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# Heat Diffusion 3D model

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## 1 Motivation

The aim is to obtain the analytical expression of the continuous component of the laser-induced temperature rise in a layer on a substrate in good thermal contact with a thermostat and in a layer in bad thermal contact with a thermostat (contact thermal resistance). The results were given in the following paper without detailed calculation:

JOURNAL OF APPLIED PHYSICS **119**, 153904 (2016)

### **Steady-state thermal gradient induced by pulsed laser excitation in a ferromagnetic layer**

S. Shihab,<sup>1</sup> L. Thevenard,<sup>1</sup> A. Lemaitre,<sup>2</sup> J.-Y. Duquesne,<sup>1</sup> and C. Gourdon<sup>1</sup>

In the case of a layer on a substrate in good thermal contact with a thermostat, the temperature rise at the surface is given by Eq. 1, which reads:

$$\Delta T(r) = \frac{P_0(1 - \Re)}{2\pi} \int_0^\infty f(u) e^{-\frac{1}{8}u^2 w^2} J_0(ru) du$$

$$f(u) = \frac{k_1 \cosh(uL_1) \sinh(uL_0) + k_0 \sinh(uL_1) \cosh(uL_0)}{k_1 (k_1 \sinh(uL_1) \sinh(uL_0) + k_0 \cosh(uL_1) \cosh(uL_0))},$$

(1)

and in the case of a single layer in bad thermal contact with a thermostat, the temperature rise is given by Eq.2, which reads:

$$\Delta T(r, z) = \frac{P_0(1 - \Re)}{2\pi} \int_0^\infty g(u, z) e^{-\frac{1}{8}u^2 w^2} J_0(ru) du,$$

$$g(u, z) = \frac{\alpha}{k(u^2 - \alpha^2)} \times \left( \frac{\cosh(uz)(\alpha e^{-Lu}(1 - kRu) + u e^{-\alpha L}(\alpha kR - 1))}{kRu \sinh(Lu) + \cosh(Lu)} + u e^{-\alpha z} - \alpha e^{-uz} \right).$$

(2)

## 2 Two-layer sample in contact with a thermostat

The heat diffusion equation is written as

$$\partial \Delta T / \partial t - D \nabla^2 \Delta T = p(r, z, t) / \rho C, \quad (2.1)$$

with  $\Delta T = T - T_{thermostat}$ ,  $D$  the diffusivity,  $p$  the absorbed power per unit volume,  $\rho$  the mass density and  $C$  the mass specific heat.  $T_{thermostat}$  is the base temperature. The continuous component of  $\Delta T$  is found as the solution

## 2 TWO-LAYER SAMPLE IN CONTACT WITH A THERMOSTAT

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of the time-independent diffusion equation in response to the time-averaged absorbed power (CW component).

Here the system under consideration is a layer of thickness  $L_1$  and thermal conductivity  $k_1$  on top of a substrate of thickness  $L_0$  and thermal conductivity  $k_0$  in contact with a thermostat (Fig. 2.1).

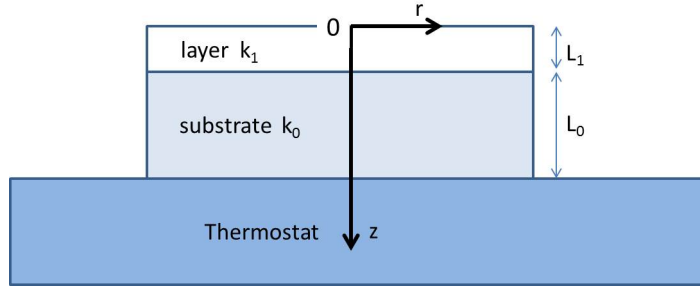


Figure 2.1:

We first assume that the energy flux from the laser transmitted at the surface ( $z=0$ ) is totally converted into heat flux within an infinitely thin depth.

The temperature and the heat flux at the layer/substrate interface are continuous functions of the depth  $z$  in the case of zero contact resistance at each interface. The sample is perfectly thermalized at the substrate-thermostat interface, i.e.,  $\Delta T = T(r, L_0 + L_1) - T_{thermostat} = 0$ .

In the continuous regime and without volume heat source the heat diffusion equation in each layer is reduced to :

$$\nabla^2 \Delta T = 0 \tag{2.2}$$

which in cylindrical coordinates translates to

$$\frac{\partial^2 \Delta T}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta T}{\partial r} + \frac{\partial^2 \Delta T}{\partial z^2} = 0 \tag{2.3}$$

$\Delta T(r, z)$  is written as a function of its Hankel transform  $\Theta(r, z)$  as  $\Delta T(r, z) = \int_0^{+\infty} \Theta(u, z) J_0(ru) u du$ , where  $J_0$  is the Bessel Function of the first kind. The Hankel transform then verifies

$$\frac{\partial^2 \Delta T}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta T}{\partial r} + \frac{\partial^2 \Delta T}{\partial z^2} = \int_0^{+\infty} \left( -u^2 \Theta(u, z) + \frac{\partial^2 \Theta(u, z)}{\partial z^2} \right) J_0(ru) u du \quad (2.4)$$

The solutions of  $\left( -u^2 \Theta(u, z) + \frac{\partial^2 \Theta(u, z)}{\partial z^2} \right) = 0$  are then

$$\Theta(u, z) = A_{1,0}(u) \exp(-uz) + B_{1,0}(u) \exp(uz) , \quad (2.5)$$

Where the two indices 1 and 0 refer to the layer and the substrate, respectively. The functions  $A$  and  $B$  are determined by the boundary conditions.

At the surface, the incoming heat flux  $\phi(r)$ , which has cylindrical symmetry is equal to the normal flux at  $z=0$ .

$$-k_1 \frac{\partial \Delta T}{\partial z} = \phi(r) \quad (2.6)$$

Taking the Hankel transform on both side, we have (condition 1)

$$-k_1 \frac{\partial \Theta}{\partial z} = \psi(u) \quad (2.7)$$

with  $\psi(u) = \int_0^{+\infty} \phi(u, z) J_0(ru) r dr$ , the Hankel transform of  $\phi$ .

Condition 2 and 3 are given by the continuity of the temperature and the heat flux at the layer-substrate interface. Condition 4 is given by the continuity of the temperature at the substrate- thermostat interface  $\Delta T(r, L_0 + L_1) = 0$ . The four conditions then lead to the following set of equations:

$$k_1 u (A_1(u) - B_1(u)) = \psi(u) \quad (2.8)$$

$$A_1(u) \exp(-u L_1) + B_1(u) \exp(u L_1) = A_0(u) \exp(-u L_1) + B_0(u) \exp(u L_1) \quad (2.9)$$

$$k_1 u [A_1(u) \exp(-u L_1) - B_1(u) \exp(u L_1)] = k_0 u [A_0(u) \exp(-u L_1) - B_0(u) \exp(u L_1)] \quad (2.10)$$

$$A_0(u) \exp(-u(L_0 + L_1)) + B_0(u) \exp(u(L_0 + L_1)) = 0 \quad (2.11)$$

Solving this system gives the functions  $A_1(u)$  and  $B_1(u)$ :

$$A_1(u) = \psi(u) \frac{e^{L_1 u} [k_1 \sinh(L_0 u) + k_0 \cosh(L_0 u)]}{2k_1 u [k_1 \sinh(L_0 u) \sinh(L_1 u) + k_0 \cosh(L_0 u) \cosh(L_1 u)]} \quad (2.12)$$

$$B_1(u) = \psi(u) \frac{e^{-L_1 u} [k_1 \sinh(L_0 u) - k_0 \cosh(L_0 u)]}{2k_1 u [k_1 \sinh(L_0 u) \sinh(L_1 u) + k_0 \cosh(L_0 u) \cosh(L_1 u)]} \quad (2.13)$$

The temperature at the surface ( $z = 0$ ) is expressed as

$$\Delta T(r, 0) = \int_0^{+\infty} (A_1(u) + B_1(u)) J_0(ru) u du \quad (2.14)$$

Since the laser spot is well fitted by a Gaussian, the incident flux is taken as  $\phi(r) = \phi_0 \exp\left(-\frac{2r^2}{w^2}\right)$ . The measured incident power is:

$$P_0 = \int_0^{+\infty} \phi_0 \exp\left(-\frac{2r^2}{w^2}\right) 2\pi r dr = \frac{\pi w^2 \phi_0}{2}. \quad (2.15)$$

The incoming flux has to be corrected by a factor  $(1 - R)$ , where  $R$  is the reflectance. Therefore the incoming flux is expressed as  $\frac{2P_0(1-R)}{\pi w^2} \exp\left(-\frac{2r^2}{w^2}\right)$  and its Hankel transform  $\psi(u) = \int_0^{+\infty} \left(\frac{2P_0(1-R)}{\pi w^2} \exp\left(-\frac{2r^2}{w^2}\right)\right) J_0(ru) r dr$  is equal to  $\frac{P_0(1-R)}{2\pi} \exp\left(-\frac{1}{8}u^2 w^2\right)$ .

Finally, combining Eqs 2.12, 2.13 with the expression of  $\psi(u)$ , one obtains:

$$\Delta T(r, 0) = \frac{P_0(1-R)}{2\pi} \int_0^{+\infty} \exp\left(-\frac{1}{8}u^2 w^2\right) f(u) J_0(ru) du, \quad (2.16)$$

with

$$f(u) = \frac{k_0 \sinh(L_1 u) \cosh(L_0 u) + k_1 \sinh(L_0 u) \cosh(L_1 u)}{k_1 (k_1 \sinh(L_0 u) \sinh(L_1 u) + k_0 \cosh(L_0 u) \cosh(L_1 u))}, \quad (2.17)$$

which is actually Eq. 1 of the paper quoted above.

### 3 Single layer and contact thermal resistance at the layer-thermostat interface

The system is now a single layer with thickness  $L$  and thermal conductivity  $k$  (Fig. 3.1). We assume that it is in bad thermal contact with a thermostat, i.e., there is a temperature discontinuity at the layer-substrate interface owing to a contact thermal resistance.

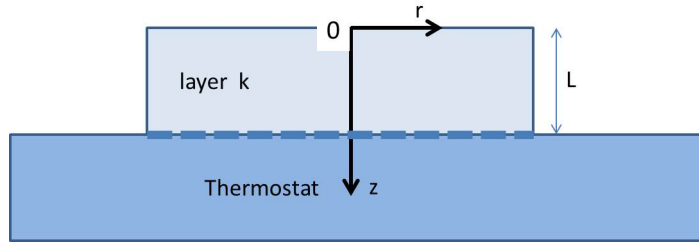


Figure 3.1:

Now we explicitly take into account the light absorption depth. We solve the heat diffusion equation  $-D\nabla^2\Delta T = p(r, z)/\rho C$ , which is also written as  $\nabla^2\Delta T = -p(r, z)/k$ . Taking the radial and depth dependence of absorbed power per unit volume as  $p(r, z) = p_0 \exp\left(-\frac{2r^2}{w^2}\right) \exp(-\alpha z)$ , we have  $P_0(1-R) = \int_0^{+\infty} p_0 \exp\left(-\frac{2r^2}{w^2}\right) \exp(-\alpha z) 2\pi r dr dz$ , therefore  $p_0 = \frac{2\alpha P_0(1-R)}{\pi w^2}$ . We have now to solve the following equation:

$$\begin{aligned} \frac{\partial^2 \Delta T}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta T}{\partial r} + \frac{\partial^2 \Delta T}{\partial z^2} &= \int_0^{+\infty} \left( -u^2 \Theta(u, z) + \frac{\partial^2 \Theta(u, z)}{\partial z^2} \right) J_0(ru) u du \\ &= -\frac{1}{k} \int_0^{\infty} \Pi(u) J_0(ru) u du, \end{aligned}$$

where  $\Pi(u, z)$  is the Hankel transform of  $p(r, z)$ , i.e.,

$$\begin{aligned} \Pi(u, z) &= \frac{\alpha P_0(1-R)}{2\pi} \exp\left(-\frac{1}{8}u^2 w^2\right) \exp(-\alpha z) \\ &= \Pi_0 \exp\left(-\frac{1}{8}u^2 w^2\right) \exp(-\alpha z) \end{aligned} \quad (3.1)$$

### 3 SINGLE LAYER AND CONTACT THERMAL RESISTANCE AT THE LAYER-THERMOSTAT INTERFACE

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with  $\Pi_0 = \frac{\alpha P_0(1-R)}{2\pi}$  . The solutions are found by solving

$$-u^2\Theta(u, z) + \frac{\partial^2\Theta(u, z)}{\partial z^2} = -\frac{1}{k}\Pi(u, z) \quad (3.2)$$

We then have

$$\Theta(u, z) = A(u) \exp(-uz) + B(u) \exp(uz) + \frac{\Pi_0}{k(u^2 - \alpha^2)} \exp\left(-\frac{1}{8}u^2w^2\right) \exp(-\alpha z) \quad (3.3)$$

The functions  $A$  and  $B$  are found by solving the boundary conditions using the Hankel transform. The flux across the interface at  $z=0$  is zero ( $-k_1 \frac{\partial\Theta}{\partial z} = 0$ ), which gives:

$$u(-A(u) + B(u)) - \frac{\alpha \Pi_0}{k(u^2 - \alpha^2)} \exp\left(-\frac{1}{8}u^2w^2\right) = 0 \quad (3.4)$$

At the layer-thermostat interface there is a temperature discontinuity owing to the contact thermal resistance  $R_c$ . We thus have:

$$\begin{aligned} T(r, z = L^+) - T(r, z = L^-) &= -R_c \left( -k \frac{\partial T}{\partial z} \Big|_{z=L} \right) \\ \Delta T(r, L^-) &= R_c \left( -k \frac{\partial \Delta T}{\partial z} \Big|_{z=L} \right) \\ \Theta(u, L^-) &= -R_c k \frac{\partial \Theta}{\partial z} \Big|_{z=L} \end{aligned}$$

$$\begin{aligned} &A \exp(-uL) + B \exp(uL) + \frac{\Pi_0}{k(u^2 - \alpha^2)} \exp\left(-\frac{1}{8}u^2w^2\right) \exp(-\alpha L) + \\ R_c k \left( -Au \exp(-uL) + Bu \exp(uL) - \frac{\alpha \Pi_0}{k(u^2 - \alpha^2)} \exp\left(-\frac{1}{8}u^2w^2\right) \exp(-\alpha L) \right) &= 0 \end{aligned} \quad (3.5)$$



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From Eqs 3.4 and 3.5, one finds for  $A(u)$  and  $B(u)$  :

$$A(u) = -\frac{e^{Lu}(\alpha e^{Lu}(kR_c u + 1) + ue^{-\alpha L}(1 - \alpha kR_c))}{ku(u^2 - \alpha^2)(e^{2Lu}(kR_c u + 1) - kR_c u + 1)} \Pi_0 \exp\left(-\frac{u^2 w^2}{8}\right) \quad (3.6)$$

$$B(u) = \frac{e^{-\alpha L}(\alpha e^{\alpha L}(1 - kR_c u) - ue^{Lu}(1 - \alpha kR_c))}{ku(u^2 - \alpha^2)(e^{2Lu}(kR_c u + 1) - kR_c u + 1)} \Pi_0 \exp\left(-\frac{u^2 w^2}{8}\right) \quad (3.7)$$

The radial and  $z$ -dependence of the temperature are then obtained as:

$$\begin{aligned} \Delta T(r, z) = & \int_0^{+\infty} \left( A(u) \exp(-uz) + B(u) \exp(uz) \right. \\ & \left. + \frac{\Pi_0}{k(u^2 - \alpha^2)} \exp\left(-\frac{1}{8}u^2 w^2\right) \exp(-\alpha z) \right) J_0(ru) u du \end{aligned} \quad (3.8)$$

which, after some algebra, gives Eq.2 of the above paper, namely:

$$\Delta T(r, z) = \frac{P_0(1 - R)}{2\pi} \int_0^\infty g(u, z) \exp\left(-\frac{u^2 w^2}{8}\right) J_0(ru) du, \quad (3.9)$$

$$\text{with } g(u, z) = \frac{\alpha}{k(u^2 - \alpha^2)} \left( \frac{\cosh(uz)(\alpha e^{-Lu}(1 - kR_c u) + ue^{-\alpha L}(\alpha kR_c - 1))}{kR_c u \sinh(Lu) + \cosh(Lu)} + ue^{-\alpha z} - \alpha e^{-uz} \right).$$