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EXISTENCE OF ENTROPY SOLUTIONS FOR MULTIDIMENSIONAL CONSERVATION LAWS WITH L^1 BOUNDARY CONDITIONS

CHRISTIAN DOGBE AND C. BIANCA

ABSTRACT. This paper deals with the construction of nonlinear boundary conditions for multidimensional conservation laws. Specifically by introducing a new type of entropy solution matching the boundary condition, the existence and uniqueness of a solution belonging to $L^\infty \cap BV$ is proved by using the Di Perna-Lions regularization method. The new entropy solution, which takes advantage by the entropy criterion introduced by Bardos-Le Roux-Nédélec for first-order quasilinear equations with boundary conditions, is based on a weaker assumption at the boundary.

1. INTRODUCTION

This paper is concerned with the definition and analysis of suitable boundary conditions for the following initial problem based on the scalar conservation law:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + \operatorname{div}_x f(x, u(x, t)) = 0 & t \geq 0, \quad x \in \Omega \\ u(x, 0) = u_0(x) \end{cases} \quad (1.1)$$

where $\Omega = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 < 0, \quad x' = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}\}$, for $d \geq 1$.

The main feature of (1.1) is the discontinuity of the flux function f in the space variable x , which yields new important difficulties in the analysis of conservation laws. Several different entropy conditions have been suggested, see among others [2, 21, 11]. Specifically it is required that the solution fulfils an additional condition on its traces at the discontinuous points of the flux function, thus requiring the existence of traces of the solution.

A well-studied boundary condition for (1.1) is the following Dirichlet type boundary condition on $\bar{\Omega}$:

$$u = \mathbf{w}(x') \quad x_1 = 0, \quad x' \in \mathbb{R}^{d-1}. \quad (1.2)$$

In what follows it is assumed that f does not depend explicitly on x , namely $f(u(x, t))$, then the usual assumptions (for the existence of L^∞ -bounds) on the divergence with respect to x are not taken into account. Bearing all above in mind the interest of this paper is focused on the following problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + \operatorname{div}_x f(u(x, t)) = 0, & x_1 < 0 \\ u(x, 0) = u_0(x) \\ u|_{\partial\Omega} = \mathbf{w}(x'). \end{cases} \quad (1.3)$$

The interest in the above defined problem is related to applications and specifically to nonlinear flow problems. Indeed many physical problems are governed by such equations, e.g. fluid mechanics and nonlinear electromagnetism (see [9], chap. 12), meteorology [22], transport of oil [17] and gas in porous media [6], interfaces (where two different equations are defined two distinct regions of space) and junction problems (where different branches join at a point, and an equation on each branch is defined, see [20, 19] and the references cited therein), traffic flow on networks [7], pedestrian dynamics models [12, 1]. The interest

in the problem investigated in the present paper and the related convergence results play also an important role in the numerical analysis of hyperbolic conservation laws [26].

The main reference for the existence and uniqueness results for entropy admissible solutions of the above system is the classical paper by (BLN) Bardos, Le Roux and Nédélec [2] which extends the analysis by Kruzhkov to initial-boundary value problems. The problem has been firstly analyzed in [2] in order to prove that if the initial datum u_0 belongs to BV and the boundary datum is C^2 -regular, then there exists a unique (weak entropy) solution of (1.3).

Recently Lions [18] has been interested in how to manage the boundary in a completely transparent way, without assumption of regularity and using only $W^{1,1}$. In particular the question is how to introduce the boundary in the language of viscosity solutions. Actually, what happens at the boundary will depend mainly on the signs. In Lions's approach, the Kruzhkov's entropy inequality is written on the closure of the set, namely multiplying by test functions and by integrating by parts on the closure of the domain, and a boundary term $f(k)$ on the boundary for some $k \in \mathbb{R}$ is recovered. The main novelty of Lions's approach compared to BLN condition is not only the very simple rewriting of the boundary conditions via an L^1 -extension but also the use of the arguments of the viscosity solutions. This paper follows the ideas of [18] and investigates the solvability of (1.1) in the L^1 -framework.

The paper is organized as follows. Section 2 deals with preliminary results concerning the formulation of this kind of boundary value problem and the notions of entropy sub- and supersolutions are given in order to establish the forthcoming results for the problem (1.3). There are two obvious possibilities to match the solution of the problem (1.3). We will present subsequently a third alternative, giving a well-posed result of the problem. In Section 3 we establish the required estimates of solution u_ε of the regularized problem (3.1). On the basis of these estimates, a wide range of results are established in Section 4. In particular, we prove the L^1 -contractivity. Finally Section 5 presents some applications.

2. PRELIMINARIES AND BACKGROUND

This section is devoted to a short overview of some of the notation used throughout the present paper.

Let $a, b \in \mathbb{R}$, then

$$a \wedge b = \min(a, b), a \vee b = \max(a, b), u^+ = \max(u, 0), u^- = \max(-u, 0),$$

and $\mathbf{sign}^+(u) = (\mathbf{sign} u)^+$ (the Heaviside function), and $\mathbf{sign}^-(u) = -(\mathbf{sign} u)^-$. Moreover $\mathbf{1}_{u>k} = 1$ if $u > k$ and 0 if $u < k$, and $(u-k)^+ := \max(u-k, 0)$ and $(u-k)^- := \min(u-k, 0)$.

A function $f \in L^1(\Omega)$ is said to have *bounded variation* in the set Ω if $\int_\Omega |\mathbf{grad} f| dx < \infty$. The set of all functions in $L^1(\Omega)$ with bounded variation is denoted by $\mathbf{BV}(\Omega)$. It is well known that BV functions are the weakest functions which have the traces. If $u \in \mathbf{BV}(]a, b[; \mathbb{R}^d)$, then the following left and right limits exist at every point $x \in]a, b[$:

$$u(x_-) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} u(y) dy, \quad u(x_+) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} u(y) dy. \quad (2.1)$$

In fact, $u(x_-)$ is well defined also in $x = b$, while $u(x_+)$ exists also in $x = a$. The left and right limits coincide *a.e.* with u and are left and right continuous, respectively.

Let $T > 0$, and $\Omega_T = \Omega \times]0, T]$. The Sobolev space $W^{1,1}(\Omega_T)$ will be considered, whose inclusion in $L^1(\Omega_T)$ is relatively compact. Moreover, $W^{1,1}(\Omega_T) \subseteq \mathbf{BV}(\Omega_T)$. In particular if $\Omega = \mathbb{R}^d \times (0, \infty)$ then $\mathbf{tr} W^{1,1}(\Omega) = L^1(\mathbb{R}^d)$ (see Gagliardo [14]). The reader interested to the notion of entropy solution to scalar conservation laws is referred to Kruzhkov [15] and Kruzhkov and Bénilan [4, 3].

2.1. A brief review of the pertinent literature. This section is devoted to a brief review on the analysis of the following problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) + \operatorname{div}_x f(x, u(x, t)) = 0, & x \in \Omega, \\ u(x, 0) = u_0(x), \\ u|_{\partial\Omega} = 0. \end{cases} \quad (2.2)$$

The existence and uniqueness of the solution result for the problem (1.3) depends on the formulation of the boundary condition (at the point $x = 0$) and on the functional space to which a solution of (2.2) belongs. The boundary condition is necessarily linked to the entropy condition. Thus the following question arises: What is the appropriate definition of an entropy solution for Eq. (2.2) on bounded domains with L^∞ or $W^{1,1}$ data? The papers of Bardos *et al.* [2], Otto [21] and the lecture of Lions [18] are the main references.

Bardos-Le Roux-Nédélec (BLN) boundary conditions (1979, [2]). Let Ω be a smooth bounded domain of \mathbb{R}^d , and $\mathbf{n}(x)$ the outward unit normal vector at $x \in \partial\Omega$. The initial value is $u_0 \in L^\infty(\mathbb{R}^d)$ and the boundary value $u^b \in L^\infty(\partial\Omega \times]0, +\infty[)$ (the boundary condition are “active” only on a part of the boundary). The following strong boundary condition has been introduced by Bardos, Le Roux & Nedelec in [2]:

$$\operatorname{sgn}(u(\bar{x}, t) - u^b(\bar{x}, t))(f(u(\bar{x}, t)) \cdot \mathbf{n}(\bar{x}) - f(k) \cdot \mathbf{n}(\bar{x})) \geq 0 \quad (2.3)$$

$$\text{for all } k \text{ in the interval with endpoints } u(\bar{x}, x) \text{ and } u^b(\bar{x}, t), \quad (2.4)$$

for almost every boundary point $(\bar{x}, t) \in \partial\Omega \times]0, +\infty[$. It has been shown that if the initial datum u_0 belongs to BV and the boundary datum is C^2 -regular, then there exists a unique entropy solution $u \in \operatorname{BV}(\Omega \times]0, +\infty[)$ of (2.2) satisfying (2.3)-(2.4) on $\partial\Omega$ and $u(x, t = 0) = u_0$. Moreover the following entropy inequality has been established: For all $k \in \mathbb{R}$, and for any C^1 test function ψ with compact support in $\bar{\Omega} \times [0, T]$ one has:

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\Omega} \left[|u(x, t) - k| \partial_t \psi + \operatorname{sgn}(u(x, t) - k) \sum \{f_j(u(x, t)) - f_j(k)\} \partial_{x_j} \psi \right. \\ & \quad \left. - \operatorname{sgn}(u(x, t) - k) \sum \partial_{x_j} f_j(k) \psi \right] dx dt \\ & + \int_{\mathbb{R}_+} \int_{\partial\Omega} \operatorname{sgn}(k) [f(\gamma|_{\partial\Omega} u) - f(k)] \mathbf{n} \cdot \psi d\gamma dt \geq 0, \end{aligned} \quad (2.5)$$

where $d\gamma$ stands for the integration with respect to d -dimensional Lebesgue measure on the boundary of Ω .

It is worth noting that the BLN condition makes sense only if the solution u admits a trace on $\partial\Omega$. It is worth stressing that to assume that u belongs to BV allows to define the solution at the boundary, in which case $f(u) = 0$ and the derivative is a measure.

Some equivalent formulations of BLN boundary conditions have been proposed in the literature.

Otto’s boundary conditions (1996). Otto has generalized the BLN result without using the bounded variation of solutions. In order to define a setting for more general data (namely L^∞ data) a new definition has been given by Otto in [21]. The approach consists in asking the boundary conditions to hold in integral form by introducing appropriate boundary entropy-entropy flux pairs. Otto takes flux that are regulars, look at a half-space and put on the boundary a function $u = \varphi$ (where φ is given) which is just assumed to be in $L^1([0, T]; L^1(\mathbb{R}^{d-1}) \cap L^\infty(\mathbb{R}^{d-1}))$ (time plays no role). In the case of L^∞ data, the author proved the existence and the uniqueness of entropy solution. In the Otto boundary condition treatment, the set BV is not employed. However some information are stated in order to ensure that we have traces on the boundary, namely when $\operatorname{div}_x f(u)$ belongs to some space, this does not implies that u has a trace at the boundary, but it tells us that $f(u)$ has a

trace at the boundary, thus the convergence of $f(u)$ when x goes to the boundary. The information on a vector field gives information on its divergence and its normal component. In the concept of definition of Lions, this normal component will be denoted f_1 . Following Otto, let $u_0 \in L^\infty(\Omega)$. A function $u \in L^\infty(\Omega_T)$ is a weak entropy solution of problem (1.3), if and only if u satisfies

- for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ for an entropy-pair (η, q)

$$\int_{\Omega_T} \{\eta(u)\partial_t \varphi + q(u) \cdot \nabla \varphi\} dx dt \geq 0. \quad (2.6)$$

- For all $\varphi \in L^1(\partial\Omega)$, $\varphi \geq 0$ and boundary entropy-entropy flux pair (H, Q) , $Q = (Q_1, \dots, Q_n)$, where $H \in C^2(\mathbb{R}^2)$, $Q \in (C^2(\mathbb{R}^2))^n$

$$\text{ess} \lim_{s \rightarrow 0^-} \int_{\partial\Omega} Q(u(r + s\nu(r)), 0) \cdot \nu(r) \beta(r) dr \geq 0 \quad (2.7)$$

for all $\beta \in L^1(\partial\Omega)$, $\beta \geq 0$ almost everywhere, and all boundary entropy-entropy flux pair (H, Q) ;

- the initial condition $u_0 \in L^\infty(\Omega)$ in the sense

$$\text{ess} \lim_{t \rightarrow 0^+} \int_{\Omega} |u(x, t) - u_0(x)| dx = 0.$$

Otto's Definition (2.6)-(2.7) generalizes BLN Definition(2.3)-(2.5).

Dubois and LeFloch's boundary conditions (1988). The authors of [11] have introduced an equivalent condition (2.3)-(2.4) of BLN whenever the solution of (1.3) has strong trace on $\partial\Omega \times [0, T]$

$$[q(u) \cdot \mathbf{n}]^t - q(u^b) \cdot \mathbf{n} - \eta'(u^b)[(f(u) - f(u^b)) \cdot \mathbf{n}]^t \geq 0, \quad (2.8)$$

where q^t denotes the trace of q on $\partial\Omega \times [0, T]$, with convex function η and related entropy flux defined by $q' = f'\eta'$.

The present paper is interested to the following question: Can we give the Dirichlet homogeneous condition at the boundary for ensure the existence of the traditional trace on the boundary? The advantage is that one can use the theory borrowed from viscosity solutions (following Crandall-Lions approach [8]).

The BLN condition and alternative formulation: Lions (2016). The trick of Lions [18] is based on the use of Gagliardo's trace and extension theorem for BV-functions [14], which asserts that every function $f \in L^1(\mathbb{R}^d)$ is the trace of a function $u \in W^{1,1}(\mathbb{R}^d \times (0, \infty))$. The author escapes from the point of view Otto and gets rid of the boundary condition of Dirichlet (1.2) by working with a boundary condition which belongs to $L^1(\mathbb{R}^{d-1})$ through the use of an extension such that $\mathbf{w}(x') \in W^{1,1}(\mathbb{R}^{d-1})$. The advantage is to apply DiPerna-Lions's regularization methods [10] in a similar way to viscosity solutions. The idea of Lions is to work directly on the boundary of the domain itself. From there, the author defines conditions of entropy sub-and-supersolutions. Thus one has existence and uniqueness of solutions. To this end, he subtracts an extension $\Psi \in W^{1,1}(\cap L^\infty(\mathbb{R}^d))$ to get back to 0. These considerations suggest that the boundary conditions should be reformulated. By setting

$$\Psi|_{x_1=0}(x') = \mathbf{w}(x'), \quad x' \in \mathbb{R}^{d-1}, \quad (2.9)$$

one replaces the function $u(x, t)$ in Eq. (1.3) by a function $\tilde{u}(x, t)$, solution of the equation

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(x, t) + \text{div}_x f(\tilde{u}(x, t)) = 0 & x_1 < 0 \\ \tilde{u} = \mathbf{w}(x') & x_1 = 0, \quad x' \in \mathbb{R}^{d-1} \\ \mathbf{w} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \end{cases} \quad (2.10)$$

where \tilde{u} is defined by

$$\tilde{u} = \Psi + u, \quad \Psi|_{\partial\Omega} = \mathbf{w}. \quad (2.11)$$

We have now a little more regular flux; we can thus reduce the boundary condition to 0. Substituting (2.11) into (2.10), one therefore looks at the following new initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{i=1}^d \frac{\partial}{\partial x_i} (f_i(x, u)) = 0, & x \in \mathbb{R}^d \\ \tilde{u} = \Psi + u, \\ u|_{x_1=0}(x') = 0, & x' \in \mathbb{R}^{d-1} \end{cases} \quad (2.12)$$

with

$$f_i(x, z) = f_i(\Psi(x) + z) \quad \text{with} \quad \Psi \in W^{1,1}(\cap L^\infty(\mathbb{R}^d)). \quad (2.13)$$

Remark 2.1. *When we have changed our boundary condition to $x_1 = 0$, we created an x -dependency; hence, with the equation (2.12) we have introduced a x -dependency. The problem (2.12) is over-determinate since according to the outgoing and incoming characteristics, the problem is nonlinear. Therefore the problem needs to be further relaxed. In our approach, we will introduce a new kind of entropy solution to match the homogeneous boundary condition in (2.10). The advantage lies in the simplicity with the boundary conditions such that $w \in L^1(\mathbb{R}^{d-1})$ implies the existence of the function Ψ belongs to $L^1((0, T); W^{1,1}(\mathbb{R}^{d-1}))$ and such that $w \equiv 0$. One adapts the proofs without boundaries to the case with boundaries and obtains the same results. It is worth mentioning that the previous transformation introduced a x -dependence in the transport term which is $W^{1,1}(\mathbb{R}^d)$. Thus, following Di Perna-Lions's [10], we must impose that $f_i(x, z) \in W^{1,1}(\mathbb{R}^d; C(\mathbb{R}))$, that is*

$$D_x f(x, z) = f'(\Psi(x) + z) D\Psi \in L^1(\mathbb{R}^d; C(\mathbb{R})). \quad (2.14)$$

Therefore, since $D\Psi \in L^1$, we deduce that $f \in C^1(\mathbb{R}^d; C(\mathbb{R}))$. Thus one has a little bit regular flux and one can thus get rid of the boundary conditions to bring back it to $u = 0$.

We introduce a new type of entropy solution following the line of Lions [18] to match the above homogeneous boundary condition in (1.3). There are two ways to deal with the corresponding problem: The BV analysis technique and the L^1 -framework. Our task is to incorporate the boundary condition into the entropy conditions for the problem (1.3) in order to define an entropy subsolution of the equation in the open set $\{x_1 < 0\}$ (which is a local property) and on the closed half-space provided we take positive constants: $k \geq 0$. This means that the Dirichlet problem (1.3) can be understood in the sense of the following inequality:

$$\forall k \geq 0, \quad \frac{\partial}{\partial t} (u - k)^+ + \frac{\partial}{\partial x_i} \left[\{f_i(u) - f_i(k)\} \mathbf{1}_{(u > k)} \right] \leq 0, \quad (2.15)$$

in the sense of distributions for $x_1 < 0$, which is true both at the boundary and for $x_1 \leq 0$. In other words, one can perform an integration by parts, without caring about the boundary. Indeed, for $k > 0$, one has $(u - k)^+ = 0$ on $\partial\Omega$. No boundary condition is necessary. In other words, this means that what happens at the boundary will depend mainly on the signs. As discussed below, Lions's point of view integrates that of the Otto, but a way to prove comparison and uniqueness is to replace the traces of the functions with integrals on small volumes.

Equipped with these preliminaries, we now introduce an appropriate notion of entropy solution to the problem (1.3) as suggested in [18].

Definition 2.1. A function u is called an entropy subsolution of the problem (1.3) on the closure set of the half-space, if the following condition holds true:

$$\forall \varphi \geq 0, \quad \forall \varphi \in C_{compact}^\infty(\{x_1 \leq 0\}), \quad \forall k \geq 0, \\ \frac{d}{dt} \int (u - k)^+ \varphi - \sum_{i=1}^d \int (f_i(u) - f_i(k)) \mathbf{1}_{u > k} \cdot \nabla \varphi \leq 0. \quad (2.16)$$

A function u is called an entropy supersolution of the problem (1.3) on the closure set of the half-space, if the following condition holds true:

$$\forall \varphi \geq 0, \quad \forall \varphi \in C_{compact}^\infty(\{x_1 \leq 0\}), \quad \forall k \leq 0, \\ \frac{d}{dt} \int (u - k)^+ \varphi - \sum_{i=1}^d \int (f_i(u) - f_i(k)) \mathbf{1}_{u < k} \cdot \nabla \varphi \leq 0. \quad (2.17)$$

The function u is an entropy solution to (1.3) if it is both an entropy subsolution and an entropy supersolution.

Remark 2.2. The Definition 2.1 gives a method to construct a solution. Actually, if u is really zero at the boundary, for $k > 0$, then $(u - k)^+$ vanishes in neighborhood of the boundary and thus one does not see the boundary; therefore one could certainly integrate by parts more times; the distribution (2.16) or (2.17) is supported by the law of the boundary. The idea is to say that one cannot expect to have, in general, $u = 0$. The key point of the inequalities (2.16) and (2.17), is that we carried out an integration by parts in a closed half-plane without worrying about the boundary. There is no restriction on $\partial\Omega$. Outside this set, everything is zero with respect to x' or for $x_1 < 0$. This means that the inequalities (2.16) and (2.17) are written on the closure of the set. From Kruzhkov's Theorem, it is well known that the sup of two entropic sub-solutions is an entropic sub-solution; thus, thinking about an entropy subsolution means talking about $(u \vee k)$, and then taking $k > 0$, means that we really care about what's going on when $u > 0$.

Equivalent definition. It is well known that if u and v are entropy subsolutions of (1.3) without the boundary condition in a convex domain, then $\max(u, v)$ (respectively $\min(u, v)$) is also an entropy subsolution (respectively entropy supersolution). Observe that for every real k , one has

$$f(u \vee k) = (f(u) - f(k)) \mathbf{1}_{(u > k)} + f(k). \quad (2.18)$$

Thus, it is useful to reformulate (2.15) with $(u \vee k)$ in the following way:

$$\forall k \geq 0, \quad \frac{\partial}{\partial t} (u \vee k) + \partial_i [f_i(u \vee k)] - f_1(\mathbf{k}) \delta_0(x_1) \leq 0, \quad (x_1 \leq 0) \quad (2.19)$$

and in that case we recover $f_1(\mathbf{k}) \delta_0(x_1)$ as a boundary term coming from the fact that the constants are not solutions of (1.3). Actually, if $k > 0$, $u \wedge k = k$ in the neighborhood of the boundary, such that the flux is different to zero; this explains the presence of the term $f_1(k)$ in (2.19). Inequality (2.17) is equivalent to (2.19). Observe, however that inequality (2.19) does not really make sense because of the presence of the Dirac mass. We translate therefore (2.19) in the following way: $\forall \varphi \in C_{compact}^\infty(x_1 \leq 0), \forall k \geq 0$,

$$\frac{d}{dt} \int (u \vee k) \varphi - \int f_i(u \vee k) \cdot \nabla \varphi dx - f_1(k) \int_{\{x_1=0\}} \varphi(0, x') dx' \leq 0. \quad (2.20)$$

The relation (2.20) expresses the fact that the constants are not solving the problem with the boundary conditions.

In the rest of this paper we will be mostly interested in the functions u which are sub-and supersolutions of (1.3) in the domain Ω . We have two results (Theorem 3.1, Theorem 3.2)

depending on the regularity of the data. First we establish the convergence under lesser regularity on the data (see Hypothesis 2.3), then use it to prove the result for data with more regular hypothesis (see Hypothesis 2.4).

We now state our hypotheses.

Hypothesis 2.3.

- (1) The flux function f is assumed to be regular from \mathbb{R} to \mathbb{R}^d , say $f \in C^1(\mathbb{R})$ not necessary Lipschitzian, but locally a Lipschitz function.
- (2) Let $\mathfrak{w} \in L^1(\mathbb{R}^{d-1})$, and $\Psi \in W^{1,1}(\cap L^\infty(\mathbb{R}^d))$.
- (3) Let $u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Hypothesis 2.4.

- (1) Let $f \in C^1(\mathbb{R})$
- (2) Let $\mathfrak{w} \in L^1(\mathbb{R}^{d-1})$, and $\Psi \in W^{1,1}(\cap L^\infty(\mathbb{R}^d))$.
- (3) Let $u_0 \in \text{BV}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

The third part of hypothesis 2.4 ensures that u has its trace γu at almost all of the boundary point of Ω_T and $\gamma u \in L^\infty$. By the trace γu at a boundary point we mean the limit of u at this point taking along the normal.

3. THE VANISHING VISCOSITY METHOD

This section deals with the existence solution result of the problem (1.3) when the initial condition belongs to $\text{BV}(\mathbb{R}^d)$. The passage from $u_0 \in \text{BV}(\mathbb{R}^d)$ to $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ can be obtained by employing a result of comparison, uniqueness, and contraction in L^1 . As it is well known, the space $\text{BV}(\mathbb{R}^d)$ does not entail necessarily $L^\infty(\mathbb{R}^d)$ but it contains $L^1(\mathbb{R}^d)$. In order to established an existence solution result, the vanishing viscosity method is employed. The construction of the entropy solution of (1.3) is based on the classical vanishing-viscosity method. Accordingly, for any positive real ε , the following viscous problem related to (1.3) is introduced:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + \text{div}_x f(u_\varepsilon) - \varepsilon \Delta u_\varepsilon = 0 & x_1 < 0, \quad x' \in \mathbb{R}^{d-1} \\ u_\varepsilon(x, 0) = u_0(x) \\ u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \\ u_\varepsilon|_{\partial\Omega} = 0. \end{cases} \quad (3.1)$$

The existence of a function u solution of (1.3) will be provided by taking the limit on ε in Eq. (3.1). It is worth observing that the over-determination is disappeared and the solution is regular.

3.1. The entropy solution and the main result for a $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ -data. This subsection is devoted to the existence of a weak solution of the problem (1.3).

Theorem 3.1. *Assume that f, u_0 and satisfy Hypothesis 2.3. Then the sequence of solutions of (3.1) converges a.e., as $\varepsilon \rightarrow 0$, to the unique entropy solution of (1.3).*

Remark 3.1. Theorem 3.1 is proved by showing that the sequence $\{u_\varepsilon\}$ has an a.e. convergent subsequence, and its limit is an entropy solution of (1.3). We will take advantage of the fact that u has been constructed by vanishing-viscosity method. Since the entropy solution is unique [18], we conclude that the entire sequence of viscous approximations $\{u_\varepsilon\}$ converges a.e. to the unique entropy solution. Existence of an a.e. convergent subsequence of $\{u_\varepsilon\}$ is proved by establishing a uniform BV-estimate on the sequence u_ε .

Proof of Theorem 3.1. The existence and uniqueness solution result for (1.3) comes from the fact that u_ε satisfies the following property:

$$0 \wedge (\inf \text{ess } u_0) \leq u_\varepsilon \leq (\sup \text{ess } u_0) \vee 0, \quad (3.2)$$

since $x = 0$ at the boundary, independent estimate of ε . Thus constants are solutions. The estimate (3.2) gives L^∞ -bound to the problem (3.1).

On the other hand, at least formally, one has:

$$\frac{\partial}{\partial t}(u - k)^+ + \sum_{i=1}^d \frac{\partial}{\partial x_i}(f_i(u) - f_i(k))\mathbf{1}_{(u > k)} - \varepsilon \Delta(u - k)^+ \leq 0. \quad (3.3)$$

We next perform an integration by parts against a test function of the form $\mathbf{1}_{(u > k)}$:

$$\frac{\partial}{\partial t} \int (u - k)^+ \mathbf{1}_{(u > k)} + \sum_{i=1}^d \int \frac{\partial}{\partial x_i}(f_i(u) - f_i(k))\mathbf{1}_{(u > k)} - \varepsilon \int \Delta(u - k)^+ \mathbf{1}_{(u > k)} \leq 0. \quad (3.4)$$

Furthermore, after integration with respect to the variable x , and taking into account the boundary, it follows from (3.4):

$$\frac{\partial}{\partial t} \int (u - k)^+ \mathbf{1}_{(0 > k)} + (f_1(u(0)) - f_1(k))\mathbf{1}_{(u(0) > k)} - \varepsilon \frac{\partial u}{\partial x_1}(0)\mathbf{1}_{(u(0) > k)} \leq 0. \quad (3.5)$$

The terms $f_1(0)$ comes from the fact that $u = 0$ at the boundary ($x_1 = 0$). The following quantity:

$$\mathcal{T}_\varepsilon := (f_1(0) - f_1(k))\mathbf{1}_{(u(0) > k)} - \varepsilon \frac{\partial u}{\partial x_1}(0)\mathbf{1}_{(u(0) > k)} \quad (3.6)$$

is equal to zero according to the sign of k . Thus, formally, we will recover the usual signs of entropy inequalities only in the case where $k \geq 0$ because the case $0 > k$ in (3.6) kills terms for which we have no idea. For a test function $\varphi \in C_{\text{compact}}^\infty(\bar{\Omega})$ ($\varphi \geq 0$), that is with compact support in on the closure of the set, and after integrating we infer that:

$$\forall k \geq 0, \quad \frac{d}{dt} \int (u - k)^+ \varphi - \sum_{i=1}^d \int (f_i(u) - f_i(k))\mathbf{1}_{u > k} \cdot \nabla \varphi + \varepsilon \int \nabla(u - k)^+ \cdot \nabla \varphi \leq 0, \quad (3.7)$$

since the boundary term cancel out. Integrating again by part the third part of (3.7), and taking into account the fact that $u(x_1) = 0$, we obtain

$$\varepsilon \int \nabla(u - k) \nabla \varphi = \int (u - k)^+ (-\varepsilon \Delta \varphi).$$

Consequently

$$\forall k \geq 0, \quad \frac{d}{dt} \int (u - k)^+ \varphi - \sum_{i=1}^d \int (f_i(u) - f_i(k))\mathbf{1}_{u > k} \cdot \nabla \varphi + \int (u - k)^+ (-\varepsilon \Delta \varphi) \leq 0. \quad (3.8)$$

Since we have the bounds at infinity and φ is smooth, we can let $\varepsilon \rightarrow 0$, which would cancel out the third term of the left-hand side of (3.8) and we thus recover a valid formulation up to the boundary of entropy subsolution. Thanks to this and the particular structure of the problem, we obtain the definition of entropy subsolution (resp. supersolution) of the initial-boundary value problem (3.1) as defined in Definition 2.1. This concludes the proof. \square

3.2. The entropy solutions and the main result for a $BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ -data. The section is devoted to the research of a suitable homogeneous boundary condition in (2.12) when $u_0(x) \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Theorem 3.2. *Assume that f, u_0 satisfy Hypothesis 2.4. Then there exists a unique solution to the Dirichlet boundary value problem (3.1) which satisfies the following uniform estimates:*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq C_0 := \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad (3.9)$$

$$\|u(\cdot, t)\|_{BV(\mathbb{R}^d)} \leq \|u_0\|_{BV(\mathbb{R}^d)}. \quad (3.10)$$

Moreover there exists a subsequence $\{u_{\varepsilon_n}\}$ of the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ of regularized problem (3.1) which is obviously bounded in $W^{1,1}(\Omega \times]0, T])$ and thus is compact in $L^1(\Omega \times]0, T])$, converges strongly in $L^1(\Omega \times]0, T])$ and the limit function u is a entropy solution of the problem (1.3).

Proof of Theorem 3.2. The existence and uniqueness result of the problem (3.1) is obtained by the classical parabolic theory.

• **L^∞ bounds.** The first estimation of the theorem relies on a comparison principle [9], Theorem 6.3.2], given a L^∞ -estimate of the convergence rate of u_ε . Indeed, by the classical parabolic theory, the problem (3.1) has a unique solution u_ε which has L^∞ -bound independent by the ε estimate:

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^d \times]0, T])} \leq C_0 := \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad (3.11)$$

after observing that constants are solutions of (3.1) and that at the boundary one has:

$$- \|u_0\|_{L^\infty} \leq u_\varepsilon(t, x) \leq + \|u_0\|_{L^\infty}, \quad (3.12)$$

such that for the comparison principle, one has a priori estimate on u_ε by the norm L^∞ . But the L^∞ -bound it is not enough to conclude since needs to have punctual convergence, namely compactness in L^1 . Thus if we have a BV-estimate, we can deduce compactness for a subfamily which converges in L^1 to the solution u of (1.3). We now turn to (3.10).

• **Uniform BV-estimate.** We derive BV-estimates on the approximate solution. Let us stress that, since the problem is invariant by translation, the derivative with respect to the variables x' (the tangential part) does not raise any problem but remains a bit more tricky for the derivative which is the normal part at the boundary, i.e. the derivative with respect to x_1 . We drop the dependence of ε for the sake of notation and we take the derivative of Eq. (3.1) with respect to x_k :

$$\frac{\partial}{\partial t}(\partial_k u) + \partial_i(f'_i(u)(\partial_k u)) - \varepsilon \Delta \partial_k u = 0, \quad (3.13)$$

and by passing to the absolute value, we derive that:

$$\frac{\partial}{\partial t}|\nabla u| + \partial_i(f'_i(u)|\nabla u|) - \varepsilon \Delta |\nabla u| \leq 0. \quad (3.14)$$

Furthermore, after integration on the half-space, it follows from (3.14) that:

$$\frac{d}{dt} \int |\nabla u| + \int_{\mathbb{R}^{d-1}} dx' \cdot f'_1(0)|u_{x_1}| - \varepsilon |\nabla u|_{x_1} \leq 0. \quad (3.15)$$

Let us observe explicitly that

$$|\nabla u|_{x_1} = \frac{\nabla u}{|\nabla u|} \cdot \nabla u_{x_1} = \text{sgn}(u_{x_1})u_{x_1 x_1}.$$

Thus integrating by parts, yields the following complete balance laws:

$$\frac{d}{dt} \int |\nabla u| + \int_{\mathbb{R}^{d-1}} dx' \cdot f_1'(0)|u_{x_1}| - \varepsilon|u|_{x_1 x_1} \operatorname{sgn}(u_{x_1}) \leq 0. \quad (3.16)$$

In fact, going back to the Eq. (3.1), $u = 0$ at point $x_1 = 0$ since the tangential derivatives are equal to 0, but not the normal derivative. Indeed computing $\partial_i f_i(u_\varepsilon)$ for $i = 1$ there remains the term $f_1'(0)u_{x_1}$. Consequently, for the whole equation, we get:

$$0 = \varepsilon u_{x_1 x_1} + f_1'(0)u_{x_1}. \quad (3.17)$$

It is worth mentioning that the second and third terms of (3.17) and (3.15) are the same. The latter is only multiplied by the sign of u_{x_1} . Thus multiplying (3.17) by the sign of u_{x_1} we obtain:

$$f_1'(0)|u_{x_1}| - \varepsilon u_{x_1 x_1} \operatorname{sgn}(u_{x_1}) = 0, \quad (3.18)$$

from which we deduce the following BV-estimate:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial x_i}(x, t) \right| dx \leq 0. \quad (3.19)$$

Thanks to our BV-estimate (3.19), we can obtain robust compactness results that serve to prove existence of our solutions in a BV-framework.

The existence of a strongly convergent subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ follows from (3.9), (3.10) and the Kolmogorov's Theorem. As a consequence, for $\varepsilon > 0$, u_ε belongs to a bounded set of $W^{1,1}(\Omega \times]0, T[)$ and then a sequence $u_{\varepsilon_{(n)}}$ can be extracted, with $\varepsilon_{(n)}$ tending to zero as n tends to infinity, which converges in $L^1(\Omega \times]0, T[)$ towards a function u lying in $\mathbf{BV}(\Omega \times]0, T[)$.

It remains to prove that u satisfies the inequality (2.15). To this end, we analyze the formulation (2.15) for the variable x_1 . Let u be as asserted by Theorem 3.2. We need to show that the function u satisfies (2.15). We introduce the test function χ_ε defined as follows:

$$\chi_\varepsilon(x) := \left(1 + \frac{x_1}{\varepsilon}\right)^+ \quad (3.20)$$

in Eq. (2.19) in the sense of distributions. This leads, for $k \geq 0$, to

$$\frac{d}{dt} \int_{-\varepsilon < x_1 < 0} (u - k)^+ \chi_\varepsilon - \frac{1}{\varepsilon} \int_{-\varepsilon < x_1 < 0} (f_1(u) - f_1(k)) \mathbf{1}_{(u > k)} \leq 0. \quad (3.21)$$

All the tangential terms from integration by part vanish. Integrating (3.21) in time, it is easily seen that:

$$\left[\int_{-\varepsilon < x_1 < 0} (u - k)^+ \chi_\varepsilon \right]_0^t - \frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} (f_1(u) - f_1(k)) \mathbf{1}_{(u > k)} dx' \leq 0. \quad (3.22)$$

The first term on the left-hand side is obviously bounded and tends uniformly to 0 when $\varepsilon \rightarrow 0$. Then it is follows by letting $\varepsilon \rightarrow 0$ that:

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} (f_1(u) - f_1(k)) \mathbf{1}_{(u > k)} dx' \geq 0, \quad (3.23)$$

uniformly in t , giving information on the boundary. Then (3.23) is reduced to the following strong entropy solution formulation.

Definition 3.3. A function u is the subsolution of (1.3) if

$$f_1(u) - f_1(k) \geq 0 \quad \text{if } u \geq k, \quad \forall k \geq 0, \quad (3.24)$$

and v is supersolution (1.3) if

$$f_1(v) - f_1(k) \leq 0 \quad \text{if } v \leq k, \quad \forall k \leq 0. \quad (3.25)$$

It is worth noticing that these conditions correspond exactly to the condition of [2], expressing the fact that one has translated a point information by (3.24) and (3.25) from an information on the integrals. Going back to (3.21) and multiplying by the function test χ_ε in x_1 and a positive function φ with compact support in \mathbb{R}^{d-1} , that is, a function of x' -variable, we get

$$\begin{aligned} & \left[\int_{-\varepsilon < x_1 < 0} \varphi(x')(u-k)^+ \chi_\varepsilon \right]_0^t - \frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} (f_1(u) - f_1(k)) \mathbf{1}_{(u>k)} \nabla \varphi \, dx' \\ & - \frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} \sum_{i=2}^d (f_i(u) - f_i(k)) \mathbf{1}_{(u>k)} \nabla \varphi \, dx' \leq 0. \end{aligned} \quad (3.26)$$

Next, we take the limit when $\varepsilon \rightarrow 0$ in the last expression to obtain

$$\left[\int_{-\varepsilon < x_1 < 0} \varphi(x')(u-k)^+ \chi_\varepsilon \right]_0^t \xrightarrow{\varepsilon \rightarrow 0} 0,$$

and

$$-\frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} \sum_{i=2}^d (f_i(u) - f_i(k)) \mathbf{1}_{(u>k)} (\varphi)_{x_i} \, dx' \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Consequently

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \int_{-\varepsilon < x_1 < 0} (f_1(u) - f_1(k)) \mathbf{1}_{(u>k)} \varphi(x') \, dx' \geq 0, \quad \forall \varphi \geq 0. \quad (3.27)$$

That means we have a punctual, but not integral information. This concludes the proof. \square

The present paper is mostly interested in function u which is subsolution of (1.3) in Ω and v supersolution of (1.3) in Ω according to definition 2.1. We are now able to establish a comparison theorem for (1.3).

Theorem 3.4. Let $u \in C([0, T], L^1(\mathbb{R}^d)) \cap L^\infty(0, T); \mathbb{R}^d$ be an entropy subsolution for $x_1 < 0$ for the problem (1.3), i.e. (3.24) holds and $v \in C([0, T], L^1(\mathbb{R}^d)) \cap L^\infty(0, T); \mathbb{R}^d$, be an entropy supersolution for Eq. (1.3), i.e. (3.25) holds for $T < \infty$ fixed. Then

- (1) The following L^1 -contraction property is true:

$$\frac{d}{dt} \int (u-v)^+ \, dx \leq 0, \quad (3.28)$$

which states that the semigroup operator associated to (1.3) is a nonlinear contraction in $L^1(\mathbb{R}^d)$.

- (2) For every $u_0 \in L^1 \cap L^\infty$, the entropy solution $u \in C([0, T], L^1) \cap L^\infty$ of problem (1.3) is unique.
- (3) If u_0 has bounded variation, then $u(t) \in BV$ for all t , and $\|u(t)\|_{BV}$ is a decreasing function.

3.3. Characterization of subsolutions with the traces.

In the previous sections, the assumption $u \in \text{BV}$ ensures the existence of trace of the function u . In what follows we will characterize the sub-solutions in the case where this trace $u(0, x')$ exists. We will now make a proof of the comparison, the uniqueness by really making traces; in these proofs, it will be necessary to replace these traces everywhere by thickened integrals.

We claimed that

$$\frac{d}{dt} \int (u - v)^+ \leq 0. \quad (3.29)$$

Proof of (3.29). If we have (3.29), that means that we have a comparison principle, we have the uniqueness and then we have everything we want. Note that we want to do it with traces. Since this is an entropy subsolution and supersolution, it means that we can write the entropy equations on $(u - v)^+$ with the flux that is what it can be. The Kruzhkov's method of doubling variables implies:

$$\frac{\partial}{\partial t} (u - v)^+ + \partial_{x_i} \left(\frac{f_i(u) - f_i(v)}{u - v} (u - v)^+ \right) \leq 0, \quad (3.30)$$

in the sense of distributions. One doubles the variables and one multiplies by $\varrho_\varepsilon(x - y)$ to have (3.30). If we want to keep local information instead of writing a report on the integrals. Now if we have traces for u and v , we can take stock of the area of inequality (3.30) and infer that:

$$\frac{d}{dt} \int (u - v)^+ + \int (f_1(u) - f_1(v)) \mathbf{1}_{(u > v)} dx' \leq 0, \quad (3.31)$$

which is a mixture of the Otto's proof [21] and Bardos *et al.* [2] and the main novelty being the formulation which is very elementary. In virtue of (3.31), the inequality (3.29) would be automatic provided that

$$(f_1(u) - f_1(v)) \mathbf{1}_{(u > v)} \geq 0. \quad (3.32)$$

We need to check the inequality (3.32). Thus we are automatically in a situation where $u > v$. We will argue "almost everywhere":

- If $v(x_0) \geq 0$, thanks to the Definition 3.24, the inequality (3.32) is verified.
- If $u(x_0) \leq 0$, thanks to the Definition 3.25, the inequality (3.32) is verified.
- Thus the only case that we cannot immediately infer that (3.32) is satisfied, is the case when $u(x_0) > 0 > v(x_0)$; at that moment, we will go through the intermediate value which is 0 to deduce that the formulation in u :

$$f_1(u(x_0)) \geq f_1(0) \geq f_1(v(x_0)). \quad (3.33)$$

Consequently, in all cases, the inequality (3.32) is true. When $u > v$, then $f_1(u) > f_1(v)$, from the set of previous inequalities. \square

4. EXISTENCE AND UNIQUENESS. PROOF OF THE MAIN THEOREM

Proof of Theorem 3.4. We will prove this theorem by the traces. Our technique is inspired by a method introduced by Kruzhkov [16] to prove L^1 -contraction for entropy solutions for of (3.1). The main ingredients of the proof is to replace the traces by the integrals of small volumes inspired by the Otto's method. We divide the proof into three steps.

First Step. In what follows, we let u depend on $(t, x) \in \Omega$ and v depend on $(s, y) \in \Omega$.

We first use the "doubling variables" method of Kruzhkov to prove the uniqueness of the solution of (3.1). Since $u = u(x, t)$ is an entropy solution of (1.3), then (2.15) holds.

By choosing $k = v(s, y)$ in (2.15) and then integrating over (s, y) , the Kruzhkov's result indicates that $u - v$ satisfies the following entropy inequality:

$$\frac{\partial}{\partial t}(u(x, t) - v(y, t))^+ + \nabla_{x+y} \cdot (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > v(y))}) \leq 0 \quad (4.1)$$

where we have introduced the operator $\nabla_{x+y} := \nabla_x + \nabla_y$. Let us observe that in (4.1), the tangential derivatives do not raise any difficulties, since we can integrate by part without problem. Then no boundary value condition is necessary. Thus we will focus on the difficult variable x_1 which sees the boundary. To this end, we write

$$\frac{\partial}{\partial t}(u(x, t) - v(y, t))^+ + \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > v(y))}) \leq 0, \quad (4.2)$$

in the sense of distributions. The inequality (4.2) allows to have uniqueness, comparison and contraction in L^1 . Similar to the Cauchy problem, provided one can integrate by parts, the crucial step of the proof is to check that

$$\frac{d}{dt} \int (u - v)^+ + \int (f_1(u) - f_1(v)) \mathbf{1}_{(u > v)} dx' \leq 0. \quad (4.3)$$

Once we have this ‘‘entropy condition’’, we derive (3.28) with the help of a method introduced by Kruzhkov to prove L^1 -contraction for entropy solutions of scalar conservation laws. Observe that (4.3) is fulfilled provided that

$$\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > v(y))} \geq 0, \quad (4.4)$$

which allows to compare $u(x_0)$ and $v(x_0)$ almost everywhere.

Second Step. In order to clarify integration by part to obtain (4.3), we are going to split the second term of left-hand side of (4.2) into three parts to have

$$\begin{aligned} \mathcal{I} &:= \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > v(y))} \geq 0 \\ &\quad + \{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0 > u(x) > v(y))}) \\ &\quad + \{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > 0 > v(y))}). \end{aligned} \quad (4.5)$$

Then, we carry out the following cutting of these terms:

$$\mathcal{I} = I_1 + I_2 + I_3 + I_4 \quad (4.6)$$

where

$$I_1 = \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x) > v(y))} \geq 0) \quad (4.7)$$

$$I_2 = \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0 > u(x) > v(y))}) \quad (4.8)$$

$$I_3 = \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(0)\} \mathbf{1}_{(u(x) > 0 > v(y))}) \quad (4.9)$$

$$I_4 = \partial_{x_1+y_1} (\{f_1(0) - f_1(v(y))\} \mathbf{1}_{(u(x) > 0 > v(y))}) \quad (4.10)$$

Having (4.6) in mind, the proof of the theorem becomes very simple. In fact, it is sufficient to analyze each term of the right-hand side of (4.6). Our task now is to multiply by a suitable kernel and integrate by part on the domain, since we have regularity. Indeed:

- For the term I_1 , since $v \geq 0$ and thanks to the formulation (3.24) with $k = v(y)$, one can integrate by part with respect to x_1 since there is no boundary. The x_1 -integration for terms I_1 and I_3 is the situation where one uses the entropy subsolution formulation with $k = v_+(y)$.
- By contrast, the integration with respect to y_1 requires greater attention, since we do not see the boundary. In order to get round this difficulty, we use the following construction, which is reminiscent of the techniques using in viscosity solution and based on Soner's method [23, 24] for the state constraint problems on each of the half-lines ($y_1 \leq 0$). This technique consists to push the point y_1 to be in the interior of domain. To this end, we

use regularization method (à la Di Perna-Lions [10]) and we choose a shift kernel defined as follows:

$$\varrho_\varepsilon = \frac{1}{\varepsilon^d} \prod_{i=1}^d \varrho\left(\frac{x_i - y_i}{\varepsilon}\right), \quad \varrho \geq 0, \quad \text{supp}(\varrho) \subset]-1, 1[, \quad \int \varrho = 1, \quad (4.11)$$

such that

$$\varrho_\varepsilon^\pm(\cdot) = \varrho_\varepsilon(\cdot \mp \varepsilon), \quad (4.12)$$

such that, in the neighborhood of the boundary, $y_1 = 0$, we have $\varrho_\varepsilon \equiv 0$. Let us emphasize that we need to shift only the part of the kernel which acts on the y_1 .

With the regularization (4.12), the inequality (4.2) will be written on the closure of the set, so that ϱ_ε^\pm is an admissible test function for Eq. (3.1). The key point in our proof is to make integrations by part by means of the definitions and using the support of ϱ_ε in the place where the definition does not operate. Thus for the terms I_1 and I_3 the integration with respect to y_1 will be possible by using $\varrho_\varepsilon^+(x_1 - y_1)$.

Consequently, we can multiply these terms by $\varrho_\varepsilon^+(x_1 - y_1)$, since

$$\text{supp } \varrho_\varepsilon^+ \subset]0, 2\varepsilon[, \quad (4.13)$$

instead of being contained in $[-\varepsilon, 2\varepsilon[$. In other words, x_1 and y_1 are of the form

$$0 < x_1 - y_1 < 2\varepsilon,$$

which especially means that $y_1 < x_1$. Finally, we can take x_1 in the whole domain because it is the formulation as we moved just a little $\varrho_\varepsilon^+(x - y)$, we have not boundary on this term because y_1 is automatically always inside the domain. We can then write inequality on the closure set, we have not seen the boundary. We can thus justify all integrations by part and bring us back to the same demonstration as if there was no boundary.

Third Step. We next perform an integration by parts against a test function of the form ϱ_ε^\pm . Using $\nabla_y \varrho_\varepsilon = -\nabla_y \varrho_\varepsilon$, we observe that

$$\partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x)>v(y)) \geq 0}) = 0. \quad (4.14)$$

Thus, having in mind that y_1 has not seen the boundary, we choose the test function $\varrho_\varepsilon^+(x_1 - y_1)$ and integrate by part on the closure of the domain with respect to (x_1, y_1) to obtain

$$\begin{aligned} & \int I_1 \times \varrho_\varepsilon^+(x_1 - y_1) dx_1 dy_1 \\ &= \int \varrho_\varepsilon^+(x_1 - y_1) \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x)>v(y)) \geq 0}) dx_1 dy_1 \\ &= - \int (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(u(x)>v(y)) \geq 0}) \partial_{x_1+y_1} \varrho_\varepsilon^+(x_1 - y_1) dx_1 dy_1 = 0. \end{aligned} \quad (4.15)$$

Similarly, we use the same regularization for I_3

$$\begin{aligned} & \int I_3 \times \varrho_\varepsilon^+(x_1 - y_1) dx_1 dy_1 \\ &= - \int \varrho_\varepsilon^+(x_1 - y_1) \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0>u(x)>v(y))}) dx_1 dy_1 \\ &= - \int (\{f_1(u(x)) - f_1(0)\} \mathbf{1}_{(u(x)>0>v(y))}) \partial_{x_1+y_1} \varrho_\varepsilon^+(x_1 - y_1) dx_1 dy_1, \end{aligned} \quad (4.16)$$

where the integral on y_1 is on a small volume of size ε . Now let us focus on the terms I_2 and I_4 . Meanwhile, for these terms, we use $\varrho_\varepsilon^-(x_1 - y_1)$ and therefore we can now justify all

integrations by part and bring us back to the same proof as if there was no boundary.

$$\begin{aligned}
& \int I_2 \times \varrho_\varepsilon^-(x_1 - y_1) dx_1 dy_1 \\
&= \int \varrho_\varepsilon^-(x_1 - y_1) \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0>u(x)>v(y))}) dx_1 dy_1 \\
&= - \int (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0>u(x)>v(y))}) \partial_{x_1+y_1} \varrho_\varepsilon^-(x_1 - y_1) dx_1 dy_1, \quad (4.17)
\end{aligned}$$

and

$$\begin{aligned}
& \int I_4 \times \varrho_\varepsilon^-(x_1 - y_1) dx_1 dy_1 \\
&= \int \varrho_\varepsilon^-(x_1 - y_1) \partial_{x_1+y_1} (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0>u(x)>v(y))}) dx_1 dy_1 \\
&= - \int (\{f_1(u(x)) - f_1(v(y))\} \mathbf{1}_{(0>u(x)>v(y))}) \partial_{x_1+y_1} \varrho_\varepsilon^-(x_1 - y_1) dx_1 dy_1. \quad (4.18)
\end{aligned}$$

Summing, we have derived the following estimate:

$$\frac{d}{dt} \iint (u(x, t) - v(y, t))^+ \varrho_\varepsilon(x - y) dx dy \leq 0. \quad (4.19)$$

Therefore, taking the limit $\varepsilon \rightarrow 0$, we get the L^1 norm, namely

$$\frac{d}{dt} \iint (u(x, t) - v(y, t))^+ dx dy \leq 0, \quad (4.20)$$

which states that the semigroup operator associated to (1.3) is a nonlinear contraction in $L^1(\mathbb{R}^d)$:

$$S(t)u_0^1 \leq S(t)u_0^2 \quad \text{if } u_0^1 \leq u_0^2.$$

If $v_0 - u_0 \in L^1$, then the corresponding solutions u and v have the property that the difference $v(t) - u(t)$ remains space-integrable for every time $t > 0$ and $t \mapsto \|v(t) - u(t)\|_1$ is non-increasing:

$$\|(S(t)u_0^1 - S(t)u_0^2)^+\|_{L^1} \leq \|(u_0^1 - u_0^2)^+\|_{L^1}. \quad (4.21)$$

Obviously, for almost all $t \in]0, T[$, we have:

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}, \quad (4.22)$$

and then uniqueness by taking $u_0 = v_0$. Let us mention that: not using exactly the same ϱ_ε for all terms, induced small errors that can be compensated for with terms of integration and volume. Let us point out that the main difference with Kruzhkov proof, is that, in [16] one proves that the inequality (4.19) is true every nonnegative function $\varrho_{\varrho_\varepsilon}$, but here we have the same inequality with (4.13) since the such functions allows to obtain the convergence. This completes the proof. \square

5. APPLICATIONS

This section is concerned with some illustrating applications. We consider

$$u_t + \operatorname{div} f(b(x), u) = 0, \quad \text{in } \Omega \times (0, T). \quad (5.1)$$

The unknown is $u : \Omega \times (0, T) \rightarrow \mathbb{R}$. The flux $f(b(x), u)$ in (5.1) may have a possibly discontinuous spatial dependence through the positive coefficient $b(x)$. We assume that b is never zero since (5.1) may not be solvable if $b(x)$ is not bounded away from zero. Our interest in this model is not merely academic. A simple physical model corresponding to

(5.1) is the traffic flow on a highway [27]. Spatial variation of the coefficient b affects the maximum speed:

$$f(b(x), u) = b(x)f(u). \quad (5.2)$$

We will be mainly interested in entropy solution of the following initial-boundary value problem:

$$\begin{cases} u_t + \operatorname{div}(b(x)f(u)) = 0, & \text{in } \Omega \times (0, T) \\ u(x, 0) = u_0(x) \\ u(x, t) = 0 \end{cases} \quad \text{for } (x, t) \in \partial\Omega \times (0, T). \quad (5.3)$$

Throughout this section we make the following assumption on the vector field b . The new assumption enables us to prove some additional uniqueness result and it play important role in the existence theory.

(E1) We assume that (see for instance [5]):

$$b \in W_{loc}^{1,1}(\mathbb{R}^d), \quad b, \operatorname{div} b \in L^\infty(\mathbb{R}^d). \quad (5.4)$$

The notion of entropy solution follows.

Definition 5.1. Let $u_0 \in L^\infty(\Omega)$. An entropy solution of the mixed initial-value problem (5.3) is a function $u \in L^\infty(\Omega \times (0, T))$ such that for every $\varphi \geq 0$, $\varphi \in C_{compact}^\infty(\{x_1 \leq 0\})$, $\forall k \geq 0$:

$$\int_0^T \int_\Omega \left\{ u - k |\partial_t \varphi + (f(u) - f(k)) \mathbf{1}_{(u>k)} b(x) \cdot \nabla \varphi - \operatorname{div} \{ f(k) b(x) \} \mathbf{1}_{(u>k)} \varphi \right\} dx dt. \quad (5.5)$$

One makes an integration by parts without caring about the boundary condition. In other words, the boundary is regarded as an interior point. Let u be subsolution and v supersolution of (5.3); the aforementioned definition 5.1 yields:

$$\frac{d}{dt} \int (u - v)^+ \leq 0. \quad (5.6)$$

Obviously, inequality (5.6) implies in particular the uniqueness of solution. On the other hand, the comparison property can be easily obtained by the uniqueness property we will prove. Indeed, since every entropy solution of (5.3) with smooth initial data is the limit, as $\varepsilon \rightarrow 0$, of the solutions of the regularized problem from (5.3) and by the Maximum Principle [13, 25] the comparison holds for the regularized problem and for (5.3).

Proof of claim (5.6). We are going to make a proof with the traces, by using a mixture of BLN and the Otto's proofs, but with a much more elementary formulation. Since u is a subsolution and v a supersolution, this allows to write a transport equation on $(u - v)^+$. We follow familiar ground (cf. the textbook [9]). From the Kruzhkov's method of doubling of variables we obtain:

$$\frac{\partial}{\partial t} (u - v)^+ + \partial_{x_1+y_1} (b(x) \{ f_1(u(x)) - f_1(v(y)) \}) \mathbf{1}_{u>v} \leq 0, \quad (5.7)$$

in the sense of distributions. As a consequence, if u and v have traces, the inequality (5.7) reduces to

$$\frac{d}{dt} \int (u - v)^+ + \int b(x) \{ f_1(u(x)) - f_1(v(y)) \} \mathbf{1}_{(u>v)} dx' \leq 0. \quad (5.8)$$

We want to prove the claim (5.8), which is the cornerstone of the proof. In view of (5.8), we prove that inequality (5.6) amounts showing that

$$b(x)(f_1(u) - f_1(v)) \mathbf{1}_{(u>v)} \geq 0, \quad (5.9)$$

thanks to the collection of inequalities (3.24) and (3.25). Thus, we are automatically in a situation where $u > v$. Since the coefficient $b(x)$ is positive, this will not affect the signs. Upon rewriting the left-hand side of (3.32) in the following form:

$$\begin{aligned} b(x)(f_1(u) - f_1(v))\mathbf{1}_{(u>v)} &= (b(x)\{f_1(u(x)) - f_1(v(y))\}\mathbf{1}_{(u(x)>v(y))\geq 0} \\ &\quad + b(x)\{f_1(u(x)) - f_1(v(y))\}\mathbf{1}_{(0\geq u(x)>v(y))} \\ &\quad + b(x)\{(f_1(u(x)) - f_1(v(y))\}\mathbf{1}_{(u(x)\geq 0>v(y))}). \end{aligned} \quad (5.10)$$

We will treat each term in a slightly different way. To this end, we proceed to the following cutting:

$$\begin{aligned} b(x)(f_1(u) - f_1(v))\mathbf{1}_{(u>v)} &= \left(b(x)\{f_1(u(x)) - f_1(v(y))\}\mathbf{1}_{(u(x)>v(y))\geq 0} \right. \\ &\quad + b(x)\{f_1(u(x)) - f_1(v(y))\}\mathbf{1}_{(0\geq u(x)>v(y))} \\ &\quad \left. + b(x)\{(f_1(u(x)) - f_1(0)) + (f_1(0) - f_1(v(y)))\}\mathbf{1}_{(u\geq 0>v)} \right). \end{aligned} \quad (5.11)$$

Remember that $k = v(y)$. Then (5.11) can be equivalently rewritten as follows:

$$\begin{aligned} b(x)(f_1(u) - f_1(k))\mathbf{1}_{(u>k)} &= (b(x)\{f_1(u(x)) - f_1(k)\}\mathbf{1}_{(u(x)>k\geq 0)} \\ &\quad + b(x)\{f_1(u(x)) - f_1(k)\}\mathbf{1}_{(0\geq u(x)>k)} \\ &\quad + b(x)\{(f_1(u(x)) - f_1(0)) + (f_1(0) - f_1(k))\}\mathbf{1}_{(u(x)\geq 0>k)}). \end{aligned} \quad (5.12)$$

We argue almost everywhere for $x \in \mathbb{R}^d$ in three steps:

- As a first step, we assume that v is positive or null, in other words, almost everywhere $v(x_0) \geq 0$. Since $b(x)$ is positive, then by using Definition 3.3 for equation of u , that is (3.24), we infer that inequality (3.32) is then satisfied for all k .
- Let us now consider the case where u is negative, in other words, almost everywhere $u(x_0) \leq 0$. Then by using Definition 3.3 for equation of v , that is (3.25), we deduce that inequality (3.32) is true for all k .
- If $u(x_0) > 0 > v(x_0)$, we pass through the intermediate value 0, and one deduces easily in view of the formulation satisfied by u that

$$f_1(u(x_0)) \geq f_1(0) \geq f_1(v(x_0)). \quad (5.13)$$

Thus, in all cases, inequality (3.32) is verified. In conclusion, if $u > v$, one has $f_1(u) > f_1(v)$, from inequalities (3.24) and (3.25) and our claim is proved. \square

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