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REGULARITY OF ENTROPY SOLUTIONS TO EQUATIONS OF CONSERVATION LAWS

CARLO BIANCA^{1,2,*}, CHRISTIAN DOGBE³

ABSTRACT. This paper is devoted to the analysis of solutions of scalar conservation laws where the flux function is a x -dependent function. Specifically we prove that, under less regular fluxes, the maximum between two entropy subsolutions is also an entropy subsolution. The proof of the main result is based on the entropy method of the Kruzhkov theory, P. L. Lions and P. Souganidis methods and the regularization method of Di Perna-Lions.

Key words. First order PDE's, Scalar conservation law, viscosity solutions, regularity
Mathematics Subject Classification. Primary 35F20, 35L65, 35L60, 35F10

1. INTRODUCTION

Let $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$, $(x, t) \mapsto u(x, t) = (u_1, \dots, u_n)$, be the unknown function that is sought, and $f = (f_1, f_2, \dots, f_d) : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ a known nonlinear function, called the flux function. This paper deals with the analysis of the following class of multidimensional nonlinear conservation laws with flux f and without a term source:

$$\begin{cases} \partial_t u + \operatorname{div} f(\mathbf{x}, u) = 0, & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

where $u_0(x)$ is the initial datum, $\partial_t u$ denotes the derivative of u with respect to t , $\partial_i u$ denotes the partial derivative u with respect to the i -th component x_i of \mathbb{R}^d , and $\operatorname{div} = \sum_{i=1}^d \partial_i$. The notation u_{x_i} and $u_{x_i x_i}$ will be also employed for the first and second derivatives of u_i with respect to x_i . In order to control f at infinity, it will be assumed that

$$u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). \quad (1.2)$$

A mathematical theory is well established in the case of a single conservation laws, i.e. when $d = 1$. In this case (1.1), called a ‘scalar conservation law’ (see for example [5, 1, 19]), is of great interest in several areas of physics and engineering. Specifically applications refer to porous media [4], traffic flows on highways with changing road conditions [13, 24], radar shape-from-shading problems, blood flow, gas flow in a variable duct [15], ion etching in the semiconductor industry [18], sedimentation-consolidation processes [2].

From the mathematical analysis point of view, several existence results have been obtained concerning the Cauchy problem for conservation laws bases on Kruzhkov theory (for a flux f independent of the space variable). In this context, the notion of entropy solution has been employed. However due to the nonlinearity and less regularity of the flux $f(x, \cdot)$, the solutions of (1.1) develop discontinuities in finite time, even when the initial data is smooth [12, 5]. Hence, solutions of (1.1) are sought in the sense of distributions. These weak solutions are not necessarily unique. Additional admissibility criteria or entropy conditions need to be imposed in order to select physically relevant solutions.

The main aim of this paper is to extend some existing results in the case of a less regular flux f . In the case of a smooth f , the existence and uniqueness of an entropy solution is provided by the well known method of doubling of variables of the celebrated Kruzhkov's theorem, or by using the measure valued concept by DiPerna [6]. In [16] Panov shows the existence of solution for a flux vector f assumed to be a Carathéodory vector (namely continuous with respect to u and measurable with respect to x). For BV initial data, Karlsen *et al.* [9] has proved the existence of a weak solution by using compensated compactness. A different approach to hyperbolic conservation laws is based on the so-called kinetic formulation. The approach allows for a complete existence theory in the scalar case for initial data that are only assumed to be integrable (see [17] for an extensive presentation of this theory). In this paper, by using the recent ideas developed by P. L. Lions and P. Souganidis, we are interested in the maximum and minimum between two entropy subsolutions of the problem (1.1) which, as will be shown, are also entropy subsolutions. The main difficulty is related to the flux f which here depends on the space variable, and then we do not have an a priori BV bound for the function u . Therefore the general results related to the scalar conservation laws cannot be employed.

The present paper is organized as follows. Section 2 is devoted to notations and background. In Section 3 we state our main result given by Theorem 3.2. The key for the proof of the main result is the attractiveness of solutions given by Lemma 5.2. This lemma gives an estimation for the gap between the solutions of Eq. (1.1). Its proof uses some ideas originally developed for the ODE. Since the fluxes depend on the space variable and are less regular, this requires additional attention in various crucial estimates. However the main novelty is to combine the approach from ordinary differential equation and consequently obtain a stronger result. The difficulty stems in part from the fact that constants are not solution of our problem and the L^1 -contraction property collapses. The proof is based on the combination of comparison principle with compactness arguments. In Section 4 we state and prove a technical lemma that we use to prove our main theorem. Section 5 is devoted to the proof of the main theorem in full generality. The main tool is a maximum principle and total variation estimates and we avoid the compensated compactness approach. In Section 6 we provide some motivating examples. Finally, in Appendix A we recall some technical lemmas using in our proofs.

2. PRELIMINARIES

This section is devoted to a brief summary of basic notations and definitions. Let $r, s \in \mathbb{R}$, $r \wedge s := \min(r, s)$, $r \vee s := \max(r, s)$, $r^+ := r \vee 0$ and $r^- := (-r) \vee 0$, $\mathbf{1}_E$ the characteristic function of the set E . The Banach space $C(\mathbb{R}^+, L^1(\mathbb{R}^d))$ endowed with the norm $\|\phi\|_{C(\mathbb{R}^+, L^1(\mathbb{R}^d))} = \max_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \|\phi(x, t)\| dx$ is considered. The space $\text{BV}(\mathbb{R}^d)$ is defined as follows:

$$\text{BV}(\mathbb{R}^d) = \{g \in L^1(\mathbb{R}^d) : |g|_{\text{BV}(\mathbb{R}^d)} < \infty\}$$

where $|g|_{\text{BV}(\mathbb{R}^d)}$ denotes the total variation of g , i.e., $g \in \text{BV}(\mathbb{R}^d)$ if and only if $g \in L^1(\mathbb{R}^d)$ and the first-order distributional derivatives of g are finite measures on \mathbb{R}^d . The following space is also considered:

$$L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})) := \{u \in L^1(\mathbb{R}^d) \text{ for all compact subset of } \mathbb{R}^d \text{ and continuous}\}.$$

It is worth mentioning that the space $L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R}))$ is not a Banach space since we have L^1_{loc} .

Definition 2.1. A function $u \in C(\mathbb{R}^+, L^1(\mathbb{R}^d)) \cap L_{loc}^\infty(\mathbb{R}^+, L^\infty(\mathbb{R}^d))$ is called an entropy solution of (1.1) if it satisfies (1.1) in the sense of distributions, more precisely, in $\mathcal{D}'(\mathbb{R}^d \times (0, T))$, for $k \in \mathbb{R}$:

$$\partial_t |u - k| + \operatorname{div}[\operatorname{sgn}(u - k)(f(x, u) - f(x, k))] + \operatorname{sgn}(u - k) \operatorname{div} f(x, k) \leq 0, \quad (2.1)$$

with a sufficiently smooth function f . That is, for every differentiable function with compact support $\varphi(x) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x) - k| \varphi_t + \int_{\mathbb{R}^d} [\operatorname{div}(f(x, u(x)) - f(x, k)) \operatorname{sgn}(u(x) - k), \nabla \varphi(x)] \\ + \operatorname{sgn}(u(x) - k) \operatorname{div} f(x, k) \varphi(x)] \geq 0, \end{aligned} \quad (2.2)$$

where (\cdot, \cdot) is the scalar product in \mathbb{R}^d and the function sgn is defined by $\operatorname{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$ and $\operatorname{sgn}(0) = 0$.

It is well known that the condition (2.1) can be rewritten in the following equivalent form: For any convex function $S(u) \in C^1(\mathbb{R})$ (entropy)

$$\begin{aligned} \partial_t S(u) + \operatorname{div}(\eta(x, u)) \\ + S'(u) \operatorname{div}_x f(x, u) - \operatorname{div}_x \eta(x, u) \leq 0, \end{aligned} \quad (2.3)$$

where S is the corresponding entropy flux vector defined by the relation

$$\eta(x, v) := \int_0^v S'(v') F_i(x, v') dv', \quad (2.4)$$

and

$$\begin{aligned} \operatorname{div}_x f(x, u) &:= \sum_{i=1}^d \frac{\partial f_i(x, u)}{\partial x_i}, \quad \operatorname{div} f(x, u) := \sum_{i=1}^d \frac{\partial f_i(x, u)}{\partial x_i} + \frac{\partial f_i(x, u)}{\partial u} \frac{\partial u}{\partial x_i} \\ F_i(x, v) &:= \frac{\partial f}{\partial v}(x, v) \in L_{loc}^\infty(\mathbb{R}; L^\infty(\mathbb{R}^d)). \end{aligned}$$

The reader interested in the main properties of the entropy solutions is referred to Kruzhkov [10] and Vol'pert [22] for the first-order and to Volpert and Hudjaev [23] for the second-order equations.

Bearing all above in mind, it is expected that if u and v are two subsolutions, $u \vee v$ is a subsolution.

The definition 2.1 is more convenient in the following form.

Definition 2.2. A function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ belonging to $L^\infty(\mathbb{R}^d)$ is called an entropy subsolution (respectively supersolution) of (1.1) if for all $k \in \mathbb{R}$ one has:

$$\partial_t (u - k)^+ + \operatorname{div}_x [\operatorname{sgn}^+(u - k)(f(x, u) - f(x, k))] - \mathbf{1}_{u < k} \left(\frac{\partial}{\partial x_i} f \right) \leq 0, \quad (2.5)$$

and

$$\partial_t (k - u)^+ + \operatorname{div}_x [\operatorname{sgn}^+(k - u)(f(x, u) - f(x, k))] - \mathbf{1}_{u > k} \left(\frac{\partial}{\partial x_i} f \right) \leq 0 \quad (2.6)$$

respectively, where $\mathbf{1}_{u > k} = 1$ if $u > k$ and 0 if $u < k$, and $(u - k)^+ := \max(u - k, 0)$ and $(u - k)^- := \min(u - k, 0)$. A function $u \in L^\infty(\mathbb{R}^d)$ is said to be an entropy solution of (1.1) if it is a subsolution and a supersolution of (1.1).

It is well known (see [10]) that for any initial condition $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a unique entropy solution $u \in C([0, +\infty), L^1(\mathbb{R}^d)) \cap L^\infty([0, +\infty) \times L^\infty(\mathbb{R}^d))$.

One of the main keys of the present paper is the fact that the maximum between an entropy

subsolution and a constant is also an entropy subsolution, which is a straightforward consequence of the definitions, particularly, the maximum between two entropy subsolutions is also an entropy subsolution. In the work of P.L. Lions with P. Souganidis [14] it is observed the general fact that the maximum between two entropy subsolutions is also an entropy subsolution. Hence we can rewrite (2.1) in a more convenient form.

Definition 2.3. *A function u is an entropy subsolution (respectively supersolution) of initial value problem (1.1) if*

$$\frac{\partial}{\partial t}(u \vee k) + \frac{\partial}{\partial x_i}(f_i(x_i, u \vee k)) + \mathbf{1}_{(u < k)} \left(\frac{\partial}{\partial x_i} f \right) (x, k) \leq 0. \quad (2.7)$$

Likewise, we have

$$\frac{\partial}{\partial t}(u \wedge k) + \frac{\partial}{\partial x_i}(f_i(x_i, u \wedge k)) + \mathbf{1}_{(u > k)} \left(\frac{\partial}{\partial x_i} f \right) (x, k) \leq 0, \quad (2.8)$$

in the sense of distributions $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^+)$.

It is worth to note that the inequality (2.7) is in the sense of distributions, that is, we perform an integration by parts against a test function. Specifically $\forall \varphi \geq 0$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ one has:

$$\frac{d}{dt} \int \varphi(u \vee k) dx - \sum_{i=1}^d \int f_i(x_i, u \vee k) \frac{\partial \varphi}{\partial x_i}(x) dx - \int \mathbf{1}_{(u > k)} f(x, k) \frac{\partial \varphi}{\partial x_i}(x) dx \leq 0. \quad (2.9)$$

Observe that the quantity $\frac{d}{dt} \int \varphi(u \vee k)$ is continuous in time, since u is continuous in time, then the term $(u \vee k)$ is continuous in time. Accordingly multiplying by φ the integral is continuous in time. Inequality (2.9) is like a differential equation.

3. STATEMENT OF THE MAIN RESULT

The construction of the entropy solution of (1.1) is based on the classical vanishing-viscosity method. Accordingly, we introduce, for any positive real ε , the following viscous problem:

$$\begin{cases} \partial_t u_\varepsilon + \partial_i f_i(x, u_\varepsilon) - \varepsilon \Delta u_\varepsilon = 0, & t \geq 0, \quad x \in \mathbb{R}^d \\ u(0, x) = u_0(x). \end{cases} \quad (3.1)$$

The interest is to understand the behavior of u_ε when ε vanishes. The terms $\partial_t u_\varepsilon$ and $\varepsilon \Delta u_\varepsilon$ converge to $\partial_t u$ and 0, the problem is the term $\partial_i f_i(x, u_\varepsilon)$. It is sufficient to prove that the term $f_i(x, u_\varepsilon)$ converges in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^+)$, but the nonlinearity makes this task complicated.

It is well known that in the case of the homogeneous flux f , i.e. f does not depend on x , the constants are stationary solutions of (1.1). The problem is invariant by translation: Specifically, let h be a vector and τ_h the space shift by this vector: $\tau_h g = g(\cdot + h)$; if u_ε is solution of the approximate equation (3.1), then $\tau_h u_\varepsilon$ is also the solution of the same equation with initial data $a := \tau_h u_0$. In addition one has the following L^1 -contraction property:

$$\int |u_\varepsilon - u_\varepsilon(\cdot + h)| dx \leq \int |u_0(x) - u_0(x + h)| dx, \quad \forall t \geq 0, \quad (3.2)$$

that is

$$\|\tau_h u_\varepsilon - u_\varepsilon\|_{L^1} \leq \|\tau_h a - a\|_{L^1}. \quad (3.3)$$

Since any constant is a solution of the equation, the comparison principle implies that

$$\inf u_0 \leq u(x, t) \leq \sup u_0. \quad (3.4)$$

In the spatially dependent case, the inequality (3.2) is no longer true, namely the spatial translation invariance and convergence arguments based on BV bound break down. However it can be proved that our sequences are compact with respect to the L^1_{loc} -norm.

The real question now is why there is compactness or how are we going to cope when the flux is less regular? For scalar conservation laws, there exist two techniques to obtain compactness in L^1 : The first technique consists in obtaining bounds BV and compactness in L^1 ; the second technique uses the phenomenon of compactification which is really linked to the nonlinear structure of the problem and leads to compactness by compensation [21, 20] or other regularizing effects. It is well-known that compactness arguments based on BV bounds for the existence of solutions for quasilinear hyperbolic conservation laws is limited to one-dimensional systems. Moreover, the multidimensional BV-based scalar existence theory of Kruzhkov [10] hinges in an essential manner on the translation invariance of the underlying solution operator. Lack of translation invariance excludes BV bounds (even in the L^1 -contractive 1D case). Here, the compactness framework in Lemma 5.2 is established to replace the BV compactness framework.

3.1. Structural assumptions on the vector field.

The flux f is assumed to satisfy the following assumption.

Hypothesis 3.1.

(H1) *The flux f is smooth in (x, z) , $x \in \mathbb{R}^d$. More precisely, $f \in \mathcal{C}_b^1(\mathbb{R}^d \times [-M, M])$, $\forall M > 0$.*

(H2) *It is assumed that*

- (i) $f \in W_{loc}^{1,1}(\mathbb{R}^d, \mathbb{R})$;
- (ii) $D_x f \in L^1(\mathbb{R}^d, C^0(\mathbb{R}))$, where $D_x f$ means the derivative part with respect to x .
Actually, we assume that

$$\sup_{|z| \leq M} |D_x f(x, z)| \in L^1_{loc}. \quad (3.5)$$

(H3) $f \in L^1_{loc}(\mathbb{R}^d; C^0_u(\mathbb{R}))$, and $\operatorname{div}_x f(x, 0) \in L^1$.

(H4) $\partial_u f$ exists continuously and $\partial_{x_i} f_i \in L^1_{loc}(\mathbb{R}^d; C^0_u(\mathbb{R}))$.

(H5) $\partial_u \nabla f$ exists continuously and

$$\operatorname{div}_x f(x, z) \operatorname{sgn} z \geq -C_0(1 + |z|) \quad \text{almost everywhere in } x, \text{ for all } z. \quad (3.6)$$

(H6) (i) *The xz -derivative of f is bounded, that is, since u is bounded*

$$\partial_{xz}^2 f \in L^\infty(\mathbb{R}^d \times [-M, +M]) \quad \forall M < \infty \quad (3.7)$$

(ii) *and*

$$\frac{\partial}{\partial x}(\operatorname{div}_x f) \in L^1(\mathbb{R}^d; C^0([-M, +M])), \quad \forall M < \infty. \quad (3.8)$$

Comments on these assumptions.

Several remarks can be made concerning the hypotheses 3.1.

- (i) If f is smooth enough then the above assumptions reduce to only (H1). Concerning hypothesis (H2), if we restrict to the fluxes with separated variables, i.e. of the type $f(x, u) = V(x)g(z)$, this means $V \in W^{1,1}(\mathbb{R}^d)$. This hypothesis is a natural assumption that will justify the doubling of variables.
- (ii) The hypothesis (H3) is rather reasonable because if $u \in L^1$, at infinity u is equal to 0 and we do not want the variation at infinity or the $\operatorname{div}_x f(x, 0)$ to be integrable otherwise it will mean that we are really adding a lot of mass to the infinity. This hypothesis thus allows to control the mass at infinity.

- (iii) The hypothesis (H4) is the minimal regularity that will ensure uniqueness and the contractiveness property. This allows to prove that the maximum of two subsolutions is a subsolution in $W^{1,1}(\mathbb{R}^d)$. The main point here is that constants are not solution to equation (1.1).
- (iv) The sublinearity condition (H5) states that there exists a constant $C_0 \geq 0$ such that, for all z , the previous inequality is true almost everywhere in x . The Hypothesis (H5) ensures the existence of L^∞ bounds.
- (v) Assumptions (H4)-(H5) have key importance. The bound on $\partial_u \nabla f$ ensures the finite propagation speed of the solution. The bound on $\operatorname{div}_x f$ ensures that the solutions are bounded, similarly to the role of sublinearity in the ordinary differential equation.
- (v) Let us point out that, one has a BV-estimate provided that the assumption (H6) is carried out. This is an essential point in the proof of our main theorem. We emphasize that an assumption like (H6) is essential to our analysis. It is a necessary and sufficient conditions in order to obtain existence and uniqueness of the solution.

The statement of the main theorem of this paper follows.

Theorem 3.2. (Less regular case). *Suppose that the flux f satisfies (H2)-(H6) and $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. If u and v are two entropy subsolutions (supersolutions) of (1.1), then the $\max(u, v)$ ($\min(u, v)$) is also an entropy supersolution.*

Remark 3.3. The main theorem has the following meaning.

- (1) In the independent spatial case, the existence of solutions is an easy consequence of the uniqueness and the L^1 -contraction property; in the heterogeneous case where f is smooth regular, the situation is also easy, but the difficult in general case considering that the constants are no longer solutions to the problem. This is quite natural if we think in (1.1) to a linear equation, $f(x, u) = b(x)u$ with $b \in W^{1,1}(\mathbb{R}^d)$ (see [7]); the L^∞ bound will depend on how the flux tends to move away trajectories, or if there are points where trajectories get closer; namely the divergence of the flow controls the L^∞ -norm (well known for the conservative Liouville equation). These results have been first observed by Lions and Souganidis [14].
- (2) The notion of subsolution (respectively supersolution) is different (at least in the formulation) from the usual definition. The main difference from the classical proof of Kruzhkov is the presence of the term $\mathbf{1}_{(u < k)} \partial_{x_i} f(x, k)$ owing to the fact that k is not any more solution of the equation of (1.1). One pays it with the presence of this additional term. This term is not well-defined in the case of non-smooth flux f . But we will prove that for a sufficiently large class of condition on f the uniqueness of solutions can be proved, which is equivalent to the Kruzhkov entropy solution in case that f is smooth.
- (3) Actually, we want to prove that $u \vee v$ is an entropy subsolution and in order to check this, we must look at $(u \vee v) \vee k$, $\forall k \in \mathbb{R}$. But we have

$$(u \vee v) \vee k = (u \vee k) \vee (v \vee k), \quad \forall k \in \mathbb{R},$$

where $(u \vee k)$ and $(v \vee k)$ are two entropy subsolutions. This is not clear since from the inequality (2.7) $u \vee k$ entropy subsolution means the $u \vee k$ is subsolution in the sense of distributions, and it is well-known that distributional solution are not entropy solutions. In fact, if u is an entropy subsolution, then $u \vee k$ is an entropy subsolution not only in sense of distributions. The $u \vee v$ is entropy subsolution means that we have to take the maximum with any other constant ℓ , namely we have to take $u \vee k \vee \ell$ and check that this is a subsolution in the sense of distributions. But

observe that

$$u \vee k \vee \ell = \begin{cases} u \vee k & \text{if } k > \ell \\ u \vee \ell & \text{if } k < \ell \end{cases}$$

in which case we apply again the definition and deduce that u is a subsolution in the sense of distributions. The explanation comes from the fact that we have maximized with the constants and the maximums are trivial. Through the properties of the operation \max , we have automatically that if we have an entropy subsolution, then $u \vee k$ checks the inequalities according to (2.7), and verifies also all the inequalities that come from the entropy subsolutions.

4. TECHNICAL LEMMA

Let $\varrho \in C_0^\infty(\mathbb{R}^d)$ be a smoothing kernel such that $\varrho \geq 0$ and

$$\int_{\mathbb{R}^d} \varrho(z) dz = 1, \quad (4.1)$$

and, for $\delta > 0$, we define the two families of functions $\varrho_\delta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\varrho_\delta(z) = \frac{1}{\delta^d} \varrho\left(\frac{z}{\delta}\right), \quad \varrho_\varepsilon(z) := \varepsilon^{-d} \varrho\left(\frac{z}{\varepsilon}\right)$$

which provide an approximation of the Dirac mass δ_0 . Moreover, we impose that the support of ϱ is included in the ball $B_1 \equiv \{x \in \mathbb{R}^d \mid |x| < 1\}$. We denote for any $g \in L^1_{loc}(\mathbb{R}^d)$

$$g_\delta(x) \equiv (g \star \varrho_\delta)(x) \equiv \int_{\mathbb{R}^d} g(x-y) \varrho_\delta(y) dy.$$

In particular, we define $u_{0\delta} \equiv u_0 \star \varrho_\delta$, where \star denotes the convolution operator.

The following Lemma will be useful in the proof of our main theorem and propositions. The Lemma 4.1 below is strongly reminiscent of Lemma II.1 in [7] and serves the same purpose.

Lemma 4.1. (Commutator estimate) *Assume that (H2)-(H6) hold true. Let R_ε and D_ε the two commutators define as follows:*

$$R_\varepsilon := - \int (f_i(x, u(s) \vee v(s)) - f_i(y, u(s) \vee v(s))) (\partial_i \varrho_\varepsilon)(x-y) ds \quad (4.2)$$

$$D_\varepsilon := - \int (\mathbf{1}_{(u < v)} \partial_i f_i(x, v) + \mathbf{1}_{(v < u)} \partial_i f_i(y, v)) \varrho_\varepsilon(x-y) dy. \quad (4.3)$$

Then

$$\|R_\varepsilon - D_\varepsilon\|_{L^1} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (4.4)$$

and

$$\|R\|_{L^1} \leq C_0 \sup_{|z| \leq M} |D_x f(x, z)| \|L^1. \quad (4.5)$$

Proof. The proof is divided into two parts: The first part is dedicated to the case where the flux is smooth enough, the second part for the case where the flux is less regular. The two terms have not the same difficulty.

Step 1. The case where the flux f is regular. Since $\partial_i f(x, v)$ is regular the term D_ε does not raise any difficulty. The flux f is integrable, hence one can use the Lebesgue's derivation theorem. Thus, the term D_ε converges to $(\partial_i f)(x, u \vee v)$ and it is a convergence in L^1 . The indicator functions disappear and one has:

$$- \int [\mathbf{1}_{(u < v)} \partial_i f_i(x, v) - \mathbf{1}_{(v < u)} \partial_i f_i(y, v)] \varrho_\varepsilon(x-y) dy \xrightarrow[\varepsilon \rightarrow 0]{L^1 \text{ p.p.}} (\partial_i f)(x, u \vee v), \quad (4.6)$$

by definition of indicator functions. The term R_ε is an oscillation of functions, but the $\partial_i \varrho_\varepsilon$ is obviously much more irregular. In its derivation, one recovers an extra $1/\varepsilon$ which is compensated by $f_i(x) - f_i(y)$. Thus, this term will be analyzed by the well-known commutator lemma. Indeed, formally, since $x - y < \varepsilon$ with support of ϱ_ε in the ball of radius ε , then we can approximate $f(x) - f(y)$ by

$$f(x) - f(y) \sim \partial_j f(x) \times (x_j - y_j),$$

such that

$$(f_i(x) - f_i(y)) (\partial_i \varrho_\varepsilon) (x - y) dy = \int \partial_j f(x) \int \frac{x_j - y_j}{\varepsilon} \frac{1}{\varepsilon^d} \partial_i \varrho_\varepsilon \left(\frac{x - y}{\varepsilon} \right) dy.$$

Performing the change of variable $z = \frac{x - y}{\varepsilon}$, one has:

$$\int \frac{x_j - y_j}{\varepsilon} \frac{1}{\varepsilon^d} \partial_i \varrho_j \left(\frac{x - y}{\varepsilon} \right) = \int z_j \partial_i \varrho(z) dz = -\delta_{ij} = \begin{cases} 0 & i \neq j \\ \int_{\mathbb{R}^d} \varrho & i = j. \end{cases}$$

Step 2. The case when the flux is less regular. By elementary properties of mollifiers, one has:

$$D_\varepsilon = - \int (\mathbf{1}_{(u < v)} \partial_i f_i(x, v) + \mathbf{1}_{(v < u)} \partial_i f_i(y, v)) \varrho_\varepsilon(x - y) dy \xrightarrow[\varepsilon \rightarrow 0]{L^1_{loc}} - \partial_i f_i(x, u \vee v). \quad (4.7)$$

By (4.4), it is therefore enough to prove

$$R_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} - \partial_i f_i(x, u \vee v) \quad \text{in } L^1_{loc}(\mathbb{R}^d). \quad (4.8)$$

Due to the Hypothesis 3.5, R_ε is bounded in $L^1_{loc}(\mathbb{R}^d)$. Next, keeping in mind that v depends on y , that is

$$f_i(x, u \vee v) - f_i(y, u \vee v) = f_i(x, u \vee v(x - \varepsilon z)) - f_i(x - \varepsilon z, u \vee v(x - \varepsilon z)),$$

and by using the Taylor formula

$$\varphi(y) - \varphi(x) = \int_0^1 \nabla \varphi(x + s(y - x))(y - x) ds$$

for smooth functions and performing the change of variable $x - y$ to $z = \frac{x - y}{\varepsilon}$, yields:

$$\begin{aligned} R_\varepsilon &:= \int (f_i(x, u \vee v) - f_i(y, u \vee v)) (\partial_i \varrho_\varepsilon) (x - y) dy \\ &= - \int_{B_1} dz \int_0^1 ds z_j \partial_j f_i(x - \varepsilon s z, u \vee v) (\partial_i \varrho_\varepsilon)(z) dz. \end{aligned} \quad (4.9)$$

Indeed, the powers of ε disappeared thanks to the increase of $(f(x, u \vee v) - f(y, u \vee v))$. It remains to show that (4.9) is bounded in L^1 . Therefore, by the Vitali's convergence theorem, one has:

$$R_\varepsilon \rightarrow - \left(\int_{B_1} z_j (\partial_i \varrho_\varepsilon)(z) dz \right) \partial_j f_i(x, u \vee v) = - \partial_i f_i(x, u \vee v). \quad (4.10)$$

It remains to prove the bound (4.5). Since u and v are bounded: $|u| \leq M$, and $|v| \leq M$ almost everywhere, $\partial_i \varrho_\varepsilon$ is bounded, one has:

$$\begin{aligned} &\left| - \int_{B_1} dz \int_0^1 ds z_j \partial_j f_i(x - \varepsilon s z, u \vee v) (\partial_i \varrho)(z) dz \right| \\ &\leq C_0 \int_{B_1} dz \int_0^1 \sup_{|\xi| \leq M} |\partial_j f_i(x + \varepsilon z, \xi)|, \end{aligned} \quad (4.11)$$

where C_0 is a constant independent of ε . By integrating the right-hand side in the space variable (justified by the regularity of the terms we have written), and by the invariance of the Lebesgue integral by translation, one has:

$$\int_{\mathbb{R}^d} C_0 \int_{B_1} dz \int_0^1 \sup_{|\xi| \leq M} |\partial_j f_i(x + \varepsilon z, \xi)| dx = \int \sup_{|z| \leq M} |D_x \cdot f|(x, z) dx. \quad (4.12)$$

Hence

$$\int (f_i(x, u \vee v) - f_i(y, u \vee v)) (\partial_i \varrho_\varepsilon)(x - y) ds = \int \sup_{|z| \leq M} |D_x \cdot f|(x, z) dx. \quad (4.13)$$

This means, the L^1 norm of R_ε is estimates by a constante independent of ε :

$$\|R\|_{L^1} \leq C_0 \left\| \sup_{|z| \leq M} |D_x f(x, z)| \right\|_{L^1}.$$

This completes the proof. \square

In what follows the case of the homogeneous conservation law, i.e. f does not depend on x , is discussed and in particular the following Cauchy problem:

$$\begin{cases} u_t + \operatorname{div} f(u) = 0 & \text{in } \mathbb{R}^d \times (0, \infty) \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^d. \end{cases} \quad (4.14)$$

We have the following results.

Proposition 4.2. (Homogeneous case) *Let $u = u(x, t)$ and $v = v(x, t)$ be two entropy subsolutions (respectively, supersolutions) of (4.14). Then $w(x) := \max(u(x), v(x))$ (respectively $\min(u, v)$) is still an entropy subsolution (respectively supersolution) of (4.14).*

For the general scalar conservation laws (1.1), we will prove the following result.

Proposition 4.3. (Heterogeneous case and regular case)

- (1) *Assume (H1) and let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. Let u and v be two entropy subsolutions of (1.1). Then $\max(u, v)$ is still an entropy subsolution. If u and v be two entropy supersolutions of (1.1), then $\min(u, v)$ is still an entropy supersolution.*
- (2) *Assume (H2). Let u and v be two entropy subsolutions of (1.1). Then $\max(u, v)$ is still an entropy subsolution. If u and v are two entropy supersolutions of (1.1), then $\min(u, v)$ is still an entropy supersolution.*

Proof of proposition 4.2. The proof is based on the Kruzhkov's doubling of variables technique. It is worth stressing that a function $u = u(x, t) \in L^\infty$ is a entropy supersolution of problem (1.1) if and only if $-u$ is a entropy subsolution of the Cauchy problem for the equation $u_t + \operatorname{div}_x(-f(-u)) = 0$ with the initial function $-u_0(x)$; hence, the second assertion of the proposition 4.2 follows from the first.

In the case of first-order homogeneous fluxes f , it is enough to check that the function $w(x) = \max(u(x), v(x))$ satisfies the Definition 2.1. The core of the proof is the regularization by convolution of problem (4.14). Let $u(x, t)$ and $v(y, s)$ be entropy subsolutions to the problem (4.14). Since

$$\begin{aligned} \max(u, k) &:= u \vee k = (k + (u - k)^+), & (u - k)^+ &= \max(u, k) - k, \\ 2 \max(u, k) &= |u - k| + (u + k), \\ 2f(\max(u, k)) &= (f(u) - f(k)) \operatorname{sgn}(u - k) + (f(u) + f(k)), \end{aligned}$$

we then write the entropy inequality (2.7) for the solution $u \vee v$ at point (y, s) as follows:

$$\frac{\partial}{\partial t} (u \vee v(y, s)) + (f_i(u \vee v(y, s)))_{x_i} \leq 0, \quad (4.15)$$

in the sense of distributions, more precisely, in $\mathcal{D}'(\mathbb{R}^d \times (0, T))$ almost everywhere in (y, s) . Exchanging the functions u and v and the variables (x, t) and (y, s) , we obtain the similar inequality

$$\frac{\partial}{\partial s}(u(x, t) \vee v) + (f_i(u(x, t) \vee v))_{y_i} \leq 0, \quad (4.16)$$

in the sense of distributions $\mathcal{D}'(\mathbb{R}^d \times (0, T))$ and almost everywhere in (x, t) . Summing up (4.15) and (4.16), the resulting equation reads:

$$\frac{\partial}{\partial t}(u \vee v(y, s)) + \frac{\partial}{\partial s}(u(x, t) \vee v) + (f_i(u(x, t) \vee v))_{y_i} + (f_i(u \vee v(y, s)))_{x_i} \leq 0 \quad (4.17)$$

which can be rewritten as follows:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s} \right) [(u \vee v(y, s)) + (u(x, t) \vee v)] \\ & + (\operatorname{div}_x + \operatorname{div}_y)[f_i(u(x, t) \vee v) + (f_i(u \vee v(y, s)))] \leq 0. \end{aligned} \quad (4.18)$$

Relation (4.17) or (4.18) is the keystone of the proof. We will prove that inequality (4.17) implies that $w = u \vee v$ is a subsolution, in the sense of \mathcal{D}' , of the problem (1.1). But we want to prove that $u \vee v$ is an entropy subsolution. To force the variables to be the same, we use a test function (approximation of the unit): $\varrho_{\delta, \varepsilon} := \varrho_\delta(t - s)\varrho_\varepsilon(x - y)$, where $\varrho_\delta(t - s)$ is dimension 1 and $\varrho_\varepsilon(x - y)$ is d -dimension. Obviously, it is clear that the sequence of functions $\varrho_\varepsilon(z)$ converges (which is meant in the sense of distributions $\mathcal{D}'(\mathbb{R})$) to the Dirac δ -measure as $\varepsilon \rightarrow 0$. Next, we take $k = v(y, s)$ in the first equation and integrate with respect to (s, y) ; then we take $\ell = u(x, t)$ in the second and integrate with respect to (t, x) . Summing up the results and rearranging the terms therein, one has:

$$\begin{aligned} & \frac{\partial}{\partial t} \int (u \vee v(y, s))\varrho_\varepsilon(x - y)\varrho_\delta(t - s)dsdy + \frac{\partial}{\partial x_i} \int \varrho_\varepsilon f_i(u \vee v(y, s))\varrho_\delta(t - s)dsdy \\ & - \int (\varrho_\varepsilon)_{x_i}(x - y)f_i((u \vee v)(y, s))\varrho_\varepsilon(t - s)dyds \\ & - \int f_i(u, \vee v(y, s))\varrho_\delta(t - s)\frac{\partial}{\partial y_i}(\varrho_\varepsilon(x - y))dyds \leq 0, \end{aligned} \quad (4.19)$$

since the terms containing the derivatives of $\varrho_\delta(t - s)$ cancel out. It remains to analyze two terms. The first one which we rewrite as follows:

$$\begin{aligned} & \int \varrho_\varepsilon(x - y)\varrho_\delta(t - s) \left[\frac{\partial}{\partial x_i}(f_i(u \vee v(y, s))) \right] dsdy \\ & = \frac{\partial}{\partial x_i} \int \varrho_\varepsilon(x - y)f_i(u \vee v(y, s))\varrho_\delta(t - s)dsdy \\ & - \int \frac{\partial}{\partial x_i}[\varrho_\varepsilon(x - y)]f_i((u \vee v)(y, s))\varrho_\varepsilon(t - s)dyds, \end{aligned} \quad (4.20)$$

and the second term which we integrate by part to have:

$$\begin{aligned} & \int \varrho_\varepsilon(x - y)\varrho_\delta(t - s) \left[\frac{\partial}{\partial y_i}f_i(u(x, t) \vee v) \right] dsdy \\ & = - \int f_i(u \vee v(y, s))\varrho_\delta(t - s)\frac{\partial}{\partial y_i}[\varrho_\varepsilon(x - y)] dsdy. \end{aligned} \quad (4.21)$$

One thereby sees that, in the first term of (4.19), as $\delta \rightarrow 0$, $\varrho_\delta \rightarrow \delta$ and since $v(y, s) \in L^1$ then $v(y, s) \rightarrow v(y, t)$. Hence,

$$\frac{\partial}{\partial t} \int (u \vee v(y, s))\varrho_\varepsilon(x - y)\varrho_\delta(t - s)dsdy \xrightarrow{\delta \rightarrow 0} \frac{\partial}{\partial t} \int (u(x, t) \vee v(y, t))\varrho_\varepsilon(x - y)dy. \quad (4.22)$$

Likewise

$$\begin{aligned} & \int (\varrho_\varepsilon)_{x_i}(x-y) f_i((u \vee v)(y, s)) \varrho_\delta(t-s) dy ds \\ & - \int f_i(u, \vee v(y, s)) \varrho_\delta(t-s) \frac{\partial}{\partial y_i} (\varrho_\varepsilon(x-y)) dy ds \\ & \xrightarrow{\delta \rightarrow 0} \int (\varrho_\varepsilon)_{x_i}(x-y) f_i((u \vee v)(y, s)) dy ds - \int f_i(u, \vee v(y, s)) \frac{\partial}{\partial y_i} (\varrho_\varepsilon(x-y)) dy ds. \end{aligned}$$

Finally, taking the limit as $\delta \rightarrow 0$ in (4.19), we reduce this inequality into the following form:

$$\begin{aligned} & \frac{\partial}{\partial t} \int (u(x, t) \vee v(y, t)) \varrho_\varepsilon(x-y) dy + \frac{\partial}{\partial x_i} \int f_i(u(x, t) \vee v(y, t)) \varrho_\varepsilon(x-y) dy \quad (4.23) \\ & - \int \left(\frac{\partial}{\partial x_i} \varrho_\varepsilon \right) (x-y) f_i((u \vee v)(y, s)) dy ds - \int f_i(u, \vee v(y, s)) \frac{\partial}{\partial y_i} (\varrho_\varepsilon(x-y)) dy ds \leq 0. \end{aligned}$$

Since $\partial y_i (\varrho_\varepsilon(x-y)) = -[\partial x_i \varrho_\varepsilon(x-y)]$, the two terms of (4.23) cancel each other out and we get

$$\frac{\partial}{\partial t} \int (u(x, t) \vee v(y, t)) \varrho_\varepsilon(x-y) dy + \frac{\partial}{\partial x_i} \int f_i(u(x, t) \vee v(y, t)) \varrho_\varepsilon(x-y) dy \leq 0, \quad (4.24)$$

in the sense of distributions. Then taking the limit as $\varepsilon \rightarrow 0$, we have:

$$\partial_t (u(x, t) \vee v(y, t)) + \operatorname{div} f(u(x, t) \vee v(y, t)) \leq 0, \quad (4.25)$$

that is, for the arbitrary test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi \geq 0$:

$$\int (u(x, t) \vee v(y, t)) \varphi_t dx + \int f(u(x, t) \vee v(y, t)) \nabla \varphi(x) dx \geq 0. \quad (4.26)$$

This justifies that $w(x) = \max(u(x), v(x))$ is an entropy subsolution. \square

Proof of Proposition 4.3. We will accomplish the proof in two steps.

First case. We are going to show that u is actually an entropy solution to (1.1), that is $u \vee v$ is a subsolution to (1.1) in the sense of distributions from Definition 2.3. To this end, we proceed exactly as in the proof of Proposition 4.2. We take $k = v(y, s)$ and $\ell = u(x, t)$ in (2.6). We then write the entropy inequality (2.7) for the solution $u \vee v$ at point (x, t) , noting that the dependence of f on the space variable x produces additional terms:

$$\frac{\partial}{\partial t} (u \vee v(y, s)) + f_i(x_i, u \vee v(y, s))_{x_i} - \mathbf{1}_{(u(x) < v(y))} \partial_i f_i(x, v(y)) \leq 0. \quad (4.27)$$

Likewise for equation $u \vee v$ at point (y, s) :

$$\frac{\partial}{\partial s} (u(x, t) \vee v) + f_i(y_i, u(x, t) \vee v)_{y_i} - \mathbf{1}_{(v(y) < u(x))} \partial_i f_i(y, u(x)) \leq 0. \quad (4.28)$$

Summing up the last two inequalities, one obtains the key equation:

$$\begin{aligned} & \frac{\partial}{\partial t} (u(x, t) \vee v(y, t)) + f_i(x_i, u \vee v)_{x_i} + f_i(y_i, u \vee v)_{y_i} \\ & - \mathbf{1}_{(u(x) < v(y))} \partial_i f_i(x, v(y)) - \mathbf{1}_{(v(y) < u(x))} \partial_i f_i(y, u(x)) \leq 0. \quad (4.29) \end{aligned}$$

It remains to prove that the inequality (4.29) leads to show that $u \vee v$ is a subsolution in the sense of $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^+)$ to Eq. (1.1), namely:

$$\frac{\partial}{\partial t} (u \vee v) + \operatorname{div} f(x, u \vee v) \leq 0. \quad (4.30)$$

The regularization is employed. For the same reason as in the previous proposition 4.2, we take the following regularization i.e. a test function $\varrho_{\delta,\varepsilon} := \varrho_\delta(t-s)\varrho_\varepsilon(x-y)$. This leads to the relation:

$$\begin{aligned} & \frac{\partial}{\partial t} \int (u \vee v) \varrho_\varepsilon(x-y) dy + \frac{\partial}{\partial x_i} \left(\int f(x, u \vee v) \varrho_\varepsilon(x-y) dy \right) \\ & + \frac{\partial}{\partial y_i} \left(\int f_i(y, u \vee v) \varrho_\varepsilon(x-y) dy \right) - \int (f_i(x, u \vee v) - f_i(y, u \vee v)) (\partial_i \varrho_\varepsilon(x-y)) dy \\ & - \int (\mathbf{1}_{(u < v)} \partial_i f_i(x, v) - \mathbf{1}_{(v < u)} \partial_i f_i(y, v)) \varrho_\varepsilon(x-y) dy \leq 0 \\ & := J_1 + J_2 + J_3 + J_4 + J_5 \leq 0. \end{aligned} \quad (4.31)$$

The convergence of the terms J_i , for $i = 1, 2, 3$ is exactly the same as in Proposition 4.2. Due to the step 1 of Lemma 4.1, the sum of the fourth and fifth terms vanishes and one has (4.30).

Second case. The proof comes down by analyzing the two terms of inequality (6.22) in the case where f is less regular.

$$\begin{aligned} C & := - \int (f_i(x, u \vee v) - f_i(y, u \vee v)) (\partial_i \varrho_\varepsilon)(x-y) dy \\ D & := - \int (\mathbf{1}_{(u < v)} \partial_i f_i(x, v) + \mathbf{1}_{(v < u)} \partial_i f_i(y, v)) \varrho_\varepsilon(x-y) dy. \end{aligned}$$

There are two difficulties in passing to the limit. The first one is the lack of compactness in space since this term is an oscillations of functions, which can be solved by using Lemma 4.1. The second one is the lack of compactness in time, due to the presence of high oscillations in time (acoustic waves). Observe that all the derivatives are in L^1 and the divergence $\partial_i f_i \in L^1$. It is then straightforward to deduce that:

$$- \int (\mathbf{1}_{(u < v)} \partial_i f_i(x, v) + \mathbf{1}_{(v < u)} \partial_i f_i(y, v)) \varrho_\varepsilon(x-y) dy \xrightarrow[\varepsilon \rightarrow 0]{L^1_{loc}} -\partial_i f_i(x, u(x, t) \vee v(x, t)).$$

It remains for us to show that:

$$- \int (f_i(x, u \vee v) - f_i(y, u \vee v)) (\partial_i \varrho_\varepsilon)(x-y) dy - \partial_i f_i(x, u \vee v) \xrightarrow[\varepsilon \rightarrow 0]{L^1_{loc}} 0, \quad (4.32)$$

uniformly with respect to ε . This is done by the second part of Lemma 4.1 and justifies how in the regular case $w(x) = \max(u(x), v(x))$ is an entropy subsolution of (1.1) and conclude the proof of Proposition 4.3. \square

5. PROOF OF THE MAIN THEOREM

The proof proceeds through a series of lemmas, which shows that a subsequence of u_ε of Eq. (3.1) converges a.e. as $\varepsilon \rightarrow 0$.

5.1. A priori estimates.

To achieve the desired compactness properties we need to establish several a priori estimates. The key point in proving the Theorem 3.2 is to obtain a priori estimates and almost everywhere convergence of a subsequence $\{u_\varepsilon\}$ solution to (3.1). These uniform bounds in ε will allow us to establish the necessary compactness in order to pass into the limit $\varepsilon \rightarrow 0$ obtaining the global existence of the solutions of the original problem (1.1).

Lemma 5.1. *Let $u_0 \in L^1 \cap L^\infty$. The solution u_ε of the approximate problem (3.1) satisfies, for $t \geq 0$, the following uniform estimates:*

$$\|u_\varepsilon\|_{C([0,T];L^1(\mathbb{R}^d))} \leq \|u_0\|_{L^1(\mathbb{R}^d)}. \quad (5.1)$$

$$\|u_\varepsilon(x, t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0(\cdot)\|_{L^\infty(\mathbb{R}^d)}. \quad (5.2)$$

$$\int_{\mathbb{R}^d} \left| \frac{\partial u_\varepsilon}{\partial x_i}(x, t) \right| dx \leq \int_{\mathbb{R}^d} \left| \frac{\partial u_0}{\partial x_i}(x) \right| dx. \quad (5.3)$$

It is worth stressing that the estimation (5.3) explains how we can get a BV type estimates, which means that the gradient belongs to the space of bounded measures $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^+)$. This estimate provides compactness in order to prove existence.

Proof of lemma 5.1. In what follows, the superscript ε is omitted except when the emphasis is necessary. In the case of homogeneous flux, these bounds follows easily from the comparison principle. However, in the present case, this argument does not hold, since constants are not solutions of (1.1).

- **L^1 bound.** The proof of the L^1 -bound is standard and exploits the comparison principle (the maximum principle) with appropriate subsolution. Since the system is conservative one has:

$$\|u_\varepsilon\|_{C([0, T]; L^1(\mathbb{R}^d))} \leq \|u_0\|_{L^1(\mathbb{R}^d)}. \quad (5.4)$$

- **L^∞ bounds.** We analyze the L^∞ bounds when the flux is space dependent. The computations can be justified in terms of approximations. Unlike the x -independence solution, the L^∞ bounds are not always true as in L^1 theory. To see that the solution associated with the problem (1.1) is uniformly bounded, we use firstly the chain rule in (1.1), obtaining

$$\operatorname{div} f(\mathbf{x}, u) = \frac{\partial f_i}{\partial z}(x, u) \frac{\partial u}{\partial x_i} + (\operatorname{div}_x f)(\mathbf{x}, u).$$

Therefore equation (1.1) can also be written as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial f_i}{\partial z}(x, u) \frac{\partial u}{\partial x_i} + (\operatorname{div}_x f)(\mathbf{x}, u) = 0. \quad (5.5)$$

By hypothesis **(H3)**, since the derivative $\frac{\partial f_i}{\partial z}(\mathbf{x}, u) \frac{\partial u}{\partial x_i}$ vanishes in an extremum point, using the maximum principle we get

$$\frac{d}{dt} \|u\|_{L^\infty} \leq C_0(1 + \|u\|_{L^\infty}). \quad (5.6)$$

Then we can apply Gronwall's inequality to conclude. We thus obtain a L^∞ bound.

- **Uniform spatial BV estimate.** The next step is to prove an uniform BV type estimate for the entropy solutions u of (1.1). We will make a formal demonstration, but it can be made rigorously by introducing a sequences of smooth functions $\operatorname{sgn}_\eta(\xi)$ that converges, when $\eta \rightarrow 0$, to the function $\operatorname{sgn}(\xi)$.

Let us differentiate the approximate equation (3.1) with respect to x_k , ($1 \leq k \leq d$) and integrate over \mathbb{R}^d . Setting $D_k u = \frac{\partial u}{\partial x_k}$, one has:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x_k} \right) \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx + \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_k} f_i(\mathbf{x}, u) + \frac{\partial f_i}{\partial z}(\mathbf{x}, u) u_k \right) \operatorname{sgn} \left(\frac{\partial u}{\partial x_i} \right) dx \\ &\quad - \varepsilon \int_{\mathbb{R}^d} \frac{\partial}{\partial x_k} (\Delta u) \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx, \end{aligned} \quad (5.7)$$

almost everywhere in $(0, T) \times \mathbb{R}^d$. The first term of the right-hand side of (5.7) is equal to:

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x_k} \right) \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx = \frac{d}{dt} \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial x_k} \right| dx.$$

On the other hand, let us observe that for the x -derivative of the third term of (5.7) gives:

$$\partial_i(\partial_{x_k} f_i(\mathbf{x}, u)) = \partial_{x_i x_k}^2 f_i(\mathbf{x}, u) + \partial_{x_k z}^2 f_i(\mathbf{x}, u) \frac{\partial u}{\partial x_i};$$

thus introducing the above expression in (5.7) and passing to the absolute value we get:

$$\frac{\partial}{\partial t} |Du| + \partial_i \left(\frac{\partial f_i}{\partial z} |Du| \right) + \left| \left(\partial_{x x_k}^2 f(x, u) + \partial_{x_k z}^2 f_i \frac{\partial u}{\partial x_i} \right) \frac{\partial_k u}{|Du|} \right| \leq 0. \quad (5.8)$$

For the second term, one has:

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} (f_i(\mathbf{x}, u)) \left(\frac{\partial u}{\partial x_i} \right) \operatorname{sgn} \left(\frac{\partial u}{\partial x_i} \right) &= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \left(f(x, u) \frac{\partial u}{\partial x_i} \right) \operatorname{sgn} \left(\frac{\partial u}{\partial x_i} \right) \\ &= - \int_{\mathbb{R}^d} f(x, u) \frac{\partial u}{\partial x_k} \frac{\partial}{\partial x_i} \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx. \end{aligned}$$

and observe that, formally, one has:

$$\frac{\partial u}{\partial x_k} \frac{\partial}{\partial x_i} \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) = 0.$$

Finally, the fourth term of (5.7) is positive, because an integration by part shows:

$$\begin{aligned} -\varepsilon \int_{\mathbb{R}^d} \frac{\partial}{\partial x_k} (\Delta u) \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx &= -\varepsilon \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \sum \frac{\partial^2 u}{\partial x_i^2} \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx \\ &= -\varepsilon \int_{\mathbb{R}^n} \sum \frac{\partial}{\partial x_i} \frac{\partial^2 u}{\partial x_k \partial x_i} \operatorname{sgn} \left(\frac{\partial u}{\partial x_k} \right) dx \\ &= \varepsilon \int_{\mathbb{R}^n} \operatorname{sgn}' \left(\frac{\partial u}{\partial x_k} \right) \left(\frac{\partial^2 u}{\partial x_i \partial x_k} \right)^2 dx, \end{aligned}$$

and this last expression is positive because $\operatorname{sgn}' \left(\frac{\partial u}{\partial x_k} \right)$ is a positive distribution. Thus, we obtain:

$$\frac{d}{dt} \int |Du| \leq \int \left| \frac{\partial}{\partial x} (\operatorname{div}_x f)(x, u) \right| dx + \int \left| \frac{\partial^2}{\partial x \partial z} f \right| |Du|. \quad (5.9)$$

Thanks to the Hypothesis 3.1-(H6), the conclusion follows from Gronwall inequality giving a BV type estimate. This ends the proof. \square

A careful examination of the above proofs reveals that in the smooth regular cases, the existence, uniqueness and the comparison principle are obtained without the assumption of regularity (besides hypotheses (H1)-(H6)). Thus, a possible strategy will be to try to estimate how the solution depends on the fluxes, with the hope that from uniqueness result, the solution will depend continuously on the flux, and therefore, using the argument for uniqueness, we can obtain solutions from the more smooth situations with more hypotheses. Bearing all above in mind, we will be particularly interested in the gap between two solutions.

Nevertheless, there two difficulties. The first is that: even we have a good framework of solutions, for a given flux f , can we able to use then uniqueness of solutions to construct Cauchy sequences and deduce that all solutions approach these sequences? The second one is that since in our case the fluxes are not the same, we cannot use the technique of doubling of variable since the estimations we will obtain, inevitably going to reveal gaps on the solutions. To overcome these difficulties it is necessary to develop a convenient tools in order to get rid of this problem. This is done as in Lemma 5.2, since the convolution provides compactness in space and the continuity equation provides compactness in time.

5.2. Gaps between two solutions.

Key tools in this section is the following lemma related to controlling the gap between two solutions.

- We define the following functional $W_\delta^p(u, v)$:

$$W_\delta^p(u, v) := \iint |u(x) - v(y)|^p \varrho_\delta(x - y) dx dy \quad 1 \leq p < \infty. \quad (5.10)$$

It is a gap between two solutions u and v except that it does not follow from a norm, where ϱ_δ is a regularizing kernel. This object naturally appears in the Kruzhkov demonstrations.

Lemma 5.2. *Let W_δ be the functional defined in (5.10). Then, one has:*

$$W_\delta^1(u, v) \geq \|u - v * \varrho_\delta\|_{L^1}; \quad \|W_\delta^1(u, v)\| \leq \|u - v\| + W_\delta^1(v, v). \quad (5.11)$$

Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Then

$$\limsup_{\delta \rightarrow 0} \limsup_{n, m} W_\delta^1(u_n, u_m) = 0; \quad \limsup_{\delta \rightarrow 0} \limsup_{n, m} W_\delta^2(u_n, u_m) = 0 \quad (5.12)$$

that is

$$W_\delta^2(u_n, u_m) \leq \omega_\delta(n, m) + \alpha(\delta)$$

with

$$\omega_\delta(n, m) \xrightarrow{n, m \rightarrow \infty} 0, \quad \alpha(\delta) \xrightarrow{\delta \rightarrow 0} 0.$$

It is worth stressing that the information in Lemma 5.2 allows to pass to the limit in all terms in entropy formulation of Eq. (1.1). The key for the convergence statement of Lemma 5.2 hinges on a method inherited from ordinary differential equations.

Proof of Lemma 5.2. The proof of Lemma 5.2 relies on the following observation:

$$W_\delta^p(u, v) := \int |u(x)|^p dx \int \varrho_\delta(x - y) dx.$$

Putting $u = v$, we deduce that $W_\delta^p \neq 0$ unless u is constant; thus W_δ^p is not a distance. At the same time, for fixed u, v , as $\delta \rightarrow 0$ in (5.10), this leads to:

$$W_\delta^p \xrightarrow{\delta \rightarrow 0} \|u - v\|_{L^p}^p. \quad (5.13)$$

Some facts concerning the cases $p = 1$ and $p = 2$.

- We begin with the simpler case $p = 1$. We are interested in estimating the L^1 distance between v and u . Clearly, one has:

$$W_\delta^1(u, v) \geq \|u - v * \varrho_\delta\|_{L^1} \quad (5.14)$$

or symmetrically

$$W_\delta^1(u, v) \geq \|u - v * \varrho_\delta\|_{L^1} \vee \|u * \varrho_\delta - v\|_{L^1}.$$

Indeed

$$\iint |u(x) - v(y)| \varrho_\delta(x - y) \geq \int dx \left| \int (u(x) - v(y)) \varrho_\delta(x - y) dy \right|,$$

and

$$\int u(x) \varrho_\delta(x - y) dy = u(x), \quad \int v(y) \varrho_\delta(x - y) dy = v * \varrho_\delta,$$

given the L^1 -norm of $(u - v)$. Meanwhile

$$\iint |u(x) - v(x)| \varrho_\delta(x - y) \leq \iint \left(|u(x) - v(y)| + |v(x) - v(y)| \right) \varrho_\delta(x - y) dx dy,$$

and then

$$\begin{aligned} \|W_\delta^1(u, v)\| &\leq \iint |u(x) - v(y)| \varrho_\delta(x - y) dx dy + \iint |v(x) - v(y)| \varrho_\delta(x - y) \\ &= \|u - v\| + W_\delta^1(v, v), \end{aligned}$$

which tends to 0 when $\delta \rightarrow 0$. Symmetrically, we have

$$\|W_\delta^1(u, v)\| \leq \|u - v\| + W_\delta^1(v, v) \wedge W_\delta^1(u, u).$$

• For $p = 2$, it is easy to work in the Fourier space (we denote by \widehat{u} the Fourier transforms of u in the x variable). One has:

$$\iint |u(x) - v(y)|^2 \varrho_\delta(x - y) dx dy = \int u^2(x) + v^2(y) - 2 \iint u(x)v(y) \varrho_\delta(x - y) dx dy$$

which in terms of Fourier transform, after normalization, gives

$$\iint |u(x) - v(y)|^2 \varrho_\delta(x - y) dx dy = \int |\widehat{u}|^2 + |\widehat{v}|^2 - 2\widehat{u}\widehat{v}\widehat{\varrho},$$

which shows that if $\widehat{\varrho} = 1$, then

$$\iint |u(x) - v(y)|^2 \varrho_\delta(x - y) dx dy = \int |\widehat{u}|^2 + |\widehat{v}|^2 - 2\widehat{u}\widehat{v},$$

given L^2 -norm. But since one has: $\varrho \geq 0$, then

$$|\widehat{\varrho}| \leq \widehat{\varrho}(0) = 1,$$

such that

$$\int |\widehat{u}|^2 + |\widehat{v}|^2 - 2\widehat{u}\widehat{v}\widehat{\varrho} \geq \int (|\widehat{u}| - |\widehat{v}|)^2,$$

given the idea to how the functional W_δ^2 controls the gap between the two solutions. \square

The next Lemma claims that we have a good estimate on $W_\delta^1(u_n, u_m)$ where u_n is solution of approximate equation; that is $\{u_n\}$ is a Cauchy sequence. Loosely speaking, contrary to the operation $\limsup_{\beta} \limsup_{\gamma}$ which commutates, interchanging the limiting operations $\limsup_{\beta} \limsup_{\gamma}$ is not possible. The convergence criterion on the sequence $\{u_n\}$ must be introduced.

Lemma 5.3. (Cauchy sequence) *Let (u_n) be the approximate solution of the problem (1.1). Then*

$$\limsup_{\delta \rightarrow 0} \limsup_{n, m} W_\delta^1(u_n, u_m) = 0. \quad (5.15)$$

Proof. We will work in L^2 -norm using the fact that:

$$(\sqrt{a} - \sqrt{b})^2 \leq (a - b)^2, \quad a, b \geq 0,$$

which allows to go from a gap W_δ^1 to a gap W_δ^2 and also in order to have a weak-convergence we need to pass to the limit. Due to the estimate (5.2), one has:

$$\|u_n\|_{L^\infty} \leq C.$$

Since we need weak-star convergence, we will work with L^2 -norm. The relation (5.15) means that

$$W_\delta^2(u_n, u_m) \leq \omega_\delta(n, m) + \alpha(\delta),$$

where $\omega_\delta(\cdot, \cdot)$ denotes a modulus of continuity, satisfying the following relation:

$$\omega_\delta(n, m) \xrightarrow{n, m \rightarrow \infty} 0 \quad \text{and} \quad \alpha(\delta) \xrightarrow{\delta \rightarrow 0} 0,$$

for n and m fixed such that

$$\limsup_{\delta \rightarrow 0} \limsup_{n,m} W_\delta^2(u_n, u_m) = 0.$$

Let u be a weak limit of the subsequence (u_m)

$$u_m \rightharpoonup u \quad \text{weakly-}^* \text{ in } L^2.$$

On the one hand, from (5.10), (where $v = u_m$), the functional W_δ^p is a convex expression when $\varrho_\delta > 0$. Hence, by weak-star topology of L^2 , one deduces that

$$W_\delta^2(u_n, u) \leq \omega_\delta(n) + \alpha(\delta)$$

In others words, one has a fixed u . On the other hand, by applying Jensen's inequality one has

$$\omega_\delta(n) + \alpha(\delta) \geq \|u_n - u \star \varrho_\delta\|_{L^2}.$$

Thus

$$\|u_n - u\|_{L^2} \leq \omega_\delta(n) + \alpha(\delta) + \|u - u \star \varrho_\delta\|_{L^2}. \quad (5.16)$$

But since

$$\alpha(\delta) + \|u - u \star \varrho_\delta\|_{L^2} \xrightarrow{\delta} 0,$$

as a result, the difference in the left-right hand side of (5.16) goes to 0 uniformly in n as $\delta \rightarrow 0$, since u_n is weakly compact in $L^1((0, T) \times \mathbb{R}^d)$. This implies that u_n converge strongly to u in L^2 (which is also true in L^1):

$$u_n \longrightarrow u \quad \text{strongly in } L^2$$

and completes the proof. \square

Now the Theorem 3.2 can be proved. With the above lemmas, it is clear that the essential problem in this strategy is to study the compactness of the approximate solutions with respect to $\varepsilon \rightarrow 0$ of (3.1) in order to obtain global entropy solutions

Proof of Theorem 3.2. The proof of Theorem 3.2 is based on Kruzhkov's idea of doubling variables, but in the rest of the proof different ideas have to be used to handle the non-linear terms. We follow the strategy first proposed by Lions [14]. At the crucial point in the proof, we will apply Lemma 5.2 using techniques developed in [14]. For simplicity we drop the index ε . Let $n \in \mathbb{N}$ and let χ_n be the characteristic function of the ball $B_n(0) \subset \mathbb{R}^d$. We regularize the initial data as follows:

$$u_0^n \equiv (u_0 \chi_n) \star \varrho_{1/n}.$$

We now obtain the existence of some $u_n \in C((0, T); L^1(\mathbb{R}^d))$, solving the following problem:

$$\begin{cases} \partial_t u_n + \operatorname{div} f(x, u_n) - \varepsilon \Delta u_n = 0 \\ u_n(0, \cdot) = u_0^n \end{cases} \quad (5.17)$$

The proof is presented in two steps. First we establish the existence of entropy solutions of (1.1) in the sense of Definition 2.2 and then, in comparison with the arguments used in ODE (see in appendix), we show a kind of attractiveness of the solutions of equation (1.1). The novelty is not so much in the particular arguments but rather in the identification of the additional term which makes the proof even more interesting.

Step 1. Existence of solution with initial datum assuming to satisfy $u_0^n \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Let $\{u_n\}$ be the sequence of solutions to the approximate problem (5.17) with initial data $u_0^n \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and such that

$$\|u_0 - u_0^n\|_{L^1(\mathbb{R}^d)} \xrightarrow{n \rightarrow \infty} 0. \quad (5.18)$$

Let $f_n(x, u)$ be the fluxes for which, thanks to the BV-estimates (5.3), one has the solutions u_n . For Lemma 5.1 these solutions u_n are bounded in the spaces $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$:

$$\|u_n\|_{C([0, \infty[; L^1(\mathbb{R}^d))} \leq C \quad \text{and} \quad \|u_n\|_{L^\infty([0, \infty[\times \mathbb{R}^d)} \leq C. \quad (5.19)$$

It is worth to note that we have any estimates on solutions u_n in BV. Choose a smooth $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Granted these two a priori bounds (5.19), since the sequence $u_n(x, t)$ satisfies (5.19), then $f(x, u_n)$ is weakly sequentially continuous with respect to the sequence u_n .

The next step consists in constructing a classical solution of the problem as limit of a Cauchy sequence in L^1 of approximate solutions of the problem. Since it is not possible to obtain BV estimates for the approximate solutions u_n , the idea is to prove that (u_n) is a Cauchy sequence. To be more precise, we want to prove that the sequence (u_n) is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R}))$ hence $u \in L^1_{\text{loc}}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R}))$ and u_n converges toward u in $L^1_{\text{loc}}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R}))$. To this end, we need to have an estimation on the term which raises the problem, that is the integral:

$$\iint \|u_n(x) - u_m(y)\| \varrho_\delta(x - y) dx dy, \quad (5.20)$$

in the sense of distributions describing a small gap between the approximate solutions and the solution we want to build. We exploit Kruzhkov-Kuznetsov's idea of doubling variables [10, 11]. Let us denote $u = u(x, t)$ and $v = v(y, s)$ for $(x, t), (y, s) \in \mathbb{R}^d \times \mathbb{R}_+^*$. Let m and n be two integers, and let u_n and u_m denote two entropy solutions corresponding to u_0^n and u_0^m respectively of the Cauchy problem (1.1). We apply the Definition 2.3 to u_n and u_m . Since u_n solves Eq. (1.1) point-wise, for each (y, s) , we can write, in the sense of distributions:

$$\frac{\partial}{\partial t} (u_n \vee u_m(y, s) - u_m) + f_i(x_i, u_n \vee u_m(y, s))_{x_i} - \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f_n(x, u_m(y)) \leq 0. \quad (5.21)$$

Note that it is the x dependency that introduces the last term of (5.21). Interchanging the roles of u_n and u_m we similarly obtain, for any fixed point (x, t) in $\mathbb{R}^d \times [0, +\infty)$:

$$\frac{\partial}{\partial s} (u_n(x, t) \vee u_m - u_n) + f_i(y_i, u(x, t) \vee u_m)_{y_i} - \mathbf{1}_{u_m(y) < u_n(x)} \operatorname{div}_y f_m(y, u_n(x)) \leq 0. \quad (5.22)$$

Summing up the obtained inequalities to bring back at the same time, one finds that:

$$\begin{aligned} & \frac{\partial}{\partial t} (u_n(x, t) - u_m(y, t))^+ + (f_i(x_i, u_n \vee u_m(y, s))_{x_i} + f_i(y_i, u(x, t) \vee u_m)_{y_i}) \\ & - (\mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f_n(x, u_m(y)) - \mathbf{1}_{u_m(y) < u_n(x)} \operatorname{div}_y f_m(y, u_n(x))) \leq 0. \end{aligned} \quad (5.23)$$

Following [8], we compute the time derivative after multiplying the equation (5.23) by $\varrho_\delta(x - y)$. Then integrating over (x, y) (which is justified by the regularity of the terms

we have written), the resulting inequality reads:

$$\begin{aligned}
& \frac{d}{dt} \iint (u_n(x, t) - u_m(y, t))^+ \varrho_\delta(x - y) dx dy \\
& - \frac{\partial}{\partial x_i} \left(\iint f_i(x_i, u_n \vee u_m(y, s))_{x_i} \varrho_\delta(x - y) dx dy \right) \\
& + \iint \left\{ \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f_n(x, u_m(y)) + \mathbf{1}_{u_m(y) < u_n(x)} \operatorname{div}_x f_m(y, u_n(x)) \right\} \varrho_\delta(x - y) dx dy \\
& + \iint \left\{ f_i^n(x, u_n(x) \vee u_m(y)) - f_i^m(y, u_n(x) \vee u_m(y)) \right\} \frac{\partial}{\partial x_i} \varrho_\delta(x - y) dx dy \leq 0. \quad (5.24)
\end{aligned}$$

As a result,

$$\begin{aligned}
& \frac{d}{dt} \iint |u_n(x) - u_m(y)| \varrho_\delta(x - y) dx dy \quad (5.25) \\
& \leq \left| \frac{\partial}{\partial x_i} \left(\iint f_i(x_i, u_n \vee u_m(y, s))_{x_i} \varrho_\delta(x - y) dx dy \right) \right. \\
& \quad + \iint \left[\left\{ \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f_n(x, u_m(y)) + \mathbf{1}_{u_m(y) < u_n(x)} \operatorname{div}_x f_m(y, u_n(x)) \right\} \varrho_\delta(x - y) \right. \\
& \quad \left. \left. + \iint \left\{ f_i^n(x, u_n(x) \vee u_m(y)) - f_i^m(y, u_n(x) \vee u_m(y)) \right\} \frac{\partial}{\partial x_i} \varrho_\delta(x - y) dx dy \right] \right|.
\end{aligned}$$

Step 2. Passing to the limit.

Passing to the limit in nonlinear terms requires convergence in a strong topology. The difficulty in passing to the limit in (5.25) is that the fluxes are of order $1/\delta$ due to the presence of ϱ_δ . Terms in right-hand side require more work. This difficulty is overcome by using the commutator Lemma 4.1.

Now we let $n \rightarrow \infty$ by using (H3)-(H5). For δ fixed, one realizes that, due to (3.6) and (5.6), one has strong convergence, i.e. convergence in L^1_{loc} :

$$f_n \xrightarrow[n \rightarrow \infty]{} f, \quad \text{and} \quad f_m \xrightarrow[m \rightarrow \infty]{} f, \quad \text{in } L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})). \quad (5.26)$$

Since u_n and u_m are uniformly bounded thanks to (5.19), we can pass to the limit in the terms of divergence to get:

$$\operatorname{div}_x f_n(x, u_m(y)) \xrightarrow[n \rightarrow \infty]{} \operatorname{div}_x f(x, u_m(y)) \quad \text{in } L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})). \quad (5.27)$$

Similar arguments yield:

$$\operatorname{div}_y f_m(y, u_n(x)) \xrightarrow[m \rightarrow \infty]{} \operatorname{div}_y f(y, u_n(x)) \quad \text{in } L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})) \quad (5.28)$$

and also for the term:

$$f_i^n(x, u_n(x) \vee u_m(y)) \xrightarrow[n \rightarrow \infty]{} f_i(x, u_n(x) \vee u_m(y)) \quad \text{in } L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})). \quad (5.29)$$

Likewise

$$f_i^m(y, u_n(x) \vee u_m(y)) \xrightarrow[m \rightarrow \infty]{} f_i(y, u_n(x) \vee u_m(y)) \quad \text{in } L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R})). \quad (5.30)$$

After combining these limits, that is to use the regularization and the convergence in $L^1_{loc}(\mathbb{R}^d; \mathcal{C}_u^0(\mathbb{R}))$, we conclude that:

$$\begin{aligned}
& \frac{d}{dt} \iint |u_n(x) - u_m| \varrho_\delta(x-y) dx dy \\
& \leq \left| \iint \left\{ \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f(x, u_m(y)) + \mathbf{1}_{u_m < u_n(x)} \operatorname{div}_x f(y, u_n(x)) \varrho_\delta(x-y) \right\} \right. \\
& \quad \left. + \iint \left\{ f_i(x, u_n(x) \vee u_m(y)) - f_i(y, u_n(x) \vee u_m(y)) \right\} \partial_i \varrho_\delta(x-y) dy \right| + \omega_\delta(n, m), \\
& := I_1^\delta + I_2^\delta + \omega_\delta(n, m), \tag{5.31}
\end{aligned}$$

where the term $\omega_\delta(n, m)$ is the error made by passing from f_i^n and f_i^m to f . This means that, by regularization, we are brought back to the same flows by paying $\omega_\delta(n, m)$ as error. We will consider each term in the above inequality, respectively. To be more precise, what remains is to investigate the limit of each term in (5.31), that is:

$$\limsup_{\delta \rightarrow 0} \limsup_{n, m} \iint |u_n(x) - u_m| \varrho_\delta(x-y) dx dy. \tag{5.32}$$

We start with the term I_1^δ . Thanks to the smoothness of $\operatorname{div}_x f$ and $\operatorname{div}_y f$, one can easily take the limit $\delta \rightarrow 0$ in I_1^δ . We can invoke Lebesgue differentiation theorem, which leads to:

$$\begin{aligned}
& \iint (\mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f(x, u_m(y)) + \mathbf{1}_{u_m < u_n(x)} \operatorname{div}_x f(y, u_n(x)) \varrho_\delta(x-y)) dx dy \\
& \xrightarrow{\delta \rightarrow 0} \operatorname{div}_x f(x, u_n(x) \vee u_m(y)) \text{ in a.e. } L^1.
\end{aligned}$$

On the other hand, we deal with I_2^δ . This term is more difficult because it is an oscillation of functions, but the $\partial_i \varrho_\delta$ is more irregular since by derivation, one brings an additional term $1/\delta$ which is compensated by $f_i(x, u_n) - f_i(y, u_m)$ and that creates $\frac{x-y}{\delta}$. We will show the convergence of this term via commutation lemma.

From (3.5) we know that I_2^δ is bounded in $L^1_{loc}(\mathbb{R}^d)$. Taking into account the Lemma 4.1, we have:

$$\iint \left\{ f_i(x, u_n(x) \vee u_m(y)) - f_i(y, u_n(x) \vee u_m(y)) \right\} \partial_i \varrho_\delta(x-y) dy \xrightarrow[\delta \rightarrow 0]{\text{a.e.}} \partial_i f_i(x, u_n \vee u_m) \text{ in } L^1.$$

It is then straightforward to deduce that:

$$\left| \frac{d}{dt} \iint |u_n(x) - u_m(y)| \varrho_\delta(x-y) dx dy \right| \leq \omega_\delta(n, m) + \alpha(\delta), \tag{5.33}$$

with

$$\omega_\delta(n, m) \xrightarrow{n, m \rightarrow \infty} 0 \quad \text{and} \quad \alpha(\delta) \xrightarrow{\delta \rightarrow 0} 0, \tag{5.34}$$

and $\alpha(\delta)$ corresponds to the fact that one has uniqueness for the limiting problem. We deduce that

$$\limsup_{\delta \rightarrow 0} \limsup_{n, m} \iint |u_n(x) - u_m| \varrho_\delta(x-y) dx dy = 0. \tag{5.35}$$

Actually, the compactness is replaced by the previous bound. In order to conclude the proof, we deal with the initial data. Thanks to the L^1 -contraction principle (3.2) for solutions of the limit equation we have, for all integers $n, m \in \mathbb{N}$, for all $t \geq 0$:

$$\int |u_n(t, x) - u_m(t, x)| \varrho_\delta dx dy \leq C \|u_0^n - u_0^m\|_{L^1(\mathbb{R}^d)}. \tag{5.36}$$

Loosely speaking, we have

$$\|u_n - u_m\|_{C^1([0,T];L^1(\mathbb{R}^d))} \leq \|u_0^n - u_0^m\|_{L^1(\mathbb{R}^d)} \longrightarrow 0, \quad \text{as } n, m \rightarrow \infty, \quad (5.37)$$

which gives, together with (6.15), that $t_0 \in [0, T]$, $u_n(t_0, x)$ forms a Cauchy sequence in $L^1(\mathbb{R}^d)$ and admits a limit $u(t_0, x)$. We conclude that sequence $\{u_n\}$ is a Cauchy sequence in $C([0, T]; L^1(\mathbb{R}^d))$; thus there exists a function $u \in C([0, T]; L^1(\mathbb{R}^d))$ such that u_n converges towards u as $n \rightarrow \infty$ in $C([0, T]; L^1(\mathbb{R}^d))$, satisfying the same distributional formula. Hence the proof is concluded. \square

6. EXTENSIONS AND APPLICATIONS

This section deals with some examples that fit into (1.1) and to which the above analytical results can be applied.

Example 6.1. *The first example is the case where the flux f depends on the space variable through a coefficient k and the dependence is of the multiplicative type:*

$$\begin{cases} \partial_t u + \operatorname{div}(k(x)f(u)) = 0, & (x, t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d \end{cases} \quad (6.1)$$

where $k = (k_1, \dots, k_d)$ is a vector field, $f = f(u)$ is a scalar function. This equation expresses that u is conserved with a flux density given by $k(x)f(u)$. Such conservation arise in a diversity of contexts ranging from models of traffic flow via models of flow in porous media, to hydrodynamic limits of nearest particle processes, and sedimentation-consolidation processes [2]. Note that the flux $f(x, u) = k(x)g(u)$ can have a possibly discontinuous spatial dependence through the coefficient k , which is allowed to have jump discontinuities, but we are not concerned with this kind of problem. A simple physical model corresponding to (6.1) is the Witham model of car traffic flow on a highway [13, 24]. The spatially varying coefficient k corresponds to changing road conditions. In a similar way, the entropy inequality for (6.1) that we impose now is:

$$\int |u - c| \phi_t + \operatorname{sgn}(u - c)[k(x)(f(u) - f(c))] \cdot \nabla \phi - \operatorname{sgn}(u - c) \operatorname{div} k(x)f(c)\phi \, dx \geq 0. \quad (6.2)$$

Using Definition 2.7 we write:

$$\frac{\partial}{\partial t}(u \vee c) + \frac{\partial}{\partial x_i}[k_i(x)f_i(u \vee c)] - \mathbf{1}_{(u < c)} \left(\frac{\partial}{\partial x_i}(k(x)f(c)) \right) \leq 0. \quad (6.3)$$

Let $u, v \in C((0, T); L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0, T))$ be two entropy solutions of (6.1) with initial data

$$u_0, v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad (6.4)$$

respectively. Throughout this section we make the following assumption on the vector field k .

Hypothesis 6.2.

(E1) *We assume that (see for instance [3])*

$$k \in W_{loc}^{1,1}(\mathbb{R}^d), \quad k, \operatorname{div} k \in L^\infty(\mathbb{R}^d). \quad (6.5)$$

Let us emphasize that the Hypothesis 6.2 is essential to our analysis. We start with the BV type estimation. To this end, observe that

$$\operatorname{div}_x(k(x)f(u)) = k_i \frac{\partial f}{\partial z} \frac{\partial u_k}{\partial x_k} + f \operatorname{div} k(x). \quad (6.6)$$

Also we have

$$\partial_i \left[k(x) \frac{\partial f}{\partial z} \frac{\partial u_k}{\partial x_k} + f \operatorname{div} k_i \right] = \partial_i \left(k_i \frac{\partial f}{\partial z} \frac{\partial u_k}{\partial x_k} \right) + \partial_i (f \operatorname{div} k_i). \quad (6.7)$$

Now, observe that the second term of the right-hand side of (6.7) can be rewritten as follows:

$$\partial_i (f \operatorname{div} k_i) = \frac{\partial u_k}{\partial x_k} \frac{\partial f}{\partial z} \cdot \operatorname{div}_x k(x) + f(u) \frac{\partial}{\partial x} (\operatorname{div}_x k). \quad (6.8)$$

Consequently, and passing to the absolute value we get:

$$\frac{\partial}{\partial t} |Du| + \partial_i \left(k_i \frac{\partial f}{\partial z} |Du| \right) + \frac{\partial f}{\partial z} \cdot \operatorname{div}_x k(x) |Du| + \left(f(u) \frac{\partial}{\partial x} (\operatorname{div}_x k) \right) \frac{Du}{|Du|} \leq 0 \quad (6.9)$$

and integrating with respect to t , one obtains the BV type estimate. Theorem 3.2 then gives that any two entropy solutions (by writing equation in x by setting $c = v(y, s)$) satisfy:

$$\frac{\partial}{\partial t} (u \vee v(y, s)) + \frac{\partial}{\partial x_i} [k_i(x) f_i(u \vee v(y, s))] - \mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(x) f(v(y, s)) \leq 0. \quad (6.10)$$

We write equation in y :

$$\frac{\partial}{\partial s} (u(x, t) \vee v) + \frac{\partial}{\partial y_i} [k_i(y) f_i(u(x, t) \vee v)] - \mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(y) f(u(x, t)) \leq 0. \quad (6.11)$$

Summing the two obtained inequalities yields the following inequality:

$$\begin{aligned} \frac{\partial}{\partial t} (u(x, t) \vee v(y, s)) + \frac{\partial}{\partial x_i} [k_i(x) f_i(u(x, t) \vee v(y, s))] + \frac{\partial}{\partial y_i} [k_i(y) f_i(u(x, t) \vee v(y, s))] \\ - \mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(x) f(v(y)) - \mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(y) f(u(x)) \leq 0. \end{aligned} \quad (6.12)$$

Multiplying (6.11) by $\varrho_\varepsilon(x - y)$ and integrating with respect to y , one has:

$$\begin{aligned} \frac{\partial}{\partial t} \int (u(x, t) \vee v(y, s)) \varrho_\varepsilon(x - y) dy + \frac{\partial}{\partial x_i} \left(\int [k_i(x) f_i(u \vee v)] \varrho_\varepsilon(x - y) dy \right) \\ - \int (k_i(x) f_i(u \vee v) - k_i(y) f_i(u \vee v)) (\partial_i \varrho_\varepsilon(x - y)) dy \\ - \int \left[\mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(x) f(v(y)) + \mathbf{1}_{(u(x) < v(y))} \operatorname{div} k(y) f(u(x)) \right] \varrho_\varepsilon(x - y) dy \leq 0. \end{aligned} \quad (6.13)$$

Since it is not possible to obtain BV estimates for the approximate solutions u_n , the idea is to prove that (u_n) is a Cauchy sequence. We repeat everything up to (5.25) and find that:

$$\begin{aligned} \frac{d}{dt} \iint |u_n(x) - u_m(y)| \varrho_\delta(x - y) dx dy \\ \leq \left| \frac{\partial}{\partial x_i} \left(\iint k_i(x) f_i^n(u_n(x) \vee u_m(y)) - k_i(y) f_i^m(u_n(x) \vee u_m(y)) \right) (\partial_i \varrho_\delta(x - y)) \right. \\ \left. + \iint \left[\left\{ \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x k(x) f_n(u_m(y)) + \operatorname{div}_y k(y) f_m(u_n(x)) \right\} \varrho_\delta(x - y) \right. \right. \\ \left. \left. + \iint \left\{ \left(k_i(x) f_i^n(u_n(x) \vee u_m(y)) - k_i(y) f_i^m(u_n(x) \vee u_m(y)) \right) \frac{\partial}{\partial x_i} \varrho_\delta(x - y) dx dy \right\} \right] \right|. \end{aligned} \quad (6.14)$$

Taking into account the convergence (5.26)-(5.30), the final result takes the form:

$$\left| \frac{d}{dt} \iint |u_n(x) - u_m(y)| \varrho_\delta(x - y) dx dy \right| \leq \omega_\delta(n, m) + \alpha(\delta), \quad (6.15)$$

with $\omega_\delta(n, m)$ and $\alpha(\delta)$ defined in (5.34). Therefore (u_n) is a Cauchy sequence, hence it converges to a unique limit u of entropy solution of the problem (6.1).

Example 6.3. *The natural extension of the Example 6.1 is about nonlinear conservation laws problem in L^1 of the following type:*

$$\begin{cases} \partial_t u + \operatorname{div} f(k(x), u) = 0, & x \in \mathbb{R}^d, \quad t \geq 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (6.16)$$

Equations of the above type arise in fluid flows in heterogeneous media such as in two phases flow in a porous medium with changing rock types that arise in the petroleum industry. The BV regularity can be done as follows. Observe that:

$$\frac{\partial}{\partial x_i} f_i(k(x), u) = \operatorname{div} k(x) (\operatorname{div}_x f)(k(x), u) + \frac{\partial f_i}{\partial z}(k(x), u) \frac{\partial u}{\partial x_k}. \quad (6.17)$$

Taking the derivative with respect to x_k in (6.17) we obtain:

$$\partial_i (\partial_{x_k} f_i(k(x), u)) = \partial_i [\operatorname{div} k(x) (\operatorname{div}_x f)(k(x), u)] + \partial_i \left[\frac{\partial f_i}{\partial z}(k(x), u) \frac{\partial u}{\partial x_k} \right]. \quad (6.18)$$

On the one hand, the first term in the right-hand side of (6.18) can be replaced by

$$\begin{aligned} & \partial_i [\operatorname{div} k(x) (\operatorname{div}_x f)(k(x), u)] \\ &= \frac{\partial}{\partial x} (\operatorname{div} k(x)) (\operatorname{div}_x f)(k(x), u) + \operatorname{div} k(x) \partial_i (\operatorname{div}_x f)(k(x), u). \end{aligned}$$

On the other hand

$$\partial_i (\operatorname{div}_x f(k(x), u)) = \operatorname{div} k(x) \partial_{x_i x_k}^2 f(k(x), u) + \partial_{x_k z}^2 f(k(x), u) \frac{\partial u}{\partial x_i}. \quad (6.19)$$

Hence

$$\begin{aligned} \partial_i [\operatorname{div} k(x) (\operatorname{div}_x f)(k(x), u)] &= \frac{\partial}{\partial x} (\operatorname{div} k(x)) (\operatorname{div}_x f)(k(x), u) \\ &\quad + (\operatorname{div} k(x))^2 \partial_{x_i x_k}^2 f(k(x), u) + \operatorname{div} k(x) \partial_{x_k z}^2 f(k(x), u) \frac{\partial u}{\partial x_i}, \end{aligned}$$

and passing to the absolute value we get:

$$\begin{aligned} \frac{\partial}{\partial t} |Du| + \partial_i \left(\frac{\partial f_i}{\partial z} |Du| \right) + \left| \left(\frac{\partial}{\partial x} (\operatorname{div} k(x)) (\operatorname{div}_x f)(k(x), u) + (\operatorname{div} k(x))^2 \partial_{x_i x_k}^2 f(k(x), u) \right. \right. \\ \left. \left. + \operatorname{div} k(x) \partial_{x_k z}^2 f(k(x), u) \frac{\partial u}{\partial x_i} \right) \frac{\partial_k u}{|Du|} \right| \leq 0. \end{aligned}$$

The entropy solution theory breaks down when $k(x)$ is discontinuous. By a standard limiting argument, (2.1) implies that the Kruzhkov-type entropy condition:

$$|u - c|_t + \operatorname{div} \left[\operatorname{sgn}(u - c) (f(k(x), u) - f(k(x), c)) \right] + \operatorname{sgn}(u - c) f(k(x), c)_x \leq 0,$$

holds for all $c \in \mathbb{R}$ in the sense of distributions on \mathbb{R}^d . That is, for $0 \leq \phi(x) \in C_0^\infty$:

$$\int (|u - c| \phi_t + \operatorname{sgn}(u - c) (f(k(x), u) - f(k(x), c)) \phi_x) dx - \int \operatorname{sgn}(u - c) f(k(x), c)_x \phi dx \geq 0, \quad (6.20)$$

holds in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$, for all $c \in \mathbb{R}$ and we used the shorthand notation:

$$f(k(x), c)_x := k'(x) \cdot f_k(k(x), c) = \sum_{j=1}^d k'_j(x) f_{k_j}(k(x), c).$$

Inequality (2.7) reads then as follows:

$$\frac{\partial}{\partial t} (u \vee c) + \frac{\partial}{\partial x_i} [f_i(k_i(x), u \vee c)] - \mathbf{1}_{(u < c)} \left(\frac{\partial}{\partial x_i} f(k(x), c) \right) \leq 0. \quad (6.21)$$

Finally, take $c = v(y, s)$ in (6.21) and multiplying by ϱ_ε and integrating with respect to y , which can be easily justified, yields:

$$\begin{aligned} & \frac{\partial}{\partial t} \int (u \vee v) \varrho_\varepsilon(x-y) dy + \frac{\partial}{\partial x_i} \left(\int f(k(x), u \vee v) \varrho_\varepsilon(x-y) dy \right) \\ & - \int (f_i(k(x), u \vee v) - f_i(k(y), u \vee v)) (\partial_i \varrho_\varepsilon(x-y)) dy \\ & + \int (\mathbf{1}_{(u < v)} \partial_i f_i(k(x), v) + \mathbf{1}_{(v < u)} \partial_i f_i(k(y), v)) \varrho_\varepsilon(x-y) dy \leq 0. \end{aligned} \quad (6.22)$$

Again, as in (5.25), we find that:

$$\begin{aligned} & \frac{d}{dt} \iint |u_n(x) - u_m(y)| \varrho_\delta(x-y) dx dy \\ & \leq \left| \frac{\partial}{\partial x_i} \left(\iint f_i^n(k_i(x), u_n(x) \vee u_m(y)) - f_i^m(k_i(y), u_n(x) \vee u_m(y)) \right) (\partial_i \varrho_\delta(x-y)) \right. \\ & \quad + \iint \left[\left\{ \mathbf{1}_{u_n(x) < u_m(y)} \operatorname{div}_x f_n(k(x), u_m(y)) + \operatorname{div}_y f_m(k(y), u_n(x)) \right\} \varrho_\delta(x-y) \right. \\ & \quad \left. \left. + \iint \left\{ \left(f_i^n(k_i(x), u_n(x) \vee u_m(y)) - f_i^m(k_i(y), u_n(x) \vee u_m(y)) \right) \frac{\partial}{\partial x_i} \varrho_\delta(x-y) dx dy \right\} \right] \right|. \end{aligned} \quad (6.23)$$

Using the same passages as in the previous section, we conclude that the whole sequence $\{u_n\}$ converge to the unique solution of (6.16).

A. APPENDIX

A.1. Auxiliary facts: Attractiveness of solutions of the ODE.

In this section we provide a few facts having an auxiliary character which we use in this paper. It would be interesting to study the links between our problem with the properties of the ordinary differential equation (ODE)

$$\frac{d}{dt} x(t) = b_0(x(t)), \quad t > 0; \quad x(0) = x_0, \quad (A.1)$$

where we assume that b_0 is Lipschitz. The two most classical and widely used results on the well-posedness of (A.1) are as follows:

- Cauchy-Lipschitz: If b_0 is locally Lipschitz continuous, (A.1) has a unique local (in time) solution for any initial value $x_0 \in \mathbb{R}^d$.
- Peano: If b_0 is continuous, for every $x_0 \in \mathbb{R}^d$, there exists at least a local solution of (A.1).

It is well known that there exists a unique flow associated to (A.1). We proceed by regularization. In fact, the most direct way to establish the existence of such flow is of course through a simple approximation procedure. That means taking smooth vector fields $\{b_n\}_n$ with $b_n \rightarrow b_0$, a.e., b_n is uniformly bounded in L^∞ which enables to solve,

$$\frac{d}{dt} x_n(t) = b_n(x_n(t)), \quad x_n(0, x) = x, \quad (A.2)$$

by the usual Cauchy-Lipschitz Theorem. To pass to the limit in (A.2) and obtain (A.1), it is enough to have compactness in some strong sense (in L^1_{loc} for instance) for the sequence x_n . One can prove that the sequences (x_n) are Cauchy sequences without applying the existence of a solution of the limit problem. Actually, all the trajectories of the approximate flows will

converge towards the unique trajectory of the limiting problem, which attracts them all. To this end, it is enough to write, for arbitrarily fixed natural numbers n, m that:

$$\frac{d}{dt}(x_n - x_m) = (b_0(x_n) - b_0(x_m)) + (b_n(x_n) - b_0(x_n)) - (b_m(x_m) - b_0(x_m)). \quad (\text{A.3})$$

Hence we infer, by using the uniformly convergent on \mathbb{R}^d , the following inequality:

$$\frac{d}{dt}|x_n - x_m| \leq C_0|x_n - x_m| + \varpi(n, m), \quad (\text{A.4})$$

where

$$\varpi(n, m) = |b_n(x_n) - b_0(x_n)| + |b_m(x_m) - b_0(x_m)|.$$

Therefore, taking into account that the sequence $\{b_n\}$ is uniform convergence on \mathbb{R}^d , we conclude that:

$$\varpi(n, m) \xrightarrow[n, m \rightarrow \infty]{} 0.$$

From the above estimate, we deduce that (x_n) is a real Cauchy sequence. It is exactly the same principle we have used in the proof of the Theorem 3.2. One has the attractivity of the solutions of (A.1). Our proof follows the same strategy. \square

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