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## NONLINEAR MODAL ANALYSIS OF THE BAR IN UNILATERAL CONTACT VIA ANALYTICAL WEAK-SOLUTIONS TO THE WAVE EQUATION

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### ABSTRACT

Nonlinear modal analysis is commonly performed by detecting families of periodic solutions of nonlinear conservative and autonomous systems. For a unilaterally constrained bar, nonlinear modes were detected using numerical solvers. In the present work, the analytical solution is investigated instead. Results show that infinitely many periodic solutions with the same energy and period might co-exist.

**Keywords:** Unilateral Contact, Nonlinear Modal Analysis, Vibration, Signorini, Wave Equation, Characteristics

### INTRODUCTION

*Vibro-impact* systems have generated interest as the introduction of contact constraints affects greatly the behavior of structures during resonance [1]. To predict nonlinear vibratory resonance, the concept of linear normal modes has been extended to nonlinear systems as nonlinear normal modes (NNMs) [4]. Nonlinear modal analysis (NMA) corresponds to finding manifolds of periodic motions of a dynamic structure [4]. NNMs of a bar under unilateral frictionless contact constraint were found using the wave finite element method [8] where a few closed-form solutions support numerical results. In the present work, using an analytical treatment of the boundary conditions (BCs), a set of conditions is derived for obtaining periodic solutions of the bar with frictionless unilateral contact. This set of conditions allows for determining any possible periodic solution of the investigated structure. The findings provide an extension to the solution space reported in [7].

### PROBLEM STATEMENT

The investigated bar, of length  $L$ , is deformable along a single spatial direction  $x$ . The partial differential equation (PDE)

describing the deformation of the bar at time  $t$  is [7]

$$\mathbf{q}_t + \mathbf{M}\mathbf{q}_x = 0 \quad \forall t > 0, \quad \forall x \in ]0; L[ \quad (1)$$

$$\mathbf{q}(x, t) \equiv \begin{pmatrix} v(x, t) \\ s(x, t) \end{pmatrix}, \quad \mathbf{M} \equiv \begin{bmatrix} 0 & -E/\rho \\ -1 & 0 \end{bmatrix} \quad (2)$$

where  $E$  represents Young's modulus and  $\rho$ , the mass density per unit length. The velocity of deformation and strains are presented using  $v = u_t$  and  $s = u_x$ , respectively, where  $u(x, t)$  stands for the displacement field within the bar. Partial derivatives in time and space are denoted using subscripts  $\circ_x$  and  $\circ_t$ , respectively. The bar is clamped at the origin such that

$$u(0, t) = 0, \quad v(0, t) = 0 \quad \forall t > 0. \quad (3)$$

A rigid wall is set at a distance  $g$  from the tip of the bar at rest.

### TREATMENT OF CONTACT CONSTRAINT

For the mathematical formulation of unilateral frictionless contact, the *Signorini condition* is considered [6]. When the tip of the bar and the wall are separated, the deformable structure is free to move in space and a homogeneous Neumann-BC

$$s(L, t) = 0 \quad \forall t \text{ such that } u(L, t) \leq g \quad (4)$$

applies. Once  $u(L, t) = g$  holds, contact is activated and the strain at  $x = L$  has to remain negative since the bar is being repulsed by the wall, as described by the Dirichlet BC

$$u(L, t) = g \text{ and } v(L, t) = 0 \quad \forall t \text{ such that } s(L, t) \leq 0. \quad (5)$$

Hence, the motion of the bar in unilateral contact for any initial state  $\mathbf{q}^0(x) \equiv \mathbf{q}(x, 0)$  is defined by equations (1) to (5).

### DEFINITION OF A PERIODIC SOLUTION

For a periodic solution, it is required to find  $\mathbf{q}^0$ , such that there exist  $T$  for which  $\mathbf{q}^0 = \mathbf{q}(x, T, \mathbf{q}^0)$ . In this work, periodic

solutions with a single activation of contact per period are of interest. Since the Signorini conditions imply a change of BCs within the period of motion,  $\mathbf{q}$  can be described by two separate motions. Let  $\mathbf{q}^{\text{DD}}(x, t, \mathbf{q}^0)$  (DD-solution) and  $\mathbf{q}^{\text{DN}}(x, t, \mathbf{q}^1)$  (DN-solution) represent bar's motion at active and inactive contact, respectively, such that:

$$\begin{cases} \mathbf{q}_t^{\text{DD}} + \mathbf{M}\mathbf{q}_x^{\text{DD}} = 0 \\ v^{\text{DD}}(0, t) = 0 \\ v^{\text{DD}}(L, t) = 0 \\ \mathbf{q}^{\text{DD}}(x, 0, \mathbf{q}^0) = \mathbf{q}^0(x) \end{cases} \quad \text{and} \quad \begin{cases} \mathbf{q}_t^{\text{DN}} + \mathbf{M}\mathbf{q}_x^{\text{DN}} = 0 \\ v^{\text{DN}}(0, t) = 0 \\ s^{\text{DN}}(L, t) = 0 \\ \mathbf{q}^{\text{DN}}(x, 0, \mathbf{q}^1) = \mathbf{q}^1(x) \end{cases} \quad (6)$$

$\forall t > 0$  and  $\forall x \in [0; L]$ . The motion  $\mathbf{q}$  is then described by:

$$\mathbf{q}(x, t, \mathbf{q}^0) = \begin{cases} \mathbf{q}^{\text{DD}}(x, t, \mathbf{q}^0) & \forall t \in [0; t_c] \\ \mathbf{q}^{\text{DN}}(x, t - t_c, \mathbf{q}^1) & \forall t \in [t_c; t_c + t_f] \end{cases} \quad (7)$$

$$\mathbf{q}^1(x) \equiv \lim_{t \rightarrow t_c^+} \mathbf{q}^{\text{DD}}(x, t, \mathbf{q}^0) \quad \forall x \in [0; L]. \quad (8)$$

where  $t_c$  and  $t_f$  denote durations of active and inactive contact, respectively. While  $\mathbf{q}$  satisfies (1) and (3),  $\mathbf{q}^{\text{DD}}$  and  $\mathbf{q}^{\text{DN}}$  satisfy only the equalities in (5) and (4), respectively. Hence, additional constraints must apply to  $\mathbf{q}$  such that the Signorini conditions are fully satisfied. Condition (5) requires:

$$u(L, 0) = \int_0^L s^0(x) dx = g \quad (9)$$

$$s^{\text{DD}}(L, t, \mathbf{q}^0) \leq 0 \quad \forall t \in [0; t_c]. \quad (10)$$

At time  $t_c$ , it is presumed that the bar is released from contact. To inactivate contact,

$$\lim_{t \rightarrow t_c^+} s^{\text{DD}}(L, t, \mathbf{q}^0) = s^1(L) \geq 0 \quad (11)$$

must hold. As well, to fully satisfy (4), the condition

$$\int_0^t v^{\text{DN}}(L, \mu, \mathbf{q}^1) d\mu \leq 0 \quad \forall t \in [0; t_f] \quad (12)$$

must hold guaranteeing the bar does not penetrate the wall. For a periodic solution to exist,

$$\mathbf{q}(x, T, \mathbf{q}^0) = \mathbf{q}^{\text{DN}}(x, t_f, \mathbf{q}^1) = \mathbf{q}^0(x) \quad \forall x \in [0; L] \quad (13)$$

must hold, where the period is  $T = t_c + t_f$ . Hence,  $\mathbf{q}^0$ ,  $t_c$  and  $t_f$  satisfying equation (13) with constraints (9) to (12) generate a periodic solution. Note that discontinuities are expected to propagate in  $v$  and  $s$  [7]. In the theory of PDEs, a solution that satisfies the PDE except at discontinuities is referred to as the weak solution [2]. Therefore, an analytical solution of (9) to (13) requires an analytical weak-solution to  $\mathbf{q}^{\text{DD}}$  and  $\mathbf{q}^{\text{DN}}$  in the form of traveling waves.

## TRAVELING WAVE SOLUTION

To formulate the traveling wave solution, the method of characteristics is used [2]. The characteristics of (1) are obtained by

performing an eigen-decomposition of the matrix  $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$  and introducing  $\mathbf{r}$  such that

$$\mathbf{q} = \mathbf{P}\mathbf{r}, \quad \mathbf{P} \equiv \frac{1}{\sqrt{1+c^2}} \begin{bmatrix} c & c \\ -1 & 1 \end{bmatrix} \quad (14)$$

where  $c = \sqrt{E/\rho}$  and re-writing (1) as

$$\mathbf{r}_t + \mathbf{D}\mathbf{r}_x = \mathbf{0}, \quad \mathbf{D} \equiv \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}. \quad (15)$$

This system is solved through the method of characteristics as

$$\mathbf{r}(x, t) = \begin{pmatrix} r_1^0(x - ct) \\ r_2^0(x + ct) \end{pmatrix} \quad (16)$$

where  $\mathbf{r}^0(x) \equiv \mathbf{r}(x, 0) = \mathbf{P}^{-1}\mathbf{q}^0(x)$ . Note that, at the points  $x - ct < 0$  and  $x + ct > L$ ,  $\mathbf{r}$  cannot be defined as  $\mathbf{r}^0(x)$  exists exclusively in  $x \in [0; L]$ . According to [2],  $\mathbf{r}$  is extended for  $x - ct < 0$  and  $x + ct > L$  using different configurations of  $\mathbf{r}^0$  to satisfy the BCs. In the present work, a systematic approach for extending  $\mathbf{r}$  is derived using the BCs and definition (14). For example,  $r_2(x + ct > L)$  can be determined for Neumann-BC at the tip:

$$s(L, t) = 0 \rightarrow r_1(L - ct) - r_2(L + ct) = 0 \quad \forall t > 0 \quad (17)$$

where  $r_2(L + ct)$  is unknown and  $r_1(L - ct) = r_1^0(L - ct) \forall t \in [0; L/c]$ . This treatment of BCs is used to obtain a closed-form weak-solution of  $\mathbf{q}^{\text{DD}}$  and  $\mathbf{q}^{\text{DN}}$ .

## DERIVATION OF DD- AND DN-SOLUTIONS

The DD-solution,  $\mathbf{r}^{\text{DD}} = \mathbf{P}^{-1}\mathbf{q}^{\text{DD}}$ , is defined using the BCs:

$$v(0, t) = 0 \rightarrow r_1^{\text{DD}}(ct) = -r_2^{\text{DD}}(-ct) \quad \forall t < 0 \quad (18)$$

$$v(L, t) = 0 \rightarrow r_2^{\text{DD}}(ct) = -r_1^{\text{DD}}(2L - ct) \quad \forall t > L/c \quad (19)$$

These guiding rules allow formulation of the weak-solution  $\mathbf{q}^{\text{DD}}$  based solely on the initial state  $\mathbf{r}^0(x) \forall x \in [0; L]$

$$\mathbf{q}^{\text{DD}}(x, t, \mathbf{q}^0) = \mathbf{P} \begin{pmatrix} r_1^{\text{DD}}(x - ct, \mathbf{q}^0) \\ r_2^{\text{DD}}(x + ct, \mathbf{q}^0) \end{pmatrix} \quad (20)$$

with

$$r_1^{\text{DD}}(s, \mathbf{r}^0) = \begin{cases} r_1^0(s) & s \in [0; L] \\ -r_2^0(-s) & s \in [-L; 0] \\ r_1^0(2L + s) & s \in [-2L; -L] \end{cases} \quad (21)$$

$$r_2^{\text{DD}}(s, \mathbf{r}^0) = \begin{cases} r_2^0(s) & s \in [0; L] \\ -r_1^0(2L - s) & s \in [L; 2L] \\ r_2^0(s - 2L) & s \in [2L; 3L] \end{cases} \quad (22)$$

where  $\mathbf{q}^{\text{DD}}$  is  $2L/c$  periodic in  $t$ . Replacing (19) with (17) yields

$$\mathbf{q}^{\text{DN}}(x, t, \mathbf{q}^0) = \mathbf{P} \begin{pmatrix} r_1^{\text{DN}}(x - ct, \mathbf{q}^0) \\ r_2^{\text{DN}}(x + ct, \mathbf{q}^0) \end{pmatrix} \quad (23)$$

which is  $4L/c$  periodic in  $t$ , with

$$r_1^{\text{DN}}(s, \mathbf{r}^0) = \begin{cases} r_1^0(s) & s \in [0; L] \\ -r_2^0(-s) & s \in [-L; 0] \\ -r_1^0(s + 2L) & s \in [-2L; -L] \\ r_2^0(-s - 2L) & s \in [-3L; -2L] \\ r_1^0(s + 4L) & s \in [-4L; -3L] \end{cases} \quad (24)$$

$$r_2^{\text{DN}}(s, \mathbf{r}^0) = \begin{cases} r_2^0(s) & s \in [0; L] \\ r_1^0(2L - s) & s \in [L; 2L] \\ -r_2^0(s - 2L) & s \in [2L; 3L] \\ -r_1^0(4L - s) & s \in [3L; 4L] \\ r_2^0(s - 4L) & s \in [4L; 5L] \end{cases} \quad (25)$$

### EXTRACTION OF PERIODIC SOLUTIONS

Rather than solving (9) to (13) in terms of  $\mathbf{q}^0$ , the conditions will be solved in terms of  $\mathbf{r}^0$ . In [5], the continuum of periodic solutions in  $T$  was proven to consist of piece-wise constant functions. Thus, it was chosen to turn conditions (10), (12) and (13) into a set of finite linear equations and inequalities by representing  $r_1^0$  and  $r_2^0$  with  $N$  piecewise-constant functions:

$$\begin{pmatrix} r_1^0(x) \\ r_2^0(x) \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \forall x \in [x_{i-1}, x_i] \quad (26)$$

where  $\Delta x = L/N$ ,  $x_i = i \Delta x$ , and  $i = 1, 2, \dots, N$ . Arrays  $\mathbf{a}$  and  $\mathbf{b}$  are of size  $1 \times N$ . In this representation,  $\mathbf{r}$  can be determined using linear mappings. For example,

$$\begin{pmatrix} r_1^{\text{DD}}(s) \\ r_2^{\text{DD}}(s) \end{pmatrix} = \begin{pmatrix} a_{j+2N}^{\text{DD}} \\ b_{j+2N}^{\text{DD}} \end{pmatrix} \quad \forall s \in [s_{j-1}, s_j] \quad (27)$$

for  $j = -2N + 1, -2N, \dots, 2N - 2, 2N - 1$  is defined by

$$\begin{pmatrix} \mathbf{a}^{\text{DD}} \\ \mathbf{b}^{\text{DD}} \end{pmatrix} = \begin{bmatrix} \mathbf{A}^{\text{DD}} \\ \mathbf{B}^{\text{DD}} \end{bmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \quad (28)$$

where  $\mathbf{A}^{\text{DD}}$  and  $\mathbf{B}^{\text{DD}}$  are arrays of size  $4N \times 2N$  formulated to satisfy (21) and (22) while conforming with definition (27).

Equation (20) requires depiction of  $\mathbf{r}^{\text{DD}}$  from domains  $x - ct$  and  $x + ct$ . By choosing time instants  $t_n = n \Delta x / c \equiv n \Delta t$  and integer  $n$ , the solution  $\mathbf{r}(x \pm ct_n)$  can be represented using (26) and a linear operation applied on  $\mathbf{a}$  and  $\mathbf{b}$ . For example, the DD-solution is expressed as

$$\begin{pmatrix} r_1^{\text{DD}}(x - ct_n) \\ r_2^{\text{DD}}(x + ct_n) \end{pmatrix} = \begin{pmatrix} a_{i-n+2N}^{\text{DD}} \\ b_{i+n+2N}^{\text{DD}} \end{pmatrix} \quad \forall x \in [x_{i-1}; x_i]. \quad (29)$$

Since  $\mathbf{r}^{\text{DN}}$  can be represented in a similar manner,  $\mathbf{r}(x \pm ct_n)$  can be defined by a linear mapping acting on  $\mathbf{a}$  and  $\mathbf{b}$ . However, the use of  $t_n$  restricts the choice of  $t_f$  and  $t_c$  to integer multiples  $n_f$  and  $n_c$  of  $\Delta t$ , respectively. Thus, equation (13) can be expressed as the linear system of equations

$$\left( \mathbf{W}(n_c, n_f) - \mathbf{1} \right) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = 0 \quad (30)$$

where  $\mathbf{1}$  represents the identity matrix of size  $2N$  and  $\mathbf{W}$  is a square matrix of size  $2N$  mapping  $\mathbf{a}$  and  $\mathbf{b}$  to the final state. Similarly, condition (9) is expressed as:

$$\frac{\Delta x}{\sqrt{c^2 + 1}} \sum_{i=1}^N b_i - a_i = g \quad (31)$$

and constraints (10), (11), and (12) are formulated as a linear system of inequalities in  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\mathbf{C}(n_c, n_f) \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \leq 0 \quad (32)$$

where  $\mathbf{C}(n_c, n_f)$  is an array of size  $(n_c + 1 + n_f)N \times 2N$ . Hence, a periodic solution is found by solving for  $\mathbf{a}$  and  $\mathbf{b}$  for (30) to (32) for specified  $n_f$  and  $n_c$ . In addition, admissible solutions are those which satisfy  $s(x, t) > -1$ ,  $\forall x \in [0; L]$  and  $\forall t \in [0; T]$  [3]. This constraint was not surpassed for the extracted periodic solutions exposed below. Findings show that, for  $t_c$  and  $t_f$  where periodic solutions were found [7], the linear system of equations (30) and (31) is under-defined for specific  $N$  and a solution space spanning multiple dimensions can be found. Constraints (32) are satisfied by trial and error as no systematic solution procedures to solve (in)equations (30) to (32) seem to exist. In this work, periodic motions have been extracted for  $L = 1$  m,  $c = 1$  m/s,  $g = 1$  mm and  $N = 300$ . Fig. 1 represents a frequency-energy plot (FEP) in which every point on the curve(s) represents a periodic solution of the system and each continuous curve implies existence of a NNM. In fact,

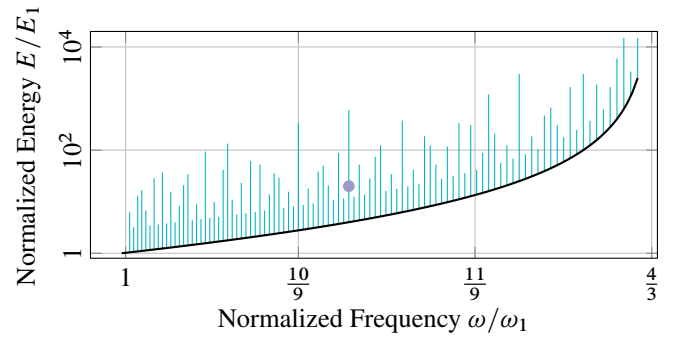


Figure 1: FEP normalized by  $\omega_1 = \pi/2$  rad/s and  $E_1 = 0.5 \mu\text{J}$ . Black: NNM found in [7], Pale Blue: NNMs found in this paper. Gray:  $(8/7\omega_1, 20E_1)$ .

since the system of equations in  $\mathbf{a}$  and  $\mathbf{b}$  is under-determined, several continua of distinct periodic solutions can be detected for energy-frequency points in the FEP. For example, in the presented results, for  $n_c = 150$  and  $n_f = 900$ , the kernel of the linear operator formed by (30) and (31) on  $\mathbf{a}$  and  $\mathbf{b}$  is of 149 dimensions with a null space  $\mathbf{a}(\mathbf{k})$  and  $\mathbf{b}(\mathbf{k})$  defined by  $\mathbf{k} \in \mathbb{R}^{149}$ . Findings show that the null space spans multiple dimensions even when restricting the motion to a specific energy. For example, Fig. 2 represents a continuum  $\mathbf{k}(\lambda)$ ,  $\forall \lambda \in [18, 72]$  such that all resulting  $\mathbf{a}(\lambda)$  and  $\mathbf{b}(\lambda)$  generate motions of  $10 \mu\text{J}$

and satisfy (32). For sake of conciseness, the formulation of  $\mathbf{k}(\lambda)$  is not detailed. More continua exist in this frequency-energy point and in other points on the detected FEP lines.

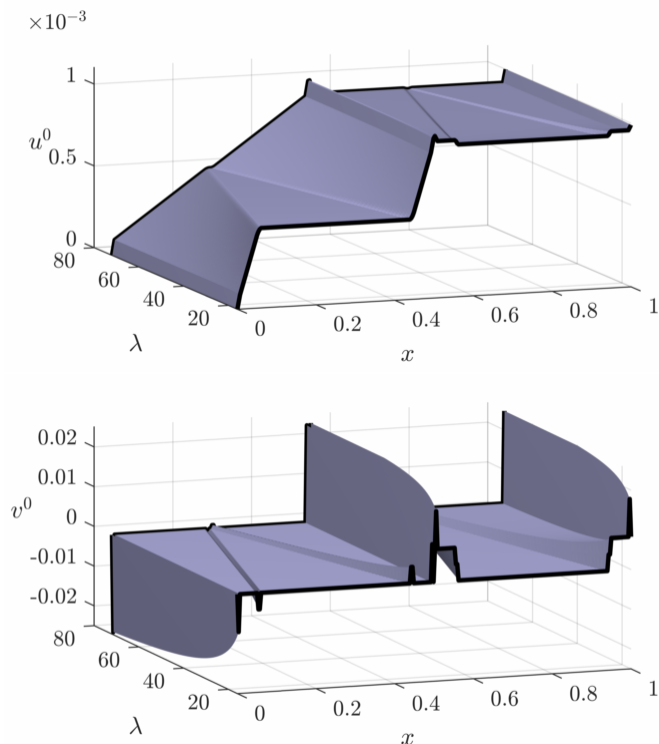


Figure 2:  $\lambda$ -parametrized continuum of initial conditions generating periodic solutions with energy  $20E_1$  and frequency  $8/7\omega_1$  in Fig. 1. Black: Initial conditions used for generating Fig. 3.

## CONCLUSIONS

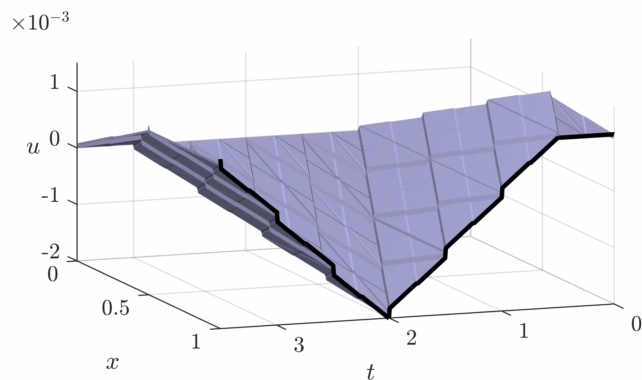
Using an analytical treatment of boundary conditions, derived from the method of characteristics, the weak solution to a clamped bar with frictionless unilateral contact and the conditions for periodic solutions to exist were derived. For specific periods  $T$  (such that  $cT/L = (n_c + n_r)/N$  is rational), continua of periodic solutions, spanning multiple dimensions, were detected. It was also found that such continua exist even for a specified energy. Implication of these results to resonance behavior and extension of this treatment to multi-dimensional vibro-impact structures are currently under investigation.

## ACKNOWLEDGMENTS

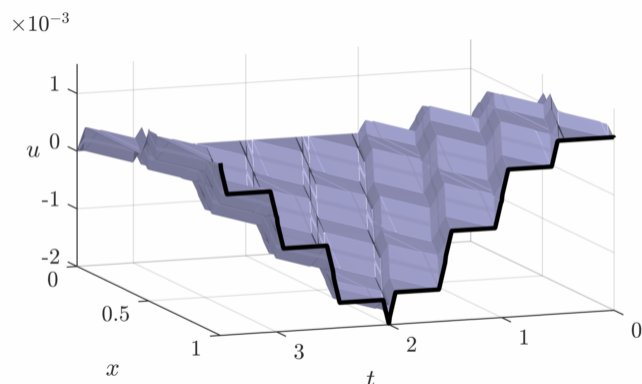
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(a)  $u(x, t)$  generated for  $\mathbf{q}^0(x, \lambda = 72)$



(b)  $u(x, t)$  generated for  $\mathbf{q}^0(x, \lambda = 18)$

Figure 3: Time-evolution for two distinct initial conditions chosen from the continuum in Fig. 2. Black:  $u(L, t)$ .

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