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# A uniform continuity property of the winding number of self-mappings of the circle

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July 1st, 2019

*Dedicated to Haim Brezis, with friendship and admiration on his 75th birthday*

## Abstract

Let  $u : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ . When  $u$  is continuous, it has a winding number  $\deg u$ , which satisfies  $\deg u = \deg v$  if  $u, v \in C^0(\mathbb{S}^1; \mathbb{S}^1)$  and  $\|u - v\|_{L^\infty} < 2$ . In particular,  $u \mapsto \deg u$  is uniformly continuous for the sup norm.

The winding number  $\deg u$  can be naturally defined, by density, when  $u$  is merely *VMO*. For such  $u$ 's, the winding number  $\deg$  is continuous with respect to the *BMO* norm.

Let  $1 < p < \infty$ . In view of the above and of the embedding  $W^{1/p,p}(\mathbb{S}^1) \hookrightarrow VMO$ , maps in  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  have a well-defined winding number, continuous with respect to the  $W^{1/p,p}$  norm. However, an example due to Brezis and Nirenberg yields sequences  $(u_n), (v_n) \subset W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  such that  $\|u_n - v_n\|_{W^{1/p,p}} \rightarrow 0$  as  $n \rightarrow \infty$  and  $\deg u_n \neq \deg v_n, \forall n$ . Thus  $\deg$  is not uniformly continuous with respect to the  $W^{1/p,p}$  norm (and, a fortiori, with respect to the *BMO* norm).

The above sequences satisfy  $\|u_n\|_{W^{1/p,p}} \rightarrow \infty, \|v_n\|_{W^{1/p,p}} \rightarrow \infty$ . We prove that a similar phenomenon cannot occur for bounded sequences. More specifically, we prove the following uniform continuity result. Given  $1 < p < \infty$  and  $M > 0$ , there exists some  $\delta = \delta(p, M) > 0$  such that

$$[\|u\|_{W^{1/p,p}} \leq M, \|u - v\|_{W^{1/p,p}} \leq \delta] \implies \deg u = \deg v.$$

## 1 Introduction

If  $u \in C^0(\mathbb{S}^1; \mathbb{S}^1)$ , then  $u$  has a winding number  $\deg u$ , which is continuous with respect to the uniform convergence, and in particular is a homotopic invariant. In fact,  $\deg$  is “better than just continuous”: it is uniformly continuous, since

$$[u, v \in C^0(\mathbb{S}^1; \mathbb{S}^1), \|u - v\|_{L^\infty} < 2] \implies \deg u = \deg v. \tag{1.1}$$

The winding number can still be “naturally” defined (i.e., by density, starting from smooth maps) when  $u$  is slightly less than continuous, more specifically when  $u \in VMO(\mathbb{S}^1; \mathbb{S}^1)$ . This has been first noticed by Boutet de Monvel and Gabber [3, Appendix] for the space  $H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ , and then extended and investigated in depth by Brezis and Nirenberg for maps  $u \in VMO(\mathbb{S}^1; \mathbb{S}^1)$  [5]. While, in this setting, the winding number is still continuous with respect to norm convergence,

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and thus provides a homotopic invariant, there is no *global* analogue of (1.1), even if we replace the *BMO* norm by one of the *stronger* norms  $W^{1/p,p}$ ,  $1 < p < \infty$ . More specifically, a construction from [5] yields two sequences of smooth maps  $u_n, v_n : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  such that

$$\|u_n - v_n\|_{W^{1/p,p}} \rightarrow 0 \text{ as } n \rightarrow \infty, \forall 1 < p < \infty, \text{ and } \deg u_n \neq \deg v_n, \forall n \quad (1.2)$$

(see [4, Lemma 6.4]). Here,  $\| \cdot \|_{W^{1/p,p}} = \| \cdot \|_{L^p} + | \cdot |_{W^{s,p}}$ , where  $| \cdot |_{W^{s,p}}$  stands for the Gagliardo seminorm, given for  $0 < s < 1$ ,  $1 \leq p < \infty$  and  $\Omega$ , an  $N$ -dimensional Lipschitz bounded domain or compact embedded manifold, by

$$|u|_{W^{s,p}}^p = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy.$$

(The fact that the  $W^{1/p,p}$  norm is stronger than the *BMO* one on  $\mathbb{S}^1$  follows from the embedding  $W^{1/p,p} \hookrightarrow BMO$ , valid in 1D.)

We prove that the above phenomenon can occur only when  $u_n, v_n$  “escape to infinity”, that is, we have uniform continuity of the degree on bounded sets:

**Theorem 1.** Let  $1 < p < \infty$  and  $M > 0$ . Then there exists some  $\delta = \delta(p, M) > 0$  such that

$$[u, v \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1), |u|_{W^{1/p,p}} \leq M, |u - v|_{W^{1/p,p}} \leq \delta] \implies \deg u = \deg v. \quad (1.3)$$

This provides a partial (positive) answer to [4, Open Problem 3].

**Remark 1.** The new contribution of this note concerns the case where  $p > 2$ . When  $1 < p \leq 2$ , Theorem 1 is a special case of [4, Proposition 7.9]. This result asserts that, when  $N \geq 1$  and  $1 < p \leq N + 1$ , the degree of maps in  $u \in W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$  is uniformly continuous on bounded sets.

Theorem 1 settles thus completely the case of the dimension one. We point out that its analogue for  $W^{N/p,p}(\mathbb{S}^N; \mathbb{S}^N)$  maps, with  $N \geq 2$  and  $p > N + 1$ , is widely open.

**Remark 2.** When  $1 < p \leq 2$ , the proof of [4, Proposition 7.9] yields an *explicit*  $\delta$  (in terms of  $M$  and  $p$ ) in (1.3). A similar explicit estimate eludes us when  $p > 2$ .

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## 2 Proof

We first recall some basic properties of the winding number. Given  $p \in (1, \infty)$ ,  $\deg u$  for  $u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  is defined as follows. To start with,  $C^\infty(\mathbb{S}^1; \mathbb{S}^1)$  is dense in  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  [5, Lemmas A.11 and A.12], and the map  $u \mapsto \deg u$ , initially defined for smooth maps  $u \in C^\infty(\mathbb{S}^1; \mathbb{S}^1)$ , is continuous with respect to the *BMO* norm [5, Theorem 1]. In view of the embedding  $W^{1/p,p} \hookrightarrow VMO$  in 1D, we find that  $\deg$  has a unique continuous extension to  $W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ . In particular, it suffices to work in (1.3) with *smooth maps*.

For further use, let us note the property

$$\deg(uv) = \deg u + \deg v, \forall u, v \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1). \quad (2.1)$$

As a consequence of (2.1), we have

$$\deg \bar{u} = -\deg u, \forall u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1). \quad (2.2)$$

These properties are well known if  $u$  and  $v$  are smooth. The general case is obtained by density, using the continuity of the degree with respect to the  $W^{1/p,p}$  norm and the following easy and standard fact (“ $W^{s,p} \cap L^\infty$  is an algebra”).

**Lemma 1.** Let  $0 < s < 1$  and  $1 \leq p < \infty$ . Let  $\Omega$  be an  $N$ -dimensional Lipschitz bounded domain or compact embedded manifold. If  $(u_n), (v_n) \subset W^{s,p}(\Omega; \mathbb{C})$  and  $u, v \in W^{s,p}(\Omega; \mathbb{C})$  satisfy

$$|u_n - u|_{W^{s,p}} \rightarrow 0, |v_n - v|_{W^{s,p}} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (2.3)$$

$$u_n \rightarrow u \text{ and } v_n \rightarrow v \text{ a.e. as } n \rightarrow \infty, \quad (2.4)$$

$$\|u_n\|_{L^\infty} \leq C, \|v_n\|_{L^\infty} \leq C, \forall n, \quad (2.5)$$

then

$$|u_n v_n - uv|_{W^{s,p}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.6)$$

*Proof.* We have

$$\begin{aligned} |[(u_n - u)v](x) - [(u_n - u)v](y)|^p &\lesssim |(u_n - u)(x)|^p |v(x) - v(y)|^p \\ &\quad + |(u_n - u)(x) - (u_n - u)(y)|^p |v(y)|^p, \end{aligned} \quad (2.7)$$

so that

$$\begin{aligned} |u_n v - uv|_{W^{s,p}}^p &\lesssim \int_{\Omega} \int_{\Omega} \frac{|(u_n - u)(x)|^p |v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\quad + \int_{\Omega} \int_{\Omega} \frac{|(u_n - u)(x) - (u_n - u)(y)|^p |v(y)|^p}{|x - y|^{N+sp}} dx dy \rightarrow 0 \text{ as } n \rightarrow \infty; \end{aligned}$$

here, we use dominated convergence for the first integral, and the facts that  $v$  is bounded and  $|u_n - u|_{W^{s,p}} \rightarrow 0$  as  $n \rightarrow \infty$  for the second integral. Thus  $|u_n v - uv|_{W^{s,p}} \rightarrow 0$  as  $n \rightarrow \infty$ . Similarly,  $|u v_n - uv|_{W^{s,p}} \rightarrow 0$  as  $n \rightarrow \infty$ . In view of the identity

$$u_n v_n - uv = (u_n v - uv) + (u v_n - uv) + (u_n - u)(v_n - v),$$

it remains to prove that  $|(u_n - u)(v_n - v)|_{W^{s,p}} \rightarrow 0$  as  $n \rightarrow \infty$ . This follows, by dominated convergence, from

$$\begin{aligned} |(u_n - u)(v_n - v)|_{W^{s,p}}^p &\lesssim \int_{\Omega} \int_{\Omega} \frac{|(u_n - u)(x)|^p |(v_n - v)(x) - (v_n - v)(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\quad + \int_{\Omega} \int_{\Omega} \frac{|(u_n - u)(x) - (u_n - u)(y)|^p |(v_n - v)(y)|^p}{|x - y|^{N+sp}} dx dy. \end{aligned} \quad \square$$

*Proof of Theorem 1.* Given  $u : \mathbb{S}^1 \rightarrow \mathbb{C}$  measurable and bounded, we let  $T_u : \mathbb{D} \rightarrow \mathbb{C}$  denote its harmonic extension to the unit disc  $\mathbb{D}$ . We set

$$c'_p := \inf\{|u|_{W^{1/p,p}}^p; u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1), T_u(0) = 0\}. \quad (2.8)$$

(Clearly, the inf is achieved in (2.8), but we will not need this fact.) We have [6, Section 2, item 5]

$$c'_p > 0. \quad (2.9)$$

We will prove (1.3) by complete (strong) induction on the integer part  $L$  of  $\frac{M^p}{c'_p}$ .

*Step 1.* Proof of (1.3) when  $L = 0$

Let  $\delta > 0$  be such that  $(M + \delta)^p := c < c'_p$ . If  $u, v$  are smooth and as in (1.3), then  $|u|_{W^{1/p,p}}^p \leq c$  and  $|v|_{W^{1/p,p}}^p \leq c$ , and therefore there exists some  $\varepsilon > 0$  such that  $|T_u(x)| \geq \varepsilon$ ,  $|T_v(x)| \geq \varepsilon$ ,  $\forall x \in \mathbb{D}$  [6, Section 2, items 8 and 9]. Thus  $u, v : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  are smooth functions with smooth non vanishing extensions to  $\overline{\mathbb{D}}$ . It follows that  $\deg u = \deg v = 0$ .

*Step 2.* Proof of (1.3) for  $L \geq 1$  (assuming that (1.3) holds for  $0, \dots, L-1$ )

Argue by contradiction. Then there exist  $c < (L+1)c'_p$  and sequences  $(u_n), (v_n) \subset C^\infty(\mathbb{S}^1; \mathbb{S}^1)$  such that  $|u_n|_{W^{1/p,p}}^p \leq c$ ,  $|v_n|_{W^{1/p,p}}^p \leq c$ ,  $\deg u_n \neq \deg v_n$ ,  $\forall n$ , and  $|u_n - v_n|_{W^{1/p,p}} \rightarrow 0$  as  $n \rightarrow \infty$ .

Since at least one of the integers  $\deg u_n, \deg v_n$  is non zero, we may assume that  $\deg u_n \neq 0$ ,  $\forall n$ . Therefore,  $T_{u_n}$  has to vanish at some point  $a_n \in \mathbb{D}$ .

Let  $M_a(z) = \frac{a-z}{1-\bar{a}z}$ ,  $\forall a \in \mathbb{D}$ ,  $\forall z \in \overline{\mathbb{D}}$ , denote the (normalized) Möbius transformation vanishing at  $a$ . Let  $N_a : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  be the restriction of  $M_a$  to  $\mathbb{S}^1$ . We note the following properties, valid for each  $a \in \mathbb{D}$  and each  $u \in C^0(\mathbb{S}^1; \mathbb{S}^1)$  [6, Section 2, item 1].

$$\deg(u \circ N_a) = \deg u, \quad (2.10)$$

$$T_{u \circ N_a} = (T_u) \circ M_a, \quad (2.11)$$

$$T_{u \circ N_a}(0) = T_u(a). \quad (2.12)$$

In addition, we have [6, Section 2, item 2]

$$|f \circ N_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}}, \quad \forall 1 < p < \infty, \forall f \in W^{1/p,p}(\mathbb{S}^1; \mathbb{C}). \quad (2.13)$$

We consider  $U_n := u_n \circ N_{a_n}$  and  $V_n := v_n \circ N_{a_n}$ . By properties (2.10)–(2.13) above and by the assumptions on  $u_n$  and  $v_n$ , we have

$$|U_n|_{W^{1/p,p}}^p \leq c \text{ and } |V_n|_{W^{1/p,p}}^p \leq c, \quad (2.14)$$

$$T_{U_n}(0) = 0, \quad \forall n, \quad (2.15)$$

$$\deg U_n \neq \deg V_n, \quad \forall n, \quad (2.16)$$

$$|U_n - V_n|_{W^{1/p,p}} \rightarrow 0 \quad (2.17)$$

and, possibly up to subsequences still denoted  $(U_n)$  and  $(V_n)$ ,

$$U_n \rightarrow U \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1) \text{ a.e. and weakly in } W^{1/p,p} \text{ as } n \rightarrow \infty, \quad (2.18)$$

$$V_n \rightarrow V \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1) \text{ a.e. and weakly in } W^{1/p,p} \text{ as } n \rightarrow \infty, \quad (2.19)$$

$$U - V = C \in \mathbb{C}, \quad (2.20)$$

$$T_U(0) = 0. \quad (2.21)$$

The last property follows, by dominated convergence, from (2.15), (2.18) and the fact that  $T_v(0) = \int_{\mathbb{S}^1} v(x) dx$ ,  $\forall v \in L^1(\mathbb{S}^1; \mathbb{C})$ .

*Claim 1.* We have  $C = 0$ , and thus

$$V_n \rightarrow U \text{ a.e. as } n \rightarrow \infty. \quad (2.22)$$

Granted Claim 1, we continue as follows. Since  $U_n \rightarrow U$  and  $V_n \rightarrow U$  a.e. as  $n \rightarrow \infty$ , we have [6, Section 2, item 10]

$$|U_n|_{W^{1/p,p}}^p = |U|_{W^{1/p,p}}^p + |U_n \overline{U}|_{W^{1/p,p}}^p + o(1) \text{ as } n \rightarrow \infty, \quad (2.23)$$

$$|V_n|_{W^{1/p,p}}^p = |U|_{W^{1/p,p}}^p + |V_n \overline{U}|_{W^{1/p,p}}^p + o(1) \text{ as } n \rightarrow \infty. \quad (2.24)$$

Combining (2.21), the fact that  $U \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$  and the definition (2.8) of  $c'_p$ , we obtain

$$|U|_{W^{1/p,p}}^p \geq c'_p. \quad (2.25)$$

From (2.23)–(2.25), (2.1) and (2.2), we find that  $\tilde{U}_n := U_n \bar{U}$  and  $\tilde{V}_n := V_n \bar{U}$  satisfy

$$|\tilde{U}_n|_{W^{1/p,p}}^p \leq c - c'_p + o(1) \text{ as } n \rightarrow \infty, \quad (2.26)$$

$$|\tilde{V}_n|_{W^{1/p,p}}^p \leq c - c'_p + o(1), \quad (2.27)$$

$$\deg \tilde{U}_n = \deg U_n - \deg U \neq \deg V_n - \deg U = \deg \tilde{V}_n, \quad \forall n. \quad (2.28)$$

*Claim 2.* We have

$$|\tilde{U}_n - \tilde{V}_n|_{W^{1/p,p}} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.29)$$

Granted Claim 2 and using (2.26)–(2.28) together with the fact that  $c - c'_p < L c'_p$ , we find that the sequences  $(\tilde{U}_n)$ ,  $(\tilde{V}_n)$  contradict, for large  $n$ , the induction hypothesis.

In order to complete the proof of the theorem, it thus remains to justify Claims 1 and 2.

*Proof of Claim 1.* Since  $|U| = 1$  and  $|V| = 1$  a.e., (2.20) implies that  $U$  takes values, a.e., in the set  $\mathbb{S}^1 \cap (C + \mathbb{S}^1)$ . When  $C \neq 0$ , this set contains at most two points. Since the essential range of  $U$  is connected [5, Section I.5, Comment 2], we find that  $U$  is constant a.e. Thus  $T_U$  is a constant of modulus 1, which is impossible, by (2.21).

*Proof of Claim 2.* Let us note that  $\bar{U} \in W^{1/p,p} \cap L^\infty$ . Claim 2 is then a consequence of (2.22) and Lemma 1, applied with  $u_n = U_n - V_n, v_n = \bar{U}, u = 0$  and  $v = \bar{U}$ .  $\square$

**Remark 3.** The idea of “extracting” information concerning a map  $u : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  from the behavior of an appropriate extension of  $u$  to  $\mathbb{D}$  (which is at the heart of the asymptotic analysis in Step 2) originates in [1], where the harmonic extension is considered; see also [2] for an extension by averages.

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