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# Reconstructed drill-bit motion for sonic drillstring dynamics

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In tunnel excavation that integrates Resonant Sonic Head Drilling (RSHD) machine, one of the important stability problem to deal with is represented by the necessity to assign axial vibration amplitude induced by the RSHD to a resonant vibration mode. In general, a control law depends on system variables which are partially available for measurements. In drilling operation, the drillstring variables at the tip boundary aren't easy, if not impossible, to be measured. The sonic drillstring infinite dimension dynamics, derived in this paper, will play an important role in PDE (Partial Derivative Equation) observer design for the drill bit motion. The well-posedness problem is addressed, and the designed PDE-observer stability is detailed using the resolvent method. The wave peak amplitude is determined from the resonant drillstring mode shape, and the system's stability is analyzed around a practical frequency mode.

## 1. Introduction

As hard rock drilling induces sever friction phenomena around the drillstring system and also at the bit, sever vibration variations are inherent for the equipment and may affect Rates Of Penetration (ROP). ROP are the principal monitoring factor like performance parameter of drilling. In drilling environment, larger machines are designed to drill harder formations; these machine vibrate, rotate and inject a fluid for cooling and removal of waste cuttings. Machines equipped with head rotary are useful in oilwell drilling. However, for tunnel drilling support where the depth is limited, machines that integrate axial vibration tool are considered more efficient. The variation of frequency by the operator and the drill bit weight to match the material he is going through, ensure the best penetration rate and most accurate sampling is obtained.

Sonic drilling has been used in industry for many years [17]. The majority of the research has been performed by private industry where they have kept the knowhow that they have developed internal and proprietary [3]. Most of the model, that describes the sonic drillstring found in the literature, did not take into account the damping of the soil along the length of the drill and they model poorly the interaction between the sonic drillstring bit and the crown [7]. The quantification of the sonic drill system variables integrates the under estimated friction from the drillstring and the crown. The essence of sonic drilling is high-frequency vibration

drilling where energy waves are transmitted though the drillstring to the end string and then reflected back. A sonic drill head works by sending high frequency resonant vibrations down the drillstring to the drill bit, as shown in Fig. 1, while the operator controls these frequencies to suit the specific conditions of the soil/rock geology. Vibrations may also be generated within the drill head. The frequency is generally between 50 and 150 Hz (cycles per second) and can be varied by the operator. Resonance magnifies the amplitude of the drill bit, which fluidizes the soil particles at the bit face, allowing for fast and easy penetration through most geological formations. An internal spring system isolates these vibrational forces from the rest of the drill rig.

Despite the performance of the axial vibration controllers proposed in [6,7] by means of adequate simulation results, implementation on real machine may be impractical. Actually, the downhole system's variables measurements, are unrealizable during drilling in underground tunneling unlike in the oil well rotary drilling field. Observability is a property of a dynamical system which means that we can determine exactly the state of the system from the observer of its input and output on a sufficiently long time interval. Note that an observer for a given system is an another dynamical system that produces an estimate of the current state of the given system based on past observations. The construction of observers is important in control applications for infinite-dimensional systems. The location of measurement and actuation plays an important role in the design of the observer for PDEs. If the sensor and the actuator are placed at the opposite boundaries, we call this an anti-collocated setup, otherwise, when the sensor and the actuator are located at the same side, it is called collocated setup. In

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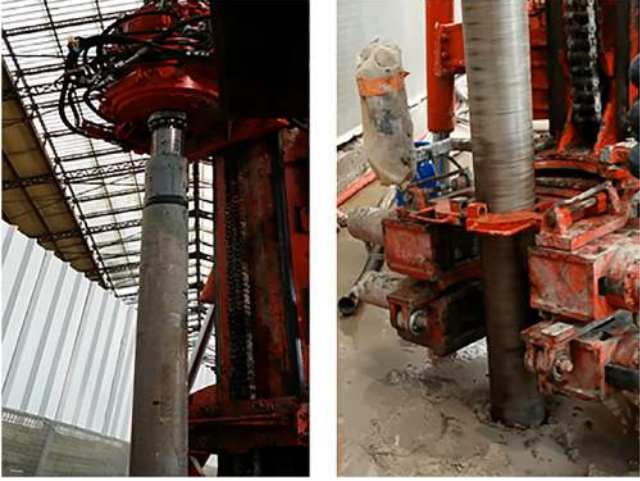


Fig. 1. Drilling machine (RSHD) tip and top boundary conditions.

general, two approaches are used to design a convergent boundary observer for a distributed parameter systems. The first approach is based on the backstepping technique: in [19], backstepping-based observers for a class of linear parabolic integro-differential equations was presented. An observer design for general periodic quasi-linear parabolic PDEs in one space dimension is used in [5]. For the case of a linear first-order hyperbolic system, an estimation of boundary parameters has been proposed in [11]. The continuum backstepping approach was applied to a hyperbolic PDEs in [8] to derive a collocated observer. The second approach is Lyapunov. Using this theory, boundary observer is designed for a motorized Euler-Bernoulli Beam in [13], for a towed seismic cable in [14] and for a flexible-link manipulator in [20].

In the present paper the main contribution here is to propose a collocated infinite dimensional observer for axial vibrations with sensing restricted to the boundary. To the author's knowledge, all previous works considering the vibrations control problem were carried out in the field of oil well drilling, where the variables in the bottom hole can be measured as there are sensors fixed on the drill collars which is not the case in the tunneling field. The estimation problem of the unmeasured parameters related to vibrations has not been found in the literature. The only results deal with estimating the pressure and temperature conditions.

The paper is organized as follows. The second section is concerned by the system model, we present a distributed parameter model of the drill string axial vibrations and we state the observer problem. In Section 3, we define the collocated observer design. Sections 5 and 6 address the well-posedness and the stability problems, respectively. Finally, simulation results are shown in Section 7, and a conclusion will end the paper.

## 2. Mathematical model

To reduce the complexity of the system and thus derive a mathematical model, it is necessary to make some initial assumptions and simplifications of the system when the boundary conditions are selected [16]. It is assumed that the drill string is a long pipe having a uniform cross-sectional area  $A$  and assumed to have a negligible effect of the torsional vibrations due to the pipe limited length (3–18 m) and the natural applied input forces (axial). The forces that excite the drill string are assumed to act on the top of the sonic drill. In oil and gas field, a rotary table is deployed, and where the pipe length is very important (km). Hence, torsional vibration can not be neglected. Thus, this phenomena has intensively

studied in the literature. One may refer to Saldivar et al. [18] and Zhao [21] and references therein.

Because damping along the length is very low, the sonic drill operator has to be very careful not to overstress the drill string at resonance when the bottom drill tip is not engaged in drilling. The damping at the drill bit is the most important variable of the drilling system, as it defines the drilling work that is taking place.

The governing differential equations of motion for the sonic drill is derived from force balance. We denote by  $u(x, t)$  the longitudinal displacement of a rod's section  $A$ , that is a distance  $x$  from the vertices at time  $t$  as shown in Fig. 2.

$$\rho A dx \frac{\partial^2 u(x, t)}{\partial t^2} + 2b A dx \frac{\partial u(x, t)}{\partial t} + a A dx u(x, t) + \sigma A - \left( \sigma A + \frac{d}{dx} (\sigma A) dx \right) = 0$$

where  $\rho$  is the pipe density,  $E$  is the Young modulus,  $a$  and  $b$  are respectively the coupling and damping constants along the length of the drillstring, and  $\sigma$  is the stress given by  $\sigma = E \frac{\partial u(x, t)}{\partial x}$ . So, we get

$$\rho A dx \frac{\partial^2 u(x, t)}{\partial t^2} + 2b A dx \frac{\partial u(x, t)}{\partial t} + a A dx u(x, t) - E A dx \frac{\partial^2 u(x, t)}{\partial x^2} = 0 \quad (1)$$

Dividing by  $\rho A dx$ , we obtain

$$\frac{\partial^2 u(x, t)}{\partial t^2} + \frac{2b}{\rho} \frac{\partial u(x, t)}{\partial t} + \frac{a}{\rho} u(x, t) - \frac{E}{\rho} \frac{\partial^2 u(x, t)}{\partial x^2} = 0 \quad (2)$$

We define the speed of the sound through the steel drill  $c$  by  $c = \sqrt{\frac{E}{\rho}}$ .

Eq. (1) became

$$\frac{\partial^2 u(x, t)}{\partial t^2} + \frac{2b}{\rho} \frac{\partial u(x, t)}{\partial t} + \frac{a}{\rho} u(x, t) - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = 0 \quad (3)$$

It remains to define the boundary conditions of the drillstring dynamics given above.

In order to calculate the drill string natural frequencies, the boundary conditions at the ends of the string must be known when the frequency functions are derived. The sonic driver mass, the input force from the sonic driver, and the air spring all reside, where  $x$  is equal to zero. The sonic driver mass and the air spring are always boundary conditions. At the drill tip of the string, where  $x$  is equal to the drillstring length  $L$ , a boundary condition caused by coupling of the sonic drill tip to the material being drilled through exists. All boundary conditions are located on the ends of the drill string and because of this, all the conditions have to equal the apparent forces at the end conditions. The forces for the ends are found by taking the drill string's elastic constant  $E$  multiplied by the cross sectional area of the drill string  $A$  and also multiplied by the partial derivative of the local deflection  $u$  with respect to the location in space  $x$  and setting this equal to the boundary condition, as shown in Fig. 2.

Top boundary condition:

$$EA \frac{\partial u(0, t)}{\partial x} = m_{sh} \frac{\partial^2 u(0, t)}{\partial t^2} + c_{sh} \frac{\partial u(0, t)}{\partial t} - U(t) + k_{sh} u(0, t) \quad (4)$$

where  $m_{sh}$  is the mass of the sonic head,  $k_{sh}$  and  $c_{sh}$  are respectively the spring and the damping rates of the air spring on top of the sonic drill.

The input force is typically fixed, as it is dependent on the size of the eccentrics and on the square of the angular frequency.  $U(t) = m_{ec} r_{ec} (2\pi\xi)^2 \sin(2\pi\xi t)$  where  $m_{ec}$ ,  $r_{ec}$  and  $\xi$  are respectively the mass of the two eccentrics, eccentricity of the eccentrics, and the temporal frequency in Hz.

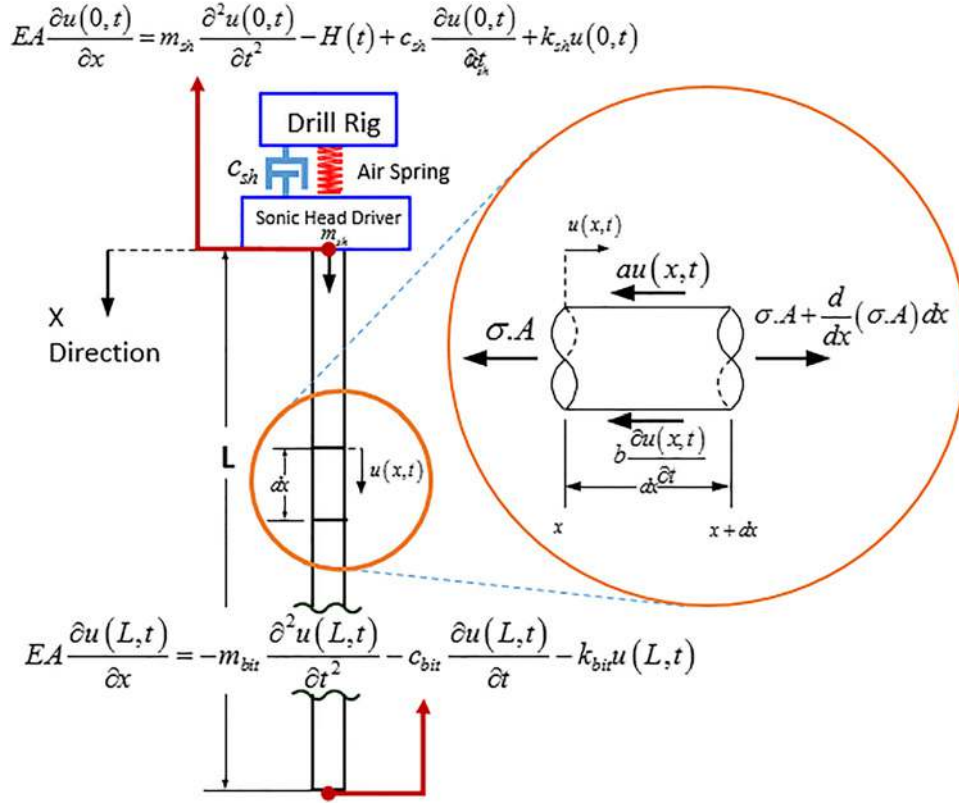


Fig. 2. Sonic drill model [10].

Tip boundary condition:

$$EA \frac{\partial u(L,t)}{\partial x} = -m_{bit} \frac{\partial^2 u(L,t)}{\partial t^2} - c_{bit} \frac{\partial u(L,t)}{\partial t} - k_{bit} u(L,t) \quad (5)$$

where  $m_{bit}$  is the mass of the sonic drill bit,  $k_{bit}$  and  $c_{bit}$  are respectively the spring and the damping rates of the drill bit while drilling.

Despite the performance of the axial vibration controllers proposed in [6,7] by means of adequate simulation results, implementation on real machine may be impractical. Actually, the down hole system's variables measurements, are unrealizable during drilling in underground tunneling unlike in the oil well rotary drilling field.

The problem is to design an observer for the system with only boundary measurements in the top of the drillstring (i.e.  $x=0$ ) available, to estimate the drill bit parameters not accessible by measurements.

### 3. Observer design

We consider the axial vibrations model (3)–(5) represented by a damped wave equation and actuated in the top extremity of the drillstring ( $x=0$ ) by the boundary control input  $U(t)$ .

The investigation of the observer error dynamics's (9)–(11) stability is really defiant, crucial and innovative in the underground tunneling field.

In the following distributed parameter observer design, we assume the availability of  $u(0,t)$  for measurement, as described above. Consider the following distributed parameter observer.

$$\frac{\partial^2 \hat{u}(x,t)}{\partial t^2} + \frac{2b}{\rho} \frac{\partial \hat{u}(x,t)}{\partial t} + \frac{a}{\rho} \hat{u}(x,t) - c^2 \frac{\partial^2 \hat{u}(x,t)}{\partial x^2} = 0 \quad (6)$$

$$EA \frac{\partial \hat{u}(0,t)}{\partial x} = m_{sh} \frac{\partial^2 \hat{u}(0,t)}{\partial t^2} + c_{sh} \frac{\partial \hat{u}(0,t)}{\partial t} - U(t) + k_{sh} \hat{u}(0,t) - G(\hat{u}(0,t) - u(0,t)) \quad (7)$$

$$EA \frac{\partial \hat{u}(L,t)}{\partial x} = -m_{bit} \frac{\partial^2 \hat{u}(L,t)}{\partial t^2} - c_{bit} \frac{\partial \hat{u}(L,t)}{\partial t} - k_{bit} \hat{u}(L,t) \quad (8)$$

where  $\hat{u}$  is the observed value of  $u$  and  $G$  is the observer gain parameter.

Let  $\bar{u} = \hat{u} - u$  the error dynamic and subtracting (6)–(8) by (3)–(5) gives the observer error model:

$$\frac{\partial^2 \bar{u}(x,t)}{\partial t^2} + \frac{2b}{\rho} \frac{\partial \bar{u}(x,t)}{\partial t} + \frac{a}{\rho} \bar{u}(x,t) - c^2 \frac{\partial^2 \bar{u}(x,t)}{\partial x^2} = 0 \quad (9)$$

with the boundary conditions

$$EA \frac{\partial \bar{u}(0,t)}{\partial x} = m_{sh} \frac{\partial^2 \bar{u}(0,t)}{\partial t^2} + c_{sh} \frac{\partial \bar{u}(0,t)}{\partial t} + \tilde{G} \bar{u}(0,t) \quad (10)$$

$$EA \frac{\partial \bar{u}(L,t)}{\partial x} = -m_{bit} \frac{\partial^2 \bar{u}(L,t)}{\partial t^2} - c_{bit} \frac{\partial \bar{u}(L,t)}{\partial t} - k_{bit} \bar{u}(L,t) \quad (11)$$

where  $\tilde{G} = k_{sh} - G > 0$ .

### 4. Well-posedness

In this section, we will prove the global existence and the uniqueness of the solution of problem (3)–(5). For this purpose we will use a semigroup formulation of the initial-boundary value problem (6)–(8).

If we denote  $V := (u, u_t, u(0), u_t(0), u(L), u_t(L))^T$ , we define the energy space:

$$\mathcal{H} = \{(u, v, w_1, w_2, z_1, z_2) \in H^1(0, L) \times L^2(0, L) \times \mathbb{R}^4, w_1 = u(0), z_1 = u(L)\}.$$

Clearly,  $\mathcal{H}$  is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle V_1, V_2 \rangle_{\mathcal{H}} &= \frac{a}{\rho} \int_0^L u^1 u^2 dx + c^2 \int_0^L u_x^1 u_x^2 dx + \int_0^L v^1 v^2 dx \\ &\quad + c^2 \frac{\tilde{G}}{EA} w_1^1 w_1^2 + c^2 \frac{m_{sh}}{EA} w_2^1 w_2^2 \\ &\quad + c^2 \frac{k_{bit}}{EA} z_1^1 z_1^2 + c^2 \frac{m_{bit}}{EA} z_2^1 z_2^2 \end{aligned} \quad (12)$$

for  $V_1 = (u^1, v^1, w_1^1, w_2^1, z_1^1, z_2^1)^T$ ,  $V_2 = (u^2, v^2, w_1^2, w_2^2, z_1^2, z_2^2)^T$ .

Therefore, if  $U \in L_{comp}^2(0, +\infty)$ ,<sup>1</sup> the problem (3)–(5) is formally equivalent to the following abstract evolution equation in the Hilbert space  $\mathcal{H}$ :

$$\begin{cases} V'(t) = AV(t) + BU(t), & t > 0, \\ V(0) = V^0 := (u^0, v^0, w_1^0, w_2^0, z_1^0, z_2^0)^T, \end{cases} \quad (13)$$

where  $'$  denotes the derivative with respect to time  $t$ ,  $B := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m_{sh}} \\ 0 \\ 0 \end{pmatrix}$

and the operator  $\mathcal{A}$  is defined by:

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w_1 \\ w_2 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} v \\ c^2 u_{xx} - \frac{2b}{\rho} v - \frac{a}{\rho} u \\ w_2 \\ \frac{EA}{m_{sh}} u_x(0) - \frac{c_{sh}}{m_{sh}} w_2 - \frac{\tilde{G}}{m_{sh}} w_1 \\ z_2 \\ -\frac{EA}{m_{bit}} u_x(L) - \frac{c_{bit}}{m_{bit}} z_2 - \frac{k_{bit}}{m_{bit}} z_1 \end{pmatrix}.$$

The domain of  $\mathcal{A}$  is given by

$$\mathcal{D}(\mathcal{A}) = \{(u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{H}; u \in H^2(0, L), v \in H^1(0, L), w_2 = v(0), z_2 = v(L)\}.$$

We have the following.

**Theorem 4.1.** *The operator  $\mathcal{A}$  generates a  $C_0$  semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  on  $\mathcal{H}$ .*

**Proof.** According to the Lumer–Phillips' theorem, we should prove that the operator  $\mathcal{A}$  is  $m$ -dissipative.

Let  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$ . By definition of the operator  $\mathcal{A}$  and the scalar product of  $\mathcal{H}$ , we have:

$$\begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= \frac{a}{\rho} \int_0^L v(x)u(x) dx + c^2 \int_0^L v_x(x)u_x(x) dx \\ &\quad + \int_0^L \left( c^2 u_{xx}(x) - \frac{2b}{\rho} v(x) - \frac{a}{\rho} u \right) v(x) dx + c^2 \frac{\tilde{G}}{EA} w_2 w_1 \\ &\quad + c^2 \frac{m_{sh}}{EA} \left( \frac{EA}{m_{sh}} u_x(0) - \frac{c_{sh}}{m_{sh}} w_2 - \frac{\tilde{G}}{m_{sh}} w_1 \right) w_2 + c^2 \frac{k_{bit}}{EA} z_1 z_2 \\ &\quad + c^2 \frac{m_{bit}}{EA} \left( -\frac{EA}{m_{bit}} u_x(L) - \frac{c_{bit}}{m_{bit}} z_2 - \frac{k_{bit}}{m_{bit}} z_1 \right) z_2. \end{aligned}$$

By Green's formula we obtain:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = -\frac{2b}{\rho} \int_0^L v^2(x) dx - c^2 \frac{c_{sh}}{EA} w_2^2 - c^2 \frac{k_{bit}}{EA} z_2^2 \leq 0. \quad (14)$$

Thus the operator  $\mathcal{A}$  is dissipative.

Now we want to show that for  $\lambda > 0$ ,  $\lambda I - \mathcal{A}$  is surjective.

For  $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ , let  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$  solution of

$$(\lambda I - \mathcal{A})V = F,$$

which is:

$$\lambda u - v = f_1, \quad (15)$$

$$\lambda v - c^2 u_{xx} + \frac{2b}{\rho} v + \frac{a}{\rho} u = f_2, \quad (16)$$

$$\lambda w_1 - w_2 = f_3, \quad (17)$$

$$\lambda w_2 - \frac{EA}{m_{sh}} u_x(0) + \frac{c_{sh}}{m_{sh}} w_2 + \frac{\tilde{G}}{m_{sh}} w_1 = f_4 \quad (18)$$

$$\lambda z_1 - z_2 = f_5 \quad (19)$$

$$\lambda z_2 + \frac{EA}{m_{bit}} u_x(L) + \frac{c_{bit}}{m_{bit}} z_2 + \frac{k_{bit}}{m_{bit}} z_1 = f_6. \quad (20)$$

To find  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$  solution of the system (15)–(20), we suppose  $u$  is determined with the appropriate regularity. Then from (15), (17) and (19), we get respectively:

$$v = \lambda u - f_1, \quad w_2 = \lambda u(0) - f_3, \quad z_2 = \lambda u(L) - f_5. \quad (21)$$

Consequently, knowing  $u$ , we may deduce  $v, w_1 = u(0), w_2, z_1 = u(L), z_2$  by (21).

We recall that since  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$  we automatically get  $w_2 = v(0)$  and  $z_2 = v(L)$ .

From Eqs. (16), (18), (20), and (21),  $u$  must satisfy:

$$\lambda^2 u - c^2 u_{xx} + \frac{2b}{\rho} \lambda u + \frac{a}{\rho} u = f_2 + \frac{2b}{\rho} f_1 + \lambda f_1, \quad \text{in } (0, L) \quad (22)$$

with the boundary conditions

$$\lambda^2 u(0) - \frac{EA}{m_{sh}} u_x(0) + \lambda \frac{c_{sh}}{m_{sh}} u(0) + \frac{\tilde{G}}{m_{sh}} u(0) = f_4 + \lambda f_3 + \frac{c_{sh}}{m_{sh}} f_3, \quad (23)$$

$$\lambda^2 u(L) + \frac{EA}{m_{bit}} u_x(L) + \lambda \frac{c_{bit}}{m_{bit}} u(L) + \frac{k_{bit}}{m_{bit}} u(L) = f_6 + \lambda f_5 + \frac{c_{bit}}{m_{bit}} f_5.$$

The variational formulation of problem (22), (23) is to find  $(u, w_1, z_1) \in H := \{(u, w_1, z_1); \omega \in H^1(0, L), w_1 = u(0), z_1 = u(L)\}$  such that:

$$\begin{aligned} &\int_0^L \left\{ \left( \lambda^2 + \frac{2b}{\rho} \lambda + \frac{a}{\rho} \right) u \omega + u_x \omega_x \right\} dx \\ &\quad + c^2 \frac{m_{sh}}{EA} \left( \lambda^2 + \lambda \frac{c_{sh}}{m_{sh}} + \frac{\tilde{G}}{m_{sh}} \right) u(0) w(0) \\ &\quad + c^2 \frac{m_{bit}}{EA} \left( \lambda^2 + \lambda \frac{c_{bit}}{m_{bit}} + \frac{k_{bit}}{m_{bit}} \right) u(L) w(L) \\ &= \int_0^L \left( f_2 + \left( \lambda + \frac{2b}{\rho} \right) f_1 \right) \omega dx \\ &\quad + c^2 w(0) \left( f_4 + \lambda f_3 + \frac{c_{sh}}{m_{sh}} f_3 \right) \\ &\quad + c^2 w(L) \left( f_6 + \left( \lambda + \frac{c_{bit}}{m_{bit}} \right) f_5 \right), \end{aligned} \quad (24)$$

<sup>1</sup>  $L_{comp}^2(0, +\infty) = \{f \in L^2(0, +\infty); \text{the support of } f \text{ is compact}\}.$

for any  $(\omega, \xi_1, \xi_2) \in H$ . Since  $\lambda > 0$ , the left hand side of (24) defines a coercive bilinear form on  $H$ . Thus by applying the Lax–Milgram theorem, there exists a unique  $(u, w_1, z_1) \in H$  solution of (24). Now, choosing  $\omega \in C_c^\infty$ ,  $(u, w_1, z_1)$  is a solution of (22) in the sense of distribution and therefore  $u \in H^2(0, L)$ . Thus using Green’s formula and exploiting Eq. (22) on  $(0, L)$ , we obtain finally:

$$\begin{aligned} & c^2 \frac{m_{sh}}{EA} \left( \lambda^2 + \lambda \frac{c_{sh}}{m_{sh}} + \frac{\tilde{G}}{m_{sh}} \right) w(0)u(0) \\ & + c^2 \frac{m_{bit}}{EA} \left( \lambda^2 + \lambda \frac{c_{bit}}{m_{bit}} + \frac{k_{bit}}{m_{bit}} \right) w(L)u(L) \\ & = c^2 \frac{m_{sh}}{EA} \left( f_4 + \left( \lambda + \frac{c_{sh}}{m_{sh}} \right) f_3 \right) w(0) \\ & + c^2 \frac{m_{bit}}{EA} \left( f_6 + \left( \lambda + \frac{c_{bit}}{m_{bit}} \right) f_5 \right) w(L). \end{aligned}$$

So  $u \in H^2(0, L)$  verifies (18), (20) and we recover  $u, w_1 = u(0), z_1 = u(L)$  and  $v \in H^1(0, L)$  and thus by (21), we obtain  $w_2 = v(0), z_2 = v(L)$ , we have found  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$  solution of  $(I - \mathcal{A})V = F$ .

This completes the proof of Theorem 4.1.  $\square$

We have, in particular, that the problem (6)–(8), which can be rewritten as follows:

$$\begin{cases} Z'(t) = \mathcal{A}Z(t), & t > 0, \\ Z(0) = Z^0 = (u^0, v^0, w_1^0, w_2^0, z_1^0, z_2^0)^T \end{cases} \quad (25)$$

admits for all  $Z^0 \in \mathcal{H}$  a unique solution  $Z(t) = e^{t\mathcal{A}}Z^0 \in C(\mathbb{R}_+; \mathcal{H})$ . Moreover, for  $Z^0 \in \mathcal{D}(\mathcal{A})$ , the system (25) admits an unique solution

$$Z(t) = (u(t), u_t(t), u(0), u_t(0), u(L), u_t(L)) \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A}))$$

and satisfies the following energy identity:

$$\begin{aligned} E(t) - E(0) &= -\frac{2b}{\rho} \int_0^L u_t^2(x) dx - c^2 \frac{c_{sh}}{EA} u_t^2(0, t) \\ &\quad - c^2 \frac{k_{bit}}{EA} u_t^2(L, t), \quad \forall t \geq 0, \end{aligned} \quad (26)$$

where

$$E(t) := \frac{1}{2} \|Z(t)\|_{\mathcal{H}}^2, \quad \forall t \geq 0. \quad (27)$$

Thus the well-posedness of problem (3)–(5) is ensured by:

**Proposition 4.2.** *Let  $U \in L_{comp}^2(0, +\infty), V^0 \in \mathcal{H}$ , then there exists a unique mild solution  $V(t) = e^{t\mathcal{A}}V^0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{B}U(s) ds \in C(\mathbb{R}_+; \mathcal{H})$  of problem (13).*

## 5. Stability

Recall the following frequency domain theorem for exponential stability from [4,15] of a  $C_0$ -semigroup of contractions on a Hilbert space:

**Theorem 5.1** [4,15]. *Let  $A$  be the generator of a  $C_0$ -semigroup of contractions  $S(t)$  on a Hilbert space  $X$ . Then,  $e^{tA}$  is exponentially stable, i.e., for all  $t > 0$ ,*

$$\|e^{tA}\|_{\mathcal{L}(X)} \leq C e^{-\delta t},$$

for some positive constants  $C$  and  $\delta$  if and only if

$$\rho(A) \supset \{i\gamma \mid \gamma \in \mathbb{R}\} \equiv i\mathbb{R}, \quad (28)$$

and

$$\limsup_{|\gamma| \rightarrow +\infty} \|(i\gamma I - A)^{-1}\|_{\mathcal{L}(X)} < \infty, \quad (29)$$

where  $\rho(A)$  denotes the resolvent set of the operator  $A$ .

We are now in a position to state the first main result of this section:

**Theorem 5.2.** *There exist  $C, \delta > 0$  such that*

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\delta t}, \quad \forall t > 0.$$

Our first concern is to show that  $i\gamma$  is not on the spectra of  $\mathcal{A}$  for any real number  $\gamma$ , which clearly implies (28). We have the following:

**Lemma 5.3.** *The spectrum of  $\mathcal{A}$  contains no point on the imaginary axis.*

**Proof.** Since the resolvent of  $\mathcal{A}$  is compact, its spectrum  $\sigma(\mathcal{A})$  only consists of eigenvalues of  $\mathcal{A}$ . We will show that the equation

$$\mathcal{A}V = i\beta V \quad (30)$$

with  $V = (u, v, w_1, w_2, z_1, z_2)^T \in \mathcal{D}(\mathcal{A})$  and  $\beta \in \mathbb{R}$  has only the trivial solution (i.e.  $V = (0, 0, 0, 0, 0, 0)^T$ ).

By taking the inner product of (30) with  $V$  and using

$$\Re \langle \mathcal{A}V, V \rangle_{\mathcal{H}} = -\frac{2b}{\rho} \int_0^L v^2(x) dx - c^2 \frac{c_{sh}}{EA} w_2^2 - c^2 \frac{k_{bit}}{EA} z_2^2, \quad (31)$$

since  $\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \Re(i\beta \|V\|_{\mathcal{H}}^2) = 0$ , then we obtain that  $v = 0, w_2 = 0, z_2 = 0$ .

Next, we get the following ordinary differential equation:

$$\begin{cases} i\beta u = 0, & (0, L), \\ -c^2 u_{xx} + \frac{a}{\rho} u = 0, & (0, L), \\ i\beta u(0) = 0, \\ -\frac{EA}{m_{sh}} u_x(0) + \frac{\tilde{G}}{m_{sh}} w_1 = 0, \\ i\beta u(L) = 0, \\ \frac{EA}{m_{bit}} u_x(L) + \frac{k_{bit}}{m_{bit}} z_1 = 0. \end{cases} \quad (32)$$

• If  $\beta = 0$  then

$$\begin{aligned} 0 &= \int_0^L \left( -c^2 u_{xx} + \frac{a}{\rho} u \right) dx \\ &= c^2 \int_0^L |u_x(x)|^2 dx + \frac{a}{\rho} \int_0^L |u(x)|^2 dx + c^2 \frac{\tilde{G}}{m_{sh}} |u(0)|^2 \\ &\quad + c^2 \frac{k_{bit}}{m_{bit}} |u(L)|^2 \end{aligned}$$

$\Downarrow$

$$u = 0, w_1 = u(0) = 0, z_1 = u(L) = 0.$$

Which implies that  $V \equiv 0$ .

• If  $\beta \neq 0$  then  $u = 0, w_1 = u(0) = 0$  and  $z_1 = u(L) = 0$ . So  $V \equiv (0, 0, 0, 0, 0, 0)^T$ .

We deduce that the system (32) has only the trivial solution.  $\square$

**Proof of Theorem 5.2.** We suppose that condition (29) does not hold. This gives rise, thanks to Banach–Steinhaus Theorem<sup>2</sup> (see [2, Theorem 2.2]), to the existence of a sequence of real numbers  $\gamma_n \rightarrow \infty$  and a sequence of vectors  $V_n = (u_n, v_n, w_n, p_n, z_n, q_n)^T \in \mathcal{D}(\mathcal{A})$  with  $\|V_n\|_{\mathcal{H}} = 1$  such that

$$\|(i\gamma_n I - \mathcal{A})V_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (33)$$

i.e.,

$$i\gamma_n u_n - v_n \equiv f_n \rightarrow 0 \quad \text{in } H^1(0, L), \quad (34)$$

<sup>2</sup> Let  $E$  and  $F$  be two Banach spaces and  $(T_i)_{i \in I}$  be a family (not necessarily countable) of continuous operators from  $E$  into  $F$ . Assume that  $\sup_{i \in I} \|T_i x\|_F < \infty \forall x \in E$ . Then  $\sup_{i \in I} \|T_i\|_{\mathcal{L}(E, F)} < \infty$ .

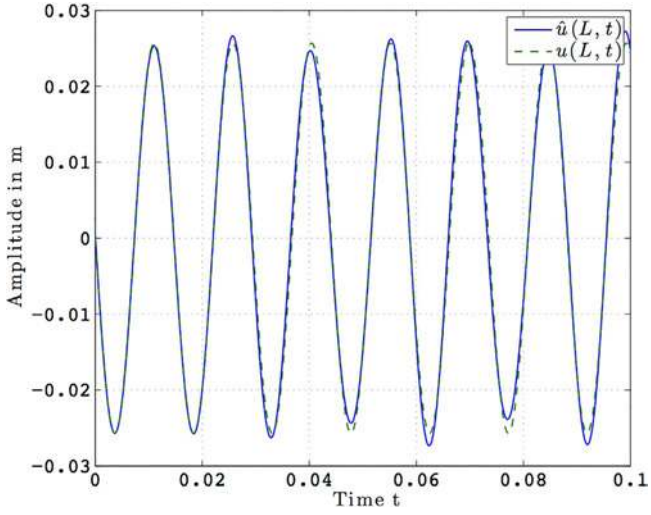


Fig. 3. Practical and estimated axial displacements at the tip boundary ( $x=L$ ),  $G = 1.62 * 10^6$ .

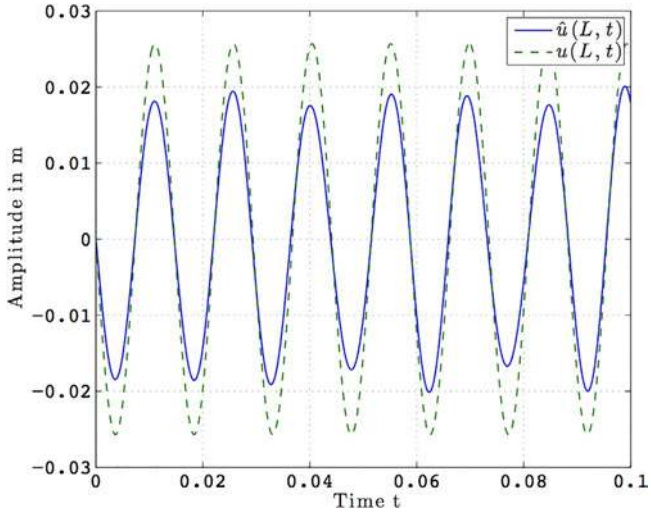


Fig. 4. Practical and estimated axial displacements at the tip boundary ( $x=L$ ),  $G = 1.5 * 10^6$ .

$$i\gamma_n u_n - c^2(u_n)_{xx} + \frac{2b}{\rho} v_n + \frac{a}{\rho} u_n \equiv g_n \rightarrow 0 \text{ in } L^2(0, L), \quad (35)$$

$$i\gamma_n w_n - p_n = a_n \rightarrow 0 \text{ in } \mathbb{C}, \quad (36)$$

$$i\gamma_n p_n - \frac{EA}{m_{sh}}(u_n)_x(0) + \frac{c_{sh}}{m_{sh}} p_n + \frac{\tilde{G}}{m_{sh}} w_n \equiv b_n \rightarrow 0 \text{ in } \mathbb{C}, \quad (37)$$

$$i\gamma_n z_n - q_n = r_n \rightarrow 0 \text{ in } \mathbb{C}, \quad (38)$$

$$i\gamma_n q_n + \frac{EA}{m_{bit}}(u_n)_x(L) + \frac{c_{bit}}{m_{bit}} q_n + \frac{k_{bit}}{m_{bit}} z_n \equiv s_n \rightarrow 0 \text{ in } \mathbb{C}. \quad (39)$$

The ultimate outcome will be convergence of  $\|V_n\|_{\mathcal{H}}$  to zero as  $n \rightarrow \infty$ , which contradicts the fact that  $\forall n \in \mathbb{N}$ ,  $\|V_n\|_{\mathcal{H}} = 1$ .

Firstly, since

$$\begin{aligned} \|(i\gamma_n I - A)V_n\|_{\mathcal{H}} &\geq |\Re(\langle (i\gamma_n I - A)V_n, V_n \rangle_{\mathcal{H}})| = -\Re(\langle AV_n, V_n \rangle_{\mathcal{H}}) \\ &= \frac{2b}{\rho} \int_0^L |v_n(x)|^2 dx + c^2 \frac{c_{sh}}{EA} |p_n|^2 + c^2 \frac{k_{bit}}{EA} |q_n|^2, \end{aligned}$$

it follows from (33) that

$$v_n \rightarrow 0, \rightarrow 0 \text{ in } L^2(0, L) \text{ and } p_n \rightarrow 0, q_n \rightarrow 0 \text{ in } \mathbb{C}. \quad (40)$$

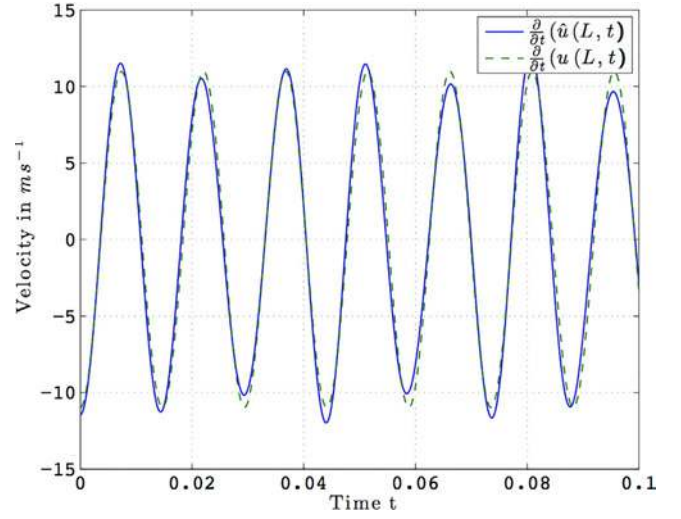


Fig. 5. Drill-bit axial displacements at the boundary  $x=L$  under  $G = 1.7 * 10^6$ .

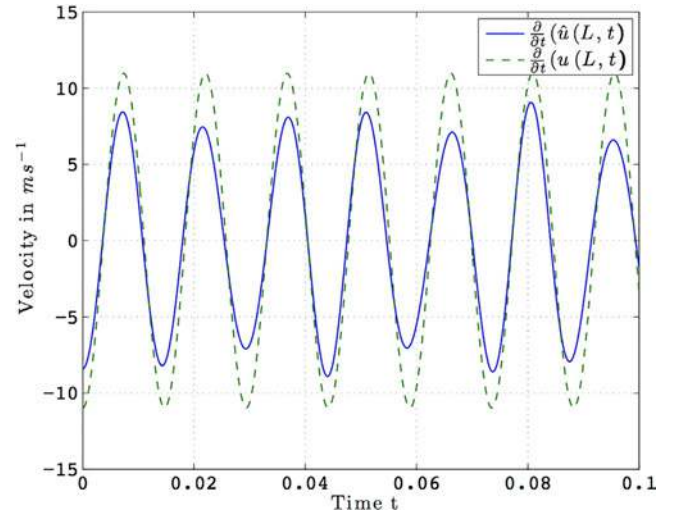


Fig. 6. Drill-bit axial velocities at the boundary  $x=L$  under  $G = 1.7 * 10^6$ .

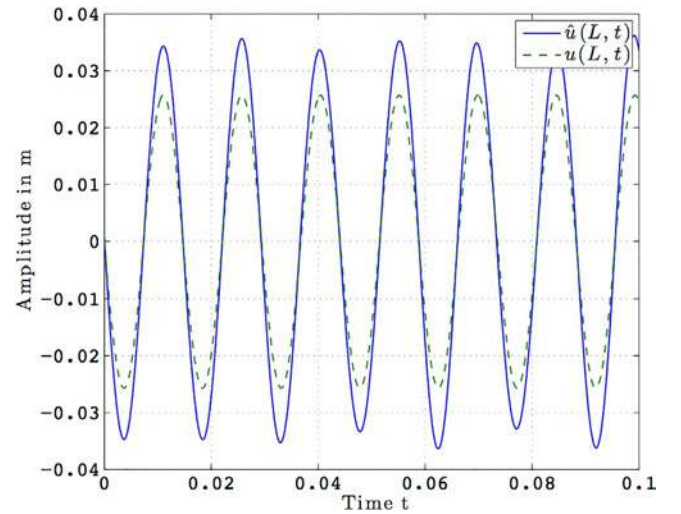


Fig. 7. Practical and estimated axial positions at the tip boundary ( $x=L$ ),  $G = 1.5 * 10^6$ .

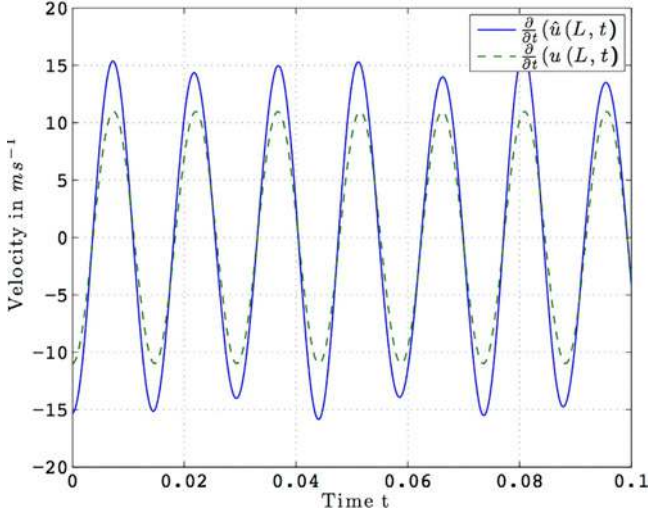


Fig. 8. Practical and estimated axial velocities at the tip boundary ( $x = L$ ),  $G = 1.5 * 10^6$ .

Therewith

$$w_n \rightarrow 0, z_n \rightarrow 0 \text{ in } \mathbb{C}. \quad (41)$$

Now, let us take the inner product of (35) with  $u_n$ . A straightforward computation gives

$$\begin{aligned} & c^2 \int_0^L |(u_n)_x|^2 dx + \frac{a}{\rho} \int_0^L |u_n|^2 dx \\ &= - \int_0^L i\gamma_n u_n v_n dx + \int_0^L g_n u_n dx - \frac{2b}{\rho} \int_0^L v_n u_n dx \\ & - c^2 \frac{\tilde{G}}{m_{sh}} |w_n|^2 - c^2 \frac{k_{bit}}{m_{bit}} |p_n|^2 - \left( i\gamma_n w_n p_n + \frac{c_{sh}}{m_{sh}} p_n w_n \right) \\ & - \left( i\gamma_n z_n p_n + \frac{c_{bit}}{m_{bit}} q_n p_n \right) \rightarrow 0. \end{aligned} \quad (42)$$

In the light of (40), (41) and (42), we conclude that  $\|V_n\|_{\mathcal{H}} \rightarrow 0$  which was our objective.

Lastly, the sufficient conditions of Theorem 5.1 are fulfilled and the proof of Theorem 5.2 is completed.  $\square$

Table 1  
Numerical values used for simulations [12].

|                |  |           |                                |
|----------------|--|-----------|--------------------------------|
| $L$            | 3 m  | $\rho$    | 7850 Kg/m <sup>3</sup>         |
| $E$            | $2.1 * 10^{11}$ Pa                             | $A$       | $8.6 * 10^{-3}$ m <sup>2</sup> |
| $m_{sh}$       | 453.6 Kg                                       | $m_{bit}$ | 8 Kg                           |
| $k_{sh}$       | 84040034.023 N/m                               | $c_{sh}$  | 10 N s/m                       |
| $c^2 = E/\rho$ | $2.6752 * 10^7$ m <sup>2</sup> /s <sup>2</sup> | $m_{ec}$  | 28.4 Kg                        |
| $r_{ec}$       | 0.06 m   | $k_{bit}$ | 1194.519 N/m                   |
| $c_{bit}$      | 1 N s/m  | $b$       | 0 N s/m <sup>4</sup>           |
| $a$            | 2334.434 N/m <sup>4</sup>                      | $\xi$     | 68 Hz                          |

## 6. Simulation results

Using the obtained analytical form of the sonic drill PDE model and integrating the boundaries, at first, we have computing the different model resonant frequencies. For this, we are referred to the work of Lieu and Jovanovic [9] and the Matlab Chebfun/tool in order to solve the analytical form in a frequency domain. The detailed frequency analysis was addressed in [12] where the obtained  $\xi = 68\text{Hz}$  matches the value requested in practice in tunnel reinforcing domain [7]. Recall here, in order to achieve the simulation procedure, we used the Fourier transform to compute the frequency operator of the system. This permits to quantifying the system's performance in the presence of a stimulus, and it characterizes the steady-state response of a stable system to persistent harmonic forcing. Consequently, the model integrating the observer is transformed in  $(x, \xi)$  domain (Fourier transform w.r.t. time  $t$  and temporal frequency  $\xi$ ). In a second step, we study the temporal response using the *intbvp* and *trapz* Matlab functions. Numerical results are performed with the initial model (3)–(5), and the observer scheme (6)–(8). For this, the physical parameters are given in Table 1.

The results are sketched in Figs. 3–9 from which the proposed observer scheme is shown stable/practically stable in terms of the drill-bit displacements and velocities. For example, shows the stability behavior of the observed and the real wave signals. The system's total energy is represented in a frequency domain for  $G = 1.62 * 10^6$  where we prove an important quantity of energy at the resonance  $\omega = 68\text{Hz}$  (Fig. 9 (left)). Fig. 9 (right) shows a less quantity of energy under the observer gain parameter  $G = 0.5 * 10^6$ . Under this last value, the system losses the resonance (figure is not given here). Hence, we may conclude, the adequate choice of the

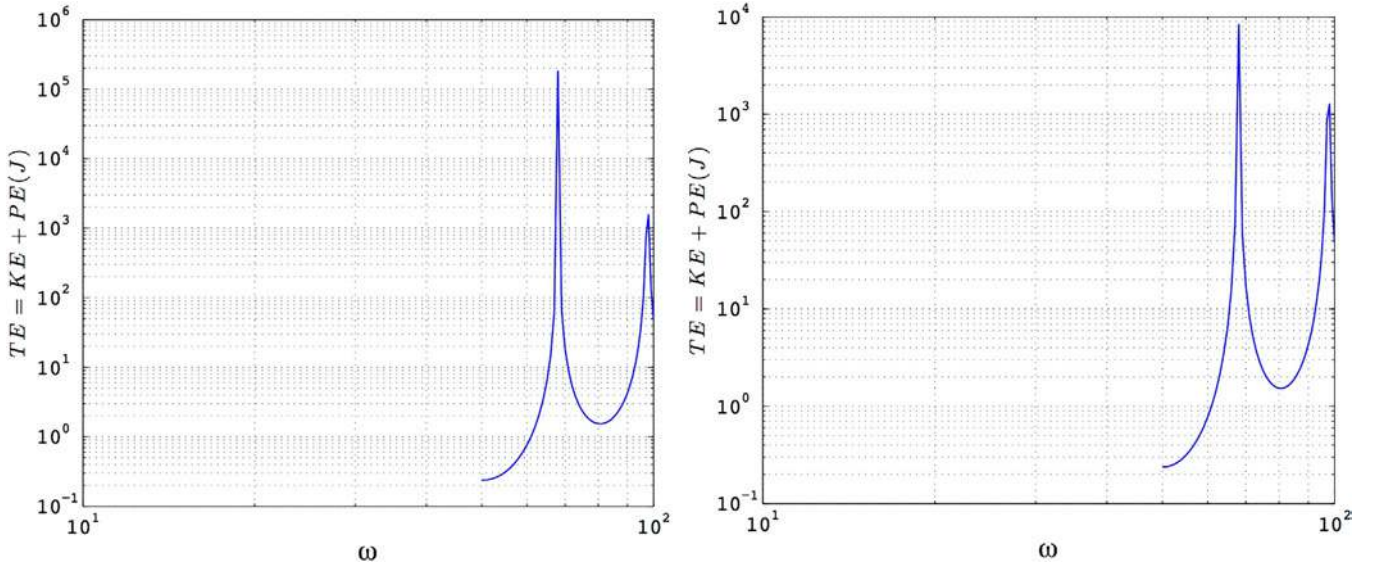


Fig. 9. Total energy, the maximum is emphasized at the resonance ( $\omega = 68$  Hz),  $G = 1.62 * 10^6$  (left) and  $G = 0.5 * 10^6$  (right).



gain parameter will guarantee that the observer works at the drillstring resonance mode. Other parameters related to non homogeneous underground layers can be studied under a control based observer.

## 7. Conclusion

From the idea of being not able to measure the drill bit amplitude at the bottom, defined here as the top boundary, a PDE observer was performed from the drillstring dynamics. Theoretical computations were conducted toward the well-posedness problem and the stability sufficient conditions of the observer. In general, sonic drill operators will monitor hydraulic pressure of motors that drive at resonant rotation of eccentrics and produce the maximum axial vibrations. Note that the operator has to adjust the drillstring behavior and fluid injection till a resonant value. Consequently, to implement axial vibration control and reach an autonomous drilling operation, and thereby the improvement of safety level on construction sites and also reducing the drilling time, a PDE observer is developed for the tip boundary. Future results require a new control input elaboration in agreement with the designed observer. Possibility of integrate and evaluate the effect of moving boundary on the system stability, as for example due to the presence of the cuttings in the crown (The wave equation case is studied in [1]).

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