Interval-valued state estimation for linear systems: the tightest estimator and its relaxations ⋆

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Abstract

This paper discusses an interval-valued state estimation framework for linear dynamic systems. In particular, we derive an expression of the tightest possible interval estimator in the sense that it is the intersection of all interval-valued estimators for the system of interest. However, from a numerical implementation perspective, this estimator might suffer from a high complexity, at least in the general setting. Therefore, practical implementation might require some over-approximations which would yield a good trade-off between computational complexity and tightness. We discuss a number of such over-approximations. We also consider the general estimation scenario when the system parameters, the initial state, the input signal and the measurement are all uncertain.

Key words: state estimation, interval estimator.

1 Introduction

State estimation is a key problem in control engineering and more generally, in decision-making systems. One approach to this estimation problem is that of interval observers. Contrary to classical observers which generate single valued state estimates [19,1], interval observers produce set-valued estimates of the state for uncertain dynamical systems. The philosophy of the approach is inspired by the so-called set-membership estimation framework [14,7]. However, as stated in [16], the proper concept of interval observer is surprisingly recent and can be traced back to [12]. More precisely, interval observers are dynamical systems which provide an interval-valued trajectory (defined by lower and upper bounds) which contains all possible state trajectories of a given uncertain system. The model uncertainties may arise from input disturbances, sensor noises or unknown initial conditions. Also, the uncertainties are considered from a deterministic point of view, that is, no stochastic significance is assigned to them. In these settings, many contributions have been made for different classes of systems: continuous-time Linear Time Invariant (LTI) [15,5,8,17], discrete-time LTI/LTV systems [9,16], Linear Parameter Varying (LPV) systems [11], nonlinear systems [21,18]. For more on the interval observer literature we refer to a recent survey reported in [10].

Although the existing literature covers a large variety of systems, a question of major importance that has not received much attention so far is that of the size (or volume) of the estimated interval-valued state trajectory. In effect, there exist in principle infinitely many interval estimators that satisfy the outer-bounding condition for the state trajectories of the system of interest. But ideally, one would like to find the smallest possible interval set (in some sense) among all those which enclose the states. Hence we ask the question of how to characterize the tightest interval estimator in the sense that the upper and lower bounds are closest componentwise. A second matter of interest concerns how to tackle the challenging estimation scenario where in addition to usual uncertainty about the signals, the system parameters (matrices) are only known to belong to interval sets. Again this issue has not received a sufficient treatment in the

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existing literature.

The current paper intends to address these two questions. For this purpose, a new approach is proposed to tackle the problem of designing an interval-valued estimator. For simplicity of exposition we restrict attention to continuous-time LTI systems but the proposed approach can be extended at a moderate effort to Linear Time-Varying (LTV) systems. The key ingredient of the proposed framework is a parametrization of the interval set in the form of a center jointly with a radius which measures the width (size) of the interval set. Then we show that simple maximization techniques allow to construct the tightest enclosing interval set for the state. Note however that computing numerically such an interval-valued estimate of the state might be expensive in general. We therefore discuss some approximation strategies illustrating the trade-off between quality (tightness indeed) of the estimate and the computational price to pay for it. Note that another aspect of the quest for tightness in interval estimation was discussed in [20] and [22]. However, these references considered cost function-based notions of tightness over the whole time domain and for a pre-specified class of estimators. In contrast, the current paper elaborates on a property of strict tightness which holds for any time.

The first part of the material to be presented here already appeared in the preliminary work [2]. That work concerned only the situation where the input signal, the measurements and the initial state of the system are uncertain and the parameters are considered to be exactly known. The second part extends the discussion to the more general and challenging situation where both the parameters and the signals are uncertain.

Outline. The remainder of the paper is organized as follows. In Section 2, we set up the estimation problem and present the technical material employed for designing the estimator. In Section 3 we discuss estimators in open-loop, that is, estimators that result only from the simulation of the state transition equation without any use of the measurement. Section 4 discusses a systematic way of transforming a classical linear observer into an interval-valued estimator. The general case when both parameters and signals are uncertain is treated in Section 5. Section 6 reports some numerical results confirming tightness of the proposed estimator. We conclude the paper in Section 7.

Notation. $\mathbb{R}$ (resp. $\mathbb{R}_+$) is the set of real (resp. nonnegative real) numbers. For a real number x, $|x|$ will refer to the absolute value of x. For $x = [x_1 \ \cdots \ x_n]^\top \in \mathbb{R}^n$, $\|x\|_p$ will denote the p-norm of x defined by $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$, for $p \geq 1$. In particular for $p = \infty$, $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$. For an n-dimensional vector x, $\mathrm{diag}(x)$ will denote the $n \times n$ square matrix having the entries of x on its main diagonal. If $X \in \mathbb{R}^{n\times n}$, then $\mathrm{diag}(X)$ is the diagonal matrix obtained from X by setting its off-diagonal entries to 0. For a matrix $A \in \mathbb{R}^{n\times m}$, $\|A\|_p$ is the matrix norm induced by the vector norm $\|\cdot\|_p$, $\|A\|_p = \max_{\|x\|_p \leq 1} \|Ax\|_p$.
If $A = [a_{ij}] \in \mathbb{R}^{n\times m}$ and $B = [b_{ij}] \in \mathbb{R}^{n\times m}$ are real matrices of the same dimensions, the notation $A \leq B$ will be understood as an elementwise inequality on the entries, i.e., $a_{ij} \leq b_{ij}$ for all (i, j). $|A|$ corresponds to the matrix $[|a_{ij}|]$ obtained by taking the absolute value of each entry of A. In case A and B are real square matrices, $A \succeq B$ (resp. $A \succ B$) means that $A - B$ is positive semi-definite (resp. positive definite). A square matrix A is called *Hurwitz* if all its eigenvalues have negative real parts. It is called *Metzler* if all its off-diagonal entries are nonnegative.
For a positive integer n, we use the notation $\mathcal{L}^n(\mathbb{R}_+) = \{s : \mathbb{R}_+ \to \mathbb{R}^n\}$ to refer to the set of n dimensional vector-valued functions on $\mathbb{R}_+$. $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^n)$ is the subset of $\mathcal{L}^n(\mathbb{R}_+)$ containing bounded and measurable functions.

2 Preliminaries

2.1 Estimation problem settings

Consider a Linear Time Invariant (LTI) system described by

$$\dot{x}(t) = Ax(t) + Bw(t) \tag{1a}$$
$$y(t) = Cx(t) + v(t) \tag{1b}$$

where $t \in \mathbb{R}_+$ refers to time, $x \in \mathcal{L}^n(\mathbb{R}_+)$ is the state of the system with $x(0)$ termed the initial state; w and v are (possibly unknown) external signal respectively in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_w})$ and $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_y})$, $y \in \mathcal{L}^{n_y}(\mathbb{R}_+)$ is the measured output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$ and $C \in \mathbb{R}^{n_y\times n}$ are some real matrices.

First of all, we define intervals of $\mathbb{R}^n$. Let $\underline{x}$ and $\overline{x}$ be two vectors in $\mathbb{R}^n$ such that $\underline{x} \leq \overline{x}$ with the inequality holding componentwise. An interval of $\mathbb{R}^n$, denoted $[\underline{x}, \overline{x}]$ is the subset defined by

$$[\underline{x}, \overline{x}] = \left\{x \in \mathbb{R}^n : \underline{x} \leq x \leq \overline{x}\right\} \tag{2}$$

Now we consider the following assumption.

Assumption 1 *There exist (known) bounded signals $(\underline{w}, \overline{w})$ and $(\underline{v}, \overline{v})$ respectively in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_w})$ and in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_y})$ such that $\underline{w}(t) \leq w(t) \leq \overline{w}(t)$ and $\underline{v}(t) \leq v(t) \leq \overline{v}(t)$ for all $t \in \mathbb{R}_+$.*

Here again the inequalities are understood componentwise.

We consider in this paper the problem of synthesizing an *interval estimator* for the state of the LTI system (1). The notion of estimator is understood here in the general sense of a map (also called an operator) between function spaces which may or may not be realizable as the solution of a differential equation (see Remark 1 and Definition 1 below). Hence the interval estimator to be designed will not necessarily be in the more usual form of observer. Considering that the initial state $x(0)$ of (1) lives in an interval of the form $[\underline{x}(0), \overline{x}(0)] \subset \mathbb{R}^n$ and that the external signals w and v satisfy Assumption 1, we want to estimate upper and lower bounds $\overline{x}(t)$ and $\underline{x}(t)$, $t \in \mathbb{R}_+$, for all possible state trajectories of the uncertain system (1).

Remark 1 *A dynamical system is viewed here as a map $\Sigma : \mathbb{R}^{n_x} \times \mathcal{L}^{n_\omega}(\mathbb{R}_+) \to \mathcal{L}^{n_z}(\mathbb{R}_+)$ between function spaces with $\mathbb{R}_+$ being called the time domain of the system. Σ associates to a pair $(X_0, \omega) \in \mathbb{R}^{n_x} \times \mathcal{L}^{n_\omega}(\mathbb{R}_+)$ of a vector X_0 called the initial state and a signal ω called the input signal, a signal $z \triangleq \Sigma(X_0, \omega)$ called the output signal. It can be conveniently described by its so-called input-output map $f_\Sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n_x} \times \mathcal{L}^{n_\omega}(\mathbb{R}_+) \to \mathcal{L}^{n_z}(\mathbb{R}_+)$ defined by $z(t) = f_\Sigma(t_0, t, X_0, \omega)$. The so-defined $z(t)$ is hence the output of the system Σ at time t when it starts at time t_0 from state X_0 and is driven by the input ω. Also, some dynamical systems can be described through the solutions of a differential equation called state-space realization (similar to the differential equation in (1)). See e.g., [23] for more on this formalism.*

Definition 1 (Interval estimator) *Consider the system (1) and let $\omega(t) = [\underline{w}(t)^\top \ \overline{w}(t)^\top \ \underline{v}(t)^\top \ \overline{v}(t)^\top \ y(t)^\top]^\top$ and $X_0 = [\underline{x}(t_0)^\top, \overline{x}(t_0)^\top]^\top$ for some t_0. Consider a dynamical system Σ_e with output $(\underline{x}, \overline{x})$ defined by its input-output maps $(\underline{f}_{\Sigma_e}, \bar{f}_{\Sigma_e})$ as*

$$\begin{aligned} \underline{x}(t) &= \underline{f}_{\Sigma_e}(t_0, t, X_0, \omega) \\ \overline{x}(t) &= \bar{f}_{\Sigma_e}(t_0, t, X_0, \omega) \end{aligned} \tag{3}$$

where $\underline{f}_{\Sigma_e} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{2n} \times \mathcal{L}^{n_\omega}(\mathbb{R}_+) \to \mathbb{R}^n$ and $\bar{f}_{\Sigma_e}$ is defined similarly.
The system defined by (3) is called an interval estimator for system (1) if:

- *(a) Any state trajectory x of (1) satisfies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq t_0$, whenever $\underline{x}(t_0) \leq x(t_0) \leq \overline{x}(t_0)$*
- *(b) (3) is Bounded Input-Bounded Output (BIBO) stable, i.e. $(\underline{x}, \overline{x})$ is bounded whenever X_0 and ω are bounded.*

Here the signals b_w, b_v, y and the initial state vector X_0 are viewed as the inputs of system (3). Boundedness is understood in the sense of the infinity norm being finite.

We will discuss two types of interval estimators: open-loop interval estimators (or simulators) where (3) does not depend on the measurements y and the measurement noise b_v ; and closed-loop interval estimators where measurement is fed back to the estimator.
There are in principle infinitely many estimators that qualify as interval estimators in the sense of Definition 1. It is therefore desirable to define a performance index (measuring e.g. the size of the estimator) which selects the best estimator among all. We will be interested here in the smallest interval estimator in the following sense.

Definition 2 *Let $\mathcal{S}$ denote a subset of $\mathbb{R}^n$. An interval $\mathcal{I}_\mathcal{S} \subset \mathbb{R}^n$ is called the tightest interval containing $\mathcal{S}$ if $\mathcal{S} \subset \mathcal{I}_\mathcal{S}$ and if for any interval $\mathcal{J}$ of $\mathbb{R}^n$, $\mathcal{S} \subset \mathcal{J} \Rightarrow \mathcal{I}_\mathcal{S} \subset \mathcal{J}$.*

In other words, the tightest interval $\mathcal{I}_\mathcal{S}$ "generated" by $\mathcal{S}$ is the intersection of all intervals containing $\mathcal{S}$.

2.2 Preliminary material on interval representation

An important observation for future developments of the paper is that the interval $[\underline{x}, \overline{x}]$ can be equivalently represented by

$$\mathcal{C}(c_x, p_x) = \left\{ c_x + P_x \alpha : \alpha \in \mathbb{R}^n, \ \|\alpha\|_\infty \leq 1 \right\} \tag{4}$$

where

$$c_x = \frac{\overline{x} + \underline{x}}{2}, \quad P_x = \operatorname{diag}\left(p_x\right), \quad p_x = \frac{\overline{x} - \underline{x}}{2} \tag{5}$$

Here, the notation $\|\cdot\|_\infty$ refers to the infinite norm of vectors and $\operatorname{diag}(v)$ is as defined in the Notation section. We will call the so-defined c_x the center of the interval $[\underline{x}, \overline{x}]$ and p_x its radius (or width). To sum up, the interval set can be equivalently represented by the pairs $(\underline{x}, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(c_x, p_x) \in \mathbb{R}^n \times \mathbb{R}^n_+$ i.e., $[\underline{x}, \overline{x}] = \mathcal{C}(c_x, p_x)$. Finally, it will be useful to keep in mind that then $\underline{x} = c_x - p_x$ and $\overline{x} = c_x + p_x$.

The following lemma states a key result for later uses.

Lemma 1 *Let $(c_z, p_z) \in \mathbb{R}^m \times \mathbb{R}^m_+$ and $(c_w(t), p_w(t)) \in \mathbb{R}^{n_w} \times \mathbb{R}^{n_w}_+$ be center-radius representations of some intervals $[\underline{z}, \overline{z}]$ and $[\underline{w}(t), \overline{w}(t)]$ where c_w is in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_w})$ and p_w in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_w}_+)$. Let $F \in \mathbb{R}^{n \times m}$ be a fixed value matrix and H be a matrix function in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n \times n_w})$. Consider the subset $\mathbb{I}$ of $\mathbb{R}^n$ defined by*

$$\begin{aligned} \mathbb{I} = \Bigg\{ & Fz + \int_{t_0}^{t_1} H(\tau) w(\tau) d\tau : \\ & z \in [\underline{z}, \overline{z}], \ w \text{ measurable}, w(\tau) \in [\underline{w}(\tau), \overline{w}(\tau)] \Bigg\} \end{aligned} \tag{6}$$

with $[t_0, t_1]$ being some interval of $\mathbb{R}_+$. Finally, consider

the pair (c, p) defined by:

$$c = Fc_z + \int_{t_0}^{t_1} H(\tau)c_w(\tau)d\tau \tag{7}$$

$$p = |F|\, p_z + \int_{t_0}^{t_1} |H(\tau)|\, p_w(\tau)d\tau \tag{8}$$

Then, $[c-p, c+p]$ is the tightest interval set enclosing $\mathbb{I}$ in the sense of Definition 2.

PROOF. We first show that $\mathbb{I} \subset [c-p, c+p]$. Let $x \in \mathbb{I}$. Then x can be written in the form

$$x = Fz + \int_{t_0}^{t_1} H(\tau)w(\tau)d\tau,$$

where z and w obey the conditions in the definition of $\mathbb{I}$. As discussed in Section 2.2 we can describe the uncertain vector z and uncertain signal w by $z = c_z + P_z\alpha_z$ and $w(\tau) = c_w(\tau) + P_w(\tau)\alpha_w(\tau)$ respectively with $\alpha_z \in \mathbb{R}^n$ and $\alpha_w(\tau) \in \mathbb{R}^{n_w}$ such that $\|\alpha_z\|_\infty \leq 1$ and $\|\alpha_w(\tau)\|_\infty \leq 1$ for all τ and $P_z = \mathrm{diag}(p_z)$, $P_w(\tau) = \mathrm{diag}(p_w(\tau))$. It follows, by plugging these representations in the expression of x, that $x = c + \psi$ with c expressed as in (7) and

$$\psi = FP_z\alpha_z + \int_{t_0}^{t_1} H(\tau)P_w(\tau)\alpha_w(\tau)d\tau.$$

Now consider the vectors $r^+ \in \mathbb{R}^n$ and $r^- \in \mathbb{R}^n$ defined for all $i = 1, \ldots, n$, by

$$r_i^+ = \max\Big\{\psi_i : \|\alpha_z\|_\infty \leq 1,\ \|\alpha_w(\tau)\|_\infty \leq 1,\ \tau \in [t_0, t_1]\Big\}$$

$$r_i^- = \min\Big\{\psi_i : \|\alpha_z\|_\infty \leq 1,\ \|\alpha_w(\tau)\|_\infty \leq 1,\ \tau \in [t_0, t_1]\Big\}$$

By denoting the i-th row of F with $f_i^\top$ and that of $H(\tau)$ with $h_i^\top(\tau)$, we see that the maximizing values of the decision variables are $\alpha_z^* = \mathrm{sign}(f_i)$ and $\alpha_w^*(\tau) = \mathrm{sign}(h_i(\tau))$, $t_0 \leq \tau \leq t_1$ hence leading to

$$r_i^+ = \big|f_i^\top\big|p_z + \int_{t_0}^{t_1} \big|h_i^\top(\tau)\big|p_w(\tau)d\tau.$$

Here sign refers to the sign function operating componentwise. Hence for all $i = 1, \ldots, n$, r_i^+ is equal to the i-th entry of p defined in (8) and so $r^+ = p$. Similarly it can be seen that $r^- = -p$. By definition of r^+ and r^-, it is obvious that $c + r^- \leq x \leq c + r^+$. Hence $\mathbb{I} \subset [c-p, c+p]$.
For clarity of the rest of the proof, we additionally observe that since $-p_i = \min_{x\in\mathbb{I}}(x-c)_i$ and $p_i = \max_{x\in\mathbb{I}}(x-c)_i$ are minimum and maximum values respectively, they are attainable for some elements s^i and $\tilde{s}^i$ of $\mathbb{I}$, i.e., $(s^i)_i = c_i - p_i$ and $(\tilde{s}^i)_i = c_i + p_i$.
Proof of tightness. We are now left with proving that $[c-p, c+p]$ is the tightest enclosing interval set for $\mathbb{I}$. For this purpose, consider another interval set $[\underline{g}, \overline{g}]$ such that $\mathbb{I} \subset [\underline{g}, \overline{g}]$. Pose $p_g = (\overline{g} - \underline{g})/2$ and $c_g = (\overline{g} + \underline{g})/2$ and consider $x \in \mathbb{I}$. Since x lies in the intersection of $[c-p, c+p]$ and $[\underline{g}, \overline{g}]$, there is α and α_g all with infinity norm less than 1, such that $x = c + \mathrm{diag}(p)\alpha = c_g + \mathrm{diag}(p_g)\alpha_g$, which translates componentwise into

$$-p_{g,i} - c_i + c_{g,i} \leq p_i\alpha_i \leq p_{g,i} - c_i + c_{g,i} \tag{9}$$

for $i = 1, \ldots, n$. From the first part of the proof, we know that for any $i = 1, \ldots, n$, there exists $(s^i, \tilde{s}^i) \in \mathbb{I}^2$ such that $(s^i)_i = c_i - p_i$ and $(\tilde{s}^i)_i = c_i + p_i$. Since x is an arbitrary element of $\mathbb{I}$, the inequalities (9) must hold for both particular instances $x = s^i$ and $x = \tilde{s}^i$. And for these values of $x \in \mathbb{I}$, α_i in (9) clearly takes the values -1 and $+1$ respectively. It follows that

$$-p_{g,i} - c_i + c_{g,i} \leq \pm p_i \leq p_{g,i} - c_i + c_{g,i}$$

from which we see that $c_{g,i} - p_{g,i} \leq c_i - p_i$ and $c_i + p_i \leq c_{g,i} + p_{g,i}$. Hence $[c-p, c+p] \subset [\underline{g}, \overline{g}]$. This shows that $[c-p, c+p]$ is the tightest interval containing $\mathbb{I}$.

Remark 2 *The definition of $\mathbb{I}$ through* (6) *implicitly assumes that z and all the values $w(\tau)$, $\tau \in [t_0, t_1]$, are not related to each other. They are thus treated in the proof (namely in the maximization part) as completely independent variables.*

3 Open-loop interval estimator for LTI systems

3.1 Open-loop simulation: the best interval estimator

We first discuss a simulation of the state trajectory of the LTI system in (1a) under uncertain perturbation w and when the initial state $x(0)$ belongs to a known interval set. In this first step, we do not make use of the measurement equation (1b). To proceed, we will assume that the matrices A and B in (1a) have fixed and known values. We use the notations $(c_w(t), p_w(t)) \in \mathbb{R}^{n_w} \times \mathbb{R}_+^{n_w}$ and $(c_x(t), p_x(t)) \in \mathbb{R}^n \times \mathbb{R}_+^n$, $t \in \mathbb{R}_+$, to denote the center-radius representations for the intervals $[\underline{w}(t), \overline{w}(t)]$ and $[\underline{x}(t), \overline{x}(t)]$ respectively.

Theorem 2 *Let the initial conditions and the uncertain input sets of system* (1a) *be described respectively by $(c_x(0), p_x(0))$ and $(c_w(t), p_w(t))$ for $t \in \mathbb{R}_+$. Assume that system* (1) *is stable, i.e., A is Hurwitz.*
Then the interval trajectory $[\underline{x}, \overline{x}]$ defined by

$$\underline{x}(t) = c_x(t) - p_x(t) \quad \text{and} \quad \overline{x}(t) = c_x(t) + p_x(t) \tag{10}$$

with

$$c_x(t) = e^{At}c_x(0) + \int_0^t e^{A(t-\tau)}Bc_w(\tau)d\tau \tag{11}$$

$$p_x(t) = \left|e^{At}\right| p_x(0) + \int_0^t \left|e^{A(t-\tau)}B\right|p_w(\tau)d\tau, \tag{12}$$

is the tightest interval-valued estimator for system (1) *(in open loop) in the sense of Definition 1.*

PROOF. That $[\underline{x}, \overline{x}]$ defined in (10) is the tightest interval-valued trajectory containing the trajectories of (1) is a statement that follows directly from Lemma 1. It suffices to note that the solution of (1) takes the form

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bw(\tau)d\tau \tag{13}$$

and apply the lemma. As to condition (b) of Definition 1, it is an immediate consequence of the assumption that the matrix A of system (1) is Hurwitz. □

Framed differently, the theorem states that the interval estimator (10)-(12) is the intersection of all enclosing intervals for the state trajectories generated by the uncertain system (1) when no measurement is available. Now the question we ask is how to compute the proposed estimates. Of course, a direct implementation of the equations (10)-(12) might be overly expensive in finite-time and unfeasible when the time horizon considered for estimation goes to infinity. We will therefore be searching, when possible, for a finite dimensional state-space realization for the signals c_x and p_x. To begin with, note that c_x can be simply realized as $\dot{c}_x = Ac_x + Bc_w$. So the challenge is rather related to the realization of p_x. In the sequel, we discuss a few particular cases where a finite dimensional realization exists.

3.2 On the realization of the tightest estimator

We start by observing that if A is a Metzler matrix and if B is either nonpositive or nonnegative, then p_x in (12) can be simply realized by $\dot{p}_x(t) = Ap_x(t)+|B|p_w(t)$. This follows from the fact that e^{At} is a nonnegative matrix for all $t \geq 0$ whenever A is a Metzler matrix. Consequently, one can drop the absolute value symbols in (12) hence yielding the simple realization displayed above.

A second remark concerns the scenario where p_w is constant. In this latter case, a simple realization of p_x can be obtained as stated in the following proposition.

Proposition 3 *Assume that $p_w(t) = p_w(0)$ for all $t \in \mathbb{R}_+$, i.e., p_w is constant. Then the signal p_x in* (12) *can be realized as follows:*

$$\begin{cases} \dot{M}(t) = AM(t), \quad M(0) = I_n \\ \dot{r}(t) = |M(t)B|\, p_w(0), \quad r(0) = 0 \\ p_x(t) = |M(t)|\, p_x(0) + r(t) \end{cases} \tag{14}$$

with state $(M(t), r(t)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$ and I_n being the identity matrix of order n.

The proof of the proposition follows by simple calculations. We will show below (see Section 3.3.3) that even though p_w is not constant in general, we can rely on this proposition to construct a nice over-approximation of the tightest interval estimator.

Finally let us comment shortly on a possible *linear time-invariant realization* of system (12) (whose input and output are respectively p_w and p_x) independently of the class of inputs. The question is then that of finding a set of matrices $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \phi_0) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times n_w} \times \mathbb{R}^{n\times d} \times \mathbb{R}^d$ for some finite dimension d and such that the solution ϕ of the differential equation $\dot{\phi}(t) = \mathcal{A}\phi(t) + \mathcal{B}p_w(t)$, $\phi(0) = \phi_0$, satisfies $p_x(t) = \mathcal{C}\phi(t)$ for all $t \in \mathbb{R}_+$. Indeed this is true if and only if $\mathcal{C}e^{\mathcal{A}t}\tilde{\mathcal{B}} = \mathcal{H}(t)$ where $\tilde{\mathcal{B}} = [\mathcal{B}\ \phi_0]$ and $\mathcal{H}$ is the *impulse response* of the system (12) defined by $\mathcal{H}(t) \triangleq \left[|e^{At}B|\ |e^{At}|p_x(0)\right]$. If $\mathcal{H}$ satisfies some regularities conditions, one can resort to realization algorithms to compute a minimal realization $(\mathcal{A}, \tilde{\mathcal{B}}, \mathcal{C})$. The reader is referred to, e.g., [6,3] for more on the realization theory of LTI systems. Unfortunately, the conditions for the existence of LTI realizations are unlikely to hold in general here due to the nonsmooth nature of the absolute values in the expression of $\mathcal{H}(t)$.

3.3 Some approximations of the tightest interval estimator

As it turns out, apart from some special situations, implementing the tight estimator (10)-(12) in the most general case is intractable in practice. We therefore consider in this section the question of whether one could over-approximate p_x by a more easily realizable signal $\hat{p}_x$. In order to discuss this question, let us recall some basic mathematical facts that will be useful.

Lemma 4 *Let A and B be matrices of compatible dimensions. Then the following properties hold:*

$$|A + B| \leq |A| + |B| \tag{15a}$$

$$|AB| \leq |A|\,|B| \tag{15b}$$

$$|A| \leq B \quad \Rightarrow \quad \|A\|_2 \leq \|B\|_2 \tag{15c}$$

$$\|A\|_2 = \||A|\|_2 \ \textit{(i.e., the 2-norms of } A \textit{ and } |A| \textit{ are equal)} \tag{15d}$$

$$\left|e^A\right| \leq e^{\psi(A)} \leq e^{|A|} \tag{15e}$$

In (15e) $\psi(A)$ *is a matrix defined by* $[\psi(A)]_{ij} = |A_{ij}|$ *if* $i \neq j$ *and* $[\psi(A)]_{ij} = A_{ij}$ *if* $i = j$.

Indeed $\psi(A)$ is the matrix obtained from A by taking the absolute value of the off-diagonal elements and leaving entries on the main diagonal unchanged: $\psi(A) = \text{diag}(A) + |A - \text{diag}(A)|$. Hence for any square real matrix A, $\psi(A)$ is a Metzler matrix. The facts (15a)-(15d) which were stated in [13, Chap. 8] are straightforward to check. As for (15e), a proof can be found in [2].

In order to reduce the potential complexity associated with the implementation of (12), we discuss three over-approximation methods.

3.3.1 Over-estimating p_x

The following proposition allows to over-estimate p_x with a signal $\hat{p}_x$ whose computation is cheaper. More specifically, we can avoid numerical evaluation of integrals on unbounded time intervals thanks to the following proposition.

Proposition 5 *Let* $T \in \mathbb{R}_+$. *Let* $\hat{p}_x : \mathbb{R}_+ \to \mathbb{R}_+^n$ *be defined by:* $\hat{p}_x(t) = p_x(t)$ *for all* $t \in [0, T[$ *where* p_x *is defined as in* (12), *and*

$$\hat{p}_x(t) = |e^{AT}|\hat{p}_x(t-T) + \int_{t-T}^{t} |e^{A(t-\tau)}B|p_w(\tau)d\tau \quad (16)$$

for all $t \geq T$ *with* A *and* B *being the matrices of system* (1) *and* p_w *as in Theorem 2. Then* $p_x(t) \leq \hat{p}_x(t)\, \forall t \in \mathbb{R}_+$ *and hence the state trajectories generated by system* (1) *satisfy*

$$c_x(t) - \hat{p}_x(t) \leq x(t) \leq c_x(t) + \hat{p}_x(t)\, \forall t \in \mathbb{R}_+. \quad (17)$$

with c_x *defined as in* (11).

PROOF. The solution to (1) can be written as

$$x(t) = e^{AT}x(t-T) + \int_{t-T}^{t} e^{A(t-\tau)}Bw(\tau)d\tau.$$

Now, by applying Lemma 1, it is immediate that (17) holds if we can establish that $\mathcal{C}(c_x(t-T), \hat{p}_x(t-T))$ is an enclosing interval for $x(t-T)$. This in turn is true if $p_x(t-T) \leq \hat{p}_x(t-T)$ for all t. Hence let us show that $p_x(t) \leq \hat{p}_x(t)$ for all t. For this purpose, write t in the form $t = q(t)T + r(t)$ where $q(t)$ is a nonnegative integer and $r(t) \in \mathbb{R}_+$ with $0 \leq r(t) < T$. Then by applying repeatedly (16) leads to

$$\hat{p}_x(t) = |e^{AT}|^{q(t)}\hat{p}_x(r(t)) + \\ + \sum_{j=1}^{q(t)} \int_{t-jT}^{t-(j-1)T} \left|e^{AT}\right|^{(j-1)} \left|e^{A(t-(j-1)T-\tau)}B\right| p_w(\tau)d\tau$$

By applying (15b), we see that

$$\hat{p}_x(t) \geq |e^{ATq(t)}|\hat{p}_x(r(t)) + \int_{t-q(t)T}^{t} \left|e^{A(t-\tau)}B\right| p_w(\tau)d\tau$$

On the other hand $\hat{p}_x(r(t)) = p_x(r(t)) = |e^{Ar(t)}|p_x(0) + \int_0^{r(t)} \left|e^{A(t-\tau)}B\right| p_w(\tau)d\tau$. Plugging this in the last inequality above and applying again (15b) show that $\hat{p}_x(t) \geq p_x(t)$. □

Note that if A is Hurwitz, then T can be chosen sufficiently large so that $|e^{AT}|$ is Schur stable [1]. For such a T, $(c_x, \hat{p}_x)$ defines an interval estimator for system (1) in the sense of Definition 1. As shown by Proposition 5, the interval-valued estimate defined by $(c_x, \hat{p}_x)$ is only an over-estimate of the one resulting from (c_x, p_x). As T gets larger, the two interval estimators will get closer but then the complexity increases. And in the extreme case where $T = t$, we recover $\hat{p}_x = p_x$. The appealing fact with this estimator is that it allows to balance, through the selection of T, computational price and tightness.

3.3.2 Approximation using a Metzler matrix

A second simple approximation can be obtained directly from Lemma 4. In effect, by applying the facts (15a)-(15e) above, we find that $p_x(t) \leq \check{p}_x(t)$, where

$$\check{p}_x(t) \triangleq e^{\psi(A)t}p_x(0) + \int_0^t e^{\psi(A)(t-\tau)}|B|p_w(\tau)d\tau. \quad (18)$$

Although $\check{p}_x$ is a looser estimate of p_x (than e.g., (16)) its benefit lies in the fact that it is much easier to compute. In effect, the new signal $\check{p}_x$ can be realized very simply in the form $\dot{\check{p}}_x(t) = \psi(A)\check{p}_x(t) + |B|p_w(t)$ with $\check{p}_x(0) = p_x(0)$. However, for $(c_x, \check{p}_x)$ to be an interval estimator in the sense of Definition 1, we must require additionally that $\psi(A)$ be Hurwitz.

3.3.3 Over-estimating p_w by a constant vector

Another over-estimate of p_x can be obtained from Proposition 3 as follows. By Assumption 1, p_w is bounded. Therefore, let δ^o be the vector in $\mathbb{R}^{n_w}$ whose i-th entry δ_i^o is defined by $\delta_i^o = \sup_{t\in\mathbb{R}_+} p_{w,i}(t)$ where $p_{w,i}(t)$

[1] i.e., its spectral radius is less than 1.

refers to the i-th entry of $p_w(t)$. Then by letting δ be a signal defined by $\delta(t) = \delta^o$ for all $t \geq 0$, w satisfies $c_w(t) - \delta(t) \leq w(t) \leq c_w(t) + \delta(t)$ and hence (c_w, δ) is a valid (but larger) interval representation for the input signal w which fulfills the condition of Proposition 3. As a consequence, replacing $p_w(0)$ in (14) with δ^o gives a computable realization of an interval estimator for the state of system (1).
For an empirical comparison of the estimators discussed here, see Section 6.

4 Closed-loop state estimator for LTI systems

In case the system (1) is not stable, let us assume it to be observable (or just detectable). Then it is possible to find a matrix gain L such that $A-LC$ is Hurwitz. We can then construct an interval observer from the classical observer form. As we did in open-loop, we can of course write the best estimator (10)-(11) also in closed-loop for a given L or compute its over-approximations discussed in Section 3.3. However here we choose to study further the type of approximation given in (18). Although this type of estimator is not the tightest one, it has the advantage of computational simplicity.

4.1 A systematic design method

In this section we discuss a systematic way of constructing linear interval observers employing an output injection. Departing from the structure of the classical Luenberger observer, it is easy to see that the state of system (1) satisfies

$$\dot{x}(t) = (A - LC)x(t) + Gs(t), \tag{19}$$

where

$$G = [B \ L \ -L] \text{ and } s(t) = [w(t)^\top \ y(t)^\top \ v(t)^\top]^\top \tag{20}$$

with L being the gain of the observer. For a *given matrix* $L \in \mathbb{R}^{n\times n_y}$ such that $A - LC$ is Hurwitz one can apply (11)-(12) to obtain the tightest interval-valued estimator [2] . Similarly, the over-estimates discussed are generically obtainable in the closed-loop configuration as well. In particular it follows from the discussion in Section 3.3.2 that $(c_x^{\mathrm{cl}}, p_x^{\mathrm{cl}})$ given by

$$\begin{cases} \dot{c}_x^{\mathrm{cl}} = (A - LC)c_x^{\mathrm{cl}}(t) + Gc_s(t), \ \ c_x^{\mathrm{cl}}(0) = c_x(0) \\ \dot{p}_x^{\mathrm{cl}} = \psi(A - LC)p_x^{\mathrm{cl}}(t) + |G|\, p_s(t), \ p_x^{\mathrm{cl}}(0) = p_x(0), \end{cases} \tag{21}$$

with $(c_s(t), p_s(t)) \in \mathbb{R}^{n_s} \times \mathbb{R}^{n_s}_+$, $n_s = n_w + 2n_y$, being a center-radius representation of $s(t)$, defines an enclosing interval estimate for the state of system (1).

[2] The question of selecting L to optimize some global tightness-based cost in the spirit of [20] is not discussed here.

Further, the systems (21) yield an interval estimator (observer) for system (1) provided that both $A - LC$ and $\psi(A-LC)$ are Hurwitz. By Lemma 6 stated below, this stability condition is satisfied if and only if $\psi\,(A - LC)$ is Hurwitz.

Lemma 6 ([2]) *Let $A, A_1, A_2 \in \mathbb{R}^{n\times n}$ and $\mathcal{P} \in \mathbb{R}^{n\times n}_+$. Let ψ be the function defined in Lemma 4. Then the following implications hold:*

(a) $\psi(A)$ is Hurwitz $\Rightarrow$ A is Hurwitz.
(b) $\psi(A) + \mathcal{P}$ is Hurwitz $\Rightarrow$ $\psi(A)$ is Hurwitz
(c) $\psi(A_1) \leq \psi(A_2) \Rightarrow 0 \leq e^{\psi(A_1)} \leq e^{\psi(A_2)}$
(d) If $\psi(A_1) \leq \psi(A_2)$, then $\psi(A_1)$ is Hurwitz whenever $\psi(A_2)$ is Hurwitz

The question now is how to effectively select a matrix gain $L \in \mathbb{R}^{n\times n_y}$ so as to fulfill the condition $\psi\,(A - LC)$ is Hurwitz. An answer is provided by the following lemma.

Lemma 7 ([2]) *The following statements are equivalent:*

(1) There exists $L \in \mathbb{R}^{n\times n_y}$ such that $\psi(A - LC)$ is Hurwitz.
(2) There exist a diagonal positive definite matrix $P \in \mathbb{R}^{n\times n}$ and some matrices $Y \in \mathbb{R}^{n\times n_y}$, $X \in \mathbb{R}^{n\times n}$ satisfying the conditions:

$$\begin{array}{l} X^\top + X + 2\,\mathrm{diag}(S) \prec 0 \\ |S - \mathrm{diag}(S)| \leq X \end{array} \tag{22}$$

where $S = PA - YC$. In case the statements hold, L is given by $L = P^{-1}Y$.

Lemma 7 shows that one can compute the observer gain L efficiently by solving a feasibility problem which is expressible in terms of Linear Matrix Inequalities (LMI) [4]. In comparison to classical results we do not require $A - LC$ to be Metzler since $\psi(A - LC)$ is naturally Metzler. Hence the only constraint associated with the search for the gain L is the Hurwitz stability of $\psi(A - LC)$.

5 Interval estimation in the presence of uncertain parameters

Consider now the general situation where the matrices (A, B) together with the initial state $x(0)$ and the signals (w, v) in (1) are all uncertain.

5.1 A technical lemma

Let us use the following notational convention. For an uncertain matrix $A \in \mathbb{R}^{n\times m}$, let $[\underline{A}, \overline{A}]$ be its interval

representation. To be precise, recall that the interval set $[\underline{A}, \overline{A}]$ for matrices in $\mathbb{R}^{n\times m}$ is defined as

$$[\underline{A}, \overline{A}] = \left\{A \in \mathbb{R}^{n\times m} : \underline{A} \leq A \leq \overline{A}\right\}$$

where all inequalities hold componentwise. Similarly as in the case of vectors, $[\underline{A}, \overline{A}]$ admits a center-radius representation $(C_A, \mathcal{P}_A) \in \mathbb{R}^{n\times m} \times \mathbb{R}_+^{n\times m}$ such that

$$[\underline{A}, \overline{A}] = \left\{C_A + \mathcal{P}_A \odot S_A : S_A \in \mathbb{R}^{n\times m}, \|\mathrm{vec}(S_A)\|_\infty \leq 1\right\} \tag{23}$$

where the symbol $\odot$ refers to the Hadamard product (componentwise product of matrices) and $\mathrm{vec}(\cdot)$ is the vectorization operator.

To proceed, we will need an extension of Lemma 1 to the general estimation scenario.

Lemma 8 *Let $(c_z, p_z) \in \mathbb{R}^m \times \mathbb{R}_+^m$ and $(c_w(t), p_w(t)) \in \mathbb{R}^{n_w} \times \mathbb{R}_+^{n_w}$ be center-radius representations of some intervals $[\underline{z}, \overline{z}]$ and $[\underline{w}(t), \overline{w}(t)]$ where c_w in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}^{n_w})$ and p_w in $\mathcal{L}^\infty(\mathbb{R}_+, \mathbb{R}_+^{n_w})$.*
Let $(\mathcal{C}_F, \mathcal{P}_F) \in \mathbb{R}^{n\times m} \times \mathbb{R}_+^{n\times m}$ and $(\mathcal{C}_H(t), \mathcal{P}_H(t)) \in \mathbb{R}^{n\times n_w} \times \mathbb{R}_+^{n\times n_w}$, $t \in \mathbb{R}_+$, be center-radius representations of $[\underline{F}, \overline{F}]$ and $[\underline{H}(t), \overline{H}(t)]$ respectively.
Consider the set $\mathbb{J} \subset \mathbb{R}^n$ defined by

$$\begin{aligned}\mathbb{J} = \Big\{ Fz + \int_{t_0}^{t_1} H(\tau)w(\tau)d\tau : \\ z \in [\underline{z}, \overline{z}],\ F \in [\underline{F}, \overline{F}], H, w \text{ measurable} \\ w(\tau) \in [\underline{w}(\tau), \overline{w}(\tau)],\ H(\tau) \in [\underline{H}(\tau), \overline{H}(\tau)] \Big\}\end{aligned} \tag{24}$$

with $[t_0, t_1]$ being some fixed interval of $\mathbb{R}_+$.
Finally, let $(c, p) \in \mathbb{R}^n \times \mathbb{R}_+^n$ be defined by

$$c = C_F c_z + \int_{t_0}^{t_1} C_H(\tau)c_w(\tau)d\tau \tag{25}$$

and $p = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}^\top$ with

$$\begin{aligned} p_i = \Lambda_i\left(C_F, \mathcal{P}_F, c_z, p_z\right) \\ + \int_{t_0}^{t_1} \Lambda_i\left(C_H(\tau), \mathcal{P}_H(\tau), c_w(\tau), p_w(\tau)\right) d\tau \end{aligned} \tag{26}$$

for $i = 1, \ldots, n$, with[3]
$\Lambda_i\left(C_F, \mathcal{P}_F, c_z, p_z\right) = \sum_{j=1}^m \nu\left(C_F^{ij} p_z^j, \mathcal{P}_F^{ij} c_z^j, \mathcal{P}_F^{ij} p_z^j\right)$ and $\nu : \mathbb{R}^3 \to \mathbb{R}$ being the function defined by

$$\nu(a_1, a_2, a_3) = \max_{|s|\leq 1, |r|\leq 1} \left(a_1 s + a_2 r + a_3 sr\right) \tag{27}$$

[3] Notation: C_F^{ij} refers to the (i,j)-th entry of the matrix $\mathcal{C}_F$ and p_z^j is the j-th entry of the vector p_z.

Then (c, p) is a center-radius representation of the tightest interval set of $\mathbb{R}^n$ containing $\mathbb{J}$.

PROOF. For any $\varphi \in \mathbb{J}$ there is $(F, z) \in [\underline{F}, \overline{F}] \times [\underline{z}, \overline{z}]$, $(H(\tau), w(\tau)) \in [\underline{H}(\tau), \overline{H}(\tau)] \times [\underline{w}(\tau), \overline{w}(\tau)]$ such that

$$\varphi = Fz + \int_{t_0}^{t_1} H(\tau)w(\tau)d\tau \tag{28}$$

Moreover, there exist $\alpha_z \in \mathbb{R}^n$, $S_F \in \mathbb{R}^{n\times m}$, $S_H(\tau) \in \mathbb{R}^{n\times n_w}$, $\alpha_w(\tau) \in \mathbb{R}^{n_w}$ satisfying $\|\alpha_z\|_\infty \leq 1$, $\|\mathrm{vec}(S_F)\|_\infty \leq 1$, $\|\alpha_w(\tau)\|_\infty \leq 1$, $\|\mathrm{vec}(S_H(\tau))\|_\infty \leq 1$ and such that

$$\begin{aligned} F &= C_F + \mathcal{P}_F \odot S_F, \\ z &= c_z + P_z \alpha_z, \\ H(\tau) &= C_H(\tau) + \mathcal{P}_H(\tau) \odot S_H(\tau), \\ w(\tau) &= c_w(\tau) + P_w(\tau)\alpha_w(\tau), \end{aligned}$$

with $P_z = \mathrm{diag}(p_z)$, $P_w(\tau) = \mathrm{diag}(p_w(\tau))$. Plugging these in (28) yields $\varphi = c + \xi$ with c as in (25) and

$$\begin{aligned} \xi = C_F P_z \alpha_z + (\mathcal{P}_F \odot S_F)c_z + (\mathcal{P}_F \odot S_F)P_z\alpha_z \\ + \int_{t_0}^{t_1} \big[C_H(\tau)P_w(\tau)\alpha_w(\tau) + (\mathcal{P}_H(\tau) \odot S_H(\tau))c_w(\tau) \\ + (\mathcal{P}_H(\tau) \odot S_H(\tau))P_w(\tau)\alpha_w(\tau)\big]d\tau \end{aligned}$$

Proceeding similarly as in the proof of Lemma 1, we just need to show that $p_i = \max \xi_i$ for all $i = 1, \ldots, n$, where the maximization is taken over all S_F, α_z and $S_H(\tau)$, $\alpha_w(\tau)$, $t_0 \leq \tau \leq t_1$ as defined above, with the magnitude constraint on their entries. For a matrix A, let for convenience A^{ij} denote the (i,j)-th entry of A. Using this convention, note that

$$\begin{aligned} &\max \xi_i \\ &= \max_{\substack{S_F, \alpha_z, \\ S_H(\tau), \alpha_w(\tau) \\ t_0 \leq \tau \leq t_1}} \Big[\sum_{j=1}^m C_F^{ij} p_z^j \alpha_z^j + \mathcal{P}_F^{ij} S_F^{ij} c_z^j + \mathcal{P}_F^{ij} S_F^{ij} p_z^j \alpha_z^j \\ &\qquad + \int_{t_0}^{t_1} \big(\sum_{j=1}^{n_w} C_H^{ij} p_w^j \alpha_w^j + \mathcal{P}_H^{ij} S_H^{ij} c_w^j + \mathcal{P}_H^{ij} S_H^{ij} p_w^j \alpha_w^j \big) d\tau \Big] \\ &= \sum_{j=1}^m \max_{|\alpha_z^j|\leq 1, |S_F^{ij}|\leq 1} \Big(C_F^{ij} p_z^j \alpha_z^j + \mathcal{P}_F^{ij} S_F^{ij} c_z^j + \mathcal{P}_F^{ij} S_A^{ij} p_z^j \alpha_z^j \Big) \\ &+ \int_{t_0}^{t_1} \Big(\sum_{j=1}^{n_w} \max_{S_H(\tau), \alpha_w(\tau)} \big(C_H^{ij} p_w^j \alpha_w^j + \mathcal{P}_H^{ij} S_H^{ij} c_w^j \\ &\qquad + \mathcal{P}_H^{ij} S_H^{ij} p_w^j \alpha_w^j \big) \Big) d\tau \end{aligned}$$

Note that it is possible here to introduce the maximum in the summand (and integrand) because all decision variables (i.e., the entries of S_F, S_H, α_z, α_w) are independent here so that all terms involved in the summands

(and integrand) can be maximized separately.

By now recalling the definition of the function ν in (27), we can write

$$\max \xi_i = \sum_{j=1}^{m} \nu\big(C_F^{ij} p_z^j, \mathcal{P}_F^{ij} c_z^j, \mathcal{P}_F^{ij} p_z^j\big) + \int_{t_0}^{t_1} \big[\sum_{j=1}^{n_w} \nu\big(C_H^{ij} p_w^j, \mathcal{P}_H^{ij} c_w^j, \mathcal{P}_H^{ij} p_w^j\big)\big] d\tau$$

and hence $\max \xi_i$ is equal to p_i defined in (26). This conclude the proof. □

Below, we state an explicit way of computing the values of the function ν in (27).

Lemma 9 *Let a_i, $i = 1, \ldots, 3$ be three real numbers and $\sigma : \{1,2,3\} \to \{1,2,3\}$ be a permutation such that $\big|a_{\sigma(1)}\big| \geq \big|a_{\sigma(2)}\big| \geq \big|a_{\sigma(3)}\big|$. Then the function ν defined in (27) satisfies*

$$\nu(a_1, a_2, a_3) = \big|a_{\sigma(1)}\big| + \big|a_{\sigma(2)}\big| + a_{\sigma(3)} \operatorname{sign}(a_{\sigma(1)}) \operatorname{sign}(a_{\sigma(2)}). \tag{29}$$

PROOF. Without loss of generality, assume that $\sigma(i) = i$. Note then that if a_1 or a_2 is equal to zero, then $a_3 = 0$ as well and (29) holds clearly. Hence assume that $a_1 a_2 \neq 0$ and let us proceed by contradiction for the rest. For this purpose, assume that for all $(a_1, a_2, a_3) \in \mathbb{R}^3$, there exist $(s, r) \in [-1, 1] \times [-1, 1]$ such that

$$a_1 s + a_2 r + a_3 sr > |a_1| + |a_2| + a_3 \operatorname{sign}(a_1) \operatorname{sign}(a_2).$$

Then

$$a_3 \left(\operatorname{sign}(a_1) \operatorname{sign}(a_2) - sr\right) < (a_1 s - |a_1|) + (a_2 r - |a_2|) \leq 0$$

Let $\delta = \operatorname{sign}(a_1) \operatorname{sign}(a_2) \in \{-1, +1\}$. Since a_1 and a_2 are arbitrary, δ is also arbitrary. The proof hypothesis can therefore be reformulated as: for all $\delta \in \{-1, +1\}$ and for all $a_3 \in \mathbb{R}$, there exist $(s, r) \in [-1, 1] \times [-1, 1]$ such that $a_3(\delta - sr) < 0$. This is not true since for example $(a_3, \delta) = (1, 1)$ yields a contradiction. Hence for all $(s, r) \in [-1, 1] \times [-1, 1]$

$$a_1 s + a_2 r + a_3 sr \leq |a_1| + |a_2| + a_3 \operatorname{sign}(a_1) \operatorname{sign}(a_2).$$

Moreover the left-hand side member of this inequality is attained for $s = \operatorname{sign}(a_1)$ and $r = \operatorname{sign}(a_2)$. In conclusion (29) holds as stated. □

It is interesting to observe that Lemma 1 can be easily recovered from Lemma 8. For example, under the conditions of Lemma 1, $\mathcal{C}_F = F$, $\mathcal{P}_F = 0$, $\mathcal{C}_H(\tau) = H(\tau)$, $\mathcal{P}_H(\tau) = 0$ so that p_i in (26) reduces to

$$\begin{aligned} p_i &= \sum_{j=1}^{m} \nu\big(F^{ij} p_z^j, 0, 0, \big) + \int_{t_0}^{t_1} \big[\sum_{j=1}^{n_w} \nu\big(H^{ij}(\tau) p_w^j(\tau), 0, 0\big)\big] d\tau \\ &= |f_i|^\top p_z + \int_{t_0}^{t_1} |h_i^\top(\tau)| p_w(\tau) d\tau \end{aligned}$$

with $f_i^\top$ and $h_i^\top(\tau)$ being the i-th rows of F and $H(\tau)$ respectively. The expression above coincides with the i-th entry of p defined in (8).

5.2 Construction of an interval estimator

To derive an interval-valued state estimator, the idea now is to apply Lemma 8 to Eq. (13). However the expression of the best estimator might, in this case, be of excessive formal complexity due to the necessity to find the smallest interval set containing the matrix exponential. We shall therefore consider a more readable overestimate. One ingredient for this purpose is the following lemma.

Lemma 10 *Let $A \in \mathbb{R}^{n \times n}$, $\mathcal{P} \in \mathbb{R}_+^{n \times n}$ and $S \in [-1, 1]^{n \times n}$ be some matrices. Then*

$$\left|e^{A + \mathcal{P} \odot S} - e^A\right| \leq e^{\psi(A) + \mathcal{P}} - e^{\psi(A)} \tag{30}$$

where ψ is the function defined in Lemma 4 and $\odot$ refers to the Hadamard matrix product.

PROOF. We first prove that for any two square matrices A and B of the same dimensions,

$$\left|(A + B)^k - A^k\right| \leq (C + |B|)^k - C^k \tag{31}$$

for any matrix C satisfying $|A| \leq C$ and for all $k \geq 1$. We proceed by induction on k. Clearly, the inequality (31) holds for $k = 1$. By noting that

$$(A + B)^{k+1} - A^{k+1} = A\left[(A + B)^k - A^k\right] + B(A + B)^k,$$

and making use of the inequalities (15a) and (15b), we easily see that if (31) is true for k, then it is also true for $k + 1$. In conclusion (31) holds true as claimed, for any $k \geq 1$.

We now turn to proving (30). For this purpose, let $\alpha > 0$ be such that $\alpha I + \operatorname{diag}(A) \geq 0$. It follows that $|\alpha I + A| = |(\alpha I + \operatorname{diag}(A)) + (A - \operatorname{diag}(A))| \leq \alpha I + \psi(A)$. We can write

$$\begin{aligned} &\left|e^{\alpha I + A + \mathcal{P} \odot S} - e^{\alpha I + A}\right| \\ &\quad \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left|(\alpha I + A + \mathcal{P} \odot S)^k - (\alpha I + A)^k\right| \end{aligned}$$

which, by applying (31), gives

$$e^{\alpha}\left|e^{A+\mathcal{P}\odot S}-e^{A}\right| \leq e^{\alpha}\left(e^{\psi(A)+|\mathcal{P}\odot S|}-e^{\psi(A)}\right)$$

This in turn yields (30) since $0 \leq |\mathcal{P}\odot S| \leq \mathcal{P}$. □

By applying Lemma 10, we see that if $A \in [\underline{A}, \overline{A}] \subset \mathbb{R}^{n\times n}$ as defined in (23), then for all $t \in \mathbb{R}_+$, e^{At} lies in the matrix interval $\mathcal{C}(C_F(t), \mathcal{P}_F(t))$ defined by

$$C_F(t) = e^{C_A t} \quad \text{and} \quad \mathcal{P}_F(t) = e^{(\psi(C_A)+\mathcal{P}_A)t} - e^{\psi(A)t}.$$

Likewise, it can be observed that if $A \in [\underline{A}, \overline{A}] \subset \mathbb{R}^{n\times n}$ and if $B \in [\underline{B}, \overline{B}] \subset \mathbb{R}^{n\times n_w}$, then $M(t) \triangleq e^{At}B$ lies in $\mathcal{C}\left(C_M(t), \mathcal{P}_M(t)\right)$ where $C_M(t) = e^{C_A t}C_B$ and the entries of the matrix $\mathcal{P}_M(t)$ are defined by $\mathcal{P}_M^{ij}(t) = \Lambda_i\left(C_F(t), \mathcal{P}_F(t), [C_B]_{:j}, [\mathcal{P}_B]_{:j}\right)$ with Λ_i defined as in Lemma 8 and $[C_B]_{:j}$ denoting the j-th column of C_B.

Then, given the uncertainty sets $(c_x(0), p_x(0))$, $(c_w(t), p_w(t))$, $t \in \mathbb{R}_+$, $(\mathcal{C}_A, \mathcal{P}_A)$, $(\mathcal{C}_B, \mathcal{P}_B)$, associated respectively with the initial state, the system input and the parameter matrices A and B, the interval-valued trajectory $\mathcal{C}(\tilde{c}_x(t), \tilde{p}_x(t))$, with $\tilde{c}_x$ and $\tilde{p}_x$ defined by

$$\tilde{c}_x(t) = C_F(t)c_x(0) + \int_0^t C_M(\tau)c_w(t-\tau)d\tau \qquad (32)$$

and $\tilde{p}_x(t) = \left[\tilde{p}_x^1(t) \ \cdots \ \tilde{p}_x^n(t)\right]^\top$,

$$\begin{aligned}\tilde{p}_x^i(t) = &\Lambda_i\left(C_F(t), \mathcal{P}_F(t), c_x(0), p_x(0)\right)\\ &+ \int_0^t \Lambda_i\big(C_M(\tau), \mathcal{P}_M(\tau), c_w(t-\tau), p_w(t-\tau)\big)d\tau\end{aligned} \qquad (33)$$

would constitute an interval-valued estimator for system (1) if we can guarantee the stability condition. Note again that $\tilde{c}_x$ in (32) can be realized by $\dot{\tilde{c}}_x = C_A\tilde{c}_x(t) + C_B c_w(t)$, $\tilde{c}_x(0) = c_x(0)$.

Theorem 11 *Consider system* (1) *where the initial state* $x(0)$*, the input* w *and the parameter matrices* A *and* B *are all uncertain and described as above. If* $\psi(C_A) + \mathcal{P}_A$ *is Hurwitz, then* $\mathcal{C}(\tilde{c}_x(t), \tilde{p}_x(t))$ *with* $\tilde{c}_x(t)$ *and* $\tilde{p}_x(t)$ *defined respectively as in* (32) *and* (33) *is an interval estimator for system* (1)*.*

PROOF. From the above discussions we already know that the bounding condition is satisfied. Hence we just need to prove the stability condition of Definition 1. By Lemma 6, the system defining $\tilde{c}_x$ in (32) is BIBO stable if $\psi(C_A) + \mathcal{P}_A$ is Hurwitz. It remains to prove that $\tilde{p}_x$ is bounded as well whenever $p_x(0)$, c_w and p_w are bounded.

For this purpose we start by noting that the function ν in (27) satifies $\nu(a_1, a_2, a_3) \leq |a_1| + |a_2| + |a_3|$. As a consequence,

$$\begin{aligned}\Lambda_i\left(C_F(t), \mathcal{P}_F(t), c_x(0), p_x(0)\right) \leq \\ \left(\big(|C_F(t)| + \mathcal{P}_F(t)\big)p_x(0) + \mathcal{P}_F(t)|c_x(0)|\right)_i\end{aligned}$$

Applying this to the whole expression of $\tilde{p}_x(t)$ gives

$$\begin{aligned}\tilde{p}_x(t) \leq &\big(|C_F(t)| + \mathcal{P}_F(t)\big)p_x(0) + \mathcal{P}_F(t)|c_x(0)| +\\ &+ \int_0^t \left[\big(|C_M(\tau)| + \mathcal{P}_M(\tau)\big)p_w(t-\tau) + \mathcal{P}_M(\tau)|c_w(t-\tau)|\right]d\tau\end{aligned} \qquad (34)$$

where $\mathcal{P}_M(t) \leq \big(|C_F(t)| + \mathcal{P}_F(t)\big)\mathcal{P}_B + \mathcal{P}_F(t)\,|C_B|$ for all $t \in \mathbb{R}_+$. Using now (15d) and the submultiplicativity and subadditivity properties of the Euclidean norm, we get

$$\begin{aligned}\|\tilde{p}_x(t)\|_2 \leq &\left(\|C_F(t)\|_2 + \|\mathcal{P}_F(t)\|_2\right)\|p_x(0)\|_2\\ &+ \|\mathcal{P}_F(t)\|_2\,\|c_x(0)\|_2\\ &+ \int_0^t \Big[\left(\|C_M(\tau)\|_2 + \|\mathcal{P}_M(\tau)\|_2\right)\|p_w(t-\tau)\|_2\\ &\qquad + \|\mathcal{P}_M(\tau)\|_2\,\|c_w(t-\tau)\|_2\Big]d\tau\end{aligned} \qquad (35)$$

with $\|\mathcal{P}_M(t)\|_2 \leq \left(\|C_F(t)\|_2 + \|\mathcal{P}_F(t)\|_2\right)\|\mathcal{P}_B\|_2 + \|\mathcal{P}_F(t)\|_2\,\|C_B\|_2$. Recall the definitions of $C_F(t)$, $\mathcal{P}_F(t)$, $C_M(t)$ given above. Under the assumption that $\psi(C_A) + \mathcal{P}_A$ is Hurwitz, we know by Lemma 6 that C_A and $\psi(C_A)$ are also Hurwitz. As a consequence, there exist some strictly positive numbers $\lambda_1, \lambda_2, \lambda_3$ and some constant c_1, c_2, c_3 such that $\|C_F(t)\|_2 \leq c_1 e^{-\lambda_1 t}$, $\|\mathcal{P}_F(t)\|_2 \leq c_2 e^{-\lambda_2 t}$, $\|C_M(t)\|_2 \leq c_3 e^{-\lambda_3 t}$ for all $t \geq 0$. From these and the above inequality on $\|\mathcal{P}_M(t)\|_2$ there also exist $\lambda_4 > 0$ and $c_4 \geq 0$ such that $\|\mathcal{P}_M(t)\|_2 \leq c_4 e^{-\lambda_4 t}$. Plugging these inequalities in (35) yields the conclusion that $\tilde{p}_x$ is bounded whenever $p_x(0)$ and p_w are bounded. This completes the proof of the stability condition. □

5.2.1 A cheap over-estimation of $\tilde{p}_x$

Exploiting the inequality (34), we can establish by straightforward manipulations that $\tilde{p}_x(t) \leq p_x^u(t) - p_x^c(t)$ where p_x^u and p_x^c can be realized in the state-space form as

$$\begin{aligned}\dot{p}_x^u(t) = &\left(\psi(C_A) + \mathcal{P}_A\right)p_x^u(t)\\ &+ \left(\mathcal{P}_B + |C_B|\right)\left(p_w(t) + |c_w(t)|\right)\end{aligned} \qquad (36)$$

$$\dot{p}_x^c(t) = \psi(C_A)p_x^c(t) + |C_B||c_w(t)| \qquad (37)$$

with $p_x^u(0) = p_x(0) + |c_x(0)|$ and $p_x^c(0) = |c_x(0)|$. As it turns out, this last system is BIBO stable when $\psi(C_A) + \mathcal{P}_A$ is Hurwitz, the same condition as in Theorem 11. It is

interesting to note that the estimator (18) corresponding to the scenario where only the initial state and the input are uncertain can be recovered from (36)-(37).

5.2.2 A comment on the extension to closed-loop

To begin with, note that Eq. (19) still holds for given values of (A, B, C) and given trajectories for w and v. For simplicity, assume that C is a fixed-value matrix. Let $(C_A, \mathcal{P}_A)$, $(C_B, \mathcal{P}_B)$, $(C_G, \mathcal{P}_G)$ denote the center-radius representations of $[\underline{A}, \overline{A}]$, $[\underline{B}, \overline{B}]$ and $[\underline{G}, \overline{G}]$ respectively and $(c_s(t), p_s(t))$ denote that of $[\underline{s}(t), \overline{s}(t)]$. Then, since L and C are fixed-valued matrices, $(C_A - LC, \mathcal{P}_A)$ is a center-radius representation of $A - LC$. By the same observation, it follows from the expression of G in (19) that

$$C_G = [C_B \ L \ -L] \text{ and } \mathcal{P}_G = [\mathcal{P}_B \ 0_{n,n_y} \ 0_{n,n_y}]$$

with $0_{n,n_y}$ denoting a $n \times n_y$ matrix filled with zeros. Similarly, since $y(t)$ takes fixed values, we have

$$\begin{aligned} c_s(t) &= [c_w(t)^\top \ y(t)^\top \ c_v(t)^\top]^\top \\ p_s(t) &= [p_w(t)^\top \ 0_{n_y}^\top \ p_v(t)^\top]^\top. \end{aligned}$$

With these remarks, the radius estimate (36)-(37) extends naturally to the closed-loop scenario as follows

$$\begin{aligned} \dot{p}_x^{u,\mathrm{cl}}(t) =& \left(\psi(C_A - LC) + \mathcal{P}_A\right) p_x^{u,\mathrm{cl}}(t) \\ &+ \left(\mathcal{P}_G + |C_G|\right)\left(p_s(t) + |c_s(t)|\right) \end{aligned} \tag{38}$$

$$\dot{p}_x^{c,\mathrm{cl}}(t) = \psi(C_A - LC) p_x^{c,\mathrm{cl}}(t) + |C_G||c_s(t)| \tag{39}$$

with $p_x^{u,\mathrm{cl}}(0) = p_x(0) + |c_x(0)|$ and $p_x^{c,\mathrm{cl}}(0) = |c_x(0)|$. Let us recapitulate.

Theorem 12 *Consider system* (1) *where the matrices* (A, B) *are uncertain along with the signals* w, v *and the initial state* $x(0)$. *If there exists a matrix* L *such that* $\psi(C_A - LC) + \mathcal{P}_A$ *is Hurwitz, then*

$$[c_x^{cl} - (p_x^{u,cl} - p_x^{c,cl}),\ c_x^{cl} + (p_x^{u,cl} - p_x^{c,cl})],$$

with $\dot{c}_x^{cl} = (C_A - LC)c_x^{cl} + C_G c_s$, $c_x^{cl}(0) = c_x(0)$ *and* $p_x^{u,cl}, p_x^{c,cl}$ *as in* (38)-(39), *defines an interval estimator for system* (1).

Remark 3 *By following a similar line of reasoning as in the proof of Lemma 7 (see [2]), it is possible to prove the following: Existence of* $L \in \mathbb{R}^{n \times n_y}$ *such that* $\psi(C_A - LC) + \mathcal{P}_A$ *is Hurwitz, is equivalent to the existence of* (P, X, Y) *with the same characteristics an in Lemma 7 such that* $X^\top + X + 2\,\mathrm{diag}(S) + \mathcal{P}_A^\top P + P\mathcal{P}_A \prec 0$ *and* $|S - \mathrm{diag}(S)| \leq X$, *where* $S = PC_A - YC$. *Hence we can find numerically a matrix* L *fulfilling the requirement of Theorem 12 by solving a convex feasibility problem.*

6 Numerical results

This section reports some simulation results that illustrate the performances of some of the interval estimators discussed in this paper.

6.1 Open-loop estimation

For concision, we just consider the open-loop configuration under the assumption that the system matrices (A, B) are known and fixed. Consider an instance of system (1) with fixed-values state transition matrices defined by

$$A = \begin{bmatrix} -3 & 1.5 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{40}$$

The input w is such that $w(t) \in \mathcal{C}\big(c_w(t), p_w(t)\big)$ for all t where $c_w(t) = 1 + 5\cos(2\pi\nu_c t)$ and $p_w(t) = |2\sin(2\pi\nu_p t)|$ with $\nu_c = 0.3$ Hz and $\nu_p = 50$ Hz. As to the initial state, it lives in an interval $\mathcal{C}\big(c_x(0), p_x(0)\big)$ with $c_x(0) = [-2 \ 2]^\top$, $p_x(0) = [3 \ 2.2]^\top$. Note that in order to be able to test all the estimators in open-loop (in particular the one suggested in Section 3.3.2), the matrix A in (40) has been selected such that $\psi(A)$ is Hurwitz.

For this example, Figure 1 compares the tightest estimator proposed in (10)-(12) with three estimators from the family described in Eqs (16)-(17) for a time horizon $T \in \{0.01, 0.1, 1\}$, expressed in seconds. The integral quantities present in the expressions of these estimators are numerically implemented here using the Simpson method. Two comments can be made. First, these simulation results provide an empirical evidence supporting our claim that the estimator proposed in (10)-(12) is indeed the tightest possible. Second, the over-approximation given in (16)-(17) gets tighter as the horizon T increases. Finally, it is interesting to observe that T needs not be too large for $\hat{p}_x$ in (16) to provide a good approximation of p_x; here we get a good match between p_x and $\hat{p}_x$ for a value as small as $T = 1$.

The second figure (Fig. 2) compares the estimator (10)-(12) to its over-approximations discussed in Sections 3.3.2 and 3.3.3. A specificity of these estimators is that they are computationally less expensive to implement as they can be realized by finite dimensional state-space representations (with state lengths equal to $2n$ and $n(n+1)$ respectively). It follows from the empirical results that in the current settings, the over-approximation using the Metzler matrix $\psi(A)$ is the cheapest but also the least tight.

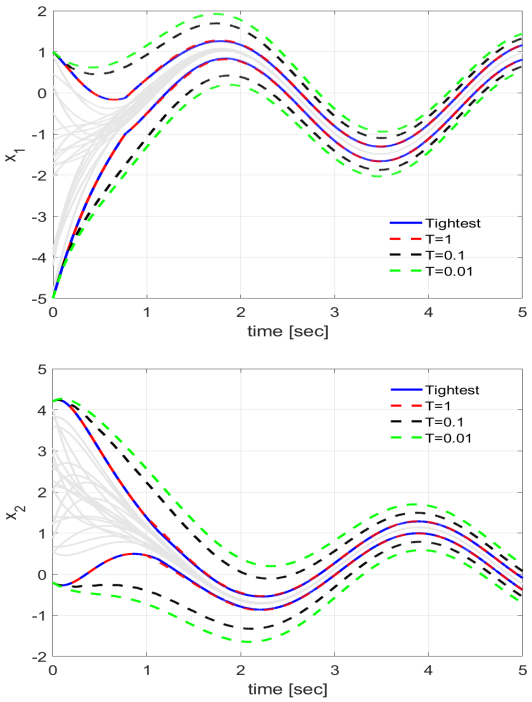

Figure 1. Comparison of open-loop interval estimators (16)-(17) for $T = 0.01$ (green), $T = 0.1$ (black), $T = 1$ (red), $T = t$ (blue). In gray are represented the state trajectories of the system generated from different initial conditions and different inputs with values on the allowed intervals.

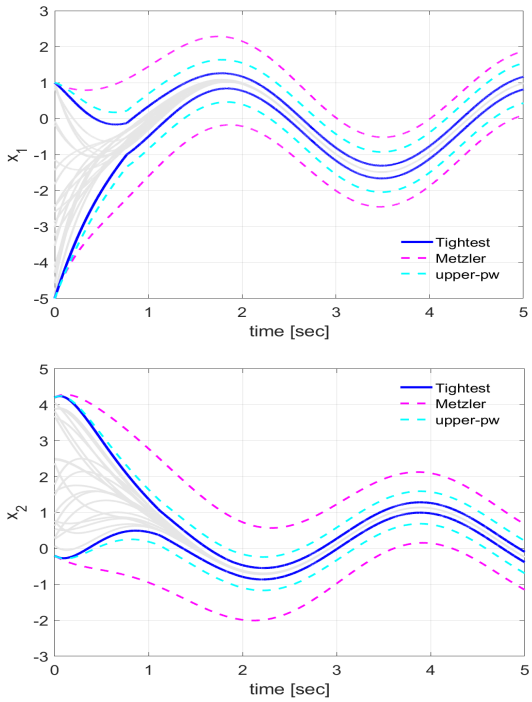

Figure 2. Comparison of open-loop interval estimators: tightest (blue), estimator (14) (cyan) obtained by upper-bounding p_w with 2, approximation using a Metzler matrix (magenta). In gray are represented the state trajectories of the system generated from different initial conditions and different inputs.

6.2 Closed loop estimation in the presence of parametric uncertainties

We consider now the closed-loop estimation scenario where in addition to uncertainties on the input and on the initial state, the parameters may also be uncertain. Using the same notational convention as in the beginning of Section 5.1, we set $C_A = A$, $C_B = B$ with A and B defined in (40), $\mathcal{P}_B = 0$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and

$$\mathcal{P}_A = \begin{bmatrix} 1.75 & 1.07 \\ 0.40 & 0.55 \end{bmatrix}.$$

The input and initial state uncertainties $(c_x(0), p_x(0))$ and (c_w, p_w) are set to the same values as in the previous case (see Section 6.1). As to the measurement noise v, it is defined by $\big(c_v(t), p_v(t)\big) = (0, 0.1)$ for all t. Then applying the method described in Remark 3 successfully returns a gain L that renders $\psi(C_A - LC) + \mathcal{P}_A$ Hurwitz. The corresponding implementation of the estimator in Theorem 12, gives the results represented in Figure 3. As we can see, this illustrates our claim that the estimator displayed in Theorem 12 is an interval-valued estimator for the uncertain linear system.

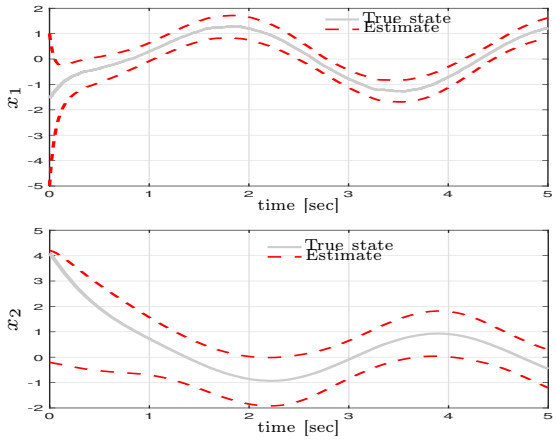

Figure 3. Closed-loop interval estimates in the scenario where the initial state x_0, the input w and the parameters are uncertain. The estimator of Theorem 12 was applied with a gain $L = \begin{bmatrix} 8.685 & -2.00 \end{bmatrix}^\top$ which was obtained according to the method described in Remark 3.

7 Conclusion

In this paper we have presented a new approach to the interval-valued state estimation problem. The proposed framework is mainly discussed for the case of continuous-time linear systems but it is generalizable (to some ex-

tent) to LTV systems and probably to some other classes of systems. The main contribution of this work consists in the derivation of the tightest interval-valued estimator which enclose all the possible state trajectories generated by an uncertain LTI system. A numerical implementation of this estimator requires however some tradeoff between tightness and computational load. Therefore some relaxations on tightness have been discussed along with extensions to the general situations where the system parameters and the external signals are all uncertain and interval-valued.

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