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# LONG TIME BEHAVIOR OF THE SOLUTIONS OF NLW ON THE D-DIMENSIONAL TORUS

JOACKIM BERNIER, ERWAN FAOU, AND BENOÎT GRÉBERT

ABSTRACT. We consider the non linear wave equation (NLW) on the  $d$ -dimensional torus with an smooth nonlinearity of order at least two at the origin. We prove that, for almost all mass, small smooth solutions of high Sobolev indices are stable up to arbitrary long times with respect to the size of the initial data. To prove this result we use a normal form transformation decomposing the dynamics into low and high frequencies with weak interactions. While the low part of the dynamics can be put under classical Birkhoff normal form, the high modes evolves according to a time dependent linear Hamiltonian system. We then control the global dynamics by using polynomial growth estimates for high modes and the preservation of Sobolev norms for the low modes. Our general strategy applies to any semi-linear Hamiltonian PDEs whose linear frequencies satisfy a very general non resonance condition. In particular it also applies straightforwardly to a full dispersion Whitham-Boussinesq system in water waves theory.

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## 1. INTRODUCTION

Let us consider the nonlinear wave equation set on the  $d$ -dimensional torus  $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$  with  $d \geq 2$ ,

$$u_{tt} - \Delta u + mu + f(u) = 0, \quad x \in \mathbb{T}^d, \quad (\text{NLW})$$

satisfied by a real valued function  $u(t, x)$  with given initial data  $u(0, x)$  and  $\dot{u}(0, x) = \partial_t u(0, x)$ . The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is analytic on a neighborhood of the origin and is at least of order 2 at the origin. For small smooth initial data  $u(0) \in H^s(\mathbb{T}^d)$  and  $\dot{u}(0) \in H^{s-1}(\mathbb{T}^d)$  with large  $s$ , we are interested in a description of the long time behavior of  $u(t)$  solution of (NLW).

In dimension  $d = 1$ , it is known that if  $\varepsilon$  measures the size of the initial data, the solution is controlled for arbitrary polynomial times with respect to  $\varepsilon$ , and for almost all  $m$  away from zero. More precisely, it has been proved (see for instance [BG06]) that for almost all  $m \in (m_0, m_1)$  with respect to the Lebesgue measure, and for all  $r$ , then there exists  $s_*(r)$  such that for  $s > s_*(r)$  there exists  $\varepsilon_0(r, s)$  such that for all  $\varepsilon < \varepsilon_0$

$$\|(u(0), \dot{u}(0))\|_{H^s \times H^{s-1}} \leq \varepsilon \implies \|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq 2\varepsilon, \quad t \leq \varepsilon^{-r}. \quad (1)$$

The crucial tool to obtain this result is to show that for a large set of parameter  $m$ , the frequencies  $\omega_j = \sqrt{|j|^2 + m}$  of the linear wave operator satisfy a non-resonance condition of the following form :

Fix  $r \geq 3$ , there exists  $\gamma > 0$  such that for  $\mathbf{k} = (k_1, \dots, k_p) \in (\mathbb{Z}^d)^p$ ,  $\ell = (\ell_1, \dots, \ell_q) \in (\mathbb{Z}^d)^q$  with  $p + q \leq r$  we have

$$|\omega_{k_1} + \dots + \omega_{k_p} - \omega_{\ell_1} - \dots - \omega_{\ell_q}| \geq \frac{\gamma}{\mu_3(\mathbf{k}, \ell)^\alpha}, \quad (\mu_3)$$

unless  $\{|k_1|, \dots, |k_p|\} = \{|\ell_1|, \dots, |\ell_q|\}$ , and where  $\mu_3(\mathbf{k}, \ell)$  denotes the third largest number amongst the collection  $(|k_i|, |\ell_j|)_{i,j}$ , and  $\alpha$  depends on  $p$  and  $q$ .

This condition, introduced in [BG06], allows to treat all the terms in the Hamiltonian of the perturbation (depending on  $F$  a primitive of  $f$  in (NLW)) involving at most two high Fourier modes via a Birkhoff normal form procedure. On the other hand, it is known since [Bam03] that we can neglect all the monomials involving more than three high modes (see for instance [Gre07] or [Bam07] for a simple presentation of these two facts). So once we have  $(\mu_3)$  we can expect a control of the Sobolev norms similar to (1).

Notice that  $(\mu_3)$  is close from the so-called *second-order* Melnikov non-resonance condition<sup>1</sup> that says in a formulation allowing comparison with  $(\mu_3)$ :

Fix  $n \geq 2$ , there exists  $\gamma > 0$  such that for  $\mathbf{k} = (k_1, \dots, k_p) \in (\mathbb{Z}^d)^p$ ,

<sup>1</sup>This terminology refers to the original papers [Mel65, Mel68] where similar conditions were introduced for proving the stability of low-dimensional invariant tori in Hamiltonian dynamics, and popularized in the KAM literature [Mos67, Eli88, Bou97, XY01] and later while extending these results to Hamiltonian PDEs [Kuk93, Pös96].

$\ell = (\ell_1, \dots, \ell_q) \in (\mathbb{Z}^d)^q$  with  $|k_i|, |\ell_i| \leq n$  and for  $(j_1, j_2) \in (\mathbb{Z}^d)^2$  with  $|j_1|, |j_2| > n$

$$|\omega_{k_1} + \dots + \omega_{k_p} - \omega_{\ell_1} - \dots - \omega_{\ell_q} + \omega_{j_1} - \omega_{j_2}| \geq \frac{\gamma}{r^\alpha},$$

unless  $\{|k_1|, \dots, |k_p|\} = \{|\ell_1|, \dots, |\ell_q|\}$ , where  $r = p + q$ .

We note that in the Melnikov case, the “length” of the resonance  $(r + 2)$  is free but the number of “interior” modes (here  $(2n + 1)^d$ ) is fixed while it is exactly the converse in  $(\mu_3)$ . So the two conditions are not equivalent; nevertheless as far as we know, we do not know a context where one of these two conditions is true and not the other.

The condition  $(\mu_3)$  applies to many situations, including the one dimensional wave equations, the one dimensional nonlinear Schrödinger equation with an external potential [Bam03, Bam07, BG06], the  $d$ -dimensional nonlinear Schrödinger equation with a convolution potential [BG03, BG06], plane waves stability for non-linear Schrödinger equation [FGL13], wave equations on Zoll manifolds [BDGS07] or quantum harmonic oscillator on  $\mathbb{R}^d$  [GIP09].

The main difficulty of the higher dimensional case for (NLW) is that the frequencies do not satisfy the second-order Melnikov condition for a large set of parameters  $m$ , as already noted for instance in [Bou95, Del09]. In fact, in dimension  $d \geq 2$   $(\mu_3)$  is not satisfied when for instance  $|k_1| \geq |\ell_1| \geq \mu_3(\mathbf{k}, \ell)$  for which the small denominator becomes

$$\omega_{k_1} - \omega_{\ell_1} + \mathcal{O}(\mu_3(\mathbf{k}, \ell)). \quad (2)$$

For  $d = 1$ , if  $k_1$  and  $\ell_1$  are large, the difference  $\omega_{k_1} - \omega_{\ell_1}$  is close to be an integer and the parameter  $m$  can be chosen in a large set so that  $(\mu_3)$  holds. However, in dimension  $d \geq 2$ ,  $\omega_{k_1} - \omega_{\ell_1}$  describes a dense set at the scale  $|k_1|$  which prevents  $(\mu_3)$  to hold. In other words, we can only prove that for a large set of parameters  $m$ , the following condition (which is related to the so-called *first-order* Melnikov condition) holds

$$|\omega_{k_1} + \dots + \omega_{k_p} - \omega_{\ell_1} - \dots - \omega_{\ell_q}| \geq \frac{\gamma}{\mu_2(\mathbf{k}, \ell)^\alpha} \quad (\mu_2)$$

unless  $\{|k_1|, \dots, |k_p|\} = \{|\ell_1|, \dots, |\ell_q|\}$  where  $\mu_2(\mathbf{k}, \ell)$  is the second largest index in the multi-index  $(\mathbf{k}, \ell)$ . By using the convolution structure of the nonlinearity and the growth of the  $\omega_k$  with  $k$  it is however possible to obtain a control of the denominator of the form  $(\mu_3)$  when the two largest integer are both in  $\mathbf{k}$  or in  $\ell$  but the previous difficulty remains an important obstruction for extending results of the form (1) to higher dimensions.

This known problem also holds in various situations like water wave problems or numerical discretization of Hamiltonian PDEs. Despite this obstacle, some results have been proved concerning the existence of quasi-periodic solutions, where the loss of derivative is controlled by the use of KAM-Newton schemes, see [Bou95] for the case of wave equations and [BBHM18] for the gravity waves case.

Concerning the control of large open sets of solutions, some results can be found in [DS06] and [Del09] but the time control depends on the shape on the nonlinearity inducing restriction on the index  $r$  (essentially driven by the annulation degree of the nonlinearity in 0).

Another situation where  $(\mu_2)$  appears in a natural way is given by numerical discretization of Hamiltonian PDEs. For example standard splitting methods applied to wave equations in dimension  $d = 1$  induces numerical resonances destroying the property  $(\mu_3)$  and degenerating to  $(\mu_2)$  even for generic time discretization parameters<sup>2</sup>. In this case, it is however possible to control the solution by playing with the time integrator or with the space discretization, see [CHL08, FGP10a, FGP10b, Fao12, FGL14].

In this paper, we propose a new method to overcome this difficulty by a careful examination of the normal form induced by  $(\mu_2)$  and a control in mixed Sobolev norm inspired by some tools used in numerical analysis. In particular it can be seen as a nonlinear extension of [DF07], here in a continuous in time setting.

As byproduct of this method, we prove the following. In dimension  $d \geq 2$ , for almost all  $m \in (m_0, m_1)$ ,  $r$  and  $s > d/2$ , there exists  $s_* = s_*(r, s)$  and  $\varepsilon_0(r, s)$  such that for all  $\varepsilon < \varepsilon_0$  the solution to (NLW) satisfies

$$\|(u(0), \dot{u}(0))\|_{H^{s_*} \times H^{s_*-1}} \leq \varepsilon \implies \|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq 2\varepsilon, \quad t \leq \varepsilon^{-r}. \quad (3)$$

In other words, (1) holds up to a finite loss of derivative in the initial condition.

The previous property (3) is in fact a corollary of a stronger abstract result proved in Theorem 2.8. The main idea is to decompose the dynamics into low and high frequencies according to some large threshold depending on  $\varepsilon$ , and then to try to conjugate the system to a normal form whose dynamics can be described and controlled.

When the  $(\mu_3)$  condition holds, this normal form approach allows to conjugate all the frequencies to a flow preserving the  $H^s$  up to terms that are arbitrarily small. When the first condition  $(\mu_2)$  only is satisfied, this cannot be done, and linear terms remains in the dynamics of high modes, coupled with the low modes. Fortunately, these terms can be put under a *symmetric* Hamiltonian quadratic form. Hence, despite the linear nature of their dynamics, the  $L^2$  norm of the high modes can be proved to be preserved over long times in the normal form analysis. This crucial information allows to initiate an effective decomposition between the low and high modes. To prove the almost preservation of higher Sobolev norms of the high modes we use a sort of pseudo-differential argument (or commutator Lemma) that

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<sup>2</sup>The fundamental reason is that times numerical schemes require the control of small divisors of the form  $e^{ih\Omega(\mathbf{k}, \boldsymbol{\ell})} - 1$  instead of  $\Omega(\mathbf{k}, \boldsymbol{\ell})$  as defined in  $(\mu_3)$ , where  $h$  is the time discretization parameter. Hence numerical resonances can occurs when for instance  $h(\omega_{k_1} - \omega_{\ell_1})$  -see (34)- is close to an arbitrary large multiple of  $2\pi$  (see for instance [Sha00, HLW06] for a finite dimensional analysis of symplectic integrators).

allows to gain one derivative (see (48) and (49)). Then for given indices  $s$  and  $s_0$  with  $s \gg s_0 > d/2$ , we can control the low modes in a Sobolev norm  $H^s$  and show a polynomial growth in time of a Sobolev norm  $H^{s_0}$  of the high modes of order  $\mathcal{O}(t^{s_0})$ . By choosing a smoother initial condition such that  $\|(u(0), \dot{u}(0))\|_{H^{2s} \times H^{2s-1}} \leq \varepsilon$  we then obtain

$$\|u(t)_{\leq N_\varepsilon}\|_{H^s} \leq 2\varepsilon \quad \text{and} \quad \|u(t)_{> N_\varepsilon}\|_{H^{s_0}} \leq \varepsilon^r, \quad t \leq \varepsilon^{-\frac{r}{s_0+1}}, \quad (4)$$

in a regime where  $s - s_0$  is large with respect to  $r$ . Here  $u(t)_{\leq N_\varepsilon}$  and  $u(t)_{> N_\varepsilon}$  denote the low and high modes parts, according to the threshold  $N_\varepsilon = \varepsilon^{-\frac{r}{s-s_0}}$ . When  $s$  is large, we obtain (3) after a change of indices, but Theorem 2.8 gives more precise informations. It shows that the dynamics of the low modes preserves the super-actions<sup>3</sup>, *i.e.* the quantities  $J_n(t) = \sum_{|k|=n} |u_k(t)|^2$ ,  $n \leq N_\varepsilon$  over very long times, where  $u_k(t)$ ,  $k \in \mathbb{Z}^d$  denote the Fourier coefficients of  $u(t)$ . However, such a result does not hold for high modes where the interaction between two close large modes cannot be eliminated but only controlled. This result thus expresses the fact that the condition  $(\mu_2)$  -much more general than  $(\mu_3)$ - is enough to ensure a decoupling of the dynamics of low and high modes for very long times.

In the previous estimate,  $s_0$  has only to satisfy the condition<sup>4</sup>  $s_0 > d/2$ . It also typically corresponds to what is numerical observed for initial data taken as trigonometric polynomials<sup>5</sup> for which the dynamics does not exhibit energy exchanges over long times. Such two-stage norms with different Sobolev scaling were previously used in the context of numerical analysis of splitting methods for Schrödinger equations in the linear case, see [DF07] and [DF09] where again the preservation of the  $L^2$  norm of the high-modes was crucial to obtain a global control of the dynamics.

Finally, we give an example of application of this technics which turns out to be very general under the condition  $(\mu_2)$ . In particular, we prove that it can be straightforwardly applied to a Hamiltonian Whitham-Boussinesq system in water-wave theory [DDK19] which has the advantage to be Hamiltonian, semi-linear and to carry over the dispersion relation of the water wave problem,  $\omega_j = \sqrt{j \tanh(hj)}$ ,  $j \in \mathbb{Z}$ , where  $h$  represent the depth of the two-dimensional fluid. In this case the condition  $(\mu_2)$  was proved in [BBHM18] for a large set of parameter  $h$ . We believe that the extension of our technics to other systems in water wave theory involving reversible dynamics (see for instance the review in [Lan13]) is a promising field of study.

We also believe that results of the form (3) or (4) involving mixed Sobolev norms provides a natural setting for numerical discretization, for which  $(\mu_2)$

<sup>3</sup>Already used in [BG06] for (NLW) in dimension  $d = 1$  in a periodic setting.

<sup>4</sup>Condition that could probably be refined in the critical case using Galigardo-Nirenberg inequality.

<sup>5</sup>Note that in the present result, all the constants depends on the Sobolev indices  $s$  and  $s_0$ . An optimization of these constant could be done in an analytic context, following the technics used in [FG13].

is the generic control of non-resonance condition. It might for instance allow to weaken the usual CFL conditions required, or to derive low-order integrators following the analysis of [HS17, OS18].

## 2. ABSTRACT STATEMENT

**2.1. Hamiltonian formalism.** We recall in this subsection the formalism used in [Gre07, FG13, BFG] to deal with infinite dimensional Hamiltonians and flows depending on a infinite set of symplectic variables  $(q, p) = (q_a, p_a)_{a \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$  equipped with the usual  $\ell_s^2(\mathbb{Z}^d, \mathbb{R}^2)$  norm defined<sup>6</sup> as

$$\|(q, p)\|_s^2 = \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s} (p_a^2 + q_a^2), \quad \langle a \rangle^2 = 1 + |a|^2.$$

As explained in [Gre07], for  $\mathcal{U}$  an open set of  $\ell_s^2$ , for a function  $H(q, p)$  such that  $H \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{R})$ , with  $\ell^2$  gradient  $\nabla_{(q,p)} H \in \mathcal{C}^\infty(\mathcal{U}, \ell_s^2)$ , we can define the flow of a Hamiltonian system

$$\forall a \in \mathbb{Z}^d, \quad \dot{q}_a = \frac{\partial H}{\partial p_a}(q, p), \quad \dot{p}_a = -\frac{\partial H}{\partial q_a}(q, p). \quad (5)$$

To easily deal with normal form transformation, it is convenient to use the complex representation  $(\xi_a)_{a \in \mathbb{Z}^d} = (\frac{1}{\sqrt{2}}(q_a + ip_a))_{a \in \mathbb{Z}^d}$  in  $\mathbb{C}^{\mathbb{Z}^d}$  equipped with the  $\ell_s^2(\mathbb{Z}^d, \mathbb{C})$  norm. Then with the notation

$$\frac{\partial}{\partial \xi_a} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q_a} - i \frac{\partial}{\partial p_a} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{\xi}_a} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial q_a} + i \frac{\partial}{\partial p_a} \right),$$

the real Hamiltonian system is equivalent to the complex system,

$$\forall a \in \mathbb{Z}^d, \quad \dot{\xi}_a = -i \frac{\partial H}{\partial \bar{\xi}_a}(\xi, \bar{\xi}) =: (X_H(\xi, \bar{\xi}))_a \quad (6)$$

where  $H(q, p) = H(\xi, \bar{\xi}) \in \mathbb{R}$  is a real Hamiltonian. The notation  $X_H(\xi, \bar{\xi}) = \{(X_H(\xi, \bar{\xi}))_a, a \in \mathbb{Z}^d\}$  thus denote the Hamiltonian vector field associated with the Hamiltonian  $H$ . If we associate with  $(\xi, \bar{\xi})$  a complex function  $\psi$  on  $\mathbb{T}^d$ , through the formula

$$\psi(x) = \sum_{a \in \mathbb{Z}^d} \xi_a e^{iax}, \quad (7)$$

then the Sobolev norm  $\|\psi\|_{H^s}$  is equivalent to the norm

$$\|\xi\|_s^2 = \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s} |\xi_a|^2.$$

The symplectic structure is written

$$\sum_{a \in \mathbb{Z}^d} dq_a \wedge dp_a = -i \sum_{a \in \mathbb{Z}^d} d\xi_a \wedge d\bar{\xi}_a. \quad (8)$$

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<sup>6</sup>with the usual notation  $|a|^2 = a_1^2 + \dots + a_d^2$  for  $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$ .

and the Poisson bracket in complex notation is given by

$$\{F, G\} = -i \sum_{a \in \mathbb{Z}^d} \frac{\partial F}{\partial \xi_a} \frac{\partial G}{\partial \bar{\xi}_a} - \frac{\partial F}{\partial \bar{\xi}_a} \frac{\partial G}{\partial \xi_a}. \quad (9)$$

**Definition 2.1.** For a given  $s > d/2$  and a domain  $\mathcal{U}$  containing 0 in  $\ell_s^2 := \ell_s^2(\mathbb{Z}^d, \mathbb{C})$ , we denote by  $\mathcal{H}_s(\mathcal{U})$  the space of real Hamiltonians  $P(\xi, \bar{\xi}) \in \mathbb{R}$  satisfying

$$P \in C^\infty(\mathcal{U}, \mathbb{R}), \quad \text{and} \quad X_P \in C^\infty(\mathcal{U}, \ell_s^2),$$

where  $X_P$  is defined in (6). We will use the shortcut  $F \in \mathcal{H}_s$  to indicate that there exists a domain  $\mathcal{U}$  containing 0 in  $\ell_s^2$  such that  $F \in \mathcal{H}_s(\mathcal{U})$ .

Notice that for  $F$  and  $G$  in  $\mathcal{H}_s$  the formula (9) is well defined in a neighborhood of 0. With a given Hamiltonian function  $H \in \mathcal{H}_s$ , we associate the Hamiltonian system (6), and we can naturally defines its flow.

**Proposition 2.2.** Let  $s > d/2$ . Any Hamiltonian in  $\mathcal{H}_s$  defines a local flow in  $\ell_s^2$  which is a symplectic transformation.

**2.2. Polynomial Hamiltonians.** To algebraically deal with polynomials depending on  $(\xi, \bar{\xi})$ , we identify  $\mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d} \simeq \mathbb{C}^{\mathbb{U}_2 \times \mathbb{Z}^d}$  where  $\mathbb{U}_2 = \{\pm 1\}$  and use the convenient notation  $(\xi, \bar{\xi}) = z = (z_j)_{j \in \mathbb{U}_2 \times \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{U}_2 \times \mathbb{Z}^d}$  where

$$j = (\delta, a) \in \mathbb{U}_2 \times \mathbb{Z}^d \implies \begin{cases} z_j = \xi_a & \text{if } \delta = 1, \\ z_j = \bar{\xi}_a & \text{if } \delta = -1. \end{cases} \quad (10)$$

We define the  $\ell_s^2$  the norm of an element  $z = (\xi, \bar{\xi})$  to be

$$\|z\|_s^2 := \sum_{j \in \mathbb{U}_2 \times \mathbb{Z}^d} \langle j \rangle^{2s} |z_j|^2 = 2 \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s} |\xi_a|^2 = 2 \|\xi\|_s^2.$$

where for  $j = (\delta, a) \in \mathbb{U}_2 \times \mathbb{Z}^d$  we set  $\langle j \rangle^2 = \langle a \rangle^2$ . With this notation, the Hamiltonian system (6) can be written

$$\dot{z} = X_H(z), \quad \text{where} \quad (X_H(z))_{(\delta, a)} := -i\delta(X_H(\xi, \bar{\xi}))_a, \quad z = (\xi, \bar{\xi}),$$

where  $X_H(\xi, \bar{\xi})$  denote the vector field in (6). Another way to formulate this notation is to say that with the identification (10), the vector field  $X_H(\xi, \bar{\xi})$  is naturally extended as  $(X_H(z))_j = -i\delta \frac{\partial H}{\partial z_{\bar{j}}}(z)$ , for  $j = (\delta, a) \in \mathbb{U}_2 \times \mathbb{Z}^d$ , where  $\bar{j} = (-\delta, a)$ .

For  $\mathbf{k} = (k_1, \dots, k_m) = (\delta_i, a_i)_{i=1}^m \in (\mathbb{U}_2 \times \mathbb{Z}^d)^m$  we denote the momentum

$$\mathcal{M}(\mathbf{k}) = \sum_{i=1}^m \delta_i a_i,$$

and we define the conjugate  $\bar{\mathbf{k}} = (-\delta_i, a_i)_{i=1}^m \in (\mathbb{U}_2 \times \mathbb{Z}^d)^m$ . We set

$$\mathcal{M}_m = \{\mathbf{k} \in (\mathbb{U}_2 \times \mathbb{Z}^d)^m \mid \mathcal{M}(\mathbf{k}) = 0\}. \quad (11)$$

For a given  $\mathbf{k} \in \mathcal{M}_m$  we write

$$z_{\mathbf{k}} = z_{k_1} \cdots z_{k_m}.$$



We also set

$$\mathcal{R}_m = \{(\delta_j, a_j)_{j=1}^m \in \mathcal{M}_m \mid \exists \sigma \in \mathfrak{S}_m, \forall j = 1, \dots, m, \delta_j = -\delta_{\sigma_j} \text{ and } \langle a_j \rangle = \langle a_{\sigma_j} \rangle\} \quad (12)$$

the set of **resonant** multi-indices. Note that by construction if  $m$  is odd then  $\mathcal{R}_m$  is empty and that if  $\mathbf{k} = (\delta_j, a_j)_{j=1}^m \in \mathcal{R}_m$  is associated with a permutation  $\sigma$  then we have

$$z_{\mathbf{k}} = \prod_{\substack{1 \leq j \leq m \\ \delta_j = 1}} \xi_{a_j} \bar{\xi}_{a_{\sigma_j}}. \quad (13)$$

**Definition 2.3.** We say that  $P(\xi, \bar{\xi})$  is a homogeneous polynomial of order  $m$  if it can be written with  $z = (\xi, \bar{\xi})$ ,

$$P(z) = P[c](z) = \sum_{\mathbf{j} \in \mathcal{M}_m} c_{\mathbf{j}} z_{\mathbf{j}}, \quad \text{with } c = (c_{\mathbf{j}})_{\mathbf{j} \in \mathcal{M}_m} \in \ell^\infty(\mathcal{M}_m), \quad (14)$$

and such that the coefficients  $c_{\mathbf{j}}$  satisfy  $c_{\bar{\mathbf{j}}} = \overline{c_{\mathbf{j}}}$ .

Note that the last condition ensures that  $P$  is real, as the set of indices are invariant by the application  $\mathbf{j} \mapsto \bar{\mathbf{j}}$ . Following [FG13, BFG] but in a  $\ell_s^2$  framework, we get the following Proposition. It turns out to be a consequence of the more general Lemma 5.1 proved below.

**Proposition 2.4.** Let  $s > d/2$ . A homogeneous polynomial,  $P[c]$ , of degree  $m \geq 2$  belongs to  $\mathcal{H}_s(\ell_s^2)$  and we have

$$\|X_{P[c]}(z)\|_s \leq (C_s)^m \|c\|_{\ell^\infty} \|z\|_s^{m-1}, \quad z = (\xi, \bar{\xi}), \quad (15)$$

for some constant  $C_s$  depending only on  $m$ . Furthermore for two homogeneous polynomials,  $P[c]$  and  $P[c']$ , of degree respectively  $m$  and  $n$ , the Poisson bracket is a homogeneous polynomial of degree  $m+n-1$ ,  $\{P[c], P[c']\} = P[c'']$  and we have the estimate

$$\|c''\|_{\ell^\infty} \leq 2mn \|c\|_{\ell^\infty} \|c'\|_{\ell^\infty}. \quad (16)$$

We now make our hypothesis on the Hamiltonian  $H$  that we will consider.

**Hypothesis 2.5.** The Hamiltonian  $H$  can be written

$$H = H_2 + P = \sum_{a \in \mathbb{Z}^d} \omega_a |\xi_a|^2 + P \quad (17)$$

where  $\omega = (\omega_a)_{a \in \mathbb{Z}^d}$  is a collection of real numbers,  $P \in \mathcal{H}_s$  is analytic on  $\ell_s^2$ , and of order at least 3 in  $z$  (which means that  $P$  and its differential up to the order 2 vanish at 0).

Analyticity means here that  $P$  can be expanded into a convergent sum  $P = \sum_{m \geq 3} P[c_m]$  with  $\|c_m\|_{\ell^\infty} \leq R^m$  for some constant  $R$ . In view of the previous Lemma, the vector field  $X_P$  defined above is analytic on  $\ell_s^2$  under this assumption. Note that in the applications, the frequencies  $\omega_a$  will

not be uniformly bounded with respect to  $a$ , and hence the quadratic part  $\sum_{a \in \mathbb{Z}} \omega_a |\xi_a|^2$ , does not belong to  $\mathcal{H}_s$ . Nevertheless it generates a continuous flow which maps  $\ell_s^2$  into  $\ell_s^2$  explicitly given for all time  $t$  and for all indices  $a$  by  $\xi_a(t) = e^{-i\omega_a t} \xi_a(0)$ . Furthermore this flow has the group property. By standard arguments (see for instance [Caz03] and [BFG] in a similar framework), this is enough to define the local flow of  $\dot{z} = X_H(z)$  in  $\ell_s^2$  which is symplectic.

**2.3. Non resonance condition.** We also assume that the family of frequencies  $\{\omega_a, a \in \mathbb{Z}^d\}$  is non resonant in the following sense.

**Definition 2.6.** *A family of frequencies  $\omega = \{\omega_a, a \in \mathbb{Z}^d\}$  is **non resonant**, if there exist  $(\alpha(r))_{r \geq 1} \in (\mathbb{R}_+^*)^{\mathbb{N}}$  and  $(\gamma_r)_{r \geq 1} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ , such that for all  $r \geq 1$ , all  $N \geq 1$  and all  $\mathbf{k} = (\delta_i, a_i)_{i=1}^r \in (\mathbb{U}_2 \times \mathbb{Z}^d)^r$  satisfying  $\langle a_i \rangle \leq N$ , for  $i = 1, \dots, r$ , we have*

$$|\delta_1 \omega_{a_1} + \dots + \delta_r \omega_{a_r}| \geq \gamma_r N^{-\alpha(r)}, \quad \text{when } \mathbf{k} \notin \mathcal{R}_r, \quad (\text{H1})$$

$$|\delta_1 \omega_{a_1} + \dots + \delta_r \omega_{a_r} + \omega_b| \geq \gamma_{r+1} N^{-\alpha(r+1)}, \quad \forall \langle b \rangle > N \text{ with } \mathcal{M}(\mathbf{k}) + b = 0, \quad (\text{H2})$$

$$|\delta_1 \omega_{a_1} + \dots + \delta_r \omega_{a_r} + \omega_{b_1} + \omega_{b_2}| \geq \gamma_{r+2} N^{-\alpha(r+2)}, \quad \forall \langle b_1 \rangle, \langle b_2 \rangle > N \quad (\text{H3})$$

with  $\mathcal{M}(\mathbf{k}) + b_1 + b_2 = 0$ .

We notice that conditions (H1)-(H2) are equivalent to condition  $(\mu_2)$  introduced in the introduction while conditions (H1)-(H2)-(H3) are not equivalent to  $(\mu_3)$  since in (H3) we are not considering the case where the two high frequencies have opposite sign.

*Remark 2.7.* Note that in (H2) using the momentum condition,  $\langle b \rangle$  is in fact bounded by  $rN$ . Hence (H2) is a trivial consequence of (H1). Similarly, as in many applications  $\omega_a \sim |a|^\nu$  when  $a \rightarrow \infty$ , for some  $\nu > 0$ , (H3) is also a consequence of (H1) as we can restrict to a set of  $(b_1, b_2)$  that are polynomially bounded by  $N^\nu$  for some  $\nu$ . However the crucial condition (H3) with a minus sign *i.e.* with  $\omega_{b_1} - \omega_{b_2}$ , which would allow a direct application of the Birkhoff reduction in [BG06], does not hold for the wave equation in dimension  $d \geq 2$ .

**2.4. Statement.** Let us start with the following notation: For  $\xi \in \ell_s^2$  and a given number  $N$ , we decompose  $\xi = \xi_{\leq N} + \xi_{> N}$  where for all  $j \in \mathbb{U}_2 \times \mathbb{Z}^d$ ,

$$(\xi_{\leq N})_j = \begin{cases} \xi_j & \text{for } \langle j \rangle \leq N, \\ 0 & \text{for } \langle j \rangle > N, \end{cases} \quad \text{and} \quad \xi_{> N} = \xi - \xi_{\leq N}.$$

Given a function  $\psi \in H^s(\mathbb{T}^d)$  with Fourier coefficients  $\xi_a$ ,  $a \in \mathbb{Z}^d$ , and a number  $N \geq 1$  the previous decomposition induces naturally the decomposition  $\psi = \psi_{\leq N} + \psi_{> N}$  with

$$\psi_{\leq N}(x) = \sum_{\langle a \rangle \leq N} \xi_a e^{ia \cdot x} \quad \text{and} \quad \psi_{> N}(x) = \sum_{\langle a \rangle > N} \xi_a e^{ia \cdot x}.$$

Similarly, we note  $z_{\geq N}$  and  $z_{>N}$  the decomposition induced with the notation (10).

**Theorem 2.8.** *Let  $H$  be a Hamiltonian satisfying Hypothesis 2.5 with  $\omega = \{\omega_a, a \in \mathbb{Z}^d\}$  a family of non resonant frequencies. Then for all  $r \geq 2$  and all  $s > s_0 > d/2$  satisfying*

$$s - s_0 \geq s_*(r) := 6r^2\alpha(3r) + 2dr \quad (18)$$

then there exists  $\varepsilon_0(d, r, s, s_0, \omega) > 0$  such that for all  $\varepsilon < \varepsilon_0(d, r, s, s_0, \omega)$  the solution  $\xi(t)$  generated by the flow of  $H$  of the Hamiltonian system (6) issued from a small initial datum  $\xi(0) \in \ell_{2s}^2$  with  $\|\xi(0)\|_{2s} \leq \varepsilon$  exists for all time  $t \leq \varepsilon^{-\frac{r}{s_0+1}}$  and satisfies

$$\begin{aligned} \forall \langle a \rangle \leq N_\varepsilon := \varepsilon^{-\frac{r}{s-s_0}}, \\ \forall t \leq \varepsilon^{-\frac{r}{s_0+1}}, \end{aligned} \quad \left| \sum_{\substack{b \in \mathbb{Z}^d \\ \langle b \rangle = \langle a \rangle}} |\xi_b(t)|^2 - \sum_{\substack{b \in \mathbb{Z}^d \\ \langle b \rangle = \langle a \rangle}} |\xi_b(0)|^2 \right| \leq \varepsilon^3 \langle a \rangle^{-2s} \quad (19)$$

and

$$\begin{cases} \|y(t)_{\leq N_\varepsilon}\|_s^2 = \sum_{\langle a \rangle \leq N_\varepsilon} \langle a \rangle^{2s} |y_a(t)|^2 \leq 4\varepsilon^2 \\ \|y(t)_{> N_\varepsilon}\|_{s_0}^2 \sum_{\langle a \rangle > N_\varepsilon} \langle a \rangle^{2s_0} |y_a(t)|^2 \leq \varepsilon^{2r} \end{cases} \quad \text{for } t \leq \varepsilon^{-\frac{r}{s_0+1}}.$$

Note that by playing with the indices, we can obtain the following corollary which essentially proves that arbitrary high regularity small solutions to (NLW) are controlled over arbitrary long times.

**Corollary 2.9.** *Let  $H$  be a Hamiltonian satisfying Hypothesis 2.5 with  $\omega = \{\omega_a, a \in \mathbb{Z}^d\}$  a family of non resonant frequencies. For all  $r \geq 2$  and  $s > d/2$ , there exists  $s_*(r, s)$  and  $\varepsilon_0(r, s, \omega) > 0$  such that for all  $\varepsilon < \varepsilon_0(r, s, \omega)$ ,*

$$\|\xi(0)\|_{s_*} \leq \varepsilon \implies \|\xi(t)\|_s \leq 2\varepsilon \quad t \leq \varepsilon^{-r}.$$

*Proof.* It is a consequence of the previous Theorem by replacing  $r$  by  $r(s_0 + 1)$ ,  $s_0$  by  $s$  and assuming that  $s_*$  is large enough with respect to  $r$  and  $s$ . Note that in practice, we have  $s_*(r, s) \simeq (rs)^q$  for some  $q \geq 2$ .  $\square$

### 3. APPLICATIONS

**3.1. Application to (NLW) on  $\mathbb{T}^d$ .** Introducing  $v = u_t \equiv \dot{u}$  we rewrite (NLW) as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -\Lambda^2 u - f(u), \end{cases} \quad (20)$$

where  $\Lambda = (-\Delta + m)^{1/2}$ . When  $m > 0$ , we can define

$$\psi = \frac{1}{\sqrt{2}} (\Lambda^{1/2} u + i\Lambda^{-1/2} v), \quad (21)$$

and we get that  $(u, v) \in H^s(\mathbb{T}^d, \mathbb{R}) \times H^{s-1}(\mathbb{T}^d, \mathbb{R})$  is solution of (20) if and only if  $\psi \in H^{s+1/2}(\mathbb{T}^d, \mathbb{C})$  is solution of

$$i\dot{\psi} = \Lambda\psi + \frac{1}{\sqrt{2}}\Lambda^{-1/2}f\left(\Lambda^{-1/2}\left(\frac{\psi + \bar{\psi}}{\sqrt{2}}\right)\right). \quad (22)$$

Then, endowing the space  $L^2(\mathbb{T}^d, \mathbb{C})$  with the standard real symplectic structure  $-id\psi \wedge d\bar{\psi}$  equation (22) reads as a Hamiltonian equation

$$i\dot{\psi} = \frac{\partial H}{\partial \bar{\psi}} \quad (23)$$

where  $H$  is the hamiltonian function

$$H(\psi, \bar{\psi}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} (\Lambda\psi)\bar{\psi} dx + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F\left(\Lambda^{-1/2}\left(\frac{\psi + \bar{\psi}}{\sqrt{2}}\right)\right) dx.$$

where  $F$  is a primitive of  $f$  with respect to the variable  $u$ :  $f = \partial_u F$ . The linear operator  $\Lambda$  is diagonal in the complex Fourier basis<sup>7</sup>  $\{e^{ia \cdot x}\}_{a \in \mathbb{Z}^d}$ , with eigenvalues

$$\omega_a = \sqrt{|a|^2 + m}, \quad a = (a_1, \dots, a_d) \in \mathbb{Z}^d, \quad |a|^2 = a_1^2 + \dots + a_d^2. \quad (24)$$

Decomposing  $\psi$  in Fourier variables with Fourier coefficients  $(\xi_a)_{a \in \mathbb{Z}^d}$  as in (7), (22) takes the form (6) where the Hamiltonian function  $H$  is given by  $H = H_2 + P$  with,

$$H_2(\xi, \bar{\xi}) = \sum_{a \in \mathbb{Z}^d} \omega_a |\xi_a|^2,$$

$$P(\xi, \bar{\xi}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F\left(\sum_{a \in \mathbb{Z}^d} \frac{\xi_a e^{ia \cdot x} + \bar{\xi}_a e^{-ia \cdot x}}{\sqrt{2\omega_a}}\right) dx.$$

As  $F$  is analytic the function  $P$  is in  $\mathcal{H}_s$  and we can define its flow. Moreover,  $P$  is analytic. Finally  $(u, v) \in H^s(\mathbb{T}^d, \mathbb{R}) \times H^{s-1}(\mathbb{T}^d, \mathbb{R})$  is a solution of (20) if and only if  $\xi \in \ell_{s-1/2}^2$  is a solution of the Hamiltonian system associated with the Hamiltonian  $H$ .

In order to apply Theorem 2.8, we need the following result

**Proposition 3.1** ([Del09], Theorem 2.1.1). *For almost all  $m \in (0, +\infty)$  the family of frequencies*

$$\omega_a(m) = \sqrt{|a|^2 + m}, \quad a \in \mathbb{Z}^d$$

*associated with (NLW) is non resonant.*

A direct proof of this proposition can also be done by following the arguments given in [Bam03, EGK16] or [FGL13]. By following these proofs, one can relatively easily show that  $\alpha(r)$  is of order  $\mathcal{O}(r^3)$ .

As a consequence, Theorem 2.8 applies. By scaling back to the variable  $(u, v)$ , we can formulate in particular Corollary 2.9 as follows.

<sup>7</sup>Here for  $a = (a_1, \dots, a_d) \in \mathbb{Z}^d$  and  $x = (x_1, \dots, x_d) \in \mathbb{T}^d$  we set  $a \cdot x = a_1 x_1 + \dots + a_d x_d$

**Theorem 3.2.** *Let  $f$  be a  $C^\infty$  function with zero at least 2 at the origin. Then for almost all  $m \in (0, +\infty)$  and all  $r$ , and  $s > (d+1)/2$ , there exists  $s_1(r, s)$  such that for all  $s_* \geq s_1(r, s)$ , there exists  $\varepsilon_0(r, s, s_*, m)$  such that for all  $\varepsilon < \varepsilon_0(r, s, s_*, m)$ , if  $(u(0), \dot{u}(0)) \in H_0^{s_*} \times H_0^{s_*-1}(\mathbb{T})$  satisfies  $\|(u(0), \dot{u}(0))\|_{H^{s_*} \times H^{s_*-1}} \leq \varepsilon$ , then the system (NLW) admits a solution over a time  $T \geq \varepsilon^{-r}$ , and we have*

$$\|(u(t), \dot{u}(t))\|_{H^s \times H^{s-1}} \leq 2\varepsilon, \quad t \leq \varepsilon^{-r}.$$

### 3.2. Application to a full dispersion Whitham-Boussinesq system.

We consider now the Whitham-Boussinesq system introduced in [DDK19] and studied in [DT]. This model describes the evolution of a fluid layer with depth  $h > 0$ , surface elevation  $\eta$  and fluid velocity  $v$  depending on a periodic dimensional variable  $x \in \mathbb{T}$ . Let  $\eta \in H_0^s(\mathbb{T})$  and  $v \in H_0^{s+1/2}(\mathbb{T})$  be two functions with zero average on  $\mathbb{T}$ . With the notation  $D = -i\partial_x$  and  $K = \sqrt{\tanh(hD)/hD}$ , the system is written<sup>8</sup>

$$\begin{cases} \partial_t \eta + h\partial_x v &= -K^2 \partial_x (\eta v), \\ \partial_t v + K^2 \partial_x \eta &= -K^2 \partial_x (v^2/2). \end{cases} \quad (25)$$

We can write this system under the form

$$\begin{cases} \partial_t \eta = -R \nabla_v H(\eta, v), \\ \partial_t v = R^* \nabla_\eta H(\eta, v), \end{cases} \quad (26)$$

with  $R = K^2 \partial_x$  and

$$H(\eta, v) = \frac{1}{4\pi} \int_{\mathbb{T}} (\eta^2 + hvK^{-2}v + \eta v^2).$$

We have the relation

$$\begin{aligned} & \begin{pmatrix} (R^*R)^{-3/4}R^* & 0 \\ 0 & (R^*R)^{-1/4} \end{pmatrix} \begin{pmatrix} 0 & -R \\ R^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (R^*R)^{3/4}R^{-1} & 0 \\ 0 & (R^*R)^{1/4} \end{pmatrix}. \end{aligned}$$

Hence setting

$$\begin{cases} p &= h^{-1/4}K^{1/2}(R^*R)^{-3/4}R\eta = h^{-1/4}K^{-1/2}|D|^{-3/2}\partial_x \eta \quad \text{and} \\ q &= h^{1/4}K^{-1/2}(R^*R)^{-1/4}v = h^{1/4}K^{-3/2}|D|^{-1/2}v \end{cases}$$

the system can be written under the canonical Hamiltonian form

$$\begin{cases} \partial_t p = -\nabla_q \mathcal{H}(p, q), \\ \partial_t q = \nabla_p \mathcal{H}(p, q), \end{cases}$$

with

$$\mathcal{H}(p, q) = H(h^{1/4}K^{1/2}|D|^{3/2}\partial_x^{-1}p, h^{-1/4}K^{3/2}|D|^{1/2}q).$$

<sup>8</sup>The gravity constant  $g$  plays no role in the result and has been normalized to 1.

Note that  $p$  and  $q$  are real functions. This Hamiltonian is given by

$$K(p, q) = \frac{1}{4\pi} \int_{\mathbb{T}} p\Omega p + q\Omega q - ph^{-1/4}K^{1/2}|D|^{3/2}\partial_x^{-1}(K^{3/2}|D|^{1/2}q)^2$$

with  $\Omega = h^{1/2}K|D| = |D|^{1/2}\sqrt{\tanh(h|D|)}$ . The cubic part of the Hamiltonian can be written

$$\begin{aligned} & \frac{1}{h^2 4\pi} \int_{\mathbb{T}} p|\tanh(hD)|^{1/4}|D|^{1/4}(|D|\partial_x^{-1})(|\tanh(hD)|^{3/4}|D|^{-1/4}q)^2 \\ &= \sum_{k+\ell+m=0} |\tanh(hk)|^{1/4}|\tanh(h\ell)|^{3/4}|\tanh(hm)|^{3/4} \frac{|k|^{1/4}(-i\text{sgn}(k))}{2h^2|\ell|^{1/4}|m|^{1/4}} \hat{p}_k \hat{q}_\ell \hat{q}_m \end{aligned}$$

and we verify that the coefficients are uniformly bounded, as  $|k| \leq |m| + |\ell| \leq 2|m||\ell|$  using the fact that  $m$  and  $\ell$  are non zero. Eventually, we define the complex function  $\psi = \frac{1}{\sqrt{2}}(p + iq)$  and we verify that it is the solution of a Hamiltonian system of the form (6) with a Hamiltonian  $H = H_2 + P$  satisfying the assumption (2.5).

The frequencies of the linear operator are written

$$\omega_a(h) = |a|^{1/2}\sqrt{\tanh(h|a|)}, \quad a \in \mathbb{Z}^*.$$

These frequencies are the same as the gravity water wave problem. It has been proved in [BBHM18] that for almost all  $h \in \mathcal{I}$  a fixed closed interval the frequencies  $\omega_a(h)$  are non resonant (see Eqn. (4.20) in [BBHM18], the measure estimates being a consequence of Proposition 3.4 of this reference and [Rüs01]). Note that here we do not impose any parity condition on the solution, but only a zero average condition which is clearly propagated by the dynamics. Hence Theorem 2.8 applies. In particular, we can formulate Corollary 2.9 as follows:

**Theorem 3.3.** *Let  $\mathcal{I} \subset \mathbb{R}_+^*$  be a closed interval. Then for almost  $h \in \mathcal{I}$  and all  $r$ , and  $s > 1/2$ , there exists  $s_1(r, s)$  such that for all  $s_* \geq s_1(r, s)$ , there exists  $\varepsilon_0(r, s, s_*, h)$  such that for all  $\varepsilon < \varepsilon_0(r, s, s_*, h)$ , if  $(\eta(0), v(0)) \in H_0^{s_*} \times H_0^{s_*+1/2}(\mathbb{T})$  satisfies  $\|(\eta(0), v(0))\|_{H^{s_*} \times H^{s_*+1/2}} \leq \varepsilon$ , then the system (25) admits a solution over a time  $T \geq \varepsilon^{-r}$ , and we have*

$$\|(\eta(t), v(t))\|_{H^s \times H^{s+1/2}} \leq 2\varepsilon, \quad t \leq \varepsilon^{-r}.$$

#### 4. NORMAL FORM

The strategy used to prove Theorem 2.8 is to transform the original Hamiltonian (17) into a normal form eliminating some interactions between the low and high frequencies. By using Taylor expansion, the Hamiltonian  $H$  can be written

$$H = H_2 + \sum_{m=3}^r P_m + R_{r+1}, \quad \text{with} \quad H_2 = \sum_{a \in \mathbb{Z}^d} \omega_a |\xi_a|^2, \quad (27)$$

where  $(P_m)_{m=3}^r$  are homogeneous polynomials of degree  $m$ , and where  $R_{r+1} \in \mathcal{H}_s$  is analytic and of order  $r+1$  which means that its differential vanish up to the order  $r$ . In particular, we have

$$\|X_{R_{r+1}}(z)\|_s \leq (C_s)^r \|z\|_s^r$$

for some constant depending only on  $s$  and  $z$  small enough in  $\ell_s^2$ .

Let us define the following notation,  $\forall \mathbf{j} \in \mathcal{M}_m$ ,  $\mu_n(\mathbf{j})$  denote  $n$ -th largest number amongst the collection  $\langle j_i \rangle_{i=1}^m$ , *i.e.* we have

$$\mu_1(\mathbf{j}) \geq \mu_2(\mathbf{j}) \geq \cdots \geq \mu_m(\mathbf{j}).$$

By convention, we will also set  $\mu_0(\mathbf{j}) = +\infty$ .

Let  $N$  be a fixed number and  $H$  satisfying Hypothesis 2.5. For a given  $r$ , this number induces a decomposition of the Hamiltonian  $H$  in (27) as follows:

$$H = H_2 + \sum_{m=3}^r (P_m^{(\circ)} + P_m^{(i)} + P_m^{(ii)} + P_m^{(iii)}) + R_{r+1},$$

where for all  $m$ ,

- the polynomial  $P_m^{(\circ)} = P[c_m^{(\circ)}]$ , depends only of low modes; for all  $\mathbf{j} \in \mathcal{M}_m$ ,

$$(c_m^{(\circ)})_{\mathbf{j}} \neq 0 \implies \mu_1(\mathbf{j}) \leq N. \quad (28)$$

- $P_m^{(i)} = P[c_m^{(i)}]$  contains only one high mode; for all  $\mathbf{j} \in \mathcal{M}_m$ ,

$$(c_m^{(i)})_{\mathbf{j}} \neq 0 \implies \mu_1(\mathbf{j}) > N \geq \mu_2(\mathbf{j}), \quad (29)$$

- $P_m^{(ii)} = P[c_m^{(ii)}]$  contains only two high modes; for all  $\mathbf{j} \in \mathcal{M}_m$ ,

$$(c_m^{(ii)})_{\mathbf{j}} \neq 0 \implies \mu_2(\mathbf{j}) > N \geq \mu_3(\mathbf{j}), \quad (30)$$

- and  $P_m^{(iii)} = P[c_m^{(iii)}]$  contains at least three high modes; for all  $\mathbf{j} \in \mathcal{M}_m$ ,

$$(c_m^{(iii)})_{\mathbf{j}} \neq 0 \implies \mu_3(\mathbf{j}) > N. \quad (31)$$

**Theorem 4.1.** *Assume that the frequencies  $\omega = (\omega_a)_{a \in \mathbb{Z}^d}$  are non resonant, and let  $r$  be given. There exists a constant  $C$  depending on  $r$ , such that for all  $N$ , there exists a Hamiltonian*

$$\chi = \sum_{m=3}^r (\chi_m^{(\circ)} + \chi_m^{(i)} + \chi_m^{(ii)}) \quad (32)$$

such that  $\chi_m^{(\circ)} = P[a_m^{(\circ)}]$ ,  $\chi_m^{(i)} = P[a_m^{(i)}]$ ,  $\chi_m^{(ii)} = P[a_m^{(ii)}]$ , contain zero, one and two high modes respectively (that is satisfy (28), (29) and (30) for the number  $N$ ), with coefficients satisfying

$$\|a_m^{(\circ)}\|_{\ell^\infty} + \|a_m^{(i)}\|_{\ell^\infty} + \|a_m^{(ii)}\|_{\ell^\infty} \leq CN^{r\alpha(r)}, \quad (33)$$

and such that

$$H \circ \Phi_\chi^1 = H_2 + \sum_{m=3}^r (Z_m^{(\circ)} + S_m^{(i)} + \tilde{P}_m^{(iii)}) + \tilde{R}_{r+1}, \quad (34)$$

where  $Z_m^{(\circ)} = P[b_m^{(\circ)}]$ ,  $S_m^{(ii)} = S[b_m^{(ii)}]$ ,  $\tilde{P}_m^{(iii)} = P[b_m^{(iii)}]$ , contain zero, two and at least three high modes respectively (that is satisfy (28), (30), (31)), with coefficients satisfying

$$\|b_m^{(\circ)}\|_{\ell^\infty} + \|b_m^{(ii)}\|_{\ell^\infty} + \|b_m^{(iii)}\|_{\ell^\infty} \leq CN^{r\alpha(r)}.$$

Moreover

- $Z_m^{(\circ)}$  contains only resonant monomials, which means that

$$\forall \mathbf{j} \in \mathcal{M}_m \quad \mathbf{j} \notin \mathcal{R}_m \implies (b_m^{(\circ)})_{\mathbf{j}} = 0.$$

- $S_m^{(ii)}$  contains terms that are symmetric in the high modes which means that if  $b_{\mathbf{j}}^{(ii)} \neq 0$  for  $\mathbf{j} \in \mathcal{M}_m$ , the two highest modes are of opposite signs: they are of the form  $(\delta, a)$  and  $(-\delta, b)$  for some  $a$  and  $b \in \mathbb{Z}^d$ .
- The remainder term

$$\tilde{R}_{r+1} = R_{r+1} \circ \Phi_\chi^1 + \int_0^1 (1-s)^{r+1} P[b_{r+1}] \circ \Phi_\chi^s ds \quad (35)$$

where  $P[b_{r+1}]$  defines a homogeneous polynomial of order  $r+1$  with coefficients bounded by  $\|b_{r+1}\|_{\ell^\infty} \leq CN^{r\alpha(r)}$ .

Finally, for all  $s > d/2$  and  $\Phi_\chi^1$  is locally invertible in  $\ell_s^2$  near 0.

*Proof.* The proof is standard and use the non resonant Birkhoff normal form procedure (see [BG06, BDGS07, Gre07]). We follow here the construction made in [FG13]. By using the formal series expansions  $H = H_2 + \sum_{m \geq 3} P_m$ ,  $N = \sum_{m \geq 0} N_m$  and  $\chi = \sum_{m \geq 3} \chi_m$  in homogeneous polynomials, the formal normal form problem is to find  $\chi$  and  $X$  (under normal form) such that

$$H \circ \Phi_\chi^1 = \sum_{k \geq 0} \text{ad}_\chi^k (H_2 + P) = H_2 + X.$$

In the formal series algebra, this problem is equivalent to a sequence of homological equation of the form

$$\forall m \geq 3, \quad \{H_2, \chi_m\} = Q_m - X_m,$$

where  $Q_m$  depends on the function  $P_k$ ,  $\chi_k$  and  $X_k$ ,  $k < m$  previously constructed, and is obtained by iterated Poisson brackets preserving the homogeneity of polynomial and boundedness of coefficients (see (16)). Formula for  $Q_m$  can be found in [FG13], Eq. (3.4).

Now assume that  $Q_m = \sum_{\mathbf{j} \in \mathcal{M}_m} q_{\mathbf{j}} z_{\mathbf{j}}$  is given. For a given  $N$  we can decompose it into terms containing zero, one, two and at least three high modes:  $Q_m = Q_m^{(\circ)} + Q_m^{(i)} + Q_m^{(ii)} + Q_m^{(iii)}$  and a similar decomposition for the coefficients  $q_{\mathbf{j}}$ . The normal form term  $X_m$  is then the sum of the resonant term in  $Q_m^{(i)}$ , the symmetric part of  $Q_m^{(ii)}$  (contributing to the term  $S_m^{(ii)}$ ),



and the term  $Q_m^{(iii)}$  (contributing to the term  $\tilde{P}_m^{(iii)}$ ). By noting that

$$\{H_2, z_{\mathbf{k}}\} = i\Omega(\mathbf{k})z_{\mathbf{k}} \quad \text{with} \quad \Omega(\mathbf{k}) = -\sum_{i=1}^m \delta_i \omega_{a_i}, \quad \mathbf{k} = ((\delta_i, a_i))_{i=1}^m \in \mathcal{M}_m,$$

we then solve the other terms by setting

$$\chi_m = \sum_{\mathbf{j} \in \mathcal{M}_m \setminus \mathcal{R}_m} \frac{q_{\mathbf{j}}^{(o)}}{i\Omega(\mathbf{j})} z_{\mathbf{j}} + \sum_{\mathbf{j} \in \mathcal{M}_m} \frac{q_{\mathbf{j}}^{(i)}}{i\Omega(\mathbf{j})} z_{\mathbf{j}} + \sum_{\mathbf{j} \in \mathcal{M}_m \setminus \mathcal{S}_m} \frac{q_{\mathbf{j}}^{(ii)}}{i\Omega(\mathbf{j})} z_{\mathbf{j}},$$

where  $\mathcal{S}_m$  denote the set of indices with two symmetric high modes. Note that when  $\mathbf{j} \in \mathcal{M}_m \setminus \mathcal{S}_m$ , the two highest modes (larger than  $N$ ) have the same sign, and the denominator  $\Omega(\mathbf{j})$  is controlled by (H3). Similarly, the first term can be controlled using (H1) and the second using (H2). We then observe that we loose a factor  $N^{\alpha(m)}$  after each solution of the homological equation, yielding a bound of order  $N^{r\alpha(r)}$  after  $r$  iterations. Note also that all the operations (solution of the Homological equation and Poisson brackets) preserve homogeneity and the reality of the global Hamiltonians.

It is easy to see that for all  $s > s_0$  for  $z$  small enough (such that  $C_{r,s}N^{r\alpha(r)}\|z\|_s \leq 1$  for some constant  $C_{r,s}$  by using (15)) the flow  $\Phi_\chi^1(z)$  is well defined and locally invertible in  $\ell_s^2$  (is inverse being  $\Phi_\chi^{-1}$ ).

Finally to obtain (34) we use a Taylor expansion of the term  $H \circ \Phi_\chi^t$  for  $t \in (0, 1)$ .  $\square$

Note that for  $\xi \in \ell_s^2$ , we define the *pseudo-actions*:

$$J_a(\xi, \bar{\xi}) = \sum_{\substack{b \in \mathbb{Z}^d \\ \langle a \rangle = \langle b \rangle}} |\xi_b|^2 \quad (\text{and} \quad J_j = \frac{1}{2} \sum_{\substack{\ell \in \mathbb{U}_2 \times \mathbb{Z}^d \\ \langle \ell \rangle = \langle j \rangle}} |z_\ell|^2, \quad j \in \mathbb{U}_2 \times \mathbb{Z}^d). \quad (36)$$

By definition of the resonant set  $\mathcal{R}_m$  (see (12)) and the corresponding resonant monomials (see (13)) we see that for all  $m$ , the normal form terms  $Z_m^{(o)}$  can be written

$$Z_m^{(o)}(\xi, \bar{\xi}) = \sum_{\substack{\mathbf{k} \in \mathcal{R}_{2m} \\ \forall i, \langle k_i \rangle \leq N}} c_{\mathbf{k}} z_{\mathbf{k}} = \sum_{\substack{\mathbf{a}, \mathbf{b} \in (\mathbb{Z}^d)^m, \\ \forall i, \langle a_i \rangle = \langle b_i \rangle \leq N}} c_{\mathbf{a}\mathbf{b}} \prod_{i=1}^m \xi_{a_i} \bar{\xi}_{b_i}, \quad z = (\xi, \bar{\xi}).$$

We notice that a polynomials in normal form commutes with all the *pseudo-actions*:

$$\{Z_m^{(o)}, J_a\} = 0, \quad \forall a \in \mathbb{Z}. \quad (37)$$

## 5. PROOF OF THE MAIN THEOREM

To prove the main Theorem, we will need the following Lemma which controls polynomial vector in mixed Sobolev norms.

**Lemma 5.1.** *For all  $s > s_0$ , there exists a constant  $C$  such that for all  $m \geq 3$ ,  $c \in \ell^\infty(\mathcal{M}_m)$  and  $P[c]$  the polynomial with coefficients  $c$ , then for all  $N$ , the following holds: Assume that  $P[c]$  contains at least  $n$ -th high modes,  $n = 0, 1, 2, 3$ , i.e.  $c_j \neq 0 \implies \mu_n(\mathbf{j}) > N$ , then we have for  $z = (\xi, \bar{\xi})$ ,*

$$\|X_{P[c]}(z)_{\leq N}\|_s \leq \begin{cases} C^m \|z_{\leq N}\|_s^{m-1} & \text{if } n = 0 \\ C^m N^{s-s_0} \|z_{\leq N}\|_s^{m-n-1} \|z_{>N}\|_{s_0}^n & \text{if } n \geq 1, \end{cases} \quad (38)$$

and

$$\|X_{P[c]}(z)_{>N}\|_{s_0} \leq \begin{cases} 0 & \text{if } n = 0, \\ C^m N^{s_0-s} \|z_{\leq N}\|_s^{m-n-1} & \text{if } n = 1, \\ C^m \|z_{\leq N}\|_s^{m-n} \|z_{>N}\|_{s_0}^{n-1} & \text{if } n \geq 2. \end{cases} \quad (39)$$

*Proof.* For  $z = (\xi, \bar{\xi}) \in \ell_s^2$ , we have

$$\|X_{P[c]}(z)_{\leq N}\|_s^2 \leq \sum_{\langle \ell \rangle \leq N} \langle \ell \rangle^{2s} \left| \frac{\partial P[c]}{\partial z_\ell}(z) \right|^2 \leq m^2 \|c\|_{\ell^\infty}^2 \sum_{\langle \ell \rangle \leq N} \langle \ell \rangle^{2s_0} P_\ell(z)^2,$$

where

$$P_\ell(z) = \langle \ell \rangle^{s-s_0} \sum_{\substack{\mathcal{M}(\mathbf{j})=0 \\ \mu_n(\mathbf{j}, \ell) > N}} |z_{j_1}| \cdots |z_{j_{m-1}}|. \quad (40)$$

Let  $\mathbf{j} = (j_1, \dots, j_m)$  be given such that  $\mathcal{M}(\mathbf{j}, \ell) = 0$ . We have

$$\langle \ell \rangle^{s-s_0} \leq (\langle j_1 \rangle + \cdots + \langle j_{m-1} \rangle)^{s-s_0} \leq m^{s-s_0} \langle j_1 \rangle^{s-s_0} \cdots \langle j_{m-1} \rangle^{s-s_0}.$$

If  $P[c]$  contains no high modes, i.e. if  $n = 0$  then we can define  $\tilde{z}_j = \langle j \rangle^{s-s_0} z_j$  and we obtain

$$P_\ell(z) \leq \sum_{\substack{\mathcal{M}(\mathbf{j}, \ell)=0 \\ \mu_1(\mathbf{j}) \leq N}} |\tilde{z}_{j_1}| \cdots |\tilde{z}_{j_{m-1}}|.$$

Now denoting  $f(x) = \sum_{\langle \ell \rangle \leq N} \tilde{z}_\ell e^{i\delta a \cdot x}$  we have

$$\sum_{\substack{\mathcal{M}(\mathbf{j}, \ell)=0 \\ \mu_1(\mathbf{j}) \leq N}} |\tilde{z}_{j_1}| \cdots |\tilde{z}_{j_{m-1}}| = (\widehat{f^{m-1}})_{-\delta a}, \quad \ell = (\delta, a) \in \mathbb{U}_2 \times \mathbb{Z}^d$$

for some constant  $C$  depending on  $m$ , and where if  $g$  a function on the torus, we denote  $g_b = (1/2\pi)^d \int_{\mathbb{T}^d} g(x) e^{-ib \cdot x} dx$  the Fourier transform, for  $b \in \mathbb{Z}^d$ . So we get for some generic constant  $A$  depending on  $s$  and  $s_0$  but independent on  $m$

$$\begin{aligned} \|X_{P[c]}(z)_{\leq N}\|_s^2 &\leq Am^{s-s_0+2} \|c\|_{\ell^\infty}^2 \sum_{a \in \mathbb{Z}^d} \langle a \rangle^{2s_0} |(f^{m-1})_a|^2 \\ &= Am^{s-s_0+2} \|c\|_{\ell^\infty}^2 \|f^{m-1}\|_{H^{s_0}}^2 \leq AC^m \|c\|_{\ell^\infty}^2 \|f\|_{H^{s_0}}^{2m-2}, \end{aligned}$$

where  $\|\cdot\|_{H^{s_0}}$ ,  $s_0 > d/2$  is the usual Sobolev norm on  $\mathbb{T}^d$  equivalent to the  $\ell_{s_0}^2$  norm. Here  $C$  depends on  $s$  and  $s_0$  but not on  $m$ . We then note that  $\|f\|_{H^{s_0}} = \|z_{\leq N}\|_s$  which shows the first equation in (38).

To prove the second in the case  $n \geq 1$ , we simply bound  $\langle \ell \rangle^{s-s_0}$  by  $N^{s-s_0}$  and we obtain

$$P_\ell(y) \leq N^{s-s_0} \sum_{\substack{\mathcal{M}(\mathbf{j}, \ell)=0 \\ \mu_n(\mathbf{j}) > N}} |z_{j_1}| \cdots |z_{j_{m-1}}| \leq (\widehat{f^{m-n} g^n})_{-\delta a}$$

for  $\ell = -\delta a$  with the same notation as before, where

$$f(x) = \sum_{\langle \ell \rangle \leq N} |z_\ell| e^{i\delta a \cdot x} \quad \text{and} \quad g(x) = \sum_{\langle \ell \rangle > N} |z_\ell| e^{i\delta a \cdot x},$$

and we conclude as in the previous case.

To show (39) we use

$$\|X_{P[c]}(z)_{>N}\|_{s_0}^2 \leq m^2 \|c\|_{\ell^\infty}^2 \sum_{\langle \ell \rangle > N} \langle \ell \rangle^{2s_0} P_\ell(z)^2,$$

where

$$P_\ell(z) = \sum_{\substack{\mathcal{M}(\mathbf{j})=0 \\ \mu_n(\mathbf{j}, \ell) > N}} |z_{j_1}| \cdots |z_{j_{m-1}}|, \quad (41)$$

In the case  $n = 1$ , we have no high mode in the sum. However due to the zero momentum condition, there exists one mode greater than  $N/(2m)$ , hence with the same notation as before, we have

$$P_\ell(z) \leq AN^{s_0-s} \sum_{\substack{\mathcal{M}(\mathbf{j})=0 \\ \mu_n(\mathbf{j}, \ell) > N}} |\tilde{z}_{j_1}| \cdots |\tilde{z}_{j_{m-1}}|,$$

and we conclude as before. The proof of the last estimate in (39) is also the same as in the previous case.  $\square$

**Proof of Theorem 2.8.** Let  $\varepsilon > 0$ ,  $s \geq s_0 > d/2$  and  $r$  be given, and  $z(0) \in \ell_{2s}^2$  such that  $\|z(0)\|_{2s} \leq \varepsilon$ . We apply the normal form Theorem (4.1) at the order  $3r$  and set

$$N = N_\varepsilon = \varepsilon^{-\frac{r}{s-s_0}}.$$

Under the hypothesis (18), we have the bound

$$N^{3r\alpha(3r)+d} \leq \varepsilon^{-1/2}. \quad (42)$$

In particular, all the coefficients of the transformation  $\chi$  and of the Hamiltonian  $H \circ \Phi_\chi^1$  are bounded by  $CN^{3r\alpha(3r)} \leq C\varepsilon^{-\frac{1}{2}}$ . Hence(15) and (33), for  $\varepsilon$  small enough (in a way depending on  $r$  and  $s$ ), we have that  $y(0) = \Phi_\chi^{-1}(z(0))$  is in  $\ell_{2s}^2$  and satisfies  $\|y(0)\|_{2s} \leq \frac{5}{4}\varepsilon$ .

This implies in particular that  $\|y(0)_{\leq N}\|_s \leq \frac{5}{4}\varepsilon$  and

$$\begin{aligned} \|y(0)_{>N}\|_{s_0}^2 &= \sum_{\langle j \rangle > N} \langle j \rangle^{2s_0} |y_j|^2 \leq N^{2s_0-4s} \sum_{\langle j \rangle > N} \langle j \rangle^{4s} |y_j|^2 \\ &\leq 4\varepsilon^2 N^{4(s_0-s)} \leq 4\varepsilon^{4r+2}. \end{aligned}$$

Now we need to control the dynamics of  $y(t)$  the solution of the Hamiltonian system associated with the Hamiltonian (34). We define

$$T := \sup\{t > 0 \mid \|y(t)_{\leq N}\|_s \leq \frac{3}{2}\varepsilon \quad \text{and} \quad \|y(t)_{>N}\|_s \leq \varepsilon^{r+1}\}. \quad (43)$$

By classical results for the definition of mild solutions of semi-linear problems in Sobolev spaces with index greater than  $d/2$ ,  $T$  is strictly positive, in particular because  $\|y(0)_{>N}\|_{s_0} \leq 2\varepsilon^{2r+1}$ . Let us prove that if  $t \leq \min(T, \varepsilon^{-\frac{r}{4(s_0+1)}})$  then  $\|y(t)_{\leq N}\|_s \leq \frac{3}{2}\varepsilon$  and  $\|y(t)_{>N}\|_s \leq \varepsilon^{r+1}$ . We will then conclude by a continuity argument that  $T \geq \varepsilon^{-\frac{r}{s_0+1}}$ .

Control of the transformation. In view of Lemma 5.1, under bootstrap hypothesis, the vector field  $X_\chi(y(t))$  satisfies the estimates

$$\|X_\chi(y)_{\leq N}\|_s \leq CN^{3r\alpha(3r)}(\varepsilon^2 + N^{s-s_0}\varepsilon^{r+1}) \leq C\varepsilon^{\frac{3}{2}}$$

as  $N^{s-s_0} = \varepsilon^{-r}$  using (42), and

$$\|X_\chi(y)_{\leq N}\|_{s_0} \leq CN^{3r\alpha(3r)}(N^{s_0-s}\varepsilon^2 + \varepsilon^{r+1}) \leq C\varepsilon^{r+\frac{3}{2}}.$$

Hence this shows that if  $y$  satisfies the estimate (43) then for  $\varepsilon$  small enough, we have for all  $s \in (0, 1)$

$$\|\Phi_\chi^s(y)_{\leq N}\|_s \leq 2\varepsilon \quad \text{and} \quad \|\Phi_\chi^s(y)_{>N}\|_{s_0} \leq 2\varepsilon^{r+1}. \quad (44)$$

Let us write the Hamiltonian (34) as  $H \circ \Phi_\chi^1 = H_2 + Z^{(o)} + S^{(ii)} + \tilde{P}^{(iii)} + \tilde{R}_{3r+1}$  by gathering together the terms with different homogeneity.

Control of the low modes  $y(t)_{\leq N}$ . For  $j \in \mathbb{U}_2 \times \mathbb{Z}^d$ , let  $J_j(t) = J_j(y(t))$  with the definition (36). As  $Z^{(o)}$  and  $H_2$  commute with  $J_a$  for all  $a \in \mathbb{Z}^d$ , we have for  $t \leq \min(T, \varepsilon^{-\frac{r}{s_0+1}})$  and  $\langle j \rangle \leq N$ ,

$$\langle j \rangle^{2s} |\dot{J}_j| = \langle j \rangle^{2s} |\{J_j, \tilde{H}\}| \leq \sum_{\langle \ell \rangle = \langle j \rangle} \langle \ell \rangle^s \left| \frac{\partial(S^{(ii)} + \tilde{P}^{(iii)} + \tilde{R}_{3r+1})(z)}{\partial y_{\bar{\ell}}} \right| \langle \ell \rangle^s |y_{\bar{\ell}}|.$$

By summing in  $\langle j \rangle \leq N$ , and using Cauchy-Schwarz inequality, we obtain

$$\sum_{\langle j \rangle \leq N} \langle j \rangle^{2s} |\dot{J}_j| \leq \|X_{S^{(ii)} + \tilde{P}^{(iii)} + \tilde{R}_{2r+1}}(y)_{\leq N}\|_s \|y_{\leq N}\|_s$$

Now under bootstrap hypothesis, we have with the previous Lemma

$$\|X_{S^{(ii)} + \tilde{P}^{(iii)}}(y)_{\leq N}\|_s \leq CN^{3r\alpha(3r)} N^{s-s_0} \varepsilon^{2r+2} \leq C\varepsilon^{r+\frac{3}{2}}.$$

Now in the terms defining the remainder term  $\tilde{R}_{r+1}$ , we see that the terms involve combination of analytic functions with the flow  $\Phi_\chi^s$  preserving the

bootstrap hypothesis. Now in view of Lemma 5.1, by expanding analytic Hamiltonians into convergent sums of homogeneous polynomial, we see that their flow preserve the bootstrap assumption for  $\varepsilon$  small enough (depending on the constant  $C$  of Lemma (5.1)). Hence we have

$$\|X_{\tilde{R}_{3r+1}}(y)_{\leq N}\|_s \leq CN^{3r\alpha(3r)}\varepsilon^{3r}.$$

Hence we obtain

$$\sum_{\langle j \rangle \leq N} \langle j \rangle^{2s} |\dot{J}_j| \leq C(\varepsilon^{3r+1} + \varepsilon^{r+\frac{5}{2}}).$$

Therefore for  $t \leq \min(T, \varepsilon^{-r+1})$  we have

$$\sum_{\langle j \rangle \leq N} \langle j \rangle^{2s} |J_j(t) - J_j(0)| \leq \varepsilon^{\frac{7}{2}}$$

for  $\varepsilon$  small enough. By using estimates on the vector field  $X_\chi$ , we deduce that  $z(t) = \Phi_\chi^1(y(t))$  satisfies (19).

Control of the high modes. The Hamiltonian  $S^{(ii)}$  can be written for  $y = (\xi, \bar{\xi})$ ,

$$S_m^{(ii)}(y) = \sum_{\substack{a, b \in \mathbb{Z}^d \\ \langle a \rangle > N, \langle b \rangle > N}} B_{ab}(y_{\leq N}) \xi_a \bar{\xi}_b, \quad z = (\xi, \bar{\xi}).$$

As the Hamiltonian is real, for all  $y = (\xi, \bar{\xi})$ , we have  $\overline{B_{ab}(y_{\leq N})} = B_{ba}(y_{\leq N})$ . Moreover, we have

$$B_{ab}(y_{\leq N}) = \sum_{m=1}^{2r-2} \sum_{\substack{j \in \mathcal{M}_m \\ \mu_1(j) \leq N \\ \mathcal{M}(j) = a-b}} b_{ab,j} y_j$$

where the coefficients  $b_{ab,j}$  are uniformly bounded by  $CN^{3r\alpha(3r)}$ . Hence for  $s > d/2$  and  $y$  satisfying the bootstrap assumption, we obtain that

$$\forall \langle a \rangle, \langle b \rangle > N \quad |B_{ab}(y_{\leq N})| \leq CN^{3r\alpha(3r)} \|y_{\leq N}\|_s. \quad (45)$$

When writing the dynamics of  $y = (\xi, \bar{\xi})$ , we thus find

$$\dot{\xi}_a = -i\omega_a \xi_a - i \sum_{b \in \mathbb{Z}^d} B_{ab}(y_{\leq N}) \bar{\xi}_b - iQ_a(y), \quad \langle a \rangle > N$$

where  $Q_a = \frac{\partial}{\partial \xi_a} (\tilde{P}^{(iii)} + \tilde{R}_{3r+1})$ . Using the previous Lemma, we thus see that

$$\|Q(y)_{>N}\|_{s_0} \leq CN^{3r\alpha(3r)} (\varepsilon^{2r+2} \varepsilon + \varepsilon^{3r}). \quad (46)$$

So using the fact that the operator  $B_{ab}$  is hermitian, we get for  $t \leq T$  (recall that we assume that  $\|y(0)_{>N}\|_0 \leq \varepsilon^{2r+1}$ )

$$\|y(t)_{>N}\|_0 = 2\|\xi(t)_{>N}\|_0 \leq 2\|\xi(0)_{>N}\|_0 + tC\varepsilon^{2r+\frac{3}{2}} \leq C(1+t)\varepsilon^{2r+1}. \quad (47)$$

Let us define  $D$  the diagonal operator from  $\ell_s^2(\mathbb{Z}_{>N}^d)$  into  $\ell_{s-2}^2(\mathbb{Z}_{>N}^d)$  given by

$$D = \text{diag}(\langle a \rangle^2, \langle a \rangle > N),$$

and let  $A(t) = (B_{ab}(y_{\leq N})_{\langle a \rangle, \langle b \rangle > N})$  be the hermitian operator acting on  $\ell_s^2(\mathbb{Z}_{>N}^d)$ . With the notation  $(\eta, \xi)_{>N} = \sum_{\langle a \rangle > N} \bar{\eta}_a \xi_a$ , we have for  $p \leq s_0$

$$\frac{d}{dt}(\xi, D^p \xi)_{>N} = -i(\xi, [D^p, A(t)]\xi)_{>N} + \text{Im}(Q(y), D^p \xi)_{>N}$$

with for two operators  $A = (A_{ab})_{a, b \in \mathbb{Z}^d}$  and  $B = (B_{ab})_{a, b \in \mathbb{Z}^d}$ , the commutator  $[A, B] = AB - BA$ . Hence by bootstrap hypothesis (see (46))

$$\left| \frac{d}{dt}(\xi, D^p \xi)_{>N} \right| \leq |(\xi, [D^p, A(t)]\xi)_{>N}| + C\varepsilon^{3r+\frac{5}{2}}.$$

Now we note that using the zero momentum condition defining the operator  $B_{ab}(t) := B_{ab}(y(t)_{>N})$ , we have  $B_{ab}(t) = 0$  when  $|a - b| > 3rN$ . Hence we have

$$\begin{aligned} |(\xi, [D^p, A(t)]\xi)_{>N}| &= \left| \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} B_{ab}(y) (\langle a \rangle^{2p} - \langle b \rangle^{2p}) \bar{\xi}_a \xi_b \right| \\ &\leq \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} 2r |B_{ab}(t)| |\langle a \rangle - \langle b \rangle| (\langle a \rangle^{2p-1} + \langle b \rangle^{2p-1}) |\bar{\xi}_a \xi_b|. \end{aligned}$$

Using the bound (45) and the fact that  $|\langle a \rangle - \langle b \rangle| \leq \langle a - b \rangle \leq CN$ ,

$$|(\xi, [D^p, A]\xi)| \leq CN^{3r\alpha(3r)} \varepsilon \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} \langle a \rangle^{p-1} \langle b \rangle^p |\bar{\xi}_a \xi_b| \quad (48)$$

where we used that for  $|a - b| \leq 3rN$  we have  $\langle a \rangle \leq \langle b \rangle + 3rN$ . Now we have by Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} \langle a \rangle^{p-1} \langle b \rangle^p |\bar{\xi}_a \xi_b| &\leq \\ &\left( \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} \langle a \rangle^{2p-2} |\bar{\xi}_a|^2 \right)^{\frac{1}{2}} \left( \sum_{\substack{\langle a \rangle, \langle b \rangle \geq N \\ |a-b| \leq 3rN}} \langle b \rangle^{2p} |\xi_b|^2 \right)^{\frac{1}{2}} \leq CN^d \|y_{>N}\|_{p-1} \|y_{>N}\|_p \end{aligned}$$

where we used that for  $|a - b| \leq 3rN$  we have  $\langle a \rangle \leq \langle b \rangle + 3rN$  with  $a \in \mathbb{Z}^d$ . Thus, using (42), we obtain

$$\frac{d}{dt} \|y(t)_{>N}\|_p^2 \leq C \|y(t)_{>N}\|_{p-1} \|y(t)_{>N}\|_p + C\varepsilon^{3r+\frac{5}{2}}. \quad (49)$$

Now we recall a *small* lemma proved in [FG13]:

**Lemma 5.2.** *let  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  a continuous function, and  $x : \mathbb{R} \rightarrow \mathbb{R}_+$  a differentiable function satisfying the inequality*

$$\forall t \in \mathbb{R}, \quad \frac{d}{dt}x(t) \leq 2f(t)\sqrt{x(t)}.$$

*Then we have the estimate*

$$\forall t \in \mathbb{R}, \quad \sqrt{x(t)} \leq \sqrt{x(0)} + \int_0^t f(s) ds.$$

Let us use this lemma to prove by induction that for  $p = 0, \dots, s_0$

$$\|y(t)_{>N}\|_p \leq C\varepsilon^{2r+1}(1 + t^{p+1}). \quad (50)$$

This estimate holds true for  $p = 0$  and if it holds true up  $p - 1$  then by (49)

$$\frac{d}{dt}\|y(t)_{>N}\|_p^2 \leq C(1 + t^p)\varepsilon^{2r+1}\|y(t)_{>N}\|_p + C\varepsilon^{3r+\frac{5}{2}}.$$

So if we denote  $x(t) = \|y(t)_{>N}\|_p^2 + \varepsilon^{3r+\frac{5}{2}}$  we have

$$\frac{d}{dt}x(t) \leq C(1 + t^p)\varepsilon^{2r+1}\sqrt{x(t)}$$

and therefore, applying Lemma 5.2 we get (recall that  $\|y(0)_{>N}\|_{s_0} \leq \varepsilon^{2r+1}$ )

$$\|y(t)_{>N}\|_p \leq \sqrt{\|y(0)_{>N}\|_p^2 + \varepsilon^{3r+\frac{5}{2}}} + C\varepsilon^{2r+1}(t + t^{p+1}) \leq C\varepsilon^{2r+1}(1 + t^{p+1}).$$

Thus (50) holds true up to  $p = s_0$ :

$$\|y(t)_{>N}\|_{s_0} \leq C\varepsilon^{2r+1}(1 + t^{s_0+1})$$

and hence for  $t \leq \min(T, \varepsilon^{-\frac{r}{(s_0+1)}})$  and  $\varepsilon$  small enough

$$\|y(t)_{>N}\|_{s_0} \leq \varepsilon^{r+1}.$$

Hence by continuity argument  $T \geq \varepsilon^{-\frac{r}{(s_0+1)}}$  which finishes the proof.

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