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# Optimal iso-area 4 x 4 checkerboard

David Dureisseix

## Abstract

Iso-area origami models and checkered patterns are highly symmetrical designs. Some challenging issues concern the double-faced models (that can be reversed), and the optimality design (folding from an as small as possible initial square sheet of paper). This study focuses on the case of a 4 x 4 checkerboard exhibiting probably the highest degree of symmetry among all flat-folded models and the best-known optimality degree.

This is an English version of a preprint of “Un échiquier 4 x 4 reverso optimal”, published in *Le Pli*, 154, pp. 12-15, 2019, the journal of the Mouvement Français des Plieurs de Papier (MFPP), the French paperfolding association.

## Introduction

Color-change has for main goal to make a pattern appear on a folded model, using a sheet with one color per face. Amazing and famous designs are John Montroll's zebra [1], Satoshi Kamiya's tiger [2], Xiaoxian Huang's A-diamond [3], Tanaka Masashi's dice [4]... and many more.

Geometric patterns, usually flat folded, leads to many challenges, puzzles... Checkered patterns with many color shifts, are among the more demanding ones, see e.g. [6]. Starting from a square sheet of paper is a strong constraint (folding a strip is much easier); therefore, chessboard patterns have been studied for several years [7], [8], [9].

If the challenge is not hard enough, additional constraints can be added, such as optimality: minimize the size of the unfolded sheet, for a given folded  $n \times n$  checkered model. For instance, the  $n = 4$  case designed by Max Hulme [10], Figure 1, starts with a 8 x 8 square. You may try to improve it... The classical chessboard ( $n = 8$ ) has numerous solutions, and up to now, optimal versions use a 32 x 32 sheet [11], [12], [13].

What happens when increasing  $n$ ? To estimate the required size of the paper square sheet, one may assume that a color change along a line segment consumes the same amount of paper edge length of the initial sheet perimeter [11], [22]. Nevertheless, 2009 produced a “coup de théâtre” [14], exemplifying that this assumption was too strong and can be relaxed. A smaller sheet can be used for a better solution... at least for  $n = 16$  and beyond... good luck to fold it! (Robert Lang did fold the 8 x 8 version and said: « Wow, this was not one of the easiest things I've done! »)

Can we increase complexity? Of course! Additional constraints can be considered, as: seamless design (each small square of the board should be made with a single continuous surface) [5], [14], pixel-matrix (can we change the color of each board square independently with a single fold?) [14], [15]...

Herein, we focus on another challenge: design an iso-area (double faced) model: the pattern is the same on each face of the flat-folded model. Indeed, other one already thought about this! For instance, Jeremy Shafer designed a 4 x 4 pattern from a 18 x 18 square sheet (doubling the paper edge size with respect to Max Hulme) [16], [17].

Now, can we do it with a smaller square sheet?

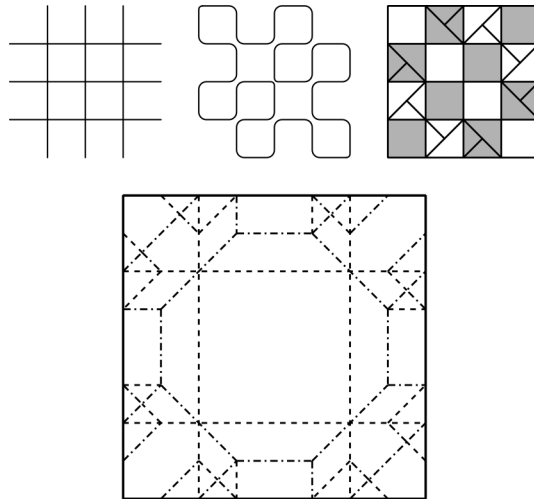


Figure 1: Max Hulme's design [10]. Top: edge graph, edge diagram and folded model. Bottom: crease pattern.

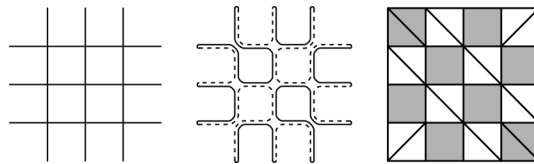


Figure 2: proposed model. Edge graph, edge diagram (recto: plain lines, verso: dotted lines) and folded model.

### Edge diagram

The edge diagram is the trace on the flat-folded model of the edge of the initial sheet of paper. For small checkerboards, it happens to be an interesting tool for design, so let's use it here (though it relies on the assumption of using the edge for the color-change which is too strong as said before...). Then, imagine a path on a regular grid that constitutes a continuous line, and a closed loop, that should separate the squares on the board. This line should do it on one face (plain line) and on the reverse face (dotted line), Figure 2.

We therefore may well improve the design (with respect to optimality) for attempting using a 12 x 12 initial sheet: some segments on the perimeter of the edge diagram are no more mandatory for having a continuous line. Now, how to design the double-faced model?

### Iso-area technique

Geometric origami often relies on symmetries. Here also, symmetries are of interest, especially the one of the iso-area technique [18], [19]. Looking at the crease pattern, if its mountain and valley folds are permuted (i.e. if the paper is reversed recto-verso, and a mirror symmetry applied) the crease pattern is unchanged, eventually up to an isometry (rotation, translation or mirror symmetry). This technique has been used for the  $n = 2$  case [20], and also, for instance, for several tessellations [21]. As an illustration, Figure 3 compares the windmill base and its iso-area version.

Now we need to fold the model, prescribing the edge of the paper to follow the previous edge diagram.

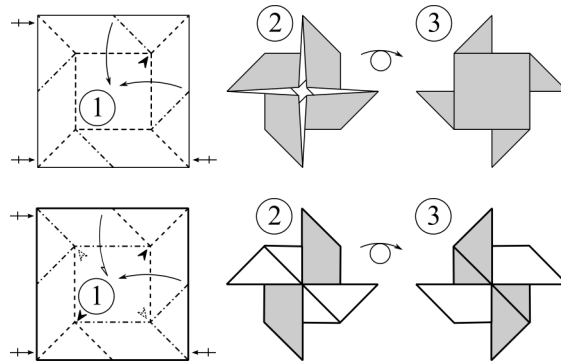


Figure 3: top: classical windmill base. Bottom: its iso-area version.

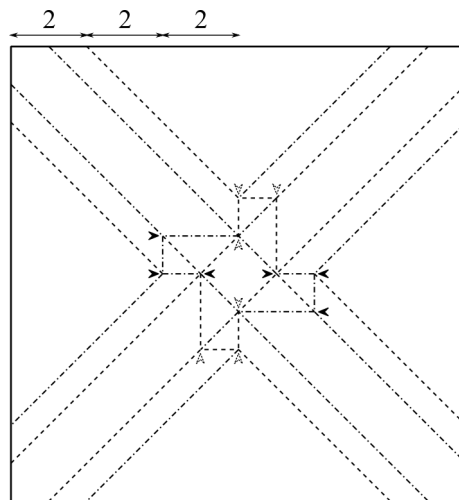


Figure 4: crease pattern of an iso-area X pattern.

### Proposed solution

Let start with a 12 x 12 square of a dual-colored paper, not too thick, Figure 5.

Step 1: mark the creases.

Tessellation addicts would probably pre-crease all the sheet for increasing precision. To get all the 12 x 12 square boards and their diagonals, this would require nevertheless 112 folds with a cumulative length close to 4.3 m for a 12 cm side sheet...

A typical difficulty of the iso-area technique concerns the strongly coupled movements during folding: it is therefore difficult to split the movement in several successive steps. Usually, it leads to a collapse of the crease pattern in a single step. Note that, though the final model is flat-folded, this step is often not feasible while respecting the rigid folding issue (keeping paper rigid between the folds) [21].

Steps 2 and 3: performing the X pattern with iso-area technique is not so easy (though feasible with the crease pattern in Figure 4), so we cheat a bit by breaking the symmetry for sake of simplicity.

Step 4: This one is really an iso-area fold. The central square rotates by 90 degrees during the collapse [23].

Step 5: the most difficult task has already been done. The remaining steps are finishing touches.

## Want more?

Beyond the  $n = 4$  case, the obvious question is: what happens when increasing  $n$ ? The simplest cases may well be when  $n$  is even, but anyway, the model complexity increases rapidly with  $n$  [15]. Who wants to set a record?

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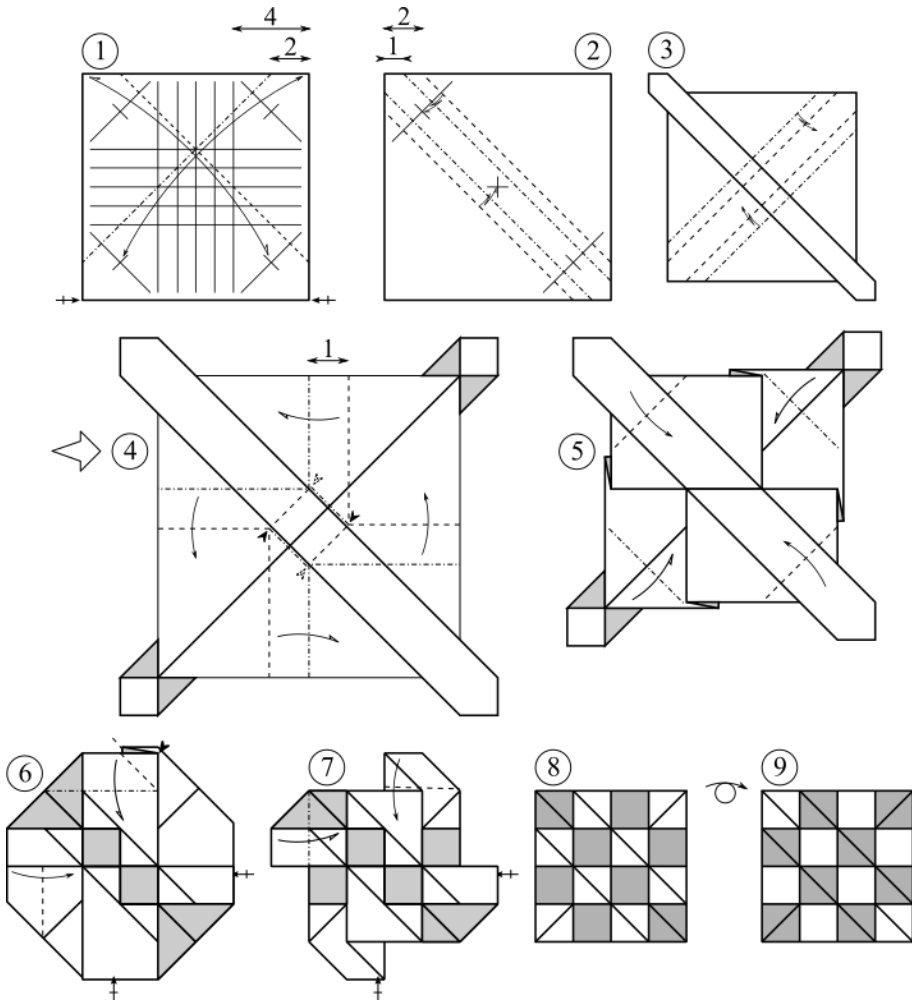


Figure 5: folding the optimal iso-area 4 x 4 checkerboard.