussion in [14, Section 2.2].
- It is likely that the restriction to the lowest Landau level (a fortiori the emergence of Laughlin's function) is valid only on length-scales large compared to the interparticle distance. At short distances, specific correlations could be formed to avoid the singular part of the interaction. See [20] for results in this direction in a related situation.
- The Jastrow factors in (1.1) should kill the singularity of the true Coulomb interaction, allowing to replace it by an effective, smoother, potential W .

In any event, as indicated in our title, our main concern is the stability of the Laughlin phase against the long-range part of the interaction, which is certainly present in the coulombic one, but neglected in the derivation of the ansatz. See [24] for mathematical results on 2D electron gases with the true Coulomb interaction.

Perturbations on mesoscopic length scales. For simplicity we consider a model with potentials scaled in such a way that they affect the shape of the electron droplet on macroscopic length scales, comparable to the size of the full system. Physically it would be relevant to allow for modifications on much smaller length scales (with the limitation that they should still be much larger than the typical interparticle distance). See [35, Section 5] for comments in this direction. The main bottleneck for a mathematical analysis of such situations is to improve the incompressibility estimates of [23]. For the pure Laughlin state $F = 1$, the results of [18, 19, 3, 2] go in this direction.

2.3. Sketch of proof. Let us now list the main ingredients of the proofs of Theorem 2.3 and Corollary 2.4. There are three main steps:

- The incompressibility estimate (2.2), which holds for any correlation factor F , suggests that the flocking energy (2.8) is a lower bound to the full energy $E(N, \lambda)$.
- For small enough λ we prove that the flocking minimizers are in the solid phase, i.e. their density takes only the values 0 or $B/(2\pi\ell)$.
- For any solid flocking minimizer⁵ μ_m we can construct a recovery sequence of "Laughlin plus quasi-holes" wave functions of the form (1.7) whose (rescaled) density converges to μ_m .

The first point shows that $E(N, \lambda) \gtrsim NE^{\text{flo}}$. The second and third taken together show that, for λ small enough, $e(N, \lambda) \lesssim NE^{\text{flo}}$. Since obviously $E(N, \lambda) \leq e(N, \lambda)$ this gives the scheme of proof for Theorem 2.3. Corollary 2.4 requires some more tools, in particular the classical de Finetti-Hewitt-Savage theorem [13], and in fact its constructive proof due to Diaconis and Freedman [7].

The first point is the most important. In fact, we need not only justify (2.2) in a rigorous way, but also understand why a mean-field approximation is valid for a lower bound, leading to the flocking energy (2.8). The key is to prove that the empirical density of a state of the form (1.4) satisfies (2.2) with large probability. This is an improvement over the incompressibility estimates derived in [22, 23]. The latter only gave a bound in average, as stated informally in (2.2). We supplement this information with deviations estimate in Theorem 4.1 below and deduce the

⁵Actually, any function taking only the values 0 and $B/(2\pi\ell)$

Theorem 2.5 (Lower bound to the many-body energy).

Suppose Assumptions 2.1 and 2.2 hold. For any Ψ_F , we have

$$\mathcal{E}_N[\Psi_F] \geqslant NE^{\text{flo}}(1 - CN^{(-3+\sqrt{5})/4+\gamma}) \quad (2.13)$$

for any $\gamma > 0$.

The second point of the overall strategy, concerning the flocking minimizers, is reminiscent of results from [5, 10]. We shall prove the following:

Theorem 2.6 (Solid phase of the flocking problem).

Suppose Assumptions 2.1 and 2.2 hold. Then:

- (i) There exists a minimizer for the problem (2.8).
- (ii) There exists $\lambda_0 > 0$ such that, for any $|\lambda| \leqslant \lambda_0$, any minimizer μ_m is in the solid phase, i.e.

$$\mu_m = \frac{B}{2\pi\ell} \mathbb{1}_\Sigma \quad a.e.$$

for some open set $\Sigma \subset \mathbb{R}^2$.

Theorem 2.6 is proven in Section 3. The main idea is that on any patch of liquid phase $0 < \mu_m < B/(2\pi\ell)$, the external potential and the mean-field potential must for optimality add up to a constant. But for small enough λ the variations of the mean-field potential cannot possibly cancel those of the external potential, because of the flocking constraint.

The third point of the overall strategy is

Theorem 2.7 (Upper bound to the many-body energy).

Let μ^{sol} be any fixed probability measure on \mathbb{R}^2 in the solid phase, i.e., whose only values are zero and $B/2\pi\ell$. Then there exists (a sequence of) polynomial(s) f_δ , indexed by a positive parameter $\delta \xrightarrow{N \rightarrow \infty} 0$, such that

$$e(N, \lambda) \leqslant \mathcal{E}_N[\Psi_{f_\delta}] \leqslant N\mathcal{E}^{\text{MF}}[\mu^{\text{sol}}] \left(1 + CN^{-1/4+\gamma}\right). \quad (2.14)$$

for any $\gamma > 0$.

Basically the above follows by adapting the methods of [35], but we have to handle in addition the two-body term of the energy, absent in the aforementioned reference. We remark that requiring μ^{sol} to be in the solid phase is crucial. Indeed, essentially any trial state of the form (1.7) constructed from uncorrelated quasi-holes leads to a density of such a form.

We will prove (2.13) in Section 4 and (2.14) in Section 5. We explain here how combining Theorems 2.5-2.6-2.7 yields the proof of Theorem 2.3. Corollary 2.4 requires other ingredients, provided in Section 6.

Proof of Theorem 2.3. First, recall that $E(N, \lambda) \leqslant e(N, \lambda)$ as already stated in (1.9). Moreover, under the assumptions of Theorem 2.3, there exists by Theorem 2.6 a minimizer μ_m of the flocking problem in the solid phase. Hence, we are allowed to choose $\mu^{\text{sol}} = \mu_m$ in (2.14), and we have

$$\mathcal{E}^{\text{MF}}[\mu^{\text{sol}}] = \mathcal{E}^{\text{MF}}[\mu_m] = E^{\text{flo}}$$

By combining (2.13) and (2.14) we deduce

$$NE^{\text{flo}}(1 - CN^{(-3+\sqrt{5})/4+\gamma}) \leq E(N, \lambda) \leq e(N, \lambda) \leq NE^{\text{flo}}(1 + CN^{-1/4+\gamma}), \quad (2.15)$$

which proves Theorem 2.3. \square

3. SOLID-PHASE MINIMIZERS OF THE FLOCKING PROBLEM

This Section is devoted to the proof of Theorem 2.6. Recall that we are considering the functional

$$\mathcal{E}^{\text{MF}}[\mu] = \int_{\mathbb{R}^2} v(x)\mu(x)dx + \frac{\lambda}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y)\mu(x)\mu(y)dxdy$$

on the variational set

$$\mathcal{M} = \left\{ \mu \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \mid 0 \leq \mu \leq \frac{B}{2\pi\ell}, \int_{\mathbb{R}^2} \mu = 1 \right\}$$

and the minimization problem

$$E^{\text{flo}} = \inf \{ \mathcal{E}^{\text{MF}}[\mu] \mid \mu \in \mathcal{M} \}.$$

3.1. Existence of minimizers.

Lemma 3.1 (Boundedness from below).

For any $\mu \in \mathcal{M}$ we have

$$\mathcal{E}^{\text{MF}}[\mu] \geq -C$$

for a constant C depending only on w .

Proof. Simply use $v \geq 0$, $w \geq -C$, and $\int \mu = 1$. \square

We recall that a sequence of probability measures $(\mu_n)_{n \in \mathbb{N}}$ converges weakly- \star to a Radon measure μ if

$$\int_{\mathbb{R}^2} f \mu_n \rightarrow \int_{\mathbb{R}^2} f \mu \quad (3.1)$$

for any continuous f with compact support.

Lemma 3.2 (Lower semicontinuity).

For any sequence μ_n of probability measures converging weakly- \star to μ , with $\int \mu = 1$, we have

$$\liminf_{n \rightarrow \infty} \mathcal{E}^{\text{MF}}[\mu_n] \geq \mathcal{E}^{\text{MF}}[\mu].$$

Proof. Recall that, by Assumption 2.2, w is lower semicontinuous and bounded. Thus, there exists $K \geq 0$ such that $w + K$ is lower semicontinuous and positive, and therefore there exists an *increasing* sequence of continuous functions $(w_m)_{m \in \mathbb{N}}$ such that $w_m(x, y)$ converges pointwise to $w(x - y) + K$ and $w_m(x, y)$ has compact support in $\mathbb{R}^2 \times \mathbb{R}^2$. We can thus write

$$\liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y)\mu_n(x)\mu_n(y)dxdy \geq \liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w_m(x,y)\mu_n(x)\mu_n(y)dxdy - K.$$

Since w_m is positive the weak- \star convergence of μ_n implies (by Fatou's lemma)

$$\liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y)\mu_n(x)\mu_n(y)dxdy \geq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w_m(x,y)\mu(x)\mu(y)dxdy - K.$$

We can finally take the limit $m \rightarrow \infty$ by monotone convergence and deduce, using also $\int \mu = 1$,

$$\liminf_{n \rightarrow \infty} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu_n(x) \mu_n(y) dx dy \geq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu(x) \mu(y) dx dy.$$

For the summand of \mathcal{E}^{MF} containing v we proceed in the same way, by recalling that v is continuous and positive by assumption. \square

Let us present the proof of existence of minimizers.

Proof of Theorem 2.6, Item (i). Consider a minimizing sequence $(\mu_n)_{n \in \mathbb{N}}$. We extract (without relabeling) a subsequence converging weakly- \star as Radon measure to a candidate minimizer μ_∞ . Lemma 3.2 implies

$$\mathcal{E}^{\text{MF}}[\mu_\infty] \leq \inf\{\mathcal{E}^{\text{MF}}[\mu] \mid \mu \in \mathcal{M}\}, \quad (3.2)$$

showing that μ_∞ is a minimizer if it belongs to \mathcal{M} . The bounds $0 \leq \mu_\infty \leq B/2\pi\ell$ are easily deduced, and since the L^1 -norm is lower semicontinuous, we only need prove

$$\int \mu_\infty \geq 1.$$

For $R > 0$, define a continuous and positive function

$$\chi_R(x) = \begin{cases} 1 & |x| \leq R \\ 0 & |x| \geq 2R \\ \in [0, 1] & R \leq |x| \leq 2R, \end{cases}$$

and let $\chi_R^c = 1 - \chi_R$. Notice that, by Assumption 2.1, $\lim_{|x| \rightarrow \infty} v(x) = +\infty$. This implies that, for R large enough,

$$\int_{\mathbb{R}^2} \chi_R^c(x) \mu_n(x) dx \leq \frac{1}{v(R)} \int_{\mathbb{R}^2} \chi_R^c(x) v(x) \mu_n(x) dx \leq \frac{1}{v(R)} (\mathcal{E}^{\text{MF}}[\mu_n] + C)$$

having also used boundedness of w . Moreover, since the sequence μ_n is minimizing, we have $\mathcal{E}^{\text{MF}}[\mu_n] = \mathcal{E}^{\text{MF}}[\mu_\infty] + o(1)$ as $n \rightarrow \infty$, and we deduce

$$\int_{\mathbb{R}^2} \chi_R^c(x) \mu_n(x) dx \leq \frac{1}{v(R)} (\mathcal{E}^{\text{MF}}[\mu_\infty] + C). \quad (3.3)$$

Notice also that, by (3.1),

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} \chi_R \mu_n = \int_{\mathbb{R}^2} \chi_R \mu_\infty \quad (3.4)$$

for any $R > 0$. We then write

$$\begin{aligned} \int_{\mathbb{R}^2} \chi_R \mu_\infty - 1 &= \int_{\mathbb{R}^2} \chi_R (\mu_\infty - \mu_n) + \int_{\mathbb{R}^2} \chi_R \mu_n - \int_{\mathbb{R}^2} \mu_n \\ &= \int_{\mathbb{R}^2} \chi_R (\mu_\infty - \mu_n) - \int_{\mathbb{R}^2} \chi_R^c \mu_n \\ &\geq \int_{\mathbb{R}^2} \chi_R (\mu_\infty - \mu_n) - \frac{1}{v(R)} (\mathcal{E}^{\text{MF}}[\mu_\infty] + C). \end{aligned}$$

By (3.4) we can take the limit $n \rightarrow \infty$, and the first term on the right converges to zero. The subsequent limit $R \rightarrow \infty$ yields

$$\int_{\mathbb{R}^2} \mu_\infty \geq 1, \quad (3.5)$$

which shows that μ_∞ is indeed a minimizer. \square

3.2. Solid phase. We now prove that, for small enough coupling constants, the minimizers are in the solid phase. For any $\mu \in \mathcal{M}$, define the total potential

$$\Phi_\mu = v + \lambda w * \mu. \quad (3.6)$$

Clearly Assumption 2.2 implies the

Lemma 3.3 (Bounds on the mean-field potential).

For $w \in W^{2,\infty}(\mathbb{R}^2)$ and $\mu \in L^1(\mathbb{R}^2)$ we have

$$\begin{aligned} \|w * \mu\|_\infty &\leq C \|\mu\|_{L^1} \\ \|\nabla w * \mu\|_\infty &\leq C \|\mu\|_{L^1} \\ \left\| \partial_{\hat{y}}^2 w * \mu \right\|_\infty &\leq C \|\mu\|_{L^1} \end{aligned} \quad (3.7)$$

for any unit vector \hat{y} .

Proof. The statements follow immediately by extracting the L^∞ norms of w , ∇w , and $\partial_{\hat{y}}^2 w$. \square

Lemma 3.4 (Variational inequalities).

Let μ be a local minimizer of \mathcal{E}^{MF} on \mathcal{M} . Then there exists $\gamma \in \mathbb{R}$ such that, for almost every x ,

$$\begin{aligned} \Phi_\mu(x) &\leq \gamma \quad \text{if } \mu(x) = \frac{B}{2\pi\ell} \\ \Phi_\mu(x) &= \gamma \quad \text{if } 0 < \mu(x) < \frac{B}{2\pi\ell} \\ \Phi_\mu(x) &\geq \gamma \quad \text{if } \mu(x) = 0. \end{aligned} \quad (3.8)$$

Proof. See, e.g., [5, Lemma 4.2]. \square

We now complete the proof of Theorem 2.6.

Proof of Theorem 2.6, Item (ii). Pick a global minimizer μ_m and define

$$A := \left\{ x \in \mathbb{R}^2 \mid 0 < \mu_m(x) < \frac{B}{2\pi\ell} \right\}.$$

We will show that A has measure zero for a range of coupling constants. Assume first that A does not accumulate to any critical point of v , i.e., there exists c_1 such that

$$|x - x_k| > c_1 > 0 \quad \forall x \in A,$$

where $(x_k)_k$ is the *finite* set of critical points of v . This implies the existence of c_2 such that

$$|\nabla v(x)| > c_2 > 0 \quad \forall x \in A.$$

Let us assume that A is an open set. Then, since by (3.8) Φ_{μ_m} is constant on A , we can differentiate and obtain

$$\nabla v + \lambda \nabla w * \mu_m = 0 \quad \text{on } A. \quad (3.9)$$

By taking $\lambda'_0 = c_2 / \|\nabla w * \mu_m\|_\infty$ we see that last two formulae are in contradiction for any $\lambda \in [0, \lambda'_0]$ due to

$$c_2 < |\nabla v(x)| = \lambda |\nabla w * \mu_m| \leq \lambda'_0 |\nabla w * \mu_m| \leq \frac{c_2}{\|\nabla w * \mu_m\|_\infty} |\nabla w * \mu_m| \leq c_2.$$

This excludes the possibility of A being open and non-empty, and more generally of A containing an open set.

We still have to rule out the case in which A has positive measure without containing open sets. However, if a function belongs to $W_{\text{loc}}^{1,1}$, then its gradient vanishes almost everywhere on the pre-image of any point (see, e.g., [21, Theorem 6.19] or [10, Proposition 4]). It is easy to see that Φ_{μ_m} belongs to $W_{\text{loc}}^{1,1}$ since $v \in C^2$ and the term containing w is controlled in $W^{1,\infty} \subset W_{\text{loc}}^{1,1}$, using (3.7). Hence we have that (3.9) still holds almost everywhere on A , and we encounter the same contradiction as above for $\lambda \in [0, \lambda'_0]$.

Consider now the case where A might accumulate to critical points $\{x_1, \dots, x_M\}$ of v . The latter are non-degenerate by Assumption 2.1. Then, for $k \in \{1, \dots, M\}$, at the critical point x_k there exists a direction \hat{y}_k along which the second derivative of v at x_k is different from zero. Since $v \in C^2$, this is true in a neighborhood of each x_k , i.e., there exist $c_{3,k}, c_{4,k} > 0$ such that

$$\left| \frac{\partial^2}{\partial \hat{y}_k^2} v(x) \Big|_{x=x_k} \right| > c_{3,k}, \quad \forall x \in D(x_k, c_{4,k}) \quad (3.10)$$

for any $k = 1, \dots, M$, where $D(a, R)$ is the disk of center a and radius R .

Define the set

$$A' = A \setminus \bigcup_{k=1}^M (D(x_k, c_{4,k}) \cap A),$$

i.e. the set A from which we removed a neighborhood of each critical point of v to which A accumulates. On A' ,

$$|x - x_k| > c_4 := \min_k c_{4,k} > 0,$$

which implies $|\nabla v(x)| > c_5 > 0$ for some c_5 . We can then replicate the above argument and deduce that A' has zero measure for $0 \leq \lambda \leq \lambda''_0$ with $\lambda''_0 = c_5 / \|\nabla w * \mu_m\|_\infty$.

On $D_{x_k}(c_{4,k}) \cap A$, in turn, we can differentiate (3.8) twice along the \hat{y}_k direction, thus obtaining

$$\frac{\partial^2}{\partial \hat{y}_k^2} v(x) + \lambda \frac{\partial^2}{\partial \hat{y}_k^2} w * \mu_m = 0. \quad (3.11)$$

This formula is valid almost everywhere even if $D(x_k, c_{4,k}) \cap A$ does not contain an open set since, as above, it only requires $\Phi_{\mu_m} \in W_{\text{loc}}^{2,1}$ (see again [10, Proposition 4]). Notice that $\partial_{\hat{y}_k}^2 \Phi_{\mu_m} \in L_{\text{loc}}^1$ follows because $v \in C^2$ and $\partial_{\hat{y}_k}^2 w * \mu_m \in L^\infty \subset L_{\text{loc}}^1$ due to (3.7). By choosing

$$\lambda'''_{0,k} = \frac{c_{3,k}}{\left\| \frac{\partial^2}{\partial \hat{y}_k^2} w * \mu_m \right\|_\infty}$$

we see that (3.10) and (3.11) are in contradiction for $\lambda \in [0, \lambda_{0,k}''']$ and hence $B_{x_k}(c_{4,k}) \cap A$ must have zero measure. The proof is concluded by taking

$$\lambda_0 = \min \{ \lambda_0', \lambda_0'', \lambda_{0,1}''', \dots, \lambda_{0,M}''' \},$$

because, by what we discussed, A must have zero measure for $\lambda \in [0, \lambda_0]$. Notice that $\lambda_0', \lambda_0'', \lambda_{0,k}''' > 0$, whence $\lambda_0 > 0$. \square

3.3. Slightly perturbed problems. In Section 4 we will need to consider the minimization of \mathcal{E}^{MF} with a perturbed upper constraint. We now prove a result of continuous dependence on such perturbations. Define

$$\mathcal{M}_\varepsilon = \left\{ \mu \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \mid 0 \leq \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell}, \int_{\mathbb{R}^2} \mu = 1 \right\}$$

and

$$E_\varepsilon^{\text{flo}} = \inf \{ \mathcal{E}^{\text{MF}}[\mu] \mid \mu \in \mathcal{M}_\varepsilon \}. \quad (3.12)$$

Lemma 3.5 (Dependence on the flocking constraint).

There exists a constant $C > 0$ such that for any $\varepsilon > 0$ small enough we have

$$E_\varepsilon^{\text{flo}} \leq E^{\text{flo}} \leq E_\varepsilon^{\text{flo}} + C\varepsilon$$

Proof. The inequality

$$E_\varepsilon^{\text{flo}} \leq E^{\text{flo}}$$

is trivial in view of the definition of the variational sets.

To prove the opposite inequality, let us fix a minimizer μ_ε for (3.12). We will construct a suitable trial function for the problem with $\varepsilon = 0$ by starting from μ_ε and removing all the mass that exceeds the upper constraint $B/2\pi\ell$. Define

$$\Omega_\varepsilon = \left\{ \mu_\varepsilon > \frac{B}{2\pi\ell} \right\}$$

Notice that

$$1 = \int_{\mathbb{R}^2} \mu_\varepsilon \geq \int_{\Omega_\varepsilon} \mu_\varepsilon > \frac{B}{2\pi\ell} |\Omega_\varepsilon|$$

and therefore, by defining

$$f_\varepsilon = \mu_\varepsilon - \frac{B}{2\pi\ell},$$

we have

$$\int_{\Omega_\varepsilon} f_\varepsilon \leq |\Omega_\varepsilon| \left((1 + \varepsilon) \frac{B}{2\pi\ell} - \frac{B}{2\pi\ell} \right) \leq \varepsilon.$$

Let K be a measurable compact set such that

$$\mu_\varepsilon \leq \frac{B}{2\pi\ell} - \frac{1}{|K|} \int_{\Omega_\varepsilon} f_\varepsilon$$

on K . Clearly such a set must exist, in view of the normalization of μ_ε . Define

$$\tilde{\mu} = \mu_\varepsilon - \mathbb{1}_{\Omega_\varepsilon} f_\varepsilon + \frac{\mathbb{1}_K}{|K|} \int_{\Omega_\varepsilon} f_\varepsilon.$$

By construction it satisfies

$$\int \tilde{\mu} = 1 - \int_{\Omega_\varepsilon} f_\varepsilon + \int_{\Omega_\varepsilon} f_\varepsilon = 1$$

and

$$\tilde{\mu} = \begin{cases} \mu_\varepsilon + \frac{1_K}{|K|} \int_{\Omega_\varepsilon} f_\varepsilon \leq \frac{B}{2\pi\ell} & \text{on } \mathbb{R}^2 \setminus \Omega_\varepsilon \\ \frac{B}{2\pi\ell} & \text{on } \Omega_\varepsilon \end{cases}$$

It is thus a suitable trial function for the minimization problem with $\varepsilon = 0$. Hence

$$\begin{aligned} E^{\text{flo}} \leq \mathcal{E}^{\text{MF}}[\tilde{\mu}] &= \mathcal{E}^{\text{MF}}[\mu_\varepsilon] - \int_{\Omega_\varepsilon} v f_\varepsilon + \left(\int_{\Omega_\varepsilon} f_\varepsilon \right) \frac{1}{|K|} \int_K v \\ &+ \lambda \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \left[-\mu_\varepsilon(x) f_\varepsilon(y) \mathbb{1}_{\Omega_\varepsilon}(y) + \left(\int_{\Omega_\varepsilon} f_\varepsilon \right) \mu_\varepsilon(x) \frac{\mathbb{1}_K(y)}{|K|} \right. \\ &\left. - \left(\int_{\Omega_\varepsilon} f_\varepsilon \right) f_\varepsilon(x) \mathbb{1}_{\Omega_\varepsilon}(x) \frac{\mathbb{1}_K(y)}{|K|} + 2f_\varepsilon(x) \mathbb{1}_{\Omega_\varepsilon}(x) f_\varepsilon(y) \mathbb{1}_{\Omega_\varepsilon}(y) + 2 \left(\int_{\Omega_\varepsilon} f_\varepsilon \right)^2 \frac{\mathbb{1}_K(x) \mathbb{1}_K(y)}{|K|^2} \right] dx dy. \end{aligned}$$

It is easy to see, using (3.7) for the w -terms, that every summand except for $\mathcal{E}^{\text{MF}}[\mu_\varepsilon]$ can be estimated from above by

$$C \int_{\Omega_\varepsilon} f_\varepsilon \leq C\varepsilon.$$

This implies

$$E^{\text{flo}} \leq \mathcal{E}^{\text{MF}}[\mu_\varepsilon] + C\varepsilon = E_\varepsilon^{\text{flo}} + C\varepsilon,$$

which concludes the proof. \square

For the minimization problem with perturbed upper constraint, we will also need to consider a functional with an external potential which is modified at large distances. For any $L > 0$ let us denote by \mathcal{S}_L the square of side $2L$ centered at the origin,

$$\mathcal{S}_L = [-L, L]^2$$

By our assumption (2.9), if L is large enough, we have $v \geq U_L$ with

$$U_L(x) = \begin{cases} v(x) & x \in \mathcal{S}_L \\ |x| & x \notin \mathcal{S}_L. \end{cases}$$

Let us define

$$\mathcal{E}_L^{\text{MF}}[\mu] = \int U_L(x) \mu(x) dx + \frac{\lambda}{2} \int w(x-y) \mu(x) \mu(y) dx dy.$$

We have the following result, proving that such a modification of the external potential does not play a role. Neither does the removal of the upper constraint at large distances.

Lemma 3.6 (Perturbed problem at large distances).

For L large enough we have

$$\inf \left\{ \mathcal{E}_L^{\text{MF}}[\mu] \mid \mu \geq 0, \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell} \text{ on } \mathcal{S}_L, \int_{\mathbb{R}^2} \mu = 1 \right\} = E_\varepsilon^{\text{flo}}.$$

Proof. We have

$$\begin{aligned} & \inf \left\{ \mathcal{E}_L^{\text{MF}}[\mu] \mid \mu \geq 0, \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell} \text{ on } \mathcal{S}_L, \int_{\mathbb{R}^2} \mu = 1 \right\} \\ & \leq \inf \left\{ \mathcal{E}^{\text{MF}}[\mu] \mid \mu \geq 0, \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell} \text{ on } \mathcal{S}_L, \int_{\mathbb{R}^2} \mu = 1 \right\} \\ & \leq \inf \left\{ \mathcal{E}^{\text{MF}}[\mu] \mid 0 \leq \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell}, \int_{\mathbb{R}^2} \mu = 1 \right\} = E_\varepsilon^{\text{flo}}, \end{aligned}$$

where the first step follows from $U_L \leq v$ and the second one is due to the fact that adding the constraint at large distances reduces the variational set.

Let us prove the opposite inequality. Let $\mu_m^{(L)}$ be a minimizer for the L -dependent problem, i.e.,

$$\inf \left\{ \mathcal{E}_L^{\text{MF}}[\mu] \mid \mu \geq 0, \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell} \text{ on } \mathcal{S}_L, \int_{\mathbb{R}^2} \mu = 1 \right\} = \mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}]. \quad (3.13)$$

The existence of such a minimizer follows by the same arguments as previously. Assume that, for L large enough, the support of $\mu_m^{(L)}$ is entirely contained in \mathcal{S}_L . Then we deduce

$$\mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}] = \mathcal{E}^{\text{MF}}[\mu_m^{(L)}] \geq E_\varepsilon^{\text{flo}},$$

where the first equality is due to $v = U_L$ on \mathcal{S}_L and the second one is the variational principle. This completes the proof in view of (3.13).

We thus aim at proving that, for L large enough, the support of $\mu_m^{(L)}$ is entirely contained in \mathcal{S}_L . Assume for contradiction that, for a sequence of L 's accumulating at $+\infty$, there exists a (sequence of) set(s) Σ_L of positive measure, entirely contained in $\mathbb{R}^2 \setminus \mathcal{S}_L$, and such that

$$\int_{\Sigma_L} \mu_m^{(L)} > 0.$$

We will show that moving mass from Σ_L to the interior of \mathcal{S}_L decreases the energy, hence violating minimality of $\mu_m^{(L)}$. To do so, we first show that $\mu_m^{(L)}$ does not saturate the upper constraint on the whole \mathcal{S}_L . Indeed, if there exists $c \in (0, (1 + \varepsilon)B/2\pi\ell]$ such that the measure of the set $\{\mu_m^{(L)} \geq c\}$ tends to infinity as $L \rightarrow \infty$, then integrability of $\mu_m^{(L)}$ is impossible.

Hence, for every $c \in (0, (1 + \varepsilon)B/2\pi\ell]$,

$$|\{\mu_m^{(L)} \geq c\}| \leq C \quad \text{uniformly in } L. \quad (3.14)$$

Let us fix $c \in (0, (1 + \varepsilon)B/2\pi\ell]$. As a consequence of (3.14), for every L large enough there exists a set R_L with the following properties:

- (1) $\mu_m^{(L)} < c$ on R_L ;
- (2) $|R_L| = C$ independent of L ;
- (3) each R_L is entirely contained in a ball whose radius is a constant independent of L , i.e., there exists $C > 0$ independent of L such that

$$R_L \subset \{x \in \mathbb{R}^2 \mid |x| \leq C\} \quad \forall L. \quad (3.15)$$

The catch is that, even though the set on which $\mu_m^{(L)}$ approximates its upper bound might ‘pulsate’ when the value of L changes, by occupying different regions of \mathcal{S}_L , still, there will

always be a set where $\mu_m^{(L)}$ is far from the upper constraint and this set can always be picked within the same fixed bounded region.

Define the function

$$\tilde{\mu} = \mu_m^{(L)} - \mathbb{1}_{\Sigma_L} \mu_m^{(L)} + \frac{\mathbb{1}_{R_L}}{|R_L|} \int_{\Sigma_L} \mu_m^{(L)}.$$

By definition we have

$$\int_{\mathbb{R}^2} \tilde{\mu} = 1 - \int_{\Sigma_L} \mu_m^{(L)} + \int_{\Sigma_L} \mu_m^{(L)} = 1.$$

Moreover,

$$\tilde{\mu} = \begin{cases} \mu_m^{(L)} & \text{on } \mathbb{R}^2 \setminus (\Sigma_L \cup R_L) \\ 0 & \text{on } \Sigma_L \\ \mu_m^{(L)} + \frac{\mathbb{1}_{R_L}}{|R_L|} \int_{\Sigma_L} \mu_m^{(L)} < c + \frac{\mathbb{1}_{R_L}}{|R_L|} \int_{\Sigma_L} \mu_m^{(L)} & \text{on } R_L. \end{cases}$$

We may now choose Σ_L small enough so that $\tilde{\mu} \leq (1+\varepsilon)B/2\pi\ell$, which makes it an admissible trial function for the minimization problem (3.13). Therefore

$$\mathcal{E}_L^{\text{MF}}[\tilde{\mu}] \geq \mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}]. \quad (3.16)$$

However we also have

$$\begin{aligned} \mathcal{E}_L^{\text{MF}}[\tilde{\mu}] &= \mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}] - \int_{\Sigma_L} U_L \mu_m^{(L)} + \frac{\int_{\Sigma_L} \mu_m^{(L)}}{|R_L|} \int_{R_L} U_L \\ &+ \lambda \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \left[-\mu_m^{(L)}(x) \mu_m^{(L)}(y) \mathbb{1}_{\Sigma_L}(y) + \left(\int_{\Sigma_L} \mu_m^{(L)} \right) \mu_m^{(L)}(x) \frac{\mathbb{1}_{R_L}(y)}{|R_L|} \right. \\ &\quad - \left(\int_{\Sigma_L} \mu_m^{(L)} \right) \mu_m^{(L)}(x) \mathbb{1}_{\Sigma_L}(x) \frac{\mathbb{1}_{R_L}(y)}{|R_L|} \\ &\quad + 2\mu_m^{(L)}(x) \mathbb{1}_{\Sigma_L}(x) \mu_m^{(L)}(y) \mathbb{1}_{\Sigma_L}(y) \\ &\quad \left. + 2 \left(\int_{\Sigma_L} \mu_m^{(L)} \right)^2 \mu_m^{(L)}(x) \frac{\mathbb{1}_{R_L}(x)}{|R_L|} \mu_m^{(L)}(y) \frac{\mathbb{1}_{R_L}(y)}{|R_L|} \right] dx dy. \end{aligned}$$

Using (3.7) is easy to see that every term containing w can be bounded from above by

$$C \int_{\Sigma_L} \mu_m^{(L)}.$$

Moreover, we know that $U_L(x) = |x| \geq L$ for $x \in \Sigma_L$ and that $U_L(x) = v(x) \leq C$ for $x \in R_L$ by (3.15). We then deduce

$$\mathcal{E}_L^{\text{MF}}[\tilde{\mu}] \leq \mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}] - L \int_{\Sigma_L} \mu_m^{(L)} + C \int_{\Sigma_L} \mu_m^{(L)},$$

which, for L large enough, implies

$$\mathcal{E}_L^{\text{MF}}[\tilde{\mu}] < \mathcal{E}_L^{\text{MF}}[\mu_m^{(L)}].$$

This contradicts (3.16), and the proof is complete. \square

4. ENERGY LOWER BOUND

The aim of this Section is the proof of Theorem 2.5. Given $\Psi_F \in \mathcal{L}_{\ell, B}^N$, let μ_F be the associated N -body probability measure in rescaled units of lengths, i.e.,

$$\mu_F(x_1, \dots, x_N) = N^N |\Psi_F(\sqrt{N}x_1, \dots, \sqrt{N}x_N)|^2. \quad (4.1)$$

Define the μ_F -probability of a Borel subset (event) $\Gamma \subset \mathbb{R}^{2N}$ as

$$\mathbb{P}_F(\Gamma) = \int_{\Gamma} \mu_F.$$

For any configuration $X_N = (x_1, \dots, x_N) \in \mathbb{R}^{2N}$ we will denote the corresponding empirical measure by

$$\text{Emp}_{X_N} = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}.$$

The following statement gives an estimate on the probability of configurations whose empirical measure violates the incompressibility bound (2.2).

Theorem 4.1 (Probability of violating the incompressibility bound).

Let Ω denote any open set with Lipschitz boundary, and, for any $\alpha > (-3 + \sqrt{5})/4$, let Ω_r denote its dilation around some origin by a factor $r = N^\alpha$. Let μ_F be the probability measure associated to $\Psi_F \in \mathcal{L}_{\ell, B}^N$. Then, for any $\varepsilon > 0$,

$$\mathbb{P}_F \left(\left\{ X_N \in \mathbb{R}^{2N} \mid \int_{\Omega_r} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|\Omega_r|}{2\pi\ell} \right\} \right) \leq e^{-C\varepsilon N^{\sqrt{5}-1}}. \quad (4.2)$$

The constant C in (4.2) depends on the geometrical details of the unscaled set Ω . A key ingredient for the proof is the incompressibility bound [23, Theorem 2.1], which in the notation of Theorem 4.1 reads

$$\int_{\Omega_r} \mu_F^{(1)} \leq \frac{B|\Omega_r|}{2\pi\ell} (1 + o_N(1)), \quad (4.3)$$

where

$$\mu_F^{(1)}(x) = \int \mu_F(x, x_2, \dots, x_N) dx_2 \dots dx_N.$$

Note that (4.3) says that

$$\mathbb{E} \left(\int_{\Omega_r} \text{Emp}_{X_N} \right) \leq \frac{B|\Omega_r|}{2\pi\ell} (1 + o_N(1))$$

and that our new estimate (4.2) is a deviation bound for this expectation bound.

Remark 4.2 (Scales on which incompressibility holds).

During the preparation of this work we became aware of an imprecision in the paper [23]. It is claimed in Theorem 2.1 there, that the incompressibility bound, in the rescaled version (4.3), holds for scales up to $\alpha > 1/4$. This does not however follow from the proof of Theorem 2.1 which is presented in [23].

More precisely, after (5.50) there, one should actually deduce (5.51) in the form

$$\gamma = \frac{1}{2 + 2\alpha'} + \frac{(2 + \delta)\beta}{2 + 2\alpha'}.$$

After choosing α' and β arbitrarily close to α , and δ arbitrarily small, one finds that γ can be picked arbitrarily close to $(1 + 2\alpha)/(2 + 2\alpha)$. Hence, the remainder terms in the right hand side of (5.50) are of order $O(N^{-1/(2+2\alpha)+\varepsilon})$. By comparing this with the main term, which is of order $N^{-1+2\alpha}$, one deduces that the remainders are actually negligible only for $\alpha > (-1 + \sqrt{5})/4$. By rescaling units of lengths one deduces that (4.3) holds precisely for the range of α which is considered in Theorem 4.1, i.e., $\alpha > (-3 + \sqrt{5})/4$.

This small imprecision is only due to the localization procedure in [23, Section 5.2]. The bounds of [23, Section 5.1], valid under small additional localization assumptions, are correct as stated. \diamond

Proof. Let us consider the case in which $|\Omega_r|$ is a disk of radius r , and let χ_r be the corresponding characteristic function. To deduce the result for more general sets one simply replaces characteristic functions of disks with those of general open sets. We denote by $\chi_{r,\delta}$ the regularization of χ_r on a scale δ , chosen so that

$$\|\Delta\chi_{r,\delta}\|_\infty \leq C\delta^{-2} \quad (4.4)$$

and

$$\chi_r \leq \chi_{r,\delta} \leq \chi_{r+\delta}. \quad (4.5)$$

We first rewrite μ_F as a Gibbs measure. By the definition (4.1) we notice that

$$\mu_F(x_1, \dots, x_N) = \frac{1}{\mathcal{Z}} \exp(-N\mathbb{H}(x_1, \dots, x_N)), \quad (4.6)$$

with

$$\mathbb{H}(x_1, \dots, x_N) = \sum_{j=1}^N |x_j|^2 - \frac{4\ell}{BN} \sum_{1 \leq i < j \leq N} \log |x_i - x_j| - \frac{2}{N} \log F(\sqrt{N}x_1, \dots, \sqrt{N}x_N)$$

and the normalization factor

$$\mathcal{Z} = \int_{\mathbb{R}^{dN}} \exp(-N\mathbb{H}(x_1, \dots, x_N)) dx_1 \dots dx_N$$

is the corresponding partition function. Hence μ_F is the Gibbs measure for \mathbb{H} with temperature N^{-1} .

Let us define the perturbed Hamiltonian

$$\mathbb{H}_s(x_1, \dots, x_N) = \mathbb{H}(x_1, \dots, x_N) - \frac{s}{N^2} \sum_{i=1}^N \chi_{r,\delta}(x_i)$$

and the associated partition function at temperature N^{-1}

$$\mathcal{Z}_s = \int \exp(-N\mathbb{H}_s(x_1, \dots, x_N)) dx_1 \dots dx_N.$$

Let us also introduce the free-energy functional

$$\mathcal{F}_s[\mu] = \int \mathbb{H}_s(X_N) \mu(X_N) + \frac{1}{N} \int \mu(X_N) \log \mu(X_N)$$

with

$$F_s = \inf \left\{ \mathcal{F}_s[\mu] \mid \int \mu = 1, \mu \geq 0 \right\}.$$

Of course, $F_s = -N^{-1} \log \mathcal{Z}_s$. We denote by μ_s the Gibbs measure corresponding to the Hamiltonian \mathbb{H}_s at temperature N^{-1} . Notice that according to this notation $\mu_0 = \mu_F$.

Let us show that $\mu_s^{(1)}$ satisfies an incompressibility bound analogous to (4.3). By adding and subtracting to \mathbb{H}_s the term

$$\frac{s}{4N^2} \|\Delta \chi_{r,\delta}\|_\infty \sum_{j=1}^N |x_j|^2$$

we write

$$\mathbb{H}_s(x_1, \dots, x_N) = \left(1 + \frac{s}{4N^2} \|\Delta \chi_{r,\delta}\|_\infty\right) \left[\sum_{j=1}^N |x_j|^2 - \frac{4\tilde{\ell}}{BN} \sum_{i<j} \log |x_i - x_j| + W(x_1, \dots, x_N) \right],$$

where $\tilde{\ell} = (1 + \frac{s}{4N^2} \|\Delta \chi_{r,\delta}\|_\infty)^{-1} \ell$ and

$$W(x_1, \dots, x_N) = -\frac{2}{N} \log F(\sqrt{N}x_1, \dots, \sqrt{N}x_N) - \frac{s}{N^2} \sum_{j=1}^N \left(\chi_{r,\delta}(x_j) + \frac{1}{4} \|\Delta \chi_{r,\delta}\|_\infty |x_j|^2 \right)$$

is superharmonic in each variable because the last two terms taken together are. We recognize that, apart from an irrelevant multiplicative factor, \mathbb{H}_s has the same structure that \mathbb{H} has, with a modified ℓ and a different superharmonic function. Hence, one can replicate the proof of (4.3) from [23] so as to get

$$\int_{\Omega_r} \mu_s^{(1)} \leq \frac{B}{2\pi\tilde{\ell}} |D_r| \left(1 + C \frac{s}{\delta^2 N^2} + o_N(1)\right). \quad (4.7)$$

Now, we notice that, for any $s > 0$,

$$\begin{aligned} & \mathbb{P}_F \left(\left\{ X_N \in \mathbb{R}^{2N} \mid \int_{\Omega_r} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|\Omega_r|}{2\pi\tilde{\ell}} \right\} \right) \\ & \leq e^{-s \frac{B|\Omega_r|}{2\pi\tilde{\ell}} (1+\varepsilon)} \int \exp \left(s \int_{\Omega_r} \text{Emp}_{X_N} \right) d\mu_F(X_N) \\ & \leq e^{-s \frac{B|\Omega_r|}{2\pi\tilde{\ell}} (1+\varepsilon)} \int \exp \left(s \int \chi_{r,\delta} \text{Emp}_{X_N} \right) d\mu_F(X_N), \end{aligned} \quad (4.8)$$

where the first inequality follows from the bound $1 \leq e^{sx}$, which holds whenever $sx > 0$, and the second inequality comes from (4.5). We will focus on estimating the integral on the right.

Within the above notation we recognize that

$$\int \exp \left(s \int \chi_{r,\delta} \text{Emp}_{X_N} \right) d\mu_F(X_N) = \frac{\mathcal{Z}_s}{\mathcal{Z}_0} = e^{-N(F_s + F_0)}. \quad (4.9)$$

We estimate F_s by writing

$$\begin{aligned} F_s &= \int \mathbb{H}\mu_s - \frac{s}{N} \int \chi_{r,\delta} \mu_s^{(1)} + \frac{1}{N} \int \mu_s \log \mu_s \\ &\geq F_0 - \frac{s}{N} \int \chi_{r,\delta} \mu_s^{(1)} \\ &\geq F_0 - \frac{s}{N} \int \chi_{r+\delta} \mu_s^{(1)} \\ &\geq F_0 - \frac{s}{N} \int_{\Omega_{r+\delta}} \mu_s^{(1)}, \end{aligned}$$

where the first inequality holds by minimality of F_0 and the second one follows from (4.5). Hence (4.9) yields, using (4.7),

$$\begin{aligned} \int \exp\left(s \int \chi_{r,\delta} \text{Emp}_{X_N}\right) d\mu_F(X_N) &\leq \exp\left(s \int_{\Omega_{r+\delta}} \mu_s^{(1)}\right) \\ &\leq \exp\left(s \frac{B(r+\delta)^2}{2\ell} (1 + C \frac{s}{\delta^2 N^2} + o_N(1))\right) \quad (4.10) \\ &\leq \exp\left(s \frac{B|\Omega_r|}{2\pi\ell} (1 + o_N(1))\right), \end{aligned}$$

provided we choose

$$r \gg \delta \quad \text{and} \quad 1 \gg \frac{s}{\delta^2 N^2}.$$

Since $r = N^\alpha$, the above relations are satisfied for $s = N^\beta$ with $\beta < 2\alpha + 2$. Plugging (4.10) into (4.8) yields

$$\mathbb{P}_F\left(\left\{X_N \in \mathbb{R}^{2N} \mid \int_{\Omega_r} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|\Omega_r|}{2\pi\ell}\right\}\right) \leq \exp\left(-\varepsilon N^\beta \frac{B|\Omega_r|}{2\pi\ell} (1 + o_N(1))\right).$$

By recalling that $|\Omega_r| = \pi N^{2\alpha}$ and $\alpha > (-3 + \sqrt{5})/4$, we deduce that we can always choose β , depending on α , so that

$$\mathbb{P}_F\left(\left\{X_N \in \mathbb{R}^{2N} \mid \int_{\Omega_r} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|\Omega_r|}{2\pi\ell}\right\}\right) \leq e^{-C\varepsilon N^{\sqrt{5}-1}},$$

which proves the result. \square

We now provide the

Proof of Theorem 2.5. Let us divide \mathbb{R}^2 into squares of side N^α with $0 > \alpha > (-3 + \sqrt{5})/4$, and denote the k -th square by S_k . Let us consider a number $L > 0$ of the form

$$L = nN^\alpha$$

for some $n \in \mathbb{N}$, and define by \mathcal{S}_L the square $[-L, L]^2$ of side $2L$ and centered at the origin. Notice that the number of S_k -squares within the set \mathcal{S}_L is $4n^2$. We will later choose L large enough but independent on N , which boils down to suitably choosing the (large) integer n . Let us also define

$$A_L = \left\{X_N \in \mathbb{R}^{2N} \mid \int_{S_k} \text{Emp}_{X_N} \leq (1 + \varepsilon) \frac{B|S_k|}{2\pi\ell} \quad \forall k \text{ such that } S_k \subset \mathcal{S}_L\right\},$$

and $A_L^c = \mathbb{R}^2 \setminus A_L$.

We start by estimating the measure of the set A_L^c . We have

$$\mu_F(A_L^c) = \mathbb{P}_F \left(\bigcup_{S_k \subset \mathcal{S}_L} \left\{ \int_{S_k} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|S_k|}{2\pi\ell} \right\} \right),$$

i.e., the measure of A_L^c is the probability of the union of all events of the type “the integral over the k -th square is bigger than $(1 + \varepsilon)B|S_k|/2\pi\ell$ ”. Hence, by the union bound and by Theorem 4.1,

$$\begin{aligned} \mu_F(A_L^c) &\leq \sum_{k=1}^{4n^2} \mathbb{P}_F \left(\int_{S_k} \text{Emp}_{X_N} > (1 + \varepsilon) \frac{B|S_k|}{2\pi\ell} \right) \\ &\leq \sum_{k=1}^{4n^2} \exp(-C\varepsilon N^{\sqrt{5}-1}) \\ &= CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}). \end{aligned} \tag{4.11}$$

Now, by a straightforward computation we can express the many-body energy in the form

$$\begin{aligned} N^{-1} \mathcal{E}_N[\Psi_F] &= \int \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) - \frac{\lambda}{2N} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu_F^{(2)}(x,y) dx dy - \lambda \frac{w(0)}{2N} \\ &\geq \int \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) - CN^{-1}, \end{aligned}$$

having used $\|w\|_\infty \leq C$ and $\int \mu_F^{(2)} = 1$ in the second step. We split the integral by writing

$$\begin{aligned} \int \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) &= \int_{A_L} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) + \int_{A_L^c} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) \\ &\geq \int_{A_L} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) - C\mu_F(A_L^c), \end{aligned}$$

where we use Lemma 3.1 in the second step. Therefore, using (4.11),

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq \int_{A_L} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) - CN^{-1} - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}). \tag{4.12}$$

By (2.9), if L is large enough, we have $v \geq U_L$ with

$$U_L(x) = \begin{cases} v(x) & x \in \mathcal{S}_L \\ |x| & x \notin \mathcal{S}_L. \end{cases}$$

As in Section 3, we denote by $\mathcal{E}_L^{\text{MF}}$ the functional \mathcal{E}^{MF} with v replaced by U_L . Since $v \geq U_L$ we can write

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq \int_{A_L} \mathcal{E}_L^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) - CN^{-1} - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}).$$

We now perform a Riemann sum approximation on the whole of \mathbb{R}^2 , by replacing the empirical measure Emp_{X_N} with the piece-wise constant function $\tilde{\mu}^{(X_N)}$ whose value in the

k -th square is

$$\tilde{\mu}_{|S_k}^{(X_N)} := \frac{\int_{S_k} \text{Emp}_{X_N}}{|S_k|}.$$

Notice that $\tilde{\mu}^{(X_N)}$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{\mu}^{(X_N)} &= 1 \\ \tilde{\mu}^{(X_N)} &\geq 0 \\ \tilde{\mu}^{(X_N)} &\leq (1 + \varepsilon) \frac{B}{2\pi\ell} \quad \text{on every } S_k \subset [-L, L]^2. \end{aligned}$$

This yields (recall that L is independent of N)

$$\begin{aligned} N^{-1} \mathcal{E}_N[\Psi_F] &\geq \int_{A_L} \mathcal{E}_L^{\text{MF}}[\tilde{\mu}^{(X_N)}] d\mu_F(X_N) - CN^\alpha (\|\nabla w\|_\infty + \|\nabla U_L\|_\infty) \\ &\quad - CN^{-1} - CL^2 N^{-2\alpha} \exp(-C\varepsilon N) \\ &= \int_{A_L} \mathcal{E}_L^{\text{MF}}[\tilde{\mu}^{(X_N)}] d\mu_F(X_N) - CN^\alpha - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}). \end{aligned} \quad (4.13)$$

Due to the above properties of $\tilde{\mu}^{(X_N)}$, we have

$$\mathcal{E}_L^{\text{MF}}[\tilde{\mu}^{(X_N)}] \geq \inf \left\{ \mathcal{E}_L^{\text{MF}}[\mu] \mid \mu \geq 0, \mu \leq (1 + \varepsilon) \frac{B}{2\pi\ell} \text{ on every } S_k \subset \mathcal{S}_L, \int \mu = 1, \right\} = E_\varepsilon^{\text{flo}},$$

the equality having been proven in Lemma 3.6 for L large enough but finite. Recall that $E_\varepsilon^{\text{flo}}$ is the flocking energy with perturbed upper constraint defined in (3.12). Recall also that by Lemma 3.5

$$E_\varepsilon^{\text{flo}} \geq E^{\text{flo}} - C\varepsilon,$$

and therefore

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq E^{\text{flo}} \mu_F(A_L) - CN^\alpha - C\varepsilon - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}).$$

Using $\mu_F(A_L) = 1 - \mu_F(A_L^c)$ and then (4.11) we deduce

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq E^{\text{flo}} - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}) - CN^\alpha - C\varepsilon.$$

We choose $\varepsilon = N^{-1}$. In this way we can neglect the exponential because L is large but fixed. Since $\alpha > (-3 + \sqrt{5})/4$ we obtain

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq E^{\text{flo}} - CN^{(-3+\sqrt{5})/4+\gamma}$$

for any $\gamma > 0$. This concludes the proof. \square

5. ENERGY UPPER BOUND

Now we prove Theorem 2.7. Let us write a generic polynomial f as

$$f(z) = c_N \prod_{j=1}^J (z - \sqrt{N} a_j)^{Nq_j/2}$$

Given the corresponding Ψ_f of the form (1.7), the associated N -particle probability density in rescaled units of length μ_f (see (4.1)) can be written as

$$\mu_f(x_1, \dots, x_N) = \frac{1}{\mathcal{Z}} e^{-N\mathcal{H}_N(x_1, \dots, x_N)}, \quad (5.1)$$

where the partition function \mathcal{Z} is a normalization factor and

$$\mathcal{H}_N(x_1, \dots, x_N) = \sum_{i=1}^N \left(\sum_{j=1}^J q_j \log \frac{1}{|x_i - a_j|} + \frac{B}{2} |x_i|^2 \right) + \frac{2\ell}{N} \sum_{k < \ell} \log \frac{1}{|x_k - x_\ell|}. \quad (5.2)$$

Hence μ_f is the Gibbs state for the Hamiltonian \mathcal{H}_N at temperature $T = 1/N$. \mathcal{H}_N is the energy of N two-dimensional particles interacting among themselves and with J fixed point particles a_j through the 2D-Coulomb kernel $-\log |\cdot|$, and trapped by an external harmonic potential.

The mean-field functional associated to the electrostatic Hamiltonian \mathcal{H}_N is, for a given probability measure μ ,

$$\mathcal{E}_f^{\text{el}}[\mu] = \int_{\mathbb{R}^2} \left(\sum_{j=1}^J q_j \log \frac{1}{|x - a_j|} + \frac{B}{2} |x|^2 \right) \mu(x) dx + \ell \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mu(x) \log \frac{1}{|x - y|} \mu(y) dx dy. \quad (5.3)$$

As discussed in [35, Sect. 3-4] and [32], as $N \rightarrow \infty$, there exists a unique normalized minimizer μ_f^{el} satisfying

$$E_f^{\text{el}} := \inf \left\{ \mathcal{E}_f^{\text{el}}[\mu] \mid \mu \geq 0, \int \mu = 1 \right\} = \mathcal{E}_f^{\text{el}}[\mu_f^{\text{el}}]. \quad (5.4)$$

Moreover, μ_f^{el} only assumes the values zero and $B/2\pi\ell$.

For σ any finite measure with $\int |\log |x|| |\sigma(x)| dx < \infty$ we define the 2D electrostatic energy

$$D(\sigma, \sigma) = \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \sigma(x) \log \frac{1}{|x - y|} \sigma(y) dx dy.$$

Lemma 5.1 (Bounds using the electrostatic energy).

Let μ_1 and μ_2 be probability densities. Then, for any test-function $\chi_1(x)$,

$$\left| \int_{\mathbb{R}^2} (\mu_1(x) - \mu_2(x)) \chi_1(x) dx \right| \leq \|\nabla \chi_1\|_{L^2} D(\mu_1 - \mu_2, \mu_1 - \mu_2)^{1/2}, \quad (5.5)$$

and, for any test-function $\chi_2(x, y)$,

$$\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_1(x)\mu_1(y) - \mu_2(x)\mu_2(y)) \chi_2(x, y) dx dy \right| \leq CD(\mu_1 - \mu_2, \mu_1 - \mu_2)^{1/2} \times \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^2} \quad (5.6)$$

Proof. Assume χ_1 is differentiable and denote

$$V = -\frac{1}{2\pi} \log |\cdot| \star (\mu_1 - \mu_2).$$

Then, since

$$-\Delta V = \mu_1 - \mu_2$$

$$\int_{\mathbb{R}^2} (\mu_1(x) - \mu_2(x))\chi_1(x)dx = \int \nabla\chi_1(x) \cdot \nabla V(x)dx,$$

and by Cauchy-Schwarz

$$\begin{aligned} \left| \int_{\mathbb{R}^2} (\mu_1(x) - \mu_2(x))\chi_1(x)dx \right| &\leq \|\nabla\chi_1\|_{L^2} \|\nabla V\|_2 \\ &= C\|\nabla\chi_1\|_{L^2} \left[\int_{\mathbb{R}^2} \left(\nabla_x \int_{\mathbb{R}^2} \frac{1}{|x-y|} (\mu_1(y) - \mu_2(y))dy \right) \right. \\ &\quad \left. \times \left(\nabla_x \int_{\mathbb{R}^2} \frac{1}{|x-z|} (\mu_1(z) - \mu_2(z))dz \right) \right]^{1/2} \\ &= C\|\nabla\chi_1\|_{L^2} \left[\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{|x-y|} (\mu_1(y) - \mu_2(y))dy \right. \\ &\quad \left. \times \left(\Delta_x \int_{\mathbb{R}^2} \frac{1}{|x-z|} (\mu_1(z) - \mu_2(z))dz \right) \right]^{1/2} \\ &= \|\nabla\chi_1\|_{L^2} D(\mu_1 - \mu_2, \mu_1 - \mu_2)^{1/2}. \end{aligned}$$

The result is then extended by density to test functions with $\nabla\chi_1 \in L^2$.

To prove (5.6) we can safely assume $\chi_2(x, y) = \chi_2(y, x)$, because the contribution of the antisymmetric part would integrate to zero. We have

$$\begin{aligned} &\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_1(x)\mu_1(y) - \mu_2(x)\mu_2(y))\chi_2(x, y)dxdy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_1(x) + \mu_2(x))(\mu_1(y) - \mu_2(y))\chi_2(x, y)dxdy, \end{aligned}$$

and hence, by (5.5),

$$\begin{aligned} &\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_1(x)\mu_1(y) - \mu_2(x)\mu_2(y))\chi_2(x, y)dxdy \right| \\ &\leq D(\mu_1 - \mu_2, \mu_1 - \mu_2)^{1/2} \left\| \nabla \int \chi_2(\cdot, x)(\mu_1(x) + \mu_2(x))dx \right\|_{L^2}. \end{aligned}$$

The Cauchy-Schwarz inequality together with $\int \mu_1 = \int \mu_2 = 1$ yields

$$\left\| \nabla \int \chi_2(\cdot, x)(\mu_1(x) + \mu_2(x))dx \right\|_{L^2} \leq C \sup_x \|\nabla\chi_2(\cdot, x)\|_{L^2},$$

which concludes the proof. \square

The first main result of this Section is the following Proposition, showing that μ_f approximates μ_f^{el} in the sense of reduced densities.

Proposition 5.2 (Mean-field approximation of Ψ_f).

Let f be a polynomial and $\mu_f^{(1)}, \mu_f^{(2)}$ be the one- and two-body densities associated to Ψ_f of the form (1.7). Then, for any test functions χ_1 and χ_2 ,

$$\left| \int_{\mathbb{R}^2} (\mu_f^{(1)}(x) - \mu_f^{\text{el}}(x))\chi_1(x)dx \right| \leq C \left(\frac{\log N}{N} \right)^{1/2} \|\nabla\chi_1\|_{L^2} + CN^{-1/2} \|\nabla\chi_1\|_{L^\infty}, \quad (5.7)$$

and

$$\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{(2)}(x, y) - \mu_f^{\text{el}}(x) \mu_f^{\text{el}}(y)) \chi_2(x, y) dx dy \right| \leq C \left(\frac{\log N}{N} \right)^{1/2} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^2} + CN^{-1/2} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^\infty}. \quad (5.8)$$

Moreover, we have the pointwise bounds

$$\begin{aligned} 0 &\leq \mu_f^{(1)}(x) \leq C e^{-NC(|x|^2 - \log N)} \\ 0 &\leq \mu_f^{(2)}(x, y) \leq C e^{-NC(|x|^2 + |y|^2 - \log N)} \end{aligned} \quad (5.9)$$

Proof of Proposition 5.2. The bound (5.7) corresponds exactly to [35, Eq. (4.1)], which in turn is based on [32, Theorem 3.2] and we refer to those papers for the complete proof. The argument is based on the free-energy functional

$$\mathcal{F}_f^{\text{el}}[\mu] = \mathcal{E}_f^{\text{el}}[\mu] + \frac{1}{N} \int \mu(x) \log \mu(x) dx \quad (5.10)$$

at temperature N^{-1} associated with the electrostatic energy functional $\mathcal{E}_f^{\text{el}}$ (5.3). Let us denote by μ_f^{Gibbs} the unique minimizer of $\mathcal{F}_f^{\text{el}}$ among probability measures.

In order to prove (5.8), we notice that an intermediate step for the proof of (5.7) is the bound

$$D\left(\mu_f^{\text{Gibbs}} - \mu_f^{\text{el}}, \mu_f^{\text{Gibbs}} - \mu_f^{\text{el}}\right) \leq CN^{-1}. \quad (5.11)$$

This corresponds to [35, Eq. (4.4)]. We also import, directly from [32, Remark 3.3] (see also [30, Lemma 7.4])

$$\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{(2)}(x, y) - \mu_f^{\text{Gibbs}}(x) \mu_f^{\text{Gibbs}}(y)) \chi_2(x, y) dx dy \right| \leq C \left(\frac{\log N}{N} \right)^{1/2} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^2}^2 + CN^{-1/2} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^\infty}. \quad (5.12)$$

Moreover, by (5.6) and (5.11),

$$\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{\text{Gibbs}}(x) \mu_f^{\text{Gibbs}}(y) - \mu_f^{\text{el}}(x) \mu_f^{\text{el}}(y)) \chi_2(x, y) dx dy \right| \leq CN^{-1/2} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^2}. \quad (5.13)$$

Hence, (5.8) is proven by decomposing

$$\begin{aligned} &\iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{(2)}(x, y) - \mu_f^{\text{el}}(x) \mu_f^{\text{el}}(y)) \chi_2(x, y) dx dy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{(2)}(x, y) - \mu_f^{\text{Gibbs}}(x) \mu_f^{\text{Gibbs}}(y)) \chi_2(x, y) dx dy \\ &\quad + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_f^{\text{Gibbs}}(x) \mu_f^{\text{Gibbs}}(y) - \mu_f^{\text{el}}(x) \mu_f^{\text{el}}(y)) \chi_2(x, y) dx dy \end{aligned}$$

and then using (5.12) and (5.13).

We are only left with the proof of (5.9), which is obtained analogously to the proof of [35, Eq. (4.2)]. There one first obtains (cf. [35, Eq. (4.14)])

$$0 \leq \mu_f^{\text{Gibbs}}(x) \leq C e^{-NC|x|^2}. \quad (5.14)$$

This estimate is then carried over to one- and two-particle densities as in the proof of [32, Eq. (3.16)]. More precisely, at the end of Section 3.3 of [32] the following formula is obtained

$$\mu_f(x_1, \dots, x_N) \leq e^{CN \log N} \prod_{j=1}^N \mu_f^{\text{Gibbs}}(x_j).$$

By integrating with respect to all but one or two variables one gets

$$\begin{aligned} \mu_f^{(1)}(x) &\leq e^{CN \log N} \mu_f^{\text{Gibbs}}(x) \\ \mu_f^{(2)}(x, y) &\leq e^{CN \log N} \mu_f^{\text{Gibbs}}(x) \mu_f^{\text{Gibbs}}(y), \end{aligned}$$

which directly imply (5.9) due to (5.14). \square

Having at hand Proposition 5.2 which shows proximity of $\mu_f^{(1)}$ and μ_f^{el} , the missing ingredient is a link between μ_f^{el} and the measure μ^{sol} of Theorem 2.7. This is provided by the following result.

Proposition 5.3 (Inverse electrostatic problem).

Let μ^{sol} be, as in Theorem 2.7, a probability measure whose only values are 0 and $B/2\pi\ell$. There exists a (sequence of) polynomial(s) f_δ indexed by a N -dependent parameter $\delta > 0$ such that, for any test-functions χ_1 and χ_2 ,

$$\left| \int_{\mathbb{R}^2} (\mu_{f_\delta}^{\text{el}}(x) - \mu^{\text{sol}}(x)) \chi(x) dx \right| \leq CN^{-1/4} \|\nabla \chi\|_{L^2} \quad (5.15)$$

and

$$\left| \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\mu_{f_\delta}^{\text{el}}(x) \mu_{f_\delta}^{\text{el}}(y) - \mu^{\text{sol}}(x) \mu^{\text{sol}}(y)) \chi_2(x, y) dx dy \right| \leq CN^{-1/4} \sup_y \|\nabla \chi_2(\cdot, y)\|_{L^2}. \quad (5.16)$$

Proof. Proposition 5.3 is proven analogously to [35, Prop. 3.1], the main difference being that the role of the bathtub minimizer ρ_0 is now played by μ^{sol} . One can anyway repeat all the steps and deduce

$$D(\mu^{\text{sol}} - \mu_{f_\delta}^{\text{el}}, \mu^{\text{sol}} - \mu_{f_\delta}^{\text{el}}) \leq CN^{-1/2}.$$

The result is then brought to the form (5.15) and (5.16) using Lemma 5.1. \square

We can now provide the

Proof of Theorem 2.7. Let us introduce χ_{in} and χ_{out} , smooth partition of unity with χ_{in} supported in the disk $D(0, 2 \log N)$ and χ_{out} identically zero in the disk $D(0, \log N)$. We can choose them so that

$$\|\nabla \chi_{\text{in}}\|_\infty + \|\nabla \chi_{\text{out}}\|_\infty \leq C \log N,$$

which implies, by the growth assumption (2.9),

$$\begin{aligned} \|\nabla(\chi_{\text{in}} v)\|_p &\leq CN^\theta \\ \|\nabla(\chi_{\text{in}} w)\|_p &\leq CN^\theta \end{aligned} \quad (5.17)$$

for any $\theta > 0$ and $p \in [2, +\infty]$.

Let f_δ be the sequence of polynomials provided by Proposition 5.3. We have

$$\begin{aligned}
\frac{e(N, \lambda)}{N} &\leq \frac{\mathcal{E}_N[\Psi_{f_\delta}]}{N} \\
&= \int \chi_{\text{in}}(x)v(x)\mu^{\text{sol}}(x)dx + \frac{\lambda}{2} \int \int \chi_{\text{in}}(x)\chi_{\text{in}}(y)w(x-y)\mu^{\text{sol}}(x)\mu^{\text{sol}}(y)dxdy \\
&\quad + \int \chi_{\text{in}}(x)v(x)\left(\mu_{f_\delta}^{(1)}(x) - \mu^{\text{sol}}(x)\right)dx \\
&\quad + \frac{\lambda}{2} \int \chi_{\text{in}}(x)\chi_{\text{in}}(y)w(x-y)\left(\mu_{f_\delta}^{(2)}(x,y) - \mu^{\text{sol}}(x)\mu^{\text{sol}}(y)\right)dxdy \\
&\quad + \int \chi_{\text{out}}(x)v(x)\mu_{f_\delta}^{(1)}(x)dx \\
&\quad + \lambda \int \chi_{\text{in}}(x)\chi_{\text{out}}(y)w(x-y)\mu_{f_\delta}^{(2)}(x,y)dxdy \\
&\quad + \frac{\lambda}{2} \int \chi_{\text{out}}(x)\chi_{\text{out}}(y)w(x-y)\mu_{f_\delta}^{(2)}(x,y)dxdy.
\end{aligned} \tag{5.18}$$

Since μ^{sol} is a fixed probability measure with compact support, the first line in the right-hand side of (5.18) coincides with $\mathcal{E}^{\text{MF}}[\mu^{\text{sol}}]$ for N large enough. To estimate the second and third lines we use (5.7), (5.8), (5.15), (5.16), together with (5.17), thus obtaining

$$\begin{aligned}
&\left| \int \chi_{\text{in}}(x)v(x)\left(\mu_{f_\delta}^{(1)}(x) - \mu^{\text{sol}}(x)\right)dx \right| \leq CN^{-1/4+\gamma} \\
&\frac{\lambda}{2} \left| \int \chi_{\text{in}}(x)\chi_{\text{in}}(y)w(x-y)\left(\mu_{f_\delta}^{(2)}(x,y) - \mu^{\text{sol}}(x)\mu^{\text{sol}}(y)\right)dxdy \right| \leq CN^{-1/4+\gamma}
\end{aligned}$$

for any $\gamma > 0$. The remaining three terms are estimated using (5.9). For example

$$\left| \int \chi_{\text{out}}(x)v(x)\mu_{f_\delta}^{(1)}(x)dx \right| \leq \int_{|x| \geq \log N} e^{-CN(|x|^2 - \log N)} |x|^s$$

is exponentially small as $N \rightarrow \infty$, and the same for the other terms. We obtained

$$\frac{e(N, \lambda)}{N} \leq \frac{\mathcal{E}_N[\Psi_{f_\delta}]}{N} \leq \mathcal{E}^{\text{MF}}[\mu^{\text{sol}}](1 + N^{-1/4+\gamma})$$

for any $\gamma > 0$, which concludes the proof. \square

6. CONVERGENCE OF DENSITIES

6.1. The Hewitt-Savage and Diaconis-Freedman Theorems. We start by recalling the Hewitt-Savage Theorem [13, 7] for N -body states, see [29, 28, 27] for more details. We will denote by $\mathcal{P}(\Sigma)$ the set of probability measures over a set Σ . We shall use the fact that the de Finetti-Hewitt-Savage can be approximated by the Diaconis-Freedman construction: given a symmetric probability measure μ_N over \mathbb{R}^{dN} , $d \geq 1$, define the probability measure over $\mathcal{P}(\mathbb{R}^2)$

$$P_{\mu_N}(\sigma) := \int_{\mathbb{R}^{2N}} \delta_{\sigma = \text{Emp}_{X_N}} d\mu_N(x_1, \dots, x_N). \tag{6.1}$$

We then have the

Theorem 6.1 (Hewitt-Savage in large N limit).

Let μ_N be a symmetric probability measure over \mathbb{R}^{dN} . Assume that its first marginal $\mu_N^{(1)}$ is tight:

$$\limsup_{R \rightarrow \infty} \sup_{N \in \mathbb{N}} \left(1 - \mu_N^{(1)}(B(0, R))\right) = 0. \quad (6.2)$$

Extract a subsequence such that $\mu_N^{(1)} \rightharpoonup \mu \in \mathcal{P}(\mathbb{R}^d)$ as measures. Then, along this (not-relabeled) subsequence,

- (1) There exists a unique probability measure $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ such that for any fixed $k \in \mathbb{N}$,

$$\mu_N^{(k)} \rightharpoonup \int_{\mathcal{P}(\mathbb{R}^d)} \mu^{\otimes k} dP(\mu). \quad (6.3)$$

- (2) Let P_{μ_N} be defined in (6.1) and P the measure such that (6.3) holds. We have that

$$P_{\mu_N} \rightharpoonup P \quad (6.4)$$

weakly as measures on $\mathcal{P}(\mathbb{R}^d)$.

Elements of proof. If \mathbb{R}^d is replaced by a compact subset, the proof is contained in [29, 28, Chapter 2]. For the benefit of the non french-reader we reproduce the argument of [27] allowing to remove the compactness assumption.

Let $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ be the one-point Alexandrov compactification of \mathbb{R}^d . Since this is a compact space we deduce there exists a unique probability measure on $\mathcal{P}(\overline{\mathbb{R}^d})$ such that

$$\mu_N^{(k)} \rightharpoonup \int_{\mathcal{P}(\overline{\mathbb{R}^d})} \mu^{\otimes k} dP(\mu).$$

We want to prove that P is actually concentrated on $\mathcal{P}(\mathbb{R}^d)$ only. Let us decompose $\mathcal{P}(\overline{\mathbb{R}^d})$ as the disjoint union

$$\mathcal{P}(\overline{\mathbb{R}^d}) = \mathcal{P}(\mathbb{R}^d) \cup \{\delta_\infty\} \cup R,$$

where δ_∞ is the Dirac delta measure concentrated on $\{\infty\}$ and $R = \mathcal{P}(\overline{\mathbb{R}^d}) \setminus (\mathcal{P}(\mathbb{R}^d) \cup \{\delta_\infty\})$.

By tightness of $\mu_N^{(1)}$ we deduce

$$1 = \int_{\mathcal{P}(\overline{\mathbb{R}^d})} \mu(\mathbb{R}^d) dP(\mu) = \int_{\mathcal{P}(\mathbb{R}^d)} \mu(\mathbb{R}^d) dP(\mu) + \delta_\infty(\mathbb{R}^d) P(\infty) + \int_R \mu(\mathbb{R}^d) dP(\mu).$$

Now, we know that $\mu(\mathbb{R}^d) = 1$ for every $\mu \in \mathcal{P}(\mathbb{R}^d)$. Moreover, we know that $\delta_\infty(\mathbb{R}^d) = 0$. Furthermore, $\mu(\mathbb{R}^d) < 1$ for any $\mu \in R$, for otherwise such a μ would belong to $\mathcal{P}(\mathbb{R}^d)$. Assume for contradiction that $P(R) > 0$. Then

$$1 < \int_{\mathcal{P}(\mathbb{R}^d)} dP(\mu) + \int_R dP(\mu) \leq 1,$$

which is absurd. Hence $P(R) = 0$, which implies $P(\mathcal{P}(\mathbb{R}^d)) = 1$. The above argument also implies that $(\mu_N^{(k)})_N$ is tight for all k .

As for Item 2 of the statement, see again [29, 28, Chapter 2] for the case where \mathbb{R}^d is compactified. First extract a subsequence along which

$$P_N \rightharpoonup P'$$

for some probability measure P' on $\mathcal{P}(\overline{\mathbb{R}^d})$. We recall the Diaconis-Freedman bound

$$\left\| \mu_N^{(k)} - \tilde{\mu}_N^{(k)} \right\|_{\text{TV}} \leq 2 \frac{k(k-1)}{N}. \quad (6.5)$$

where we used the total variation norm

$$\|\mu - \nu\|_{\text{TV}} = 2 \sup \{ |\mu(A) - \nu(A)| \mid A \in \Sigma \},$$

and denoted

$$\tilde{\mu}_N = \int_{\mathcal{P}(\mathbb{R}^d)} \mu^{\otimes N} dP_N(\mu). \quad (6.6)$$

It follows that $P' = P$. Indeed, passing to the limit in (6.5), the restriction of P' to $\mathcal{P}(\mathbb{R}^d)$ must be P . But the latter is a probability measure, so P' must charge only $\mathcal{P}(\mathbb{R}^d)$. \square

6.2. Energy estimates on the support of the limit measure. As a preparatory step for the proof of Corollary 2.4 we will show in this Subsection that the limit measure P which Theorem 6.1 associates to a many-body minimizing sequence only charges probability measures whose energy is smaller than E^{flo} .

For any $\delta > 0$ define the function $U : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R}$ as

$$U_\delta(\mu) = \begin{cases} 1 & \text{if } \mathcal{E}^{\text{MF}}[\mu] > E^{\text{flo}} + \delta \text{ or if } \mathcal{E}^{\text{MF}}[\mu] \text{ is not defined} \\ 0 & \text{if } \mathcal{E}^{\text{MF}}[\mu] \leq E^{\text{flo}} + \delta. \end{cases} \quad (6.7)$$

Notice that U_δ is also defined on probability measures that do not satisfy the upper constraint $\mu \leq B/2\pi\ell$ of the flocking problem 2.8. The following Lemma will later be useful in order to apply a weak version of Fatou's Lemma.

Lemma 6.2 (Weak- \star lower semicontinuity of U_δ).

For any sequence of probability measures μ_n converging weakly- \star as Radon measures to μ we have

$$\liminf_{n \rightarrow \infty} U_\delta(\mu_n) \geq U_\delta(\mu).$$

Proof. By Lemma 3.2 we know that

$$\liminf_{n \rightarrow \infty} \mathcal{E}^{\text{MF}}[\mu_n] \geq \mathcal{E}^{\text{MF}}[\mu]$$

and hence

$$\mathcal{E}^{\text{MF}}[\mu_n] - E^{\text{flo}} \geq \mathcal{E}^{\text{MF}}[\mu] - E^{\text{flo}} + o_n(1). \quad (6.8)$$

Depending on the value of $U_\delta(\mu)$ we distinguish two cases. If $U_\delta(\mu) = 1$, then, by definition of U_δ , there exists $\delta' > \delta$ such that

$$\mathcal{E}^{\text{MF}}[\mu] - E^{\text{flo}} = \delta'.$$

This implies, using (6.8), that for n large enough

$$\mathcal{E}^{\text{MF}}[\mu_n] - E^{\text{flo}} \geq \delta' > \delta$$

for some δ'' . Hence, for n large enough,

$$U_\delta(\mu_n) = 1 = U_\delta(\mu),$$

which in particular proves lower semicontinuity.

The other situation is when $U_\delta(\mu) = 0$, in which case, by positivity of U_δ ,

$$\liminf_{n \rightarrow \infty} U_\delta(\mu_n) \geq 0 = U_\delta(\mu).$$

□

Proposition 6.3 (The limit measure of minimizing sequences charges low energies).

Let Ψ_F be a minimizing sequence for the many-body energy (1.5), i.e.,

$$\mathcal{E}_N[\Psi_F] = E(N, \lambda) + o(N)$$

as $N \rightarrow \infty$. Let P be the limit measure associated to $|\Psi_F|^2$ via Theorem 6.1. Then

$$\mathcal{E}^{\text{MF}}[\mu] \leq E^{\text{flo}} \quad P - a.e.$$

Proof. Denote P_{Ψ_F} the measure over $\mathcal{P}(\mathbb{R}^2)$ associated to $|\Psi_F|^2$ via (6.1). We first claim that, for any fixed $\delta > 0$,

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP_{\Psi_F}(\mu) \leq (\delta^{-1} + 1)o_N(1). \quad (6.9)$$

Let us define the set

$$\Xi_\delta = \{X_N \in \mathbb{R}^{2N} \mid \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] > E^{\text{flo}} + \delta\}.$$

From the definition (6.1) of P_{Ψ_F} we deduce

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP_{\Psi_F}(\mu) = \mu_F(\Xi_\delta). \quad (6.10)$$

In order to obtain an estimate for $\mu_F(\Xi_\delta)$, our proof now goes through computations similar to those in the proof of (2.13) at the end of Section 4. In particular, importing directly (4.12), we have

$$\begin{aligned} N^{-1}\mathcal{E}_N[\Psi_F] &\geq \int_{A_L \cap \Xi_\delta} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) + \int_{A_L \cap \Xi_\delta^c} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) \\ &\quad - CN^{-1} - CL^2 N^{-2\alpha} \exp(-C\varepsilon N^{\sqrt{5}-1}). \end{aligned} \quad (6.11)$$

where

$$A_L = \left\{ X_N \in \mathbb{R}^{2N} \mid \int_{S_k} \text{Emp}_{X_N} \leq (1 + \varepsilon) \frac{B|S_k|}{2\pi\ell} \quad \forall k \text{ such that } S_k \subset \mathcal{S}_L \right\},$$

\mathcal{S}_L is the square $[-L, L] \times [-L, L]$ and S_k is the k -th square of side N^α of a tiling of \mathcal{S}_L . We now use the definition of Ξ_δ in the integral on $A_L \cap \Xi_\delta$ to get

$$\int_{A_L \cap \Xi_\delta} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) > (E^{\text{flo}} + \delta)\mu_F(A_L \cap \Xi_\delta).$$

Notice also that

$$\mu_F(A_L \cap \Xi_\delta) = \mu_F(\Xi_\delta) - \mu_F(A_L^c \cap \Xi_\delta) \geq \mu_F(\Xi_\delta) - \mu_F(A_L^c),$$

and we can use (4.11) to estimate $\mu(A_L^c)$. Furthermore, in the integral on $A_L \cap \Xi_\delta^c$, we perform a Riemann sum approximation as explained in the proof of (2.13) when passing from (4.12) to (4.13). After repeating the same steps, this procedure yields

$$\int_{A_L \cap \Xi_\delta^c} \mathcal{E}^{\text{MF}}[\text{Emp}_{X_N}] d\mu_F(X_N) \geq E^{\text{flo}} \mu_F(A_L \cap \Xi_\delta^c) - C\varepsilon - CN^\alpha.$$

Hence, we can bring (6.11) to the form

$$N^{-1} \mathcal{E}_N[\Psi_F] \geq E^{\text{flo}} + \delta \mu_F(\Xi_\delta) - C\delta L^2 N^{-2\alpha} e^{-C\varepsilon N^{\sqrt{5}} - 1} - CN^\alpha - C\varepsilon.$$

Since we chose a minimizing sequence Ψ_F we have, by the upper bound (2.14),

$$N^{-1} \mathcal{E}_N[\Psi_F] \leq E^{\text{flo}} + o_N(1)$$

as $N \rightarrow \infty$. Combining the last two inequalities we find that we can choose $\varepsilon = N^{-1}$ and L large enough such that

$$\delta \mu_F(\Xi_\delta) \leq (1 + \delta) o_N(1).$$

Using (6.10), this completes the proof of (6.9).

Next, since P_{Ψ_F} converges weakly as measures to P as in (6.4), the integral

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP_{\Psi_F}(\mu)$$

has a positive and weakly lower semicontinuous function U_δ (see Lemma 6.2) integrated against a weakly converging sequence of measures. We are within the assumptions of [4, Chapter 1, Section 2, Problem 2.6] and [8, Theorem 1.1] which yield the Fatou-like Lemma

$$\liminf_{N \rightarrow \infty} \int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP_{\Psi_F}(\mu) \geq \int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP(\mu),$$

whence

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP_{\Psi_F}(\mu) \geq \int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP(\mu) - o_N(1).$$

Combining with (6.9) we deduce

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP(\mu) \leq (1 + \delta^{-1}) o_N(1),$$

whence, passing to the limit $N \rightarrow \infty$,

$$\int_{\mathcal{P}(\mathbb{R}^2)} U_\delta(\mu) dP(\mu) \leq 0$$

The subsequent limit $\delta \rightarrow 0$ concludes the proof. \square

6.3. Incompressibility estimate for the limit measure and proof of Corollary 2.4.

Proposition 6.4 (Incompressibility estimate for the limit measure).

Let Ψ_F be a minimizing sequence for the many-body energy (1.5), i.e.,

$$\mathcal{E}_N[\Psi_F] = E(N, \lambda) + o(N)$$

as $N \rightarrow \infty$. Let $P \in \mathcal{P}(\mathcal{P}(\mathbb{R}^2))$ be the limit measure associated to $|\Psi_F|^2$ by Theorem 6.1. Then, P -almost everywhere,

$$\mu \leq \frac{B}{2\pi\ell}.$$

Proof. For any $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^2$, define the set

$$J_{k,\Omega} = \left\{ X_N \in \mathbb{R}^{2N} \mid \left(\int_{\Omega} \text{Emp}_{X_N} \right)^k > \left((1 + \varepsilon) \frac{B|\Omega|}{2\pi\ell} \right)^k \right\}.$$

Since $J_{k,\Omega} = J_{1,\Omega}$, Theorem 4.1 implies immediately

$$\mu_F(J_{k,\Omega}) \leq e^{-C\varepsilon N^{\sqrt{5}-1}}.$$

Notice that the C on the right of the last inequality depends on the set Ω , which we will however keep fixed. We write

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \left(\int_{\Omega} \text{Emp}_{X_N} \right)^k d\mu_F(X_N) &= \int_{J_{k,\Omega}^c} \left(\int_{\Omega} \text{Emp}_{X_N} \right)^k d\mu_F(X_N) \\ &\quad + \int_{J_{k,\Omega}} \left(\int_{\Omega} \text{Emp}_{X_N} \right)^k d\mu_F(X_N) \quad (6.12) \\ &\leq \left((1 + \varepsilon) \frac{B|\Omega|}{2\pi\ell} \right)^k + e^{-C\varepsilon N^{\sqrt{5}-1}}, \end{aligned}$$

having used $\int_{\Omega} \text{Emp}_{X_N} \leq 1$ for the second term.

Let P be the measure on $\mathcal{P}(\mathbb{R}^2)$ that Theorem 6.1 associates to μ_F . We will show that

$$\int_{\mathcal{P}(\mathbb{R}^2)} \left(\int_{\Omega} \mu \right)^k dP(\mu) \leq \left(\frac{B|\Omega|}{2\pi\ell} \right)^k. \quad (6.13)$$

Let us denote by χ_{Ω} the characteristic function of Ω , and let $\chi_{\Omega,\vartheta}$ be its regularization on a small scale ϑ . Let Ω_{ϑ} be the dilation of Ω by a factor $(1 + \vartheta)$ and let $\chi_{\Omega_{\vartheta}}$ be the corresponding characteristic function. We can pick these functions so as to have

$$\chi_{\Omega} \leq \chi_{\Omega,\vartheta} \leq \chi_{\Omega_{\vartheta}}. \quad (6.14)$$

Then

$$\begin{aligned} \int_{\mathcal{P}(\mathbb{R}^2)} \left(\int_{\Omega} \mu \right)^k dP(\mu) &\leq \int_{\mathcal{P}(\mathbb{R}^2)} \left(\int \chi_{\Omega,\vartheta} \mu \right)^k dP(\mu) \\ &= \int \chi_{\Omega,\vartheta}^{\otimes k} \mu_F^{(k)} + o_N(1) \\ &\leq \int_{\Omega_{\vartheta}^k} \mu_F^{(k)} + o_N(1), \end{aligned}$$

where the first step is due to (6.14), the second is due to Theorem 6.1 (and the bound (6.5), notice that $\chi_{\Omega,\delta}$ is continuous and compactly supported), and the third again to (6.14). We remark that the error $o_N(1)$ might depend badly on ϑ . Now, by (6.5) we have

$$\int_{\Omega_{\vartheta}^k} \mu_F^{(k)} = \int_{\Omega_{\vartheta}^k} \tilde{\mu}_F^{(k)} + o_N(1).$$

Furthermore, notice that

$$\int_{\Omega_{\vartheta}^{\times k}} \tilde{\mu}_F^{(k)} = \int_{\mathbb{R}^{2N}} \left(\int_{\Omega_{\vartheta}} \text{Emp}_{X_N} \right)^k d\mu_F(X_N).$$

By putting together the last three estimates we get

$$\begin{aligned} \int_{\mathcal{P}(\mathbb{R}^2)} \left(\int_{\Omega} \mu \right)^k dP(\mu) &\leq \int_{\mathbb{R}^{2N}} \left(\int_{\Omega_{\vartheta}} \text{Emp}_{X_N} \right)^k d\mu_F(X_N) + o_N(1) \\ &\leq \left((1 + \varepsilon) \frac{B|\Omega_{\vartheta}|}{2\pi\ell} \right)^k + e^{-C\varepsilon N\sqrt{\varepsilon}-1} + o_N(1), \end{aligned}$$

having used (6.12) for the second step. We can choose, for example, $\varepsilon = N^{-1}$, which allows to take the limit $N \rightarrow \infty$ first. The subsequent limit $\vartheta \rightarrow 0$ finally yields (6.13).

The proof is then completed by an argument already appearing in [9, Theorem 2.6]. Assume for contradiction that there exists a set $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^2)$ such that $P(\mathcal{K}) > 0$ and $\mu(\Omega) > |\Omega|B/2\pi\ell$ for some set $\Omega \subset \mathbb{R}^2$ and for every $\mu \in \mathcal{K}$. We have, by (6.13),

$$\int_{\mathcal{K}} \left(\frac{\int_{\Omega} \mu}{\frac{B|\Omega|}{2\pi\ell}} \right)^k dP(\mu) \leq 1.$$

Since

$$\int_{\Omega} \mu > \frac{B|\Omega|}{2\pi\ell}$$

on the whole of \mathcal{K} , the above inequality is absurd for large enough k and therefore it must be that

$$\mu(\Omega) \leq \frac{B|\Omega|}{2\pi\ell} \quad P \text{ - a.e..}$$

This is actually equivalent to having the density of μ bounded from above by $B/2\pi\ell$ almost everywhere, and the proof is complete. \square

We are finally ready to prove Corollary 2.4.

Proof of Corollary 2.4. Let Ψ_F be a minimizing sequence as in (2.11). By Theorem 6.1 there exists P such that, for every $k \in \mathbb{N}$,

$$\mu_F^{(k)} \rightharpoonup \int_{\mathcal{P}(\mathbb{R}^2)} \mu^{\otimes k} dP(\mu)$$

weakly as measures as $N \rightarrow \infty$. The fact that P is a probability measure follows from the second part of Theorem 6.1 if we show that the sequences of densities $\mu_F^{(k)}$ are tight. For fixed $\varepsilon > 0$, let $B(0, R_\varepsilon)$ be the ball of radius R_ε and centered at the origin. We have

$$\mu_F^{(2)}(B(0, R_\varepsilon) \times B(0, R_\varepsilon)) \geq 1 - 2 \int_{B(0, R_\varepsilon)^c} \mu_F^{(1)}.$$

However, since by our assumption (2.9) the external potential v grows at infinity, we have

$$\int_{B(0, R_\varepsilon)^c} \mu_F^{(1)} \leq \frac{1}{v(R_\varepsilon)} \int_{B(0, R_\varepsilon)^c} v \mu_F^{(1)} \leq \frac{E^{\text{flo}} + C}{v(R_\varepsilon)}$$

having used in the second step that (recall that $w \in L^\infty$)

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} w(x-y) \mu_F^{(2)}(x, y) dx dy \geq -C$$

together with the fact that the sequence μ_F minimizes the energy. Hence, for R_ε large enough, $(E^{\text{flo}} + C)/v(R_\varepsilon) < \varepsilon$ and we have

$$\mu_F^{(2)}(B(0, R_\varepsilon) \times B(0, R_\varepsilon)) \geq 1 - 2\varepsilon$$

for all N . Tightness of all other marginals follows.

By Proposition 6.3 and Proposition 6.4 we know that P must be concentrated on

$$\left\{ \mu \mid \mathcal{E}^{\text{MF}}[\mu] \leq E^{\text{flo}}, \int_{\mathbb{R}^2} \mu = 1 \right\} \cap \left\{ \mu \mid 0 \leq \mu \leq \frac{B}{2\pi\ell} \text{ a.e.}, \int_{\mathbb{R}^2} \mu = 1 \right\}.$$

This is equivalent to saying that P is concentrated on the set of minimizers of the flocking problem (2.8), which completes the proof. \square

REFERENCES

- [1] AROVAS, S., SCHRIEFFER, J., AND WILCZEK, F. Fractional statistics and the quantum Hall effect. *Phys. Rev. Lett.* 53, 7 (1984), 722–723.
- [2] BAUERSCHMIDT, R., BOURGADE, P., NIKULA, M., AND YAU, H.-T. The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. arXiv:1609.08582, 2016.
- [3] BAUERSCHMIDT, R., BOURGADE, P., NIKULA, M., AND YAU, H.-T. Local density for two-dimensional one-component plasma. *Communications in Mathematical Physics* 356, 1 (2017), 189–230.
- [4] BILLINGSLEY, P. *Convergence of probability measures*, 2nd ed. Wiley Series in Probability and Statistics. John Wiley & Sons Inc., New York, 1999.
- [5] BURCHARD, A., CHOKSI, R., AND TOPALOGLU, I. Nonlocal shape optimization via interactions of attractive and repulsive potentials. *Indiana Univ. J. Math.* (2017).
- [6] DE PICCIOTTO, R., REZNIKOV, M., HEIBLUM, M., UMANSKY, V., BUNIN, G., AND MAHALU, D. Direct observation of a fractional charge. *Nature* 389 (1997), 162–164.
- [7] DIACONIS, P., AND FREEDMAN, D. Finite exchangeable sequences. *Ann. Probab.* 8, 4 (1980), 745–764.
- [8] FEINBERG, E., KASYANOV, P., AND ZADOIANCHUK, N. Fatou’s lemma for weakly converging probabilities. *Theory of Probability & Its Applications* 58, 4 (2014), 683–689.
- [9] FOURNAIS, S., LEWIN, M., AND SOLOVEL, J.-P. The semi-classical limit of large fermionic systems. arXiv:1510.01124, 2015.
- [10] FRANK, R. L., AND LIEB, E. H. A liquid-solid phase transition in a simple model for swarming. *Indiana Univ. J. Math.* (2017).
- [11] GIRVIN, S. Introduction to the fractional quantum Hall effect. *Séminaire Poincaré* 2 (2004), 54–74.
- [12] GOERBIG, M. O. Quantum Hall effects. arXiv:0909.1998, 2009.
- [13] HEWITT, E., AND SAVAGE, L. J. Symmetric measures on Cartesian products. *Trans. Amer. Math. Soc.* 80 (1955), 470–501.
- [14] JAIN, J. K. *Composite fermions*. Cambridge University Press, 2007.
- [15] LAUGHLIN, R. B. Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.* 50, 18 (May 1983), 1395–1398.
- [16] LAUGHLIN, R. B. Elementary theory : the incompressible quantum fluid. In *The quantum Hall effect*, R. E. Prange and S. E. Girvin, Eds. Springer, Heidelberg, 1987.
- [17] LAUGHLIN, R. B. Nobel lecture: Fractional quantization. *Rev. Mod. Phys.* 71 (Jul 1999), 863–874.
- [18] LEBLÉ, T. Local microscopic behavior for 2D Coulomb gases. *Probability Theory and Related Fields* 169, 3-4 (2017), 931–976.
- [19] LEBLÉ, T., AND SERFATY, S. Fluctuations of two-dimensional Coulomb gases. arXiv:1609.08088, 2016.
- [20] LEWIN, M., AND SEIRINGER, R. Strongly correlated phases in rapidly rotating Bose gases. *J. Stat. Phys.* 137, 5-6 (Dec 2009), 1040–1062.
- [21] LIEB, E. H., AND LOSS, M. *Analysis*, 2nd ed., vol. 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [22] LIEB, E. H., ROUGERIE, N., AND YNGVASON, J. Rigidity of the Laughlin liquid. arXiv:1609.03818, 2016.

- [23] LIEB, E. H., ROUGERIE, N., AND YNGVASON, J. Local incompressibility estimates for the Laughlin phase. arXiv:1701.09064, 2017. to be published in *Comm. Math. Phys.*
- [24] LIEB, E. H., SOLOVEJ, J.-P., AND YNGVASON, J. Ground states of large quantum dots in magnetic fields. *Phys. Rev. B* 51 (Apr 1995), 10646–10665.
- [25] LUNDHOLM, D., AND ROUGERIE, N. Emergence of fractional statistics for tracer particles in a Laughlin liquid. *Phys. Rev. Lett.* 116 (2016), 170401.
- [26] MARTIN, J., ILANI, S., VERDENE, B., SMET, J., UMANSKY, V., MAHALU, D., SCHUH, D., ABSTREITER, G., AND YACOBY, A. Localization of fractionally charged quasi-particles. *Science* 305 (2004), 980–983.
- [27] MISCHLER, S. Estimation quantitative et uniforme en temps de la propagation du chaos et introduction aux limites de champ moyen pour des systèmes de particules. Cours de l’Ecole doctorale EDDIMO, 2011.
- [28] ROUGERIE, N. De Finetti theorems, mean-field limits and Bose-Einstein condensation. arXiv:1506.05263, 2014. LMU lecture notes.
- [29] ROUGERIE, N. *Théorèmes de De Finetti, limites de champ moyen et condensation de Bose-Einstein*. Les cours Peccot. Spartacus IDH, Paris, 2016. Cours Peccot, Collège de France : février-mars 2014.
- [30] ROUGERIE, N., AND SERFATY, S. Higher-dimensional Coulomb gases and renormalized energy functionals. *Communications on Pure and Applied Mathematics* 69, 3 (2016), 519.
- [31] ROUGERIE, N., SERFATY, S., AND YNGVASON, J. Quantum Hall states of bosons in rotating anharmonic traps. *Phys. Rev. A* 87 (Feb 2013), 023618.
- [32] ROUGERIE, N., SERFATY, S., AND YNGVASON, J. Quantum Hall phases and plasma analogy in rotating trapped Bose gases. *J. Stat. Phys.* 154 (2014), 2–50.
- [33] ROUGERIE, N., AND YNGVASON, J. Incompressibility estimates for the Laughlin phase. *Comm. Math. Phys.* 336 (2015), 1109–1140.
- [34] ROUGERIE, N., AND YNGVASON, J. Incompressibility estimates for the Laughlin phase, part II. *Comm. Math. Phys.* 339 (2015), 263–277.
- [35] ROUGERIE, N., AND YNGVASON, J. The Laughlin liquid in an external potential. *Letters in Mathematical Physics* 108, 4 (2018), 1007–1029.
- [36] SAMINADAYAR, L., GLATTLI, D. C., JIN, Y., AND ETIENNE, B. Observation of the $e/3$ fractionally charged Laughlin quasiparticle. *Phys. Rev. Lett.* 79 (Sep 1997), 2526–2529.
- [37] STÖRMER, H., TSUI, D., AND GOSSARD, A. The fractional quantum Hall effect. *Rev. Mod. Phys.* 71 (1999), S298–S305.

UNIVERSITÉ GRENOBLE-ALPES & CNRS, LPMMC, F-38000 GRENOBLE, FRANCE
E-mail address: `alessandro.olgiati@lpmmc.cnrs.fr`

UNIVERSITÉ GRENOBLE-ALPES & CNRS, LPMMC, F-38000 GRENOBLE, FRANCE
E-mail address: `nicolas.rougerie@lpmmc.cnrs.fr`