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Modeling Consonance and Its Relationships with Temperament and Harmony (CDT-10)

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Abstract

After revising the concepts of consonance/dissonance, a respective mathematic-computational model is described, based on Helmholtz’s consonance theory and also considering the partials intensity. It is then applied to characterize five scale temperaments, as well as some minor and major triads. In spite of the simplicity of this model, a good agreement is often observed between the obtained consonances/dissonances and the typically observed properties of scales and chords. The representation of temperaments as graphs where links correspond to consonance (or dissonance) is also presented and used to compare distinct temperaments, allowing the identification of two main groups of approaches.

‘Geometry is in the humming of strings.’

Pythagoras.

1 Introduction

Sound and music are intrinsically human concepts and activities. In other words, there are no definitions to be found in nature regarding what sound and music are, or how they should be. These definitions and properties are *relative* to us, humans. As such, these concepts are inherently subjective, and they may vary in time and space. Indeed, the history of music has witnessed big changes in instruments, styles, and theories.

Nevertheless, there is an aspect somewhat stable in sound and music, and this has to do with *consonance* and *dissonance*. These concepts have permeated the history of tonal music, while also playing important roles in many other musical traditions. Basically, consonance has to do with our sensation of ‘harmony’ between two simultaneous sounds, or notes. As such, consonance/dissonance is at the very basis of *sound combinations* along time and space, which underlies *music*.

One of the first mathematic approaches to consonance was developed by Pythagoras (e.g. [1]), by considering ratios between the length of strings. Afterwards, the historic records mostly go back to monophony, especially

Gregorian chant. With time, drones were added, initiating a very simple kind of harmony. Subsequent developments would lead to two parallel voices, contrary/oblique voices, and then to ever increasing types of polyphony. With the incorporation of more and more voices, the problem of combining sounds in time and space became ever more critical.

To make a long (and interesting) history short, the progression to polyphony followed, or even implied, the development of *harmony* and *counterpoint*, which can be informally understood as the activity of combining simultaneous or subsequent sounds, respectively (e.g. [2, 3]). One of the first more formal approaches to consonance was developed by J. P. Rameau, who argued strongly against the current view overlooking differences between consonance and dissonance. To him, consonance was a kind of *repose*, so that dissonance would imply a depart from that state (e.g. [4]).

Because of its central role in sound and music, consonance motivated several theories, such as based on interval ratios or interaction between harmonics/partial. In particular, the latter was developed mainly by H. L. F. Helmholtz (1821–1894). Backed by careful experiments, Helmholtz advanced the idea that consonance derives from interrelationships between partials, producing constructive combination of frequencies or, alternatively, unpleasant *beats* (e.g. [5]).

Recall that, given a fundamental tone, the accompa-

nying higher frequency tones are called *partials*. In case these partials follow the harmonic series (i.e. string wavelengths of 1, 1/2, 1/3, 1/4, ...), they are called *harmonics*.

In this text, we approach the interesting issues of consonance/dissonance by considering Helmholtz’s respective theory. More specifically, we derive a simple mathematical-computational model (e.g. [6]) capable of, given two sounds, providing degrees of consonance and dissonance. In addition to considering the frequencies of the involved partials, the respective amplitudes are also taken into account their magnitudes. Then, we use this model to revisit some of the main temperaments (informally speaking, ways of assigning frequencies to notes) as well as some aspects of basic harmony, especially triad consonance.

2 On the Mixture of Sinusoidals with the Same Frequency

A first interesting issue to be addressed regards what happens when two sinusoidal sounds with the *same* frequency $\omega = 2\pi f$, but distinct amplitudes (a_1, a_2) and phases (θ_1, θ_2), are combined, i.e. $a_1\cos(\omega t + \theta_1) + a_2\cos(\omega t + \theta_2)$.

This interesting question can be simply addressed by considering the phasor representation (e.g. [7]) of these two sounds, respectively $a_1e^{i\theta_1}$ and $a_2e^{i\theta_2}$, as illustrated in Figure 1.

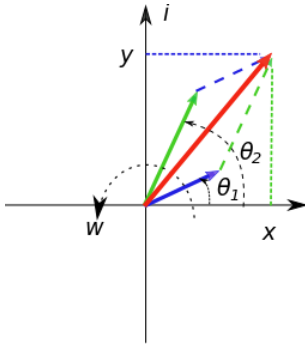


Figure 1: Phasor representation of two sinusoidal sounds with the same frequency $f = 2\pi f$ but distinct amplitudes (a_1, a_2) and phases (θ_1, θ_2). Because these signals have the same frequency, this can be omitted, so that the phasors consider only the respective phases, yielding $a_1e^{i\theta_1}$ (shown in blue) and $a_2e^{i\theta_2}$ (shown in green). The addition of these two sounds can be implemented vectorially in the complex plane (and then taking the real projection), yielding the phasor shown in red.

Phasor representation allows us to add, in complex or vector fashion, the involved phasors, resulting the phasor shown in red in Figure 1, which has real and imaginary parts, as well as its magnitude a and phase θ , easily calculated as

$$\begin{aligned} x &= a_1\cos(\theta_1) + a_2\cos(\theta_2) \\ y &= a_1\sin(\theta_1) + a_2\sin(\theta_2) \\ a &= \sqrt{x^2 + y^2} \\ \theta &= \text{atan}\left(\frac{y}{x}\right). \end{aligned}$$

So, we have that

$$a_1\cos(\omega t + \theta_1) + a_2\cos(\omega t + \theta_2) = a\cos(\omega t + \theta).$$

Mixtures of sines and cosines can also be handled by using Equation 1 while recalling that $\sin(\alpha) = \cos(\alpha - \pi/2)$

Thus, the linear combination of any number of sines and cosines with the same frequency, no mattering how different are their magnitudes and phases are, will always yield a simple cosine with effective magnitude a and phase θ as a result. In the next section we show that more interesting results arise when linearly combining two or more sinusoidals with *different* frequencies.

3 Mixture of Sinusoidal Signals with Distinct Frequencies

Given two sinusoidal tones with respective distinct frequencies $\omega_1 = 2\pi f_1$ and $\omega_2 = 2\pi f_2$, $\omega_1 < \omega_2$, it immediately becomes interesting to quantify the relationship between these frequencies. One possibility is in terms of their *ratio* f_2/f_1 , and another is in terms of the measurement called *cent*, which is defined as

$$\text{cent}(\omega_2, \omega_1) = \text{cent}(f_2, f_1) = 1200 \log_2\left(\frac{\omega_2}{\omega_1}\right). \quad (1)$$

Observe that 100 cents equate a equally-tempered *half-tone*, corresponding to each subsequent note along a scale.

Let’s sum these two signals as

$$s(t) = a_1\cos(\omega_1 t + \theta_1) + a_2\cos(\omega_2 t + \theta_2).$$

Again, the phasor representation turns out to be particularly useful in analysing this signal addition. Figure 2 illustrates this situation.

Because the two frequencies are now different, for simplicity’s sake we can keep the slower-rotating phasor (shown in blue) parallel to the real axis. Observe that the original phases θ_1 and θ_2 are therefore lost, but this does not matter for our following analysis.

Observe that the second phasor (shown in red) now rotates with angular velocity $\delta\omega = \omega_2 - \omega_1$. So, as it moves,

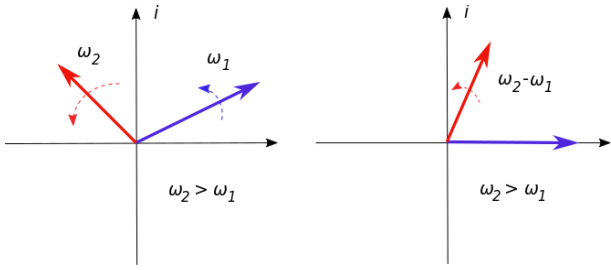


Figure 2: (a) Phasor representation of two sinusoidal functions with distinct frequencies, amplitudes and phases. (b) It is possible to keep the phasor corresponding to the smaller frequency signal parallel to the real axis while analysing the behavior of the other phasor, which now rotates with angular velocity (or frequency) $\omega_2 - \omega_1$.

the magnitude of the resulting phasor (projection onto the real axis) varies from a maximum value of a_1 , for $\delta\omega = 0$, to its smallest value, observed at delta $\delta\omega = \pi$. This oscillating effect on the amplitude takes place with angular frequency $\omega_2 - \omega_1$. Figure 3 illustrates the amplitude oscillation produced by the addition of two signals with respective frequencies $f_1 = 440Hz$ and $f_2 = 450Hz$ and identical unit magnitudes.

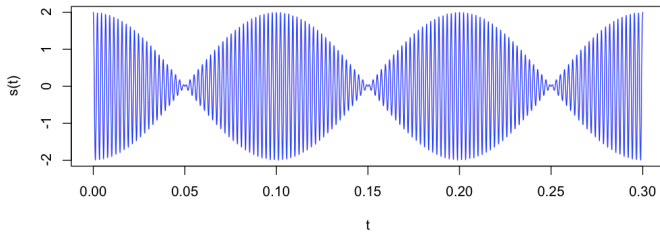


Figure 3: Signal resulting from the sum of two sinusoidsals $f_1 = 440Hz$ and $f_2 = 450Hz$. Observe the amplitude *modulation* taking place with frequency $10Hz$. Both original signals have unit amplitude.

It is interesting to observe that, though there is a difference of angular velocities, the sum of the two signals only incorporates the two original frequencies ω_1 and ω_2 , so that $\omega_2 - \omega_1$ does not belong to the respective frequency spectrum. This can be readily appreciated by taking into account this addition in the Fourier domain, where only these two frequencies can be present.

Now, we come to an interesting issue: the way in which *humans* perceive this frequency difference *depends on its magnitude*. For $\delta\omega$ very small, (e.g. $< 2Hz$, approximately), we tend to perceive the two sounds as a single, *tuned* note. For slightly larger difference values (typically $10Hz > \delta\omega \geq 2Hz$), a single frequency with a relatively slow varying amplitude tend to be heard. Indeed, difference in this range are smaller than our minimal frequency limit of $20Hz$.

For larger values of $\delta\omega$ (e.g. $60Hz > \delta\omega > 10$), the added sinusoidal signals tend to cause an unpleasant sensation, as the resulting sound cannot be distinguished be-

Table 1: .

interval	just ratio	just cents	type of cons.
unison	1/1	0	absolute cons.
perfect octave	1/2	111.73	absolute cons.
fifth	2/3	203.91	perfect cons.
forth	3/4	315.64	medium cons.
major sixth	3/5	386.31	medium cons.
major third	4/5	498.04	medium cons.
minor third	5/6	582.51	imperfect cons.
minor sixth	5/8	701.96	imperfect cons.
major second	8/9	813.69	diss.
major seventh	8/15	884.36	diss.
minor seventh	9/16	996.09	diss.
minor second	15/16	1088.27	diss.
tritone	32/45	1200	diss.

tween a single or double frequency stimulus. The two latter situations are often described as the *beat* effect, and are typically avoided in music and instrument construction and tuning. Recall that this phenomenon, jointly with tuned sounds, provided some of the main motivation for Helmholtz’s consonance theory.

The situation where $\delta\omega > 60Hz$ typically leads to two separate sounds, which are typically called *intervals*. We will discuss this concept further in the next section.

4 Intervals

Table 1 presents the main intervals often considered in tonal music, as well as the respective just ratio (in cents) and equal-temperament counterpart. The type of consonance typically perceived for each case is also provided in the last column of this table.

It should be observed that this classification of consonances is not definitive, and some of these intervals have been perceived differently along time and space. This may have to do with the fact that consonance, as we will see further on this text, depends on the partial features of each type of sound, as well as on the type of assumed temperament.

The just intonation, commonly used to tune some types of instruments, represents the basis for some temperament systems, such as the pythagorean.

Going back to the previous table, we have that only three or four intervals – namely those corresponding to unison, octave, and fifth – are understood as presenting more definite consonance. It is thus hardly surprising that the fifth, and to a lesser extent the forth, represent such an important reference in occidental music composition.

Four intervals are understood as having intermediate consonance, and the remainder five are typically understood to be dissonant, with emphasis on the tritone (e.g. $C - F\#$).

Classification of consonance/dissonance has traditionally relied on relatively subjective human appreciation. In the following we will describe Helmholtz’s experimental/mathematical approach to consonance and then consider it for developing a simple, and yet relatively effective, model capable of automatically estimating degrees of consonance and dissonance.

5 Helmholtz’s Theory of Consonance, and a Simple Model

Helmholtz’s theory of consonance was presented mainly in the second part of his *Sensations of Tone* (e.g. [5]). The two involved sounds are understood in terms of their respective harmonic series, and these are pairwise compared in order to identify respective tuning and beats. The resulting sensation of consonance/dissonance would be related to the intensity in which these constructive interactions and beats occur given each the two original sounds. Interestingly, we have that two pure sounds (i.e. without partials) could hardly be compared regarding their consonance.

Here, we consider Helmholtz’s main concepts to develop a simple mathematic-computational model, capable of estimating in a quantitative way the consonance/dissonance between two sounds $s_1(t)$ and $s_2(t)$ with respective fundamental frequencies $f_{0,1}$ and $f_{0,2}$.

A first important issue regards the frequency distribution of partials. For simplicity’s sake, and without loss of generality, we assume that these follow the *harmonic series*, leading to partials of $f_0, 2f_0, 3f_0, 4f_0, \dots$. This is not to say that real sounds necessarily follow this structure, but other partial models can be immediately considered in the described model.

As there are virtually infinite harmonics, we also have to delimit the frequency extent of these sounds. For this purpose, we consider the fact that the partials of most sounds decay steadily with frequency (and also along time), so that too high harmonic partials result inaudible. For simplicity’s sake, we consider that the partials decay in terms of a negative exponential. Figure 4 illustrates the adopted partial decay with respect to the frequency index i corresponding to $f_0 = 1Hz$.

Though other configurations can be adopted, we henceforth assume a partial decay scheme as in Figure 4, incorporating a maximum of $N = 50$ harmonic partials.

Now, given two signals represented in terms of their partial content, we need to devise some means to com-

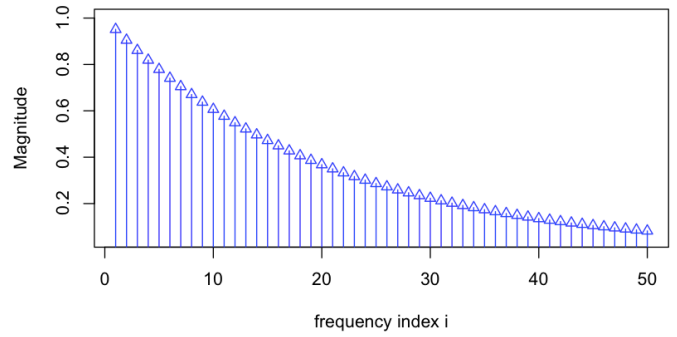


Figure 4: The adopted negative exponential way in which the magnitude of the harmonic partials are assumed to decay with the frequency index i (relative to fundamental $f_0 = 1Hz$).

pare these partials in pairwise fashion and then estimate, for each case, the respective degrees of consonance and dissonance.

Figure 5 illustrate the spectrum of two sounds to be compared $X(f)$ and $Y(f)$ (red and blue, respectively). Observe that $X(f)$ has higher fundamental frequency, implying in more widely spaced partial intervals as well as effective frequency extent.

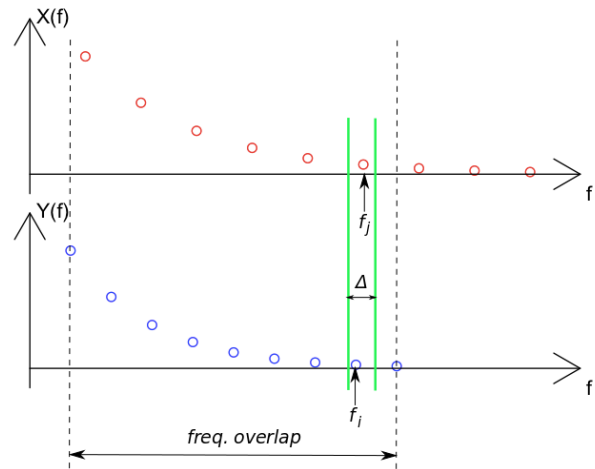


Figure 5: .

Let’s assume the lower fundamental signal $Y(f)$ to be our reference. For each of its partials, with respective frequency f_i , the closest partial f_j belonging to the other signal, $X(f)$, is identified. The frequency difference $\delta f_{i,j} = |f_j - f_i|$ is then taken, being then classified as: (i) consonance if $\delta f_{i,j} < f_c$; (ii) dissonance if $f_c \leq \delta f_{i,j} < f_d$; and (iii) neutral for $\delta f_{i,j} \geq f_d$.

So, many consonance or dissonance matches, as well as several neutral cases, can be identified for a same pair of sounds. In order to derive an overall degree of consonance and dissonance between the two original tones X and Y , we can respectively apply

$$\text{consonance}(X, Y) = \sum_{\text{cons.}(i,j)} M(i) M(j) \delta_{i,j} \quad (2)$$

$$\text{dissonance}(X, Y) = \sum_{\text{diss.}(i,j)} M(i) M(j) \delta_{i,j} \quad (3)$$

Observe that the sum takes place over the pairs (i, j) of consonant harmonic partials in the case of Equation 2, and dissonant pairs of partials in Equation 3. Other ways of combining the partial relationships are possible and can be immediately adapted into the developed model.

So, we end up with a model that, as illustrated in Figure 6, is capable of automatically assigning degrees of consonance or dissonance to pairs of presented sounds represented in terms of partials and magnitudes (spectral).

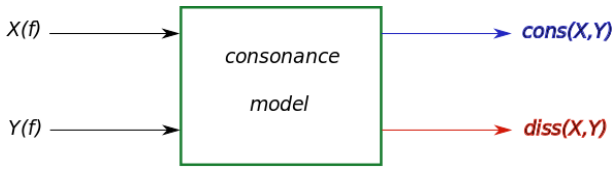


Figure 6: A ‘consonance machine,’ incorporating a simple model based on Helmholtz’s respective theory. The two sounds to be combined are input into the system, which provides degrees of consonance and dissonance as output.

Observe that, according to the adopted assumptions, the same pair of sounds (interval) has intrinsic levels of both consonance *and* dissonance (this happens rarely, though). It is also possible to adopt a threshold for classifying an interval as being consonant/dissonant. In the following we adopt 5 and 4 as thresholds for consonance and dissonance. An exponential decay of 0.08, as well as $f_c = 10Hz$ and $F_d = 60Hz$ have been also considered, as these tended to yield more appropriate results.

Some interesting implications can be observed about the above outlined approach to consonance. First, we have that consonance tends to decrease for larger difference of fundamental frequencies because of resulting smaller overlap between the partials. We also have that consonance and dissonance are not commutative, as a consequence of the above definition using the lower fundamental signal as reference.

All in all, we also have that consonance/dissonance ultimately depends on a series of factors, including temperament type, distribution of partials (other than harmonic series), and different decay profiles of partial magnitudes.

Nevertheless, the developed model allows us, to some limited degree of precision that also depends on the configuration of the involved parameters and roles (which can be adapted), to study several issues in sound and music. In the remainder of this text, we apply the simple developed model to investigate temperaments and basic harmony, especially triads.

6 Temperaments

Because scales are so important in music, several such systems have been proposed along centuries. Here, we will be limited to five representative cases: (i) equal temperament; (ii) pythagorian; (iii) just major; (iv) mean-tone; and (v) Weckmeister.

Figure 7 illustrates the bipartite graph of consonance (blue) and dissonance (red) relationships between the 12 tones. This scale is characterized by having fixed relationship between each subsequent half-tone, in the sense that $f_{i+1}/f_i = 2^{1/12}$. Therefore, any minor or major modes will be characterized by identical intervals, providing uniform subsidy for modulations and transpositions.

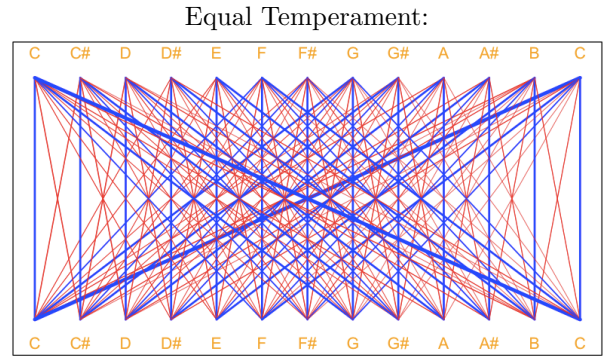


Figure 7: The consonance/dissonances obtained for the equal temperament scale starting at central C by using the described model. The consonances are shown in blue, while the dissonances are presented in red. Intervals with higher levels of consonance and dissonance are shown in black. The consonance/dissonance intensities are reflected in the respective edge widths. Observe the almost perfectly uniform pattern of consonances/dissonances obtained for the equal temperament approach.

The effect of the uniform interval characteristic of this type of scale becomes evident in this figure. Except for the fact that the consonance tends to change as one of the two frequencies becomes larger than the other, the obtained consonance/dissonance patterns are mostly similar for every note, confirming the properties often expected from this scale. However, observe that the interval relationships are mostly dissonant, with the main exceptions of the major fourth and major fifth. So, interval uniformity is achieved at the expense of consonance.

The consonance/dissonance results obtained for the *pythagorean* scale are shown in Figure 8(a). This scale is founded on the $2/3$ ratio interval, corresponding to the pure perfect fifth, possibly the third most consonant interval after the unison and octave.

Interestingly, the consonance/dissonance patterns turn out to be very similar to those obtained for the equal temperament, but the consonances related to the major fourth and fifth now alternates for almost every node, sometimes vanishing or being exchanged for other intervals.

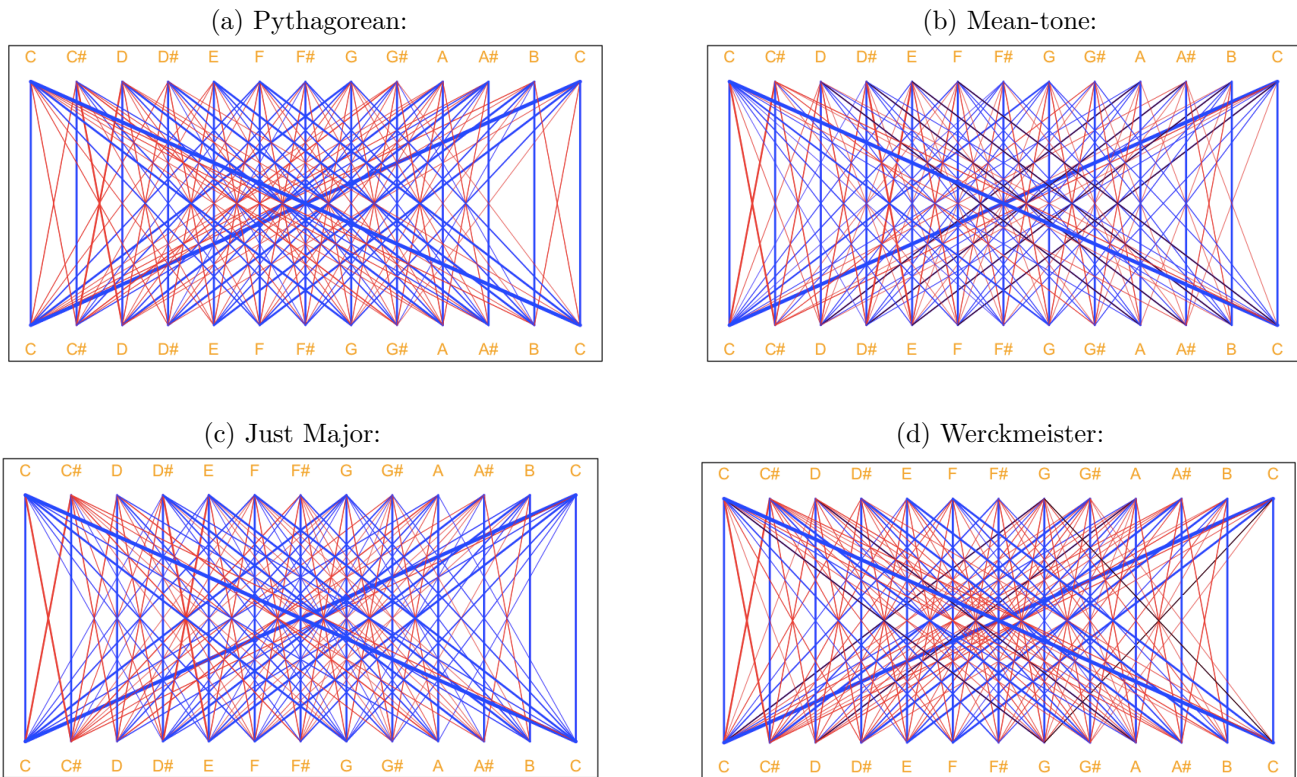


Figure 8: Consonance/dissonance patterns obtained for the pythagorean (a), mean-tone (b), just major (c) and Werckmesiter scales (d).

Figure 8(b) presents the consonance/dissonance patterns obtained for the *mean-tone* temperament. The presence of a substantially increased number of consonances can be readily observed. For instance, most of the intervals become consonant in the case of C . The intervals involving G , an important tone, also tend to be consonant. However, the aforementioned effect is achieved at the expense of consonance uniformity. Indeed, the intervals based on $C\#$ are mostly dissonant, but they become slightly more consonant for D . It is also interesting to observe the variation of the intensities of the dissonances, represented by the edges width. Particularly strong dissonances are observed for the intervals $D-D\#$ and $D\#-E$. These result are in agreement with the fact that the mean-tone will provide best results for the first modes, e.g. C -major and G -major, becoming less suitable for further mode transpositions.

An even more consonant overall pattern is observed for the *just major* scale, depicted in Figure 8(d). Indeed, this results is similar to that observed for the mean-tone case.

The consonance/dissonance patterns obtained for the intervals of the *Werckmeister* temperament, shown in Figure 8(e) seem to be similar to those obtained for the equal temperament and pythagorean.

7 Scales as Graphs

Visualizing the consonances as graphs (e.g. [8]) where each node corresponds to a single node can provide further insights about the properties and relationships between diverse temperaments. Figure 10 presents the graphs respective to the results discussed in the previous section.

Remarkably, this type of visualization allows a nice complementation of the information provided by the previously considered bipartite interrelationships. Now, it becomes evident that we have two main groups of scales: one similar to the equal temperament (also including the pythagorean and Werckmeister), and the other corresponding to the mean-tone and just major.

The fact that the mean-tone and just major scales tend to promote overall consonance is directly reflected in the greater interconnectivity exhibited by the respective graphs. The more uniform interval distributions underlying the other three types of scales define a chain of consonances between several notes, with a cluster of connections being observed between C , G , $C2$, and F .

In the Werckmeister temperament, this cluster also includes A and E . The consonances between non-adjacent notes in these three temperaments tend to be little consonant, with the pythagorean scale providing possibly improved consonance in this case.

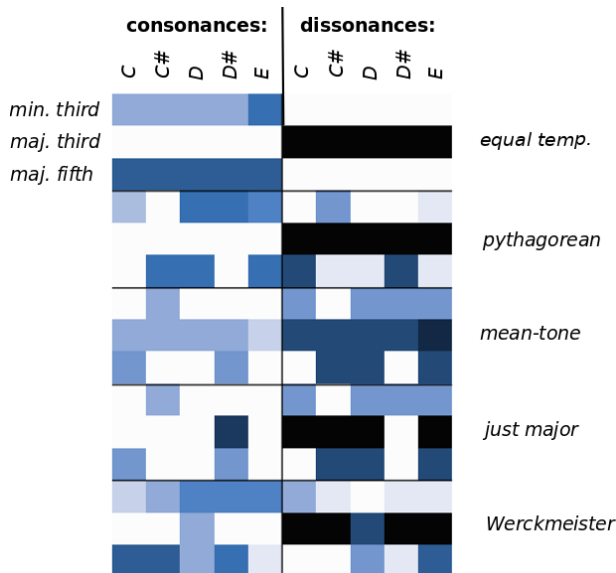


Figure 9: The consonance (left columns) and dissonances (right columns), with respective intensity levels coded from black to white, obtained for the minor and major triads considering the five temperaments.

8 Triads and Harmony

Harmony, the art of combining simultaneous sounds, is largely related to the consonance/dissonance properties characteristic of the involved intervals. The minor (major) *triads*, corresponding to the basis tone plus the minor (major) triad and fifth, provides one of the most important foundations of traditional harmony.

Let's also apply the developed simple model of consonance/dissonance to characterize the first minor and major triads obtained for the considered five temperament types. Figure 9 present the respective consonance (left part of the table) and dissonance (right portion) levels.

The intrinsic uniformity of the equal temperament approach is evident from these results, characterized by constant horizontal intensities. The other cases are characterized by the respective triads having varying consonance/dissonance patterns. This would be reflected in a different effects while transposing or modulating along a music piece played according to these temperaments.

9 Concluding Remarks

We have briefly addressed the interesting issues of consonance and dissonance in sound and music, with special focus on Helmholtz's consonance approach. A simple model was developed that allowed us to quantify the levels of consonance and dissonance between two tones according to comparisons of their respective partials.

The potential of this approach was then illustrated with respect to an analysis of temperaments and triads, and in-

teresting results were obtained that are often in agreement with what is commonly believed. In particular, the visualization of consonance relationships estimated for five types of temperaments resulted to be particularly useful for identifying specific features and comparing different scales.

It should be kept in mind that the obtained results are preliminary and dependent on the model parameter configurations. Substantial more efforts need to be developed in further assessing the simple proposed approach, and new results should be reported in the future.

We believe that, to any extent, this work motivated the potential of consonance/dissonance models with respect to sound and music aspects.

In addition to the study of temperaments and triads, it would be also possible to consider a large number of possible applications including: provide subsidy to design of scales and composition, musical instrument construction and characterization, acoustics, design and characterization of electronic amplifiers (especially regarding the unavoidable nonlinear effects) and filters, to name but a few possibilities.

It would be also promising to try to quantify the intrinsic sound consonance of a single note by considering interval relationships between its harmonics through an adaptation of the described modeling approach.

Acknowledgments.

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CDTs intend to be a halfway point between a formal scientific article and a dissemination text in the sense that they: (i) explain and illustrate concepts in a more informal, graphical and accessible way than the typical scientific article; and, at the same time, (ii) provide more in-depth mathematical developments than a more traditional dissemination work.

It is hoped that CDTs can also provide integration and new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

Though CDTs are intended primarily for those who have some preliminary experience in the covered concepts, they can also be useful as summary of main topics and concepts to be learnt by other readers interested in the respective CDT theme.

Each CDT focuses on a few interrelated concepts. Though attempting to be relatively self-contained, CDTs also aim at being relatively short. Links to related material are provided in order to complement the covered subjects.

The complete set of CDTs can be found at: <https://www.researchgate.net/project/Costas-Didactic-Texts-CDTs>.

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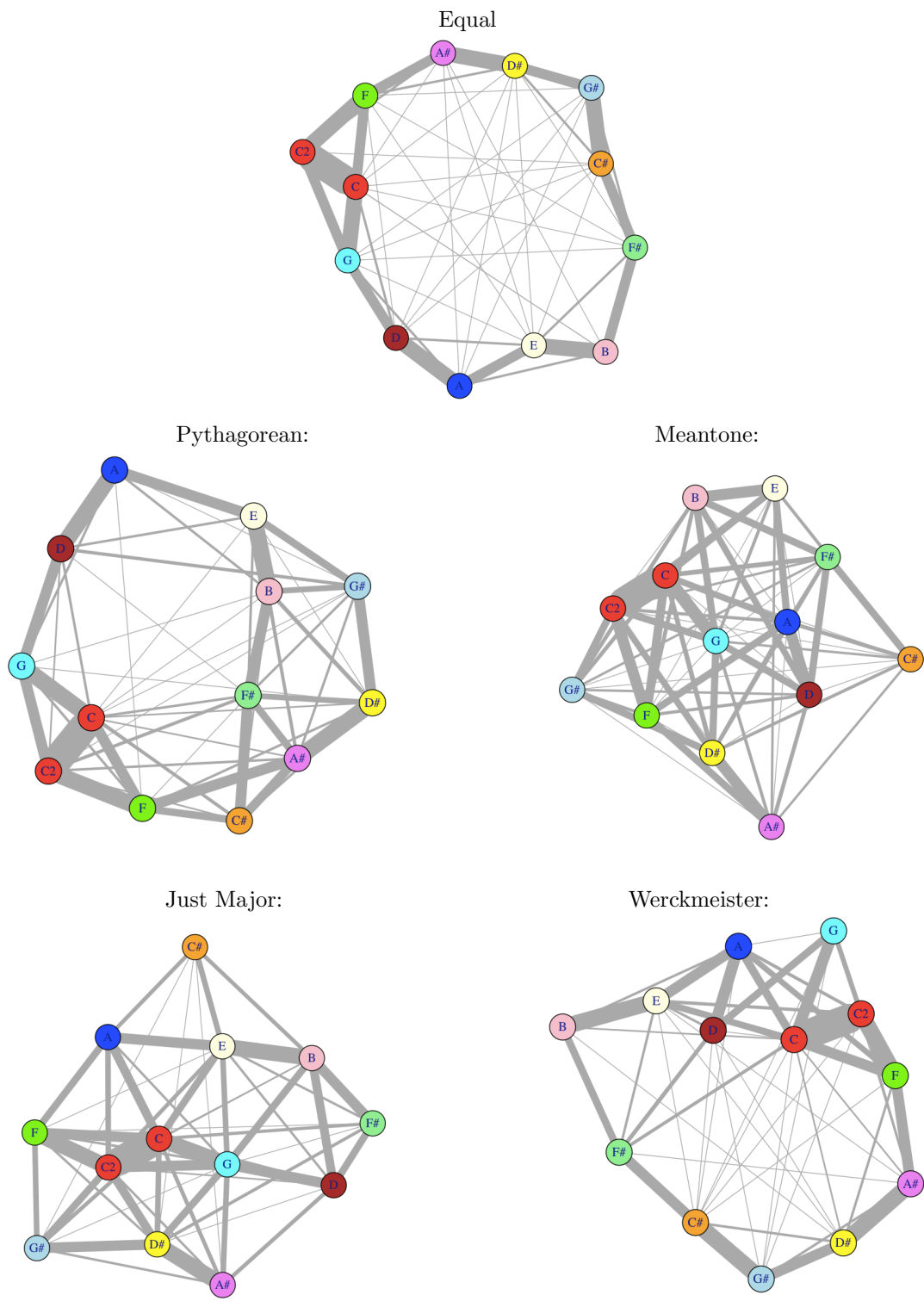


Figure 10: The consonance patterns obtained for the considered temperaments are shown in this figure, allowing further insights about the respective properties and relationships, as discussed in the main text. The intensities of the consonances are reflected in the edge widths.