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Sampled Boundary Observer for Strict-Feedback Nonlinear ODE Systems with Parabolic PDE Sensor

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Abstract. We design an observer for ODE-PDE cascades where the ODE is nonlinear of strict-feedback structure and the PDE is a linear and of parabolic type. The observer provides online estimates of the (finite-dimensional) ODE state vector and the (infinite-dimensional) state of the PDE, based only on sampled boundary measurements. A design that simultaneously addresses nonlinear ODEs and boundary measurement sampling is the paper's key contribution. Our observer design combines the backstepping design approach and the high-gain observer methodology and our analysis employs a Lyapunov-Krasovskii type functional to establish exponential convergence. Our sufficient conditions for convergence involve the maximum sampling interval and the PDE domain length.

Keywords: ODE-PDE cascades, nonlinear systems, parabolic PDEs sampled-data observer, high-gain observer.

1. INTRODUCTION

Designing boundary observers for cascade systems involving PDE sensing and/or actuation has recently become a highly active research topic. Linear ODE-PDE cascades with first-order hyperbolic PDEs have been dealt with in (Krstic and Smyshlyaev, 2008). The proposed boundary observer is a copy of the system with additional innovation terms involving domain-dependent gains which are obtained through the solution of a first order ODE. The innovation terms provide the observer with a feedback structure and introduce a coupling between its ODE and PDE parts. Exponential stability of the resulting error system has been analysed using an appropriate error coordinate backstepping transformation and a quadratic Lyapunov functional. This study has been extended in (Ahmed-Ali et al., 2017a) to the case where the system ODE part is nonlinear of globally Lipschitz type and with a triangular structure. The ODE nonlinearity is accounted for in the observer design by using the high-gain formalism, especially in the part devoted to the ODE state observation. The system nonlinearity gives rise to a limitation on the admissible length D of the PDE domain: the larger the Lipschitz constant, the smaller the admissible values of D . This limitation has been coped with in (Ahmed-Ali et al., 2017b) by using a cascade observer involving a number, say m , of partial observers each one providing the estimates of the PDE state on a subdomain of length D/m . Accordingly, the observer (error system) exponential stability can be guaranteed, however large the Lipschitz constant of the ODE nonlinearity, provided the number m is taken sufficiently large.

Observer design for linear ODE-PDE cascades that involve parabolic PDEs has been introduced in (Krstic, 2009). The observer involves domain position dependent gains that are governed by second order ODEs. Again, exponential stability of the corresponding observation error system is established using a quadratic Lyapunov functional. In (Ahmed-Ali et al., 2015), we have extended the result of (Krstic, 2009) to ODE-PDE systems where the ODE part is nonlinear Lipschitz with triangular structures. Observer design for nonlinear multivariable systems with parabolic PDE sensor dynamics and application to output feedback control was studied in (Wu and Wang, 2013).

Boundary observer design for linear PDE-ODE cascade systems with first-order hyperbolic PDEs has been studied in e.g. (Hasan et al., 2016). The observer is constructed in a collocated setup, which means that both sensing and actuation are located at the same boundary. The observer gains are computed analytically by solving Goursat-type PDEs in terms of Bessel function of the first kind. Finally, boundary observer design for linear coupled PDE-ODE, with hyperbolic PDE, has been considered in e.g. (Tang and Xie, 2010, 2011).

A common characteristic of the above mentioned observers is that they all require the system boundary outputs to be continuously accessible to measurements. As only sampled measurements are available in practice, continuous-time measurement based observers may not meet their theoretical performances. This explains the great deal of interest that has been paid to sampled-data based observer design over the last few years. However, quite a few sampled-measurement observers have been developed for systems described by PDEs, see e.g. (Fridman and Blighovsky, 2012; Ahmed-Ali et al., 2016a) for parabolic PDEs. The problem of sampled-data observer design for linear ODE-PDE cascades has only recently been investigated in (Ahmed-Ali et al., 2016b, 2017b) considering a specific class of parabolic PDEs and boundary conditions. The observer design developed there can be viewed as a sampled version of that in (Krstic, 2009) as it relied on a coordinate transformation of the ODE and PDE states. However, the error system exponential stability has been investigated using a small-gain (input-output) stability analysis, rather than Lyapunov functional analysis in (Krstic, 2009).

In this paper, we present a new sampled-measurement based observer design for ODE-PDE cascades where the ODE is nonlinear of the Lipschitz type and with a triangular structure, the PDE is parabolic with various boundary conditions. Our observer design combines ideas from the high-gain observer technique and the backstepping transformation principle. It turns out that the observer gain depends on a matrix-valued function (of the PDE spatial variable) that is the solution of a second order ODE. The observation error system is analyzed using a suitable quadratic Lyapunov-Krasovski functional, that constitutes another novelty of this work. Accordingly, upper bounds on the sampling interval and the PDE domain length are

established for the error system to be exponentially stable. Compared to the authors' previous works, the present result extends that of (Ahmed-Ali et al., 2015) in at least two directions: (i) the class of considered (parabolic) PDEs and their associated boundary conditions are much larger; (ii) the present observer is sampled-output (while it was continuous-time output measurement based in the previous work). The present work is also an extension of (Ahmed-Ali et al., 2016b, 2017b) due to the ODE nonlinearity. It is also the complete version of the conference paper (Karafyllis et al., 2017) where most technical proofs were missing.

The paper is organized as follows: the first observer problem under study is formulated in Section 2 and the corresponding observer design and analysis are dealt with in Sections 3 and 4; an extension of this observer design to a larger class of systems is presented in Section 5; well-posedness analysis of all considered systems and observers is discussed in Section 6; some concluding remarks end the paper.

Notation. $H^1(0, D)$ (with D any positive real constant) is the Sobolev space of absolutely continuous functions $\eta: [0, D] \rightarrow \mathbf{R}$ with $d\eta/d\xi \in L^2[0, D]$. $H^2(0, D)$ denotes the Sobolev space of scalar functions $\eta: [0, D] \rightarrow \mathbf{R}$ with absolutely continuous $d\eta/d\xi \in L^2[0, D]$ and $d^2\eta/d\xi^2 \in L^2[0, D]$. Let $w: [0, D] \times \mathbf{R}_+ \rightarrow \mathbf{R}; (x, t) \rightarrow w(x, t)$ be any function. Then, $w[t]$ and $w_x[t]$ refer to the functions defined on $0 \leq x \leq D$ by $(w[t])(x) = w(x, t)$ and $(w_x[t])(x) = \partial w(x, t) / \partial x$. Consider a scalar function $\varphi \in H^1(0, D)$ such that $\varphi(0) = 0$ or $\varphi(D) = 0$. Then, one has the following Wirtinger's inequality:

$$\int_0^D \varphi^2(\xi) d\xi \leq \frac{4D^2}{\pi^2} \int_0^D \left(\frac{d\varphi}{d\xi}(\xi) \right)^2 d\xi \quad (1)$$

3. OBSERVER PROBLEM STATEMENT

3.1 Class of observed systems

We are considering continuous-time nonlinear systems that can be modelled by the following ODE-PDE cascade:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)), \text{ for } t \geq 0 \quad (2)$$

$$u_t(x, t) = au_{xx}(x, t) - cu(x, t), \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \quad (3)$$

$$u_x(0, t) = qu(0, t), \text{ for all } t \geq 0 \quad (4a)$$

$$u(D, t) = CX(t), \text{ for all } t \geq 0 \quad (4b)$$

with,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \in \mathbf{R}^{n \times n} \quad (5a)$$

$$C = (1 \ 0 \ \dots \ 0) \in \mathbf{R}^{1 \times n} \quad (5b)$$

$$f(X) = \begin{pmatrix} f_1(X_1, v) \\ f_2(X_1, X_2, v) \\ \vdots \\ f_n(X_1, \dots, X_n, v) \end{pmatrix} \text{ and } f_i: \mathbf{R}^{i+m} \rightarrow \mathbf{R} \quad (5c)$$

where $X(t) \in \mathbf{R}^n$ denotes the state vector of the finite-dimensional subsystem, described by (2) with any initial condition $X(0) = X_0 \in \mathbf{R}^n$; $v(t) \in \mathbf{R}^m$ (for some known $m \geq 1$) is an external known bounded signal of class $C^2(\mathbf{R}_+; \mathbf{R}^m)$; $u(x, t) \in \mathbf{R}$ designates the state of the infinite-dimensional subsystem described by the parabolic type PDE (3) with boundary conditions (4) and any initial condition $u(x, 0) = (u[0])(x) \in \mathbf{R}$ for $x \in [0, D]$; $a > 0$, $c \in \mathbf{R}$ and $q \geq 0$ are known real parameters. Letting $f_X(X, v)$ denote the Jacobian matrix of $f(\cdot)$ w.r.t X , it is supposed that:

$$\exists \beta_f > 0, \forall X \in \mathbf{R}^n: \|f_X(X, v)\| \leq \beta_f \quad (6)$$

where β_f might depend on the external signal v . The pair (A, C) is observable and the whole system is observed through a ZOH sampling of the signal $u(0, t)$, i.e. the system output is:

$$y(t) = u(0, t_k), \text{ for all } t \in [t_k, t_{k+1}) \text{ and } k = 0, 1, 2, \dots \quad (7a)$$

where $\{t_k\}_{k=0}^\infty$ denotes the sampling time sequence. This is a partition of \mathbf{R}_+ i.e. $\{t_k\}_{k=0}^\infty$ is increasing with $t_0 = 0$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$. It is also supposed that $0 < h < +\infty$ with,

$$h \stackrel{\text{def}}{=} \sup_{k \in \mathbf{N}} (t_k - t_{k-1}) \quad (7b)$$

We seek an observer that provides accurate online estimates of both the (finite-dimensional) state vector $X(t)$ and the distributed state $u(x, t)$, $0 \leq x \leq D$, based only the measurements of the external signals $v(t)$ and $y(t)$.

Remark 1. Sampled-output observers have recently been developed in (Ahmed-Ali et al., 2016b, 2017b) for subclasses of (2)-(4) corresponding to $f(\cdot) = 0$. Then, Luenberger type linear observers were used, while high-gain observer is necessary to deal with the present problem. The considered strict-feedback structure of the system and the globally Lipschitz assumption on $f(\cdot)$ are precisely considered to make the high-gain observer design applicable, see e.g. (Khalil, 2015, p. 271). In this respect, note that all uniformly observable SISO systems are diffeomorphic to the strict-feedback form. Accordingly, considering this form entails covering all uniformly observable SISO systems. Owing to the global Lipschitz assumption made on $f(\cdot)$, although this is quite usual, it actually entails limitation of the set of admissible nonlinearities.

4. OBSERVER DESIGN AND ANALYSIS

Consider the following backstepping transformation:

$$p(x, t) = u(x, t) - CM^{-1}(D)M(x)X(t), \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \quad (8)$$

where $M(x) \in \mathbf{R}^{n \times n}$ is defined by the following ODE and boundary conditions:

$$\frac{d^2 M}{dx^2}(x) = a^{-1}M(x)(A + cI) \quad (9)$$

$$M(0) = I, \quad \frac{dM}{dx}(0) = qI \quad (10)$$

where $I \in \mathbf{R}^{n \times n}$ denotes the identity matrix. The solution of the problem (9)-(10) is analytic and expressed by the following globally convergent series

$$M(x) = (1 + qx)I + \sum_{l=1}^{\infty} a^{-l} (A + cI)^l \left(\frac{x^{2l}}{(2l)!} + q \frac{x^{2l+1}}{(2l+1)!} \right) \quad (11)$$

The transformation (8) also involves the inverse of $M(D)$ which implicitly entails an additional assumption on A . This issue is no longer investigated in this paper. Using (9), (2) and (3), it follows from (8) that $p(x, t)$ is governed by the following PDE, for all $(x, t) \in (0, D) \times (0, +\infty)$:

$$p_t(x, t) = ap_{xx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(X) \quad (12)$$

For convenience, the new system representation in terms of the states $(X(t), p(x, t))$ is rewritten:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)), \text{ for all } t \geq 0 \quad (13)$$

$$p_t(x, t) = ap_{xx}(x, t) - cp(x, t) - CM^{-1}(D)M(x)f(X, v), \quad \text{for all } (x, t) \in (0, D) \times (0, +\infty) \quad (14)$$

$$p_x(0, t) = qp(0, t), \text{ for all } t \geq 0 \quad (15)$$

$$p(D, t) = 0, \text{ for all } t \geq 0 \quad (16)$$

$$u(x, t) = p(x, t) + CM^{-1}(D)M(x)X(t) \quad (17)$$

where (17) and the boundary conditions (15)-(16) are derived from (8) using (4) and (10). The (X, p) -system representation (13)-(17) mainly differs from the initial (X, u) -representation in that the boundary conditions (15)-(16) of the former are X -free leading to homogeneous boundary conditions (unlike those of the initial representation, see (2)-(4)). Homogeneous boundary conditions are required in the existence/uniqueness theorems for PDEs. For convenience, the well posedness issue will be discussed latter (see Remark 4).

To get online estimates $\hat{X}(t)$ and $\hat{p}(x, t)$ of the unmeasurable states $X(t)$ and $p(x, t)$, we propose the following sampled-output observer:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + f(\hat{X}(t), v(t)) - \theta M(D)\Delta^{-1}L\tilde{y}(t_k), \quad \text{for all } t \in [t_k, t_{k+1}) \text{ and } k = 0, 1, 2, \dots \quad (18)$$

$$\hat{p}_t(x, t) = a\hat{p}_{xx}(x, t) - c\hat{p}(x, t) - CM^{-1}(D)M(x)f(\hat{X}(t), v(t)), \quad \text{for all } (x, t) \in (0, D) \times (0, +\infty) \quad (19)$$

$$\hat{p}_x(0, t) - q\hat{p}(0, t) = \hat{p}(D, t) = 0, \text{ for all } t \geq 0 \quad (20)$$

$$\hat{u}(x, t) = \hat{p}(x, t) + CM^{-1}(D)M(x)\hat{X}(t), \quad \text{for all } (x, t) \in (0, D) \times (0, +\infty) \quad (21)$$

$$\tilde{y}(t) = \hat{y}(t) - y(t), \text{ with } \hat{y}(t) = \hat{u}(0, t) \quad (22)$$

with

$$\Delta \stackrel{\text{def}}{=} \text{diag} \left\{ 1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}} \right\} \in \mathbf{R}^{n \times n}, \quad (23)$$

where the real scalars K and $\theta > 1$ are design parameters and $L \in \mathbf{R}^n$ is any vector such that $A - LC$ is Hurwitz (this is not an issue since the pair (A, C) is observable).

Remark 2. In the case of linear systems (i.e. if $f(\cdot) = 0$ then) the use of a high-gain observer is not necessary. In such a case, one can consider simpler sampled-data observers like those in (Ahmed-Ali et al., 2016b, 2017b; Karafyllis et al.,

2017). Presently, this amounts to letting $\theta = 1$ and $\Delta = I$ in the ODE part of the observer (given par (18)). In all mentioned observers, the ‘measurement delay’ caused by the heat PDE sensor dynamics is compensated for in the observer ODE part by the matrix $M(D)$ (Krstic, 2008; Ahmed-Ali et al., 2016b, 2017b; Karafyllis et al., 2017). In a sense, $M(D)$ acts as a ‘transition’ matrix between the measurement point, located at $x = 0$, and the output of the ODE part, located at $x = D$. In the case of no PDE sensor dynamics (i.e. if $D = 0$), $M(D) = M(0) = I$ and the observer part (18) boils down to a standard high-gain observer. To analyse the proposed observer, we introduce the estimation errors:

$$\tilde{X}(t) = \hat{X}(t) - X(t), \quad (24a)$$

$$\tilde{p}(x, t) = \hat{p}(x, t) - p(x, t), \quad \tilde{u}(x, t) = \hat{u}(x, t) - u(x, t) \quad (24b)$$

Using (10), (22) and (24a), one immediately gets by subtracting (17) to (21) and letting there $x = 0, t = t_k$:

$$\tilde{y}(t_k) = \tilde{p}(0, t_k) + CM^{-1}(D)\tilde{X}(t_k) \quad (25)$$

Using the observer equations (18)-(22), the system equations (13)-(17), and definitions (24a-b), we get the following error equations, for all $x \in [0, 1]$, $t \in [t_k, t_{k+1})$ and $k \in \mathbf{N}$:

$$\begin{aligned} \dot{\tilde{X}}(t) &= A\tilde{X}(t) + f(\hat{X}(t), v(t)) - f(X(t), v(t)) \\ &\quad - \theta M(D)\Delta^{-1}LCM^{-1}(D)\tilde{X}(t_k) \\ &\quad - \theta M(D)\Delta^{-1}L\tilde{p}(0, t_k), \end{aligned} \quad (26)$$

$$\begin{aligned} \tilde{p}_t(x, t) &= a\tilde{p}_{xx}(x, t) - c\tilde{p}(x, t) \\ &\quad - CM^{-1}(D)M(x)(f(\hat{X}(t), v(t)) - f(X(t), v(t))) \end{aligned} \quad (27)$$

$$\tilde{p}_x(0, t) - q\tilde{p}(0, t) = \tilde{p}(D, t) = 0, \text{ for all } t \geq 0, \quad (28)$$

$$\tilde{u}(x, t) = \tilde{p}(x, t) + CM^{-1}(D)M(x)\tilde{X}(t), \quad (29)$$

The exponential stability of the error system (26)-(29) is stated in the following theorem:

Theorem 1 (main result). Consider the class of systems defined by equations (2)-(4) with parameters $a > 0$, $c \in \mathbf{R}$ and $q \geq 0$. Consider the output-sampled observer defined (18)-(22), with the scalar parameter $\theta > 0$ arbitrary, the gain $L \in \mathbf{R}^n$ such that the matrix $A - LC \in \mathbf{R}^{n \times n}$ is Hurwitz, and the matrix function $M(x)$ defined by (9)-(11). Then, there exists $\theta^* > 0$ such that for any $\theta > \theta^*$, there exist $0 < D^* < 1/\sqrt{\theta^*}$ and $h^* > 0$ so that if (the domain length D of the PDE (3) and the diameter h of the time partition $\{t_k\}_{k=0}^{\infty}$ are such that) $0 < D < D^*$ and $0 < h < h^*$ then:

1) The observer (18)-(22) is exponentially convergent in the sense that,

$$\|\tilde{X}(t)\| \text{ and } \int_0^D \tilde{u}^2(x, t) dx \quad (30)$$

exponentially converge to zero, whatever the initial conditions $X(0), \hat{X}(0) \in \mathbf{R}^n$, $u[0], \hat{p}[0] \in C^2([0, D]; \mathbf{R})$, with $\hat{p}_x0 - q\hat{p}0 = \hat{p}[0](D) = 0$, $(u_x[0])(0) = q(u[0])(0)$, and $(u[0])(D) = CX(0)$.

2) In the case of linear systems, i.e. if the Lipschitz coefficient β_f in (6) is zero then, D can be arbitrarily large.

Proof. Introduce the following notation:

$$Z = \Delta M^{-1}(D)\tilde{X} \quad (31)$$

$$\chi(D, t) = \int_0^1 f_X(X(t) + s\tilde{X}(t), v(t))ds \quad (32)$$

Then, differentiating both sides of (31), one obtains using (26) and applying the mean-value theorem:

$$\begin{aligned} \dot{Z}(t) &= \Delta M^{-1}(D)A\tilde{X}(t) + \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \\ &\quad - \theta LC\Delta^{-1}Z(t_k) - \theta L\tilde{p}(0, t_k), \\ &= \theta AZ(t) - \theta LCZ(t_k) - \theta L\tilde{p}(0, t_k) \\ &\quad + \Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t), \end{aligned} \quad (33)$$

for all $x \in [0, D]$, $t \in [t_k, t_{k+1})$, $k \in \mathbf{N}$

where the second equality is obtained using the following equalities which are direct consequences of the following properties:

$$C\Delta = C, \quad (34)$$

$$\Delta A = \theta A\Delta, \quad (35)$$

$$AM(x) = M(x)A, \quad AM^{-1}(x) = M^{-1}(x)A, \quad (36)$$

for all $x \in [0, D]$.

In turn, (27) is rewritten as follows:

$$\begin{aligned} \tilde{p}_t(x, t) &= a\tilde{p}_{xx}(x, t) - c\tilde{p}(x, t) \\ &\quad - CM^{-1}(D)M(x)\chi(D, t)M(D)\Delta^{-1}Z(t) \end{aligned} \quad (37)$$

where the last term on the right side is obtained using (34). Equation (37) is completed with the boundary conditions (28)-(29). To analyse the (Z, \tilde{p}) -system, we consider the following Lyapunov-Krasovskii functional:

$$\begin{aligned} V(Z, \tilde{p}, t) &= Z^T(t)PZ(t) + \frac{\alpha_0}{2} \int_0^D \tilde{p}^2(x, t)dx \\ &\quad + \frac{\alpha_1}{2} \int_0^D \tilde{p}_x^2(x, t)dx + \frac{q\alpha_1}{2} \tilde{p}^2(0, t) \\ &\quad + \alpha_2 \int_{t_k}^t (s - t + h) \|\dot{Z}(s)\|^2 ds \end{aligned} \quad (38)$$

for $t_k \leq t < t_{k+1}$, $k \in \mathbf{N}$, where $\alpha_0 > 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$ are real scalars such that:

$$a\alpha_0 + c\alpha_1 > 0 \quad (39a)$$

and P any positive definite matrix satisfying the inequality, $P(A - LC) + (A - LC)^T P \leq -\mu I$ (39b)

for some $\mu > 0$, where I denotes the identity matrix of $\mathbf{R}^{n \times n}$. Note that P exists (whatever $\mu > 0$) because $A - LC$ is Hurwitz. Also, from (28) one has the following relations which will prove to be useful:

$$Z(t_k) = Z(t) - \int_{t_k}^t \dot{Z}(s)ds, \quad \tilde{p}(0, t) = -\int_0^D \tilde{p}_x(x, t)dx \quad (40)$$

Using (33), (37) and (39b), it follows differentiating $V(Z, \tilde{p}, t)$ with respect to time:

$$\begin{aligned} \dot{V}(Z, \tilde{p}, t) &= 2Z^T(t)P(\theta AZ(t) - \theta LCZ(t_k) - \theta L\tilde{p}(0, t_k)) \\ &\quad + 2Z^T(t)P\Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \\ &\quad + \alpha_0 \int_0^D \tilde{p}(x, t)(a\tilde{p}_{xx}(x, t) - c\tilde{p}(x, t))dx \\ &\quad - \alpha_0 \int_0^D \tilde{p}(x, t)(CM^{-1}(D)M(x)\chi(D, t)M(D)\Delta^{-1}Z(t))dx \\ &\quad + h\alpha_2 \|\dot{Z}(t)\|^2 - \alpha_2 \int_{t_k}^t \|\dot{Z}(s)\|^2 ds + \alpha_1 \int_0^D \tilde{p}_x(x, t)\tilde{p}_{xt}(x, t)dx \\ &\quad + q\alpha_1 \tilde{p}(0, t)\tilde{p}_t(0, t), \text{ for } t_k \leq t < t_{k+1}, k \in \mathbf{N} \end{aligned} \quad (41)$$

where the last couple of terms are obtained using the Leibnitz's derivation rule. Using the relation equality

$Z^T(t)P(A - LC)Z(t) = Z^T(t)(A - LC)^T PZ(t)$ (which holds because P is symmetric), equality (41) immediately develops further as follows, for $t_k \leq t < t_{k+1}$, $k \in \mathbf{N}$:

$$\begin{aligned} \dot{V}(Z, \tilde{p}, t) &= \theta Z^T(t)(P(A - LC) + (A - LC)^T P)Z(t) \\ &\quad + 2\theta Z^T(t)PLC \int_{t_k}^t \dot{Z}(s)ds \\ &\quad + 2\theta Z^T(t)PL \int_0^D \tilde{p}_x(x, t_k)dx \\ &\quad + 2Z^T(t)P\Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \\ &\quad + \alpha_0 \int_0^D \tilde{p}(x, t)(a\tilde{p}_{xx}(x, t) - c\tilde{p}(x, t))dx \\ &\quad - \alpha_0 \int_0^D \tilde{p}(x, t)(CM^{-1}(D)M(x)\chi(D, t)M(D)\Delta^{-1}Z(t))dx \\ &\quad + \alpha_1 \int_0^D \tilde{p}_x(x, t)\tilde{p}_{xt}(x, t)dx + q\alpha_1 \tilde{p}(0, t)\tilde{p}_t(0, t) \\ &\quad + h\alpha_2 \|\dot{Z}(t)\|^2 - \alpha_2 \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \end{aligned} \quad (42)$$

Existence and continuity of \tilde{p}_{xt} on the right side of (42) can be proved as follows: replacing x by $1 - x$ and using the regularity properties of all functions, it follows from Proposition 2.12 in (Karafyllis and Krstic, 2017a) that \tilde{p}_{xt} exists and is continuous for all $x \in [0, D]$ and all $t > 0$, except for the t_k 's. Now, the various terms on the right side of (42) are examined in order and, some of them, bounded from above. Using (39b), the first term is bounded as follows:

$$\theta Z^T(t)(P(A - LC) + (A - LC)^T P)Z(t) \leq -\theta\mu \|Z(t)\|^2 \quad (43)$$

The second term on the right side of (42) is developed as follows, using Young's inequality:

$$\begin{aligned} 2\theta Z^T(t)PLC \int_{t_k}^t \dot{Z}(s)ds &\leq \theta\zeta \|Z(t)\|^2 + \frac{\theta\|PLC\|^2}{\zeta} \left\| \int_{t_k}^t \dot{Z}(s)ds \right\|^2 \\ &\leq \theta\zeta \|Z(t)\|^2 + \frac{h\theta\|PLC\|^2}{\zeta} \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \end{aligned} \quad (44)$$

for any $\zeta > 0$, where the last inequality is obtained using Jensen's inequality. The third term on the right side of (42) is bounded from above as follows, using Young's inequality:

$$\begin{aligned} 2\theta Z^T(t)PL \int_0^D \tilde{p}_x(x, t_k)dx \\ \leq \theta\zeta \|Z(t)\|^2 + \frac{\theta D\|PL\|^2}{\zeta} \int_0^D \tilde{p}_x^2(x, t_k)dx \end{aligned} \quad (45)$$

for any $\zeta > 0$. Owing to the fourth term on the right side of (42), this can be rewritten as follows:

$$\begin{aligned} 2Z^T(t)P\Delta M^{-1}(D)\chi(D, t)M(D)\Delta^{-1}Z(t) \\ = 2Z^T(t)P\Delta M^{-1}(D)\Delta^{-1}\Delta\chi(D, t)\Delta^{-1}\Delta M(D)\Delta^{-1}Z(t) \end{aligned}$$

In view of (5c) and (23) that, for all $\theta > 1$, one has:

$$\|\Delta\chi(D, t)\Delta^{-1}\| = \left\| \Delta \left(\int_0^1 f_X(X(t) + s\tilde{X}(t), v(t)) \right) \Delta^{-1} \right\| \leq \beta_f, \quad (46)$$

using (6). Also, using (35) and (6), it readily follows from (11) that, for all $x \in [0, D]$:

$$\begin{aligned} \|\Delta M(x)\Delta^{-1}\| &= \left\| \left((1 + qx)I + \sum_{l=1}^{\infty} a^{-l}(\theta A + cI)^l \left(\frac{x^{2l}}{(2l)!} + q \frac{x^{2l+1}}{(2l+1)!} \right) \right) \right\| \\ &\leq (1 + qD) + \sum_{l=1}^{\infty} a^{-l} \left\| A + \frac{c}{\theta} I \right\|^l \left(\frac{(\theta D^2)^l}{(2l)!} + q \frac{(\theta D^2)^l D}{(2l+1)!} \right) \end{aligned} \quad (47)$$

Assuming $\theta D^2 < 1$, one immediately gets from (47) that:

$$\|\Delta M(x)\Delta^{-1}\| \leq \beta_M, \text{ for all } x \in [0, D] \quad (48)$$

with

$$\beta_M = 1 + qD + \max(1, q) \left\| \exp(a^{-1}(A + cI)) - I \right\| \sum_{l=1}^{\infty} \frac{a^{-l} \|A + cI\|^l}{l!}$$

is independent of θ . Using similar argument, it is shown that if $\theta D^2 < 1$ then:

$$\|\Delta M^{-1}(x)\Delta^{-1}\| \leq \gamma_M, \text{ for all } x \in [0, D] \quad (49)$$

for some real constant γ_M independent of θ .

Using (46), (48) and (49), one gets:

$$\|2Z^T(t)P\Delta M^{-1}(D)\chi(D,t)M(D)\Delta^{-1}Z(t)\| \leq 2\beta_f\beta_M^2\|P\| \|Z(t)\|^2 \quad (50)$$

The fifth term on the right side of (42) is bounded as follows:

$$\begin{aligned} & \alpha_0 \int_0^D \tilde{p}(x,t)(a\tilde{p}_{xx}(x,t) - c\tilde{p}(x,t))dx \\ & \leq -a\alpha_0 q\tilde{p}^2(0,t) - a\alpha_0 \int_0^D \tilde{p}_x^2(x,t)dx - c\alpha_0 \int_0^D \tilde{p}^2(x,t)dx \quad (51) \end{aligned}$$

where we have used an integration by parts and the boundary conditions (28). The sixth term on the right side of (42) is first bounded as follows:

$$\begin{aligned} & \left| \alpha_0 \int_0^D \tilde{p}(x,t) \left(CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t) \right) dx \right| \\ & \leq \frac{\alpha_0}{\varpi} \int_0^D \tilde{p}^2(x,t)dx + \\ & \quad + \alpha_0 \varpi \int_0^D \left(CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t) \right)^2 dx \quad (52) \end{aligned}$$

for any $\varpi > 0$. As $C\Delta = C$ the squared term in the last integral, on the right side of (50), develops as follows:

$$\begin{aligned} & CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t) \\ & = C\Delta M^{-1}(D)\Delta^{-1}\Delta M(x)\Delta^{-1}\Delta\chi(D,t)\Delta^{-1}\Delta M(D)\Delta^{-1}Z(t) \end{aligned}$$

Then, it follows using (45), (47) and (48), one gets:

$$\|CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t)\| \leq \beta_f\beta_M\gamma_M\|Z(t)\| \quad (53)$$

Using (53), it follows from (52) that, for $t_k \leq t < t_{k+1}$, $k \in \mathbf{N}$:

$$\begin{aligned} & \left| \alpha_0 \int_0^D \tilde{p}(x,t) \left(CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t) \right) dx \right| \\ & \leq \frac{\alpha_0}{\varpi} \int_0^D \tilde{p}^2(x,t)dx + \alpha_0 \varpi D (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2 \quad (54) \end{aligned}$$

The seventh and eighth terms on the right side of (42) are dealt with as follows, for $t_k \leq t < t_{k+1}$, $k \in \mathbf{N}$:

$$\begin{aligned} & \alpha_1 \int_0^D \tilde{p}_x(x,t)\tilde{p}_{xt}(x,t)dx + q\alpha_1\tilde{p}(0,t)\tilde{p}_t(0,t) \\ & = \alpha_1 \left[\tilde{p}_x(x,t)\tilde{p}_t(x,t) \right]_0^D - \alpha_1 \int_0^D \tilde{p}_{xx}(x,t)\tilde{p}_t(x,t)dx \\ & \quad + q\alpha_1\tilde{p}(0,t)\tilde{p}_t(0,t) \\ & = -\alpha_1 \int_0^D \tilde{p}_{xx}(x,t)(a\tilde{p}_{xx}(x,t) - c\tilde{p}(x,t))dx \\ & \quad + \alpha_1 \int_0^D \tilde{p}_{xx}(x,t) \left(CM^{-1}(D)M(x)\chi(D,t)M(D)\Delta^{-1}Z(t) \right) dx \\ & \leq -a\alpha_1 \int_0^D \tilde{p}_{xx}^2(x,t)dx - cq\alpha_1\tilde{p}^2(0,t) - c\alpha_1 \int_0^D \tilde{p}_x^2(x,t)dx \\ & \quad + \frac{\alpha_1}{2\eta} \int_0^D \tilde{p}_{xx}^2(x,t)dx + \frac{\alpha_1\eta D}{2} (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2 \quad (55) \end{aligned}$$

for any real $\eta > 0$, where the penultimate equality is obtained using (28), while the last inequality is obtained using an integration by part, applying Young's inequality and using (53). Using (33) and (40), the penultimate term on the right side of (42) is bounded from above as follows:

$$\begin{aligned} h\alpha_2 \|\dot{Z}(t)\|^2 & = h\alpha_2 \|\theta AZ(t) - \theta LCZ(t_k) - \theta L\tilde{p}(0,t_k) \\ & \quad + \Delta M^{-1}(D)\chi(D,t)M(D)\Delta^{-1}Z(t)\|^2 \\ & = h\alpha_2 \left\| (\theta A - \theta LC)Z(t) + \theta LC \int_{t_k}^t \dot{Z}(s)ds + \theta L \int_0^D \tilde{p}_x(x,t_k)dx \right. \\ & \quad \left. + \Delta M^{-1}(D)\Delta^{-1}\Delta\chi(D,t)\Delta^{-1}\Delta M(D)\Delta^{-1}Z(t) \right\|^2 \\ & \leq 4h\alpha_2\theta^2 \|A - LC\|^2 \|Z(t)\|^2 + 4\alpha_2 h^2 \theta^2 \|LC\|^2 \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \\ & \quad + 4h\alpha_2 D \theta^2 \|L\|^2 \left\| \int_0^D \tilde{p}_x^2(x,t_k)dx \right\|^2 \\ & \quad + 4h\alpha_2 (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2, \text{ for } t_k \leq t < t_{k+1}, k \in \mathbf{N} \quad (56) \end{aligned}$$

where the third term on the right side of (56) is got using Jensen's inequality. Using (56), (55), (54), (51), (50), (45), (44), (43), it follows from (42) that, for $t_k \leq t < t_{k+1}$, $k \in \mathbf{N}$:

$$\begin{aligned} \dot{V}(Z, \tilde{p}, t) & \leq -\theta\mu \|Z(t)\|^2 + \theta\zeta \|Z(t)\|^2 \\ & \quad + \frac{h\theta \|PLC\|^2}{\zeta} \int_{t_k}^t \|\dot{Z}(s)\|^2 ds + \theta\zeta \|Z(t)\|^2 \\ & \quad + \frac{\theta D \|PL\|^2}{\zeta} \int_0^D \tilde{p}_x^2(x,t_k)dx + 2\beta_f\beta_M^2 \|P\| \|Z(t)\|^2 \\ & \quad - a\alpha_0 q\tilde{p}^2(0,t) - a\alpha_0 \int_0^D \tilde{p}_x^2(x,t)dx - c\alpha_0 \int_0^D \tilde{p}^2(x,t)dx \\ & \quad + \frac{\alpha_0}{\varpi} \int_0^D \tilde{p}^2(x,t)dx + \alpha_0 \varpi D (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2 \\ & \quad - c\alpha_0 \int_0^D \tilde{p}^2(x,t)dx + 4h\alpha_2\theta^2 \|A - LC\|^2 \|Z(t)\|^2 \\ & \quad - a\alpha_1 \int_0^D \tilde{p}_{xx}^2(x,t)dx - cq\alpha_1\tilde{p}^2(0,t) - c\alpha_1 \int_0^D \tilde{p}_x^2(x,t)dx \\ & \quad + \frac{\alpha_1}{2\eta} \int_0^D \tilde{p}_{xx}^2(x,t)dx + \frac{\alpha_1\eta D}{2} (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2 \\ & \quad + 4\alpha_2 h^2 \theta^2 \|LC\|^2 \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \\ & \quad + 4h\alpha_2 D \theta^2 \|L\|^2 \left\| \int_0^D \tilde{p}_x^2(x,t_k)dx \right\|^2 \\ & \quad + 4h\alpha_2 (\beta_f\beta_M\gamma_M)^2 \|Z(t)\|^2 - \alpha_2 \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \\ & \leq \rho_1 \|Z(t)\|^2 - \alpha_0 \left(c - \frac{1}{\varpi} \right) \int_0^D \tilde{p}^2(x,t)dx \\ & \quad - \left[a\alpha_0 + c\alpha_1 - \frac{\theta D \|PL\|^2}{\zeta} \right] \int_0^D \tilde{p}_x^2(x,t)dx \\ & \quad - \alpha_1 \left(a - \frac{1}{2\eta} \right) \int_0^D \tilde{p}_{xx}^2(x,t)dx - (a\alpha_0 + c\alpha_1) q\tilde{p}^2(0,t) \\ & \quad - \left(\alpha_2 - \frac{h\theta \|PLC\|^2}{\zeta} - 4\alpha_2 h^2 \theta^2 \|LC\|^2 \right) \int_{t_k}^t \|\dot{Z}(s)\|^2 ds \\ & \quad + \left(4h\alpha_2 D \theta^2 \|L\|^2 + \frac{\theta D \|PL\|^2}{\zeta} \right) \int_0^D \tilde{p}_x^2(x,t_k)dx \quad (57) \end{aligned}$$

with

$$\begin{aligned} \rho_1 & = \theta\mu - \theta\zeta - \theta\zeta - 2\beta_f\beta_M^2 \|P\| - 4h\alpha_2 (\beta_f\beta_M\gamma_M)^2 \\ & \quad - \alpha_0 \varpi D (\beta_f\beta_M\gamma_M)^2 - 4h\alpha_2\theta^2 \|A - LC\|^2 \\ & \quad - \frac{\alpha_1\eta D}{2} (\beta_f\beta_M\gamma_M)^2 \quad (58) \end{aligned}$$

We need the following condition to be satisfied:

$$\rho_1 > 0 \quad (59)$$

First, let the parameter θ be selected such that:

$$\theta\mu - 2\beta_f\beta_M^2\|P\| > 0 \quad (60)$$

Then, we let the maximum sampling period h , the domain length D , and the free positive real parameters $(\varpi, \varsigma, \zeta)$ be small enough so that (59) holds. A possible choice is to let $(\varpi, \varsigma, \zeta)$ be small enough so that:

$$\theta\zeta < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (61)$$

$$\theta\varsigma < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (62)$$

$$\alpha_0\varpi D(\beta_f\beta_M\gamma_M)^2 < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (63)$$

Then, one sets the maximum sampling period h so that the following inequalities hold:

$$4h\alpha_2(\beta_f\beta_M\gamma_M)^2 < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (64)$$

$$4h\alpha_2\theta^2\|A - LC\|^2 < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (65)$$

Finally, we let the domain length D small enough so that:

$$\frac{\alpha_1\eta D}{2}(\beta_f\beta_M\gamma_M)^2 < \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{6} \quad (66)$$

The still free parameters α_0 and η are let to be large enough so that the following additional couple of conditions hold:

$$a\alpha_0 - \frac{\theta D\|PL\|^2}{\varsigma} > 0, \quad a - \frac{1}{2\eta} > 0 \quad (67)$$

Then, applying Wirtinger's inequality (1), one has:

$$\begin{aligned} & - \left[a\alpha_0 - \frac{\theta D\|PL\|^2}{\varsigma} \right] \int_0^D \tilde{p}_x^2(x, t) dx - \alpha_1 \left(a - \frac{1}{2\eta} \right) \int_0^D \tilde{p}_{xx}^2(x, t) dx \\ & \leq - \frac{\pi^2}{4D^2} \left(a\alpha_0 - \frac{\theta D\|PL\|^2}{\varsigma} \right) \int_0^D \tilde{p}^2(x, t) dx \\ & \quad - \alpha_1 \frac{\pi^2}{4D^2} \left(a - \frac{1}{2\eta} \right) \int_0^D \tilde{p}_x^2(x, t) dx \end{aligned} \quad (68)$$

Also, let the maximum sampling period h be small enough so that one has,

$$\alpha_2 - \frac{h\theta\|PLC\|^2}{\varsigma} - 4h\alpha_2\theta^2\|LC\|^2 > 0 \quad (69)$$

Using (67)-(69), it follows from (57):

$$\begin{aligned} \dot{V} & \leq -\rho_1\|Z(t)\|^2 - \rho_2 \int_0^D \tilde{p}^2(x, t) dx - \rho_3 \int_0^D \tilde{p}_x^2(x, t) dx \\ & \quad - \rho_4 \int_k^t \|\dot{Z}(s)\|^2 ds - \rho_5 q \tilde{p}^2(0, t) \\ & \quad + \rho_6 \int_0^D \tilde{p}_x^2(x, t_k) dx \end{aligned} \quad (70)$$

with

$$\rho_2 = \frac{\pi^2}{4D^2} \left(a\alpha_0 - \frac{\theta D\|PL\|^2}{\varsigma} \right) + \alpha_0 \left(c - \frac{1}{\varpi} \right) \quad (71)$$

$$\rho_3 = \alpha_1 \left(a - \frac{1}{2\eta} \right) \frac{\pi^2}{4D^2} \quad (72)$$

$$\rho_4 = \alpha_2 - \frac{h\theta\|PLC\|^2}{\varsigma} - 4h\alpha_2\theta^2\|LC\|^2 \quad (73)$$

$$\rho_5 = a\alpha_0 + c\alpha_1 \quad (74)$$

$$\rho_6 = 4h\alpha_2 D \theta^2 \|L\|^2 + \frac{\theta D \|PL\|^2}{\varsigma} \quad (75)$$

By letting D be small enough, the following properties hold:

$$\rho_i > 0, (i = 2 \dots 6) \quad (76)$$

Since ρ_1 to ρ_6 are positive, it follows from (70) that:

$$\dot{V}(Z, \tilde{p}, t) \leq -\rho V(Z, \tilde{p}, t) + \frac{2\rho_6}{\alpha_1} V(Z, \tilde{p}, t_k) \quad (77)$$

using (38), with

$$\rho = \min \left(\frac{\rho_1}{\lambda_{\max}(P)}, \frac{2\rho_2}{\alpha_0}, \frac{2\rho_3}{\alpha_1}, \frac{2\rho_5}{\alpha_1}, \frac{2\rho_4}{h\alpha_2} \right) \quad (78)$$

Now, if

$$\frac{2\rho_6}{\alpha_1} < \rho \quad (79)$$

then it follows applying Halanay's inequality to (77) (see e.g. Fridman and Blighovsky, 2010, Lemma 3):

$$V(Z, \tilde{p}, t) \leq e^{-2\gamma(t-t_0)} \sup_{-h \leq \tau \leq 0} V(Z, \tilde{p}, t_0 + \tau); \quad (t \geq t_0) \quad (80)$$

where γ is the unique solution of the algebraic equation:

$$2\gamma = \rho - \frac{2\rho_6}{\alpha_1} e^{2h\gamma} \quad (81)$$

Note that the condition $\frac{2\rho_6}{\alpha_1} < \rho$ holds if D is small

enough, because $\rho_6 \rightarrow 0$ as $D \rightarrow 0$, due to (75). It readily follows from (80) that both $\|Z(t)\|$ and $\int_0^D \tilde{p}^2(x, t) dx$ exponentially converge to zero. In view of (31) and (29), so do $\|\tilde{X}(t)\|$ and $\int_0^D \tilde{u}^2(x, t) dx$. Finally, recall that the domain length D is required (in the above analysis) to be small only to make sure that inequalities (66) and (79) do hold. It is readily seen that, in the case of a zero Lipschitz coefficient β_f , those inequalities hold irrespective of the value D . This proves Theorem 1. \triangleleft

Remark 3. 1) From (60) it follows that the minimal observer gain θ^* in Theorem 1 is such that $\theta^* > 2\beta_f\beta_M^2\|P\|/\mu$. The

larger the nonlinearity coefficient β_f the larger θ^* .

2) Inequalities (61)-(67), (69), (76), and (79) are sufficient conditions, on the maximum sampling interval h and the domain length D . It is readily checked that those conditions define nonempty sets of admissible values. Let us check it for h . To meet (61)-(63) and (the second part of) (67), let the free scalars $(\varpi, \varsigma, \zeta, \eta)$ be set as follows:

$$\zeta = \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{12\theta}, \quad \varsigma = \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{12\theta} \quad (82)$$

$$\varpi = \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{12D\alpha_0(\beta_f\beta_M\gamma_M)^2}, \quad \eta = \frac{1}{a} \quad (83)$$

To meet (64) and (65), the sampling interval is set so that:

$$h < h_0 \quad (84)$$

with

$$h_0 = \min\{h_1, h_2, h_3\} \quad (85)$$

$$h_1 = \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{24\alpha_2(\beta_f\beta_M\gamma_M)^2}, h_2 = \frac{\theta\mu - 2\beta_f\beta_M^2\|P\|}{24\alpha_2\theta^2\|A - LC\|^2} \quad (86)$$

and h_3 the unique value for which the left side of (69) is zero i.e.

$$h_3 = \frac{\alpha_2}{\theta^2} \left(\frac{12\|PLC\|^2}{\theta\mu - 2\beta_f\beta_M^2\|P\|} + 4\alpha_2\|LC\|^2 \right)^{-1} \quad (87)$$

where (83) has been used to get (87).

3) In addition to the results of Theorem 1, it is noticed that all conditions on the sampling interval, namely inequalities (64), (65) and (69), are useless if $h \rightarrow 0$ i.e. in the continuous-measurement case. In such a case, only the condition on the domain length D stands. If further the Lipschitz constant is zero then, by Part 2 of Theorem 1, it follows that also the condition on D disappears, retrieving thus a similar result as in (Krstic, 2008).

4) In practical applications, the domain length D might not be small enough to meet the related smallness requirement of Theorem 1. Then, a practical solution is to implement several sensors (say N) providing the measurements of

$u(i\frac{D}{N}, t_k)$ ($i = 0 \dots N-1$), with N selected large enough

so that the ratio D/N meets the requirement on D in Theorem 1. Doing so, the observer (18)-(22) provides the estimates of the state $X(t)$ and the distributed state

$u(x, t)$, for $\frac{(N-1)}{N}D \leq x \leq D$, based only on the

measurements of $v(t)$ and $u(\frac{N-1}{N}D, t_k)$. Then, the state

$u(x, t)$ of the rest of the subdomains $(iD/N, (i+1)D/N)$ ($i = 0 \dots N-2$) can be simultaneously

estimated using the measurements $u(i\frac{D}{N}, t_k)$ (for

$i = 0 \dots N-2$) and existing observers for parabolic PDEs like (3), see e.g. (Fridman and Blighovsky, 2012, Schaum et al., 2016).

5. OBSERVER DESIGN EXTENSION

In this section, we extend the observer design method of Section III to the following wider class of systems:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)) + g_1(v(t)), \text{ for } t \geq 0 \quad (88)$$

$$w_t(x, t) = aw_{xx}(x, t) + bw_x(x, t) - cw(x, t) + g_2(x, v(t)), \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \text{ a.e.} \quad (89)$$

$$w_x(0, t) = qw(0, t) + p_0(v(t)), \text{ for all } t \geq 0 \quad (90)$$

$$w(D, t) = CX(t) + p_1(v(t)), \text{ for all } t \geq 0 \quad (91)$$

for some functions $g_1(\cdot)$ and g_2 of class $C^1(\mathbf{R}^m; \mathbf{R}^n)$ and $C^1([0, D] \times \mathbf{R}^m; \mathbf{R})$, respectively, and some functions p_0, p_1 of class $C^2(\mathbf{R}^m; \mathbf{R})$; the exogenous signal $v(t) \in \mathbf{R}^m$ (for some integer $m \geq 1$) is accessible to measurements and of

class $C^2(\mathbf{R}_+; \mathbf{R}^m)$. All other quantities remain unchanged with respect to (2)-(4). Compared to the initial class of systems defined by (2)-(4), the new PDE equation (47) includes the convection term $bw_x(x, t)$ (with $b \in \mathbf{R}$);

We seek an exponentially convergent observer for the new system (88)-(91). To this end, introduce the state transformation,

$$u(x, t) = e^{r(x-D)} w(x, t), \text{ for } (x, t) \in [0, D] \times [0, +\infty) \quad (92)$$

with

$$r = \frac{b}{2a} \quad (93)$$

Differentiating $u(x, t)$, one gets using (40):

$$u_t(x, t) = ae^{r(x-D)} w_{xx}(x, t) + be^{r(x-D)} w_x(x, t) - ce^{r(x-D)} w(x, t) + e^{r(x-D)} g(x, v(t)) \quad (94)$$

$$u_x(x, t) = re^{r(x-D)} w(x, t) + e^{r(x-D)} w_x(x, t) \quad (95)$$

$$u_{xx}(x, t) = r^2 e^{r(x-D)} w(x, t) + 2re^{r(x-D)} w_x(x, t) + e^{r(x-D)} w_{xx}(x, t) \quad (96)$$

for $(x, t) \in [0, D] \times [0, +\infty)$. It follows from (94)-(96) that:

$$u_t(x, t) = au_{xx}(x, t) - (c + \frac{b^2}{4a})u + e^{r(x-D)} g_2(x, v(t)) \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \text{ a.e.} \quad (97)$$

Similarly, the following boundary conditions are readily obtained from (92) and (95), using (90) and (91):

$$u(D, t) = w(D, t) = CX(t) + p_1(v(t)), \text{ for all } t \geq 0 \quad (98)$$

$$u_x(0, t) = (q + \frac{b}{2a})u(0, t) + e^{-rD} p_0(v(t)), \text{ for all } t \geq 0 \quad (99)$$

The transformed system modelled by equations (88) and (97)-(99), is rewritten here for convenience:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)) + g_1(v(t)), \text{ for } t \geq 0 \quad (100)$$

$$u_t(x, t) = au_{xx}(x, t) - (c + \frac{b^2}{4a})u + e^{r(x-D)} g_2(x, v(t)) \quad \text{for } (x, t) \in (0, D) \times (0, +\infty) \text{ a.e.} \quad (101)$$

$$u(D, t) = CX(t) + p_1(v(t)), \text{ for all } t \geq 0 \quad (102)$$

$$u_x(0, t) = (q + \frac{b}{2a})u(0, t) + e^{-rD} p_0(v(t)), \text{ for } t \geq 0 \quad (103)$$

$$w(x, t) = e^{-r(x-D)} u(x, t) \quad (104)$$

where the last equation is obtained from (92) to make the link between the new system model (100)-(103) and its initial model (88)-(91). Except for the terms involving the external signal $v(t)$, equations (100)-(103) fit the model structure (2)-(4). In the latter the parameter c (resp. q) is replaced by

$$c + \frac{b^2}{4a} \text{ (resp. } q + \frac{b}{2a} \text{)}. \text{ Accordingly, it is supposed that:} \quad a > 0, q + \frac{b}{2a} \geq 0 \quad (105)$$

The system (100)-(104) is observed through the sampled measurements $y(t) = u(0, t_k)$, for $t \in [t_k, t_{k+1})$ and $k = 0, 1, 2, \dots$ (this amounts to assuming $w(0, t_k) = e^{rD} u(0, t_k)$ accessible to measurements). Again, consider the transformation (8), with $M(x) \in \mathbf{R}^{n \times n}$ defined by the ODE

(9) and initial conditions (10) replacing there c (resp. q) by $c + \frac{b^2}{4a}$ (resp. $q + \frac{b}{2a}$). It turns out that $M(x)$ is defined by,

$$\frac{d^2 M}{dx^2}(x) = a^{-1} M(x) \left(A + \left(c + \frac{b^2}{4a} \right) I \right) \quad (106)$$

$$M(0) = I, \quad \frac{dM}{dx}(0) = \left(q + \frac{b}{2a} \right) I \quad (107)$$

Then, the system (100)-(103) rewrites as follows in terms of the (X, p) -coordinates:

$$\dot{X}(t) = AX(t) + f(X(t), v(t)) + g_1(v(t)), \quad \text{for } t \geq 0 \quad (108)$$

$$\begin{aligned} p_t(x, t) &= ap_{xx}(x, t) - \left(c + \frac{b^2}{4a} \right) p(x, t) + e^{r(x-D)} g_2(x, v(t)) \\ &\quad - CM^{-1}(D)M(x)(f(X(t), v(t)) + g_1(v(t))), \\ &\text{for } (x, t) \in (0, D) \times (0, +\infty) \end{aligned} \quad (109)$$

$$\begin{aligned} p_x(0, t) &= \left(q + \frac{b}{2a} \right) p(0, t) + e^{-rD} p_0(v(t)), \\ &\text{for all } t \geq 0 \end{aligned} \quad (110)$$

$$p(D, t) = p_1(v(t)), \quad \text{for all } t \geq 0 \quad (111)$$

$$u(x, t) = p(x, t) + CM^{-1}(D)M(x)X(t) \quad (112)$$

$$w(x, t) = e^{-r(x-D)} u(x, t) \quad (113)$$

Then, inspired by the observer (65)-(70), we propose the following observer for the system (108)-(113):

$$\begin{aligned} \dot{\hat{X}}(t) &= A\hat{X}(t) + f(\hat{X}(t), v(t)) + g_1(v(t)), \\ &\quad - L(\hat{y}(t_k) - y(t_k)), \text{ for } t \in [t_k, t_{k+1}), k \geq 0 \end{aligned} \quad (114)$$

$$\begin{aligned} \hat{p}_t(x, t) &= a\hat{p}_{xx}(x, t) - \left(c + \frac{b^2}{4a} \right) \hat{p}(x, t) + e^{r(x-D)} g_2(x, v(t)) \\ &\quad - CM^{-1}(D)M(x)(f(\hat{X}, v) + g_1(v(t))), \\ &\text{for } (x, t) \in (0, 1) \times (0, +\infty) \end{aligned} \quad (115)$$

$$\begin{aligned} \hat{p}_x(0, t) &= \left(q + \frac{b}{2a} \right) \hat{p}(0, t) + e^{-rD} p_0(v(t)) = 0, \\ &\text{for } t \geq 0 \end{aligned} \quad (116)$$

$$\hat{p}(D, t) = p_1(v(t)), \quad \text{for all } t \geq 0 \quad (117)$$

$$\begin{aligned} \hat{u}(x, t) &= \hat{p}(x, t) + CM^{-1}(D)M(x)\hat{X}(t), \\ &\text{for } (x, t) \in [0, 1] \times [0, +\infty) \end{aligned} \quad (118)$$

$$\hat{w}(x, t) = e^{-r(x-D)} \hat{u}(x, t) \quad \text{with } r = \frac{b}{2a} \quad (119)$$

$$\hat{y}(t) = \hat{u}(0, t) \quad (120)$$

Note that all terms involving the external signal $v(t)$, on the right side of equations (114)-(117), disappear when subtracting those equations from the corresponding equations in (108)-(111), leading to an error system similar to (26)-(28)

except for that c (resp. q) in the latter is replaced by $c + \frac{b^2}{4a}$

(resp. $q + \frac{b}{2a}$) in the former. Then, applying Theorem 1 to

the new error system, we obtain the following result:

Corollary 1. Consider the class of systems defined by equations (100)-(104) with parameters $a > 0$, $c \in \mathbf{R}$ and $q \geq -\frac{b}{2a}$. Consider the output sampled observer defined

(114)-(119), where the scalar parameter $\theta > 0$ is arbitrary, the gain $L \in \mathbf{R}^n$ is such that the matrix $A - LC \in \mathbf{R}^{n \times n}$ is Hurwitz, and the matrix function $M(x)$ is defined either by (106)-(107). There exists a $\theta^* > 0$ such that for any $\theta > \theta^*$, there exist $D^* > 0$ and $h^* > 0$ so that if $0 < D < D^*$ and $0 < h < h^*$ then the observer (108)-(113) is exponentially convergent in the sense described in Theorem 1.

Remark 4. The existence and uniqueness of solutions of the systems considered above can be analysed in many ways.

1) Existence and uniqueness of solutions for (108)-(111). Following e.g. the approach developed in (Karafyllis and Krstic, 2017a), especially Theorem 2.1, we can show that for every $p[0] \in C^2([0, 1])$ with $(p[0])(1) - p_1(v(0)) = 0$ and

$$(p[0])'(0) - D \left(q + \frac{b}{2a} \right) (p[0])(0) - D \exp(-rD) p_0(v(0)) = 0,$$

there exists a unique mapping $p \in C^0([0, 1] \times \mathbf{R}_+) \cap C^1([0, 1] \times (0, +\infty))$, with $p[t] \in C^2([0, 1])$ for all $t \geq 0$, that satisfies (123), for all $t > 0$ and $z \in [0, 1]$ and (124)-(125, for all $t \geq 0$.

A similar result holds for (2)-(4).

2) Existence and uniqueness of solutions for the systems (114)-(117).

Here, we assume that $D = 1$ (otherwise perform the scaling $z = Dx$). Using Theorem 2.1 in (Karafyllis and Krstic, 2017a), we can show the result:

For every increasing sequence $\{t_k \geq 0\}_{k=0}^\infty$ with $t_0 = 0$ and $\lim_{k \rightarrow +\infty} (t_k) = +\infty$, for every $\hat{X}_0 \in \mathbf{R}^n$, $\hat{p}_0 \in C^2([0, 1])$, with

$$\hat{p}'_0(0) - \left(q + \frac{b}{2a} \right) \hat{p}_0(0) - \exp(-r) p_0(v(0)) = 0$$

$$\hat{p}_0(1) - p_1(v(0)) = 0,$$

and for every locally bounded function $y: \mathbf{R}_+ \rightarrow \mathbf{R}$, there exist unique mappings $\hat{X} \in C^0(\mathbf{R}_+) \cap C^1(I)$, where

$$I = \bigcup_{k=0}^\infty (t_k, t_{k+1}), \quad \hat{p} \in C^0(\mathbf{R}_+ \times [0, 1]) \cap C^1(I \times [0, 1]) \quad \text{with}$$

$\hat{p}[t] \in C^2([0, 1])$, for all $t \geq 0$, $\hat{X}(0) = \hat{X}_0$, $\hat{p}[0] = \hat{p}_0$ and $\hat{X} \in C^0(\mathbf{R}_+) \cap C^1(I)$ being right-differentiable on \mathbf{R}_+ , so that (114) holds for all $t \in [t_k, t_{k+1})$ and $k = 0, 1, 2, \dots$, (115) holds for all $(x, t) \in [0, 1] \times I$ and (116), (117) hold for $t \geq 0$.

A similar result can be stated for system (18)-(20).

5. SIMULATION

To illustrate the observer design, we consider the ODE-PDE cascade (2)-(4) with

$$n = 2, a = D = 1, c = -1, q = 0 \quad (121)$$

$$f(X, v) := [0 \quad -g(X, v)]^T \quad (122)$$

where $g(X, v) := v + 5X_2 + X_1 + X_1 / (1 + X_1^2)$. The ODE part (2) represents a mass-spring system with viscous force and nonlinear restoring force. The mass-spring system is subject to the effect of an external force. For this case, the matrix $M(x)$ defined by (9), (10) is given by

$$M(x) = \begin{bmatrix} \cos(x) & x \sin(x)/2 \\ 0 & \cos(x) \end{bmatrix}, \text{ for } x \in [0,1] \quad (123)$$

Using $L = [2 \ 1]^T$, $\theta = 2$, the observer (18), (19), (20), (21), (22), (23) is expressed by the equations

$$\begin{aligned} \frac{d}{dt} \hat{X}_1 &= \hat{X}_2 - 2[2 \cos(1) + \sin(1)](\hat{u}(0, t_k) - y(t_k)) \\ \frac{d}{dt} \hat{X}_2 &= -g(\hat{X}, v) - 4 \cos(1)(\hat{u}(0, t_k) - y(t_k)) \end{aligned} \quad (124)$$

$$\begin{aligned} \hat{u}(0, t) &= \hat{p}(0, t) + \frac{1}{\cos(1)} \hat{X}_1(t) - \frac{\sin(2)}{2 \cos^2(1)} \hat{X}_2(t) \\ \hat{p}_t(x, t) &= \hat{p}_{xx}(x, t) + \hat{p}(x, t) + \kappa(x) g(\hat{X}(t), v(t)) \\ \hat{p}_x(0, t) &= \hat{p}(1, t) = 0 \end{aligned} \quad (125)$$

where

$$\kappa(x) := \frac{\cos(1)x \sin(x) - \cos(x) \sin(1)}{2 \cos^2(1)}, \text{ for } x \in [0,1] \quad (126)$$

In order to simulate numerically system (2), (14), (15), (16), (17), (124), (125) we used an eigenfunction expansion of the solution of (125) up to 30 terms, i.e., we simulated the ODE system for $n = 0, 1, \dots, 30$

$$\begin{aligned} \frac{d \hat{a}_n}{dt}(t) &= - \left(\left(n\pi + \frac{\pi}{2} \right)^2 - 1 \right) \hat{a}_n(t) \\ &\quad - \frac{8(-1)^n (2n+1) \pi \sqrt{2}}{\left((2n+1)^2 \pi^2 - 4 \right)^2} g(\hat{X}(t), v(t)) \end{aligned} \quad (127)$$

and approximated the solution of (125) by the formula

$$\hat{p}(x, t) = \sqrt{2} \sum_{n=0}^{30} \hat{a}_n(t) \cos \left(\left(n\pi + \frac{\pi}{2} \right) x \right)$$

A similar approximation was applied for the solution of (14), (15), (16), (17).

We also have considered the presence of noise, i.e., we set $y(t) = u(0, t_k) + \xi(t)$,

$$\text{for all } t \in [t_k, t_{k+1}) \text{ and } k = 0, 1, 2, \dots \quad (128)$$

where $\xi(t)$ is the noise. We have simulated the system described by (2), (14)-(17) and (121)-(122), on the one hand, and the observer represented by (124)-(125), on the other, with initial conditions $\hat{X}_1(0) = \hat{X}_2(0) = 0$, $X_1(0) = X_2(0) = 1$,

$$p(x, 0) = \sqrt{2} \sum_{n=0}^3 \cos \left(\left(n\pi + \frac{\pi}{2} \right) x \right), \quad \hat{p}(x, 0) \equiv 0, \quad \text{input } v(t) = \cos(t).$$

The simulation is performed considering noise-free output and noisy output. In the last case, the sequence $\xi(t)$ is a zero-mean Gaussian white noise with variance 0.03. We have used a uniform sampling schedule $t_k = kh$, for all integers $k \geq 0$, with two values of the sampling interval: $h = 0.1$ and $h = 0.2$. The obtained results are illustrated by Figures 1 to 5. First, Fig. 1 shows the effect of the injected noise on the output. Fig. 2 and Fig. 3 illustrate the observer performances in the free-noise case ($\xi \equiv 0$), while Fig. 4 and Fig. 5 show the performances in the noisy case. Clearly, the observer errors converge to zero in the noise-free case (Figs 2 and 3), with greater convergence rate when the smaller sampling period $h = 0.1$ is used. Furthermore, it is readily seen on Figs 4 and 5 that the noise has only a weak effect on the observer convergence quality. This can be explained by the fact the

observer gain $\theta = 2$ is presently not too large. Finally, we have checked that the good convergence quality is preserved for sampling periods up to $h = 0.35$.

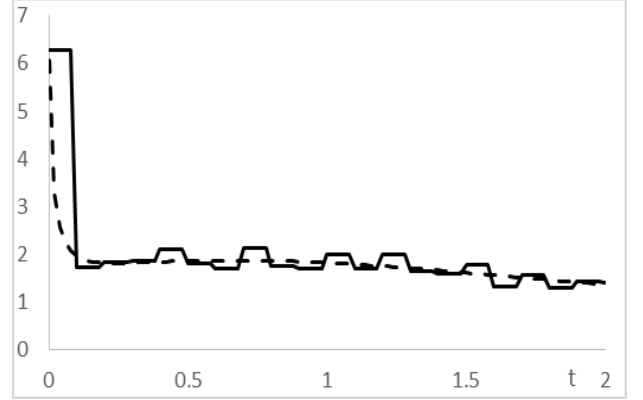


Fig. 1. Noise effect on the output $y(t)$ given by (128) for $h = 0.1$: noisy output (solid line) and noise-free output (dashed line).

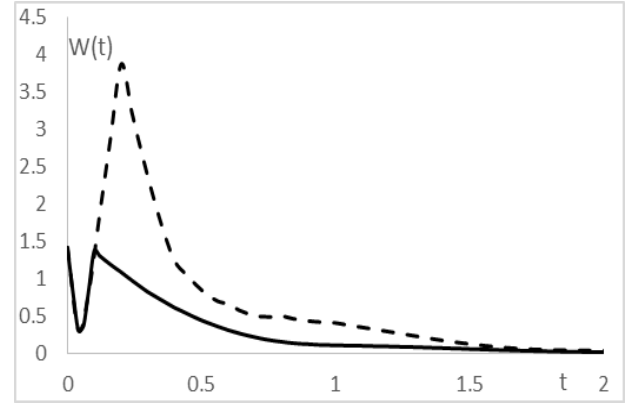


Fig. 2. The time evolution of $W(t) = |\hat{X}(t) - X(t)|$ without noise and $h = 0.1$ (solid line) and $h = 0.2$ (dashed line)

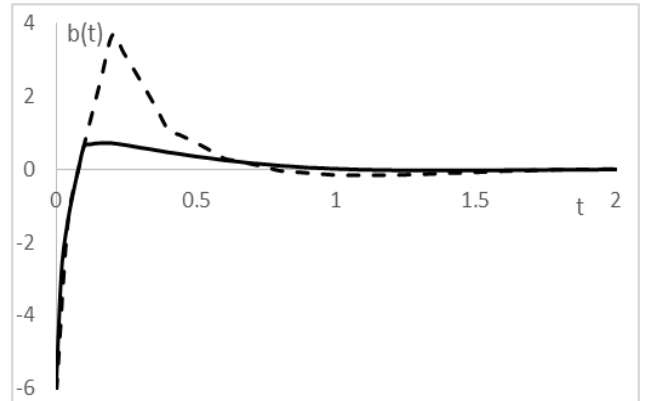


Fig. 3. The time evolution of $b(t) = \hat{u}(0, t) - u(0, t)$ without noise and $h = 0.1$ (solid line) and $h = 0.2$ (dashed line).

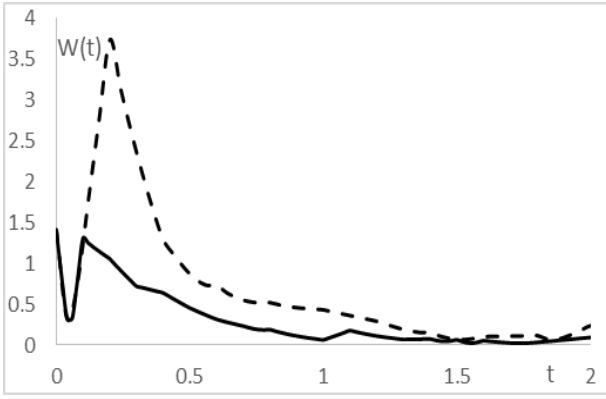


Fig. 4. Error $W(t) = |\hat{X}(t) - X(t)|$ in the presence of noise with $h = 0.1$ (solid line) and $h = 0.2$ (dashed line).

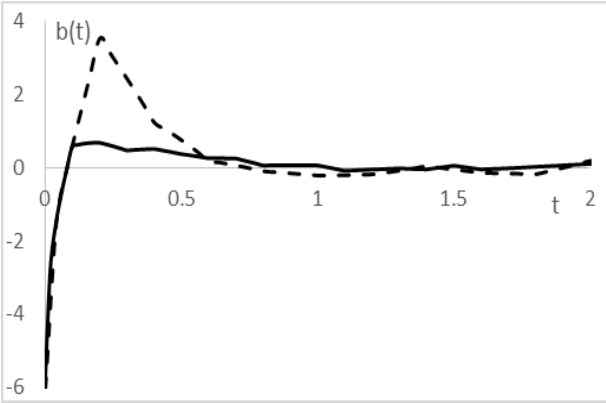


Fig. 5. The error $b(t) = \hat{u}(0,t) - u(0,t)$ in the presence of noise with $h = 0.1$ (solid line) and $h = 0.2$ (dashed line).

6. CONCLUSION

We have developed a sampled boundary observer design method that applies to the class of ODE-PDE cascades modelled by (2)-(4) and (88)-(91). The main characteristics of the new design method are: (i) the coordinate transformations (8) and (31); (ii) the matrix-valued function $M(x)$ (defined by (10)-(11)) and the high-gain matrix Δ used in the observer gains; (iii) the Lyapunov-Krasovskii type functional (38). Theorem 1 shows that exponential stability of the observation error system (26)-(29) entails conditions on the maximum (time) sampling interval and the PDE domain length. Accordingly, the maximum domain length is a decreasing function of the Lipschitz constant β_f and tends to infinity when $\beta_f \rightarrow \infty$, retrieving thus the linear case result of (Ahmed-Ali et al., 2016b, 2017b; Karafyllis et al., 2017). To the authors' knowledge, it is the first time that a sampled boundary observer design is developed for ODE-PDE cascades that involve nonlinear ODEs. The present work can be pursued in many senses including the extension to more general classes of ODEs, involving non-globally Lipschitz nonlinearities.

REFERENCES

- Ahmed-Ali T., E. Fridman, F. Giri, L. Burlion, F. Lamnabhi-Lagarigue (2016a). A new approach to enlarging sampling intervals for some sampled-data systems and observers. *Automatica*, vol. 67, pp. 244-251.
- Ahmed-Ali T., F. Giri, M. Krstic (2017a). Observer Design for Triangular Nonlinear Systems in the Presence of Arbitrarily Large Output Delay - A PDE Based Approach. *American Control Conference*, Seattle, WA, USA.
- Ahmed-Ali T., F. Giri, M. Krstic, F. Lamnabhi-Lagarigue (2015). High-Gain Observer Design for a Class of Nonlinear ODE-PDE Cascade Systems. *Systems & Control Letters*, vol. 83, pp. 19-27.
- Ahmed-Ali T., F. Giri, M. Krstic, M. Kahelras. PDE Based Observer Design for Nonlinear Systems with Large Output Delay. *Systems and Control Letters*, *Systems & Control Letters*, vol. 113, pp. 1-8, 2018.
- Ahmed-Ali T., I. Karafyllis, F. Giri, M. Krstic and F. Lamnabhi-Lagarigue (2016b) Stability Result for a Class of Sampled-Data Systems and Application to Observer Design for Cascade ODE-PDE systems. *22nd International Symposium on Mathematical Theory of Networks and Systems (MTNS 2016)*, Minneapolis, MN, USA.
- Ahmed-Ali T., I. Karafyllis, F. Giri, M. Krstic, and F. Lamnabhi-Lagarigue (2017b). Exponential Stability Analysis of Sampled-Data ODE-PDE Systems and Application to Observer Design. *IEEE Transactions on Automatic Control*, Vol. 62 (6), pp. 3091-3098.
- Fridman E., and A. Blighovsky (2010). Sampled-Data Stabilization of a Class of Parabolic Systems", *Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems*, pp. 1129-1134.
- Fridman E., and A. Blighovsky (2012). Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, vol. 48, pp. 826-836.
- Hasan A., O. M. Aamo, and M. Krstic (2016). Boundary observer design for hyperbolic PDE-ODE cascade systems. *Automatica*, vol. 68, pp. 75-86.
- Karafyllis I., M. Krstic (2017a). ISS in Different Norms for 1-D Parabolic PDEs with Boundary Disturbances. *SIAM Journal on Control and Optimization*, 55(3), pp. 1716-1751.
- Karafyllis I., M. Krstic (2017b). Decay estimates for 1-D parabolic PDEs with boundary disturbances. <https://arxiv.org/abs/1706.01410>
- Karafyllis I., T. Ahmed-Ali, F. Giri, M. Krstic, F. Lamnabhi-Lagarigue (2017). Stability of Sampled-Data PDE-ODE Systems and Application to Observer Design and Output-Feedback Stabilization. *IFAC World Congress*, Toulouse, France, pp. 13831-13836.
- Khalil H.K. (2015). *Nonlinear Control*. Pearson Education. NJ, USA.
- Krstic M., (2009). Compensating actuator and sensor dynamics governed by diffusion PDEs. *Systems & Control Letters*, vol. 58, pp. 372-377.
- Krstic M., A. Smyshlyaev (2008). Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays. *Systems & Control Letters*, vol. 57, pp. 750-758.
- Schaum A., J. Alvarez, T. Meurer, J.A. Moreno, (2016). Pointwise innovation-based state observation of exothermic tubular reactors. *IFAC-PapersOnLine* Vol. 49(7), pp. 955-960.
- Smyshlyaev A., Krstic M., (2010). *Adaptive control of parabolic PDEs*. Princeton University Press, NJ, USA.

- Tang S., and C. Xie (2010). Stabilization of a Coupled PDE-ODE System by Boundary Control. 49th IEEE Conference on Decision and Control, Atlanta, GA, USA.
- Tang S.X., C. Xie (2011). State and output feedback boundary control for a coupled PDE-ODE system. *Systems & Control Letters*, vol. 60(8), pp. 540-545.
- Wu H.N., J.W. Wang (2013). Observer design and output feedback stabilization for nonlinear multivariable systems with diffusion PDE-governed sensor dynamics. *Nonlinear Dynamics*, vol. 72(3), pp 615–628.