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On Pythagorean Triples

César Aguilera.

June 9, 2019

Abstract

We talk about pythagorean triples and their different representations, we permute these representations to form groups, we classify these groups with a quotient and find that this quotient is related to NSW numbers and primes of the form $2x^2 - y^2$, then we talk about some properties related to all groups, including a pythagorean triples generator, also we talk about linearity, recursion and periodicity, we finally talk about two specific groups and give some identities.

Introduction.

A Pythagorean Triple, is a triple of positive integers a, b, c such that $a^2 + b^2 = c^2$. From a geometrical point of view, a, b, c can conform a right triangle where a and b are it's legs and c the hypotenuse, and so, from pythagorean theorem, $a^2 + b^2 = c^2$.

The Pythagorean Triples are called primitives when the triple a, b, c does not share common divisors, the hypotenuses of this primitives can be composite or not, we will start with composite ones and find their different representaions.

Pythagorean Triples and their Different Representations.

The first sixteen Composite Hypotenuses of Primitive Pythagorean Triples are: 25,65,85,125,145,169,185,205,221,265,289,305,325,365,377,425.[1]

Table 1: Different representations of composite hypotenuses of primitive Pythagorean Triples.

Hypotenuse	Representations
25	$\left\{ \begin{matrix} 24 \\ 7 \end{matrix} \right\} \left\{ \begin{matrix} 20 \\ 15 \end{matrix} \right\}$
65	$\left\{ \begin{matrix} 63 \\ 16 \end{matrix} \right\} \left\{ \begin{matrix} 60 \\ 25 \end{matrix} \right\} \left\{ \begin{matrix} 56 \\ 33 \end{matrix} \right\} \left\{ \begin{matrix} 52 \\ 39 \end{matrix} \right\}$
85	$\left\{ \begin{matrix} 84 \\ 13 \end{matrix} \right\} \left\{ \begin{matrix} 77 \\ 36 \end{matrix} \right\} \left\{ \begin{matrix} 75 \\ 40 \end{matrix} \right\} \left\{ \begin{matrix} 68 \\ 51 \end{matrix} \right\}$
125	$\left\{ \begin{matrix} 120 \\ 35 \end{matrix} \right\} \left\{ \begin{matrix} 117 \\ 44 \end{matrix} \right\} \left\{ \begin{matrix} 100 \\ 75 \end{matrix} \right\}$
145	$\left\{ \begin{matrix} 144 \\ 17 \end{matrix} \right\} \left\{ \begin{matrix} 143 \\ 24 \end{matrix} \right\} \left\{ \begin{matrix} 116 \\ 87 \end{matrix} \right\} \left\{ \begin{matrix} 105 \\ 100 \end{matrix} \right\}$
169	$\left\{ \begin{matrix} 156 \\ 65 \end{matrix} \right\} \left\{ \begin{matrix} 120 \\ 119 \end{matrix} \right\}$
185	$\left\{ \begin{matrix} 176 \\ 57 \end{matrix} \right\} \left\{ \begin{matrix} 175 \\ 60 \end{matrix} \right\} \left\{ \begin{matrix} 153 \\ 104 \end{matrix} \right\} \left\{ \begin{matrix} 148 \\ 111 \end{matrix} \right\}$
205	$\left\{ \begin{matrix} 200 \\ 45 \end{matrix} \right\} \left\{ \begin{matrix} 187 \\ 84 \end{matrix} \right\} \left\{ \begin{matrix} 164 \\ 123 \end{matrix} \right\} \left\{ \begin{matrix} 156 \\ 133 \end{matrix} \right\}$
221	$\left\{ \begin{matrix} 220 \\ 21 \end{matrix} \right\} \left\{ \begin{matrix} 204 \\ 85 \end{matrix} \right\} \left\{ \begin{matrix} 195 \\ 104 \end{matrix} \right\} \left\{ \begin{matrix} 171 \\ 140 \end{matrix} \right\}$
265	$\left\{ \begin{matrix} 264 \\ 23 \end{matrix} \right\} \left\{ \begin{matrix} 247 \\ 96 \end{matrix} \right\} \left\{ \begin{matrix} 225 \\ 140 \end{matrix} \right\} \left\{ \begin{matrix} 212 \\ 159 \end{matrix} \right\}$
289	$\left\{ \begin{matrix} 255 \\ 136 \end{matrix} \right\} \left\{ \begin{matrix} 240 \\ 161 \end{matrix} \right\}$
305	$\left\{ \begin{matrix} 300 \\ 55 \end{matrix} \right\} \left\{ \begin{matrix} 273 \\ 136 \end{matrix} \right\} \left\{ \begin{matrix} 244 \\ 183 \end{matrix} \right\} \left\{ \begin{matrix} 224 \\ 207 \end{matrix} \right\}$
325	$\left\{ \begin{matrix} 323 \\ 36 \end{matrix} \right\} \left\{ \begin{matrix} 315 \\ 80 \end{matrix} \right\} \left\{ \begin{matrix} 312 \\ 91 \end{matrix} \right\} \left\{ \begin{matrix} 300 \\ 125 \end{matrix} \right\} \left\{ \begin{matrix} 280 \\ 165 \end{matrix} \right\} \left\{ \begin{matrix} 260 \\ 195 \end{matrix} \right\} \left\{ \begin{matrix} 253 \\ 204 \end{matrix} \right\}$
365	$\left\{ \begin{matrix} 364 \\ 27 \end{matrix} \right\} \left\{ \begin{matrix} 357 \\ 76 \end{matrix} \right\} \left\{ \begin{matrix} 292 \\ 219 \end{matrix} \right\} \left\{ \begin{matrix} 275 \\ 240 \end{matrix} \right\}$
377	$\left\{ \begin{matrix} 352 \\ 135 \end{matrix} \right\} \left\{ \begin{matrix} 348 \\ 145 \end{matrix} \right\} \left\{ \begin{matrix} 345 \\ 152 \end{matrix} \right\} \left\{ \begin{matrix} 273 \\ 260 \end{matrix} \right\}$
425	$\left\{ \begin{matrix} 420 \\ 65 \end{matrix} \right\} \left\{ \begin{matrix} 416 \\ 87 \end{matrix} \right\} \left\{ \begin{matrix} 408 \\ 119 \end{matrix} \right\} \left\{ \begin{matrix} 385 \\ 180 \end{matrix} \right\} \left\{ \begin{matrix} 375 \\ 200 \end{matrix} \right\} \left\{ \begin{matrix} 340 \\ 255 \end{matrix} \right\} \left\{ \begin{matrix} 304 \\ 297 \end{matrix} \right\}$

Permutations of Representations of Primitive Pythagorean Triples.

Given two different representations with their respective hypotenuses h and k.

$$\begin{Bmatrix} a \\ b \end{Bmatrix} h \begin{Bmatrix} c \\ d \end{Bmatrix} k$$

If $\sin[\arctan(\frac{a}{b})]k$ is integer, then:

$$\sin[\arctan(\frac{a}{b})]k = c \text{ and } \sin[\arctan(\frac{c}{d})]h = a$$

$$\cos[\arctan(\frac{a}{b})]k = d \text{ and } \cos[\arctan(\frac{c}{d})]h = b$$

$$\arctan(\frac{a}{b}) = \arctan(\frac{c}{d})$$

$$\text{Det} \begin{vmatrix} a & c \\ b & d \end{vmatrix} = 0.$$

Example:

Given the representation $\begin{Bmatrix} 20 \\ 15 \end{Bmatrix}$ such that $20^2 + 15^2 = 25^2$ and the set of composite hypotenuses of pythagorean triples, we have:

$$\sin[\arctan \frac{20}{15}]25 = 20 ; \cos[\arctan \frac{20}{15}]25 = 15$$

$$\sin[\arctan \frac{20}{15}]65 = 52 ; \cos[\arctan \frac{20}{15}]65 = 39$$

$$\sin[\arctan \frac{52}{39}]85 = 68 ; \cos[\arctan \frac{52}{39}]85 = 51$$

$$\sin[\arctan \frac{68}{51}]125 = 100 ; \cos[\arctan \frac{68}{51}]125 = 75$$

$$\sin[\arctan \frac{100}{75}]145 = 116 ; \cos[\arctan \frac{100}{75}]145 = 87$$

.....

they form the group of representations:

$$\begin{Bmatrix} 20 \\ 15 \end{Bmatrix} \begin{Bmatrix} 52 \\ 39 \end{Bmatrix} \begin{Bmatrix} 68 \\ 51 \end{Bmatrix} \begin{Bmatrix} 100 \\ 75 \end{Bmatrix} \begin{Bmatrix} 116 \\ 87 \end{Bmatrix} \begin{Bmatrix} 148 \\ 111 \end{Bmatrix} \begin{Bmatrix} 164 \\ 123 \end{Bmatrix} \begin{Bmatrix} 212 \\ 159 \end{Bmatrix} \begin{Bmatrix} 244 \\ 183 \end{Bmatrix} \begin{Bmatrix} 260 \\ 195 \end{Bmatrix} \begin{Bmatrix} 292 \\ 219 \end{Bmatrix} \begin{Bmatrix} 340 \\ 255 \end{Bmatrix} \begin{Bmatrix} 356 \\ 267 \end{Bmatrix}$$

Also: given the representation $\begin{Bmatrix} 24 \\ 7 \end{Bmatrix}$ such that $24^2 + 7^2 = 25^2$.

they form the group of representations:

$$\begin{Bmatrix} 24 \\ 7 \end{Bmatrix} \begin{Bmatrix} 120 \\ 35 \end{Bmatrix} \begin{Bmatrix} 312 \\ 91 \end{Bmatrix} \begin{Bmatrix} 408 \\ 119 \end{Bmatrix} \begin{Bmatrix} 600 \\ 175 \end{Bmatrix} \begin{Bmatrix} 696 \\ 203 \end{Bmatrix} \begin{Bmatrix} 888 \\ 259 \end{Bmatrix} \begin{Bmatrix} 984 \\ 287 \end{Bmatrix} \begin{Bmatrix} 1176 \\ 343 \end{Bmatrix} \begin{Bmatrix} 1272 \\ 371 \end{Bmatrix} \begin{Bmatrix} 1464 \\ 427 \end{Bmatrix} \begin{Bmatrix} 1560 \\ 455 \end{Bmatrix} \begin{Bmatrix} 1752 \\ 511 \end{Bmatrix}$$

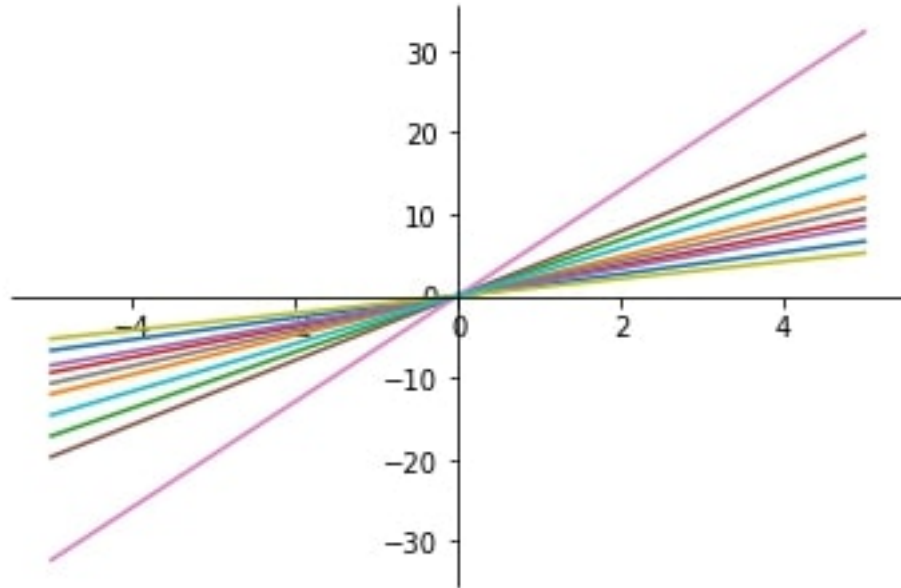


Figure 1: Groups of representations of pythagorean triples

Quotients of Groups of Representations.

Given a group G and a representation of this group $\left\{\frac{a}{b}\right\}$, we have a quotient $q = \frac{a+b}{a-b} = \frac{l}{m}$ where (l) or (m) or both are primes of the form $2x^2 - y^2$.

Also, if (q) is an integer, then this integer is an NSW number.

Examples:

$$q = \frac{60+25}{60-25} = \frac{17}{7} = \left\{\begin{matrix} 60 & 156 \\ 25 & 65 \end{matrix}\right\} \text{ where 17 and 7 are primes of the form } x^2 + y^2.$$

$$q = \frac{200+45}{200-45} = \frac{49}{31} = \left\{\begin{matrix} 200 & 1000 \\ 45 & 225 \end{matrix}\right\} \text{ where 31 is a prime of the form } x^2 + y^2.$$

$$q = \frac{120+119}{120-119} = 239 = \left\{\begin{matrix} 120 & 600 \\ 119 & 595 \end{matrix}\right\} \text{ where 239 is an NSW number.}$$

So, if we have two different representations $\left\{\frac{a}{b}\right\}$ and $\left\{\frac{a'}{b'}\right\}$ such that $\frac{a+b}{a-b} = q$

and $\frac{a'+b'}{a'-b'} = q'$ where $q=q'$ then $\text{Det}\left\|\begin{matrix} a & a' \\ b & b' \end{matrix}\right\| = 0$.

Properties of Groups of Representations.

Every Group of Representations have a series of properties that we list above.

Linearity.

Given any two different representations $\begin{Bmatrix} a \\ b \end{Bmatrix}$ and $\begin{Bmatrix} a' \\ b' \end{Bmatrix}$ then:

$$\text{Det} \begin{vmatrix} a & a' \\ b & b' \end{vmatrix} = 0$$

Recursion.

Every Group of Representations is a Pythagorean Triples Generator.

Given three consecutive representations: $\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}$ $\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix}$ $\begin{Bmatrix} a_3 \\ b_3 \end{Bmatrix}$ \dots $\begin{Bmatrix} a_k \\ b_k \end{Bmatrix}$

we have

$$a_k + a_{k-1} - a_{k-2} = a_{k+1}$$

$$b_k + b_{k-1} - b_{k-2} = b_{k+1}$$

Example:

Given the Group of Representations with quotient $(\frac{47}{23})$ and the three first

representations $\begin{Bmatrix} 175 \\ 60 \end{Bmatrix}$ $\begin{Bmatrix} 455 \\ 156 \end{Bmatrix}$ $\begin{Bmatrix} 595 \\ 204 \end{Bmatrix}$ then:

$$595 + 455 - 175 = 875$$

$$204 + 156 - 60 = 300$$

and they form the group:

$$\begin{Bmatrix} 175 \\ 60 \end{Bmatrix} \begin{Bmatrix} 455 \\ 156 \end{Bmatrix} \begin{Bmatrix} 595 \\ 204 \end{Bmatrix} \begin{Bmatrix} 875 \\ 300 \end{Bmatrix}$$

where

$$\text{Det} \begin{vmatrix} 175 & 875 \\ 60 & 300 \end{vmatrix} = 0$$

Periodicity.

Given a Group of Representations:

$$\left\{ \begin{matrix} a_k \\ b_k \end{matrix} \right\} \left\{ \begin{matrix} a_{k+1} \\ b_{k+1} \end{matrix} \right\} \left\{ \begin{matrix} a_{k+2} \\ b_{k+2} \end{matrix} \right\} \left\{ \begin{matrix} a_{k+3} \\ b_{k+3} \end{matrix} \right\} \left\{ \begin{matrix} a_{k+4} \\ b_{k+4} \end{matrix} \right\} \dots \dots \left\{ \begin{matrix} a_{k+n} \\ b_{k+n} \end{matrix} \right\}$$

then

$$[b_{k+1} - b_k = b_{k+3} - b_{k+2}] \text{ and } [b_{k+2} - b_{k+1} = b_{k+4} - b_{k+3}] [2]$$

it follows that

$$\text{Det} \left\| \begin{matrix} a_{k+1} - a_k & a_{k+2} - a_{k+1} \\ b_{k+1} - b_k & b_{k+2} - b_{k+1} \end{matrix} \right\| = 0$$

Example:

Given the Group of Representations with quotient $(\frac{23}{7})$ and the four first representations $\left\{ \begin{matrix} 75 \\ 40 \end{matrix} \right\} \left\{ \begin{matrix} 195 \\ 104 \end{matrix} \right\} \left\{ \begin{matrix} 255 \\ 136 \end{matrix} \right\} \left\{ \begin{matrix} 375 \\ 200 \end{matrix} \right\}$ then:

$$195-75=120 \text{ and } 255-195=60$$

$$104-40=64 \text{ and } 136-104=32$$

and the group have a periiodicity of:

$$120, 60, 120, 60$$

$$64, 32, 64, 32$$

it follows that:

$$\text{Det} \left\| \begin{matrix} 120 & 60 \\ 64 & 32 \end{matrix} \right\| = \text{Det} \left\| \begin{matrix} 2 & 1 \\ 2 & 1 \end{matrix} \right\| = 0$$

In particular, if quotient (q) is an NSW number:

$$\text{Given a Group of Representations } \left\{ \begin{matrix} a_k \\ b_k \end{matrix} \right\} \left\{ \begin{matrix} a_{k+1} \\ b_{k+1} \end{matrix} \right\} \left\{ \begin{matrix} a_{k+2} \\ b_{k+2} \end{matrix} \right\} \dots \left\{ \begin{matrix} a_{k+n} \\ b_{k+n} \end{matrix} \right\}$$

$$\text{and given the equation } 2x^2 = (NSW)^2 + 1$$

we have

$$\frac{\sqrt{(a_{k+1})^2 + (b_{k+1})^2} - \sqrt{(a_k)^2 + (b_k)^2}}{x} = \frac{\sqrt{(a_{k+3})^2 + (b_{k+3})^2} - \sqrt{(a_{k+2})^2 + (b_{k+2})^2}}{x}$$

and

$$\frac{\sqrt{(a_{k+2})^2 + (b_{k+2})^2} - \sqrt{(a_{k+1})^2 + (b_{k+1})^2}}{x} = \frac{\sqrt{(a_{k+4})^2 + (b_{k+4})^2} - \sqrt{(a_{k+3})^2 + (b_{k+3})^2}}{x}$$

Example:

$$2x^2 = (NSW)^2 + 1$$

$$2 \cdot 29^2 = 41^2 + 1$$

Given the Group of Representations with $q=(41)$

$$\left\{ \begin{matrix} 105 \\ 100 \end{matrix} \right\} \left\{ \begin{matrix} 273 \\ 260 \end{matrix} \right\} \left\{ \begin{matrix} 525 \\ 500 \end{matrix} \right\} \left\{ \begin{matrix} 693 \\ 660 \end{matrix} \right\} \left\{ \begin{matrix} 945 \\ 900 \end{matrix} \right\} \cdots \left\{ \begin{matrix} a_n \\ b_n \end{matrix} \right\}$$

We have:

$$\frac{\sqrt{273^2+260^2}-\sqrt{105^2+100^2}}{29} = \frac{\sqrt{693^2+660^2}-\sqrt{525^2+500^2}}{29}$$

$$\frac{377-145}{29} = \frac{957-725}{29} = \frac{232}{29} = 8$$

and

$$\frac{\sqrt{525^2+500^2}-\sqrt{273^2+260^2}}{29} = \frac{\sqrt{945^2+900^2}-\sqrt{693^2+660^2}}{29}$$

$$\frac{725-377}{29} = \frac{1305-957}{29} = \frac{348}{29} = 12$$

periodicity is:

8,12,8,12

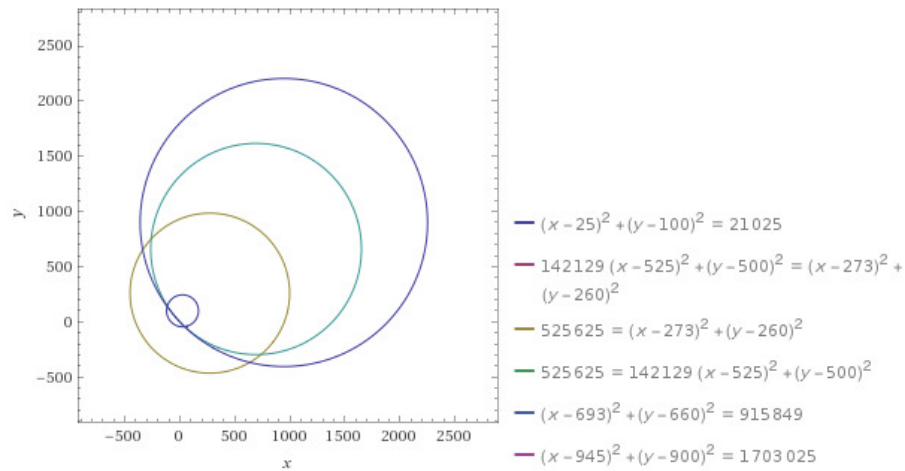


Figure 2: Circles with center at $q=41$.

Groups of Representations $q = 7$ and $q = \frac{31}{17}$.

Given the two Groups of Representations:

$$q = 7 : \left\{ \begin{matrix} a_1 \\ b_1 \end{matrix} \right\} \left\{ \begin{matrix} a_2 \\ b_2 \end{matrix} \right\} \left\{ \begin{matrix} a_3 \\ b_3 \end{matrix} \right\} \left\{ \begin{matrix} a_4 \\ b_4 \end{matrix} \right\} \cdots \left\{ \begin{matrix} a_n \\ b_n \end{matrix} \right\}$$

and

$$q = \frac{31}{17} : \left\{ \begin{matrix} c_1 \\ d_1 \end{matrix} \right\} \left\{ \begin{matrix} c_2 \\ d_2 \end{matrix} \right\} \left\{ \begin{matrix} c_3 \\ d_3 \end{matrix} \right\} \left\{ \begin{matrix} c_4 \\ d_4 \end{matrix} \right\} \cdots \left\{ \begin{matrix} c_n \\ d_n \end{matrix} \right\}$$

we have

$$a_1 - b_1 = \frac{c_2}{c_1}; a_2 - b_2 = \frac{c_3}{c_1}; a_3 - b_3 = \frac{c_4}{c_1} \cdots a_n - b_n = \frac{c_{n+1}}{c_1}$$

Given any two representations of the groups with $q = 7 = \frac{a}{b}$ and $q = \frac{31}{17} = \frac{c}{d}$ with (k) being any composite hypotenuse of a pythagorean triple.

we have

$$\sin[\arctan(\frac{a}{b}) + \arctan(\frac{c}{d})]k = \sin[\arctan(\frac{a}{b})]k$$

Given the two Groups of Representations:

$$q = 7 = \left\{ \begin{matrix} 20 \\ 15 \end{matrix} \right\} \left\{ \begin{matrix} 52 \\ 39 \end{matrix} \right\} \left\{ \begin{matrix} 68 \\ 51 \end{matrix} \right\} \left\{ \begin{matrix} 100 \\ 75 \end{matrix} \right\} \cdots \left\{ \begin{matrix} a_n \\ b_n \end{matrix} \right\}$$

and

$$q = \frac{31}{17} = \left\{ \begin{matrix} 24 \\ 7 \end{matrix} \right\} \left\{ \begin{matrix} 120 \\ 35 \end{matrix} \right\} \left\{ \begin{matrix} 312 \\ 91 \end{matrix} \right\} \left\{ \begin{matrix} 408 \\ 119 \end{matrix} \right\} \cdots \left\{ \begin{matrix} a_n \\ b_n \end{matrix} \right\}$$

then

$$\cos[\arctan(\frac{20}{15}) + \arctan(\frac{24}{7})]25 = -\cos[\arctan(\frac{20}{15})]25$$

$$\sin[\arctan(\frac{20}{15}) + \arctan(\frac{24}{7})]25 = \sin[\arctan(\frac{20}{15})]25$$

it follows that

$$2 \cdot \arctan(\frac{4}{3}) + \arctan(\frac{24}{7}) = \pi$$

therefore:

Given any triangle with sides x,y,z.

$$\arctan(\frac{4x}{3x}) + \arctan(\frac{4y}{3y}) + \arctan(\frac{24z}{7z}) = \pi$$

References

- [1] *OEIS Foundation Inc. (2019), The On-Line Encyclopedia of Integer Sequences, <http://oeis.org/A120961>.*
- [2] *Kzimirz Kuratowski. Introduction To Set Theory and Topology. Ordering Relations. Similarity. Order Types. ADDISON-WESLEY PUBLISHING COMPANY INC. Reading, Massachusetts, U.S.A. 1962.*