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ON THE EXISTENCE AND UNIQUENESS OF INVARIANT MEASURE FOR MULTIDIMENSIONAL STOCHASTIC PROCESSES

C. BIANCA AND C. DOGBE

ABSTRACT. - This paper deals with the mathematical analysis of multidimensional processes solution of a class of stochastic differential equations. Specifically the analysis is addressed to the derivation of criteria for the existence and uniqueness of the invariant probability measure and its regularity properties in the case of stochastic processes whose infinitesimal generator is uniformly elliptic or degenerate. The criteria are based on the definition of Lyapunov functions and the Hörmander's rank bracket condition. Finally the criteria are employed for characterizing the invariant probability measure in some applications, including Kolmogorov-Fokker-Planck-type operators.

Keywords. Invariant measure; Ergodicity; Elliptic equations for measures; Lyapunov functions

MSC. 60H15; 82C31; 35R60; 35H10

1. INTRODUCTION

The existence and uniqueness of an invariant probability measure has recently gained much attention considering the linking with the Markov processes theory. This paper aims at exploring the invariant probability measure for a class of partial differential equations (PDE) by employing techniques recently proposed. Specifically, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space, the present paper is concerned with a multidimensional diffusion process $X(t) \doteq X_t$ which is solution of the following stochastic differential problem:

$$\begin{cases} dX_t = b(X_t) dt + \sigma(X_t) dW_t, \\ X_0 = u_0 \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where $(W_t)_{t \in [0, T]}$ is the standard d -dimensional Brownian motion, $d \in \mathbb{N}$, the drift coefficient $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow L(\mathbb{R}^m, \mathbb{R}^d)$, $m \in \mathbb{N}$, are (at least) Lipschitz continuous function. As well known, the problem (1.1) admits a unique solution $(X_t)_{t \geq 0}$, $t \geq 0$, $x \in \mathbb{R}^d$, see, among others, Friedman [8]. The solution of (1.1) admits the following stochastic representation:

$$S_t \varphi(x) := \mathbb{E}[\varphi(X(t, x))], \quad x \in \mathbb{R}^d, \quad \varphi \in \mathcal{C}_b(\mathbb{R}^d), \quad (1.2)$$

where S_t , $t \geq 0$, is the corresponding transition semigroup, $\mathcal{C}_b(\mathbb{R}^d)$ denotes the space of all functions from \mathbb{R}^d into \mathbb{R} that are uniformly continuous and bounded, \mathbb{E} denotes the conditional expectation. If $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is a regular function, then the following function:

$$u(t, x) := (S_t u_0)(x) := \mathbb{E}[u_0(X_t)] \quad (1.3)$$

is the unique solution of the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u = 0 & \text{in } [0, +\infty) \times \mathbb{R}^d \\ u(0, x) \equiv u_0 & \text{in } \mathbb{R}^d \end{cases} \quad (1.4)$$

where \mathcal{L} is the linear, second-order, uniformly elliptic operator associated with a diffusion process in the whole space.

Let ∇ and D^2 denote the gradient and the Hessian operators with respect to the spatial variable x , respectively. The infinitesimal generator of the process (1.1) reads:

$$\mathcal{L} = -a(x) : D^2 - b(x)\nabla, \quad (1.5)$$

where the matrix $a(x) = (a_{ij}(x))$ is defined as follows:

$$a_{ij}(x) = \frac{1}{2} \sum_{\ell=1}^k \sigma_{i\ell}(x)\sigma_{j\ell}(x), \quad (1.6)$$

and

$$a(x) : D^2 = \text{trace}[aD^2] = \sum_{i,j=1}^d a_{ij}\partial_{ij}.$$

It is worth pointing out that, under some assumptions on the coefficients, for any function $u \in W^{2,1}(\mathbb{R}^d)$, the operators $\mathcal{L}u$ and \mathcal{L}^*u are defined in the generalized function set. For a function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, we have (after integration by parts):

$$\langle a_{ij}\partial_{ij}u, \varphi \rangle = - \int_{\mathbb{R}^d} \partial_j u \partial_i (a_{ij}\varphi) dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner-product in \mathbb{R}^d with the Euclidean norm for a vector x : $|x| = \sqrt{\langle x, x \rangle} = \left(\sum_i x_i^2 \right)^{1/2}$. In particular $\partial_i(a_{ij}\varphi) = \varphi\partial_i a_{ij} + a_{ij}\partial_i\varphi \in L^2(\mathbb{R}^d)$ and similarly $\partial_i(b_i u)$ and $\partial_i(a_{ij}\partial_j u)$ are meaningful. Hence the formal adjoint is defined by

$$\mathcal{L}^*\psi := - \sum_{i,j} \partial_{ij}(a_{ij}\psi) + \sum_i \partial_i(b_i\psi) \quad (1.7)$$

where the summation convention is employed if confusion does not occur.

The connection between the diffusion process and the above defined problem can be proved directly by using the Itô calculus. Accordingly, $S(t) = e^{-t\mathcal{L}}$ denotes the C_0 -semigroup (strongly continuous one-parameter semigroup) whose infinitesimal generator is $-\mathcal{L}$.

The present paper deals with the existence and uniqueness of invariant probability measure for the above defined stochastic problem. Specifically the analysis is addressed to the *nondegenerate* case (the least eigenvalue of $\sigma\sigma^T$ is bounded away from zero on every compact subset of \mathbb{R}^d) and the *degenerate* case. More precisely, if we assume that $b_i(x) \in \mathcal{C}^\infty(\mathbb{R}^d)$, and the $d \times d$ matrix a is symmetric and positive semidefinite, i.e.

$$a_{ij}(x) = a_{ji}(x) \in \mathcal{C}^\infty(\mathbb{R}^d), \quad \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for any } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d,$$

then \mathcal{L} is a degenerate elliptic operator in \mathbb{R}^d , i.e. $(a_{ij}(x))$ is not strictly positive definite everywhere. An important example is the degenerate Ornstein-Uhlenbeck operator:

$$-\mathcal{L}u = \text{trace}(AD^2u) + \langle Bx, Du \rangle, \quad x \in \mathbb{R}^d. \quad (1.8)$$

A particular case of (1.6) is the uncontrolled diffusion, i.e. $\sigma = \sigma(x)$, which leads to quasi-linear equations of the degenerate elliptic equations.

As known, for a bounded open set Ω in \mathbb{R}^d , the controlled diffusion in (1.1), is said to be nondegenerate in Ω if \mathcal{L} is uniformly elliptic in Ω , namely if the following inequality holds:

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2, \quad (1.9)$$

for some constant $\theta > 0$, $\xi \in \mathbb{R}^d$ and $x \in \Omega$. The controlled diffusion in (1.1) is nondegenerate (in \mathbb{R}^d) if it is nondegenerate on every open ball B_R . Intuitively this assumption means that each component of the state vector x is directly influenced by the random increments dW_t .

The present paper is organized into six more sections which follow this introduction. Specifically Section 2 is devoted to review the definition of the mean and the strong ergodicity and the definition of the related invariant probability measure. The existence and uniqueness of the invariant probability measure is analyzed in Section 3 in the uniformly elliptic case. Specifically the equivalence among different assumptions is studied, the related Liouville problem is analyzed and by employing the definition of (weak and strong) Lyapunov functions the main results on the existence and uniqueness of the invariant probability measure for the Markov diffusion process solution of (1.1) follows. The regularity properties of the invariant measure are also established. The degenerate framework is analyzed in Section 4 where, in particular, the main results are obtained by employing the Hörmander's rank bracket condition. Section 5 is devoted to further investigations for the one-dimensional framework. Section 6 deals with various examples and applications where the existence and uniqueness of the invariant measure can be gained by employing the methods developed in the present paper. Finally, Section 7 concludes the paper with references to future investigations.

2. PRELIMINARIES

This section collects some basic definitions and results related to the ergodicity of a diffusion process X_t . The reader interested in a more deeper understanding is referred to the lecture of Lions [16].

The existence of invariant probability measure for a diffusion process is strictly related to the notion of ergodicity. A probabilistic approach for the existence and uniqueness of an invariant distribution and the related ergodicity can be found in the work of Meyn and Tweedie [17], where the interest focuses on systems driven by jump processes.

The definition of ergodicity is based on the following time-averaged Cesàro functional:

$$C_u(t, x) := \frac{1}{t} \int_0^t u(s, x) ds, \quad t > 0.$$

Definition 2.1. (Cesàro mean ergodicity). *Let Ω be a compact set of \mathbb{R}^d . The system (1.4) (or the associated semigroup of (1.5)) is said ergodic in the sense of Cesàro, if:*

- \mathcal{L} admits a unique invariant probability measure m , namely

$$\int_{\mathbb{R}^d} e^{-t\mathcal{L}} u_0 dm = \int_{\mathbb{R}^d} u_0 dm, \quad \forall t > 0, \quad \text{and} \quad \int_{\mathbb{R}^d} m(x) dx = 1. \quad (2.1)$$

- The solution $u(t, x)$ of Eq. (1.4) satisfies the following assumption:

$$C_u(t, x) \xrightarrow[t \rightarrow \infty]{} \mathbf{c} := \int_{\mathbb{R}^d} u_0 dm, \quad \text{uniformly in } x. \quad (2.2)$$

Definition 2.2. (Strong ergodicity). *The system (1.4) is said ergodic if \mathcal{L} admits a unique invariant probability measure m such that*

$$\mathcal{L}^* m = 0 \quad \text{in } \mathbb{R}^d, \quad m \geq 0, \quad \int_{\mathbb{R}^d} dm = 1, \quad (2.3)$$

and the solution $u(t, x)$ of the Eq. (1.4) satisfies the following assumption:

$$\lim_{t \rightarrow +\infty} u(t, x) = \mathbf{c} := \int_{\mathbb{R}^d} u_0 dm, \quad (2.4)$$

uniformly in x . The term \mathbf{c} is the so-called ergodic constant.

It is worth stressing that a probability measure $m \in \mathbb{R}^d$ is an invariant measure of the diffusion process X_t if for each $u_0 \in L^\infty(\mathbb{R}^d)$, one has (see [5]):

$$\int_{\mathbb{R}^d} u(t, x) m(x) dx = \int_{\mathbb{R}^d} u_0(x) m(x) dx, \quad (2.5)$$

where $u(t, x) = \mathbb{E}_x[u_0(X_t)]$ is the solution of the parabolic Cauchy problem (1.4). In differential form (infinitesimal invariance) one has:

$$\int_{\mathbb{R}^d} \mathcal{L}f(x) m(x) dx = 0 \quad \forall f \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad (2.6)$$

where $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the set of all infinitely differentiable functions with compact support. It is worth pointing out that the above definition of invariant measure is fairly general, because no regularity for \mathcal{L} is required. However, if \mathcal{L} is regular enough, the identity (2.1) can be equivalently rewritten in a different way. Indeed by differentiation one has

$$\int_{\mathbb{R}^d} \mathcal{L}u_0 dm = 0,$$

which can be rewritten as $\langle \mathcal{L}u_0, m \rangle = 0$; going back to (1.7) and by the arbitrariness of u_0 , this becomes

$$\mathcal{L}^* u = 0. \quad (2.7)$$

Conversely, assume that (2.7) is fulfilled, then we observe that

$$\frac{d}{dt} \int_{\mathbb{R}^d} S(t)u_0 dm = - \int_{\mathbb{R}^d} \mathcal{L}u_0 dm = - \int_{\mathbb{R}^d} u_0 d\mathcal{L}^* m = 0,$$

whence we deduce that $t \mapsto \int_{\mathbb{R}^d} S(t)u_0 dm$ is constant and (2.1) follows.

It is worth mentioning that Cesàro mean ergodicity implies that for all $u_0 \in \mathcal{C}(\mathbb{R}^d)$

$$\frac{1}{T} \int_0^T u_0(X_t) dt \xrightarrow[\varepsilon \rightarrow 0]{\text{a.s.}} \int_{\mathbb{R}^d} u_0 dm \quad \text{if } t \geq +\infty. \quad (2.8)$$

This paper aims at answering to the questions: How a diffusion process is related to invariant measures? There exists and it is unique a probability measure m on \mathbb{R}^d such that

$$\mathcal{L}^* m = 0? \quad (2.9)$$

The answer is negative in general. Indeed if $b = 0$, then any solution of (2.9) with the matrix $A = I$ has harmonic density, hence cannot be a probability measure. Moreover if one takes any smooth probability density φ on \mathbb{R} such that $\varphi(0) = 0$ and $\varphi > 0$ outside 0, then the measure φdx is not unique solution of (2.9) with $A = 1$ and $b(x) = \varphi'(x)/\varphi(x)$. Indeed, let

$\psi(x) = \frac{\varphi(x)}{2}$ if $x \leq 0$ and $\psi(x) = c\varphi(x)$ if $x > 0$, where c is such that $\psi(x)$ is a probability density. Then $\psi'/\psi = \varphi'/\varphi$ and ψdx satisfies the same equation. A typical example is the function $\varphi(x) = cx^2e^{-x^2}$. It appears that the uniqueness is lost because of the singularity of b at the origin. However in general for $d \geq 2$ the smoothness of b does not guarantee the uniqueness.

Another example is the heat equation for $x \in \mathbb{R}$:

$$\frac{\partial u}{\partial t} - \frac{1}{2}a(x)\frac{\partial^2 u}{\partial x^2} = 0, \quad a > 0.$$

If $a = 1$ (weak growth) we have not ergodicity because u does not tend towards a constant which is the mean of the initial condition. Typically if the initial condition is constant at infinity, it is the limit of the constant of the value at infinity, thus it is not a measure.

It is worth noting that, in the case of uniformly bounded coefficients (or more generally that \mathcal{L} is uniformly strongly hypoelliptic in the sense of Bony) and classical solutions, the probability measure for the parabolic equation (1.4) has been investigated in the important paper by Haminskii [15], where the main objective was to study the behavior of solutions and their stabilization as $t \rightarrow \infty$.

An interesting relationship between uniqueness and ergodicity of an invariant probability measure is contained in the following proposition [16].

Proposition 2.1. *The following statements are equivalent.*

- (i) *The process X_t solution of (1.1) is ergodic.*
- (ii) *The diffusion (1.1) has an invariant probability measure.*
- (iii) *There exists a unique invariant probability measure m , if and only if, for every $u_0 \in L^\infty(\mathbb{R}^d)$*

$$C_u(t, x) \xrightarrow[t \rightarrow \infty]{} \mathbf{c} := \int_{\mathbb{R}^d} u_0 dm, \quad \text{uniformly in } x. \quad (2.10)$$

- (iv) *For any $u_0 \in L^\infty(\mathbb{R}^d)$, $u(x, t) \xrightarrow[t \rightarrow +\infty]{} \mathbf{c} := \int_{\mathbb{R}^d} u_0 m dx$, uniformly in x .*

3. THE UNIFORMLY ELLIPTIC CASE

This section is devoted to the existence and uniqueness analysis of invariant measure for the problem (1.4) under the assumption that the operator \mathcal{L} is uniformly elliptic (see (1.9)).

It is well known that the solution of equation (1.1) may explode in a finite time T . However, according to [19], if the mean return time $\tau(t, x)$ is bounded, the existence and uniqueness of a global solution, i.e. defined on $[0, T]$, is ensured. Roughly speaking, let $B_0 = \overline{B}(x_0, r_0)$ be the closed ball of center x_0 and radius r_0 . It is expected that the underlying process would return to the ball in finite times. Since, these are random times, it means that their expectation is finite. Let τ_i be the return time to the state i and τ_0 be the first time at which the path of the process X_t reaches the boundary ∂B_0 . In the probability language, non-exploding solution means that if we are in the whole space \mathbb{R}^d (i.e. a ball in \mathbb{R}^d), the ergodicity is related to the existence of a point such that the first time of reaching the ball satisfies

$$\mathbb{E}_x(\tau_0) < \infty, \quad \forall x. \quad (3.1)$$

If the process leaves from very far, it will take an increasingly long time to reach x . Hence $\mathbb{E}_x(\tau_0)$ must tend towards infinity when x tends to infinity. Let x_0 be a point of the ball B_0 .

We look at the time to reach x_0 .
Let w be the following function:

$$\begin{cases} w(x) := \mathbb{E}_x(\tau) < \infty \\ w \rightarrow \infty & |x| \rightarrow \infty. \end{cases} \quad (3.2)$$

From the PDE point of view, Eq. (3.2) means that there exists a function \tilde{w} which solves the following boundary value problem

$$\begin{cases} \mathcal{L}\tilde{w} = 1, & \text{on } B_0^c & (3.3a) \\ \tilde{w} = 0 & \text{in } \partial B_0 & (3.3b) \\ \tilde{w} \geq 0, \quad w \rightarrow \infty & |x| \rightarrow \infty. & (3.3c) \end{cases}$$

It is worth noticing that Eq. (3.3b) is the Dirichlet condition, (3.3c) is the property of the solution, while one is looking for solutions of the equation (3.3a).

A different way to write (3.3a)-(3.3c) without the boundaries conditions is the following Lyapunov assumption.

ASSUMPTION 3.1.

There exist functions f and $w \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\begin{cases} \mathcal{L}w + \chi w = 1, & x \in \mathbb{R}^d \\ w \in \mathcal{C}^0(\mathbb{R}^d) \cap \mathcal{C}^\infty(B_0^c), & w \geq 0 \\ \mathcal{C}^\infty \ni \chi > 0 \text{ with compact support, i.e.} \\ \chi \not\equiv 0 \text{ strictly positive somewhere} \end{cases} \quad (3.4)$$

and there exists a ball B_0 on which $\chi > 0$.

Roughly speaking, the ball is replaced by the choice of the function χ and the existence of such a function means the existence of a solution. Indeed χ is a particular function defined as follows:

$$\chi = \begin{cases} +\infty & \text{in } B_0 \\ 0 & \text{in } B_0^c. \end{cases}$$

Remark 3.2. The assumption “ $\chi \not\equiv 0$ strictly positive somewhere” in Eq. (3.4) can be rewritten as follows:

$$\exists x_0, \exists r_0 > 0 \quad \text{such that} \quad \chi \geq \alpha \mathbf{1}_{B_0}, \quad \alpha > 0$$

where $\mathbf{1}_{B_0}$ is the indicator function of the unit ball B . We claim that

Proposition 3.3. *The ASSUMPTION 3.1 and Eqs. (3.3a)-(3.3c) are equivalent.*

Proof of Proposition 3.3. First we prove that the ASSUMPTION 3.1 implies (3.3a)-(3.3c). The claim follows by a purely probabilistic argument which states that the solution of (3.4) is given by the following Feynman-Kac formula:

$$w(x) = \mathbb{E} \left[\int_0^\infty \exp \left(- \int_0^t \chi(x_s) ds \right) dt \right]. \quad (3.5)$$

It is worth mentioning that in the integral (3.5), when we take the first reaching time of the support of the ball B_1 (where $B_1 \supset B_0$)

$$w(x) \geq \mathbb{E} \left[\int_0^{\tau_1} \exp \left(- \int_0^t \chi(x_s) ds \right) dt \right] \quad (3.6)$$

and if we know that the support of $\text{supp}(\chi) \subset B_1$, thus for all the times up to τ_1 which is 0, we have:

$$w(x) \geq \mathbb{E} \left[\int_0^\tau \exp \left(- \int_0^t \chi(x_s) ds \right) dt \right] = \mathbb{E}[\tau_1].$$

This shows that the ball B_0 can be chosen arbitrarily and establishes the claim.

We prove now that the problem (3.3a)-(3.3c) implies (3.4). The proof is based on the fact that \tilde{w} is extendable to a continuous function. Indeed, the regularity (i.e. smoothness) of \tilde{w} in the uniformly elliptic framework allows to extend it to the whole \mathbb{R}^d , to be reduced to the following equation with some function f :

$$\begin{cases} \mathcal{L}\tilde{w} = f & \text{in } B_0^c \\ f \equiv 1, \quad \tilde{w} \geq 0 & \text{in } B_0^c \\ \tilde{w} \geq -c_0, \quad f \geq -c_0 & \end{cases}$$

with a constant c_0 . Then, the lemma is proved once we will construct a super-solution of the equation (3.4). In order to establish this result, adding a constant c to \tilde{w} and looking at the quantity

$$\mathcal{Q} := \mathcal{L}(\tilde{w} + c) + \chi(\tilde{w} + c),$$

to obtain

$$\mathcal{L}(\tilde{w} + c) + \chi(\tilde{w} + c) \geq f + \chi(c - c_0).$$

Since $\mathcal{L}\tilde{w} = f$ we have $\tilde{w} \geq -c_0$.

On the one hand, on B_0^c (outside B_0) we have $f = 1$; therefore we have the estimate on the right-hand side term:

$$f + \chi(c - c_0) = 1 + \chi(c - c_0) \geq 1 \quad \text{for } c \geq c_0.$$

On the other hand, on B_0 , one has $\chi \geq \alpha \cdot \mathbf{1}_{B_0}$, i.e. is greater than α . Since $f \geq -c_0$ on B_0 and $\chi \geq \alpha$, we have

$$f + \chi(c - c_0) \geq -c_0 + \alpha(c - c_0).$$

Moreover, for a large c we get

$$f + \chi(c - c_0) \geq -c_0 + \alpha(c - c_0) \geq 1, \quad \text{if } c \text{ is large,}$$

and thus we obtain a super-solution for Eq. (3.4). We conclude by observing that by taking \tilde{w} and adding a large constant, we obtain a super-solution. Thus, we deduce that

$$\tilde{w} + c \geq w.$$

The converse is thus proved. ■

The ASSUMPTION 3.1 can be stated more precisely as follows.

ASSUMPTION 3.4.

There exist functions f and $w \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\begin{cases} \mathcal{L}w + \chi w = f \geq 1, & \text{in } \mathbb{R}^d \\ w \geq 0, \\ w, f \rightarrow \infty, & \text{if } |x| \rightarrow \infty \end{cases} \quad (3.7)$$

for some $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$.

It is worth noting that the assumption that f tends to infinity allows us to have tightly compactness.

A key result is the following lemma, which makes precise the equivalence between Eqs. (3.3a)-(3.3c) and ASSUMPTION 3.4. This characterization was first introduced by Lions [16].

Lemma 3.5. *The following statements are equivalent.*

(i) *There exist a function f and $\tilde{w} \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that*

$$\begin{cases} \mathcal{L}\tilde{w} + \chi\tilde{w} = f \geq 1, & \text{in } \mathbb{R}^d \\ \tilde{w} \geq 0, \\ \tilde{w}, f \rightarrow \infty, & \text{if } |x| \rightarrow \infty \end{cases} \quad (3.8)$$

for some $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$.

(ii) *There exist a function f and $\tilde{w} \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that*

$$\begin{cases} \mathcal{L}\tilde{w} = f \geq 1 & \text{in } B_0^c \\ \tilde{w}|_{\partial B_0} = 0, \\ \tilde{w}, f \rightarrow \infty, & \text{if } |x| \rightarrow \infty. \end{cases} \quad (3.9)$$

Proof of Lemma 3.5. The proof is divided into two steps. Firstly we justify the limit for w tending to zero, secondly we prove the equivalence between (3.8) and (3.9).

• **First Step.** Our first interest is to look for the justification of $w \rightarrow \infty$. In order to establish this claim, we turn to the problem

$$\begin{cases} \mathcal{L}\tilde{w} = 1, & |x| \geq 1 \\ \tilde{w}|_{|x|=1} = 0, & \tilde{w} \geq 0 \end{cases} \quad (3.10)$$

which implies that there exists $\delta > 0$ such that

$$\tilde{w} \geq \delta \text{Log}|x|. \quad (3.11)$$

Since the ball plays any role, we can take in (3.10) $f = 1$ and in order to simplify the computations, we take $|x| \geq 1$, which is a ball centered at the origin with radius 1. To prove the claim (3.11), we will prove that $\delta \text{Log} x$ is a sub-solution. Indeed, we have

$$\partial_i \text{Log}|x| = \frac{x_i}{|x|^2},$$

and

$$\partial_{ij} \text{Log}|x| = \frac{\delta_{ij}}{|x|^2} - 2 \frac{x_i x_j}{|x|^4} \quad |x| \geq 1.$$

Since the second derivatives are bounded by $\frac{1}{|x|^2}$, the terms a_{ij} are bounded by x^2 , and b_i are bounded by $c_0|x|$, the following inequality holds true:

$$-a_{ij}\partial_{ij}\text{Log}|x| - b_i\partial_i\text{Log}|x| \leq \frac{c_0|x|^2}{|x|^2} + \frac{c_0|x|}{|x|} \leq 2c_0.$$

Multiplying by a small δ and choosing $\delta = \frac{1}{2c_0}$ we obtain a sub-solution with $\delta\text{Log}|x|$.

• **Second Step.** Proof of the equivalence between (3.8) and (3.9).

(ii) (\Rightarrow) (i). This proof uses an argument borrowed from [16]. It is enough to show that it remains bounded at a point, by arguing at the level of the truncations and by means of the Theorem of Harnack. The argument in the proof does not apply anymore if we can not apply Harnack inequality (see [9], Theorem 8.19). It is sufficient to consider the case $f = 1$, solution of (3.3a) to establish the equivalence between assumptions and to find a solution of (3.8), which tends to infinity.

Let $U_{R_0} = B(0, R_0)$ denote the ball in \mathbb{R}^d with center 0 and (large) radius R_0 . Assume $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ with $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ satisfying $\chi \geq 0$, and $\chi = 1$ in \bar{U}_{R_0} and has compact support, lying within \bar{U}_{R_0} and set $w^\natural(x) := w(x) + \max_{\bar{U}_{R_0}} |\mathcal{L}w| + \max_{\bar{U}_{R_0}} |w| + \max_{\bar{U}_{R_0}} |\mathcal{L}w| + 1$.

First, for $|x| \leq R_0$, observe that

$$\mathcal{L}w^\natural + \chi w^\natural \geq -\max_{\bar{U}_{R_0}} |\mathcal{L}w| + \chi(w + \max_{\bar{U}_{R_0}} |w| + \max_{\bar{U}_{R_0}} |\mathcal{L}w| + 1)$$

and then

$$\mathcal{L}w^\natural + \chi w^\natural \geq 1.$$

Second, for $|x| \geq R_0$, we have

$$\mathcal{L}w^\natural + \chi w^\natural \geq \mathcal{L}w \geq 1.$$

By collecting the above estimates, we deduce that the function w^\natural satisfies the following equation:

$$\mathcal{L}w^\natural + \chi w^\natural =: f^*(x) \geq 1 \quad \text{in } \mathbb{R}^d, \quad \lim_{|x| \rightarrow \infty} w^\natural = +\infty. \quad (3.12)$$

Let us now consider a regular partition of unity $\{\phi_i\}_{i \geq 1}$ such that $\phi_i \geq 0$, $\sum_{i=1}^{\infty} \phi_i(x) = 1$,

$\text{supp } \phi_i \subset \bar{U}_{i+1} \setminus U_{i-1}$. Denoting by $W_n(x)$ the solution corresponding to one of the partition ϕ_i satisfying

$$\mathcal{L}W_n + \chi W_n = \sum_{i=1}^n \phi_i \quad \text{in } \mathbb{R}^d, \quad 0 \leq W_n \leq w^\natural. \quad (3.13)$$

Since $W_n(x)$ is a super-solution of (3.8) and is positive, there exists a regular solution w_n to the equation (3.13). Indeed, to prove this existence, following Bensoussan [5], for $n > 0$ sufficiently large, we introduce for each $\varepsilon > 0$ and $m \geq n + 1$, the approximating PDE of the solution W_{nm}^ε of the Dirichlet problem (in a bounded region):

$$\begin{cases} (\mathcal{L} - \varepsilon\Delta)W_n^\varepsilon + \chi W_n^\varepsilon = \sum_{i=1}^n \phi_i & \text{in } U_m \\ W_n^\varepsilon = 0 & \text{on } \partial U_m \end{cases} \quad (3.14)$$

where for convenience we will omit the subscript the subscript m . The boundary value problem (3.14) has a unique solution according to the classical Perron method (see [9],

Sections 2.8, 6.3 or [12]). By standard arguments of the hypoelliptic theory, in the light of the so-called vanishing viscosity method, as $\varepsilon \rightarrow 0^+$, $W_n^\varepsilon(x)$ converges to $W_n(x)$ in the ball U_m , where W_n is the solution of the following problem:

$$\begin{cases} (\mathcal{L} - \varepsilon\Delta)W_n + \chi W_n = \sum_{i=1}^n \phi_i & \text{in } U_m \\ W_n = 0 & \text{on } \partial U_m \end{cases} \quad (3.15)$$

and the boundary condition is attained only in the viscosity sense. Also, Hörmander condition guarantees the comparison principle for (3.15). Clearly, by (3.13) the functions $w_i(x) := W_i(x) - W_{i-1}(x)$ solve

$$\mathcal{L}w_i + \chi w_i = \phi_i \quad \text{in } \mathbb{R}^d, \quad (3.16)$$

and $w(x) = \sum_{i=1}^{\infty} w_i(x) < \infty$ is a sum of a convergent series in \mathbb{R}^d . At this point, the key tools in our proof will be provided by the following claim related to the use of Harnack inequalities.

Claim. Let a_n be a sequence of real numbers such that $\sum_{i=1}^{\infty} u_i < +\infty$ with $u_i \geq 0$. Then, there exists a sequence $\{a_i\}_i$ such that $\lim_{i \rightarrow +\infty} a_i = +\infty$ and $\sum_{i=1}^{\infty} a_i u_i < +\infty$.

Proof of the claim. We construct a strictly increasing function of $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and a sequence

$$\forall n \in \mathbb{N} \quad \sum_{i=\varphi(n)}^{\varphi(n+1)-1} u_i \leq \frac{1}{(n+1)2^{n+1}}. \quad (3.17)$$

Once constructed the function $\varphi(i)$ we choose $a_i = n+1$ for $\varphi(n) \leq i \leq \varphi(n+1) - 1$. One then has

$$0 \leq \sum_{i=\varphi(n)}^{\varphi(n+1)-1} a_i u_i \leq \sum_{i=0}^n \frac{1}{2^{i+1}} \leq 1. \quad (3.18)$$

Since the increasing sequence $p \mapsto \sum_{i=\varphi(n)}^p a_i u_i$ is bounded from time to time by 1, it is always bounded, and therefore converges. \square

Next, fix x_0 and observe that the sequences $w_n(x_0)$ converge. Then, there exists a monotonic increasing sequence $\{a_i\}_i$ such that $\lim_{i \rightarrow +\infty} a_i = +\infty$, with $a_0 \geq 1$ and $w_n^b(x) := \sum_{i=1}^{\infty} a_i w_i(x) < +\infty$. By means of Harnack's theorem, we see that $\sum_{i=1}^{\infty} a_i w_i(x_0) < +\infty$. Thereby the equation (3.20) reduces to

$$\mathcal{L}w + \chi w = \sum_{i=0}^{\infty} a_i \phi_i.$$

Let $n_0 \in \mathbb{N}$ be fixed. Let us denote by $w_n^b(x) := \sum_{i=1}^{\infty} a_i w_i(0)$. In the ball U_{n_0} , the function $w_n^b(x)$ satisfies:

$$\mathcal{L}w_n^b + \chi w_n^b = \sum_{i=1}^{n_0+1} a_i \phi_i \quad w_n^b \geq 0.$$

Now the remainder of the proof simply exploits the Harnack inequality. Accordingly, there exists a constant $C(n_0)$, independent of n , such that

$$\begin{aligned} \sup_{U_{\frac{n_0}{2}}} w_n^b &\leq C(n_0) \left(\inf_{U_{\frac{n_0}{2}}} w_n^b + \sup_{U_{\frac{n_0}{2}}} \sum_{i=1}^{n_0+1} a_i \phi_i \right) \\ &\leq C(n_0) \left(\max_{\bar{U}_{R_0}} |w| + \sup_{U_{\frac{n_0}{2}}} \sum_{i=1}^{n_0+1} a_i \phi_i \right) = C^*(n_0). \end{aligned} \quad (3.19)$$

We now use the inequality (3.19) to argue that in any bounded set the function w^b is well defined, i.e. $w^b(x) := \sum_{i=1}^{\infty} a_i w_i(x) < \infty$ for every $x \in \mathbb{R}^d$. We have a function of variable x which tends to infinity, since for $x \in \llbracket n, n+1 \rrbracket$, we have $\lim_{i \rightarrow +\infty} a_i = +\infty$. We thereby see that the function $w^b \geq 0$ satisfies

$$\mathcal{L}w^b + \chi w^b = \sum_{i=1}^{\infty} a_i \phi_i := \phi. \quad (3.20)$$

By combining (3.12) and (3.20), we deduce that the function $w = w^h + w^b$ satisfies (ii) and complete the first part of proof.

(i) \Rightarrow (ii). The converse is trivial, since $f \geq 1$, the solutions w and \tilde{w} of the problems (3.8) are more large than the solution \tilde{w} in the equations (3.3a)-(3.3c). Thanks to the maximum principle they are supersolutions. \blacksquare

3.1. Liouville-type problem.

We consider now the existence of non-constant, bounded solutions of the following problem:

$$\mathcal{L}w = 0 \quad \text{on } \mathbb{R}^d, \quad w \in L^\infty(\mathbb{R}^d). \quad (3.21)$$

Any such solution belongs to $W_{loc}^{2,p}(\mathbb{R}^d)$ for all $1 < p < \infty$, $\mathcal{C}^{2,\alpha}(\mathbb{R}^d)$, for all $0 < \alpha < 1$, if b is locally Lipschitz, and is smooth if σ and b are smooth. Roughly speaking, we will prove that X_t is ergodic if 0 is a simple eigenvalue of \mathcal{L} , or equivalently if the equation (3.21) has only one solution.

Since our setting is the whole space \mathbb{R}^d , we will use a property that replaces the standard strong maximum principle of the periodic case and is the key ingredient for extending some results of [2] to the nonperiodic setting. The classical theorem states that if for a bounded \mathcal{C}^2 -function w , one has (3.21), then w is constant on \mathbb{R}^d . We recall that, since we are interested in the whole space, an assumption relative to the existence of a Lyapunov function is needed. According to [14] (see Theorem 3.2. p. 450), w is a Lyapunov function if and only if we have the non-explosion of the solution of the SDE (1.1). In order to establish that any solution $w \in L^\infty(\mathbb{R}^d)$ of (3.21) is constant, we state all the additional assumptions that we make about the Lyapunov function. This second group of assumptions deals with Lyapunov

functions which allows to obtain the uniqueness of the invariant probability measure. The definition of Lyapunov conditions to prove ergodicity for uniformly elliptic diffusions is a well-developed subject (see [11, 18]).

We assume that one of the following assumptions are fulfilled.

ASSUMPTION 3.6.

There exists a function $\bar{w} \in \mathcal{C}(\mathbb{R}^d)$, called (strong) Lyapunov function, such that:

$$\bar{w}(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty, \quad \lim_{|x| \rightarrow +\infty} \mathcal{L}\bar{w} = +\infty. \quad (3.22)$$

ASSUMPTION 3.7.

There exists a function $\bar{w} \in \mathcal{C}(\mathbb{R}^d)$, called (weak) Lyapunov function, such that:

$$\bar{w}(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty, \quad \mathcal{L}\bar{w} \geq 1 \quad \text{for } |x| \text{ large.} \quad (3.23)$$

ASSUMPTION 3.8.

There exists a function $\bar{w} \in \mathcal{C}(\mathbb{R}^d)$, called (weak) Lyapunov function, such that for some $R_0 > 0$:

$$\mathcal{L}\bar{w} \geq 0 \quad \text{for } |x| > R_0, \quad \bar{w}(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty. \quad (3.24)$$

Before stating a Liouville-type theorem, we recall the notion of viscosity solutions, which is used in the statement of the next proposition.

The notion of *viscosity solutions*, introduced in the early 1980's [7] and related to Kruzkov's theory of entropy solutions for scalar conservation laws, was used in earlier studies as a natural tool for obtaining solution to PDE. Let us recall the following definition.

Definition 3.9. (Viscosity solutions). *Let $\Omega \subset \mathbb{R}^d$ be an open set. The upper semicontinuous function $u : \Omega \rightarrow \mathbb{R}$ is a viscosity subsolution of $Lu = 0$ in Ω if whenever $\phi \in \mathcal{C}^2(\Omega)$ and $x_0 \in \Omega$ such that $(u - \phi)(x) \leq (u - \phi)(x_0)$ for all x in a neighborhood of x_0 , then we must have $L\phi(x_0) \geq 0$. A function $u : \Omega \rightarrow \mathbb{R}$ is a viscosity supersolution if $-u$ is a viscosity subsolution. A real function u is a viscosity solution of $Lu = 0$ if it is both a viscosity subsolution and supersolution.*

It is worth noting that in order to prove that u is a viscosity subsolution (supersolution), it is sufficient to use test functions $\phi \in \mathcal{C}^2(\Omega)$ such that $u - \phi$ has a strict local max (min) at x_0 . Indeed if, for example, $u - \phi$ has a maximum at x_0 and we set $\bar{\varphi}(x) = \phi(x) + \varepsilon|x - x_0|^2$, then $u - \phi - \bar{\varphi}$ has a strict maximum at x_0 . Since $L\bar{\varphi}(x_0) = L\phi(x_0) + 2\varepsilon \text{trace}(a_{ij}(x_0))$, letting $\varepsilon \rightarrow 0$ we get $L\phi(x_0) \geq 0$.

Finally, the *Strong Maximum Principle* will be employed. It asserts that any viscosity sub- or supersolution in \mathbb{R}^d that attains an interior nonnegative maximum must be constant. In addition, the generator \mathcal{L} has the Liouville property (based on the Strong Maximum Principle), i.e. any bounded sub- or supersolution of $\mathcal{L}u = 0$ is constant. Moreover we recall (see [9], Theorem 8.19) that for every non-negative solution $u \in W^{2,1}(\Omega)$ of the equation $\mathcal{L}u = 0$ (or the equation $\mathcal{L}^*u = 0$) Harnack's inequality states that for every ball B , with $\bar{B} \subset \Omega$, there exists $C(B) > 0$ such that

$$\sup_B u \leq C(B) \inf_B u.$$

In what follows the interest focuses on the problem of viscosity subsolutions and supersolutions construction for equation (3.21) that satisfy Lyapunov conditions.

Proposition 3.10. *Let $w \in L^\infty(\mathbb{R}^d)$ be solution of the following problem:*

$$\mathcal{L}w = 0 \quad \text{in } \mathbb{R}^d. \quad (3.25)$$

Assume that the ASSUMPTION 3.8 holds true. Then:

- (i) *Every bounded viscosity subsolution of (3.25) is constant.*
- (ii) *Every bounded viscosity supersolution of (3.25) is constant.*

Proof of the Proposition 3.10. The steps of the proof follow by adaptation of the method used in [20] (see also [4]).

Let $w \geq 0$ be a bounded subsolution of (3.25) and $\bar{w} \in \mathcal{C}^2(\mathbb{R}^d)$ defined in (3.24). Following [20], for any fixed $\varepsilon > 0$, define

$$w_\varepsilon(x) := w(x) - \varepsilon\bar{w}(x).$$

We fix $R > R_0$, and we claim that $w_\varepsilon(x)$ is a viscosity subsolution to (3.25) in $|x| > R$ for every $\varepsilon > 0$. Indeed consider $\bar{x} \in \mathbb{R}^d$ with $|\bar{x}| > R$ and a smooth function ψ such that $w_\varepsilon(\bar{x}) = \psi(\bar{x})$ and $w_\varepsilon - \psi$ has a strict maximum at \bar{x} . Assume by contradiction that $\mathcal{L}\psi(\bar{x}) > 0$. By the regularity of ψ and of \mathcal{L} , there exists $0 < k < R - R_0$ such that $\mathcal{L}\psi(x) > 0$ for every $|x - \bar{x}| \leq k$. Now we prove that $\varepsilon\bar{w} + \psi$ is a supersolution of (3.25) in $B(\bar{x}, k)$ and ζ such that $\varepsilon\bar{w} + \psi - \zeta$ has a minimum at \tilde{x} .

Since w is a supersolution of (3.25) in $|x| > R_0$, the linearity of the differential operator \mathcal{L} yields the following bounds:

$$0 \leq \mathcal{L} \left(\frac{1}{\varepsilon}(\zeta - \psi)(\tilde{y}) \right) = \frac{1}{\varepsilon}\mathcal{L}\zeta(\tilde{y}) + \frac{1}{\varepsilon}\mathcal{L}\psi(\tilde{y}) < \mathcal{L}\zeta(\tilde{y}),$$

where in the last inequality we used the fact that ψ is a supersolution in $B(\bar{x}, k)$. By our assumption $w - (\varepsilon\bar{w} + \psi)$ has a strict maximum at \bar{x} and $w(\bar{x}) = (\varepsilon\bar{w} + \psi)(\bar{x})$. Then, there exists $\alpha > 0$ such that

$$w(x) - (\varepsilon\bar{w} + \psi)(x) < -\alpha \quad \text{on } \partial B(\bar{x}, k).$$

A standard comparison principle gives that

$$w(x) \leq \varepsilon\bar{w} + \psi(x) - \alpha \quad \text{on } B(\bar{x}, k).$$

This contradicts our assumptions. This proves the claim: w_ε is a viscosity subsolution to (3.25) in $|x| > R$ for every $\varepsilon > 0$.

Now, observing that $w_\varepsilon(x) \rightarrow -\infty$ as $|x| \rightarrow +\infty$, for every ε we fix $M_\varepsilon > R$ such that $w_\varepsilon(x) \leq \sup_{|z|=R} w_\varepsilon(z)$ for every x such that $|x| \geq M_\varepsilon$. The maximum principle applied in $\{x, R \leq |x| \leq M_\varepsilon\}$, yields the following bound:

$$w_\varepsilon(x) \leq \sup_{|z|=R} w_\varepsilon(z) \quad \forall |x| \geq R, \quad \forall \varepsilon > 0. \quad (3.26)$$

Next we let $\varepsilon \rightarrow 0$ in (3.26) and obtain $w(x) \leq \sup_{|z|=R} w(z)$ for every x such that $|x| > R$.

Therefore w attains its global maximum at some interior point, then it is a constant (strong maximum principle).

The proof of (ii) for bounded supersolutions u is analogous, with minor changes. It is sufficient to define $u_\varepsilon(x)$ as $u(x) + \varepsilon\bar{w}(x)$ and to prove that $u_\varepsilon \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and

that it is a viscosity supersolution to (3.25) in $|x| > R$. Then the same argument holds by exchanging the role of super- and subsolutions and using the strong minimum principle. ■

ASSUMPTION 3.8 or ASSUMPTION 3.4 are sufficiently for the existence of an invariant measure. The following result on the existence of invariant measure m holds.

Theorem 3.11. *Assume that the ASSUMPTION 3.6 holds. Then the Markov diffusion process X_t solution of (1.1) admits a unique invariant probability measure m . Moreover $m \in W^{1,p}(\mathbb{R}^d)$ ($1 < p < \infty$) (thus $m \in \mathcal{C}^\alpha(\mathbb{R}^d)$ ($0 < \alpha < 1$)), m is smooth if σ and b are smooth, $m > 0$ on \mathbb{R}^d and m satisfies the following property:*

$$\int_{\mathbb{R}^d} m(\mathcal{L}\bar{w})_+ dx < \infty. \quad (3.27)$$

Furthermore, for any $u_0 \in L^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\mathbb{E}[u_0(X_t)]$ converges locally uniformly on \mathbb{R}^d , as t goes to $+\infty$, to $\int_{\mathbb{R}^d} u_0 m dx$.

Proof of Theorem 3.11. The proof is based on an approximation procedure. Accordingly the following family of increasing sets is defined. Let $\bar{w} \in \mathcal{C}^2(\mathbb{R}^d)$. We pick $\mathfrak{R} = \mathfrak{R}_n \uparrow_n +\infty$. Define $\mathcal{O}_{\mathfrak{R}}$ as

$$\mathcal{O}_{\mathfrak{R}} := \{x \in \mathbb{R}^d \text{ st. } \bar{w}(x) \leq \mathfrak{R}\},$$

and \mathfrak{R} is not a critical value of \bar{w} . Of course, such sequence exists in view of the Sard's Theorem, according to which the set of singular values $F(\Sigma)$ of the critical set Σ of a smooth map $F : X \rightarrow M$ (where X and Y are open \mathcal{C}^∞ (paracompact) manifolds) is null in M (see [1]). By coercivity¹ of \bar{w} we argue that $\bigcup_R \mathcal{O}_{\mathfrak{R}} = \mathbb{R}^d$. We will consider approximating problems, that is, the diffusion process x_t^n solving (1.1) on the domains $\mathcal{O}_{\mathfrak{R}}$ and pass to the limit as $\mathfrak{R} \rightarrow \infty$ to capture the behavior of the original problem on the whole space.

Since $\mathcal{O}_{\mathfrak{R}}$ is a bounded domain in \mathbb{R}^d , it is a smooth open set. We then consider the unique reflected diffusion process on $\mathcal{O}_{\mathfrak{R}}$ whose diffusion operator is still given by \mathcal{L} with the following boundary condition on $\mathcal{O}_{\mathfrak{R}}$:

$$a_{ij} \frac{\partial u}{\partial \nu_j} = 0 \quad \text{on } \partial \mathcal{O}_{\mathfrak{R}}, \quad (3.28)$$

where ν denotes the unit outnormal to $\partial \mathcal{O}_{\mathfrak{R}}$. In other words, x_t^n solves

$$\begin{cases} dx_t^n = \sigma(x_t^n) \cdot dW_t + b(x_t^n)dt - a(x_t^n) \cdot \nu(x_t^n) dk_t^n \\ x_0^n = x \in \bar{\mathcal{O}}_R \\ k_t^n \text{ is continuous, nondecreasing in } t \text{ and } k_t^n = \int_0^t 1_{\partial \mathcal{O}_{\mathfrak{R}}}(x_s^n) dk_s^n. \end{cases}$$

As it is well known, x_t^n is, for each n , ergodic and there exists a unique invariant probability measure $m_{\mathfrak{R}}$ on $\mathcal{O}_{\mathfrak{R}}$ that solves

$$\begin{cases} \mathcal{L}^* m_{\mathfrak{R}} = 0 & \text{in } \mathcal{O}_{\mathfrak{R}} \\ \nu_i \{\partial_j (a_{ij} m_{\mathfrak{R}}) - b_i m_{\mathfrak{R}}\} = 0 & \text{on } \partial \mathcal{O}_{\mathfrak{R}} \end{cases} \quad (3.29)$$

with Neumann condition due to the reflection that takes place at the boundary $\partial \mathcal{O}_{\mathfrak{R}}$. We will show that $m_{\mathfrak{R}}$ is not “disperse” as the domain $\mathcal{O}_{\mathfrak{R}}$ grows. As explain above, $m_{\mathfrak{R}}$ is

¹A function f is coercive if $f(x) \rightarrow +\infty$ when $|x| \rightarrow +\infty$.

bounded, uniformly in n large enough, in $W^{1,p}(B_{r_0})$ for all $1 < p < \infty$, $r_0 \in (0, \infty)$ denoting $B_{r_0} = \{|x| < r_0\}$ and thus, extracting subsequences if necessary, we may assume that $m_{\mathfrak{R}}$ converges uniformly locally in \mathbb{R}^d to some $m \geq 0$ ($m \in L^1(\mathbb{R}^d)$). We have to verify that m is a probability measure. If we able to show that $\int_{\mathbb{R}^d} m dx = 1$, then m is indeed an invariant measure for x_t . Indeed

$$x_t^n = x_t, \quad \text{if } t \leq \tau_n = \inf\{t \geq 0, \quad x_t \notin \overline{\mathcal{O}_{\mathfrak{R}}}\} \quad \text{and } \tau_n \xrightarrow[n \rightarrow +\infty]{} +\infty \text{ a.s.}$$

Therefore,

$$\mathbb{E}[u_0(x_t^n)] \xrightarrow[n \rightarrow +\infty]{} \mathbb{E}[u_0(x_t)], \quad \text{for any } t \geq 0, \quad x \in \mathbb{R}^d$$

$u_0 \in L^\infty(\mathbb{R}^d)$, and thus

$$\int_{\mathbb{R}^d} u(x, t) m(x) dx = \int_{\mathbb{R}^d} u_0(x) m(x) dx,$$

follows from the fact that $m_{\mathfrak{R}}$ is the invariant measure of x_t^n . Therefore, we only have to prove that $\int_{\mathbb{R}^d} m dx = 1$. In order to overcome this problem, we multiply (3.29) by \bar{w} (according to assumption 3.6) and integrate by parts over $\overline{\mathcal{O}_{\mathfrak{R}}}$:

$$\begin{aligned} \int_{\mathcal{O}_{\mathfrak{R}}} \partial_j (a_{ij} m_{\mathfrak{R}}) \partial_i \bar{w} - m_{\mathfrak{R}} b \cdot \nabla \bar{w} &= - \int_{\mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} (a_{ij} \partial_{ij} \bar{w} + b \cdot \nabla \bar{w}) + \int_{\mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} a_{ij} \partial_i \bar{w} \nu_j \\ &= \int_{\mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} \mathcal{L} \bar{w} + \int_{\partial \mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} a_{ij} \partial_i \bar{w} \nu_j. \end{aligned}$$

Next, we observe that, since $\bar{w} \equiv \mathfrak{R}$ on $\partial \mathcal{O}_{\mathfrak{R}}$ and $\bar{w} < \mathfrak{R}$ in $\mathcal{O}_{\mathfrak{R}}$, we get

$$\begin{cases} \partial_i \bar{w} = \frac{\partial \bar{w}}{\partial \nu} \nu_i \\ \frac{\partial \bar{w}}{\partial \nu} \geq 0, \end{cases} \quad \text{on } \partial \mathcal{O}_{\mathfrak{R}}.$$

An immediate consequence is

$$\int_{\mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} \mathcal{L} \bar{w} dx \leq 0.$$

Thanks to ASSUMPTION 3.6, $\mathcal{L} \bar{w} \rightarrow \infty$ as $|x| \rightarrow +\infty$, we deduce that $m_{\mathfrak{R}}$ satisfies for some constant $C > 0$ independent of n

$$\int_{\mathcal{O}_{\mathfrak{R}}} m_{\mathfrak{R}} (\mathcal{L} \bar{w})_+ dx \leq C,$$

and thus, more specifically we get

$$\sup_{\substack{n \geq 1 \\ (|x| \geq R) \cap \mathcal{O}_{\mathfrak{R}}}} \int m_{\mathfrak{R}} \rightarrow 0 \quad \text{as } R \rightarrow +\infty.$$

Therefore m is a probability measure on \mathbb{R}^d and yields the following bound:

$$\int_{\mathbb{R}^d} m (\mathcal{L} \bar{w})_+ dx < \infty.$$

The proof is thus completed. ■

In what follows the interest focuses on the solutions of the following equation:

$$\mathcal{L}^* m = 0 \quad \text{on } \mathbb{R}^d, \quad \int_{\mathbb{R}^d} m = 1, \quad (3.30)$$

where \mathcal{L}^* is the formal adjoint to the operator \mathcal{L} . We will prove that the existence of a Lyapunov function yields the existence of a unique solution in the class of probability measures.

Remark 3.12. *From ellipticity regularity results and the strong maximum principle, we immediately deduce (at least when σ and b are smooth) that m is smooth and $m > 0$ on \mathbb{R}^d . Indeed, if $m \geq 0$ satisfies Eq. (3.30) then $m \in W_{loc}^{1,p}(\mathbb{R}^d)$ for all $1 < p < \infty$ and thus $m \in \mathcal{C}^\alpha(\mathbb{R}^d)$ (for all $0 < \alpha < 1$) by Sobolev embeddings. In addition, if b and $\partial_j(a_{ij})$ are locally Lipschitz, then $m \in W_{loc}^{2,p}(\mathbb{R}^d)$, for all $1 < p < \infty$ and thus $m \in \mathcal{C}^{1,\alpha}(\mathbb{R}^d)$.*

The next theorem gives another characterization of ergodicity and an invariant probability measure for the process X_t .

Theorem 3.13. *Assume that the ASSUMPTION 3.8 holds. Then the Markov diffusion process X_t solution of (1.1) admits a unique invariant probability measure m on \mathbb{R}^d , which satisfies the properties of Theorem 3.11.*

Proof of the Theorem 3.13. This is straightforward since the process X_t is well-defined if and only if there is no blow up. Let $\{u(t, x); t \in \mathbb{R}^+\}$ denotes the solution of the SDE (1.1) and let $t \wedge \tau_n$ be a Markov time associated with $u(t, x)$. If we assume that there exists a function $\bar{w} \in \mathcal{C}^2(\mathbb{R}^d)$ satisfying the ASSUMPTION (3.24), the following holds:

$$\bar{w}(x_{t \wedge \tau_n}) = \bar{w}(x) + \left[\int_0^{t \wedge \tau_n} (-\mathcal{L}\bar{w})(x_s) ds \right] \quad (3.31)$$

where \wedge denotes the minimum. By taking expectations on both sides we then obtain the well-known Dynkin's Formula: for all $x \in \mathbb{R}^d$, $t \geq 0$ and $n \geq 1$

$$\mathbb{E}[\bar{w}(x_{t \wedge \tau_n})] = \bar{w}(x) + \mathbb{E} \left[\int_0^{t \wedge \tau_n} (-\mathcal{L}\bar{w})(x_s) ds \right].$$

Therefore $\tau_n \uparrow_n +\infty$ as $n \rightarrow +\infty$ and for all $t \geq 0$, $x \in \mathbb{R}^d$, $\mathbb{E}[\bar{w}(x_t)] < \infty$. Next, to obtain the uniqueness of bounded solution of (1.4) we use the maximum principle. Let u and v be two bounded solution of (1.4). For $\varepsilon > 0$, consider

$$u_\varepsilon = v + \varepsilon(\bar{w} + C(t + 1))$$

for some C to be determined later on. Obviously, we have for C large enough

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + \mathcal{L}u_\varepsilon = \varepsilon\{\mathcal{L}\bar{w} + C\} \geq 0 \\ u_\varepsilon|_{t=0} \equiv u_0 + \varepsilon(\bar{w} + C) \geq u_0. \end{cases} \quad (3.32)$$

In addition, since $\bar{w} \rightarrow +\infty$ as $|x| \rightarrow +\infty$, $u_\varepsilon > v$ for $|x|$ large enough (uniformly for t bounded). We may thus apply the maximum principle and deduce that we have:

$$u_\varepsilon \geq v \quad \text{on } \mathbb{R}^d \times [0, \infty).$$

Hence, we conclude upon letting ε go to 0^+ . ■

The following proposition provides some important equivalences for the existence of a unique invariant measure.

Proposition 3.14. *Let M be a definite positive symmetric matrix and (a_{ij}) defined as in (1.6). Suppose that*

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \left\{ |x|^{\mu-1} \left[-\text{Tr}(a \cdot M) - (\mu-1) \frac{a \langle Mx, Mx \rangle}{\langle Mx, x \rangle} - \langle b, Mx \rangle \right] \right\} = +\infty, \quad (3.33)$$

where μ is some positive number. Then the process X_t solution of (1.1) is ergodic (there exists a unique invariant probability measure m on \mathbb{R}^d). In particular, the ASSUMPTION 3.6 holds true as soon as for some $\varepsilon > 0$, we have

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \left\{ \text{Tr}(a \cdot M) - \varepsilon \frac{a \langle Mx, Mx \rangle}{\langle Mx, x \rangle} - \langle b, Mx \rangle \right\} > 0. \quad (3.34)$$

Furthermore, if a is bounded on \mathbb{R}^d , then (3.6) holds true if

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \{ \langle b, Mx \rangle - \text{Tr}(a \cdot M) \} > 0. \quad (3.35)$$

Proof of Proposition 3.14. The proof is based on the identification of a useful Lyapunov structure. We choose

$$\bar{w}(x) = \left(\frac{1}{2} \langle Mx, x \rangle \right)^\mu, \quad \mu > 1, \quad (3.36)$$

and we compute $\mathcal{L}\bar{w}$. A straightforward computation shows that

$$\mathcal{L}\bar{w} = \mu \left(\frac{1}{2} \langle Mx, x \rangle \right)^{\mu-1} \left\{ -\text{Tr}(a \cdot M) - (\mu-1) \frac{a \langle Mx, Mx \rangle}{\langle Mx, x \rangle} - \langle b, Mx \rangle \right\}. \quad (3.37)$$

Therefore the ASSUMPTION 3.6 holds if there exists $\mu > 1$, and M is a symmetric positive definite matrix such that:

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \left\{ |x|^{\mu-1} \left[-\text{Tr}(a \cdot M) - (\mu-1) \frac{a \langle Mx, Mx \rangle}{\langle Mx, x \rangle} - \langle b, Mx \rangle \right] \right\} = +\infty. \quad (3.38)$$

Specifically, for a such μ , the ASSUMPTION 3.6 holds as soon as we have some $\varepsilon > 0$, for a symmetric positive definite matrix M :

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \left\{ \text{Tr}(a \cdot M) - \varepsilon \frac{a \langle Mx, Mx \rangle}{\langle Mx, x \rangle} - \langle b, Mx \rangle \right\} > 0. \quad (3.39)$$

If a is bounded over \mathbb{R}^d , then (3.39) holds (thus the ASSUMPTION 3.6 as well) and if we have some symmetric positive definite matrix M , upon choosing μ close enough to 1, taking the limit as $|x| \rightarrow +\infty$, an immediate consequence of the attenuation inequality (3.39) is that

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \{ \langle b, Mx \rangle - \text{Tr}(a \cdot M) \} > 0. \quad (3.40)$$

Suppose we had

$$a(x) \rightarrow c_0 I \quad \text{as } |x| \rightarrow \infty \quad \text{for some } c_0 \geq 0. \quad (3.41)$$

It is easily checked that the limit (3.40) (and thus ASSUMPTION 3.6) holds if b satisfies

$$\liminf_{|x| \rightarrow +\infty} \text{ess} \{ -\langle b(x), x \rangle \} > c_0 d, \quad (3.42)$$

which completes the proof. ■

A trivial consequence of the above theorems is the following corollary, which guarantees the uniqueness of the invariant distribution.

Corollary 3.15. *Assume that the ASSUMPTION 3.6 holds true with (3.27) replaced by*

$$\lim_{|x| \rightarrow \infty} \sup_k \langle b_k(x), x \rangle = -\infty. \quad (3.43)$$

Then the Markov diffusion process X_t solution of (1.1) admits a unique invariant probability measure $m \in \mathbb{R}^d$.

Remark 3.16. The formula (3.43) is reminiscent of other similar assumptions about recurrence of diffusion processes in the whole space. Indeed if we take the following fully nonlinear degenerate operator:

$$\mathcal{L} := b(x) \cdot DV + \text{trace}(\sigma(x)\sigma^T(x)D^2V), \quad (3.44)$$

then the ASSUMPTION 3.4 is satisfied

$$\limsup_{|x| \rightarrow +\infty} [b(x) \cdot y + \text{trace}(\sigma(x)\sigma^T(x))] < 0.$$

Indeed, in this case it is sufficient to choose $w(x) = |x|^2$. Pardoux and Veretennikov [21] assume $\sigma\sigma^T$ bounded and

$$\lim_{|x| \rightarrow +\infty} b(x) \cdot x = -\infty,$$

called the recurrence condition.

4. THE DEGENERATE CASE

The results presented in the two preceding sections are based on the essential assumption of ellipticity (even uniform ellipticity). Under ellipticity and convenient smoothness assumptions on the drift and the diffusion coefficient, the process X_t admits a unique invariant measure. This section is devoted to the degenerate case, i.e. when $\sigma\sigma^T(x)$ is nowhere strictly positive.

Hypoellipticity is a condition that guarantees the existence of smooth solutions for the equation despite this degeneracy. Roughly speaking, a system is hypoelliptic if the drift terms help to spread the noise to all phase space directions, such that the system has a nondegenerate transition density. Technically, hypoellipticity requires certain conditions involving the Lie brackets (i.e. commutator) of drift and diffusion fields, known as Hörmander's conditions; when these assumptions are satisfied, the system can be shown to possess *smooth transition densities*. Observe that for any differential operator

$$P = \sum_{i,j}^d a_{ij}(x) \partial_i \partial_j$$

with $(a_{ij}(x))$ positive semi-definite matrix, the weak maximum principle holds. Moreover, if P is in divergence form and is generated by vector fields satisfying the Hörmander condition, then Strong Maximum Principle (see [6]) holds.

Going back to the linear second-order operator \mathcal{L} of the form (1.5), we define smooth vector fields X_0, X_1, \dots, X_d on \mathbb{R}^d by

$$X_0 = \sum_{i=1}^d \left(b_i - \sum_{j=1}^d \partial_j a_{ij} \right) \partial_i, \quad X_i = \sum_{j=1}^d a_{ij} \partial_j, \quad (1 \leq i \leq d) \quad (4.1)$$

We assume that

$$\begin{aligned} a(x) &= (a_{ij}(x))_{d \times d} \in \mathcal{C}^1(\mathbb{R}^d, \mathcal{S}_d), \\ b(x) &= (b_1(x), \dots, b_d(x)) \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}^d), \end{aligned} \quad (4.2)$$

where \mathcal{S}_d denotes the class of symmetric non-negative matrices; this implies that \mathcal{L} is the degenerate elliptic-parabolic operator in \mathbb{R}^d . With the notations (4.1) the operator \mathcal{L} takes the Hörmander form:

$$\mathcal{L} = \frac{1}{2} \sum_{k=1}^m X_k^2 + X_0, \quad (4.3)$$

where X_k^2 denotes the second iterative of directional derivative operator X . The form (4.3) is however much more convenient for the purpose of the study of the present section. The vector fields X_1, \dots, X_m satisfy the Strong Hörmander condition if their Lie brackets generate the tangent space. It satisfies Hörmander's condition if X_0 is allowed. We now make the assumption that combined with the Lyapunov function will induce ergodicity.

ASSUMPTION 4.1. (Hörmander's rank bracket condition)

We assume that σ and b are smooth and \mathcal{L} satisfies the Strong Hörmander condition, namely the vector fields $X_i = \sigma^i \cdot \nabla$ and a finite number of their Lie brackets generate the full algebra of vector fields on \mathbb{R}^d .

The ASSUMPTION 4.1 is a specific case of the widely known parabolic Hörmander condition that ensures the hypoellipticity of $\partial_t - \mathcal{L}^*$, which implies the existence of an invariant measure.

Theorem 4.2. *Assume that σ and b fulfilled the conditions resulting from the ASSUMPTION 4.1 and there exists a Lyapunov function. Then the process X_t admits a unique invariant probability measure m on \mathbb{R}^d .*

The reader interested in further references on the ergodicity of hypoelliptic diffusions is addressed to [13] and [3].

5. THE INVARIANT MEASURE FOR THE ONE-DIMENSIONAL DIFFUSION

This section is devoted to the analysis of (2.3) and (3.21) for $d = 1$. A regular time invariant solution m of Equation (2.3) will then satisfy the following problem:

$$-\frac{d^2}{dx^2}(a(x)m) + \frac{d}{dx}(b(x)m(x)) = 0 \quad x \in \mathbb{R}. \quad (5.4)$$

If we are looking for $m > 0$ (locally Lipschitz) which solves (5.4), given sufficient regularity on the parameters, we may then state that

$$-\frac{d}{dx}(a(x)m(x)) + b(x)m(x) = C_1, \quad x \in \mathbb{R} \quad (5.5)$$

for some $C_1 \in \mathbb{R}$. A straightforward calculation shows that if m exists, is given by

$$m = \frac{1}{a(x)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) \left\{ C_1 + C_2 \int_0^x \exp\left(-\int_0^y \frac{b(s)}{a(s)} ds\right) dy \right\} \quad (5.6)$$

for some constants C_1 and C_2 with $C_1 > 0$ since $m(0) > 0$.

Next we solve the equation (3.21). It is a second order linear equation in one variable; the set of solutions has dimension two and there exists a linearly independent solution. Clearly, if $w \in L_{loc}^\infty$ (and thus $W_{loc}^{2,\infty}$) solves (3.21), then a simple calculation shows:

$$w(x) = D_1 + D_2 \int_0^x \exp\left(-\int_0^y \frac{b(s)}{a(s)} ds\right) dy. \quad (5.7)$$

The stopping times are defined as follows:

$$\tau_c = \inf\{t \geq 0, x(t) = 0\} \quad \tau_a = \lim_{x \rightarrow a} \tau_c, \quad \text{and} \quad \tau_b = \lim_{x \rightarrow b} \tau_c.$$

Lemma 5.1. *Assume that $a < 0 < b$. If $\lim_{x \rightarrow b} w(x) = \infty$, that is*

$$\int_0^b \exp\left(-\int_0^y \frac{b(s)}{a(s)} ds\right) dy = +\infty$$

then $\mathbb{P}_x[\tau_b < \infty] = 0$. If

$$\int_a^0 \exp\left(-\int_0^y \frac{b(s)}{a(s)} ds\right) dy = +\infty$$

then $\mathbb{P}_x[\tau_a < \infty] = 0$.

Proof of Lemma 5.1. For $a < a' < x < b' < b$,

$$\mathbb{P}_x[\tau_{b'} > \tau_{a'}] = \frac{w(b') - w(x)}{w(b') - w(a')} \quad (5.8)$$

and it tends to 1 as $b' \rightarrow b$. Therefore, $\mathbb{P}_x[\tau_{b'} > \tau_{a'}] = 1$. The path from $x \rightarrow b$ has to go to a' first. But if it is go to b from a' it has cross x , and then by strong Markov property has to return to a' again. Clearly it is getting a run around and is never going to make it to b . \blacksquare

Consequently the choice of the condition w bounded implying $D_2 = 0$ (where D_2 is the constant defined in (5.7)) reduces to:

$$\int_{\mathbb{R}} \exp\left(-\int_0^y \frac{b(s)}{a(s)} ds\right) dy = +\infty, \quad (5.9)$$

and this is obviously necessary and sufficient. If we define the scale function by

$$s(x) = \exp\left(-\int_0^x \frac{b(y)}{a(y)} dy\right), \quad (5.10)$$

and

$$S(x) = \int_0^x s(y) dy \quad \text{for } x \geq 0, \quad S(x) = -\int_x^0 s(y) dy \quad \text{for } x < 0 \quad (5.11)$$

then (5.9) is equivalent to

$$\lim_{x \rightarrow +\infty} S(x) = +\infty, \quad \lim_{x \rightarrow -\infty} S(x) = -\infty \quad (5.12)$$

which is equivalent to

$$M := \int_{-\infty}^{+\infty} m(x) dx < +\infty \quad (5.13)$$

M is the normalizing constant. In this case, the unique invariant probability measure of the process can be represented as

$$m(x) = \frac{1}{Ma(x)} \exp\{-\theta(x)\} \quad \text{with} \quad \theta(x) := - \int_0^x \frac{b(s)}{a(s)} ds. \quad (5.14)$$

The problem is now to ensure that the process does not blow up in finite time. Accordingly the following assumption on the coefficients of (1.1) is made.

ASSUMPTION 5.2.

We assume (at least) that σ and b are measurable and locally bounded and $\inf_{\sigma \in [-R, +R]} \sigma > 0$, for all $R \in (0, \infty)$.

Claim. Under the ASSUMPTION 5.2, $X(t)$ is well-defined locally and the process does not blow up in finite time if one has

$$\int_0^{+\infty} \exp(\theta(x)) dx = \int_{-\infty}^0 \exp(\theta(x)) dx = +\infty. \quad (5.15)$$

Proof of Claim 5.15. Assume to the contrary that

$$\int_0^{+\infty} \exp(\theta(x)) dx < +\infty,$$

the other case is similar. We apply the Itô formula with

$$w(x) = \int_0^x \exp\left(- \int_0^t \frac{b(s)}{a(s)} ds\right) dt,$$

and first for $x > 0$

$$\mathbb{E}[w(x_{\tau_0 \wedge \tau_R})] = w(x),$$

where

$$\tau = \inf\{t \geq 0, x(t) = 0\} \quad \text{and} \quad \tau_R = \inf\{t \geq 0, x(t) = R\}, \quad x < R < \infty.$$

In particular we have

$$(w(R) - w(0))\mathbb{P}(\tau_R < \tau_0) = w(x) - w(0).$$

Letting R go to $+\infty$ and denoting by $\tau_\infty = \inf\{t \geq 0, X_t = +\infty\}$ (first time of “positive” blow up) we deduce

$$\mathbb{P}(\tau_R < \tau_0) = \frac{w(x) - w(0)}{w(+\infty) - w(0)} > 0, \quad \text{where} \quad w(+\infty) = \int_0^{+\infty} \exp\left(- \int_0^x \frac{b(y)}{a(y)} dy\right) dx.$$

The process almost surely never diverges before possibly reaching zero as soon as $S(x)$ diverges at ∞ . Hence, we have shown that if X_t is well-defined and ergodic, then necessarily (5.15) holds. We thus conclude that the process does not blow up in finite time in these cases, which ends this part of the proof. \blacksquare

Now we return to the question of the invariant probability measure and the blow up property.

Proposition 5.3. *There exists an invariant measure $m \in L^1(\mathbb{R})$ for (1.1) if and only if*

$$\frac{1}{a(x)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) \in L^1(\mathbb{R}) \quad (5.16)$$

and the above is equivalent to (5.15), and then the ASSUMPTION 3.6 holds.

Proof. The proof is similar to the one used in [22]. Specifically we choose a function $f \geq 0$ with $f \in \mathcal{C}(\mathbb{R})$ such that:

$$+\infty > \int_{-\infty}^0 \frac{f(y)}{a(y)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) dx > \int_0^{+\infty} \frac{f(x)}{a(x)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) dx \geq 0$$

and $f(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$. Upon choosing $M > 0$ such that

$$\int_{-\infty}^0 \frac{f(x)}{a(x)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) dx > M > \int_0^{+\infty} \frac{f(x)}{a(x)} \exp\left(\int_0^x \frac{b(y)}{a(y)} dy\right) dx \quad (5.17)$$

and defined \bar{w} as follows:

$$\bar{w}(x) = \int_0^x \left\{ M - \left(\int_0^\eta \frac{f(y)}{a(y)} \exp\left(\int_0^y \frac{b(s)}{a(s)} ds\right) dy \right) \right\} \exp\left(-\int_0^x \frac{b(y)}{a(y)} dy\right) d\eta, \quad (5.18)$$

a straightforward computation shows that

$$\bar{w}'(x) = \left(M - \left(\int_0^x \frac{f(y)}{a(y)} \exp\left(\int_0^y \frac{b(s)}{a(s)} ds\right) dy \right) \right) \exp\left(-\int_0^x \frac{b(y)}{a(y)} dy\right) \quad (5.19)$$

and

$$\begin{aligned} \bar{w}''(x) &= -\frac{b(x)}{a(x)} \left\{ M - \left(\int_0^x \frac{f(y)}{a(y)} \exp\left(\int_0^y \frac{b(s)}{a(s)} ds\right) dy \right) \right. \\ &\quad \left. - \frac{f(x)}{a(x)} \exp\left(\int_0^x \frac{b(s)}{a(s)} ds\right) \right\} \exp\left(-\int_0^x \frac{b(y)}{a(y)} dy\right). \end{aligned} \quad (5.20)$$

Combining (5.19) and (5.20), we obtain

$$\mathcal{L}\bar{w} = -a\bar{w}'' - b\bar{w}' = f \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty.$$

Taking into account (5.12) (or with (5.15)), we get $\bar{w}'(x) \rightarrow +\infty$ as $x \rightarrow +\infty$ and similarly, $\bar{w}'(x) \rightarrow -\infty$ as $x \rightarrow -\infty$. Consequently, $\bar{w} \rightarrow +\infty$ as $|x| \rightarrow +\infty$ and (3.6) holds, which completes the proof. \blacksquare

Bearing all above in mind, the following result holds.

Proposition 5.4. *The following statements are equivalent.*

- (i) X_t is well-defined and ergodic;
- (ii) The identity (5.15) and the condition (5.16) hold;
- (iii) ASSUMPTION 3.6 holds.

According to the above proposition, the ASSUMPTION 3.6 is necessary and sufficient in the case $d = 1$.

EXAMPLE 5.5. (Ornstein-Uhlenbeck process) Let X be the one-dimensional Ornstein-Uhlenbeck process given by

$$dX_t = -kX_t dX_t + \sigma dB_t, \quad (5.21)$$

where b and σ are positive constants. In this case $a(x) = \sigma^2 \in \mathbb{R}_+^*$ and $b(x) = -kx$ with $k \in \mathbb{R}_+^*$, given constants. We have

$$\theta(x) := - \int_0^x \frac{k}{\sigma^2} x dx = \frac{k}{\sigma^2} x^2, \quad x \in \mathbb{R},$$

leading to

$$K = \left\{ \int_{-\infty}^{+\infty} \frac{1}{a} \exp(-\theta(x)) dx \right\}^{-1} = \frac{1}{a} \left(\frac{\pi a}{k} \right)^{-\frac{1}{2}}.$$

Consequently

$$m(x) = \left(\frac{k}{\pi a} \right)^{\frac{1}{2}} \exp\left\{ -\frac{k}{a} x^2 \right\}, \quad x \in \mathbb{R},$$

which is a Gaussian density.

6. EXAMPLES AND APPLICATIONS

This section is devoted to some specific examples of stochastic processes solution of (1.1) such that the existence of the invariant measure can be proved by employing the theory discussed in the present paper.

EXAMPLE 1.

Consider the diffusion on \mathbb{R}^d solution of the following SDE:

$$dX_t^i = \gamma X_i dt + (1 + X_i^2) dW_t^i. \quad (6.1)$$

Proposition 6.1. *The ASSUMPTION 3.6 holds true for the Example 1.*

Proof. The infinitesimal generator with unbounded coefficients associated to the Eq. (6.1) is given by

$$\mathcal{L} = - \sum_{i=1}^d (1 + x_i^2) \partial_{x_i}^2 - \gamma_i x_i \partial_{x_i}, \quad \gamma_i > 1. \quad (6.2)$$

According to Section 3, it is sufficient to consider $w(x) = \sum_{i=1}^d x_i^2 := r^2$ as Lyapunov function. ■

EXAMPLE 2.

Assume that the vector field b in Eq. (1.1) is conservative (is the gradient of a scalar function V):

$$dX_t = -\nabla V(X_t) dt + dW_t. \quad (6.3)$$

This is a gradient flow perturbed by a noise. The generator is

$$\mathcal{L} = -\nabla V(x) \cdot \nabla + \Delta. \quad (6.4)$$

Proposition 6.2. *Assume that $V(x)$ in Eq. (6.4) is a smooth potential and that $e^{-V(x)} \in L^1(\mathbb{R}^d)$. Then the Markov process solution of (6.3) is ergodic.*

Proof. The equation

$$\mathcal{L}^*m = 0, \quad \int_{\mathbb{R}^d} m(x)dx = 1, \quad (6.5)$$

has a unique solution (a Gibbs distribution) given by

$$m(x) = \frac{1}{Z}e^{-V(x)}, \quad Z = \int_{\mathbb{R}^d} e^{-V(x)}dx, \quad (6.6)$$

where Z is the normalization constant. The Gibbs distribution is an invariant distribution. Indeed, Eq. (6.5) can be rewritten as follows:

$$\nabla_x \cdot (\nabla_x V(x)m(x) + \nabla_x m(x)) = 0. \quad (6.7)$$

One can immediately check that $m(x)$ given by (6.7) satisfies

$$\nabla_x V(x)m(x) + \nabla_x m(x) = 0,$$

and hence it satisfies (6.6). Furthermore, by construction we have that

$$\int_{\mathbb{R}^d} \frac{1}{Z}e^{-V(x)}dx = 1$$

and hence $m(x)$ is correctly normalized. Finally, m is the smooth density of an invariant measure, since it satisfies $\mathcal{L}^*m = 0$. Uniqueness follows by the ergodicity of the stochastic process with generator \mathcal{L} . \blacksquare

EXAMPLE 3. (Fokker-Planck equation)

Let $X, Y \in \mathbb{R}^d$, $V(X) : \mathbb{R}^d \rightarrow \mathbb{R}$ a \mathcal{C}^2 function and B_t a d -dimensional Brownian motion. Consider the following Fokker-Planck equation on $\mathbb{R}^d \times \mathbb{R}^d$ defined as follows:

$$\begin{cases} dX_t = Y_t dt \\ dY_t = \sqrt{2} dB_t - \nabla V(X_t) dt - Y_t dt \end{cases} \quad (6.8)$$

where $\sqrt{2}$ represents a normalization coefficient.

Proposition 6.3. *Assume that V in Eq. (6.8) is a smooth potential and that $e^{-V(x) - \frac{\|v\|^2}{2}} \in L^1(\mathbb{R}^d)$. Then the Markov process solution of (6.8) is hypoelliptic and admits an invariant measure.*

Proof. Define a density p_t of the law (X_t, \dot{X}_t) in $\mathbb{R}^d \times \mathbb{R}^d$. Then p is solution of the following partial differential equation associated to (6.8):

$$\frac{\partial p}{\partial t} + v \cdot \nabla_x p - \nabla V(x) \cdot \nabla_v p = \Delta_v p - v \cdot \nabla_v p, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d. \quad (6.9)$$

The generator of (6.9) given by the following partial differential operator:

$$\mathcal{L} = \Delta_v - v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x \quad (6.10)$$

is not elliptic. We rewrite (6.10) with the help of Hörmander form as follows:

$$\mathcal{L} = \sum_{i=1}^n X_i^2 + Y, \quad \text{with } X_i = \frac{\partial}{\partial v_i}, \quad Y = -v \cdot \nabla_v + \nabla V \cdot \nabla_v - v \cdot \nabla_x$$

By setting

$$X_i := \frac{\partial}{\partial v_i}, \quad X := \nabla_v, \quad Y = v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v,$$

one obtains

$$[X, Y] = \nabla_x \quad \text{or} \quad [X_i, Y] = \frac{\partial}{\partial x_i}$$

Then, the vectors $(X_1, \dots, X_n, [Y, X_1], \dots, [Y, X_n])$ form a basis of \mathbb{R}^{2d} at each point. Hence \mathcal{L} is hypoelliptic. The existence and uniqueness of an invariant measure m follows from Theorem 4.2. Moreover, as recalled in section 3, m solves $\mathcal{L}^*m = 0$ in the sense of distributions and m is given by

$$dm = e^{-V(x) - \frac{\|v\|^2}{2}} dx dv. \quad \blacksquare$$

EXAMPLE 4.

We consider a hypoelliptic diffusion. Let $x_1(t)$ (resp. $x_2(t)$) be the position (resp. the velocity) at time t of a physical system moving in \mathbb{R}^2 :

$$\begin{cases} dx_1(t) = dw_1 \\ dx_2(t) = x_1 dt. \end{cases} \quad (6.11)$$

Proposition 6.4. *The condition of Theorem 4.2 is satisfied in Example 4.*

Proof. This process is degenerate in the sense that its infinitesimal generator

$$\mathcal{L} = -\partial_{x_1}^2 - x_1 \partial_{x_2}, \quad (6.12)$$

is not elliptic. The operator \mathcal{L} is Hörmander type hypoelliptic. Indeed, on the other hand

$$X_0 = x_1 \partial_2, \quad X_1 = \partial_1, \quad \text{and} \quad X_2 = 0.$$

Thus the

$$\text{Span}\{X_1, X_2, [X_1, X_2], [X_1, X_0], [X_2, X_0]\} = \text{Span}\{\partial_1, \partial_2\} = \mathbb{R}^2.$$

This implies that \mathcal{L} and \mathcal{L}^* are hypoelliptic. Accordingly the Hörmander's condition holds and by using the Lyapunov function $w(x) = x_1^2 + x_2^2$, the existence and smoothness of invariant measure is the immediate consequence of the Theorem 4.2. \blacksquare

EXAMPLE 5.

Let us consider the stochastic differential equation (1.1) with formal generator of the diffusion generated by the Kolmogorov operator:

$$\mathcal{L}\phi = \text{Tr}(\sigma\sigma^T D^2\phi) + \langle b, D\phi \rangle, \quad \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad (6.13)$$

where $\mathcal{C}_0^\infty(\mathbb{R}^d)$ denotes the class of all infinitesimally differentiable functions on \mathbb{R}^d with compact support in \mathbb{R}^d .

Proposition 6.5. *The condition of Proposition 3.11 is satisfied in Example 5.*

Proof. The operator \mathcal{L} satisfies all the hypotheses of section 3. Taking $\bar{w}(x) = \langle x, x \rangle^\mu = |x|^{2\mu}$, for $\mu \in \mathbb{N}$, we show that $\bar{w}(x)$ is a Lyapunov function of the process x_t . We start by computing

$$D\bar{w}(x) = 2\mu \{|x|^2\}^{\mu-1} x, \quad D^2\bar{w}(x) = 2\mu \{|x|^2\}^{\mu-1} \mathbf{I} + 4\mu(\mu-1) \{|x|^2\}^{\mu-2} (x \cdot x^T).$$

Upon placing the above computation into (6.13) we obtain

$$\begin{aligned} \mathcal{L}\bar{w}(x) &= 4\mu(\mu-1)\{|x|^2\}^{\mu-2} \sum_{i,j=1}^d A_k^{ij} x_i x_j + 2\mu\{|x|^2\}^{\mu-1} \text{Tr}(A_k(x)) \\ &\quad + 2\mu\{|x|^2\}^{\mu-1} \langle x, b_k(x) \rangle \\ &\leq \langle x, x \rangle^{\mu-1} [4\mu(\mu-1)\|A_k(x)\| + 2\mu d\|A_k(x)\| + 2p\langle x, b_k(x) \rangle]. \end{aligned} \quad (6.14)$$

Observing that

$$\sup_k \int_{\mathbb{R}^d} |\mathcal{L}\bar{w}| dm \leq C, \quad (6.15)$$

with

$$C := 2 \sup_k \int_B |\mathcal{L}\bar{w}| dm, \quad (6.16)$$

where, for every $R > 0$, we set $B_R = \{x \in \mathbb{R}^d : |x| < R\}$, a centered open ball of radius R , and B is such that $\mathcal{L}\bar{w} \geq 1$ we deduce that

$$\sup_k \int_{\mathbb{R}^d} |\mathcal{L}\bar{w}(x)| m dx < \infty. \quad (6.17)$$

Thanks to (6.14) and the uniform boundedness of $\|A_k(x)\|$, we conclude that

$$\sup_k \int_{\mathbb{R}^d} \langle x, x \rangle^{\mu-1} m dx < \infty. \quad (6.18)$$

The theory can be applied if we take $A = I$ and $B(x) = -k(x)x$ where $k \in L_{loc}^\alpha(\mathbb{R}^d)$ is such that:

$$k(x)|x|^2 \rightarrow +\infty \quad \text{as } |x| \rightarrow \infty.$$

More generally, if $A = I$, then it suffices to have the weaker estimate

$$\lim_{|x| \rightarrow +\infty} \langle x, x \rangle^{\mu-1} [2(\mu-1) + d\langle b(x), x \rangle] = -\infty,$$

for $\mu \geq 1$. ■

EXAMPLE 6. (Sharp condition in \mathbb{R}^2)

Let $x_t = (x_t, y_t)^T$ be solution of the following system of SDE:

$$\begin{cases} dx_t = \tau(x_t)dt + \varrho dW_t^1 \\ dy_t = \kappa(x_t)dt + \alpha \frac{x_t}{\sqrt{1+x_t^2}} dW_t^2 \end{cases} \quad (6.19)$$

where $\varrho, \alpha > 0$, W_t^1 and W_t^2 are two independent Brownian motions.

Proposition 6.6. *The condition of Proposition 3.14 is satisfied in Example 6.*

Proof. The corresponding generator reads:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\varrho^2 \partial_{x^2}^2 - \frac{1}{2} \frac{\alpha^2 x^2}{1+x^2} \partial_{y^2}^2 - \tau(x) \partial_x - \kappa(x) \partial_y \\ &= \text{Tr}(\sigma \sigma^T D^2 \cdot) + \langle B, D \cdot \rangle \equiv \frac{1}{2} \sum_{k=1}^m (X_k)^2 + X_0, \end{aligned} \quad (6.20)$$

with

$$\sigma = \begin{pmatrix} \varrho & 0 \\ 0 & \frac{\alpha x}{\sqrt{1+x^2}} \end{pmatrix}, \quad \mathcal{A} = \sigma\sigma^T = \begin{pmatrix} \varrho^2 & 0 \\ 0 & \frac{\alpha^2 x^2}{1+x^2} \end{pmatrix}. \quad (6.21)$$

We are interested in identifying the sharp condition of the above system of SDE. Let us point out that, the matrix \mathcal{L} is degenerate. The vector fields are

$$X_0 = \tau(x_t)\partial_1 + \kappa(x_t)\partial_2, \quad X_1 = \varrho\partial_1, \quad X_2 = \alpha \frac{x_t}{\sqrt{1+x_t^2}}\partial_2.$$

The corresponding Lie brackets are

$$[X_0, X_1] = -\varrho(\tau'(x_t)\partial_1 + \kappa'(x_t)\partial_2), \quad [X_0, X_2] = \frac{\tau(x)\alpha}{(1+x^2)^{3/2}}\partial_2, \quad [X_1, X_2] = \frac{\alpha\varrho}{(1+x^2)^{3/2}}\partial_2,$$

which generates the full tangent space at each point $(x, y) \in \mathbb{R}^2$

$$\text{Lie}([X_1, [X_1, X_2]]) = \left(\varrho\partial_1, \frac{\alpha\varrho}{(1+x^2)^{3/2}}\partial_2 \right) = \mathbb{R}^2$$

and then, this satisfies the strong Hörmander condition. This implies the hypoellipticity of \mathcal{L} . The ASSUMPTION 3.6 is indeed sufficient for the ergodicity of x_t . It remains to find a Lyapunov function fulfilling the assumptions in Section 3. To this end, we will choose the Lyapunov function \bar{w} to be $\bar{w}(x) = \{\frac{1}{2}\langle x, x \rangle\}^\mu$, with $\mu > 1$. We compute

$$D\bar{w}(x) = \mu \{\frac{1}{2}\langle x, x \rangle\}^{\mu-1} x, \quad D^2\bar{w}(x) = \mu \{\frac{1}{2}\langle x, x \rangle\}^{\mu-1} \mathbf{I} + \mu(\mu-1) \{\frac{1}{2}\langle x, x \rangle\}^{\mu-2} (x \cdot x^T).$$

Therefore, we find

$$\begin{aligned} \mathcal{L}\bar{w} &= -\frac{1}{2}\varrho^2\partial_{x^2}^2\bar{w} - \frac{1}{2}\frac{\alpha^2 x^2}{1+x^2}\partial_{y^2}^2\bar{w} - \tau(x)\partial_x\bar{w} - \kappa(x)\partial_y\bar{w} \\ &= \mu\{\frac{1}{2}|x|^2\}^{\mu-1} \left\{ -\frac{1}{2}\varrho^2 - \frac{1}{2}\frac{\alpha^2 x^2}{1+x^2} - b(x) \cdot x \right. \\ &\quad \left. - (\mu-1) \left[\varrho^2 x^2 + \frac{\alpha^2 x^2 y^2}{1+x^2} \right] |x|^{-2} \right\}, \end{aligned}$$

with $b := (\tau, \kappa)$. Consequently, arguing as in Proposition 3.14, we find that ASSUMPTION (3.6) holds as soon as b satisfies

$$\liminf_{|x| \rightarrow +\infty} (-b(x) \cdot x) > \frac{1}{2}\varrho^2 + \frac{1}{2}\alpha^2, \quad (6.22)$$

upon choosing $M = I$ in (3.38), μ close enough to 1.

• *Existence of the measure m .* In order to prove the existence of m , it is enough to regularize x_t and y_t by x_t^δ and y_t^δ ($\delta > 0$) which solves

$$\begin{cases} dx_t^\delta = \tau(x_t^\delta)dt + \varrho dW_t^1 \\ dy_t^\delta = \kappa(x_t^\delta)dt + \alpha \frac{x}{\sqrt{1+x^2}} dW_t^2 + \delta dW_t^3 \end{cases} \quad (6.23)$$

for some Brownian motion W_t^3 independent of (W_t^1, W_t^2) . We may rewrite (6.23) as

$$\begin{pmatrix} dx_t^\delta \\ dy_t^\delta \end{pmatrix} = \begin{pmatrix} \tau \\ \kappa \end{pmatrix} dt + \begin{pmatrix} \varrho & 0 \\ 0 & \frac{\alpha x}{\sqrt{1+x^2}} \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{pmatrix} \quad (6.24)$$

with now

$$\sigma_\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix} \quad \sigma_\delta(\sigma_\delta)^T = \begin{pmatrix} 0 & 0 \\ 0 & \delta^2 \end{pmatrix} \quad (6.25)$$

The matrix

$$\mathcal{A}_\delta = \begin{pmatrix} \varrho^2 & 0 \\ 0 & \frac{\alpha^2 x^2}{1+x^2} + \delta \end{pmatrix} \quad (6.26)$$

is locally positive definite in \mathbb{R}^2 . The generator of this process is

$$\mathcal{L}_\delta = -\frac{1}{2}\varrho^2\partial_{x^2}^2 - \frac{1}{2}\frac{\alpha^2 x^2}{1+x^2}\partial_{y^2}^2 - \frac{\delta^2}{2}\partial_{y^2}^2 - \tau(x)\partial_x - \kappa(x)\partial_y. \quad (6.27)$$

Therefore, using again the function $\bar{w}(x) = \{\frac{1}{2}\langle x, x \rangle\}^\mu$, we obtain

$$\begin{aligned} \mathcal{L}\bar{w} &= -\frac{1}{2}\varrho^2\partial_{x^2}^2\bar{w} - \frac{1}{2}\frac{\alpha^2 x^2}{1+x^2}\partial_{y^2}^2\bar{w} - \frac{\delta^2}{2}\partial_{y^2}^2\bar{w} - \tau(x)\partial_x\bar{w} - \kappa(x)\partial_y\bar{w} \\ &= \mu\left\{\frac{1}{2}|x|^2\right\}^{\mu-1} \left\{ -\frac{1}{2}\varrho^2 - \frac{1}{2}\left(\frac{\alpha^2 x^2}{1+x^2} - \delta^2\right) - b(x) \cdot x \right. \\ &\quad \left. - (\mu-1) \left[\varrho^2 x^2 + \frac{\alpha^2 x^2 y^2}{1+x^2} + \delta^2 \right] |x|^{-2} \right\}. \end{aligned}$$

The existence of an invariant probability measure m^δ for x^δ (for $\delta > 0$ small enough) follows from (6.22) and the argument in the previous example. Since

$$\mathcal{L}_\delta\bar{w} + \frac{\delta^2}{2}\partial_{y^2}^2\bar{w} \geq \nu|x|^{2(\mu-1)},$$

for $|x|$ large, for some $\nu > 0$, uniformly in $\delta > 0$, we find for some $C \geq 0$ independent of δ ,

$$\int_{\mathbb{R}^N} m^\delta(x)|x|^{2(\mu-1)} dx \leq C. \quad (6.28)$$

Once this is done, extracting a subsequence if necessary and letting δ go to 0^+ , we obtain an invariant probability measure m for x_t . Since m is a stationary solution of the following parabolic equation:

$$\frac{\partial v}{\partial t} - \frac{1}{2}\varrho^2\partial_{x^2}^2 v - \frac{1}{2}\frac{\alpha^2 x^2}{1+x^2}\partial_{y^2}^2 v - \frac{\delta^2}{2}\partial_{y^2}^2 v + \tau(x)\partial_x + \kappa(x)\partial_y v = 0,$$

we deduce the smoothness and strict positivity of m , thereby finishing the proof. \blacksquare

EXAMPLE 7.

Consider the following multi-dimensional stochastic differential equation:

$$dx_t = \sigma(x_t)dW_t + b(x)dt \quad (6.29)$$

where $x \in \mathbb{R}^d$, W is a standard m -dimensional Brownian motion for some $m \leq d$ and σ is a fixed matrix in $\mathbb{R}^{d \times m}$.

Proposition 6.7. *The condition of Proposition 3.11 is satisfied in Example 7.*

Proof. Define the infinitesimal generator of the process x_t to be the differential operator defined (1.5). The operator \mathcal{L} is no more elliptic but still hypoelliptic. In order to use our theory, we introduce x_t^δ solving

$$dx_t^\delta = \sigma(x_t^\delta)dW_t + \delta dB_t + (b(x_t^\delta) + \delta^2 Y(x_t^\delta))dt \quad (6.30)$$

with B_t a Brownian motion independent of W_t and

$$Y(x) = -\varphi(x)\nabla \cdot \quad (6.31)$$

with a function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^+$. The generator of (6.30) reads:

$$\mathcal{L}_\delta := \frac{\sigma^2(x)}{2}\Delta_x + \frac{\delta^2}{2}\Delta_y + b \cdot \nabla + \delta^2 Y(x)\nabla \cdot \quad (6.32)$$

We now make assumption concerning $Y(x)$ that combined with the ASSUMPTION 3.6 induces ergodicity. Let $\bar{w} \in \mathcal{C}^2(\mathbb{R}^d)$ be solution of the ASSUMPTION 3.6. There exists a function $\varphi \geq 0$ and $\varphi \in \mathcal{C}(\mathbb{R}^d)$,

$$Y(x) = -\varphi(x)\nabla \bar{w}, \quad (6.33)$$

such that, for x sufficiently large, one has the inequality

$$\varphi \geq \frac{1}{2} \frac{\Delta \bar{w}}{|\nabla \bar{w}|^2}.$$

Clearly, arguing in Example 6, we find that ASSUMPTION 3.6 is satisfied uniformly in δ and a little computation reveals that

$$\mathcal{L}\bar{w} - \frac{\delta^2}{2}\Delta \bar{w} - \delta^2 \beta(x) \cdot \nabla \bar{w} = \mathcal{L}\bar{w} + \delta^2 \left(-\frac{1}{2}\Delta \bar{w} + \varphi(x)|\nabla \bar{w}|^2 \right) \geq \mathcal{L}\bar{w} \quad (6.34)$$

as required. ■

7. CONCLUSIONS AND PERSPECTIVES

The present paper has been devoted to develop criteria based on stochastic Lyapunov technique in order to establish sufficient conditions for the existence of invariant probability measures for multidimensional diffusion process. The existence and uniqueness of invariant measures investigated in the pertinent literature by other researchers, such as [10, 21, 17], is different from the analysis in this paper considering the different notion of invariant measures. Indeed usually the invariant measures is defined in terms of the transition function of a particular version of an associated diffusion process which in turn is constructed by a Girsanov transformation.

The main novelty of the present paper is that by interpreting the first time of reaching of a system underling in some closed ball we obtain new criteria that have enabled us to prove the existence and uniqueness of invariant measures for stochastic processes with degenerate diffusions in the whole space \mathbb{R}^d . The proof strategy leading to our results is also not standard: we propose an approach that combines the use of Lyapunov function techniques with an approximation of the problem on bounded sets with reflection of the diffusion at the boundary. To the best of our knowledge, the assumptions used to obtain the invariant measure are missing in many papers. In the pertinent literature, the existence is postulated by using probabilistic arguments. The present framework and result allows to clarify the notion of ergodicity in the whole space.

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